Chapter 2
V. Yankov's Contributions to Propositional Logic

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Abstract I give an exposition of the papers by Yankov published in the 1960s in which he studied positive and some intermediate propositional logics, and where he developed a technique that has successfully been used ever since.

Keywords Yankov's formula · Characteristic formula · Intermediate logic · Implicative lattice · Weak law of excluded middle · Yankov's logic · Positive logic · Logic of realizability · Heyting algebra

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2.1 Introduction

V. Yankov started his scientific career in early 1960s while writing his Ph.D. thesis under A. A. Markov's supervision. Yankov defended thesis "Finite implicative lattices and realizability of the formulas of propositional logic" in 1964. In 1963, he published three short papers Jankov (1963a, b, c) and later, in Jankov (1968a, b, c, d, 1969), he provided detailed proofs together with new results. All these papers are primarily concerned with studying *super-intuitionistic* (or super-constructive, as he called them) propositional logics, that is, logics extending the intuitionistic propositional logic Int. Throughout the present paper, the formulas are propositional formulas in the signature $\to, \wedge, \vee, \mathfrak{f}$, and as usual, $\neg p$ denotes $p \to \mathfrak{f}$ and $p \leftrightarrow q$ denotes $(p \to q) \wedge (q \to p)$; the logics are the sets of formulas closed under the rules Modus Ponens and substitution.

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To put Yankov's achievements in a historical context, we need to recall that Int was introduced by Heyting (cf. Heyting 1930[1]), who defined it by a calculus denoted by IPC as an attempt to construct a propositional logic addressing Brouwer's critique of the law of excluded middle and complying with intuitionistic requirements. Soon after, Gödel (cf. Gödel 1932) observed that Int cannot be defined by any finite set of finite logical matrices and that there is a strongly descending (relative to set-inclusion) set of super-intuitionistic logics (*si-logics* for short); thus, the set of si-logics is infinite. Gödel also noted that IPC possesses the following property: for any formulas A, B, if IPC $\vdash (A \vee B)$, then IPC $\vdash A$, or IPC $\vdash B$—the *disjunction property*, which was later proved by Gentzen.

Even though Int cannot be defined by any finite set of finite matrices, it turned out that it can be defined by an infinite set of finite matrices (cf. Jaśkowski 1936), in other words, Int enjoys the finite model property (f.m.p. for short). This led to a conjecture that every si-logic enjoys the f.m.p., which entails that every si-calculus is decidable.

At the time when Yankov started his research, there were three objectives in the area of si-logics: (a) to find a logic that has semantics suitable from the intuitionistic point of view, (b) to study the class of si-logics in more details, and (c) to construct a convenient algebraic semantics.

By the early 1960s the original conjecture that Int is the only si-logic enjoying the disjunction property and that the realizability semantics introduced by Kleene is adequate for Int were refuted: in Kreisel and Putnam (1957), it was shown that the logic of IPC endowed with axiom $(\neg p \to (q \vee r)) \to ((\neg p \to q) \vee (\neg p \to r))$ is strictly larger than Int, and in Rose (1953), a formula that is realizable but not derivable in IPC was given. Using the technique developed by Yankov, Wroński proved that in fact, there are continuum many si-logics enjoying the disjunction property (cf. Wroński 1973).

In Heyting (1941), Heyting suggested an algebraic semantics, and in 1940s, McKinsey and Tarski introduced an algebraic semantics based on topology. In his Ph.D. (Rieger 1949), which is not widely known even nowadays, Rieger essentially introduced what is called a "Heyting algebra," and in Rieger (1957), he constructed an infinite set of formulas on one variable that are mutually non-equivalent in IPC. It turned out (cf. Nishimura 1960) that every formula on one variable is equivalent in IPC to one of Rieger's formulas. We need to keep in mind that the book (Rasiowa and Sikorski 1963) was published only in 1963. In 1972, this book had been translated into Russian by Yankov, and it greatly influenced the studies in the area of si-logics.

By the 1960s, it also became apparent that the structure of the lattice of the si-logics is more complex than expected: in Umezawa (1959) it has was observed that the class of si-logics contains subsets of the order type of ω^ω; in addition, it contains infinite subsets consisting of incomparable relative to set-inclusion logics.

Generally speaking, there are two ways of defining a logic: semantically by logical matrices or algebras, and syntactically, by calculus. In any case, it is natural to ask whether two given logical matrices, or two given calculi define the same logic. More

[1] The first part was translated in Heyting (1998).

precisely, is there an algorithm that, given two finite logical matrices decides whether their logics coincide, and is there an algorithm that given two formulas A and B decides whether calculi $\mathsf{IPC} + A$ and $\mathsf{IPC} + B$ define the same logic? The positive answer to the first problem was given in Łoś (1949). But in Kuznetsov (1963), it was established that in a general case (in the case when one of the logics can be not s.i.), the problem of equivalence of two calculi is unsolvable. Note that if every si-logic enjoys the f.m.p., then every si-calculus would be decidable and consequently, the problem of equivalence of two calculi would be decidable as well.

In Jankov (1963a), Yankov considers four calculi:

(a) $\mathsf{CPC} = \mathsf{IPC} + (\neg\neg p \to p)$—the classical propositional calculus;
(b) $\mathsf{KC} = \mathsf{IPC} + (\neg p \vee \neg\neg p)$—the calculus of the weak law of excluded middle (nowadays the logic of KC is referred to as Yankov's logic);
(c) $\mathsf{BD}_2 = \mathsf{IPC} + ((\neg\neg p \wedge (p \to q) \wedge ((q \to p) \to p)) \to q)$;
(d) $\mathsf{SmC} = \mathsf{IPC} + (\neg p \vee \neg\neg p) + ((\neg\neg p \wedge (p \to q) \wedge ((q \to p) \to p)) \to q)$—the logic of SmC is referred to as Smetanich's logic and it can be also defined by $\mathsf{IPC} + ((p \to q) \vee (q \to r) \vee (r \to s))$

and he gives a criterion for a given formula to define it relative to IPC (cf. Sect. 2.7). In Jankov (1968a), Yankov studied the logic of KC, and he proved that it is the largest si-logic having the same positive fragment as Int. Moreover, in Jankov (1968d), Yankov showed that the positive logic, which is closely related to the logic of KC, contains infinite sets of mutually non-equivalent, strongly descending, and strongly ascending chains of formulas (cf. Sect. 2.6).

Independently, a criterion that determines by a given formula A whether $\mathsf{Int} + A$ defines Cl was found in Troelstra (1965). In Jankov (1968c), Yankov gave a proof of this criterion as well as a proof of a similar criterion for Johansson's logic (cf. Sect. 2.5).

In Jankov (1963b), Yankov constructed infinite sets of realizable formulas that are not derivable in IPC and that are not derivable from each other. Moreover, he presented the seven-element Heyting algebra in which all realizable formulas are valid (cf. Sect. 2.8).

Jankov (1963c) is perhaps the best-known Yankov's paper, and it is one of the most quoted papers even today. In this paper, Yankov established a close relation between syntax and algebraic semantics: with every finite subdirectly irreducible Heyting algebra $\mathbf{A}$ he associates a formula $X_{\mathbf{A}}$—a characteristic formula of $\mathbf{A}$, such that for every formula B, the refutability of B in $\mathbf{A}$ (i.e. $\mathbf{A} \not\models B$) is equivalent to $\mathsf{IPC} + B \vdash X_{\mathbf{A}}$. Jankov (1963c) is a short paper and does not contain proofs. The proofs and further results in this direction are given in Jankov (1969), and we discuss them in Sect. 2.3. Let us point out that characteristic formulas in a slightly different form were independently discovered in de Jongh (1968).

Applying the developed machinery of characteristic formulas, Yankov proved (cf. Jankov 1968b) that there are continuum many distinct si-logics, and that among them there are logics lacking the f.m.p. Because the logic without the f.m.p. presented by Yankov was not finitely axiomatizable, it left a hope that perhaps all si-calculi enjoy the f.m.p. (this conjecture was refuted in Kuznetsov and Gerčiu 1970.)

Let us start with the basic definitions used in Yankov's papers.

2.2 Classes of Logics and Their Respective Algebraic Semantics

2.2.1 Calculi and Their Logics

Propositional formulas are formulas built in a regular way from a denumerable set of propositional variables Var and connectives.

Consider the following six propositional calculi with axioms from the following formulas:

$$p \to (q \to p); \quad (p \to (q \to r)) \to ((p \to q) \to (p \to r)); \tag{I}$$
$$(p \wedge q) \to p; \quad (p \wedge q) \to q; \quad p \to (q \to (p \wedge q)); \tag{C}$$
$$p \to (p \vee q); \quad q \to (p \vee q); \quad (p \to r) \to ((q \to r) \to ((p \vee q) \to r)); \tag{D}$$
$$\mathfrak{f} \to p. \tag{N}$$

they have inference rules Modus Ponens and substitution:

Calculus	Connectives	Axioms	Description	Logic
IPC	$\to, \wedge, \vee, \mathfrak{f}$	I,C,D,N	intuitionistic	Int
MPC	$\to, \wedge, \vee, \mathfrak{f}$	I,C,D	minimal or Johansson's	Min
PPC	$\to, \wedge, \vee$	I,C,D	positive	Pos
IPC$^-$	$\to, \wedge, \mathfrak{f}$	I,C,N	$\{\to, \wedge, \mathfrak{f}\}$ – fragment of IPC	Int$^-$
MPC$^-$	$\to, \wedge, \mathfrak{f}$	I,C	$\{\to, \wedge, \mathfrak{f}\}$ – fragment of MPC	Min$^-$
PPC$^-$	$\to, \wedge$	I,C	$\{\to, \wedge, \}$ – fragment of PPC	Pos$^-$

If $\Sigma \subseteq \{\to, \wedge, \vee, \mathfrak{f}\}$, by a Σ-formula we understand a formula containing connectives only from Σ and in virtue of the Separation Theorem (cf., e.g., Kleene 1952, Theorem 49): for every $\Sigma \in \{\{\to, \wedge, \vee\}, \{\to, \wedge, \mathfrak{f}\}, \{\to, \wedge\}\}$, if A is a C-formula $\{\to, \wedge\}$-formula, IPC $\vdash A$ if and only if PPC $\vdash A$ or IPC$^- \vdash A$, or PPC$^- \vdash A$.

By a C-calculus we understand one of the six calculi under consideration, and a C-logic is a logic of the C-calculus. Accordingly, C-formulas are formulas in the signature of the C-calculus. For C-formulas A and B, by $A \overset{C}{\vdash} B$ we denote that formula B is derivable in the respective C-calculus extended by axiom B; that is, $C + A \overset{C}{\vdash} B$.

The relation between PPC and MPC (or between PPC$^-$ and MPC$^-$) is a bit more complex: for any formula $\{\to, \wedge, \vee, \mathfrak{f}\}$-formula A (or any $\{\to, \wedge, \mathfrak{f}\}$-formula A), MPC $\vdash A$ (or MPC$^- \vdash A$) if and only if PPC $\vdash A'$ (or PPC$^- \vdash A'$), where A' is a formula obtained from A by replacing all occurrences of $\mathfrak{f}$ with a propositional variable not occurring in A (cf., e.g., Odintsov 2008, Chap. 2). In virtue of the Separation Theorem, in the previous statement, PPC or PPC$^-$ can be replaced with IPC or IPC$^-$, respectively.

