Extended Vuza Canons

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Abstract. Starting from well-known constructions of aperiodic tiling rhythmic canons by G. Hajós, N.G. de Bruijn and D.T. Vuza, several generalisations are given. In this way, it is possible to find new aperiodic canons, that we call *extended Vuza canons*.

Keywords: Mathematical models for music · Vuza canons · Aperiodic factorisations of cyclic groups

1 Prelude

Canons in music have a very long tradition; among these, a few cases of *tiling rhythmic canons* (i.e. canons such that, given a fixed tempo, at every beat exactly one voice is playing) have emerged. Only in the last century, stemming from the analogous problem of factorizing finite abelian groups, *aperiodic tiling rhythmic canons* have been studied: these are canons that tile a certain interval of time in which each voice (*inner voice*) plays at an aperiodic sequence of beats, and the sequence of starting beats of every voice (*outer voice*) is also aperiodic. From the musical point of view the seminal paper was probably the four-parts article written by D.T. Vuza between 1991 and 1993 [14–17], while the mathematical counterpart of the problem was studied also before, e.g. by de Bruijn [5], Sands [13], etc., and after, e.g. by Coven and Meyerowitz [4], Jedrzejewski [9], Amiot [1], Andreatta [3], etc.

A thorough theory of the conditions of existence and the structure of aperiodic tiling rhythmic canons has not been established yet. In this paper we try to give a contribution to this fascinating field.

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2 Aperiodic Tiling Canons

We begin fixing some notations and giving the main definitions. In the following, we conventionally denote the cyclic group of remainder classes modulo n by $\mathbb{Z}_n$ and its elements with the integers $\{0, 1, \ldots, n-1\}$, i.e., identifying each class with its least non-negative member.

Definition 1. *Let $A, B \subset \mathbb{Z}_n$. Let us define the application*

$$\sigma : A \times B \to \mathbb{Z}_n, (a, b) \mapsto a + b.$$

We set $A + B \doteqdot Im(\sigma)$; if σ is bijective, we say that A and B are in direct sum*, and we write*

$$A \oplus B \doteqdot Im(\sigma).$$

If $\mathbb{Z}_n = A \oplus B$, we call (A, B) a tiling rhythmic canon *of* period *n; A is called the* inner voice *and B the* outer voice *of the canon.*

Remark 1. It is easy to see that the tiling property is invariant under translations, i.e., if A is a tiling complement of some set B, also any translate $A + z$ of A is a tiling complement of B (and any translate of B is a tiling complement of A). Thus, without loss of generality, we shall limit our investigation to rhythms containing 0 and consider equivalence classes under translation.

Definition 2. *A rhythm $A \subset \mathbb{Z}_n$ is* periodic (of period z) *if and only if there exists an element $z \in \mathbb{Z}_n$, $z \neq 0$, such that $z + A = A$. In this case, A is also called periodic modulo $z \in \mathbb{Z}_n$. A rhythm $A \subset \mathbb{Z}_n$ is* aperiodic *if and only if it is not periodic.*

Denote by $\Phi_d(x)$ the *cyclotomic polynomial of index d*. Then, tiling rhythmic canons can be characterised as follows.

Lemma 1. *Let A be a rhythm in $\mathbb{Z}_n$ and let $A(x)$ be the* characteristic polynomial *of A, that is, $A(x) = \sum_{k \in A} x^k$. Given $B \subset \mathbb{Z}_n$ and its characteristic polynomial $B(x)$, we have that*

$$A(x) \cdot B(x) \equiv \sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x - 1} = \prod_{d \mid n, d \neq 1} \Phi_d(x) \qquad \mod (x^n - 1) \tag{1}$$

if and only if $A(x)$ and $B(x)$ are polynomials with coefficients in $\{0, 1\}$ and $A \oplus B = \mathbb{Z}_n$.

As a consequence, for each $d \mid n$, with $d > 1$, we have

$$\Phi_d(x) \mid A(x) \text{ or } \Phi_d(x) \mid B(x).$$

Definition 3. *A tiling rhythmic canon* (A, B) *in* $\mathbb{Z}_n$ *is an* aperiodic tiling rhythmic canon *if both* A *and* B *are aperiodic.*

For an extensive discussion on tiling problems, we refer the reader to Amiot [2]. If we indicate the set $\{d \in \mathbb{N} : d \mid n\}$ by $\mathsf{div}(n)$, the following proposition establishes a polynomial criterion for the aperiodicity of a given rhythm.

Proposition 1. *A set* $A \subset \mathbb{Z}_n$ *is aperiodic if and only if for all* $k \mid n$*, with* $k \neq n$*, we have*

$$\frac{x^n - 1}{x^k - 1} \nmid A(x),$$

that is, if and only if for all $k \in \mathsf{div}(n) \setminus \{n\}$ *there exists* $d \in \mathsf{div}(n) \setminus \mathsf{div}(k)$ *such that* $\Phi_d(x) \nmid A(x)$.

The following result, in conjunction with Theorem 2, identifies which are the periods of aperiodic tiling rhythmic canons.

Theorem 1 (Vuza). *Let*

- $\mathcal{V} \doteqdot \{n \in \mathbb{N} : n = p_1 n_1 p_2 n_2 n_3$ *with* $\gcd(p_1 n_1, p_2 n_2) = 1$ *and* $p_1, n_1, p_2, n_2, n_3 > 1\}$, *and*
- $\mathcal{H} \doteqdot \{p^\alpha, p^\alpha q, p^2 q^2, pqr, p^2 qr, pqrs : \alpha \in \mathbb{N},\ p, q, r, s$ *distinct primes*$\}$,

then $\mathbb{N}^* = \mathcal{V} \sqcup \mathcal{H}$.

