Hypercube + Rubik's Cube + Music = HyperCubeHarmonic

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Abstract. Musical chords and chord relations can be described through mathematics. Abstract permutations can be visualized through the Rubik's cube, born as a pedagogical device [7,21]. Permutations of notes can also be heard through the CubeHarmonic, a novel musical instrument. Here, we summarize the basic ideas and the state of the art of the physical implementation of CubeHarmonic, discussing its conceptual lifting up to the fourth dimension, with the HyperCubeHarmonic (HCH). We present the basics of the hypercube theory and of the 4-dimensional Rubik's cube, investigating its potential for musical applications. To gain intuition about HCH complexity, we present two practical implementations of HCH based on the three-dimensional development of the hypercube. The first requires a laptop and no other devices; the second involves a physical Rubik's cube enhanced through augmented and virtual reality and a specifically-designed mobile app. HCH, as an augmented musical instrument, opens new scenarios for STEAM teaching and performing, allowing us to hear the "sound of multiple dimensions."

Keywords: Rubik's cube · Hypergeometry · Permutations · Chords · Tonnetz · Mobile

1 Introduction

Pythagoras advanced joint knowledge of mathematics and music. In the Middle Ages, they were part of the Quadrivium. Today, their exchanges are leading to a flourishing research field [20]. Classic examples of interactions between mathematics and music are the permutations of pitches as well as entire musical sequences, as in Mozart's dice game. Here, we connect (hyper)geometry,

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combinatorics, and music, presenting the general idea of the HyperCubeHarmonic (HCH). It is a development of the CubeHarmonic (CH) [11], that generates chords according to faces' rotations. The CH is a novel musical instrument inspired by the Rubik's cube and the abstract pitch spaces (a family of representations including the tonnetz).[1] The core idea of CH is the following:

– each facet is a note within an octave;
– each face is a chord (with nine notes, and some of them can be repeated);
– rotating the cube, the initial chords are scrambled.

For reasons of clarity of sound and simplicity, the reproduced sounds correspond to the notes on the top face of the cube. A cycle of rotations corresponds to a cycle of chords. In fact, we can investigate chord-preserving symmetries of the Rubik's cube. For example, the cube's rotational symmetry allows cyclical chord progressions. We notice that the symmetry of harmonically-played chords is bigger than the symmetry of melodically-played chords. In fact, a rotated face corresponds to the same sound of an un-rotated face if the notes are played simultaneously.

A simplification of the unscrambled cube has one pitch for all the facets in each face; twisting the cube, new chords are obtained. From the point of view of music theory, we can see CH as a tangible application of slot-machine transformations.[2] CH involves a topic of mathematical music theory, the *tonnetz* [20], a lattice of pitches and musical chords, which is first cut and glued to be adapted to the cube, and then twisted (and cut again) according to the performed rotations.

The first concept of CubeHarmonic (called CubHarmonic in [12, p. 20.1.2]) is a $4 \times 4 \times 4$ cube, used for a 4-part harmony. Twisting this cube, new chord sequences are obtained. However, all current physical CH prototypes [10,11] use a $3 \times 3 \times 3$ cube. With a 3-part harmony, the performer can still enjoy a considerable amount of combinations, with the advantages of easiness in cube's scrambling.

The very first implementation of the CH was realized by M. Mannone through a giant Rubik's cube and sound modules, allowing the user/performer to play each note separately or simultaneously,[3] see Fig. 1. It is described in [10]. The disadvantage of size was overcome by IM3D+ use [8], as described below.

The $3 \times 3 \times 3$ prototype built at the Tohoku University,[4] see Fig. 2, needs only a few markers (LC coils) for 3D motion tracking [11], while a $4 \times 4 \times 4$ cube would present more technical issues regarding physical motion capture.

[1] The tonnetz is a lattice constituted by notes and their connections as chords [20]. The CH has been thought by M. Mannone during her studies at IRCAM in 2013, and then first described in [12].

