Combinatorial Spaces

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Abstract. Combinatoriality—the property that obtains when unions of corresponding subsets within tone rows comprise aggregates—takes various forms, following the canonical operations that relate the constituent rows to one another: transposition, inversion, retrograde, and/or retrograde inversion. The mathematical field of combinatorics presents tools to answer such basic questions as: How many combinatorial sets exist in a space of a given size? To how many equivalence classes do they belong? Such enumeration procedures involve various techniques that have prior connections to music theory. In the process of answering these questions, our results reveal further aspects of combinatorial sets. For instance, no combinatorial n-chords are held invariant by a translation operation with an odd index. The set of I-invariant n-chords that are P-combinatorial is equivalent to the set of those that are I-combinatorial, and this set is precisely the set of all-combinatorial n-chords. Such information sheds new light on these intriguing structures.

Keywords: Combinatoriality · Serialism · Combinatorics · Enumeration

1 Introduction

Combinatoriality in serial music takes various forms, following the canonical operations that relate constituent tone rows to one another: prime or transposition (P), inversion (I), retrograde (R), and/or retrograde inversion (RI). Inversional combinatoriality, or I combinatoriality, is of particular historical significance, as it characterizes much of Arnold Schoenberg's twelve-tone music. Among the tone rows in his forty-two twelve-tone compositions, thirty-six (85.7%) use hexachords that produce I combinatoriality. Regarding the basic set of his *Variations for Orchestra*, op. 31, Schoenberg writes [13, p. 116]: "the inversion a fifth below of the first six tones, the antecedent, should not reproduce a repetition of one of these six tones, but should bring forth the hitherto unused six tones of the chromatic scale. Thus, the consequent of the basic set...comprises the tones of this inversion, but, of course, in a different order," as shown here in Fig. 1.

Specifically, the tone row from Schoenberg's op. 31 is combinatorial under the pitch-class operation I_5. To maintain the complement relation between the hexachords, no two pitch classes that relate by I_5 can be present in the same

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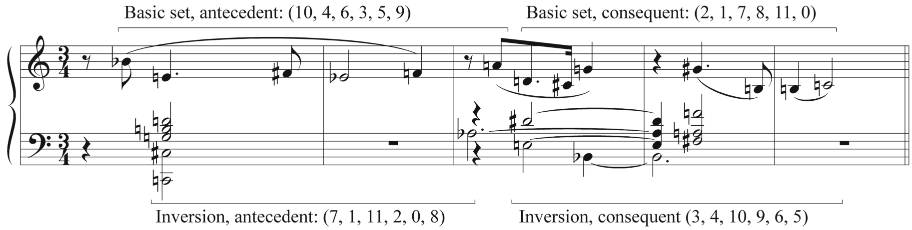

Fig. 1. Hexachordal I combinatoriality in the Thema to Schoenberg's *Variations for Orchestra*, op. 31, mm. 34–38.

hexachord, as those pitch classes map onto one another under that operation. Figure 2 depicts the members of the row's two hexachords as beads in a binary necklace; the white beads represent the pitch classes of the first hexachord and the black beads represent those of the second. We note that the necklace balances across the I_5 axis: for each pitch class c of one hexachord, a corresponding pitch class $d = 11c + 5 \pmod{12}$ from the other hexachord exists directly across the axis. We can represent any partition of the twelve-tone aggregate into I_5-combinatorial hexachords in this way. Therefore, as we find two possible positions relative to the I_5 axis for any one of the six $\{c, d\}$ pairs, we note that there exist $2^6 = 64$ hexachords that are I_5-combinatorial.

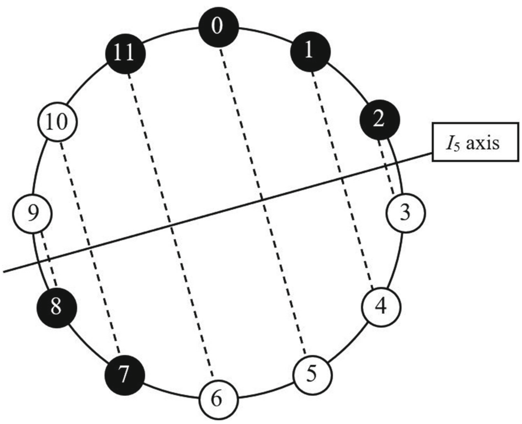

Fig. 2. The tone row of Schoenberg's *Variations for Orchestra*, op. 31, as a binary necklace, balanced across the I_5 axis (first hexachord in white, second hexachord in black).

