Computational Analysis of Musical Structures Based on Morphological Filters

Paul Lascabettes[1(✉)], Carlos Agon[1], Moreno Andreatta[1,2], and Isabelle Bloch[3]

[1] STMS-UMR9912, Ircam, Sorbonne Université, CNRS, Ministère de la Culture, Paris, France
{lascabettes,agon,andreatta}@ircam.fr
[2] CNRS-IRMA, ITI CREAA, Université de Strasbourg, Strasbourg, France
andreatta@math.unistra.fr
[3] Sorbonne Université, CNRS, LIP6, Paris, France
isabelle.bloch@sorbonne-universite.fr

Abstract. This paper deals with the computational analysis of musical structures by focusing on the use of morphological filters. We first propose to generalize the notion of melodic contour to a chord sequence with the chord contour, representing some formal intervallic relations between two given chords. By defining a semi-metric, we compute the self-distance matrix of a chord contour sequence. This method allows generating a self-distance matrix for symbolic music representations. Self-distance matrices are used in the analysis of musical structures because blocks around the diagonal provide structural information on a musical piece. The main contribution of this paper comes from the analysis of these matrices based on mathematical morphology. Morphological filters are used to homogenize and detect regions in the self-distance matrices. Specifically, the opening operation has been successfully applied to reveal the blocks around the diagonal because it removes small details such as high local values and reduces all blocks around the diagonal to a zero value. Moreover, by varying the size of the morphological filter, it is possible to detect musical structures at different scales. A large opening filter identifies the main global parts of the piece, while a smaller one finds shorter musical sections. We discuss some examples that demonstrate the usefulness of this approach to detect the structures of a musical piece and its novelty within the field of symbolic music information research.

Keywords: Symbolic music information research · Music structure · Chord contour · Self-distance matrix · Mathematical morphology

1 Introduction

Mathematical morphology is an algebraic theory that analyzes shapes and is mostly used in image analysis and understanding. However, this theory is not

This work was partly supported by the chair of I. Bloch in Artificial Intelligence (Sorbonne Université and SCAI).

M. Montiel et al. (Eds.): MCM 2022, LNAI 13267, pp. 267–278, 2022.
https://doi.org/10.1007/978-3-031-07015-0_22

very common yet in the Mathematics and Music community. The fundamental idea of this theory is to modify the shape, the size or the topological properties of objects with non-linear and non-reversible transformations. Among the few existing applications of mathematical morphology to symbolic representations of music, automatic methods have been developed in Music Information Research community (MIR) to detect approximate occurrences of musical patterns in symbolic music databases [12,13]. In this case, mathematical morphology enables to match almost identical patterns. Moreover, mathematical morphology has also been used to analyze concept lattices based on musical intervals [2], and basic operators of mathematical morphology have been adapted to find a musical meaning, allowing for example extracting harmonic components or to obtain musical transformations [14].

The main contribution of this paper is to propose a novel method, based on mathematical morphology, to extract hierarchical musical structures from the self-distance matrix. This method can be applied to any type of similarity matrix and to any type of data. In our case, the self-distance matrix is computed from symbolic music representations, using a generalization of melodic contour to chord sequences. The purpose of this method is to homogenize the different regions of the self-distance matrix in order to identify the musical structures. Two basic morphological operations, the erosion and dilation, have already been successfully used to detect the repeating patterns longer than a minimum length into a time-lag matrix (a similar representation as the self-distance matrix) [15]. However, rather than identifying segments as in [15], we demonstrate the usefulness of the morphological opening operation in order to identify blocks in the self-distance matrix. This operation eliminates small details, while flatter and homogeneous regions are obtained. In addition, it reduces all the blocks around the diagonal, which correspond to musical sections, to a zero value. We discuss the form to choose when applying an opening filter to extract information from the self-distance matrix: a constant square-shaped filter. But the size can also be adjusted to detect different musical structures. A large opening will identify the global part of the piece while a smaller one will reveal shorter sections. This idea is illustrated by detecting different musical structures in Mozart's Piano Sonata *Alla Turca*.

