Quantum-Musical Explorations on $\mathbb{Z}_n$

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Abstract. Motivated through recent applications of quantum theory to the music-theoretical conceptualisation of tonal attraction, the paper recapitulates basic facts about quantum wave functions over the finite configuration space $\mathbb{Z}_n$, and proposes a particular musical application.

After an introduction of position and momentum operators, the Fourier transform as well as the translation and ondulation operators, particular attention is plaid to the Quantum Harmonic Oscillator via its Hamilton operator and its eigenstates. In this setup the time development of chosen wave functions is applied to the control of moving sound sources in a Spatialisation scenario.

Keywords: Quantum theory · Music theory · Pitch class profiles

1 Motivation

A new quantum-theoretical approach to the study of musical tones (c.f. [2,7,9]) motivates the present attempt for an integration of other mathematical approaches into this new line of investigation. These new ideas may possibly open productive theoretical links between statistical approaches to music cognition on the one hand and structural mathematical approaches to music on the other. Up to now connections between these two areas are not yet highly sought-after and both areas suffer from deficiencies, which exhibit a remarkable complementary: Statistical approaches treat histograms of and transition matrices between possible musical events as these were already fully valid models of musical reality, while mathematicians build nice but somewhat empty spaces of musical objects, wherein no events actually happen. Under the quantum perspective pitch class profiles are interpreted as probability density functions of underlying quantum wave functions, which may inhabit the "empty" spaces of the mathematical music theorists. And this entails the possibility to gain explanatory power for the constitution of empirically derived pitch class profiles from these wave functions alongside with the Hermitian and unitary operators acting on them, and last not least from the Schrödinger equation. Needless to say, that these wave functions are not intended to be interpreted in a literal physical way. The wave functions are defined on spaces of musical tones or higher musical objects, not in physical space.

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The starting point for the new quantum-theoretical approach in [9] (and follow up papers [2,7]) is the modelling of *tonal attraction* by means of a suitable match of the Krumhansl-Kessler pitch class profiles. It turns out that the circle-of-fifths ordering of the twelve pitch classes allows to build such a match from a continuous wave function on $\mathbb{R}/\mathbb{Z}$ exemplifying cosine-similarity. The present study is further motivated by a potential conceptual bifurcation within the quantum-theoretical framework. We observed, that there is an alternative possibility to match the Krumhansl-Kessler pitch class profiles, namely by starting from a Gaussian wave function on $\mathbb{R}$, which represents the ground state of a quantum harmonic oscillator.

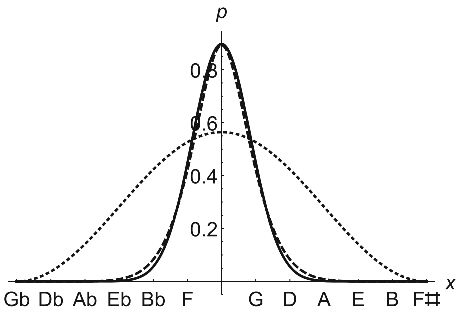

Fig. 1. Three attraction kernels $p(x) = |\psi(x)|^2$ of the Krumhansl-Kessler experimental data for C major (centered at tone C of the quint group), obtained from their respective wave functions $\psi(x)$. Solid: Gaussian wave packet, dashed: deformed cosine similarity, dotted: default cosine similarity.

In search of an analogy to the situation in physics we would view the (continuous) circle or line of fifths in the role of a configuration space for the "position representation" of wave functions ϕ. And consequently the question is on the table what the musical meaning of the associated "momentum representation" might be. We reflect about this question with the awareness that the Fourier Transform, which mediates between the two representations, already plays a productive role in recent approaches to the study of pitch classes and pitch class profiles (see [1,12–16]), for example. But while we deal with the Fourier Transform $\hat{\phi}$ of wave functions ϕ we would categorize the objects of study in these investigations as Fourier Transforms $\widehat{|\phi|^2}$ of probability density functions $|\phi|^2$. A second conceptual difference in these investigations consists in the finite configuration space $\mathbb{Z}_{12}$ as opposed to $\mathbb{R}/\mathbb{Z}$ or $\mathbb{R}$. But this difference is not an obstacle for an integration. The quantization result in [7] actually provides confidence into the suitability of the finite-dimensional approach also from within the quantum approach. Therefore, the most straight-forward first step towards an integration of the pre-established Fourier approach into the new quantum approach consists in the study and musical interpretation of wave functions on $\mathbb{Z}_n$. And particular attention has to be paid to the role of phases. And we approach this project with the idea in mind to encode the parameter of *tone width* together with the parameter of *tone height* (as studied by David Clampitt and the first author in [3]) which is further inspired by Martin Ebeling's proposal to model musical tones on the complex plane [5].