Whereas we find sixty-four I_x-combinatorial hexachords for each one of the six odd values of x, we note that there exist fewer than $64 \times 6 = 384$ I-combinatorial hexachords in total, as some of these hexachords are combinatorial under more than one I_x operator. Consulting a standard post-tonal textbook, such as [16], we count 348 pitch-class sets that have this property and note that those hexachords belong to nineteen set classes, but how may we arrive at these numbers computationally? Furthermore, composers in the twentieth century make use of additional types of combinatoriality. For example, Milton Babbitt incorporates "all-combinatorial" sets frequently in his compositions. These sets display all four of the canonical combinatorial types: P, I, R, and RI. How might we obtain similar results for these other sorts of combinatorial sets or for combinatorial sets in modular spaces of sizes other than twelve?

The mathematical field of combinatorics presents tools to answer such basic questions as: How many combinatorial sets of any type exist in a space of a given size? What are their symmetries? To how many equivalence classes do they belong? In the process of answering these questions, our results reveal further aspects of combinatorial sets. The general notion of combinatoriality is not limited to serial procedures or to twelve-element aggregates. The defining concepts that it brings together—complementation and equivalence under translation and reflection—are of broad musical interest, as both are interval-preserving when the integrant sets are of the same cardinality (per the Generalized Hexachord Theorem, see [17]). The concepts manifest in combinatoriality apply to numerous musical parameters in addition to pitch, such as rhythmic structure. Further, the procedures we use to study combinatorial structures incorporate various techniques that have prior connections to music theory (e.g., [3,6], and [7]), including the enumeration of serial structures, linking this inquiry with the investigation of other aspects of musical structure. In particular, [4, especially pp. 135–158] presents a detailed enumeration of tone rows in the standard 12-tone chromatic space; further, [4, p. 161] enumerates 12-tone tropes (following [5]) according to different types of combinatoriality (including P-, I-, R-, RI-, and all-combinatoriality), though using different combinatorial techniques from those in the present study.

2 Music-Theoretical and Mathematical Background

In this section, we give basic information that will apply to later sections. Further detailed information on the mathematical theory of musical serialism, particularly from the perspective of combinatorics, can be found in [4].

Let $\mathbb{Z}_{2n}$ be a modular space of elements (typically pitch classes), called the aggregate. Let $\mathscr{S}_{2n}$ be the set of all orderings of the $2n$ elements of that space. $\mathscr{S}_{2n}$ is of size $(2n)!$. Call $S \in \mathscr{S}_{2n}$ a $2n$-tone row, where $(s_0, s_1, ..., s_{2n-1})$ is the particular ordering of elements within the row. G is the canonical group of serial operations with an action on $\mathscr{S}_{2n}$, generated by unit transposition $T_1 := s_i \mapsto s_i + 1$; inversion $I_x := s_i \mapsto (2n-1)s_i + x$, where $x \in \mathbb{Z}_{2n}$; and order-position retrograde $R_x : s_i \mapsto s_{2n-1-i} + x$, where $x \in \mathbb{Z}_{2n}$. G is of order $8n$.

We call an unordered subset $N \subset \mathbb{Z}_{2n}$ an n-chord if $|N| = n$. $\mathscr{N}_{2n}$ is the set of all n-chords in $\mathbb{Z}_{2n}$. We call the orbit of N under the action of the group H of transposition-and-inversion operators, $H(N)$, a set class, following [2]. $\mathscr{N}_{2n}/H$ is the set of all n-chordal set classes under the action of H on $\mathscr{N}_{2n}$, and $\mathscr{N}_{2n}^h$ is the set of n-chords that are stabilized by the element $h \in H$. $\bar{N}$ is the complement of N in $\mathbb{Z}_{2n}$, and we note that $|\bar{N}| = |N|$.

We are concerned with five types of n-chordal combinatoriality: P-, I-, R-, RI-, and all-combinatoriality.

Definition 1. *Given a* $2n$*-tone row* $S \in \mathscr{S}_{2n}$*, where* $N = \{s_0, s_1, ..., s_{n-1}\}$*,* S *has the property* ***n-chordal combinatoriality*** *if and only if there exists some* $g \in G$ *such that* $\bar{N} = \{g(s_0), g(s_1), ..., g(s_{n-1})\}$*. Then, we call* N ***P-combinatorial*** *if there exists some* $x \in \mathbb{Z}_{12}$ *such that* $\bar{N} = T_x(N)$*;* N *is* ***I-combinatorial*** *if there exists some* $x \in \mathbb{Z}_{12}$ *such that* $\bar{N} = I_x(N)$*;* N *is* ***R-combinatorial*** *if there exists some* $x \in \mathbb{Z}_{12}$ *such that* $N = T_x(N)$*; and* N *is* ***RI-combinatorial*** *if there exists some* $x \in \mathbb{Z}_{12}$ *such that* $N = I_x(N)$*. Finally,* N *is* ***all-combinatorial*** *if all four of the preceding statements are true.*

Other forms of combinatoriality involve aggregates formed as unions of $m > 2$ n-chords; in these cases, the relevant space is of size mn. (For example, in $\mathbb{Z}_{12}$, we may use trichordal combinatoriality, in which the aggregate comprises four images of a set of cardinality $n = 3$; see [14,15]). In this study, however, we consider only the special case of n-chordal combinatoriality that results from the unions of two n-chords; hence, we observe that such n-chordal combinatoriality obtains only in spaces with even-parity size.