Figure 2.1 shows the relations between the introduced logics: a double edge depicts an extension of the logic without any extension of the language (e.g., Min $\subset$ Int),

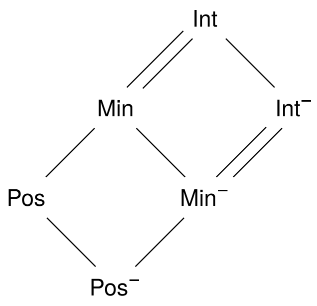


Fig. 2.1 Logics

while a single edge depicts an extension of the language but not of the class of theorems (e.g., if A is a $\{\to, \wedge, \neg\}$-formula, then $A \in \mathsf{Int}$ if and only if $A \in \mathsf{Int}^{-}$).

Let us observe that $((p \to \neg q) \to (q \to \neg p)) \in \mathsf{Min}^{-} \subseteq \mathsf{Min}$. Indeed, formula $(p \to (q \to r)) \to (q \to (p \to r))$ can be derived from the axioms (I). Hence, formula $(p \to (q \to \mathfrak{f})) \to (q \to (p \to \mathfrak{f}))$ is derivable too, that is, $(p \to \neg q) \to (q \to \neg p)$ is derivable in MPC^{-}.

We use **Ext**Int, **Ext**Min, **Ext**Pos, **Ext**Int^{-}, **Ext**Min^{-}, **Ext**Pos^{-} to denote classes of logics extending, respectively, Int, Min, Pos, Int^{-}, Min^{-}, and Pos^{-}. Thus, **Ext**Int is a class of all si-logics.

2.2.2 Algebraic Semantics

As pointed out in the Introduction, the first Yankov papers were written before the book by Rasiowa and Sikorski (1963) was published, and the terminology used by Yankov in his early papers was, as he himself admitted in Jankov (1968b), misleading. What he then called an "implicative lattice"[2] he later called a "Brouwerian algebra," and then he finally settled with the term "pseudo-Boolean algebra". We use a commonly accepted terminology, which we clarify below.

2.2.2.1 Correspondences Between Logics and Classes of Algebras.

In a meet-semilattice $\mathbf{A} = (\mathsf{A}; \wedge)$ an element c is a *complement of element* a *relative to element* b if c is the greatest element of A such that $\mathsf{a} \wedge \mathsf{c} \le \mathsf{b}$ (e.g. Rasiowa 1974a). If a semilattice $\mathbf{A}$ for any elements a and b contains a complement of a relative to b, we say that $\mathbf{A}$ is a *semilattice with relative pseudocomplementation*, and we denote the relative pseudocomplementation by $\to$.

[2] In some translations of the Yankov paper, this term was translated as "implicative structure" (e.g. Jankov 1963a).

Proposition 2.1 *Suppose that* $\mathbf{A}$ *is a meet-semilattice and* $\mathsf{a}, \mathsf{b}, \mathsf{c} \in \mathbf{A}$. *If* $\mathsf{a} \to \mathsf{b}$ *and* $\mathsf{a} \to \mathsf{c}$ *are defined in* $\mathbf{A}$*, then* $\mathsf{a} \to (\mathsf{b} \wedge \mathsf{c})$ *is defined as well and*

$$\mathsf{a} \to (\mathsf{b} \wedge \mathsf{c}) = (\mathsf{a} \to \mathsf{b}) \wedge (\mathsf{a} \to \mathsf{c}).$$

Proof Suppose that $\mathbf{A}$ is a meet-semilattice in which $\mathsf{a} \to \mathsf{b}$ and $\mathsf{a} \to \mathsf{c}$ are defined. We need to show that $(\mathsf{a} \to \mathsf{b}) \wedge (\mathsf{a} \to \mathsf{c})$ is the greatest element of $\mathbf{A}' := \{\mathsf{d} \in \mathbf{A} \mid \mathsf{a} \wedge \mathsf{d} \leq \mathsf{b} \wedge \mathsf{c}\}$.

First, we observe that $(\mathsf{a} \to \mathsf{b}) \wedge (\mathsf{a} \to \mathsf{c}) \in \mathbf{A}'$:

$$(\mathsf{a} \to \mathsf{b}) \wedge (\mathsf{a} \to \mathsf{c}) \wedge \mathsf{a} = (\mathsf{a} \wedge (\mathsf{a} \to \mathsf{b})) \wedge (\mathsf{a} \wedge (\mathsf{a} \to \mathsf{c})) \leq \mathsf{b} \wedge \mathsf{c},$$

because by the assumption, $\mathsf{a} \wedge (\mathsf{a} \to \mathsf{b}) \leq \mathsf{b}$ and $\mathsf{a} \wedge (\mathsf{a} \to \mathsf{c}) \leq \mathsf{c}$.

Next, we show that $(\mathsf{a} \to \mathsf{b}) \wedge (\mathsf{a} \to \mathsf{c})$ is the greatest element of $\mathbf{A}'$. Indeed, suppose that $\mathsf{d} \in \mathbf{A}'$. Then, $\mathsf{a} \wedge \mathsf{d} \leq \mathsf{b} \wedge \mathsf{c}$ and consequently,

$$\mathsf{a} \wedge \mathsf{d} \leq \mathsf{b} \text{ and } \mathsf{a} \wedge \mathsf{d} \leq \mathsf{c}.$$

Hence, by the definition of relative pseudocomplementation,

$$\mathsf{d} \leq \mathsf{a} \to \mathsf{b} \text{ and } \mathsf{d} \leq \mathsf{a} \to \mathsf{c},$$

which means that $\mathsf{d} \leq (\mathsf{a} \to \mathsf{b}) \wedge (\mathsf{a} \to \mathsf{c})$.

By an *implicative semilattice* we understand an algebra $(\mathsf{A}; \to, \wedge, \mathbf{1})$, where $(\mathsf{A}; \wedge)$ is a meet-semilattice with the greatest element $\mathbf{1}$ and $\to$ is a relative pseudocomplementation and accordingly, an algebra $(\mathsf{A}; \to, \wedge, \vee, \mathbf{1})$ is an *implicative lattice* if $(\mathsf{A}; \wedge\vee, \mathbf{1})$ is a lattice and $(\mathsf{A}; \to, \wedge, \mathbf{1})$ is an implicative semilattice (cf. Rasiowa 1974a). In implicative lattices, $\mathbf{0}$ denotes a constant (0-ary operation) that is the smallest element.

The logics described in the previous section have the following algebraic semantics:

Logic	Signature	Algebraic semantic	Denotation
Pos^-	$\{\to, \wedge, \mathbf{1}\}$	implicative semilattices	**BS**
Pos	$\{\to, \wedge, \vee, \mathbf{1}\}$	implicative lattices	**BA**
Min^-	$\{\to, \wedge, \mathfrak{f}, \mathbf{1}\}$	implicative semilattices with constant	**JS**
Min	$\{\to, \wedge, \vee, \mathfrak{f}, \mathbf{1}\}$	implicative semilattices with constant	**JA**
Int^-	$\{\to, \wedge, \mathbf{0}, \mathbf{1}\}$	bounded implicative semilattices	**HS**
Int	$\{\to, \wedge, \vee, \mathbf{0}, \mathbf{1}\}$	bounded implicative lattices	**HA**

As usual, in **JS** and **JA**, we let $\neg\mathsf{a} = \mathsf{a} \to \mathfrak{f}$, while in **HS** and **HA**, $\neg\mathsf{a} = \mathsf{a} \to \mathbf{0}$. Also, we use the following denotations: $\mathcal{L} := \{\mathsf{Pos}^-, \mathsf{Pos}, \mathsf{Min}^-, \mathsf{Min}, \mathsf{Int}^-, \mathsf{Int}\}$ and $\mathcal{A} := \{\mathbf{BS}, \mathbf{BA}, \mathbf{JS}, \mathbf{JA}, \mathbf{HS}, \mathbf{HA}\}$. For each $\boldsymbol{L} \in \mathcal{L}$, $\mathrm{Mod}(\boldsymbol{L})$ denotes the respective class of algebras. By a C-algebra we shell understand an algebra in the signature

$\Sigma \cup \{\mathbf{1}\}$, and we assume that Σ is always a signature of one of the six classes of logics under consideration.

Every class from $\mathcal{A}$ forms a variety. Moreover, **HS** and **HA** are subvarieties of, respectively, **JS** and **JA** defined by the identity $\mathfrak{f} \to x = \mathbf{1}$.

Remark 2.1 Let us observe that **BS** is a variety of all Brouwerian semilattices, and it was studied in detail in (cf. Köhler 1981); **BA** is a variety of all Brouwerian algebras (cf. Galatos et al. 2007); **JA** is a variety of all Johansson's algebras (j-algebras; cf. Odintsov 2008); and **HA** is a variety of all Heyting or pseudo-Boolean algebras (cf. Rasiowa and Sikorski 1963).

Let us recall the following properties of C-algebras.

Proposition 2.2 *The following holds:*

(a) every Brouwerian algebra forms a distributive lattice;
(b) every finite distributive lattice forms a Brouwerian algebra, and because it always contains the least element, it forms a Heyting algebra as well;
(c) every finite **BS***-algebra forms a Brouwerian algebra.*

(a) and (b) were observed in Rasiowa and Sikorski (1963) and Birkhoff (1948). (c) follows from the observation that in any finite **BS**-algebra **A**, for any two elements $\mathsf{a}, \mathsf{b} \in \mathbf{A}$, $\mathsf{a} \vee \mathsf{b}$ can be defined as a meet of $\{\mathsf{c} \in \mathbf{A} \mid \mathsf{a} \leq \mathsf{c}, \mathsf{b} \leq \mathsf{c}\}$.

As usual, given a formula A and a C-algebra, a map $\nu : Var \longrightarrow \mathbf{A}$ is called a *valuation* in **A**, and ν allows us to calculate a value of A in **A** by treating the connectives as operations of **A**. If $\nu(A) = \mathbf{1}$ for all valuations, we say that A is *valid* in **A**, in symbols, $\mathbf{A} \models A$. If for some valuation ν, $\nu(A) \neq \mathbf{1}$, we say that A is *refuted* in **A**, in symbols, $\mathbf{A} \not\models A$, in which case ν is called a *refuting valuation*. For a class of algebras $\mathbb{K}$, $\mathbb{K} \models A$ means that A is valid in every member of $\mathbb{K}$. Given a class of C-algebras $\mathbb{K}$, $\mathbb{K}_{fin}$ is a subclass of all finite members of $\mathbb{K}$.

For every logic $\boldsymbol{L} \in \mathcal{L}$, a respective class from $\mathcal{A}$ is denoted by $\mathrm{Mod}(\boldsymbol{L})$. A class of models $\mathbb{M}$ of logic $\boldsymbol{L}$ forms an *adequate algebraic semantics* of $\boldsymbol{L}$ if for each formula A, $A \in \boldsymbol{L}$ if and only if A is valid in all algebras from $\mathbb{M}$.

Proposition 2.3 *For every* $\boldsymbol{L} \in \mathcal{L}$ *class* $\mathrm{Mod}(\boldsymbol{L})$ *forms an adequate algebraic semantics. Moreover, each logic* $\boldsymbol{L} \in \mathcal{L}$ *enjoys the f.m.p.; that is,* $A \in \boldsymbol{L}$ *if and only if* $\mathrm{Mod}(\boldsymbol{L})_{fin} \models A$.

Proof The proofs of adequacy can be found in Rasiowa (1974a). The f.m.p. for Int follows from Jaśkowski (1936). The f.m.p. for Int^-, Pos, Pos^- follows from the f.m.p. for Int and the Separation Theorem.