The minimum period necessary for an aperiodic canon is 72, and the corresponding p_i and n_i are:

$$(p_1, n_1, p_2, n_2, n_3) = (2, 2, 3, 3, 2)\,.$$

3 Extended Vuza Canons

The canons with periods 72, 108, 120, 144 and 168 have been completely enumerated by Vuza [14], Fripertinger [7], Amiot [1], Kolountzakis and Matolcsi [11].

An exhaustive construction method for aperiodic tiling rhythmic canons is not known to date; the first method to find some of them was provided by the following result (see [8] by Hajós, Theorem 1 in [5] by de Bruijn, and Proposition 2.2 in [14] by Vuza).

Theorem 2. *Let* $n = p_1 n_1 p_2 n_2 n_3 \in \mathbb{N}$ *such that*

1. $p_1, n_1, p_2, n_2, n_3 > 1$ *and*
2. $\gcd(p_1 n_1, p_2 n_2) = 1$.

Then $\mathbb{Z}_n$ *admits an aperiodic tiling rhythmic canon.*

Example 1. In the hypotheses of Theorem 2, an example of tiling canon of $\mathbb{Z}_n$ with two aperiodic subsets is given by the following construction by F. Jedrzejewski (see Theorem 227 in [9]). Indicating with $\mathbb{I}_k$ the set $\{0, 1, \ldots, k-1\}$, let us call:

$$\begin{aligned} A_1 &= n_3 p_1 n_1 \mathbb{I}_{n_2} & A_2 &= n_3 p_2 n_2 \mathbb{I}_{n_1} \\ U_1 &= n_3 p_1 n_1 n_2 \mathbb{I}_{p_2} & U_2 &= n_3 p_2 n_2 n_1 \mathbb{I}_{p_1} \\ V_1 &= n_3 n_2 \mathbb{I}_{p_2} & V_2 &= n_3 n_1 \mathbb{I}_{p_1} \\ K_1 &= \{0\} & K_2 &= \{1, 2, \ldots, n_3 - 1\}. \end{aligned}$$

Then taking

$$\begin{aligned} A &= A_1 \oplus A_2 \\ B &= (U_1 \oplus V_2 \oplus K_1) \sqcup (U_2 \oplus V_1 \oplus K_2), \end{aligned}$$

we have the canon $\mathbb{Z}_n = A \oplus B$.

Remark 2. From now on, given p_1, n_1, p_2, n_2, and n_3, we will denote by A_1, A_2, U_1, U_2, V_1, and V_2 the sets so called in Example 1.

Many other ways of constructing aperiodic tiling canons are possible, see for example de Bruijn [5], Vuza [14], Fidanza [6], and Jedrzejewski [9]. These methods fall into a category treated by F. Jedrzejewski (Theorem 14 in [10]). We refine his result lifting the hypothesis that p_1 and p_2 are prime and proving that B is aperiodic if n_3 satisfies a simple arithmetic constraint.

Theorem 3. *Let $n = p_1 n_1 p_2 n_2 n_3 \in \mathbb{N}$ such that:*

1. $p_1, n_1, p_2, n_2, n_3 > 1$;
2. $\gcd(p_1 n_1, p_2 n_2) = 1$;
3. *there is no prime q such that $q \mid n_3$, but $q \nmid p_1 n_1 p_2 n_2$.*

Let H be the subgroup $H = n_3 \mathbb{I}_{p_1 n_1 p_2 n_2}$ of $\mathbb{Z}_n$ and let K be a complete set of cosets representatives for $\mathbb{Z}_n$ modulo H such that K is the disjoint union $K = K_1 \sqcup K_2$. Then the pair (A, B) defined by

$$\begin{aligned} A &= A_1 \oplus A_2 \\ B &= (U_1 \oplus V_2 \oplus K_1) \sqcup (U_2 \oplus V_1 \oplus K_2) \end{aligned}$$

is an aperiodic tiling rhythmic canon of $\mathbb{Z}_n$.

Proof. The proof that $A \oplus B = \mathbb{Z}_n$ and that the set A is aperiodic is the same as in Vuza (Proposition 2.2 in [14]). We are left to prove that B is aperiodic. Consider the characteristic polynomial $B(x)$:

$$B(x) = \frac{x^{n_3 p_1 n_1} - 1}{x^{n_3 n_1} - 1} \frac{x^n - 1}{x^{n_3 p_1 n_1 n_2} - 1} K_1(x) + \frac{x^{n_3 p_2 n_2} - 1}{x^{n_3 n_2} - 1} \frac{x^n - 1}{x^{n_3 p_2 n_2 n_1} - 1} K_2(x).$$

Given any $h \in \operatorname{div}(n) \setminus \{n\}$, we look for a $d \in \operatorname{div}(n) \setminus \operatorname{div}(h)$ such that $\Phi_d(x) \nmid B(x)$. Let us consider the cases:

1. if $n_3p_2n_2n_1 \nmid h$, then $\Phi_{n_3p_2n_2n_1}(x) \nmid B(x)$ since
$$\Phi_{n_3p_2n_2n_1}(x) \mid \frac{x^n - 1}{x^{n_3p_1n_1n_2} - 1}$$
but
$$\Phi_{n_3p_2n_2n_1}(x) \nmid \frac{x^{n_3p_2n_2} - 1}{x^{n_3n_2} - 1} \frac{x^n - 1}{x^{n_3p_2n_2n_1} - 1} K_2(x).$$
In particular, $\Phi_{n_3p_2n_2n_1}(x) \nmid K_2(x)$ by Lemma 4 of Rédei's paper [12].
2. if $n_3p_1n_1n_2 \nmid h$, then $\Phi_{n_3p_1n_1n_2}(x) \mid B(x)$ (symmetrically to the previous case).