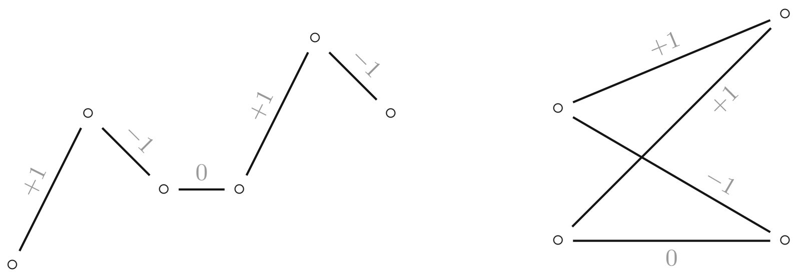
This paper details the above ideas and is organized as follows. Section 2 proposes a method to generate a self-distance matrix from symbolic music representations. We introduce the concept of chord contour (Sect. 2.1) as a generalization of the usual melodic contour, and then define a distance to compute the self-distance matrix of a chord contour sequence (Sect. 2.2). Section 3 describes how to use morphological filters in order to extract musical structures from the self-distance matrix. After providing a short introduction to mathematical morphology (Sect. 3.1) we then demonstrate the relevant applications of the opening operation in order to identify the main blocks of the self-distance matrix (Sect. 3.2). Finally, in Sect. 4, we apply our proposed method and illustrate how morphological filters can be used to extract musical structures at multiple levels of granularity.

2 Creating a Self-distance Matrix from Symbolic Music

2.1 Converting Symbolic Music to Sequence Using Chord Contour

Whether for music perception [6,23], music analysis [1] or music theory [5], *melodic contour* has become a fundamental tool in the MIR community. This tool applies on monophonic structures, i.e., musical phrases or motives in which two notes never sound at once. It is defined by the set of the directions between consecutive pitches of a melody, $+1$ and -1 indicating respectively an ascending and a descending interval. Figure 1a illustrates this idea by representing each note of a melody by a circle in a time/pitch graph. Melodic contour summarizes intervallic information and can be used to compare and classify melodic patterns or to help understand their perception. Considering the importance of melodic contour, it is not surprising that multiple extensions have been proposed. For example, two other contours were defined in [3]: the *strong contour* (melodic contour of only the notes present on the beat) and the *weak contour* (strong contour with extra information if there is a contour variation within the beat). Moreover, it was proposed in [16,20] to observe the directions at longer range, i.e., all the directions between the i^{th} and j^{th} pitches, not only between the i^{th} and $(i+1)^{th}$ pitches as for the usual melodic contour. To this purpose, both works used a matrix representation: *Morris's comparison matrix (COM-matrix)* in [16], and *combinatorial contour matrix* in [20]. In the COM-matrix, the coefficient at position (i,j) is the pitch direction between notes i and j, and for the combinatorial contour matrix this coefficient is $+1$ if the j^{th} note is higher in pitch than the i^{th} note or 0 otherwise. However, these generalizations remain in the monophonic context, and they do not handle musical chords.

We propose a generalization of the melodic contour to chord sequences, i.e., not restricted to note sequences. In the proposed definition, the direction between the pitches of two given chords is no longer a number but a matrix, called *chord contour*. The coefficient (i,j) of the chord contour is the direction between the



(a) The melodic contour is $\{1,-1,0,1,-1\}$.

(b) The chord contour is $\begin{pmatrix}1 & -1\\ 1 & 0\end{pmatrix}$.

Fig. 1. Illustration of the melodic contour and the chord contour.

i^{th} note of the first chord and the j^{th} note of the second chord, where the notes of the chords are ordered in descending pitch order. Therefore, the chord contour from an n-note chord to an m-note chord is of size $n \times m$. Figure 1b illustrates the construction of the chord contour: in this example, the two chords have two notes, so the corresponding chord contour is a 2×2 matrix. The first row corresponds to the directions from the highest note of the first chord to the notes of the next chord, and so on. The chord contour sequence of the introduction of Edvard Grieg's *March of the Dwarfs* is graphically represented in Fig. 2. It will be analyzed in the next sections in order to find the main passages or blocks of this sequence.