As we mentioned earlier, for any formula A, $A \in \mathsf{Min}$ (or $A \in \mathsf{Min}^-$) if and only if $A^f \in \mathsf{Int}$ (or $A \in Int^-$), where A^f is a formula obtained from A by replacing every occurrence of $\mathfrak{f}$ with a new variable p. Because Int (and Int^-) enjoys the f.m.p., if $A \notin \mathsf{Min}$ (or $A \notin \mathsf{Min}^-$), there is a finite Heyting algebra **A** refuting A^f (finite **HS**-algebra refuting A^f). If ν is a refuting valuation, we can convert **A** into a **JA**-algebra (or into a **JS**-algebra) by regarding A as a Brouwerian algebra (or a Brouwerian semilattice) with $\mathfrak{f}$ being $\nu(A)$. It is clear that A is refuted in such a **JA**-algebra (**JS**-algebra).

2.2.2.2 Meet-Irreducible Elements

Let $\mathbf{A} = (\mathsf{A}; \wedge)$ be a meet-semilattice and $\mathsf{a} \in \mathbf{A}$. Element a is called *meet-irreducible*, if for every pair of elements b, c, $\mathsf{a} = \mathsf{b} \wedge \mathsf{c}$ entails that $\mathsf{a} = \mathsf{b}$ or $\mathsf{a} = \mathsf{c}$. And a is called *meet-prime* if $\mathsf{a} \leq \mathsf{b} \wedge \mathsf{c}$ entails that $\mathsf{a} = \mathsf{b}$ or $\mathsf{a} = \mathsf{c}$. For formulas where $\wedge$ is a conjunction, instead of meet-irreducible or meet-prime we say *conjunctively-irreducible* or *conjunctively-prime*.

If $\mathbf{A}$ is a semilattice, then elements a, b of $\mathbf{A}$ are *comparable* if $\mathsf{a} \leq \mathsf{b}$ or $\mathsf{b} \leq \mathsf{a}$, otherwise these elements are *incomparable*. A set of mutually incomparable elements is called an *antichain*. It is not hard to see that a meet of any finite set of elements is equal to a meet of a finite subset of mutually incomparable elements.

It is clear that every meet-prime element is meet-irreducible. In the distributive lattices, the converse holds as well.

The meet-irreducible elements play a role similar to that of prime numbers: every positive natural number is a product of primes. As usual, if a is an element of a semilattice, the representation $\mathsf{a} = \mathsf{a}_1 \wedge \cdots \wedge \mathsf{a}_n$ of a as a meet of finitely many meet-prime elements $\mathsf{a}_i, i \in [1, n]$ is called a *finite decomposition* of a. This finite decomposition is *irredundant* if no factor can be omitted.

It is not hard to see that because the factors in a finite decomposition are meet-irreducible, the decomposition is irredundant if and only if the elements of its factors are mutually incomparable.

Proposition 2.4 *In any semilattice, if element* a *has a finite decomposition,* a *has a unique (up to an order of factors) irredundant finite decomposition. Thus, in finite semilattices, every element has a unique irredundant finite decomposition.*

Proof Indeed, if element a has two finite irredundant decompositions $\mathsf{a} = \mathsf{a}_1 \wedge \cdots \wedge \mathsf{a}_n$ and $\mathsf{a} = \mathsf{a}'_1 \wedge \cdots \wedge \mathsf{a}'_m$, then $\mathsf{a}_1 \wedge \cdots \wedge \mathsf{a}_n = \mathsf{a}'_1 \wedge \cdots \wedge \mathsf{a}'_m$ and

$$(\mathsf{a}_1 \wedge \cdots \wedge \mathsf{a}_n) \to (\mathsf{a}'_1 \wedge \cdots \wedge \mathsf{a}'_m) = \mathbf{1}.$$

Hence, for each $j \in [1, m]$,

$$(\mathsf{a}_1 \wedge \cdots \wedge \mathsf{a}_n) \to \mathsf{a}'_j = \mathbf{1}; \text{ that is, } (\mathsf{a}_1 \wedge \cdots \wedge \mathsf{a}_n) \leq \mathsf{a}'_j.$$

Because a'_j is meet-prime, $\mathsf{a}'_\mathsf{j} \in \{\mathsf{a}_1, \ldots, \mathsf{a}_\mathsf{n}\}$ and thus, $\{\mathsf{a}'_1, \ldots, \mathsf{a}'_m\} \subseteq \{\mathsf{a}_1, \ldots, \mathsf{a}_n\}$. By the same reason, $\{\mathsf{a}_1, \ldots, \mathsf{a}_n\} \subseteq \{\mathsf{a}'_1, \ldots, \mathsf{a}'_m\}$ and therefore,
$\{\mathsf{a}_1, \ldots, \mathsf{a}_n\} = \{\mathsf{a}'_1, \ldots, \mathsf{a}'_m\}$.

Proposition 2.5 (Jankov 1969). *If a meet-semilattice* $\mathbf{A}$ *has a top element and all its elements have a finite irredundant decomposition, then* $\mathbf{A}$ *forms a Brouwerian semilattice.*

Proof We need to define on semilattice $\mathbf{A}$ a relative pseudocomplement $\to$. Because every element of $\mathbf{A}$ has a finite irredundant decomposition, for any two elements $\mathsf{a}, \mathsf{b} \in \mathbf{A}$ one can consider their finite irredundant decompositions $\mathsf{a} = \mathsf{a}_1 \wedge \cdots \wedge \mathsf{a}_n$

and $\mathsf{b} = \mathsf{b}_1 \wedge \cdots \wedge \mathsf{b}_m$. Now, we can define $\mathsf{a} \to \mathsf{c}$, where c is a meet-prime element, and then extend this definition by letting

$$\mathsf{a} \to (\mathsf{b}_1 \wedge \cdots \wedge \mathsf{b}_m) = (\mathsf{a} \to \mathsf{b}_1) \wedge \cdots \wedge (\mathsf{a} \to \mathsf{b}_m). \tag{2.1}$$

Proposition 2.1 ensures the correctness of such an extension.

Suppose $\mathsf{c} \in \mathbf{A}$ is meet-prime and $\mathsf{a} = \mathsf{a}_1 \wedge \cdots \wedge \mathsf{a}_n$ is a finite irredundant decomposition of a. Then we let

$$\mathsf{a} \to \mathsf{c} = \begin{cases} \mathbf{1}, & \text{if } \mathsf{a}_i \leq \mathsf{c} \text{ for some } i \in [1, n]; \\ \mathsf{c}, & \text{otherwise.} \end{cases}$$

Let us show that $\mathsf{a} \to \mathsf{c}$ is a pseudocomplement of a relative to c, that is, we need to show that $\mathsf{a} \to \mathsf{c}$ is the greatest element of $\mathbf{A}' := \{\mathsf{d} \in \mathbf{A} \mid \mathsf{a} \wedge \mathsf{d} \leq \mathsf{b}\}$.

Indeed, if $\mathsf{a}_i \leq \mathsf{c}$ for some $i \in [1, n]$, then

$$\mathbf{1} \wedge \mathsf{a} = \mathsf{a} = \mathsf{a}_1 \wedge \cdots \wedge \mathsf{a}_n \leq \mathsf{a}_i \leq \mathsf{c},$$

and obviously, $\mathbf{1}$ is the greatest of $\mathbf{A}'$.

Suppose now that $\mathsf{a}_i \not\leq \mathsf{c}$ for all $i \in [1, n]$. In this case, $\mathsf{a} \to \mathsf{c} = \mathsf{c}$, it is clear that $\mathsf{a} \wedge \mathsf{c} \leq \mathsf{c}$ (i.e., $\mathsf{a} \in \mathbf{A}'$), and we only need to verify that $\mathsf{d} \leq \mathsf{c}$ for every $\mathsf{d} \in \mathbf{A}'$.

Indeed, suppose that $\mathsf{a} \wedge \mathsf{d} \leq \mathsf{c}$; that is, $\mathsf{a}_1 \wedge \cdots \wedge \mathsf{a}_n \wedge \mathsf{d} \leq \mathsf{c}$. Then, $\mathsf{d} \leq \mathsf{c}$ because c is meet prime and $\mathsf{a}_i \not\leq \mathsf{c}$ for all $i \in [1, n]$.

Immediately from Propositions 2.5 and 2.2(c), we obtain the following statement.

Corollary 2.1 *Every finite meet-semilattice* $\mathbf{A}$ *with a top element in which every element has an irredundant finite decomposition forms a Brouwerian algebra. And because* $\mathbf{A}$ *is finite and has a bottom element,* $\mathbf{A}$ *is a Heyting algebra.*

2.2.3 Lattices Ded_C *and* $\mathsf{Lind}_{(C,k)}$

On the set of all C-formulas, relation $\overset{C}{\vdash}$ is a quasiorder and hence, the relation

$$A \overset{C}{\approx} B \quad \overset{\text{def}}{\iff} \quad A \overset{C}{\vdash} B \text{ and } B \overset{C}{\vdash} A$$

is an equivalence relation. Moreover, the set of all C-formulas forms a semilattice relative to connecting formulas with $\wedge$. It is not hard to see that equivalence $\overset{C}{\approx}$ is a congruence and therefore, we can consider a quotient semilattice which is denoted by Ded_C.

For each $k > 0$, we consider the set of all formulas on variables $p_1, \dots, p_k$. This set formulas a semilattice relative to connecting two given formulas with $\wedge$. It is not hard to see that relation

$$A \overset{C}{\sim} B \quad \overset{\text{def}}{\Longleftrightarrow} \quad \overset{C}{\vdash} A \leftrightarrow B$$

is a congruence, and by $\mathsf{Lind}_{(C,k)}$ we denote a quotient semilattice.

Theorem 2.1 (Jankov 1969) *For any C and $k > 0$, semilattices $\mathsf{Lind}_{(C,k)}$ and Ded_C are distributive lattices.*

Proof For $C \in \{\mathsf{PPC}, \mathsf{MPC}, \mathsf{IPC}\}$, it was observed in Rasiowa and Sikorski (1963). If $C \in \{\mathsf{PPC}^-, \mathsf{MPC}^-, \mathsf{IPC}^-\}$, by the Diego theorem (cf., e.g., Köhler 1981), lattice $\mathsf{Lind}_{(C,k)}$ is a finite implicative semilattice and, hence, a distributive lattice.

To convert Ded_C into a lattice we need to define a meet. Given two formulas A and B, we let

$$A \vee' B \quad = \quad (A \to p) \wedge ((B' \to p) \to p),$$

where formula B' is obtained from B by replacing the variables in such a way that formulas A and B have no variables in common, and p is a variable not occurring in formulas A and B'. If $C \in \{\mathsf{PPC}, \mathsf{MPC}, \mathsf{IPC}\}$, one can take

$$A \vee' B \quad = \quad A \vee B'.$$

A proof that Ded_C is indeed a distributive lattice can be found in Jankov (1969).

Meet-prime and meet-irreducible elements in $\mathsf{Lind}_{(C,k)}$ and Ded_C are called *conjunctively prime* and *conjunctively irreducible*, and because these lattices are distributive, every conjunctively irreducible formula is conjunctively prime and vice versa.

2.2.3.1 Congruences, Filters, Homomorphisms

Let us observe that every C-algebra $\mathbf{A}$ has a $\{\to, \wedge, \mathbf{1}\}$-reduct that is a Brouwerian semilattice, and therefore, any congruence on $\mathbf{A}$ is at the same time a congruence on its $\{\to, \wedge, \mathbf{1}\}$-reduct. It is remarkable that the converse is true too: every congruence on a $\{\to, \wedge, \mathbf{1}\}$-reduct can be lifted to the algebra.