There are no other possibilities: in fact, if we had $n_3p_2n_2n_1 \mid h$ and $n_3p_1n_1n_2 \mid h$, then $h = \alpha n_3p_2n_2n_1 = \beta n_3p_1n_1n_2$ and therefore $\alpha p_2 = \beta p_1$. Since $\gcd(p_1, p_2) = 1$, it would follow $\alpha = p_1$ and $\beta = p_2$ and so $h = n$, which is a contradiction. ☐

Example 2. Consider $n = 216$; let $p_1 = 2$, $n_1 = 2$, $p_2 = 3$, $n_2 = 3$, and $n_3 = 6$. Theorem 3 ensures that, defining

$$\begin{aligned} A &= 24\mathbb{I}_3 \oplus 54\mathbb{I}_2 \\ B &= (72\mathbb{I}_3 \oplus 12\mathbb{I}_2 \oplus \{0, 106\}) \sqcup (108\mathbb{I}_2 \oplus 18\mathbb{I}_3 \oplus \{21, 43, 122, 167\}), \end{aligned}$$

$A \oplus B = \mathbb{Z}_{216}$ and (A, B) is an aperiodic tiling rhythmic canon.

In a first generalization of Theorem 3, rhythm B is the disjoint union of three sets, one being periodic both modulo n/p_1 and modulo n/p_2.

Theorem 4. *Let $n = p_1n_1p_2n_2n_3 \in \mathbb{N}$ such that:*

1. $p_1, n_1, p_2, n_2, n_3 > 1$;
2. $\gcd(p_1n_1, p_2n_2) = 1$;
3. *there is no prime q such that $q \mid n_3$, but $q \nmid p_1n_1p_2n_2$.*

Let H be the subgroup $H = n_3\mathbb{I}_{p_1n_1p_2n_2}$ of $\mathbb{Z}_n$ with $n = p_1n_1p_2n_2n_3$, K be a complete set of cosets representatives for $\mathbb{Z}_n$ modulo H such that K is the disjoint union $K = K_1 \sqcup K_2 \sqcup K_3$ with $K_1, K_2 \neq \varnothing$, and $W = n_3n_1n_2\mathbb{I}_{p_1p_2}$. Then the pair (A, B) defined by

$$\begin{aligned} A &= A_1 \oplus A_2 \\ B &= (U_1 \oplus V_2 \oplus K_1) \sqcup (U_2 \oplus V_1 \oplus K_2) \sqcup (W \oplus K_3) \end{aligned}$$

is an aperiodic tiling rhythmic canon of $\mathbb{Z}_n$.

Proof. The only case we need to consider is $K_3 \neq \varnothing$ (notice that this is possible only if $n_3 > 2$). We already know, from Theorem 2 that A is aperiodic; B is aperiodic too, since

$$B(x) = U_1(x)V_2(x)K_1(x) + U_2(x)V_1(x)K_2(x) + W(x)K_3(x)$$

and the cyclotomic polynomials $\Phi_{n_3p_2n_2n_1}$ and $\Phi_{n_3p_1n_1n_2}$ divide exactly 2 of the summands on the right hand side.

We now prove that $A \oplus B = \mathbb{Z}_n$: to this aim we make use of the following facts, proven by F. Jedrzejewski (Theorem 14 in [10]):

$$\begin{aligned} A_1 + U_1 + V_2 &= A_1 + U_1 + U_2 \\ A_2 + U_2 + V_1 &= A_2 + U_2 + U_1. \end{aligned}$$

By an easy check, we see that

$$U_1 + U_2 = n_3 n_1 n_2 \left(p_1 \mathbb{I}_{p_2} + p_2 \mathbb{I}_{p_1}\right) = n_3 n_1 n_2 \mathbb{Z}_{p_1 p_2} = W,$$

and $|U_1||U_2| = p_2 p_1 = |W|$. This means that

$$U_1 \oplus U_2 = W.$$

We obtain that

$$\begin{aligned} A + B &= (A_1 + A_2) + ((U_1 + V_2 + K_1) \sqcup (U_2 + V_1 + K_2) \sqcup (W + K_3)) \\ &= (A_1 + A_2 + U_1 + V_2 + K_1) \sqcup (A_1 + A_2 + U_2 + V_1 + K_2) \\ &\qquad \sqcup (A_1 + A_2 + W + K_3) \\ &= (A_1 + A_2 + U_1 + U_2 + K_1) \sqcup (A_1 + A_2 + U_2 + U_1 + K_2) \\ &\qquad \sqcup (A_1 + A_2 + U_1 + U_2 + K_3) \\ &= A_1 + A_2 + U_1 + U_2 + (K_1 \sqcup K_2 \sqcup K_3) \\ &= A_1 + U_1 + A_2 + U_2 + K. \end{aligned}$$