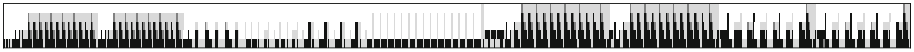


Fig. 2. Representation of the chord contour sequence of the introduction of *March of the Dwarfs*. Black, dark gray and light gray pixels map respectively to values of 1, 0 and -1.

2.2 Distance Matrix of a Chord Contour Sequence

In this section, we propose to define a distance between two chord contours. The main difficulty comes from the fact that chord contours, which are matrices, can have different sizes. First, we consider two chord contours with the same size. In this case the *Hamming distance* will be used. Let $A = (a_{i,j})$ and $B = (b_{i,j})$ be two chord contours of sizes $n \times m$, the Hamming distance $d(A, B)$ between the matrices A and B is defined as the number of coefficients which differ:

$$d(A, B) = |\{(i, j) \in [1...n] \times [1...m] \mid a_{i,j} \neq b_{i,j}\}|. \tag{1}$$

If one of the two matrices has more rows (or columns) than the other matrix, one can reduce it by deleting rows (or columns) in order to get two matrices of the same size and use the previous formula, with the addition of the number of deleted rows (or columns). The rows (or columns) to be deleted are those that minimize the distance between the two matrices. Deleting a row (respectively a column) corresponds to omitting a note in the first chord (respectively the second chord). Thus, if A and B are two matrices of size $n_1 \times m_1$ and $n_2 \times m_2$, the distance $\mathcal{D}(A, B)$ between these two matrices is defined as:

$$\mathcal{D}(A, B) = \min_{A', B'} (d(A', B')) + |n_1 - n_2| + |m_1 - m_2|, \tag{2}$$

where A' and B' are two matrices of size $\min(n_1, n_2) \times \min(m_1, m_2)$ such that A' (respectively B') is obtained by removing $n_1 - \min(n_1, n_2)$ rows and $m_1 - \min(m_1, m_2)$ columns from A (respectively B). From a mathematical point of view, the first distance d respects symmetry, identity of indiscernibles, non-negativity and triangular inequality. It is well defined as a metric on the space of matrices with the same size. On the other hand, for the second distance $\mathcal{D}$, the triangular inequality is lost; hence, it is only a semi-metric in the mathematical sense. However, since we only make pairwise comparisons, without looking for a path from one matrix to another one, the triangular inequality is not essential.

In order to visualize the musical structures, the *self-similarity matrix* was proposed in [7], as a two-dimensional representation defined by computing the similarity between any two instants. As stated in [19], self-similarity matrices have become a major concept in the study of musical structures. In addition, the dual of self-similarity matrices are *self-distance matrices* where each coefficient describes the distance between two elements. Here we will focus on self-distances matrices, but the same logic can be transcribed on self-similarity matrices. Let c_k be the k^{th} chord contour of the musical piece, i.e., from the k^{th} chord to the $(k+1)^{th}$ chord. Then the coefficient of the line i and the column j of the self-distance matrix is defined by $\mathcal{D}(c_i, c_j)$. Figure 3 displays the self-distance matrix corresponding to the example of the introduction of *March of the Dwarfs* in Fig. 2. Since $\mathcal{D}$ is symmetric, the self-distance matrix is a symmetric matrix. The musical structures can be inferred from the information near the diagonal: the different blocks around the diagonal framed in red in Fig. 3 represent the musical sections. It is possible to understand the shape of the self-distance matrix in comparison to the chord contour sequence: blocks on the diagonal correspond to sections that are visually identifiable in Fig. 2.