Any congruence on a C-algebra $\mathbf{A}$ is uniquely defined by the set $\mathbf{1}/\theta := \{\mathsf{a} \in \mathbf{A} \mid (\mathsf{a}, \mathbf{1}) \in \theta\}$: indeed, it is not hard to see that $(\mathsf{b}, \mathsf{c}) \in \theta$ if and only if $(\mathsf{b} \leftrightarrow \mathsf{c}, \mathbf{1}) \in \theta$ (cf. Rasiowa 1974a). A set $\mathbf{1}/\theta$ forms a filter of $\mathbf{A}$: a subset $\mathsf{F} \subseteq \mathbf{A}$ is a *filter* if $\mathbf{1} \in \mathsf{F}$ and $\mathsf{a}, \mathsf{a} \to \mathsf{b} \in \mathsf{F}$ yields $\mathsf{b} \in \mathsf{F}$. The set of all filters of C-algebra $\mathbf{A}$ is denoted by $\mathsf{Flt}(\mathbf{A})$. It is not hard to see that a meet of an arbitrary system of filters is a filter and hence, $\mathsf{Flt}(\mathbf{A})$ forms a complete lattice. A set-join of two filters does not need to be a filter, but a join of any ascending chain of filters is a filter.

As we saw, every congruence is defined by a filter. The converse is true too: any filter F of a C-algebra $\mathbf{A}$ defines a congruence

$$(\mathsf{a}, \mathsf{b}) \in \theta_{\mathsf{F}} \iff (\mathsf{a} \leftrightarrow \mathsf{b}) \in \mathsf{F}.$$

Moreover, the map $\mathsf{F} \longrightarrow \theta_{\mathsf{F}}$ is an isomorphism between complete lattices of filters and complete lattice of congruences (cf. Rasiowa 1974a). It is clear that any nontrivial C-algebra has at least two filters: $\{\mathbf{1}\}$ and the set of all elements of the algebra. The filter $\{\mathbf{1}\}$ is called *trivial*, and the filters that do not contain all the elements of the algebra are called *proper*. In what follows, by $\mathbf{A}/\mathsf{F}$ and a/F we understand $\mathbf{A}/\theta_{\mathsf{F}}$ and $\mathsf{c}/\theta_{\mathsf{F}}$.

If $\mathbf{A}$ is a C-algebra and $\mathsf{B} \subseteq \mathbf{A}$ is a subset of elements, there is the least filter $[\mathsf{B})$ of $\mathbf{A}$ containing B: $[\mathsf{B}) = \bigcap\{\mathsf{F} \in \mathsf{Flt}(A) \mid \mathsf{B} \subseteq \mathsf{F}\}$, and we write $[\mathsf{a})$ instead of $[\{\mathsf{a}\})$. The reader can easily verify that for any element a of a C-algebra $\mathbf{A}$, $[\mathsf{a}) = \{\mathsf{b} \in \mathbf{A} \mid \mathsf{a} \leq \mathsf{b}\}$.

Immediately from the definitions of a filter and a homomorphism, the following holds.

Proposition 2.6 *Suppose that* $\mathbf{A}$ *and* $\mathbf{B}$ *are* C*-algebras and* $\varphi : \mathbf{A} \longrightarrow \mathbf{B}$ *is a homomorphism of* $\mathbf{A}$ *onto* $\mathbf{B}$. *Then*

(a) If F *is a filter of* $\mathbf{A}$, *then* $\varphi(\mathsf{F})$ *is a filter of* $\mathbf{B}$*;*
(b) If F *is a filter of* $\mathbf{B}$, *then* $\varphi^{-1}(\mathsf{F})$ *is a filter of* $\mathbf{A}$.

A nontrivial algebra $\mathbf{A}$ is called *subdirectly irreducible* (s.i. for short) if the meet of all nontrivial filters is a nontrivial filter; or, in terms of congruences, the meet of all congruences that are distinct from the identity is distinct from the identity congruence.

Because every element a of a C-algebra $\mathbf{A}$ defines a filter $[\mathsf{a})$, the meet of all nontrivial filters of $\mathbf{A}$ coincides with $\bigcap\{[\mathsf{a}), \mathsf{a} \in \mathbf{A} \mid \mathsf{a} \neq \mathbf{1}\}$ and consequently, $\mathbf{A}$ is s.i. if and only if the set $\{\mathsf{a} \in \mathbf{A} \mid \mathsf{a} \neq \mathbf{1}\}$ contains the greatest element which is referred to as a *pretop* element or an *opremum* and is denoted by $\mathsf{m}_{\mathbf{A}}$.

Let us observe that immediately from the definition of a pretop element, if $\mathsf{m}_{\mathbf{A}}$ is a pretop element of a C-algebra $\mathbf{A}$ and F is a filter of $\mathbf{A}$, then, $\mathsf{m}_{\mathbf{A}} \in \mathsf{F}$ if and only if F is nontrivial. In terms of homomorphism, this can be stated in the following way.

Proposition 2.7 *Suppose that* $\mathbf{A}$ *is an s.i.* C*-algebra and* $\varphi : \mathbf{A} \longrightarrow \mathbf{B}$ *is a homomorphism of* $\mathbf{A}$ *into* C*-algebra* $\mathbf{B}$. *Then* φ *is an isomorphism if and only if* $\varphi(\mathsf{m}_{\mathbf{A}}) \neq \mathbf{1}_{\mathbf{B}}$.

The following simple proposition was observed in Jankov (1969) and it is very important in what follows.

Proposition 2.8 *Let* $\mathbf{A}$ *be a nontrivial* C*-algebra,* $\mathsf{a}, \mathsf{b} \in \mathbf{A}$ *and* $\mathsf{a} \not\leq \mathsf{b}$. *Then, there is a maximal (relative to* $\subseteq$*) filter* F *of* $\mathbf{A}$ *such that* $\mathsf{a} \in \mathsf{F}$ *and* $\mathsf{b} \notin \mathsf{F}$. *Furthermore,* $\mathbf{A}/\mathsf{F}$ *is an s.i.* C*-algebra with* b/F *being the pretop element.*

Proof First, let us observe that the condition $\mathsf{a} \not\leq \mathsf{b}$ is equivalent to $\mathsf{b} \notin [\mathsf{a})$. Thus, $\mathcal{F} := \{\mathsf{F} \in \mathsf{Flt}(\mathbf{A}) \mid \mathsf{a} \in \mathsf{F}, \mathsf{b} \notin \mathsf{F}\} \neq \varnothing$.

Next, we recall that the joins of ascending chains of filters are filters and therefore, $\mathcal{F}$ enjoys the ascending chain condition. Thus, by the Zorn Lemma, $\mathcal{F}$ contains a maximal element.

Let F be a maximal element of $\mathcal{F}$. We need to show that b/F is a pretop element of $\mathbf{A}/\mathsf{F}$.

Because $\mathsf{b} \notin \mathsf{F}$ (cf. the definition of $\mathcal{F}$), we know that $\mathsf{b}/\mathsf{F} \neq \mathbf{1}_{\mathbf{A}/\mathsf{F}}$.

Let $\varphi : \mathbf{A} \longrightarrow \mathbf{A}/\mathsf{F}$ be a natural homomorphism. By Proposition 2.6, for every filter F' of $\mathbf{A}/\mathsf{F}$, the preimage $\varphi^{-1}(\mathsf{F}')$ is a filter of $\mathbf{A}$. Because $\mathbf{1}_{\mathbf{A}/\mathsf{F}} \in \mathsf{F}'$,

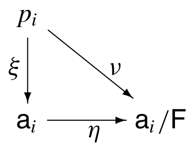
$$\mathsf{F} = \varphi^{-1}(\mathbf{1}_{\mathbf{A}/\mathsf{F}}) \subseteq \varphi^{-1}(\mathsf{F}').$$

Hence, if $\mathsf{F}' \supsetneq \mathbf{1}_{\mathbf{A}}/\mathsf{F}$, then $\mathsf{b} \in \varphi^{-1}(\mathsf{F}')$ (because F is a maximal filter not containing b), and consequently, $\mathsf{b}/\mathsf{F} \in \mathsf{F}'$. Thus, b/F is in every nontrivial filter of $\mathbf{A}/\mathsf{F}$, which means that $\mathbf{A}/\mathsf{F}$ is s.i. and that b/F is a pretop element of $\mathbf{A}/\mathsf{F}$.

Corollary 2.2 *Suppose that $A \to B$ is a C-formula refuted in a C-algebra $\mathbf{A}$. Then there is an s.i. homomorphic image $\mathbf{B}$ of algebra $\mathbf{A}$ and a valuation ν in $\mathbf{B}$ such that*

$$\nu(A) = \mathbf{1}_{\mathbf{B}} \quad \textit{and} \quad \nu(B) = \mathsf{m}_{\mathbf{B}}.$$

Proof Suppose that ξ is a refuting valuation in $\mathbf{A}$; that is, $\xi(A \to B) \neq \mathbf{1}_{\mathbf{A}}$. Let $\xi(A) = \mathsf{a}$ and $\xi(B) = \mathsf{b}$. Then, $\mathsf{a} \not\leq \mathsf{b}$ and by Proposition 2.8, there is a filter F of $\mathbf{A}$ such that $\mathsf{a} \in \mathsf{F}$, $\mathsf{b} \notin \mathsf{F}$ and $\mathbf{A}/\mathsf{F}$ is subdirectly irreducible with b/F being a pretop element of $\mathbf{A}/\mathsf{F}$. Thus, one can take a natural homomorphism $\eta : \mathbf{A} \longrightarrow \mathbf{A}/\mathsf{F}$ and let $\nu = \eta \circ \xi$.



It is not hard to see that ν is a desired refuting valuation.

Suppose that $\boldsymbol{L}$ is an extension of one of the logics from $\mathcal{L}$ and A is a formula in the signature of $\boldsymbol{L}$. We say that a C-algebra $\mathbf{A}$ in the signature of $\boldsymbol{L}$ *separates A from $\boldsymbol{L}$* if all formulas from $\boldsymbol{L}$ are valid in $\mathbf{A}$ (i.e., $\mathbf{A} \in \mathrm{Mod}(\boldsymbol{L})$), while formula A is not valid in $\mathbf{A}$, that is, if $\mathbf{A} \models \boldsymbol{L}$ and $\mathbf{A} \not\models A$.

Corollary 2.3 *Suppose that $\mathbf{L}$ is a C-logic and A is a C-formula. If a C-algebra $\mathbf{A}$ separates formula A from $\mathbf{L}$, then there is an s.i. homomorphic image $\mathbf{B}$ of $\mathbf{A}$ and a valuation ν in $\mathbf{B}$ such that $\nu(A) = \mathsf{m}_{\mathbf{B}}$.*

Proof If formula A is invalid in $\mathbf{A}$, then there is a refuting valuation ξ in $\mathbf{A}$ such that $\xi(A) = \mathsf{a} < \mathbf{1}$. By Proposition 2.8, there is a maximal filter F of $\mathbf{A}$ such that $\mathsf{a} \notin \mathsf{F}$. Then, $\mathbf{B} := \mathbf{A}/\mathsf{F}$ is an s.i. algebra, and $\nu = \eta \circ \xi$, where ν is a natural homomorphism, is a desired refuting valuation.

Let us note that because $\mathbf{B}$ is a homomorphic image of $\mathbf{A}$, the finiteness of $\mathbf{A}$ yields the finiteness of $\mathbf{B}$.

Remark 2.2 In Jankov (1969), Corollary 2.3 (the Descent Theorem) is proved only for finite algebras. Yankov, being a disciple of Markov and sharing the constructivist view on mathematics, avoided using the Zorn Lemma which is necessary for proving Proposition 2.8 for infinite algebras.

2.3 Yankov's Characteristic Formulas

One of the biggest achievements of Yankov, apart from the particular results about si-logics, is the machinery that he had developed and used to establish these results. This machinery rests on the notion of a characteristic formula that he introduced in Jankov (1963c) and studied in detail in Jankov (1969).