Again, an easy computation shows that

$$\begin{aligned} (A_1 + U_1) + (A_2 + U_2) &= n_3 p_1 n_1 \mathbb{I}_{p_2 n_2} + n_3 p_2 n_2 \mathbb{I}_{p_1 n_1} \\ &= n_3 \mathbb{I}_{p_1 n_1 p_2 n_2} \\ &= H \end{aligned}$$

and so

$$A + B = H + K = \mathbb{Z}_n.$$

Moreover, since $|A||B| = n = |H||K|$, the sum $A + B$ is direct. □

Example 3. Let us go back to $n = 216$ with the same choices of p_1, n_1, p_2, n_2, and n_3. By Theorem 4, we find a new aperiodic tiling rhythmic canon (A, B) defining

$$\begin{aligned} A &= 24\mathbb{I}_3 \oplus 54\mathbb{I}_2 \\ B &= (72\mathbb{I}_3 \oplus 12\mathbb{I}_2 \oplus \{0, 106\}) \sqcup (108\mathbb{I}_2 \oplus 18\mathbb{I}_3 \oplus \{21, 43\}) \sqcup (36\mathbb{I}_6 \oplus \{122, 167\}). \end{aligned}$$

The second generalization of Theorem 3 widens the definitions of sets A_1, A_2, V_1, and V_2. We precede it with a useful lemma.

Lemma 2. *Suppose that a subset $S \subseteq \mathbb{Z}_n$ is periodic of period $m \mid n$, i.e. $S + m = S$, and for $i = 0, \ldots, k-1$ let $S_i = \{a \in S : a \equiv i \mod k\}$ where k is a divisor of m. Then also the sets S_i are periodic of period m for every i.*

Proof. It is sufficient to observe that since m is a multiple of k the remainder classes modulo k are invariant by the translation by m, hence also $S_i + m = S_i$. □

Theorem 5. *Let $n = p_1 n_1 p_2 n_2 n_3 \in \mathbb{N}$ such that:*

1. $p_1, n_1, p_2, n_2, n_3 > 1$;
2. $\gcd(p_1 n_1, p_2 n_2) = 1$;
3. *there is no prime q such that $q \mid n_3$, but $q \nmid p_1 n_1 p_2 n_2$.*

Let H be the subgroup $H = n_3 \mathbb{I}_{p_1 n_1 p_2 n_2}$ of $\mathbb{Z}_n$, and $K = K_1 \sqcup K_2$ (with $K_1, K_2 \neq \varnothing$) be a complete set of cosets representatives for $\mathbb{Z}_n$ modulo H. Take

- $\tilde{A}_1$ *as a complete aperiodic set of coset representatives for $\mathbb{Z}_{p_2 n_2}$ modulo $n_2 \mathbb{I}_{p_2}$;*
- $\tilde{A}_2$ *as a complete aperiodic set of coset representatives for $\mathbb{Z}_{p_1 n_1}$ modulo $n_1 \mathbb{I}_{p_1}$;*
- $\tilde{V}_1^1, \ldots, \tilde{V}_1^j$ *as complete aperiodic sets of coset representatives for $\mathbb{Z}_{p_2 n_1}$ modulo $p_2 \mathbb{I}_{n_1}$;*
- $\tilde{V}_2^1, \ldots, \tilde{V}_2^h$ *as complete aperiodic sets of coset representatives for $\mathbb{Z}_{p_1 n_2}$ modulo $p_1 \mathbb{I}_{n_2}$.*

Set $K_1 = K_1^1 \sqcup \cdots \sqcup K_1^j$ and $K_2 = K_2^1 \sqcup \cdots \sqcup K_2^h$, where $K_\alpha^s = \left\{k_\alpha^{j_{s-1}+1}, \ldots, k_\alpha^{j_s}\right\}$ are non-empty subsets of K_α for $\alpha = 1, 2$. Then the pair (A, B) defined by

$$
\begin{aligned}
A &= n_3 p_1 n_1 \tilde{A}_1 \oplus n_3 p_2 n_2 \tilde{A}_2 \\
B &= \left(\left(U_1 \oplus n_3 n_1 \tilde{V}_2^1 \oplus \left\{k_1^1, \ldots, k_1^{l_1}\right\} \right) \sqcup \cdots \right. \\
&\quad \cdots \sqcup \left. \left(U_1 \oplus n_3 n_1 \tilde{V}_2^j \oplus \left\{k_1^{l_{j-1}+1}, \ldots, k_1^{|K_1|}\right\} \right) \right) \\
&\quad \sqcup \left(\left(U_2 \oplus n_3 n_2 \tilde{V}_1^1 \oplus \left\{k_2^1, \ldots, k_2^{m_1}\right\} \right) \sqcup \cdots \right. \\
&\quad \cdots \sqcup \left. \left(U_2 \oplus n_3 n_2 \tilde{V}_1^h \oplus \left\{k_2^{m_{h-1}+1}, \ldots, k_2^{|K_2|}\right\} \right) \right)
\end{aligned}
$$

is an aperiodic tiling rhythmic canon of $\mathbb{Z}_n$.

Proof. We have

- $n_3 p_1 n_1 \tilde{A}_1 + U_1 = n_3 p_1 n_1 \left(\tilde{A}_1 \oplus n_2 \mathbb{I}_{p_2} \right) = n_3 p_1 n_1 \mathbb{I}_{p_2 n_2} = A_1 + U_1$;
- $n_3 p_2 n_2 \tilde{A}_2 + U_2 = n_3 p_2 n_2 \left(\tilde{A}_2 \oplus n_1 \mathbb{I}_{p_1} \right) = n_3 p_2 n_2 \mathbb{I}_{p_1 n_1} = A_2 + U_2$;
- $A_1 + n_3 n_1 \tilde{V}_2 = n_3 n_1 \left(p_1 \mathbb{I}_{n_2} + \tilde{V}_2 \right) = n_3 n_1 \mathbb{I}_{p_1 n_2} = A_1 + V_2$;
- $A_2 + n_3 n_2 \tilde{V}_1 = n_3 n_2 \left(p_2 \mathbb{I}_{n_1} + \tilde{V}_1 \right) = n_3 n_2 \mathbb{I}_{p_2 n_1} = A_2 + V_1$.