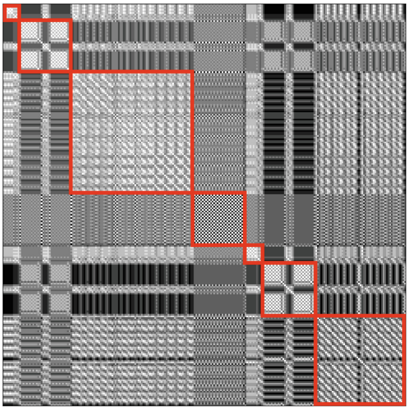


Fig. 3. Self-distance matrix of the introduction of *March of the Dwarfs* (white = 0, i.e. low distance and high similarity, black = high distance values and low similarity). (Color figure online)

3 Analysis of the Self-distance Matrix Using Morphological Operations and Filters

In this section, we propose an original method, based on mathematical morphology, to extract meaningful information from the self-distance matrix. This method can be applied to any type of self-distance or self-similarity matrix and to any type of data. The purpose of this method is to homogenize the different regions of the matrix in order to identify the musical structure.

3.1 A Short Introduction to Morphological Filters

Developed in the 1960s s by G. Matheron and J. Serra, *Mathematical Morphology* is, in its deterministic component, an algebraic theory developed initially to analyze shapes, and is widely used in image analysis. In this paper, we will rely on mathematical morphology defined on functions, typically used to analyze gray level images, making an analogy between self-distance matrices and images. Only the useful notions are recalled here, and more details can be found in [4,10,11,18,21,22]. Let $(\mathcal{F}, \leq)$ be a lattice of functions (here we consider functions from $E = \mathbb{Z}^n$ into $\mathbb{R}^+$ to handle self-distance matrices, and the lattice is complete). A *dilation* is an operation that commutes with the supremum of the lattice, and an *erosion* an operation that commutes with the infimum. Concrete forms of these operations, which are often used, rely on the notion of *structuring element*, an element B of the lattice, which can be considered as a binary relation between elements of the underlying space E, or as a spatial neighborhood in our analogy with images, or more generally as a function with bounded support. *Dilation* $\oplus$ and *erosion* $\ominus$ in the complete lattice $(\mathcal{F}, \leq)$ are extensions of *Minkowski addition* [17] and *subtraction* [9] in the binary morphological case, and are defined for any $X \in \mathcal{F}$, any structuring element $B \in \mathcal{F}$ and any $x \in E$:

$$X \oplus B(x) = \sup_{t \in E} \left(X(t) + B(x-t) \right), \qquad X \ominus B(x) = \inf_{t \in E} \left(X(t) - B(t-x) \right). \quad (3)$$

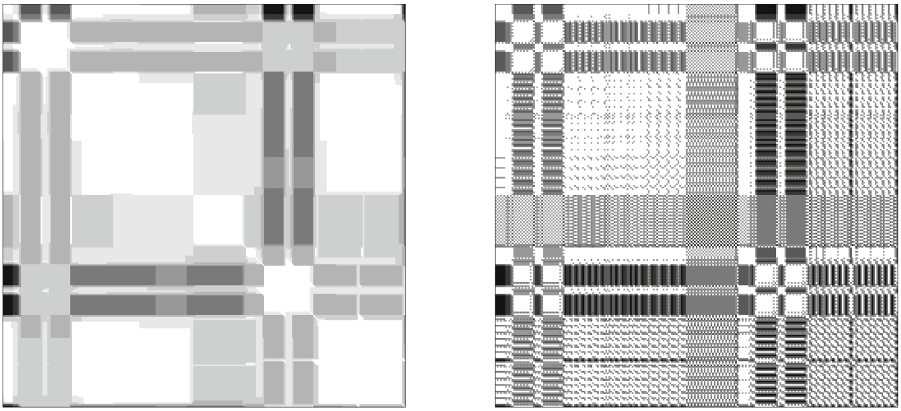
Dilation extends bright zones and reduces dark ones, while erosion does the opposite. The other two fundamental operations result from the composition of these operators. Indeed, the *opening* $\circ$ is the composition of an erosion and a dilation and the *closing* $\bullet$ is a dilation followed by an erosion:

$$X \circ B = (X \ominus B) \oplus B, \qquad X \bullet B = (X \oplus B) \ominus B. \quad (4)$$

Opening and closing are increasing and idempotent operators, hence morphological filters. They can be used to eliminate small details (having higher values than their surrounding using opening, and smaller ones using closing) according to the size and shape of the structuring element. Therefore, by using these filters, some detailed information may be lost, while more flat and homogeneous regions are obtained. This property will be used to highlight homogeneous regions in the self-distance matrix, in order to exhibit the main musical structures, as detailed in the next section.