2.3.1 Formulas and Homomorphisms

With each finite C-algebra $\mathbf{A}$ in the signature Σ we associate a formula $D_{\mathbf{A}}$ on variables $\{p_{\mathsf{a}}, \mathsf{a} \in \mathbf{A}\}$ in the following way: let $\Sigma_2 \subseteq \Sigma$ be a subset of all binary operation and $\Sigma_0 \subseteq \Sigma$ be a subset of nullary operations (constants); then

$$D_{\mathbf{A}} = \bigwedge_{\circ \in \Sigma_2} (p_{\mathsf{a}} \circ p_{\mathsf{b}} \leftrightarrow p_{\mathsf{a} \circ \mathsf{b}}) \wedge \bigwedge_{\mathfrak{c} \in \Sigma_0} (\mathfrak{c} \leftrightarrow p_{\mathfrak{c}}).$$

Example 2.1 Let $\mathbf{3} = (\{\mathsf{a}, \mathsf{b}, \mathbf{1}\}; \to, \wedge, \mathbf{1})$ be a Brouwerian semilattice, $\mathsf{a} \leq \mathsf{b} \leq \mathbf{1}$, and the operations are defined by the Cayley tables:

$\to$	a	b	$\mathbf{1}$
a	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
b	a	$\mathbf{1}$	$\mathbf{1}$
$\mathbf{1}$	a	b	$\mathbf{1}$

$\wedge$	a	b	$\mathbf{1}$
a	a	a	a
b	a	b	b
$\mathbf{1}$	a	b	$\mathbf{1}$

Then, in the Cayley tables, we replace the elements with the respective variables:

$\to$	p_{a}	p_{b}	$p_{\mathbf{1}}$
p_{a}	$p_{\mathbf{1}}$	$p_{\mathbf{1}}$	$p_{\mathbf{1}}$
p_{b}	p_{a}	$p_{\mathbf{1}}$	$p_{\mathbf{1}}$
$p_{\mathbf{1}}$	p_{a}	p_{b}	$p_{\mathbf{1}}$

$\wedge$	p_{a}	p_{b}	$p_{\mathbf{1}}$
p_{a}	p_{a}	p_{a}	p_{a}
p_{b}	p_{a}	p_{b}	p_{b}
$p_{\mathbf{1}}$	p_{a}	p_{b}	$p_{\mathbf{1}}$

and we express the above tables in the form of a formula:

$$\begin{aligned} D_3 = {} & (p_\mathsf{a} \to p_\mathsf{a}) \leftrightarrow p_\mathbf{1} \ \wedge (p_\mathsf{a} \to p_\mathsf{b}) \leftrightarrow p_\mathbf{1} \ \wedge (p_\mathsf{a} \to p_\mathbf{1}) \leftrightarrow p_\mathbf{1} \ \wedge \\ & (p_\mathsf{b} \to p_\mathsf{a}) \leftrightarrow p_\mathsf{a} \ \wedge (p_\mathsf{b} \to p_\mathsf{b}) \leftrightarrow p_\mathbf{1} \ \wedge (p_\mathsf{b} \to p_\mathbf{1}) \leftrightarrow p_\mathbf{1} \wedge \\ & (p_\mathbf{1} \to p_\mathsf{a}) \leftrightarrow p_\mathsf{a} \ \wedge (p_\mathbf{1} \to p_\mathsf{b}) \leftrightarrow p_\mathsf{b} \ \wedge (p_\mathbf{1} \to p_\mathbf{1}) \leftrightarrow p_\mathbf{1} \wedge \\ & (p_\mathsf{a} \wedge p_\mathsf{a}) \leftrightarrow p_\mathsf{a} \ \wedge (p_\mathsf{a} \wedge p_\mathsf{b}) \leftrightarrow p_\mathsf{a} \ \wedge (p_\mathsf{a} \wedge p_\mathbf{1}) \leftrightarrow p_\mathsf{a} \ \wedge \\ & (p_\mathsf{b} \wedge p_\mathsf{a}) \leftrightarrow p_\mathsf{a} \ \wedge (p_\mathsf{b} \wedge p_\mathsf{b}) \leftrightarrow p_\mathsf{b} \ \wedge (p_\mathsf{b} \wedge p_\mathbf{1}) \leftrightarrow p_\mathsf{b} \ \wedge \\ & (p_\mathbf{1} \wedge p_\mathsf{a}) \leftrightarrow p_\mathsf{a} \ \wedge (p_\mathbf{1} \wedge p_\mathsf{b}) \leftrightarrow p_\mathsf{b} \ \wedge (p_\mathbf{1} \wedge p_\mathbf{1}) \leftrightarrow p_\mathbf{1} \ \wedge \\ & \mathbf{1} \leftrightarrow p_\mathbf{1}. \end{aligned}$$

Let us note that formula D_3 is equivalent in Pos^- to a much simpler formula,

$$D' = ((p_\mathsf{b} \to p_\mathsf{a}) \to p_\mathsf{b}) \wedge p_1.$$

The importance of formula $D_\mathbf{A}$ rests on the following observation.

Proposition 2.9 *Suppose that* $\mathbf{A}$ *and* $\mathbf{B}$ *are* C*-algebras. If for valuation* ν *in* $\mathbf{B}$, $\nu(D_\mathbf{A}) = \mathbf{1}_\mathbf{B}$, *then the map*

$$\eta : \mathsf{a} \mapsto \nu(p_\mathsf{a})$$

is a homomorphism.

Proof Indeed, for any $\mathsf{a}, \mathsf{b} \in \mathbf{A}$ and any operation $\circ$, formula $p_\mathsf{a} \circ p_\mathsf{b} \leftrightarrow p_{\mathsf{a}\circ\mathsf{b}}$ is a conjunct of $D_\mathbf{A}$ and hence, $\nu(p_\mathsf{a} \circ p_\mathsf{b}) = \nu(p_{\mathsf{a}\circ\mathsf{b}})$, because $\nu(D_\mathbf{A}) = \mathbf{1_b}$. Thus,

$$\eta(\mathsf{a} \circ \mathsf{b}) = \nu(p_{\mathsf{a}\circ\mathsf{b}}) = \nu(p_\mathsf{a} \circ p_\mathsf{b}) = \nu(p_\mathsf{a}) \circ \nu(p_\mathsf{b}) = \eta(p_\mathsf{a}) \circ \eta(p_\mathsf{b}).$$

It is not hard to see that η preserves the operations and therefore, η is a homomorphism.

Let us note that using any set of generators of a finite C-algebra $\mathbf{A}$, one can construct a formula having properties similar to $D_\mathbf{A}$. Suppose that elements $\mathsf{g}_1, \ldots, \mathsf{g}_n$ generate algebra $\mathbf{A}$. Then, each element $\mathsf{a} \in \mathbf{A}$ can be expressed via generators, that is, there is a formula $B_\mathsf{a}(p_{\mathsf{g}_1}, \ldots, p_{\mathsf{g}_n})$ such that $\mathsf{a} = B_\mathsf{a}(\mathsf{g}_1, \ldots, \mathsf{g}_\mathsf{n})$. If we substitute in $D_\mathbf{A}$ each variable p_a with formula B_a, we obtain a new formula $D'_\mathbf{A}(p_{\mathsf{g}_1}, \ldots, p_{\mathsf{g}_n})$, and this formula will posses the same property as formula $D_\mathbf{A}$. Because $D'_\mathbf{A}$ depends on the selection of formulas B_a, we use the notation $D_\mathbf{A}[B_{\mathsf{a}_1}, \ldots, B_{\mathsf{a}_m}]$, provided that $\mathsf{a}_1, \ldots, \mathsf{a}_m$ are all elements of $\mathbf{A}$.

Proposition 2.10 *Suppose that* $\mathbf{A}$ *and* $\mathbf{B}$ *are* C*-algebras. If* ν *is a valuation in* $\mathbf{B}$ *and* $\nu(D_\mathbf{A}[B_{\mathsf{a}_1}, \ldots, B_{\mathsf{a}_m}]) = \mathbf{1}_\mathbf{B}$, *then the map*

$$\eta : \mathsf{a} \mapsto \nu(B_\mathsf{a})$$

is a homomorphism.

Example 2.2 Let $\mathbf{3}$ be a three-element Heyting algebra with elements $\mathbf{0}, \mathsf{a}, \mathbf{1}$. It is clear that $\mathbf{A}$ is generated by element a:

$$B_{\mathbf{0}}(p_{\mathsf{a}}) = p_{\mathsf{a}} \wedge (p_{\mathsf{a}} \to \mathbf{0}), \quad B_{\mathsf{a}} = p_{\mathsf{a}}, \quad B_{\mathbf{1}} = (p_{\mathsf{a}} \to p_{\mathsf{a}}).$$

Formula $D_{\mathbf{3}}[B_{\mathbf{0}}(p_{\mathsf{a}}), B_{\mathsf{a}}(p_{\mathsf{a}}), B_{\mathbf{1}}(p_{\mathsf{a}})]$ is equivalent in Int to the formula $(p_{\mathsf{a}} \to \mathbf{0}) \to \mathbf{0}$. It is not hard to verify that in any Heyting algebra $\mathbf{B}$, if element $\mathsf{b} \in \mathbf{B}$ satisfies condition $((\mathsf{b} \to \mathbf{0}) \to \mathbf{0}) = \mathbf{1}$ (i.e., $\neg\neg\mathsf{b} = \mathbf{1}$), then the map

$$\mathbf{0_3} \mapsto \mathsf{b} \wedge (\mathsf{b} \to \mathbf{0_B}), \quad \mathsf{a} \mapsto \mathsf{b}, \quad \mathbf{1_3} \mapsto (\mathsf{b} \to \mathsf{b}),$$

that is, the map

$$\mathbf{0_3} \mapsto \mathbf{0_B}, \quad \mathsf{a} \mapsto \mathsf{b}, \quad \mathbf{1_3} \mapsto \mathbf{1_B},$$

is a homomorphism.

2.3.2 Characteristic Formulas

Now, we are in a position to define the Yankov characteristic formulas. These formulas are instrumental in studying different classes of logics. It also turned out that characteristic formulas, and only these formulas, are conjunctively indecomposable.

Definition 2.1 Suppose that $\mathbf{A}$ is a finite s.i. C-algebra (finite s.i. algebra, for short). Then the formula

$$X_{\mathbf{A}} \;:=\; D_{\mathbf{A}} \to p_{\mathsf{m_A}}$$

is a *Yankov (or characteristic) formula* of $\mathbf{A}$.

Let us observe that the valuation $\eta : p_{\mathsf{a}} \mapsto \mathsf{a}$ refutes $X_{\mathbf{A}}$, because clearly, $\eta(D_{\mathbf{A}}) = \mathbf{1}$, while $\eta(p_{\mathsf{m_A}}) = \mathsf{m_A} \neq \mathbf{1}$. That is,

$$\mathbf{A} \not\models X_{\mathbf{A}}. \tag{2.2}$$

Proposition 2.11 *Suppose that $\mathbf{A}$ is a finite s.i. C-algebra and ν is a refuting valuation of $X_{\mathbf{A}}$ in a C-algebra $\mathbf{B}$ such that $\nu(D_{\mathbf{A}}) = \mathbf{1_B}$. Then, the map*

$$\varphi : \mathsf{a} \mapsto \nu(p_{\mathsf{a}})$$

is an isomorphism.

Proof Because $\nu(D_{\mathbf{A}}) = \mathbf{1_B}$, by Proposition 2.9, φ ia a homomorphism. Because ν refutes $X_{\mathbf{A}}$, that is, ν refutes $D_{\mathbf{A}} \to p_{\mathsf{m_A}}$, we know that $\nu(p_{\mathsf{m_A}}) \neq \mathbf{1_B}$ and consequently,

$$\varphi(\mathsf{m_A}) = \nu(p_{\mathsf{m_A}}) \neq \mathbf{1_B}.$$

Thus, by Proposition 2.7, φ is an isomorphism.