For the sake of simplicity, we now give the proof in the case $j = 1$ and $h = 1$. The general case is completely analogous. We compute

$$\begin{aligned}
A + B &= \left(n_3 p_1 n_1 \tilde{A}_1 + n_3 p_2 n_2 \tilde{A}_2\right) \\
&\qquad + \left(\left(U_1 + n_3 n_1 \tilde{V}_2 + K_1\right) \sqcup \left(U_2 + n_3 n_2 \tilde{V}_1 + K_2\right)\right) \\
&= \left(n_3 p_1 n_1 \tilde{A}_1 + n_3 p_2 n_2 \tilde{A}_2 + U_1 + n_3 n_1 \tilde{V}_2 + K_1\right) \\
&\qquad \sqcup \left(n_3 p_1 n_1 \tilde{A}_1 + n_3 p_2 n_2 \tilde{A}_2 + U_2 + n_3 n_2 \tilde{V}_1 + K_2\right) \\
&= \left(A_1 + n_3 p_2 n_2 \tilde{A}_2 + U_1 + n_3 n_1 \tilde{V}_2 + K_1\right) \\
&\qquad \sqcup \left(n_3 p_1 n_1 \tilde{A}_1 + A_2 + U_2 + n_3 n_2 \tilde{V}_1 + K_2\right) \\
&= \left(A_1 + n_3 p_2 n_2 \tilde{A}_2 + U_1 + V_2 + K_1\right) \\
&\qquad \sqcup \left(n_3 p_1 n_1 \tilde{A}_1 + A_2 + U_2 + V_1 + K_2\right) \\
&= \left(A_1 + n_3 p_2 n_2 \tilde{A}_2 + U_1 + U_2 + K_1\right) \\
&\qquad \sqcup \left(n_3 p_1 n_1 \tilde{A}_1 + A_2 + U_2 + U_1 + K_2\right) \\
&= A_1 + A_2 + U_1 + U_2 + (K_1 \sqcup K_2) \\
&= A_1 + U_1 + A_2 + U_2 + K \\
&= \mathbb{Z}_n .
\end{aligned}$$

A cardinality argument analogous to that used in Theorem 4 shows that the sum is direct.

The proof that A is aperiodic follows from Vuza's argument (Proposition 2.2 in [14]), as above. Assume now that B is periodic of period a: we can assume without loss of generality that $a = n/p$ where p is a prime number. Hypothesis 3. now implies that a is a multiple of n_3: but then by Lemma 2 also the sets $B_i = B \cap (\{i\} + n_3\mathbb{Z}_n)$ must be periodic of period a. However, the sets B_i are simply translates of $U_1 \oplus n_3 n_1 \tilde{V}_2$ by elements of K_1 or of $U_2 \oplus n_3 n_2 \tilde{V}_1$ by elements of K_2 (remember that also the elements of U_1 and U_2 are multiple of n_3): on their turn, $U_1 \oplus n_3 n_1 \tilde{V}_2$ and $U_2 \oplus n_3 n_2 \tilde{V}_1$ are indeed periodic resp. of period n/p_1 and n/p_2, but since p_1 and p_2 are coprime no common period smaller than n is possible. A contradiction follows since we assumed both K_1 and K_2 to be non-empty. □

Remark 3. Note that Theorems 3–5 hold trivially if hypothesis 3. is replaced by the condition that n_3 is prime.

Example 4. This time we choose $n = 252$; let $p_1 = 2$, $n_1 = 7$, $p_2 = 3$, $n_2 = 3$, and $n_3 = 2$. We can take e.g.

$$\begin{aligned} \tilde{A}_1 &= \{0,2,7\} & \tilde{A}_2 &= \{0,1,3,4,9,12,13\} \\ \tilde{V}_1 &= \{0,10,17\} & \tilde{V}_2 &= \{0,1\} = \mathbb{I}_{p_1} \\ K_1 &= \{0\} & K_2 &= \{1\} \end{aligned}$$

obtaining a new canon (A, B) where

$$\begin{aligned} A &= 28\tilde{A}_1 \oplus 18\tilde{A}_2 \\ &= \{0,56,196\} \oplus \{0,18,54,72,162,216,234\} \\ B &= \left(U_1 \oplus 14\tilde{V}_2 \oplus K_1\right) \sqcup \left(U_2 \oplus 6\tilde{V}_1 \oplus K_2\right) \\ &= (\{0,84,168\} \oplus \{0,14\} \oplus \{0\}) \sqcup (\{0,126\} \oplus \{0,60,102\} \oplus \{1\})\,. \end{aligned}$$

Definition 4. *We call* Vuza canons *all the canons obtained using the constructions described in Theorems 2, 3, 4, 5.*

It is possible to stretch this type of constructions even further. With the following theorem, we improve the result of Jedrzejewski (Theorem 21 in [10]).