The present poster is intended as a preparatory "mathe-musical warmup" with the purpose to get the Quantum theory on $\mathbb{Z}_7$ and/or $\mathbb{Z}_{12}$ at our fingertips. Here we postpone the search for answers to the motivating questions in favour of a plain sailing musical playground, where every wave function with moderate parameters and its time development for a given Energy operator can be realised and musically explored. In this scenario the parameters of amplitude and phase are interpreted in terms of the loudnesses and angular positions of a cycle of sound sources in a spatialisation scenario.

2 Quantum Theory on $\mathbb{Z}_n$

The mathematical foundations of quantum theory in n dimensions have been thoroughly investigated in recent years. We draw upon [4,6,10,11]. In this section we recapitulate elementary knowledge.

Quantum states are described in terms of wave functions $\psi : \mathbb{Z}_n \to \mathbb{C}$, which we will identify with vectors $\psi \in \mathbb{C}^n$. In the position representation the residue classes $0, ..., n-1 \in \mathbb{Z}_n$ denote positions, while they denote momenta in the momentum representation.

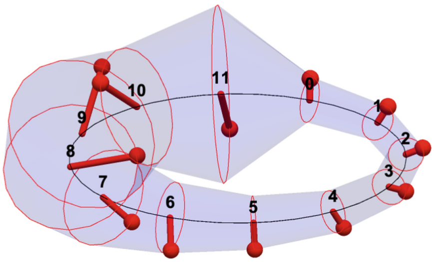

Fig. 2. Representation of a wave function over $\mathbb{Z}_{12}$. The lengths of the twelve needles represent amplitudes and their directions represent phases.

Linear operators $F : \mathbb{C}^n \to \mathbb{C}^n$ are represented through $n \times n$-matrices with complex coefficients, accordingly. The can represent active and passive transformations (i.e. active transformations of the wave functions themselves or passive coordinate transformations of one and the same wave function).

We start with the consideration of the *position operator* Q. Following [4] we define it as a diagonal $n \times n$ matrix Q_a with the n diagonal entries and eigenvalues $\{-\frac{a(n-1)}{2}, \ldots, \frac{a(n-1)}{2}\}$. The indices $j \in \{-\frac{(n-1)}{2}, \ldots, \frac{(n-1)}{2}\}$ are centered around 0 and are integers for odd n and half-integers for even n. The real scaling factor $a > 0$ is a length unit.

$$Q_a = a \cdot \begin{pmatrix} -\frac{n-1}{2} & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & -\frac{n-1}{2}+1 & \dots & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \dots & \frac{n-1}{2}-1 & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 & \frac{n-1}{2} \end{pmatrix}$$

The normalized eigenstates of Q_a are the 'δ-functions" with precisely one non-vanishing coordinate, i.e.

$$\begin{aligned} \varphi_{-\frac{n-1}{2}} &= (1, 0, \dots, 0), \\ \varphi_{-\frac{n-1}{2}+1} &= (0, 1, 0, \dots, 0), \\ &\dots \\ \varphi_0 &= (0, \dots, 0, 1, 0, \dots, 0), \\ &\dots \\ \varphi_{\frac{n-1}{2}} &= (0, \dots, 0, 1). \end{aligned}$$

The exponential $M = exp(\frac{2\pi i}{n} Q)$ is known as the associated *Modulation-* or *Undulation operator*. M is an unitary operator and its n eigenvalues are either the n-th roots of unity or the odd $2n$-th root of unity. The determinant $det(M)$ is either 1 or -1. The latter happens, when n is even and $\frac{n-1}{2}$ the half of an odd number. The Fourier transform mediates between the position representation and the momentum representation of the wave functions, and it is therefore considered to be a passive transformation. Let $\omega(k) = exp(\frac{2\pi ik}{n}), k = 0, \dots, n-1$ denote the n'th root of unity. They form the coefficients of the Fourier transform:

$$F = \frac{1}{\sqrt{n}} \begin{pmatrix} \omega(0) & \omega(0) & \dots & \omega(0) & \dots & \omega(0) & \omega(0) \\ \omega(0) & \omega(-1) & \dots & \omega(-k) & \dots & \omega(2) & \omega(1) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \omega(0) & \omega(-k) & \dots & \omega(-k^2) & \dots & \omega(2k) & \omega(k) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \omega(0) & \omega(2) & \dots & \omega(2k)) & \dots & \omega(-4) & \omega(-2) \\ \omega(0) & \omega(1) & \dots & \omega(k) & \dots & \omega(-2) & \omega(-1) \end{pmatrix}$$

F is a unitary operator, satisfying $F^* = F^{-1}$. Its eigenvalues are $i, -1, -i, 1$. Their multiplicities depend on n and can be characterized in terms of the residue n mod 4 (see [10], p. 273).