As all $2n$-tone rows are trivially combinatorial under order-position retrograde without transposition R_0 [1, p. 91], we note that the full set of combinatorial n-chords in $\mathbb{Z}_{2n}$ is equivalent to $\mathscr{N}_{2n}$ itself. Therefore, we distinguish between particular subsets of $\mathscr{N}_{2n}$: the subsets of P-combinatorial n-chords ($\mathscr{N}_{2n(P)}$), I-combinatorial n-chords ($\mathscr{N}_{2n(I)}$), R-combinatorial n-chords ($\mathscr{N}_{2n(R)}$), RI- combinatorial n-chords ($\mathscr{N}_{2n(RI)}$), and all-combinatorial n-chords ($\mathscr{N}_{2n(\text{all})}$).

Our enumeration incorporates various standard results from the mathematical fields of combinatorics, group theory, and number theory. Many of our formulae use powers of 2, which we use to count binary strings, as in our example of the 2^6 I_5-combinatorial hexachords as binary necklaces in Sect. 1 above. Concerning the powers of 2, certain of our results use the 2-adic order of n.

Definition 2. *Given a prime number* p*, the* ***p-adic order*** *of the integer* n *is the highest exponent* ν_p *such that* $p^{\nu_p} \mid n$*. (If* $p^{\nu_p} \nmid n$*, then* $\nu_p = 0$*, since* $p^0 = 1$*.)*

The formula n-choose-k counts the number of all k-subsets of an n-set.

Definition 3. ***Binomial coefficient.***

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

We use the Möbius μ-function, which eliminates redundancies in reckoning the sizes of various sets of combinatorial n-chords by incorporating 0 and -1 among its three coefficients as potential multipliers.

Definition 4. ***Möbius μ-function.***

$$\mu(n) = \begin{cases} 1, & \textit{if } n = 1 \\ 0, & \textit{if } n \textit{ has a square prime factor} \\ (-1)^r, & \textit{if } n \textit{ has } r \textit{ distinct prime factors.} \end{cases}$$

Moreover, we use the Cauchy-Frobenius Lemma (Lemma 1) to determine numbers of orbits in the action of a finite group G on a finite set S (e.g., music-theoretical set classes). Here, S^g is the set of all elements in S that are stabilized by $g \in G$.

Lemma 1. *Cauchy-Frobenius*

$$|S/G| = \frac{1}{|G|} \sum_{g \in G} |S^g|$$

Finally, we introduce two theorems that relate to each of the cases in the next section.

Theorem 1. $\left|\mathscr{N}_{2n}^{T_{2x}}\right| = \left|\mathscr{N}_{2gcd(n,x)}\right|.$

Proof. As a cyclic group, the action of the group generated by T_{2x} on $\mathbb{Z}_{2n}$, $x \in \mathbb{Z}_{2n}$, partitions $\mathbb{Z}_{2n}$ into $2\gcd(x,n)$ orbits of size $n/\gcd(x,n)$. Then, an n-set $N \subseteq \mathbb{Z}_{2n}$ is stabilized by T_{2x} if and only if N is formed by a union of $\gcd(x,n)$ of these orbits. Hence, we find $\binom{2\gcd(x,n)}{\gcd(x/n)}$ possibilities for a T_{2x}-symmetrical N, which is equivalent to $\left|\mathscr{N}_{2\gcd(n,x)}\right|$, as any $\mathscr{N}_{2n}$ contains $\binom{2n}{n}$ n-chords by definition. □

Theorem 2. $\left|\mathscr{N}_{2n}^{T_{2x+1}}\right| = 0.$

Proof. As a cyclic group, the action of the group generated by T_{2x+1} on $\mathbb{Z}_{2n}$, $x \in \mathbb{Z}_{2n}$, partitions $\mathbb{Z}_{2n}$ into $\gcd(2x+1, 2n)$ orbits of size $2n/\gcd(2x+1, 2n)$. Then, an n-set $N \subseteq \mathbb{Z}_{2n}$ is stabilized by T_{2x+1} if and only if N is formed by a union of $\gcd(2x+1, 2n)/2$ of these orbits. However, as $\gcd(2x+1, 2n)/2$ is odd, it is not possible to write N as the union of $\gcd(2x+1, 2n)/2$ orbits. □