Theorem 6. *Let $n = p_1 n_1 p_2 n_2 n_3 \in \mathbb{N}$ such that:*

1. $p_1, n_1, p_2, n_2, n_3 > 1$*;*
2. $\gcd(p_1 n_1, p_2 n_2) = 1$*;*
3. *there is no prime q such that $q \mid n_3$, but $q \nmid p_1 n_1 p_2 n_2$.*

Let H be the subgroup $H = n_3 \mathbb{I}_{p_1 n_1 p_2 n_2}$ of $\mathbb{Z}_n$. Suppose that L and K are proper subsets of $\mathbb{Z}_{n_3}$ such that $L \oplus K = \mathbb{Z}_{n_3}$ and $K = K_1 \sqcup K_2$, with $K_1, K_2 \neq \varnothing$. Then the pair (A, B) defined by

$$\begin{aligned} A &= A_1 \oplus A_2 \oplus L \\ B &= (U_1 \oplus V_2 \oplus K_1) \sqcup (U_2 \oplus V_1 \oplus K_2) \end{aligned}$$

is an aperiodic tiling rhythmic canon of $\mathbb{Z}_n$.

Proof.

$$\begin{aligned} A + B &= (A_1 + A_2 + L) + ((U_1 + V_2 + K_1) \sqcup (U_2 + V_1 + K_2)) \\ &= (A_1 + A_2 + L + U_1 + V_2 + K_1) \sqcup (A_1 + A_2 + L + U_2 + V_1 + K_2) \\ &= (A_1 + A_2 + L + U_1 + U_2 + K_1) \sqcup (A_1 + A_2 + L + U_2 + U_1 + K_2) \\ &= A_1 + A_2 + L + U_1 + U_2 + (K_1 \sqcup K_2) \\ &= A_1 + U_1 + A_2 + U_2 + L + K. \end{aligned}$$

The sum is direct because the computation of the cardinality leads to

$$|A_1||A_2||U_1||U_2||L \oplus K| = n.$$

Aperiodicity of A is immediate from Lemma 2, since $A_1 + A_2$ is aperiodic, and B is the union of the subsets B_i contained in different remainder classes modulo n_3, some of which have a period coprime with the period of the other ones (exactly as in the previous theorem).

Example 5. Choosing again $n = 216$ and the same values for p_1, n_1, p_2, n_2, and n_3 as in Example 3, we set $L = \{0, 1\}$, $K_1 = \{2\}$, and $K_2 = \{0, 4\}$. By Theorem 6, we get that

$$A = 24\mathbb{I}_3 \oplus 54\mathbb{I}_2 \oplus L$$
$$B = (72\mathbb{I}_3 \oplus 12\mathbb{I}_2 \oplus K_1) \sqcup (108\mathbb{I}_2 \oplus 18\mathbb{I}_3 \oplus K_2)$$

define an aperiodic tiling rhytmic canon.

To prove our next result we take advantage of the equivalent polynomial formulation of tilings. Using it, in [4] E.M. Coven, and A. Meyerowitz introduced two sufficient conditions for a rhythm A to be a factor of a tiling rhythmic canon. To state them we need the following definitions.

Definition 5. *$R_A \doteqdot \{d : \Phi_d(x) \mid A(x)\}$ and $S_A \doteqdot \{p^\alpha \in R_A : p$ prime$\}$.*

The *Coven-Meyerowitz conditions* are the following:

T1 $|A| = \prod_{p^\alpha \in S_A} p$;
T2 for all $p^\alpha, q^\beta, r^\gamma, \ldots \in S_A$, $p^\alpha q^\beta r^\gamma \cdots \in R_A$, where $p^\alpha, q^\beta, r^\gamma, \ldots$ are powers of distinct primes.

The polynomial approach provides a few new important properties.

Lemma 3. *Let $A(x), B(x) \in \mathbb{N}[x]$ and $n \in \mathbb{N}^*$. Then*

$$A(x)B(x) \equiv \sum_{k=0}^{n-1} x^k \mod (x^n - 1) \qquad \text{(T0)}$$

if and only if

1. *$A(x), B(x) \in \{0, 1\}[x]$, so they are the characteristic polynomials of sets A and B, and*
2. *$A \oplus B = \{r_1, \ldots, r_n\} \subset \mathbb{Z}$, with $r_i \neq r_j \mod n$ for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$.*

Lemma 4. *Let $f(x) \in \mathbb{Z}[x]$ and $n \in \mathbb{N}^*$. The following are equivalent:*

1. *$f(x) \equiv \sum_{k=0}^{n-1} x^k \mod (x^n - 1)$;*
2. *(a) $f(1) = n$ and*
   (b) for every $d \mid n$, with $d > 1$, we have $\Phi_d(x) \mid f(x)$.

Definition 6. *Let A be a subset of $\mathbb{Z}_n$ and let $S_A = \{p^\alpha, q^\beta, \ldots, r^\gamma\}$. We call the* extension of A *any rhythm $\overline{A}$ whose characteristic polynomial is*

$$\overline{A}(x) = \Phi_{p^\alpha}\left(x^{\frac{n}{p^\alpha k_p}}\right) \Phi_{q^\beta}\left(x^{\frac{n}{q^\beta k_q}}\right) \cdots \Phi_{r^\gamma}\left(x^{\frac{n}{r^\gamma k_r}}\right).$$

where $k_p, k_q, \ldots, k_r$ are divisors of n such that $p \nmid k_p, q \nmid k_q, \ldots, r \nmid k_r$.

Note that by definition clearly $S_A = S_{\overline{A}}$.

