Formal Structures of a Harmony in the Parabola

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Abstract. We develop a geometric analog of musical harmony from the group law of the affine parabola. First, we associate musical notes and intervals with points of a parabola. Immediately, we can define the usual affine and linear transformations for musical chords in module theory. Subsequently, we show that the actions of the groups T/I in PK-nets, PLR, $UTTs$, and JQZ behave identically to the circle space. Then, we propose to recreate the Planet-4D model, the study of musical distance and the DFT for subsets of points on the parabola. We believe that we have an innovative and motivational perspective to approach the parabola in a musical meaning.

Keywords: Parabola · Group law · Pitch-class set theory · Affine transformations · Neo-Riemannian theory · Music Fourier space

1 Introduction

Due to the cyclical nature of musical objects, the circle is a conventional locus to represent them. However, it is possible to define for a parabola $\mathcal{P}$ the finite ring geometric structure $\mathcal{P}(\mathbb{Z}/n\mathbb{Z})$ that behaves similarly to the classical circular pitch class space. We will see that the harmony of the circle is a kind of base layer to the harmony of the parabola. Thus, inspired by the isomorphic structure $\mathcal{P}(\mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}/n\mathbb{Z}, +)$ proved in [15], we will bijectively associate each point of the affine parabola $y = x^2$ with a musical note or interval of the chromatic scale.

2 The Group Law on the Parabola

Let Λ be a commutative ring with unity. The group law on the parabola $\mathcal{P}(\Lambda) = \{y = x^2 : x, y \in \Lambda\}$ is defined by taking a fixed point as a neutral element (the vertex of the parabola) which we denote by $N = (x_N, x_N^2) = (0, 0)$. Now, let $P = (x_P, x_P^2)$ and $Q = (x_Q, x_Q^2)$ be any two points on the parabola $\mathcal{P}$. The sum $P \oplus Q = R = (x_R, x_R^2)$ is the point of intersection with the parabola of the line parallel to PQ passing through vertex N. Algebraically, the addition of the group of points is given by

$$P \oplus Q = (x_P, x_P^2) + (x_Q, x_Q^2) = (x_P + x_Q, (x_P + x_Q)^2). \quad (1)$$

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The proof of the group axioms in Definition (1) can be found in Lemmermeyer [15, p. 42] (see Shirali [18, pp. 31–32] for a general definition whose neutral element is any point on $\mathcal{P}$).

3 Harmonic Polygons over a Parabola

For abstract musical purposes, we are interested in associating points to notes and intervals of the chromatic scale. Thus, we have $\mathcal{P}(\mathbb{Z}_{12}) = \{(0,0),(1,1),(2,4),(3,9),(4,4),(5,1),(6,0),(7,1),(8,4),(9,9),(10,4),(11,1)\}$.

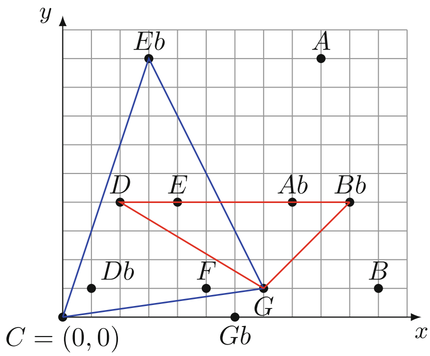


Fig. 1. Neo-Riemannian function PR, transposition by an interval of seven semitones $+(7,1)$, transform C-minor triad (blue polygon) to G-minor triad (red polygon). (Color figure online)

In Fig. 1 we describe harmonic progressions by drawing polygons on the parabola. From a geometric and metaphorical perspective, the abscissa is the base layer of the harmony in the circle $\mathbb{Z}_{12}$, while the ordinate is the harmonic layer which belongs to the parabola. We could express this idea as $(x, y) = (\text{circle}, \text{parabola})$. Therefore, the y-coordinate can be understood as a harmony added or attached to the harmony of the circle that corresponds to the x-coordinate. For example, the C-minor triad in the circle harmony $x = \{C, E\flat, G\}$ has the parabolic layer $y = \{C, A, C\sharp\}$. With this interpretation we also have two sets of intervals: between the notes of the ordinates and, between the layers of the circle and the parabola. Furthermore, this point of view allows the algoritmic composition if we consider affine transformations, e.g. $y = 2x^2 + 1$, where the C-minor triad in the parabola varies to $y = \{C\sharp, G, D\sharp\}$.