3.2 Application of Mathematical Morphology to the Self-distance Matrix

We propose to use morphological filters to identify the main blocks of the self-distance matrix. Blocks along the diagonal provide information on the musical structure of the piece since low distance values of the self-distance matrix correspond to passages with high similarity. In order to identify larger similar blocks, locally higher distance values should be removed. The opening operation is particularly well suited to this situation. To do this, the structuring element has to be constant and square-shaped in order to preserve the general organization of the matrix, which exhibits strong vertical and horizontal structures, as well as squared blocks. By using this operation, it is possible to homogenize the regions of the self-distance matrix and to reduce the blocks on the diagonal to a zero value (because the diagonal coefficients are equal to zero due to the identity of indiscernibles of the metric).



(a) Opening (b) Threshold

Fig. 4. Filtering of the self-distance matrix using a morphological opening (a). As a comparison, a simple thresholding is shown in (b).

The result of this operation on the self-distance matrix of Fig. 3 with a square structuring element of size 12 × 12 is represented in Fig. 4a. Blocks on the diagonal appear in white, which is the minimal value (equal to zero), and we can easily detect them. To compare this method with simpler methods, thresholding is shown in Fig. 4b. Here, each coefficient below half of the maximum coefficient of the matrix is set to zero. However, this method does not detect the main blocks of the self-distance matrix. The threshold operation acts globally on the matrix, with the same threshold value applied everywhere. By contrast, opening is an operator that acts locally on the coefficients of the matrix, depending on local shape and size of the distance function, not on absolute values, which fits our filtering objective better.

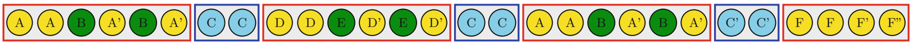
The *self-similarity matrix*, introduced in [7], is also used in audio-based approaches to the analysis of musical structures. In this case, the values of the self-similarity matrix are inverted with respect to the self-distance matrix. The diagonal coefficients are the highest values (equal to one) and the goal is to remove locally lower values and to reduce the blocks around the diagonal to the highest value of the matrix. This change can also be handled with the morphological tools because dilation and erosion (respectively opening and closing) form pairs of dual operators [4]. This means concretely that applying a dilation (respectively an opening) on a self-distance matrix is equivalent to applying an erosion (respectively a closing) on a self-similarity matrix, and vice versa.

4 Changing the Shape of the Morphological Filter to Detect Different Musical Structures

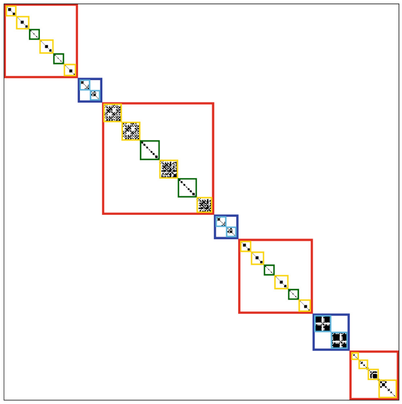
The morphological operations provide new computational tools for the analysis and identification of the overall structure of a musical piece. Moreover, it is possible to detect musical structures at different scales, for example to refine the granularity of the analysis and identify the bars of the piece. This can be done by changing the size of the structuring element, in order to detect blocks of different sizes. With a smaller structuring element, it is possible to detect smaller blocks around the diagonal, representing for instance the bars of the piece, while a larger one will allow detecting the global musical structure at a bigger scale.