[2] In music theory, slot-machine transformations are permutations. If we have three discs with three notes in each of them, they give a sequence of 3-note chords. Rotating the discs, the chord changes. For example, the vertical sequence $0-1-2$ becomes $1-2-0$ after a rotation of one of the discs [1].

[3] Video: https://tinyurl.com/3j5csh36.

[4] Video: https://www.youtube.com/watch?v=r_wNpQnsWhg.

In fact, in the current CH prototype, the IM3D+ platform [8] allows motion tracking through battery-less and wireless small lightweight identifiable passive sensors modifying a magnetic field (Fig. 2). If these sensors are embedded in HCH, more degrees of freedom are allowed: seven more Rubik's cubes and all their intertwined rotations.

The CH and HCH have a good potential not only as music instruments but also as teaching tools for STEAM (Science, Technology, Engineering, Art, and Mathematics) education [3] in the domain of mathematics [9,14,15], because of their specific features described in Sects. 2 and 3. In fact, HCH joins the abstraction of hypergeometry with the tangibility of a physical tool and the resources of computer visualization.

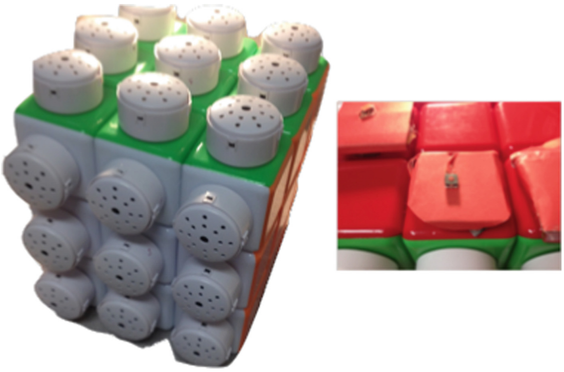


Fig. 1. First implementation of CubeHarmonic, from [10].

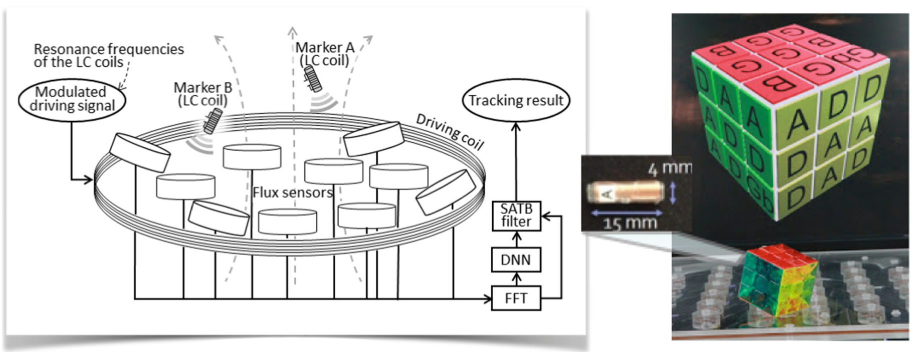


Fig. 2. IM3D+ (left) and CubeHarmonic (right) [11].

Other independently-developed applications of music to the Rubik's cube are described in a NIME article [16] and in a TED tutorial [19]. The TED tutorial concerns a similar theoretical idea, without however mentioning chord sequences or proposing the development of a physical instrument. Moreover, the tutorial dates back to November 2015, while the book where the CubeHarmonic had first been described [12] was submitted to Springer in September 2015. The article [16], published one year after the book [12], proposes a new device, with a random choice of pitch, to be used as a controller. However, the starting idea of CH is theoretical, and its implementation is physical and tangible. Also, in CH [10,11] some musical features, such as the possibility to manipulate overall loudness and pitch changes according to hands' movements, are all implemented, creating crossmodal effects. These features will be added to future versions of HCH as well.