3 Results and Applications

3.1 *I* Combinatoriality

We begin with I combinatoriality, which is the most commonly studied type because of Schoenberg's frequent incorporation of it. As such, it serves as a useful introduction to our applications. We noted in Sect. 1 that $2^6 = 64$ I-combinatorial hexachords exist for each of the six odd-indexed inversion operators in $\mathbb{Z}_{12}$, yet we find fewer than $64 \cdot 6 = 384$ I-combinatorial hexachords in total, and that the reason for this discrepancy is the fact that certain hexachords are combinatorial under several different I_x operators. The same situation exists in any space $\mathbb{Z}_{2n}$. First, I combinatoriality is possible only under inversion operators with odd indices, as even-indexed inversion operators always hold two elements of $\mathbb{Z}_{2n}$ invariant (for instance, see [16, p. 316]). Then, we find 2^n I-combinatorial n-chords for any odd-indexed inversion operator I_{2x+1}—hence, $\left|\mathscr{N}_{2n(I)}\right| \leq 2^n n$—but, again, certain of these n-chords are combinatorial under more than one inversion operator.

Ultimately, per Lemma 1, to reckon the number of set classes to which the members of the set of $\mathscr{N}_{2n(I)}$ belong, we need to determine how many I-combinatorial n-chords are stabilized by each member of the transposition-and-inversion group H. (The set classes are orbits in the action of H on $\mathscr{N}_{2n(I)}$.) The following equation, derived from [9], which counts the number of $2n$-bead balanced binary strings that are rotationally equivalent to reversed complement, determines the number of I-combinatorial N-chords that are stabilized by even-indexed transposition operators, including T_0.

$$\left|\mathscr{N}_{2n(I)}^{T_{2x}}\right| = \sum_{j|\gcd(x,n)} \sum_{k|j} \mu(k) 2^{j/k} j \tag{1}$$

As with our observation in Sect. 1 that the number of I-combinatorial n-chords that are stabilized by any one particular I_{2x+1} operator is a power of 2, a power of 2 serves also as the basis of Eq. 1. Then, the Möbius μ-function (Definition 4) eliminates redundancies from n-chords that are combinatorial under multiple values of I_{2x+1}. As an example, the following application illustrates the numbers of I-combinatorial hexachords in $\mathbb{Z}_{12}$ that are stabilized by the identity element $T_{2x=0}$.

- For $j = 6$:

$$\begin{aligned} k = 1 &: (1 \cdot 2^6) \cdot 6 &&= 384 \\ k = 2 &: (-1 \cdot 2^3) \cdot 6 &&= -48 \\ k = 3 &: (-1 \cdot 2^2) \cdot 6 &&= -24 \\ k = 6 &: (1 \cdot 2^1) \cdot 6 &&= \frac{+12}{324} \end{aligned}$$

- For $j = 3$:

$$\begin{aligned} k = 1 &: (1 \cdot 2^3) \cdot 3 &&= 24 \\ k = 3 &: (-1 \cdot 2^1) \cdot 3 &&= \frac{-6}{18} \end{aligned}$$

- For $j = 2$:

$$\begin{aligned} k = 1 &: (1 \cdot 2^2) \cdot 2 &&= 8 \\ k = 2 &: (-1 \cdot 2^1) \cdot 2 &&= \frac{-4}{4} \end{aligned}$$

– For $j = 1$:

$$k = 1 : (1 \cdot 2^1) \cdot 1 \quad = \quad 2$$

It yields 324 T_0-symmetric hexachords that are combinatorial under precisely one inversion operator, eighteen hexachords that are combinatorial under two, four hexachords that are combinatorial under three, and two hexachords that are combinatorial under all six odd-indexed inversion operators, for a total of 348, the size of $\mathscr{N}_{12(I)}$. In this way, we may determine the numbers of I-combinatorial n-chords that are stabilized by any other even-indexed transposition operator.

The next two equations determine the number of I-combinatorial n-chords that are stabilized by inversion operators with even and odd indices, respectively.

$$\left|\mathscr{N}_{2n(I)}^{I_{2x}}\right| = 2^{((n/2^{\nu_2(n)})+1)/2} \tag{2}$$

$$\left|\mathscr{N}_{2n(I)}^{I_{2x+1}}\right| = 2\alpha(n) - \left|\mathscr{N}_{2n(I)}^{I_{2x}}\right|, \tag{3}$$

$$\text{where } \begin{cases} \alpha(0) & = 1 \\ \alpha(2n) & = \alpha(n) + 2^{n-1}, \text{ for } n > 0 \\ \alpha(2n+1) & = 2^n, \text{ for } n \geq 0 \end{cases}$$

The first equation incorporates the 2-adic order of n (Definition 2). The second uses the α-function [10], which determines the number of $2n$-bead balanced binary necklaces which are equivalent to their reverse, complement, and reversed complement. In this case, we note that $2\alpha(n)$ counts the total number of I-combinatorial n-chords that are stabilized by both I_{2x} and I_{2x+1} for a specific value of $x \in \mathbb{Z}_n$, so it is necessary to subtract the number of I-combinatorial n-chords that are stabilized by the even-indexed inversion operator I_{2x} to determine the number of those stabilized by an odd-indexed inversion operator. For instance, given $n = 6$, Eq. 2 yields four I-combinatorial hexachords that are stabilized by an even-indexed inversion operator I_{2x}. For odd-indexed inversions, Eq. 3 yields eight hexachords for I_{2x+1}, as $\alpha(6) = 6$; hence, $2\alpha(6) - \left|\mathscr{N}_{12(I)}^{I_{2x}}\right| = 8$.