Proposition 2. *Let $A \oplus B = \mathbb{Z}_n$ and let B satisfy condition (T2). Then $\overline{A} \oplus B = \mathbb{Z}_n$, too.*

Proof. Since p^α is a prime power, then

$$\Phi_{p^\alpha}\left(x^{\frac{n}{p^\alpha k_p}}\right) \in \{0,1\}[x],$$

and so $\overline{A}(x) \in \mathbb{N}[x]$. Moreover,

- $\overline{A}(1)B(1) = n$ and
- $\Phi_d(x) \mid \overline{A}(x)B(x)$ for all $d \mid n$, with $d > 1$.

By Lemma 4, this means that

$$\overline{A}(x)B(x) \equiv \sum_{k=0}^{n-1} x^k \mod (x^n - 1),$$

that is, condition (T0) in Lemma 3 holds. Therefore $\overline{A}(x) \in \{0,1\}[x]$ and $\overline{A} \oplus B = \mathbb{Z}_n$, that is, $\overline{A}$ tiles with B. □

Combining Theorem 6 and Proposition 2, we are able to find new Vuza canons where L is not a subset of $\mathbb{Z}_{n_3}$.

Theorem 7. *Let $n = p_1 n_1 p_2 n_2 n_3 \in \mathbb{N}$ such that:*

1. $p_1, n_1, p_2, n_2, n_3 > 1$;
2. $\gcd(p_1 n_1, p_2 n_2) = 1$;
3. *there is no prime q such that $q \mid n_3$, but $q \nmid p_1 n_1 p_2 n_2$.*

Let H be the subgroup $H = n_3 \mathbb{I}_{p_1 n_1 p_2 n_2}$ of $\mathbb{Z}_n$. Suppose that L and K are proper subsets of $\mathbb{Z}_{n_3}$ such that $L \oplus K = \mathbb{Z}_{n_3}$ and $K = K_1 \sqcup K_2$, with $K_1, K_2 \neq \emptyset$. Let $\tilde{L}$ be an extension of L; then the pair (A, B) defined by

$$\begin{aligned} A &= A_1 \oplus A_2 \oplus \tilde{L} \\ B &= (U_1 \oplus V_2 \oplus K_1) \sqcup (U_2 \oplus V_1 \oplus K_2) \end{aligned}$$

is an aperiodic tiling rhythmic canon of $\mathbb{Z}_n$.

Proof. Since, by definition, A_1 and A_2 coincide with their own extensions, the extension of $A_1 \oplus A_2 \oplus L$ is A. By Theorem 6, $A_1 \oplus A_2 \oplus L \oplus B = \mathbb{Z}_n$, therefore Proposition 2 implies that $A \oplus B = \mathbb{Z}_n$.

We already know from Theorem 6 that B is aperiodic. To show that A is aperiodic, consider $\tilde{L}(x)$. By hypothesis 3 $S_{\tilde{L}}$ does not contain any maximal prime power dividing n, as S_{A_1} and S_{A_2}. As a consequence, $S_A = S_{A_1} \cup S_{A_2} \cup S_{\tilde{L}}$ does not contain any such prime power, either. By Proposition 1, A can not be periodic. □

Definition 7. *We call* extended Vuza canons *all the canons obtained using the constructions of Theorems 6 and 7, possibly combined with those of Theorems 2, 3, 4 and 5.*

Example 6. We show now an extended Vuza canon with period $n = 240$ ($p_1 = 2, n_1 = 2, p_2 = 5, n_2 = 3, n_3 = 4$). Set $L = \mathbb{I}_2$; then $\tilde{L} = 15\mathbb{I}_2$. Choosing $K_1 = \{2\}$ and $K_2 = \{0\}$, we obtain the canon

$$\begin{aligned} A &= A_1 \oplus A_2 \oplus \tilde{L} \\ &= 16\mathbb{I}_3 \oplus 60\mathbb{I}_2 \oplus 15\mathbb{I}_2 \\ B &= (U_1 \oplus V_2 \oplus K_1) \sqcup (U_2 \oplus V_1 \oplus K_2) \\ &= (48\mathbb{I}_5 \oplus 8\mathbb{I}_2 \oplus \{2\}) \sqcup (120\mathbb{I}_2 \oplus 12\mathbb{I}_5 \oplus \{0\}) . \end{aligned}$$

It is worth noting that it would not be possible to obtain such a canon without applying Theorem 7.

We include below a table showing the number of Vuza canons and extended Vuza canons for all the periods n with values between 72 and 280 (Table 1).

As a final comment, one could say that the recipes by Hajós, de Bruijn and Vuza to generate aperiodic tiling rhythmic canons are deceivingly simple.

Table 1. The number of aperiodic rhythms for non-Hajós values of n from 72 to 280, generated with the constructions described in Theorems 2–7.