To illustrate the notion of filtering with different structuring elements, we consider the third movement of the Piano Sonata No.11 in A Major, composed by Wolfgang Amadeus Mozart and commonly known as *Alla Turca* or *Turkish Rondo*. The structures of the piece are represented in Fig. 5a, where each letter symbolizes 8 bars. This piece is divided into four main parts represented by red rectangles and linked with blue rectangles. There are two levels of structure: the 7 colored rectangles (global structure) or the 28 letters (detailed structure). As seen previously, the structuring element has to be constant and square-shaped, the only parameter to choose being the size. We applied an opening filter with a constant square-shaped structuring element of size 3×3 and 6×6 to the self-distance matrix (computed using the chord contour sequence). The result of these opening filters is displayed in Figs. 5b and 5c. For a clearer understanding, only the diagonal blocks (detected with the flood-fill algorithm) are shown in black in this figure, i.e., zero value coefficients connected to the diagonal of the matrix. We computed the *novelty score*, introduced in [8], of these two opening diagonals. The novelty score N is the correlation along the diagonal of a matrix M with the checkerboard kernel C:

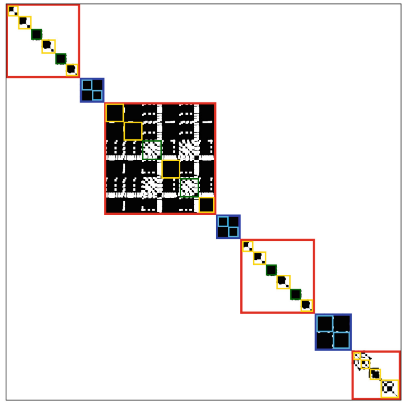
$$N(t) = \sum_{i=-L/2}^{L/2} \sum_{j=-L/2}^{L/2} C(i,j) M(i+t, j+t), \tag{5}$$



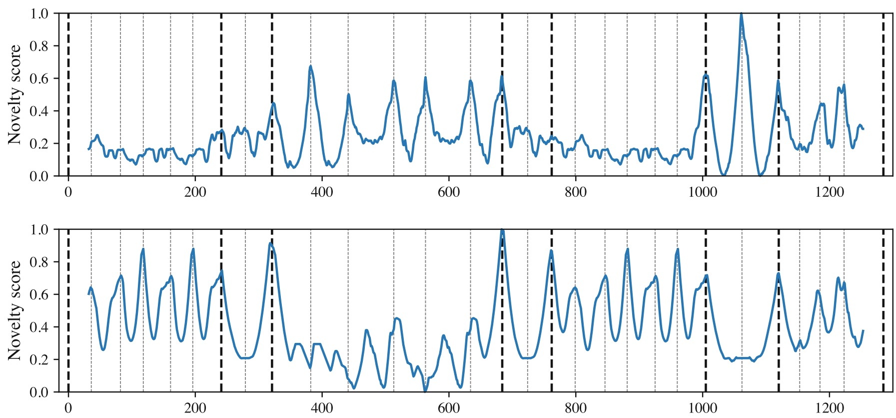
(a) Musical structures of *Alla Turca* (W.A. Mozart).



(b) Opening diagonal with a 3×3 constant square-shaped structuring element.



(c) Opening diagonal with a 6×6 constant square-shaped structuring element.



(d) Novelty score of the opening diagonal with a constant square-shaped 3×3 (top) and 6×6 (bottom) structuring element.

Fig. 5. Filtering of the self-distance matrix at different scales by the opening operation in order to obtain different musical structures. (Color figure online)

where C is the 64×64 symmetric matrix defined as

$$C = \begin{pmatrix} -1 & \cdots & -1 & 1 & \cdots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -1 & \cdots & -1 & 1 & \cdots & 1 \\ 1 & \cdots & 1 & -1 & \cdots & -1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & -1 & \cdots & -1 \end{pmatrix}. \tag{6}$$

Notice that we use the opposite of the checkerboard kernel presented in [8] because we have a self-distance matrix instead of a self-similarity matrix. This novelty score allows detecting changes, and therefore the limits of the blocks to identify. The novelty scores of the two opening diagonals are represented in Fig. 5d. We also add the boundaries of the musical structures shown in Fig. 5a with thick dotted lines (boundaries between rectangles) and thin dotted lines (boundaries between letters). The high value of the novelty score represents the boundaries of the piece. The novelty score of the opening diagonal with a 3×3 structuring element detects the boundaries between the D/D/E/D'/E/D' and C'/C' sections. While the novelty score of the opening diagonal with a 6×6 structuring element detects the boundaries between the rectangles and the A/A/B/A'/B/A' sections. With these two diagonals blocks, it is possible to detect two different structures of the piece.