Table 1 presents a summary of all the values for stabilized hexachords in the familiar example of $n = 6$ in $\mathbb{Z}_{12}$. Thus, by Lemma 1, the number of set classes to which the members of the set $\mathscr{N}_{12(I)}$ belong is the average number of hexachords stabilized by twenty-four members of the transposition and inversion group H, or nineteen (see Eq. 4).

$$\frac{348 + (2 \cdot 2) + (6 \cdot 2) + 20 + (4 \cdot 6) + (8 \cdot 6)}{24} = 19 \tag{4}$$

Finally, Table 2 shows the results of applying this enumeration to cases in which $n \leq 12$.

Table 1. Sizes of $\mathscr{N}^{h}_{12(I)}$ for each member $h \in H$.

$\left\|\mathscr{N}^{T_0}_{12(I)}\right\| = 348$	$\left\|\mathscr{N}^{T_1}_{12(I)}\right\| = 0$	$\left\|\mathscr{N}^{I_0}_{12(I)}\right\| = 4$	$\left\|\mathscr{N}^{I_1}_{12(I)}\right\| = 8$
$\left\|\mathscr{N}^{T_2}_{12(I)}\right\| = 2$	$\left\|\mathscr{N}^{T_3}_{12(I)}\right\| = 0$	$\left\|\mathscr{N}^{I_2}_{12(I)}\right\| = 4$	$\left\|\mathscr{N}^{I_3}_{12(I)}\right\| = 8$
$\left\|\mathscr{N}^{T_4}_{12(I)}\right\| = 6$	$\left\|\mathscr{N}^{T_5}_{12(I)}\right\| = 0$	$\left\|\mathscr{N}^{I_4}_{12(I)}\right\| = 4$	$\left\|\mathscr{N}^{I_5}_{12(I)}\right\| = 8$
$\left\|\mathscr{N}^{T_6}_{12(I)}\right\| = 20$	$\left\|\mathscr{N}^{T_7}_{12(I)}\right\| = 0$	$\left\|\mathscr{N}^{I_6}_{12(I)}\right\| = 4$	$\left\|\mathscr{N}^{I_7}_{12(I)}\right\| = 8$
$\left\|\mathscr{N}^{T_8}_{12(I)}\right\| = 6$	$\left\|\mathscr{N}^{T_9}_{12(I)}\right\| = 0$	$\left\|\mathscr{N}^{I_8}_{12(I)}\right\| = 4$	$\left\|\mathscr{N}^{I_9}_{12(I)}\right\| = 8$
$\left\|\mathscr{N}^{T_{10}}_{12(I)}\right\| = 2$	$\left\|\mathscr{N}^{T_{11}}_{12(I)}\right\| = 0$	$\left\|\mathscr{N}^{I_{10}}_{12(I)}\right\| = 4$	$\left\|\mathscr{N}^{I_{11}}_{12(I)}\right\| = 8$

Table 2. Numbers of I-combinatorial n-chords and their set classes in spaces $\mathbb{Z}_{2n}$, $n \leq 12$.

Space	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_6$	$\mathbb{Z}_8$	$\mathbb{Z}_{10}$	$\mathbb{Z}_{12}$	$\mathbb{Z}_{14}$	$\mathbb{Z}_{16}$	$\mathbb{Z}_{18}$	$\mathbb{Z}_{20}$	$\mathbb{Z}_{22}$	$\mathbb{Z}_{24}$
$n =$	1	2	3	4	5	**6**	7	8	9	10	11	12
$\left\|\mathscr{N}_{2n(I)}\right\|$	2	6	20	54	152	**348**	884	1974	4556	10056	22508	48636
$\left\|\mathscr{N}_{2n(I)}/H\right\|$	1	2	3	6	10	**19**	36	70	136	266	528	1043

3.2 *P* Combinatoriality

P combinatoriality results when the union of two n-chords that relate by transposition form an aggregate. Hence, there exists some value(s) of $x \in \mathbb{Z}_{2n}$ for which $T_x(N) = \bar{N}$. As with I-combinatorial n-chords, our enumeration of P-combinatorial n-chords derives from the numbers of n-chords that are stabilized by various members of the transposition-and-inversion group. Equation 5 gives the number of n-chords that are stabilized by an even-indexed transposition operator T_{2x}. It incorporates the β-function [8], which determines the number of $2n$-bead balanced binary strings that are rotationally equivalent to their complement.

$$\left|\mathscr{N}^{T_{2x}}_{2n(P)}\right| = \beta(\gcd(n, x)), \tag{5}$$

$$\text{where } \begin{cases} \beta(0) & = 1 \\ \beta(2n) & = \beta(n) + 2^{2n}, \text{ for } n > 0 \\ \beta(2n+1) & = 2^{2n+1}, \text{ for } n \geq 0 \end{cases}$$

For instance, as $\gcd(6, 0) = 6$ and $\beta(6) = 72$, we use Eq. 5 to determine that there exist 72 P-combinatorial hexachords in $\mathbb{Z}_{12}$ that are stabilized by the identity element T_0.