HCH is inspired by the multidimensional geometry, and it will allow the transformation of the instrument's structure during performance, besides adding more parameters to the existing system. This results in an innovative instrument which enables rich manipulations of tones and other musical parameters while making notes, melodies, chords, and timbre transitions interactive and appreciably tangible. Thus, the HCH should allow musicians to hear the "sound of multiple dimensions," by extending a customizable physical cube into additional dimensions in virtual reality. An object living in the 4 dimensions cannot be completely realized in our 3-dimensional world. However, augmented reality will help us gain intuition about this research.

Thus, we discuss the theory behind 4-dimensional Rubik's cube and its potential for musical application. Then, we present two first implementations. The first one implements a real hyper-Rubik's cube and allows the user/musician to play it; it is based on Mathematica^{TM}, and requires only a laptop. The second one exploits a physical cube and a mobile device. Thus, HCH can be totally mobile, because it does not require any platform. However, if small sensors are embedded in the corners of the cube, it can be used jointly with the former technology for CH, adding more degrees of freedom—and thus, more potential features. HCH can exploit all resources of musical mobile connectivity (both via cables, MIDI to computer or to sound interface, and wireless with bluetooth), and it has a potential as a controller.

The mathematical idea is described in Sect. 2; the implementations in Sect. 3; discussions on potentialities and further developments are proposed in Sect. 4.

2 Mathematical Concepts

2.1 *Ars Combinatoria*, Permutations, and the Rubik's Cube

Gottfried Wilhelm von Leibniz defined as *ars combinatoria*[5] the technique of ideas' symbolization through geometric and algebraic signs and their recombination and organization in all possible ways, finalized to create a universal map of concepts [4]. Then, the computational resources of *ars combinatoria* inspired artistic applications through the centuries, in music, visual arts, and literature [27]. Well-known examples include musical dice games (Musikalische

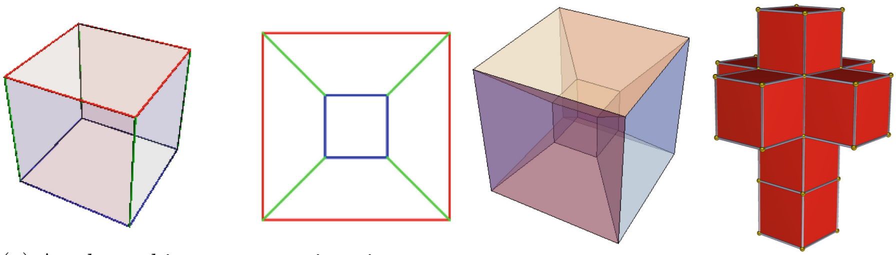
[5] It had been called *ars magna* by Ramon Llull [4].

Würfelspiele), which allowed to compose musical pieces from random combinations of pre-composed musical fragments—see some works by Llull, Haydn, and Mozart [1,17]. Mathematically, combinatorics is based on group theory. A *group* is constituted by a set and an operation such as the combination of two elements of a group gives an element of the same group. The operation verifies closure, associativity, identity, and invertibility [18]. A *permutation group* is a group whose elements are the permutations, and whose operations are the permutation compositions.

Invented as a pedagogical device to teach permutations, the Rubik's cube, thanks to its colorful aesthetic appeal and manipulative attractiveness, became one of the best-sold toys and games. The design of the Rubik's cube is based on permutation groups. The facets are the set, and their possible permutations (moves) constitute the group operations. Sets of moves are the subgroups of the Rubik's cube, and all possible moves belong to this group [26].

2.2 The Hypercube

Our hypercube has four dimensions; it is a tesseract and it can be developed with eight cubes in three dimensions [23,24]. The cubes are regarded to be located oppositely on the four axes of the Cartesian coordinates in a hyperspace. Hypercubes sparkled the interest of music theorists; hypergeometry led to dynamic visualizations of symmetries in musical pieces [2,5]. The famous way to understand the hypercube is extending the relation between 2D and 3D to 3D and 4D. For example, a square (2D) moving in the space (3D) for a length equal to its side and reaching another square, spans the space of a cube (3D). A cube, moving in space (4D) and reaching another cube, spans the space of a hypercube (4D). In short, a dimensionally-lifted square gives a cube, and a dimensionally-lifted cube gives a hypercube. Figure 3a shows a cube and its central projection, and Fig. 3b presents a hypercube through its center projection. Fig. 3c illustrates the expansion of a hypercube with eight cubes on a hypersurface; this structure



(a) A cube and its representation via a central projection [25].