4 The Ring and Field Law on the Parabola

For the rest of the analogous definitions we need a richer structure than a group. Thus, to the group $\mathcal{P}(\mathbb{Z}_n)$ we can also equip the structure of a finite ring with unity by the multiplication operation

$$P * Q = (x_P, x_P^2) * (x_Q, x_Q^2) = (x_P \cdot x_Q, x_P^2 \cdot x_Q^2). \tag{2}$$

Proposition 4.1. *The set of points* $(\mathcal{P}(\mathbb{Z}_n), \oplus, *)$ *with addition and multiplication defined by (1) and (2) forms a commutative ring with unity point* $(1, 1)$.

The proof of Proposition 4.1 is straightforward if the projection on the x-axis is established also for the multiplication. Definition (2) follows from the geometric operation [15, p. 56] which equip a field for the parabola over $\mathbb{Q}$, and we are considering the ring $\mathcal{O}_{\mathcal{P}(\mathbb{Q})}$ of such rational field. Take the fixed point $M = (1, 1)$. Let us draw a line between two points P and Q and see the intersection point R with the y-axis. Then, let us choose the intersection $S = P * Q$ of the line through R and M over $\mathcal{P}(\mathbb{Q})$. This field structure can extend the possibility of also modeling continuous spaces from a physical perspective of music if we take $\mathcal{P}(\mathbb{R})$.

5 Parabola over a Module and Affine Transformations

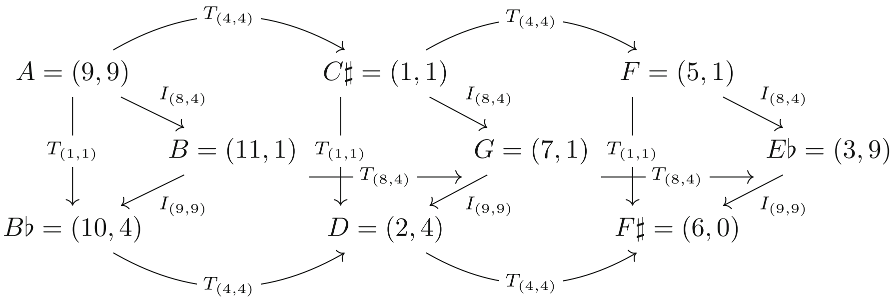
Since $\mathcal{P}(\mathbb{Z}_n)$ forms a commutative ring by Definitions (1) and (2), we can observe it as a module over itself ${}_{\mathcal{P}(\mathbb{Z}_n)}\mathcal{P}(\mathbb{Z}_n)$ or a module with scalar action $[n] \in \mathbb{Z}_n$ given by $\cdot : \Lambda \times \mathcal{P}(\Lambda) \longrightarrow \mathcal{P}(\Lambda)$, $([n], P) \longmapsto [n] \cdot P = \underbrace{P \oplus P \oplus P \oplus \cdots \oplus P}_{\text{[n]-times}}$.

Proposition 5.1. *The points of the parabola over* $\mathbb{Z}_n$ *with addition and scalar action form a* $\mathbb{Z}_n$*-module* $\mathcal{P}(\mathbb{Z}_n)$.