Regarding the numbers of P-combinatorial n-chords that are stabilized by inversion operators with even and odd indices, we note the following result, which accounts for both cases.

Theorem 3. $\mathscr{N}^{I_x}_{2n(P)} = \mathscr{N}^{I_x}_{2n(I)}$

Proof. Assume that $I_y(N) = \bar{N}$ for some $y \in \mathbb{Z}_{2n}$. Then, by definition, there exists some inversion operation I_{2x+1}, $x \in \mathbb{Z}_{2n}$, such that

$$\begin{aligned}\bar{N} &= I_{2x+1}(N)\\ &= I_{2x+1}(I_y(N))\\ &= (I_{2x+1}I_y)(N)\\ &= T_{2x+1-y}(N).\end{aligned}$$

As I_y stabilizes N and I_{2x+1} maps N to $\bar{N}$, we observe that $2x + 1 \neq y$. Therefore, $T_{2x+1-y} \neq T_0$, so N is also P-combinatorial (by definition). By the same reasoning, the reverse is true: if N is P-combinatorial, then it is also I-combinatorial. □

Corollary 1. $\mathscr{N}_{2n(all)} = \mathscr{N}^{I_x}_{2n(I)}$

Proof. Every all-combinatorial n-chord must belong to $\mathscr{N}^{I_x}_{2n(I)}$, $x \in \mathbb{Z}_{2n}$, by definition. The members of $\mathscr{N}^{I_x}_{2n(I)}$ are I-combinatorial, also by definition. Theorem 3 determines further that they are P-combinatorial. As all n-chords are trivially R-combinatorial, we may combine these facts to ascertain that the members of $\mathscr{N}^{I_x}_{2n(I)}$ are also RI-combinatorial; hence, they are all-combinatorial. □

For example, we reckon the numbers of P-combinatorial hexachords that are stabilized by the twenty-four elements of the usual transposition-and-inversion group's action on $\mathbb{Z}_{12}$ (see Table 3).

Table 3. Sizes of $\mathscr{N}^{h}_{12(P)}$ for each member $h \in H$.

$\left\|\mathscr{N}^{T_0}_{12(P)}\right\| = 72$	$\left\|\mathscr{N}^{T_1}_{12(P)}\right\| = 0$	$\left\|\mathscr{N}^{I_0}_{12(P)}\right\| = 4$	$\left\|\mathscr{N}^{I_1}_{12(P)}\right\| = 8$
$\left\|\mathscr{N}^{T_2}_{12(P)}\right\| = 2$	$\left\|\mathscr{N}^{T_3}_{12(P)}\right\| = 0$	$\left\|\mathscr{N}^{I_2}_{12(P)}\right\| = 4$	$\left\|\mathscr{N}^{I_3}_{12(P)}\right\| = 8$
$\left\|\mathscr{N}^{T_4}_{12(P)}\right\| = 6$	$\left\|\mathscr{N}^{T_5}_{12(P)}\right\| = 0$	$\left\|\mathscr{N}^{I_4}_{12(P)}\right\| = 4$	$\left\|\mathscr{N}^{I_5}_{12(P)}\right\| = 8$
$\left\|\mathscr{N}^{T_6}_{12(P)}\right\| = 8$	$\left\|\mathscr{N}^{T_7}_{12(P)}\right\| = 0$	$\left\|\mathscr{N}^{I_6}_{12(P)}\right\| = 4$	$\left\|\mathscr{N}^{I_7}_{12(P)}\right\| = 8$
$\left\|\mathscr{N}^{T_8}_{12(P)}\right\| = 6$	$\left\|\mathscr{N}^{T_9}_{12(P)}\right\| = 0$	$\left\|\mathscr{N}^{I_8}_{12(P)}\right\| = 4$	$\left\|\mathscr{N}^{I_9}_{12(P)}\right\| = 8$
$\left\|\mathscr{N}^{T_{10}}_{12(P)}\right\| = 2$	$\left\|\mathscr{N}^{T_{11}}_{12(P)}\right\| = 0$	$\left\|\mathscr{N}^{I_{10}}_{12(P)}\right\| = 4$	$\left\|\mathscr{N}^{I_{11}}_{12(P)}\right\| = 8$

Using Lemma 1, we are able to determine that the 72 P-combinatorial hexachords belong to eight set classes. Accordingly, Table 4 gives the numbers of P-combinatorial n-chords and their set classes for values $n \leq 12$.