(b) Hypercube center projection [25].

(c) Hypercube 3D development [22].

Fig. 3. Some representations of cube and hypercube.

has been used in Dalí's painting *Crucifixion (Corpus Hypercubus)* [6]. The idea of hypercube also inspired music-tech applications, not related with the Rubik's cube; as an example, see the *Sound-reactive cube*.[6]

2.3 The Hyper-Rubik's Cube

As the next step, we imagine that each cube of the hypercube is a Rubik's cube. Fig. 4 shows a 3D projection of a 4D Rubik's cube, and its constitutive eight cubes. In this subsection, we assume that all faces of each cube are painted the same color, for simplicity. However, we can paint them more generally: different colors to different subcubes or to different facets.

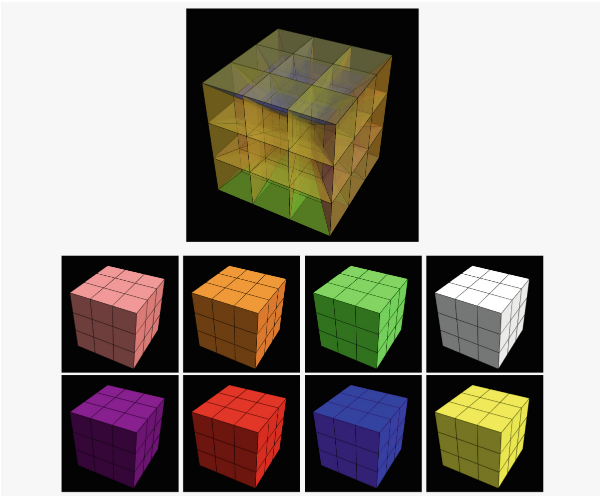


Fig. 4. Development of the Hyper Rubik's cube. Image from [25].

The traditional Rubik's cube has 27 subcubes; the center cube is not colored (ideally, it is in the inner core of the cube, where there is the rotational mechanism), and the other cubes are partially colored (according if they are corner, lateral, or central pieces). The colored pieces are "faces $\times$ facets" $= 6 \times 9 = 54$. It can be shown [25] that the hyper-Rubik's cube contains overall 81 small 4-cubes; one of the cubes is at the center and it is not colored, while the remaining 80 are partially colored. The colored subcubes in a 4-Rubik's cube are: cubes $\times$ subcubes $= 8 \times 27 = 216$. For the people in 3D, the subcube at the center of each cube is hidden so the value is 26. However, for the people in 4D, the center can be observed, maybe. This is the reason why we are considering 27 rather than 26. A rotation on one cube provokes some rotation of the adjacent cubes. According to [13,25], while in 3D an axis of rotation is a line, in 4D it

[6] https://www.youtube.com/watch?v=PmsCRypjMRI.

is a plane. E.g., the rotation matrix $R_{x,y}(\theta)$ with respect to the $(x-y)$-plane is given in Eq. (1); because there are six planes of rotations, there are six rotation matrices.

$$R_{x,y}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix} \tag{1}$$

We refer to the smallest cluster of subcubes to be rotated simultaneously as a *block*. In the case of Rubik's cube, a block is a prism of $3 \times 3 \times 1$ subcubes. Similarly, for hyper-Rubik cube, a block is a hyperprism consisting of $3 \times 3 \times 3 \times 1$ small hypercubes. The operation to rotate a block is described in [25].