Finally, we can adjust the size of the structuring element used to filter the self-distance matrix depending on the granularity that we want in the analysis of the musical structures (which enables for example to detect only few very long passages or a greater number of short passages). By varying the size of the structuring element, we can computationally grasp the segmentation process at multiple levels. In fact, every time we increase the size of the structuring element we force some segments to merge and become a new bigger segment, starting from few notes segments to the whole piece.

5 Conclusions

This article proposed a new method to visualize a piece of music and analyze its structures in an automatic way. By focusing on the pitch variations between the elements of a melodic line (notes) or harmonic progression (chords), we have proposed an original approach that generalizes the notion of melodic contour to a sequence of chords, called *chord contour*. In such a sequence, the pitch variation is described by matrices, instead of just a number as in the case of the traditional melodic contour. These matrices characterize a sequence of chords by using the direction of the pitch variation of the notes from one chord to the next one. We then introduced a proximity measure between two chord contours of any sizes with the semi-metric $\mathcal{D}$ in order to compute the self-distance matrix of a chord contour sequence. The self-distance matrix is used to analyze musical structures by leveraging the fact that the principal blocks around the diagonal correspond to the main passages of the piece. The main idea of this paper is to use filters borrowed from the domain of mathematical morphology in order to identify these blocks. Mathematical morphology is a relatively new theory in the MIR community and we are convinced that it can be particularly useful in connection with the self-distance matrix. These morphological filters have been used to homogenize and identify well-defined regions of the self-distance matrix corresponding to musical entities. The opening operation has been successfully applied to the analysis of the musical structures of a piece because it locally removes the high values. With a constant square-shaped structuring element, it reveals the horizontal and vertical blocks of the self-distance matrix. In addition, the blocks around the diagonal, which correspond to a well-defined musical structure, all have a zero value. Moreover, by varying the size of the filter, it is possible to have different filtering levels in the automatic detection of the underlying structures of the musical piece. By filtering the self-distance matrix with a large opening, one is able to identify the main global parts of the piece, while using a smaller morphological filter reveals shorter musical sections. Some promising results of applying this new method in the field of music automatic segmentation have been obtained and discussed by presenting a computational analysis of an excerpt of Edvard Grieg's *March of Dwarfs* and of Mozart's Piano Sonata *Alla Turca*.

In this paper, we demonstrated the usefulness of morphological filters to homogenize musical sections to detect the musical structure. However, homogeneity is not the only criteria for music structure analysis, and the other main approach is based on repetition. Paulus et al. argue that a combined approach (based on homogeneity, novelty and repetition) provides promising results [19]. Our method does not handle repetition, because the goal of this paper is to show the application of mathematical morphology for music structures analysis. Due to the simplicity yet powerful utility of morphological filters, we strongly believe that this method can be reuse for future algorithms for the homogeneity step. Moreover, although we have applied this method on symbolic music representations with a chord contour sequence, this method can also be applied for audio-based analysis of musical structures. For future research, we plan to test this method on a large audio database with annotated structures in a hierarchical way to validate it experimentally.

References

1. Adams, C.R.: Melodic contour typology. Ethnomusicology **20**(2), 179–215 (1976)
2. Agon, C., Andreatta, M., Atif, J., Bloch, I., Mascarade, P.: Musical descriptions based on formal concept analysis and mathematical morphology. In: Chapman, P., Endres, D., Pernelle, N. (eds.) ICCS 2018. LNCS (LNAI), vol. 10872, pp. 105–119. Springer, Cham (2018). https://doi.org/10.1007/978-3-319-91379-7_9
3. Anagnostopoulou, C., Giraud, M., Poulakis, N.: Melodic contour representations in the analysis of children's songs. In: van Kranenburg, P., Anagnostopoulou, C., Volk, A. (eds.) 3rd International Workshop on Folk Music Analysis, pp. 40–43. Amsterdam, Netherlands (2013)