Table 4. Numbers of P-combinatorial n-chords and their set classes in spaces of size $\mathbb{Z}_{2n}$, $n \leq 12$.

Space	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_6$	$\mathbb{Z}_8$	$\mathbb{Z}_{10}$	$\mathbb{Z}_{\mathbf{12}}$	$\mathbb{Z}_{14}$	$\mathbb{Z}_{16}$	$\mathbb{Z}_{18}$	$\mathbb{Z}_{20}$	$\mathbb{Z}_{22}$	$\mathbb{Z}_{24}$
$n =$	1	2	3	4	5	**6**	7	8	9	10	11	12
$\lvert \mathscr{N}_{2n(P)} \rvert$	2	6	8	22	32	**72**	128	278	512	1056	2048	4168
$\lvert \mathscr{N}_{2n(P)}/H \rvert$	1	2	2	4	4	**8**	10	20	30	56	94	180

3.3 *R* Combinatoriality

Unlike I and P combinatoriality, in which n-chords map to their complements under some member of the transposition-and-inversion group, R and RI combinatoriality result when some $h \in H$ exists that maps N to itself. Specifically, R combinatoriality occurs when h is a transposition operator. As this situation always obtains for the identity element T_0, we observe that all n-chords are trivially R-combinatorial. Instead of powers of 2—as with Eqs. 1, 2, 3, and 5—the basis for the enumeration of R-combinatorial n-chords is the binomial coefficient. The formula in Eq. 6 for determining the numbers of n-chords stabilized by even-indexed transposition operators brings the binomial coefficient together with the results of Theorem 1.

$$\left| \mathscr{N}_{2n(R)}^{T_{2x}} \right| = \binom{2n/j}{n/j}, \text{ where } j = n/\gcd(n, x) \tag{6}$$

For example, using $x = 0$ in the familiar case of $n = 6$, we find $\binom{12}{6} = 924$ R-combinatorial hexachords that are stabilized by T_0.

The numbers of R-combinatorial n-chords that are stabilized by inversion operators derive from the binomial coefficient as well. However, unlike the formula for determining numbers of n-chords stabilized by transposition operators, the formulae in Eqs. 7 and 8 differentiate between even and odd values of n.

$$\left| \mathscr{N}_{2n(R)}^{I_{2x}} \right| = \begin{cases} \binom{n-1}{n/2} + \binom{n-1}{(n/2)-1}, & \text{if } 2 \mid \text{ n} \\ 2\binom{n-1}{(n-1)/2}, & \text{if } 2 \nmid n \end{cases} \tag{7}$$

$$\left| \mathscr{N}_{2n(R)}^{I_{2x+1}} \right| = \begin{cases} \binom{n}{n/2}, & \text{if } 2 \mid n \\ 0, & \text{if } 2 \nmid n \end{cases} \tag{8}$$

An outline of a simple proof follows. Equations 7 and 8 present four cases: (1) I_{2x} with $2 \mid n$, (2) I_{2x+1} with $2 \mid n$, (3) I_{2x} with $2 \nmid n$, and (4) I_{2x+1} with $2 \nmid n$, which we take in turn.

1. For any even-indexed inversion in a space of size $2n$, where n is even, the axis of reflection runs through two fixed points: y and $y + n$. Hence, the inversionally symmetrical n-chord may exclude both these points as members, in which case there exist $n - 1$ points on either side of the axis from which to choose one half, $n/2$, of the elements of the n-chord; or the n-chord may

include both these points as members, in which case there exist $n-1$ points on either side of the axis from which to choose one less than one half, $(n/2)-1$, of the elements of the n-chord.
2. For any odd-indexed inversion in a space of size $2n$, where n is even, the axis of reflection fixes no points. Hence, there exist n points on either side of the axis from which to choose one half, $n/2$ of the elements of the inversionally symmetrical n-chord.
3. For any even-indexed inversion in a space of size $2n$, where n is odd, the axis of reflection runs through two fixed points: y and $y+n$. Hence, the inversionally symmetrical n-chord must include one or the other—but not both—of these points as members. In either of the two cases, there exist $n-1$ points on either side of the axis from which to choose one half of the remaining points, $(n-1)/2$, of the elements of the n-chord.
4. For any odd-indexed inversion in a space of size $2n$, where n is odd, the axis of reflection fixes no points. However, for the n-chord to be inversionally symmetrical, one point must be fixed. Hence, the situation fails.

Along with the Lemma 1, the above equations enable us to determine the numbers of set-classes to which the set of R-combinatorial n-chords belong. Table 5 presents this information for cases $n \leq 12$.

Table 5. Numbers of R-combinatorial n-chords and their set classes in spaces of size $\mathbb{Z}_{2n}$, $n \leq 12$.