2.4 The Hyper-Rubik's Cube with Music: The HyperCubeHarmonic

If we have 8 different Rubik's cubes, we have more degrees of freedom to be used musically. For example, all cubes could contain the same notes, but with different timbres. Alternatively, each cube can play the same note, or there can be different notes for different squares. A different choice of musical layers could lead, for example, to rhythm-combination controllers. We can choose to play all cubes together or just one after the other. In any case, eight cubes remind one of the mentioned 3D-development of the hypercube.

If each cube is a CH, adding more cubes means adding degrees of freedom, that is, more musical resources to the instrument. Figure 5 shows the concept of HCH. For the geometric discussion, see [24]. In the $3 \times 3 \times 3$ Rubik's cube used for CH, each facet is a single note; in HCH each cube of the tesseract is a CH. Thus, higher geometric dimensions allow higher musical complexity.

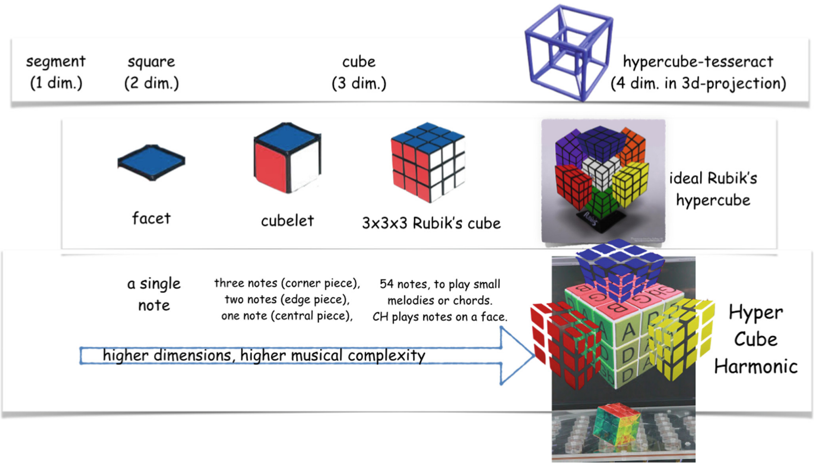


Fig. 5. The concept of HyperCubeHarmonic

3 First Implementations

To have a grasp of the musical potentialities of the Rubik's hypercube, we present here two implementations.

3.1 Implementation 1

As a first computational implementation, for the sake of simplicity we put the same note on each one of the eight cubes (e.g., creating a scale C-D-E-F-G-A-B-C octave), visually identifying each note and cube with a color. In this way, the result of each rotation will straightforwardly be visualized and heard. This implementation, coded in Mathematica^{TM} by T. Yoshino, can be accessed online.[7]

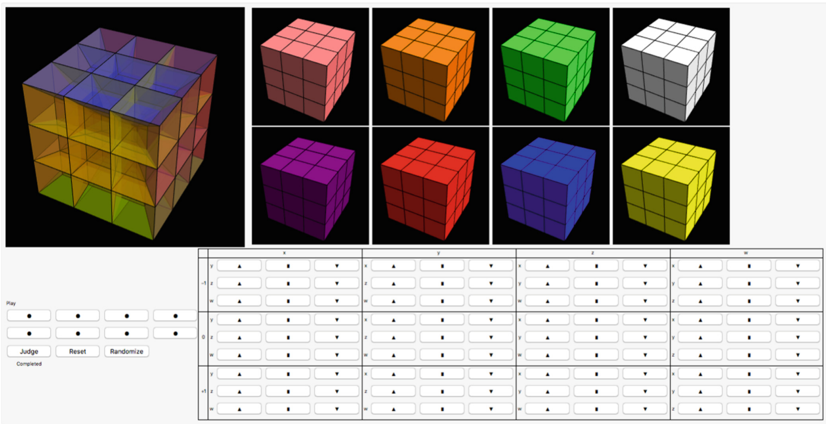


Fig. 6. Mathematica^{TM} rendition of the unscrambled HyperCubeHarmonic. Pink: C4; Orange: D4; Green: E4; White: F4; Yellow: G4; Blue: A4; Red: B4; Violet: C5. (Color figure online)