n	p_1	n_1	p_2	n_2	n_3	L	$\#K$	$\#A$			$\#B$				
Theorem:								(2)	(6)	(7)	(2)	(3)	(4)	(5)	(6)
72	2	2	3	3	2	{0}	1	3	0	0	6	0	0	0	0
108	2	2	3	3	3	{0}	1	3	0	0	180	0	72	0	0
120	2	2	3	5	2	{0}	1	16	0	0	20	0	0	0	0
	2	2	5	3	2	{0}	1	8	0	0	18	0	0	0	0
144	2	2	3	3	4	{0}	1	3	0	0	2808	1944	3888	0	0
	2	2	3	3	4	{0, 1}	2	0	312	0	0	0	0	0	6
	2	2	3	3	4	{0, 9}	2	0	0	12	0	0	0	0	6
	2	2	3	3	4	{0, 2}	4	0	156	0	0	0	0	0	12
	2	4	3	3	2	{0}	1	6	0	0	12	0	0	48	0
	4	2	3	3	2	{0}	1	6	0	0	6	0	0	30	0
168	2	2	3	7	2	{0}	1	104	0	0	14	0	0	28	0
	2	2	7	3	2	{0}	1	16	0	0	6	0	0	48	0
180	2	5	3	3	2	{0}	1	9	0	0	15	0	0	105	0
	5	2	3	3	2	{0}	1	6	0	0	6	0	0	90	0
	3	5	2	2	3	{0}	1	16	0	0	500	0	200	1100	0
	5	3	2	2	3	{0}	1	8	0	0	252	0	72	1728	0
	2	2	3	3	5	{0}	1	3	0	0	45360	77760	158112	0	0

(*continued*)

Table 1. (*continued*)

n	p_1	n_1	p_2	n_2	n_3	L	$\#K$	$\#A$			$\#B$				
Theorem:								(2)	(6)	(7)	(2)	(3)	(4)	(5)	(6)
200	2	2	5	5	2	{0}	1	125	0	0	10	0	0	50	0
216	2	4	3	3	3	{0}	1	6	0	0	180 + 540	72 + 216	0	12672	0
	2	2	3	3	6	{0, 3}	8	0	156	0	0	0	0	0	180 + 540 + 72 + 216
	2	2	3	3	6	{0, 1}	2	0	324	0	0	0	0	0	180 + 72
	2	2	3	3	6	{0}	1	3	0	0	754272	2449440	5832000	0	0
	2	2	3	3	6	{0, 1, 2}	3	0	34992	0	0	0	0	0	6
	2	2	3	3	6	{0, 2, 4}	9	0	10935	0	0	0	0	0	6 + 12
	2	2	3	9	2	{0}	1	729	0	0	6 + 12	0	0	54	0
	2	2	9	3	2	{0}	1	27	0	0	6	0	0	162	0
	4	2	3	3	3	{0}	1	6	0	0	252	0	72	5940	0
240	2	4	3	5	2	{0}	1	32	0	0	20	0	0	20 + 160	0
	2	2	3	5	4	{0, 6}	4	0	0	588	0	0	0	0	20 + 20
	2	2	3	5	4	{0, 2}	4	0	7252	0	0	0	0	0	20 + 20
	2	2	3	5	4	{0, 15}	2	0	0	64	0	0	0	0	20
	2	2	3	5	4	{0, 3}	2	0	1176	0	0	0	0	0	20
	2	2	3	5	4	{0, 1}	2	0	14504	0	0	0	0	0	20
	2	2	3	5	4	{0}	1	16	0	0	13000	9000	18000	94000	0
	2	2	5	3	4	{0}	1	8	0	0	6264	3240	5184	197856	0
	2	2	5	3	4	{0, 1}	2	0	4016	0	0	0	0	0	18
	2	2	5	3	4	{0, 5}	2	0	0	112	0	0	0	0	18
	2	2	5	3	4	{0, 15}	2	0	0	32	0	0	0	0	18
	2	2	5	3	4	{0, 2}	4	0	2008	0	0	0	0	0	12 + 24
	2	2	5	3	4	{0, 10}	4	0	0	56	0	0	0	0	12 + 24
	2	4	5	3	2	{0}	1	16	0	0	12	0	0	24 + 576	0
	4	2	3	5	2	{0}	1	32	0	0	10	0	0	290	0
	4	2	5	3	2	{0}	1	16	0	0	6	0	0	102	0
252	2	7	3	3	2	{0}	1	27	0	0	21	0	0	315	0
	7	2	3	3	2	{0}	1	9	0	0	6	0	0	618	0
	3	7	2	2	3	{0}	1	104	0	0	980	0	392	5096	0
	7	3	2	2	3	{0}	1	16	0	0	324	0	72	21312	0
	2	2	3	3	7	{0}	1	3	0	0	12830400	71383680	206126208	0	0
264	2	2	3	11	2	{0}	1	5368	0	0	22	0	0	88	0
	2	2	11	3	2	{0}	1	40	0	0	6	0	0	552	0
270	3	3	2	5	3	{0}	1	9	0	0	1125	0	450	48825	0
	3	3	5	2	3	{0}	1	6	0	0	288	0	72	48600	0

(*continued*)

Table 1. (*continued*)

n	p_1	n_1	p_2	n_2	n_3	L	$\#K$	$\#A$			$\#B$				
Theorem:								(2)	(6)	(7)	(2)	(3)	(4)	(5)	(6)
280	2	2	5	7	2	{0}	1	2232	0	0	14	0	0	112	0
	2	2	7	5	2	{0}	1	480	0	0	10	0	0	170	0

Note: In each column only the rhythms that can be generated by the corresponding theorem, but not by previous ones are counted. Grey numbers correspond to rhythms that can be generated also by the choice of parameters in the previous line. When there is no column (e.g., $\#A$ for Theorem 3) all the possible rhythms already appear in previous columns.

Their basic mechanism can be (and has indeed been) generalised in several ways; this paper gives a generalisation on its own, but Theorem 7 can certainly still be improved. Further studies should follow, aiming at lifting the hypotheses used in the present results and (hopefully) at establishing a systematic theory of aperiodic tiling rhythmic canons given by all the known constructions, and eventually of all aperiodic tiling rhythmic canons straightaway.

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