Space	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_6$	$\mathbb{Z}_8$	$\mathbb{Z}_{10}$	$\mathbb{Z}_{12}$	$\mathbb{Z}_{14}$	$\mathbb{Z}_{16}$	$\mathbb{Z}_{18}$	$\mathbb{Z}_{20}$	$\mathbb{Z}_{22}$	$\mathbb{Z}_{24}$
$n=$	1	2	3	4	5	**6**	7	8	9	10	11	12
$\lvert\mathscr{N}_{2n(R)}\rvert$	2	6	20	70	252	**924**	3432	12870	48620	184756	705432	2704156
$\lvert\mathscr{N}_{2n(R)}/H\rvert$	1	2	3	8	16	**50**	133	440	1387	4752	16159	56822

3.4 *RI* Combinatoriality

RI combinatoriality results when an n-chord maps onto itself under an inversion operation. As with R combinatoriality, we determine the numbers of RI-combinatorial n-chords by using the binomial coefficient. Moreover, the formula for reckoning the number of RI-combinatorial n-chords that are stabilized by even-indexed transposition operators (equivalent to the number of $2n$-bead balanced binary necklaces that are equivalent to their reverse [11]) also incorporates the μ-function, which again eliminates redundancies.

$$\left|\mathscr{N}_{2n(RI)}^{T_{2x}}\right| = \sum_{j|\gcd(x,n)} \sum_{k|j} \mu(k)wj, \tag{9}$$

$$\text{where } w = \begin{cases} \binom{j/k}{j/2k} + \binom{(j/k)-1}{j/2k} + \binom{(j/k)-1}{(j/2k)-1}, & \text{if } 2 \mid j/k \\ 2\binom{(j/k)-1}{((j/k)-1)/2}, & \text{if } 2 \nmid j/k \end{cases}$$

As was the case with I and P combinatorialities, the set of I_x-stabilized RI-combinatorial n-chords in any particular space $\mathbb{Z}_{2n}$ is the same as it is for R-combinatoriality.

Theorem 4. $\mathscr{N}_{2n(RI)}^{I_x} = \mathscr{N}_{2n(R)}^{I_x}$

Proof. We note that any n-chord is R-combinatorial. Therefore, an n-chord is RI-combinatorial, if and only if it is stabilized by I_x for some $x \in \mathbb{Z}_{2n}$. □

Using these results, we are now ready to apply Lemma 1 to determine the number of set classes to which the members of $\mathscr{N}_{2n(RI)}$ belong. Table 6 provides sample results for $n \leq 12$.

Table 6. Numbers of RI-combinatorial n-chords and their set classes in spaces of size $\mathbb{Z}_{2n}$, $n \leq 12$.

Space	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_6$	$\mathbb{Z}_8$	$\mathbb{Z}_{10}$	$\mathbb{Z}_{12}$	$\mathbb{Z}_{14}$	$\mathbb{Z}_{16}$	$\mathbb{Z}_{18}$	$\mathbb{Z}_{20}$	$\mathbb{Z}_{22}$	$\mathbb{Z}_{24}$
$n =$	1	2	3	4	5	**6**	7	8	9	10	11	12
$\lvert\mathscr{N}_{2n(RI)}\rvert$	2	6	8	38	52	**216**	268	1062	1232	4956	5524	21848
$\lvert\mathscr{N}_{2n(RI)}/H\rvert$	1	2	2	6	6	**20**	20	70	70	252	252	924

4 Conclusions

In this study, we have examined combinatorial n-chords using techniques from the mathematical fields of combinatorics, number theory, and group theory. Specifically, we have enumerated the sets of P-, I-, R-, and RI-combinatorial n-chords and their set classes. In the process, our results reveal further aspects of combinatorial sets. For instance, we note that the number of T_{2x}-symmetric combinatorial n-chords in a space of size $2n$ is equivalent to the total number of combinatorial n-chords in a space of size $2\gcd(n, x)$ (Theorem 1). No combinatorial n-chords are held invariant by a translation operation with an odd index (Theorem 2). The set of I-invariant n-chords that are P-combinatorial is equivalent to the set of those that are I-combinatorial (Theorem 3), and this set is precisely the set of all-combinatorial n-chords (Corollary 1). Similarly, the set of I-invariant n-chords that are R-combinatorial is equivalent to the set of those that are RI-combinatorial (Theorem 4).

Several avenues exist for future work on combinatorial n-chords and their spaces. Whereas this study is limited to aggregates formed from unions of two n-chords, its methodology could be extended to study aggregate formation that results from unions of $m > 2$ P-, I-, R-, and RI-combinatorial n-chords. Further, we can study sets of combinatorial n-chords from other mathematical perspectives that have yielded significant music-theoretical results, such as the Discrete Fourier Transform or algebraic topology and geometry. Such investigations will continue to shed new light on these intriguing structures.

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