Figure 6 shows a screenshot of this implementation. The sound(s) of each cube can be heard pressing the round buttons. The table on the right contains the commands for the rotations: the black triangle indicates a rotation angle of $\pi/2$, the black square of π, and the black inverted triangle of $-\pi/2$. As an example, Figs. 7 and 8 show the result of a rotation of $-\pi$ of the y-column and x-row. Our video shows how this app works.[8]

[7] http://random-walk.org/PrototypeHyperCH.nb.
[8] https://youtu.be/wB8VoCKHrmc.

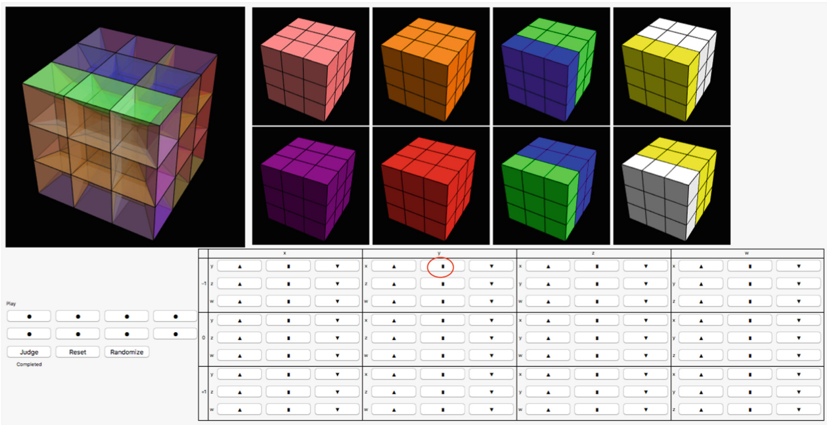


Fig. 7. HyperCubeHarmonic with the highlighted rotation.

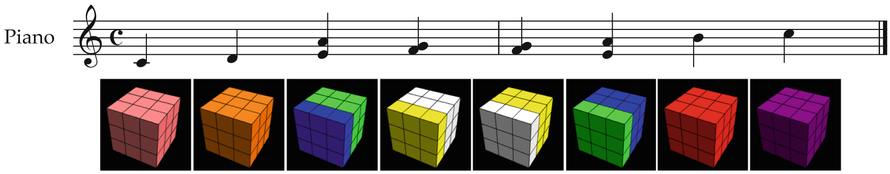


Fig. 8. Musical effect of the rotation of Fig. 7.

3.2 Implementation 2

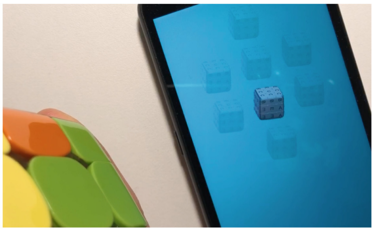
We can also simplify the HCH 3D projection and develop a manipulative device used together with an app. The user/performer has a physical device, a Rubik's cube with embedded sensors, and a mobile with the specific app coded in Unity. The physical cube is the "Go Cube,"[9] which contains embodied, inner sensors transmitting information about the position to a mobile app. The mobile app (Fig. 9a) has been specifically coded by P. Chiu for HCH, working for Android and iOS mobile operative systems. In this app, the cubes are considered as independent; this is why they are not distinguished by colors.

The Go Cube + app system implements an IoT (internet of things, connected smart-objects) approach. The app screen shows, in real time, the position of the facets of the real cube, and other additional seven cubes. The user can select specific cubes to create a local rotation. In doing so, we do the simplification of independent cubes. In future releases, these rotations will be related between them according to the hypercube's geometrical constraints [25].