Corollary 2.4 *If a characteristic formula of a finite s.i. C-algebra* **A** *is refuted in a C-algebra* **B**, *then algebra* **A** *is embedded in a homomorphic image of algebra* **B**.

The proof immediately follows from Corollary 2.2 and Proposition 2.11.

One of the most important properties of characteristic formula of C-algebra **A** is that $X_{\mathbf{A}}$ is the weakest formula refutable in **A**. More precisely, the following holds.

Theorem 2.2 (Jankov 1969, Characteristic formula theorem) *A C-formula A is refutable in a finite s.i. C-algebra* **A** *if and only if* $A \overset{C}{\vdash} X_{\mathbf{A}}$.

Proof It is clear that if $A \overset{C}{\vdash} X_{\mathbf{A}}$, then A is refuted in **A**, because $X_{\mathbf{A}}$ is refuted in **A**.

To prove the converse statement, we will do the following:

(a) using a refuting valuation of A in **A**, we will introduce a substitution σ such that formula $A' := \sigma(A)$ has the same variables as $X_{\mathbf{A}}$;

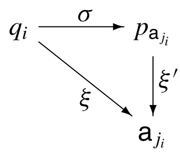
(b) we will prove that $\overset{C}{\vdash} A' \to X_{\mathbf{A}}$ by showing that formula $A' \to X_{\mathbf{A}}$ cannot be refuted in any C-algebra.

Indeed, because clearly $A \overset{C}{\vdash} A'$, (b) entails that $A \overset{C}{\vdash} X_{\mathbf{A}}$

(a) Assume that **A** is a k-element C-algebra, $\mathsf{a}_i, i \in [1, k]$ are all its elements, and that $q_1, \ldots, q_n$ are all variables occurring in A. Suppose that $\xi : q_i \mapsto \mathsf{a}_{j_i}$ is a refuting valuation of A in **A**; that is,

$$\xi(A(q_1, \ldots, q_n)) = A(\xi(q_1), \ldots, \xi(q_n)) = A(\mathsf{a}_{j_1}, \ldots, \mathsf{a}_{j_n}) \neq \mathbf{1}_{\mathbf{A}}. \quad (2.3)$$

Let us consider formula A' obtained from A by a substitution $\sigma : q_i \mapsto p_{\mathsf{a}_{j_i}}$ and a valuation $\xi' : p_{\mathsf{a}_{j_i}} \mapsto \mathsf{a}_{j_i}, i \in [1, n]$, in **A**:



Let us note that A' contains variables only from $\{p_{\mathsf{a}_i}, i \in [1, k]\}$ but not necessarily all of them. To simplify notation and without losing generality, we can assume that A' is a formula in variables $\{p_{\mathsf{a}_i}, i \in [1, k]\}$ (if p_{a_i} does not occur in A', one simply can take $A' \wedge (p_{\mathsf{a}_i} \to p_{\mathsf{a}_i})$ instead of A' and let $\xi' : p_{\mathsf{a}_i} \mapsto \mathbf{1}_{\mathbf{A}}$).

Now, if we apply ξ' to A' and take into consideration (2.3), we get

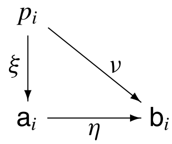
$$A'(\mathsf{a}_{j_1}, \ldots, \mathsf{a}_{j_k}) = A(\xi'(p_1), \ldots, \xi'(p_n)) = A(\mathsf{a}_{j_1}, \ldots, \mathsf{a}_{j_n}) \neq \mathbf{1}_{\mathbf{A}}. \quad (2.4)$$

(b) For contradiction, assume that $\overset{C}{\nvdash} A' \to X_{\mathbf{A}}$. Thus, $\overset{C}{\nvdash} A' \to (D_{\mathbf{A}} \to p_{\mathsf{m}_{\mathbf{A}}})$ and therefore, $\overset{C}{\nvdash} (A' \wedge D_{\mathbf{A}}) \to p_{\mathsf{m}_{\mathbf{A}}}$ Then, there is a C-algebra in which formula

$(A' \wedge D_{\mathbf{A}}) \to p_{\mathsf{m}_{\mathbf{A}}}$ is refuted, and by Corollary 2.2, there is an s.i. C-algebra $\mathbf{B}$ and a valuation ν in $\mathbf{B}$ such that $\nu((A' \wedge D_{\mathbf{A}})) = \mathbf{1_B}$ and $\nu(p_{\mathsf{m}_{\mathbf{A}}}) = \mathsf{m}_{\mathbf{B}} \neq \mathbf{1_B}$; that is,

$$A'(\mathsf{b}_1, \ldots, \mathsf{b}_k) = \mathbf{1_B} \text{ and } D(\mathsf{b}_1, \ldots, \mathsf{b}_k) = \mathbf{1_B}, \tag{2.5}$$

where $\mathsf{b}_i = \nu(p_{j_i})$, $i \in [1, k]$. Let $\eta : \mathsf{a}_i \mapsto \mathsf{b}_i$:



Then, because $D(\mathsf{b}_1, \ldots, \mathsf{b}_n) = \mathbf{1_B}$, η is a homomorphism and we can apply Proposition 2.9. Moreover, η is an isomorphism, because $\eta(\mathsf{m}_\mathsf{A}) = \nu(p_{\mathsf{m}_\mathsf{A}}) = \mathsf{m}_\mathbf{B} \neq \mathbf{1}_\mathsf{B}$, and we can apply Proposition 2.7.

We have arrived at a contradiction: on the one hand, by (2.5), $A'(\mathsf{b}_1, \ldots, \mathsf{b}_n) = \mathbf{1_B}$, while on the other hand, by (2.4), $A'(\mathsf{a}_1, \ldots, \mathsf{a}_k) \neq \mathbf{1_A}$, and because η is an isomorphism,

$$\eta(A'(\mathsf{a}_1, \ldots 3, \mathsf{a}_k)) = A'(\eta(\mathsf{a}_1), \ldots, \eta(\mathsf{a}_n)) = A'(\mathsf{b}_1, \ldots, \mathsf{b}_k) \neq \mathbf{1_B}.$$

Example 2.3 Consider three-element Heyting algebra **3** from Example 2.2. Then, $X_\mathbf{3} = D(\mathbf{A}) \to p_{\mathsf{m}_\mathbf{A}}$. It is clear that $\mathsf{m}_\mathbf{3} = \mathsf{a}$, and from Example 2.2 we know that $D(\mathbf{3})$ is equivalent to $(p_\mathsf{a} \to \mathbf{0}) \to \mathbf{0}$. Therefore,

$$X_\mathbf{3} \text{ is equivalent in } \mathsf{Int} \text{ to } ((p_a \to \mathbf{0}) \to \mathbf{0}) \to p_a \text{ or to } \neg\neg p_\mathsf{a} \to p_\mathsf{a}.$$

2.3.3 Splitting

Suppose that $\mathbf{A}$ is a finite s.i. C-algebra and $X_\mathbf{A}$ is its characteristic formula. We already know from (2.2) that $\mathbf{A} \not\models X_\mathbf{A}$. But $X_\mathbf{A}$ possesses a much stronger property.

Proposition 2.12 *Suppose that* $\mathbf{A}$ *is a finite s.i. C-algebra and* $\mathbf{B}$ *is a C-algebra. Then,*

$$\mathbf{B} \not\models X_\mathbf{A} \iff \mathbf{A} \textit{ is embedded in a homomorphic image of } \mathbf{B}.$$

Proof If $\mathbf{A}$ is embedded in a homomorphic image of $\mathbf{B}$, then $\mathbf{B} \not\models X_\mathbf{A}$, because by (2.2), $\mathbf{A} \not\models X_\mathbf{A}$.

Conversely, suppose that $\mathbf{B} \not\models X_\mathbf{A}$; that is, $\mathbf{B} \not\models (D_\mathbf{A} \to p_{\mathsf{m}_\mathbf{A}})$. Then, we can use the same argument as in the proof of Theorem 2.2(b) and conclude that $\mathbf{A}$ is embedded in a homomorphic image of $\mathbf{B}$.

Let $\mathcal{A}$ be a class of finite s.i. C-algebras. We take $\overline{\mathcal{A}}$ to be a class of all finite s.i. C-algebras not belonging to $\mathcal{A}$. Denote by $\boldsymbol{L}(\mathcal{A})$ a logic of all formulas valid in each

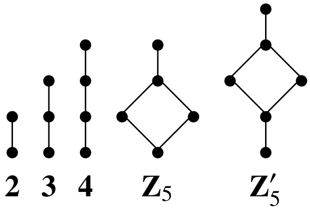


Fig. 2.2 Algebras

algebra from $\mathcal{A}$, and denote by $\overline{\boldsymbol{L}}(\mathcal{A})$ a logic defined by characteristic formulas of algebras from $\mathcal{A}$ as additional axioms, that is, the logic defined by $C + \{X_{\mathbf{A}} \mid \mathbf{A} \in \mathcal{A}\}$. If $\mathcal{A}$ consists of a single algebra $\mathbf{A}$, we omit the curly brackets and write $\boldsymbol{L}(\mathbf{A})$ and $\overline{\boldsymbol{L}}(\mathbf{A})$.

Let us observe that if a two-element algebra is not in $\mathcal{A}$, logic $\overline{\boldsymbol{L}}(\mathcal{A})$ is not trivial: no algebra $\mathbf{A}$ having more than two elements can be a subalgebra of a two-element algebra or its homomorphic image and hence, formula $X_{\mathbf{A}}$ is valid in a two-element algebra.

Corollary 2.5 *Suppose that $\mathcal{A}$ is a class of finite s.i. C-algebras. Then, logic $\overline{\boldsymbol{L}}(\mathcal{A})$ is the smallest extension of C such that algebras from $\mathcal{A}$ are not its models.*

Proof We need to prove that for every C-logic $\boldsymbol{L}'$ for which $\boldsymbol{L} \supsetneq \boldsymbol{L}'$ is a proper extension of $\boldsymbol{L}'$, there is an algebra $\mathbf{A} \in \mathcal{A}$ that is a model for $\boldsymbol{L}'$; that is, $\mathbf{A} \models A$ for every $A \in \boldsymbol{L}'$.

For contradiction, assume that $\boldsymbol{L} \supsetneq \boldsymbol{L}'$ and for each algebra $\mathbf{A} \in \mathcal{A}$ there is a formula $A_{\mathbf{A}} \in \boldsymbol{L}'$ such that $\mathbf{A} \not\models A_{\mathbf{A}}$. Then, by Theorem 2.2, $A_{\mathbf{A}} \overset{C}{\vdash} X_{\mathbf{A}}$. Hence, because $A_{\mathbf{A}} \in \boldsymbol{L}'$ and $\boldsymbol{L}'$ is closed under Modus Ponens, $X_{\mathbf{A}} \in \boldsymbol{L}'$, and subsequently, $\boldsymbol{L} \subseteq \boldsymbol{L}'$, because $\boldsymbol{L}$ is defined by $C + \{X_{\mathbf{A}} \mid \mathbf{A} \in \mathcal{A}\}$. Thus, we have arrived at a contradiction with the assumption that $\boldsymbol{L}$ is a proper extension of $\boldsymbol{L}'$.

Example 2.4 If $\mathcal{A}$ consists of two algebras $\mathbf{Z}_5$ and $\mathbf{Z}'_5$ the Hasse diagrams of which are depicted in Fig. 2.2, then $\overline{\boldsymbol{L}}(\mathcal{A})$ is Dummett's logic (cf. Idziak and Idziak 1988.)