The proof of Proposition 5.1 is straightforward. With this structure on the parabola $\mathcal{P}(\mathbb{Z}_{12})$, we can transform D-major triad into D-aug triad under a morphism that takes $((2, 4), (6, 0), (9, 9)) \mapsto ((2, 4), (6, 0), (10, 4))$, i.e., $\varphi : (P, Q, R) \mapsto (P, Q, [2]Q - P)$. In fact, we can rewrite all affine homomorphisms common in music theory. For instance, following [3], symmetries of consonance and dissonance in counterpoint, e.g. $e^{(2,4)}[5]((3, 9)) = (5, 1)$. If we consider the ring structure, we can represent counterpoint intervals as linear polynomials in $\mathcal{P}(\mathbb{Z}_{12})[X]$, for example a minor third $(7, 1) \oplus (3, 9)X$.

6 Group Actions over Parabolic Music

The musical groups T/I [10], PLR [6,7,9] and $UTTs$ [13] can act in the usual way on sets of points of the musical parabola $\mathcal{P}(\mathbb{Z}_{12})$. Consider first the elements in the T/I group that reveal underlying symmetries between notes of chords in PK-nets [17]. Transposition of a note Q of the parabola is defined as $T_P(Q) = P \oplus Q$, while inversions is given by $I_P(Q) = -Q \oplus P$. Thus, rewriting the musical PK-net analysis in [17, p. 36], we have



In the context of Neo-Riemannian theory, the composition PR acts on the E-minor chord as a $\{(4,4),(7,1),(11,1)\} \mapsto \{(7,1),(10,4),(2,4)\}$. In fact, it can be generalized on a simplicial Tonnetz model [5,22] as it is observed in Fig. 1. Suppose we have an unfolded space $\mathcal{K}[2,4,6]$, then

$$PR \cdot \{(4,4),(6,0),(10,4)\} \mapsto \{(6,0),(8,4),(0,0)\}.$$

Similarly, we can reinterpret the uniform triadic transformation of E-major triad to the A-minor triad:

$$((4,4),+) \xrightarrow{U=(-,(5,1),(10,4))} ((9,4),-).$$

Another group action, in this case non-contextual, that we can use for $\mathcal{P}(\mathbb{Z}_{12})$ is JQZ [14] redefining $J = I_{(7,1)}, Q = I_{(11,1)}$ and, $Z = I_{(4,4)}$. Then,

$$ZJZ \cdot \{(5,1),(8,4),(0,0)\} \mapsto \{(1,1),(5,1),(8,4)\}.$$

On the other hand, it would be interesting to explore algebraic or formal relationships in a three-dimensional Tonnetz [12] or in a Cube Dance [8].

7 Parabolic Planet-8D and Metric

The points of a parabola behave similarly to their numerical analogues as we can observe in the conmutative diagrams below. For a field $K = \mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$ and a ring $R = \mathbb{Z}$, $\mathbb{Z}_n$, following [15, p. 42], the morphisms ϕ_x and ψ_x can be understood as an injection into $y = x^2$, or as a geometrical projection on the x-axis. This properties would allow us to define a metric that emulates voice leading definitions [19] or the related problems for a multi-set metric [11].

$$\begin{array}{ccc} \mathcal{P}(K) & \longrightarrow & \mathcal{P}(R) \\ {\scriptstyle \phi_x}\uparrow & & {\scriptstyle \psi_x}\uparrow \\ K & \longrightarrow & R \end{array} \qquad \begin{array}{ccc} (x, x^2) & \longrightarrow & (x \mod n, x^2 \mod n) \\ \downarrow{\scriptstyle \phi_x^{-1}} & & \downarrow{\scriptstyle \psi_x^{-1}} \\ (x) & \longrightarrow & (x \mod n) \end{array} .$$

Now let us define the following isomorphism through the decomposition into direct sums of groups: $\mathcal{P}(\mathbb{Z}_{12}) \cong \mathbb{Z}_{12} \cong \mathbb{Z}_3 \bigoplus \mathbb{Z}_4 \cong \mathcal{P}(\mathbb{Z}_3) \bigoplus \mathcal{P}(\mathbb{Z}_4)$.