4. Bloch, I., Heijmans, H., Ronse, C.: Mathematical morphology. In: Aiello, M., Pratt-Hartmann, I., Van Benthem, J. (eds.) Handbook of Spatial Logics, pp. 857–944. Springer, Dordrecht (2007). https://doi.org/10.1007/978-1-4020-5587-4_14
5. Buteau, C., Mazzola, G.: Motivic analysis according to Rudolph Réti: formalization by a topological model. J. Math. Music **2**(3), 117–134 (2008)
6. Dowling, W.J.: Melodic contour in hearing and remembering melodies. In: Aiello, R., Sloboda, J.A. (eds.) Musical Perceptions, pp. 173–190. Oxford University Press, Oxford (1994)
7. Foote, J.: Visualizing music and audio using self-similarity. In: Seventh ACM International Conference on Multimedia (Part 1), pp. 77–80 (1999)
8. Foote, J.: Automatic audio segmentation using a measure of audio novelty. In: IEEE International Conference on Multimedia and Expo, ICME 2000. Latest Advances in the Fast Changing World of Multimedia, vol. 1, pp. 452–455 (2000)
9. Hadwiger, H.: Minkowskische Addition und Subtraktion beliebiger Punkt-mengen und die Theoreme von Erhard Schmidt. Mathematische Zeitschrift **53**, 210–218 (1950)
10. Heijmans, H.: Morphological Image Operators. Advances in Electronics and Electron Physics: Supplements. Academic Press (1994)
11. Heijmans, H., Ronse, C.: The algebraic basis of mathematical morphology I. dilations and erosions. Comput. Vis. Graph. Image Process. **50**(3), 245–295 (1990)
12. Karvonen, M., Laitinen, M., Lemström, K., Vikman, J.: Error-tolerant content-based music-retrieval with mathematical morphology. In: Ystad, S., Aramaki, M., Kronland-Martinet, R., Jensen, K. (eds.) CMMR 2010. LNCS, vol. 6684, pp. 321–337. Springer, Heidelberg (2011). https://doi.org/10.1007/978-3-642-23126-1_20
13. Karvonen, M., Lemström, K.: Using mathematical morphology for geometric music information retrieval. In: International Workshop on Machine Learning and Music, MML 2008, Helsinki, Finland (2008)
14. Lascabettes, P., Bloch, I., Agon, C.: Analyse de représentations spatiales de la musique par des opérateurs simples de morphologie mathématique. Journées d'Informatique Musicale (2020)
15. Lu, L., Wang, M., Zhang, H.J.: Repeating pattern discovery and structure analysis from acoustic music data. In: 6th ACM SIGMM International Workshop on Multimedia Information Retrieval, pp. 275–282 (2004)
16. Marvin, E.W., Laprade, P.A.: Relating musical contours: extensions of a theory for contour. J. Music Theory **31**(2), 225–267 (1987)
17. Minkowski, H.: Volumen und Oberfläche. Mathematische Annalen **57**, 447–495 (1903)
18. Najman, L., Talbot, H.: Mathematical Morphology: From Theory to Applications. ISTE-Wiley (2010)
19. Paulus, J., Müller, M., Klapuri, A.: State of the art report: audio-based music structure analysis. In: ISMIR, Utrecht, pp. 625–636 (2010)
20. Quinn, I.: The combinatorial model of pitch contour. Music Percept.: Interdiscip. J. **16**(4), 439–456 (1999)
21. Ronse, C., Heijmans, H.: The algebraic basis of mathematical morphology: II. openings and closings. CVGIP: Image Underst. **54**(1), 74–97 (1991)
22. Serra, J.: Image Analysis and Mathematical Morphology. Academic Press Inc., London (1982)
23. Trehub, S.E., Bull, D., Thorpe, L.A.: Infants' perception of melodies: the role of melodic contour. Child Dev. **55**(3), 821–830 (1984)