If $\boldsymbol{L}$ is a C-logic, denote by $\boldsymbol{L}_{\mathsf{f}}$ a class of all finite models of $\boldsymbol{L}$, and by $\boldsymbol{L}_{\mathsf{fsi}}$—a class of all finite s.i. models of $\boldsymbol{L}$. It should be clear that for any C-logics $\boldsymbol{L}$ and $\boldsymbol{L}'$, $\boldsymbol{L}_{\mathsf{f}} = \boldsymbol{L}'_{\mathsf{f}}$ if and only if $\boldsymbol{L}_{\mathsf{fsi}} = \boldsymbol{L}'_{\mathsf{fsi}}$. We say that two C-logics $\boldsymbol{L}$ and $\boldsymbol{L}'$ are finitely indistinguishable if $\boldsymbol{L}_{\mathsf{f}} = \boldsymbol{L}'_{\mathsf{f}}$ (in symbols, $\boldsymbol{L} \approx_f \boldsymbol{L}'$). Obviously, $\approx_f$ is an equivalence relation on the lattice $\mathbf{Ext}C$. Let us note that each $\approx_f$-equivalence class $[\boldsymbol{L}]_f$ contains the largest element, namely a logic of all formulas valid in $\boldsymbol{L}_{\mathsf{f}}$. Moreover, by Corollary 2.5, $[\boldsymbol{L}]_f$ contains the smallest element, namely $\overline{\boldsymbol{L}}(\overline{\mathcal{A}})$—the logic defined relative to C by the characteristic formulas of all algebras from $\overline{\mathcal{A}}$, where $\mathcal{A} = \boldsymbol{L}_{\mathsf{fsi}}$ Thus, if $\boldsymbol{L}$ is a C-logic and $\mathcal{A} = \boldsymbol{L}_{\mathsf{fsi}}$, then $\approx_f$-equivalence class $[\boldsymbol{L}]_f$ forms a segment

$$[\boldsymbol{L}]_f \;=\; [\overline{\boldsymbol{L}}(\overline{\mathcal{A}}), \boldsymbol{L}(\mathcal{A})].$$

Let us point out that each $\approx_f$-equivalence class, contains a unique logic enjoying the f.m.p., namely, its largest logic, which is a logic defined by all finite models. Thus, if the cardinality of an $\approx_f$-equivalence class is distinct from one, this class contains logics lacking the f.m.p. (cf. Sect. 2.4 for examples). In fact (cf. Tomaszewski 2003, Theorem 4.8), there is an $\approx_f$-equivalent class of si-logics having continuum many members. Therefore, there are continuum many si-logics lacking the f.m.p.

The case in which class $\mathcal{A}$ consists of a single algebra plays a very special role.

Corollary 2.6 *Suppose that* $\mathbf{A}$ *is a finite s.i.* C*-algebra and* $X_{\mathbf{A}}$ *is its characteristic formula. Then, the logic* $\boldsymbol{L}$ *defined by* $C + X_{\mathbf{A}}$ *is the smallest extension of* C *for which* $\mathbf{A}$ *is not a model.*

Corollary 2.6 yields that for any logic $\boldsymbol{L} \in \mathbf{Ext}C$,

$$\text{either } \boldsymbol{L} \subseteq \boldsymbol{L}(\mathbf{A}), \ \text{ or } \boldsymbol{L} \supseteq \overline{\boldsymbol{L}}(\mathbf{A}).$$

Indeed, if $\mathbf{A}$ is a model of $\boldsymbol{L}$, then $\boldsymbol{L} \subseteq \boldsymbol{L}(\mathbf{A})$; otherwise, $\mathbf{A}$ is not a model of $\boldsymbol{L}$ and by Corollary 2.5, $\boldsymbol{L} \supseteq \overline{\boldsymbol{L}}(\mathbf{A})$.

Let us recall (cf., e.g., Kracht 1999; Galatos et al. 2007) that if $\boldsymbol{L}$ is a logic, a pair of its extension $(\boldsymbol{L}_1, \boldsymbol{L}_2)$ is a *splitting pair* of $\mathbf{Ext}\boldsymbol{L}$ if

$$\boldsymbol{L}_1 \nsubseteq \boldsymbol{L}_2, \ \text{and for each } \boldsymbol{L}' \in \mathbf{Ext}\boldsymbol{L}, \ \text{either } \boldsymbol{L}_1 \subseteq \boldsymbol{L}' \text{ or } \boldsymbol{L}' \subseteq \boldsymbol{L}_2,$$

and $\mathbf{A}$ is a *splitting algebra*, while $X_{\mathbf{A}}$ is a *splitting formula*.

Example 2.5 Consider Heyting algebra $\mathbf{3}$ from Example 2.3. Algebra $\mathbf{3}$ defines a splitting: for each logic $\boldsymbol{L} \in \mathbf{Ext}\mathsf{Int}$,

$$\text{either } \boldsymbol{L} \subseteq \boldsymbol{L}(\mathbf{3}) \text{ or } \boldsymbol{L} \supseteq \overline{\boldsymbol{L}}(\mathbf{3}),$$

and $\overline{\boldsymbol{L}}(\mathbf{3})$ is defined by $\mathsf{IPC} + X_{\mathbf{3}}$. From Example 2.3, we know that formula $X_{\mathbf{3}}$ is equivalent to formula $\neg\neg p_{\mathsf{a}} \to p_a$; that is, $\overline{\boldsymbol{L}}(\mathbf{3})$ is defined by $\mathsf{IPC} + \neg\neg p_{\mathsf{a}} \to p_a$ and therefore, $\overline{\boldsymbol{L}}(\mathbf{3}) = \mathsf{Cl}$. Thus, for any formula A refuted in $\mathbf{3}$, $\mathsf{Int} + A$ defines a logic extending Cl; that is, $\mathsf{Int} + A$ is Cl or a trivial logic.

Example 2.6 Let $\mathbf{n}$ denote a linearly ordered n-element Heyting algebra. Then, each nontrivial algebra $\mathbf{n}$ is s.i. and defines a splitting pair: for logic $\boldsymbol{L} \in \mathbf{Ext}\mathsf{Int}$,

$$\text{either } \boldsymbol{L} \subseteq \boldsymbol{L}(\mathbf{n}) \text{ or } \boldsymbol{L} \supseteq \overline{\boldsymbol{L}}(\mathbf{n}),$$

and $\overline{\boldsymbol{L}}(\mathbf{n})$ is defined by $\mathsf{IPC} + X_{\mathbf{n}}$. Logic $\boldsymbol{L}(\mathbf{n})$ is the smallest logic of the $n - 2$ slice introduced in Hosoi (1967).

2.3.4 Quasiorder

On the class of all finite s.i. C-algebras we introduce the following quasiorder: for any C-algebras $\mathbf{A}$ and $\mathbf{B}$,

$$\mathbf{A} \leq \mathbf{B} \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad X_{\mathbf{A}} \stackrel{C}{\vdash} X_{\mathbf{B}}.$$

The following theorem establishes the main properties of the introduced quasiorder.

Theorem 2.3 (Jankov 1963c, 1969) *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be finite s.i.* C*-algebras. The following conditions are equivalent:*

(a) $\mathbf{A} \leq \mathbf{B}$*;*
(b) $X_{\mathbf{A}}$ *is refutable in* $\mathbf{B}$*;*
(c) every formula refutable in $\mathbf{A}$ *is refutable in* $\mathbf{B}$*;*
(d) $\mathbf{A}$ *is embedded in a homomorphic image of* $\mathbf{B}$.

Proof (a) $\Rightarrow$ (b), because by (2.2), $\mathbf{B} \not\models X_{\mathbf{B}}$ and by the definition of quasiorder, $X_{\mathbf{A}} \stackrel{C}{\vdash} X_{\mathbf{B}}$.

(b) $\Rightarrow$ (c). If a formula A is refutable in $\mathbf{A}$, then by Theorem 2.2, $A \stackrel{C}{\vdash} X_{\mathbf{A}}$. By (b), $X_{\mathbf{A}}$ is refutable in $\mathbf{B}$ and then, by Theorem 2.2, $X_{\mathbf{A}} \stackrel{C}{\vdash} X_{\mathbf{B}}$. Hence, $A \stackrel{C}{\vdash} X_{\mathbf{B}}$ and consequently, A is refutable in $\mathbf{B}$, because $X_{\mathbf{B}}$ is refutable in $\mathbf{B}$.

(c) $\Rightarrow$ (d). Characteristic formula $X_{\mathbf{A}}$ is refutable in $\mathbf{A}$ and hence, by (c), formula $X_{\mathbf{A}}$ is refutable in $\mathbf{B}$. By Corollary 2.4, $\mathbf{A}$ is embedded in a homomorphic image of algebra $\mathbf{B}$.

(d) $\Rightarrow$ (a). Characteristic formula $X_{\mathbf{A}}$ is refutable in $\mathbf{A}$. Hence, if $\mathbf{A}$ is embedded in a homomorphic image $\mathbf{B}$, formula $X_{\mathbf{A}}$ is refutable in this homomorphic image and consequently, it is refutable in $\mathbf{B}$. Then, by Theorem 2.2, $X_{\mathbf{A}} \stackrel{C}{\vdash} X_{\mathbf{B}}$, which means that $\mathbf{A} \leq \mathbf{B}$.

Corollary 2.7 *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be finite s.i.* C*-algebras such that* $\mathbf{A} \leq \mathbf{B}$ *and* $\mathbf{B} \leq \mathbf{A}$. *Then, algebras* $\mathbf{A}$ *and* $\mathbf{B}$ *are isomorphic.*

Proof Indeed, by Theorem 2.3, $\mathbf{A} \leq \mathbf{B}$ entails that $\mathbf{A}$ is a subalgebra of a homomorphic image of $\mathbf{B}$ and hence, $\mathsf{card}(\mathbf{A}) \leq \mathsf{card}(\mathbf{B})$. Likewise, $\mathsf{card}(\mathbf{B}) \leq \mathsf{card}(\mathbf{A})$. Therefore, $\mathsf{card}(\mathbf{A}) = \mathsf{card}(\mathbf{B})$ and because $\mathbf{A}$ and $\mathbf{B}$ are homomorphic images of each other, their finiteness ensures that they are isomorphic.

The following corollaries are the immediate consequences of Theorem 2.3(d).

Corollary 2.8 *For any finite s.i.* C*-algebras* $\mathbf{A}$ *and* $\mathbf{B}$*, if* $\mathbf{A} \leq \mathbf{B}$ *and* $\mathsf{card}(\mathbf{A}) = \mathsf{card}(\mathbf{B})$*, then* $\mathbf{A} \cong \mathbf{B}$.

Let us observe that Corollary 2.7 entails that $\leq$ is a partial order and that by Corollary 2.8, any class $\mathcal{A}$ of finite s.i. C-algebras enjoys the descending chain condition. Hence, the following holds.

Corollary 2.9 *Let $\mathcal{A}$ be a class of finite s.i. C-algebras. Then $\mathcal{A}$ contains a subclass $\mathcal{A}^{(m)} \subseteq \mathcal{A}$ of pairwise nonisomorphic algebras that are minimal relative to $\leq$ such that*

$$\textit{for any algbera } \mathbf{A} \in \mathcal{A}, \ \textit{there is an algebra } \mathbf{A}' \in \mathcal{A}^{(m)} \textit{ and } \mathbf{A}' \leq \mathbf{A}. \tag{2.6}$$

Proposition 2.13 *For any class of finite s.i. C-algebras $\mathcal{A}$,*

$$\overline{\boldsymbol{L}}(\mathcal{A}) = \overline{\boldsymbol{L}}(\mathcal{A}^{(m)}). \tag{2.7}$$

Proof Indeed, $\mathcal{A}^{(m)} \subseteq \mathcal{A}$ entails $\{X_{\mathbf{A}} \mid \mathbf{A} \in \mathcal{A}^{(m)}\} \subseteq \{X_{\mathbf{A}} \mid \mathbf{A} \in \mathcal{A}\}$ and subsequently, $\overline{\boldsymbol{L}}(\mathcal{A}^{(m)}) \subseteq \overline{\boldsymbol{L}}(\mathcal{A})$.