One of the models for visualization of harmonic relationships between pitch classes is Planet-4D [4]. We see the formal possibility of reconstructing the model in a space of four complex dimensions $\mathbb{C}^2 \times \mathbb{C}^2$. We define the same isomorphism of the direct product of cyclic groups and roots of unity but modified over the points of the parabola under multiplication. Thus, we have the isomorphisms

$$\mathcal{P}(\mathbb{Z}_3) \cong \{(1,1),(e^{\frac{2\pi i}{3}},e^{\frac{-2\pi i}{3}}),(e^{\frac{-2\pi i}{3}},e^{\frac{2\pi i}{3}})\}.$$

$$\mathcal{P}(\mathbb{Z}_4) \cong \{(1,1),(i,-1),(-1,1),(-i,-1)\}.$$

Consider the F note associated with the element $(2,1) \in \mathbb{Z}_3 \times \mathbb{Z}_4$. Then, on the parabolic planet we have $(2,1) \cong (2,1,1,1) \cong (e^{\frac{-2\pi i}{3}}, e^{\frac{2\pi i}{3}}, i, -1)$. The bijection of an element of the direct product $\mathcal{P}(\mathbb{Z}_3) \times \mathcal{P}(\mathbb{Z}_4)$ to return to the parabola $\mathcal{P}(\mathbb{Z}_{12})$ is defined in imitation of [2] by sending the points $(P,Q) \mapsto 4P - 3Q$.

8 The Discrete Fourier Transform in a Parabolic World

The importance of Discrete Fourier Transform for the mathematical music theory is due to the fact that it helps to reveal hidden periodic qualities behind subsets of rhythms and scales [1]; even analyze harmony from a geometric perspective [20,21]. The DFT is built over a space of distributions $\mathbb{C}^{\mathbb{Z}_n}$. The analog for points P_i of the parabola is defined by the function $\mathcal{P}(\mathbb{Z}_{12}) \to \mathbb{C}^{2n}$, $f \mapsto \big(f(P_0), f(P_1), \ldots, f(P_{n-1})\big)$. Thus, we define the DFT of a subset of points $P \subset P(\mathbb{Z}_n)$ as the transformation of its characteristic function

$$\mathcal{F}_P = \widehat{f}_k = \sum_{x_P, x_P^2 \in P} \left(e^{\frac{-2\pi i k x_P}{n}}, e^{\frac{-2\pi i k x_P^2}{n}} \right). \tag{3}$$

Note that in the Definition (3) the sum is parabolic. For example, let $P = \{C, E\flat, G\flat, B\flat\flat\}$, the fourth Fourier coefficient of P produces $\widehat{f}_4 = (1,1) + (1,1) + (1,1) + (1,1) = (4,4)$. It is immediate to rewrite the convolution product for a set of points of a parabola, which mathematically describes musical operations such as multiplication of Boulez chords, intervallic content or rhythmic canons. Let f, g be characteristic functions, i.e., $f = (P_0, P_1, \ldots, P_{n-1})$, of subsets P and Q, respectively. The circular convolution is given by

$$f * g(k) = \sum_{n_{PQ}=0}^{n_{PQ}-1} f(k - n_{PQ}) g(n_{PQ}), \tag{4}$$

for all $k \in \mathbb{Z}_{12}$. Note that n_{PQ} is the indexed position of the points in f, g. It follows analogously from (3) and (4) the identity that relates convolution and DFT for each k, $\widehat{f * g}(k) = \widehat{f}(k)\widehat{g}(k)$.

9 Conclusions

We have seen that musical harmony can be represented as polygons in a parabola. Although the geometry of the parabola, seen as a layer on top of the harmony of the circle, operates analogously in analytic approaches, it is possible to extend this perspective to layers defined by other equations maintaining a base formal structure, even with more dimensions, e.g. two ellipses whose integral points are isomorphic to the direct product $\mathbb{Z}_3 \times \mathbb{Z}_4$. In that sense, the arithmetic and geometric aspects of the affine transformations on $\mathcal{P}(\mathbb{Z}_n)$ and other curves can serve as a locus to generate musical ideas embedded in a mathematical environment. With regard to future research, the development of new geometric approaches to music theory can inspire technological and computational advances [16], which might also lead to new software developments for teaching and composition.

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