The vectors $F \cdot \varphi_k$ are the momentum representations of the position eigenstates. Analogously we have a momentum operator P, and a basis of associated eigenstates $\phi_0, \phi_1, \dots \phi_{n-1}$, which in the momentum representation take the simple form 'δ-functions"

$$F \cdot \phi_0 = (1, 0, \dots, 0),\ F \cdot \phi_1 = (0, 1, 0, \dots, 0), \quad \dots \quad , F \cdot \phi_{n-1} = (0, \dots, 0, 1)$$

and the momentum operator in the momentum representation takes the diagonal form

$$F \cdot P \cdot F^* = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & n-1 \end{pmatrix}.$$

The position representations of the eigenstates of P are the exponential circle functions:

$$\begin{aligned} \phi_0 &= \tfrac{1}{\sqrt{n}}(1, 1, \dots, 1), \\ \phi_1 &= \tfrac{1}{\sqrt{n}}(1, \omega^1, \omega^2, \dots, \omega^{n-1}), \\ &\dots \\ \phi_k &= \tfrac{1}{\sqrt{n}}(1, \omega^k, \omega^{2k}, \dots, \omega^{k(n-1)}), \\ &\dots \\ \phi_{n-1} &= \tfrac{1}{\sqrt{n}}(1, \omega^{-1}, \omega^{-2}, \dots, \omega^1). \end{aligned}$$

The exponential $T = exp(-\frac{2\pi i}{n}P)$ is known as the *Translation operator*. T is a permutation matrix, and hence orthogonal (and hence unitary). Its n eigenvalues are the n-th roots of unity. The determinant $det(T)$ is either 1 (for odd n) and -1 (for even n).

$$T = \begin{pmatrix} 0 & 0 & 0 \dots & 0 & 0 & 1 \\ 1 & 0 & 0 \dots & 0 & 0 & 0 \\ 0 & 1 & 0 \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 \dots & 0 & 1 & 0 \end{pmatrix}.$$

T and M generate the *Heisenberg group*.

3 Exploring the Finite Quantum-Harmonic Oscillator

In the context of an ongoing investigation the authors found a suitable definition for a tonal attraction kernel in terms of a Gaussian wave function (as a possible alternative to the deformed cosine kernel in [9]). This finding brings the quantum harmonic oscillator into the spotlight of interest, whose ground state is a Gaussian. The mentioned investigations assume a continuous configuration space $\mathbb{R}$. But in connection with the already established Fourier approach in music theory it seems worthwhile to explore this important and well-studied physical example also in the finite-dimensional scenario. Although there is no analogue to the Schrödinger Equation, several constructions can be based on the study of Eigenvalues and Eigenfunctions. We start by inspecting the Hamilton operator $H = \frac{1}{2}(P^2 + a^2Q^2)$ with parameter a (abstractly) measuring the impact of the potential energy against the normalised kinetic energy P^2. The excited states $\xi_0, \dots, \xi_{n-1}$ can be obtained as the eigenfunctions of H and they can be ordered in accordance with the raising positive real eigenvalues of H. We observed in the case $n = 7$, that the choice of symmetric position eigenvalues (to both sides of 0) ensures that the excited states are also eigenfunctions of the Fourier transform

F, which in turn motivates the inspection of a finite analogue for the Bargmann transformation (e.g. [8], Sect. 14.4), where the excited states ξ_k are chosen as a basis and are mapped to the associated monomials $z \mapsto z^k : \mathbb{C} \to \mathbb{C}$. Figure 2 shows the first excited state ξ_1 for the case $n = 12$.

The Hamilton-Operator gives rise to the unitary *time evolution operator* $U(t) = exp(-iHt)$ and allows the study and musical exploration of the time developments of individual wave functions. A crucial open problem for their interpretation in the music theoretical context of pitch class profiles ($n = 12$) or scale degree profiles ($n = 7$) is the interpretation of the phases. On the one hand, building on [9] it seems plausible to interpret the pitch class profiles as probability density functions of underlying wave functions. On the other hand, this would imply that the established application of the finite Fourier-Transform to pitch class profiles, is not the quantum-theoretical change of perspective from the position to the momentum representation. While these questions need to be addressed in future investigations, it is useful to explore the finite wave functions and their time developments in practical musical experiments.

An auspicious musical application of the time development of finite quantum wave functions is the control of sound sources in a spatialisation scenario. The dimension n of the wave function is the number of sound sources, which are supposed to move in a horizontal plane. As an illustration I will show some experiments with the Max/MSP library *Spat*[1] in combination with *Mathematica*[2]. The time development of a given wave function is encoded in a textfile and using a *Coll*-Object in connection with a metronome at control rate, the Max/MSP-patch interprets the magnitudes and phases at every time stamp in terms of distances and azimuths of the individual sound sources. The Spatialisateur calculates the resulting outputs for the available arrangement of a circle of loudspeakers. In conjunction with our poster presentation we will demonstrate this scenario through the usage of a binaural synthesis instead of the multichannel version. A five-dimensional application is part of a musical piece (of the first author) entitled *The Backside of the Stroboscope* which is dedicated to Jack Douthett.

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