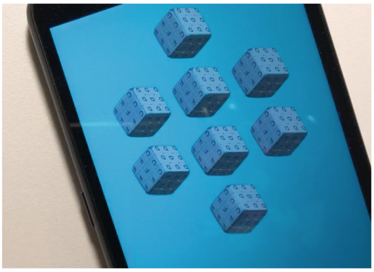
Even if the physical cube is scrambled, the cube(s) in the app can be reset to the unscrambled state. Thus, HCH does not require the ability to solve the Rubik's cube to be played. The musical parameters in the current prototype are

[9] https://getgocube.com/.

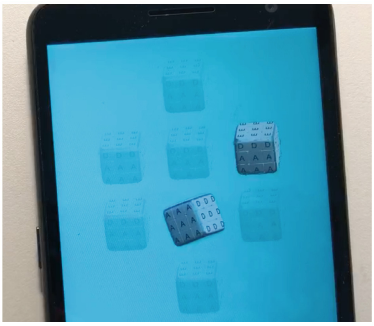
(a) Opening the app.



(b) Selection of one cube.



(c) All cubes are selected and partially scrambled.



(d) Long-tapping on it, we can scramble only one cube.

Fig. 9. HCH mobile app

the same pitches with different timbres: each cube has a different timbre. Sounds come from the sampled orchestral sounds.[10]

The top face of each cube is the one which sounds. If the rotation does not involve the top face, the sound does not change. HCH development is still ongoing, to enhance sound-movement correspondence and use easiness. Next developments will include rhythms and sound effects, to make the performance richer. Our video[11] shows how the app prototype works, with the original (unedited) sound. A single tap enables a cube to play (Fig. 9b). As default, all cubes are moving (Fig. 9c), but a long-tap on one cube disables scrambling for the other cubes, letting them keep the same notes (Fig. 9d).

4 Discussion and Conclusions

From a theoretical idea, the CubeHarmonic has been little by little shaped into a working musical instrument. We described the idea of CH and its 4-dimensional development, including two computational implementations. The presented extension to the fourth dimension broadens the scenarios in musical applications and conceptual developments. One of the proposed implementations

[10] https://philharmonia.co.uk/resources/sound-samples/.
[11] https://youtu.be/p5i6_uzIips.

of HCH has already been used in musical setups such as multi-instrument improvisations.[12] We are planning to join both prototypes to use a physical cube as a controller for the Mathematica app with the real HyperCubeHarmonic (Fig. 10).

Fig. 10. The logo of HyperCubeHarmonic project.

The main advantage of the presented simplified implementation of HCH with respect to CH is the portability. Also, it can be connected (through the mobile) wireless with bluetooth, or through Sonobus software, allowing a stable stream transmission, ideal for remote collaborative performances. HCH could also be used as a MIDI controller if the input is transmitted to an audio interface. The use of sequencers such as Bitwig, Mulab, or Ableton will allow HCH playing recordings. Further developments may include a detailed user study, and the application of HCH concept to other Rubik's Platonic Solids.

Connectivity of HCH makes it suitable for remote STEAM teaching [3,9]. In fact, HCH might have a potential in both mathematical and musical education, as it makes complex abstract objects tangible, by manipulating their parameters interactively. HCH can be creatively used to generate chords in correspondence of non-intuitive Rubik's hypercube rotations. In particular, students could recognize and master combinations of rotations along with their inverse through their musical effect. A classroom network of connected HCH might allow the teacher to give feedback on students' learning. The hour of math could be turned into an HCH orchestra rehearsal!

The progressive addition of multiple cubes as in HCH could help enhance creativity, mixing timbres and developing motor abilities, through the common ground of hearing and movement. Finally, HCH, through its multi-sensory modalities, might hopefully help students with visual or hearing impairments learn mathematics and solve the Rubik's cube.

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[12] *Night Forest*: https://www.youtube.com/watch?v=1OClm1pn7-g&t=1s, *Blues and Grays*: https://www.youtube.com/watch?v=oMldYLbfIRE, Female Laptop Orchestra (FLO).

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