On the other hand, suppose that $\mathbf{A} \in \mathcal{A}$. Then, by (2.6), there is an algebra $\mathbf{A}' \in \mathcal{A}^{(m)}$ such that $\mathbf{A}' \leq \mathbf{A}$ and by definition, $X_{\mathbf{A}'} \overset{C}{\vdash} X_{\mathbf{A}}$. Thus, $X_{\mathbf{A}} \in \overline{\boldsymbol{L}}(\mathcal{A}^{(m)})$ for all $\mathbf{A} \in \mathcal{A}$, that is, $\overline{\boldsymbol{L}}(\mathcal{A}) \subseteq \overline{\boldsymbol{L}}(\mathcal{A}^{(m)})$.

2.4 Applications of Characteristic Formulas

In Jankov (1968b), the characteristic formulas were instrumental in proving that the cardinality of **ExtInt** is continuum and that there is an si.-logic lacking the f.m.p.

2.4.1 Antichains

Suppose that $\mathcal{A}$ is a class of finite s.i. C-algebras. We say that class $\mathcal{A}$ forms an *antichain* if for any $\mathbf{A}, \mathbf{B} \in \mathcal{A}$, algebras $\mathbf{A}$ and $\mathbf{B}$ are *incomparable*; that is, $\mathbf{A} \not\leq \mathbf{B}$ and $\mathbf{B} \not\leq \mathbf{A}$.

Let us observe that for any nonempty class of algebras $\mathcal{A}$, the subclass $\mathcal{A}^{(m)}$ forms an antichain.

Let C be a C-calculus and C be a set of formulas in the signature of C. Then C is said to be *strongly independent relative to* C if $C \setminus \{A\} \overset{C}{\nvdash} A$ for each formula $A \in C$. In other words, C is strongly independent relative to C if no formula from C can be derived in C from the rest of the formulas of C.

Let us observe that if C is a strongly independent set of C-formulas, then for any distinct subsets $C_1, C_2 \subseteq C$, the logics defined by C_1 and C_2 as sets of axioms, are distinct. Hence, if there is a countably infinite set C of strongly independent C-formulas, then the set of all extensions of the C-calculus is uncountable. This property of strongly independent sets was used in Jankov (1968b) for proving that the set of si-logics is not countable (cf. Sect. 2.5).

Antichains of finite s.i. C-algebras posses the following very important property.

Proposition 2.14 *Suppose that $\mathcal{A}$ is an antichain of finite s.i. C-algebras. Then the set $\{X_{\mathbf{A}} \mid \mathbf{A} \in \mathcal{A}\}$ is strongly independent.*

Proof For contradiction, suppose that for some $\mathbf{A} \in \mathcal{A}$,

$$\{X_{\mathbf{B}} \mid \mathbf{B} \in \mathcal{A} \setminus \{\mathbf{A}\}\} \vdash_C X_{\mathbf{A}}.$$

Recall that by (2.2), $\mathbf{A} \not\models A$ and hence, there is a $\mathbf{B} \in \mathcal{A} \setminus \{\mathbf{A}\}$ such that $\mathbf{A} \not\models X_{\mathbf{B}}$. Then, by Theorem 2.3, $\mathbf{B} \leq \mathbf{A}$, and we have arrived at a contradiction.

Corollary 2.10 *If there is an infinite antichain of finite s.i. C-algebras which are models of a given C-logic $\mathbf{L}$, then*

(a) the set of extensions of $\mathbf{L}$ is uncountable;
(b) there is an extension of $\mathbf{L}$ that cannot be defined by any C-calculus; that is, it cannot be defined by a finite set of axioms and the rules of substitution and Modus Ponens;
(c) there is a strongly ascending chain of C-logics.

In fact, if $\mathcal{A} = \{\mathbf{A}_i \mid i \geq 0\}$ is an infinite antichain of finite s.i. C-algebras, then logics $\boldsymbol{L}_k$ defined by $\{X_{\mathbf{A}_i} \mid i \in [1, k]\}$ form a strongly ascending chain, and consequently, logic $\overline{\boldsymbol{L}}(\mathcal{A})$ defined by $\{X_{\mathbf{A}} \mid \mathbf{A} \in \mathcal{A}\}$ cannot be defined by any C-calculus.

2.5 Extensions of C-Logics

In Jankov (1968b), it was observed that **Ext**Int is uncountable. To prove this claim, it is sufficient to present a countably infinite antichain of finite s.i. Heyting algebras.

Let $\mathcal{A}$ be a class of all finite s.i. Heyting algebras, generated by elements $\mathsf{a}, \mathsf{b}, \mathsf{c}$ and satisfying the following conditions:

$$\neg(\mathsf{a} \wedge \mathsf{b}) = \neg(\mathsf{b} \wedge \mathsf{c}) = \neg(\mathsf{c} \wedge \mathsf{a}) = \neg\neg\mathsf{a} \to \mathsf{a} = \neg\neg\mathsf{b} \to \mathsf{b} = \neg\neg(\mathsf{a} \vee \mathsf{b} \vee \mathsf{c}) = \mathbf{1} \tag{2.8}$$

$$\neg\mathsf{a} \vee \neg\mathsf{b} \vee (\neg\neg\mathsf{c} \to \mathsf{c}) = \mathsf{d}, \tag{2.9}$$

where d is a pretop element. Class $\mathcal{A}$ is not empty; moreover, it contains infinitely many members (cf. Fig. 2.3).

Conditions (2.8) and (2.9) yield that algebra is generated by three elements a, b, and c that are distinct from $\mathbf{0}$ such that elements a and b are *regular*, that is, $\neg\neg\mathsf{a} = \mathsf{a}$ and $\neg\neg\mathsf{b} = \mathsf{b}$, while element c is neither regular nor *dense*; that is $\neg\mathsf{c} \neq \mathbf{0}$.

The goal of this section is to prove the following theorem.

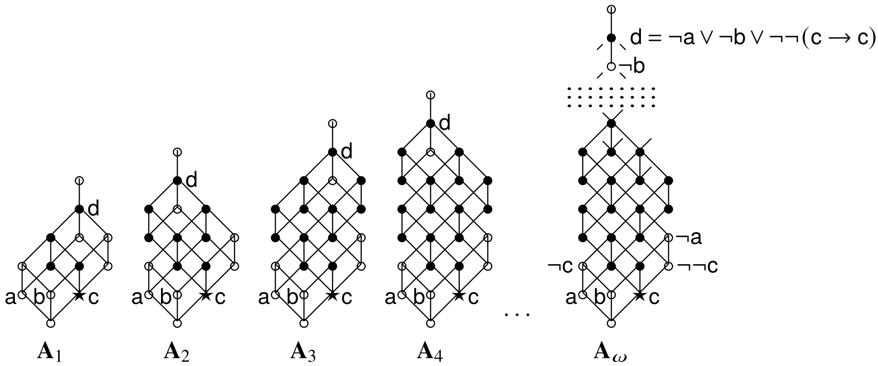


Fig. 2.3 Yankov's antichain

Theorem 2.4 *Logic* $\overline{\boldsymbol{L}}(\mathcal{A})$ *does not enjoy the finite model property.*

To show that $\overline{\boldsymbol{L}}(\mathcal{A})$ lacks the f.m.p., we will take the following formula:

$$A = \neg(p \wedge q) \wedge \neg(q \wedge r) \wedge \neg(p \wedge r) \wedge (\neg\neg p \to p) \wedge (\neg\neg q \to q) \wedge \neg\neg(p \vee q \vee r) \to \neg p \vee \neg q \vee (\neg\neg r \to r),$$

and we will prove the following two lemmas.

Lemma 2.1 $A \notin \overline{\boldsymbol{L}}(\mathcal{A})$.

Lemma 2.2 *A is valid in all finite models of* $\overline{\boldsymbol{L}}(\mathcal{A})$.

The proofs of Lemmas 2.1 and 2.2 can be found in Sect. 2.5.2.2, but first, we need to establish some properties of the algebras from $\mathcal{A}$ (cf. Sect. 2.5.1). In particular, we will prove (cf. Sect. 2.5.2.1) the following proposition, which has a very important corollaries on its own.

Proposition 2.15 *Algebras* $\{\mathbf{A}_i \mid i = 1, 2, \ldots\}$ *are minimal (relative to* $\leq$*) elements of* $\mathcal{A}$.

Corollary 2.11 *The class* $\{\mathbf{A}_i \mid i = 1, 2, \ldots\}$ *forms an antichain.*

Corollary 2.11 has three immediate corollaries, which at the time of the publication of Jankov (1968b) changed the view on the structure of **ExtInt**.

Corollary 2.12 *There are continuum many si-logics.*

Corollary 2.13 *There are si-logics that cannot be defined by an si-calculus.*

Corollary 2.14 *There exists a strictly ascending sequence* $\boldsymbol{L}_i$, $i > 0$, *of si-logics defined by si-calculi.*

Corollaries 2.12 and 2.13 follow immediately from Proposition 2.15 and Corollary 2.10. To prove Corollary 2.14, consider logics $\boldsymbol{L}_i$ defined by axioms $X_{\mathbf{A}_j}$, $j \in [1, i]$.

The rest of this section is dedicated to a proof of Proposition 2.15 and Theorem 2.4.

2.5.1 Properties of Algebras $\mathbf{A}_i$

In this section, $\mathbf{A}_i$ are algebras the diagrams of which are depicted in Fig. 2.3.

Proposition 2.16 *Each algebra* $\mathbf{A}_i$, $i \in [1, \omega]$, *contains precisely one set of three elements, namely* $\{\mathsf{a}, \mathsf{b}, \mathsf{c}\}$, *satisfying the following conditions:*

$$\neg(\mathsf{a} \wedge \mathsf{b}) = \neg(\mathsf{b} \wedge \mathsf{c}) = \neg(\mathsf{c} \wedge \mathsf{a}) = \neg\neg\mathsf{a} \to \mathsf{a} = \neg\neg\mathsf{b} \to \mathsf{b} = \mathbf{1} \quad (2.10)$$

$$\neg\mathsf{a} \neq \mathbf{1}, \quad \neg\mathsf{b} \neq \mathbf{1}, \quad \neg\neg\mathsf{c} \to \mathsf{c} \neq \mathbf{1}. \quad (2.11)$$

Proof It is not hard to see that in each $\mathbf{A}_i$, elements $\{\mathsf{a}, \mathsf{b}, \mathsf{c}\}$ satisfy conditions (2.10) and (2.11). Let us now show that there are no other elements satisfying these conditions.

It is clear that (2.11) yields that all elements $\mathsf{a}, \mathsf{b}, \mathsf{c}$ are distinct from $\mathbf{0}$, and $\mathsf{c} \neq \mathbf{1}$. Moreover, by (2.10), $\neg(\mathsf{a} \wedge \mathsf{c}) = \neg(\mathsf{b} \wedge \mathsf{c}) = \mathbf{1}$. Hence, $\mathsf{a} \wedge \mathsf{c} = \mathsf{b} \wedge \mathsf{c} = \mathbf{0}$ and therefore, $\mathsf{c} \wedge (\mathsf{a} \vee \mathsf{b}) = \mathbf{0}$. Hence, $\mathsf{a} \vee \mathsf{b} \leq \neg\mathsf{c}$ and consequently, $\neg\mathsf{c} \neq \mathbf{0}$.

Let us observe that in each algebra $\mathbf{A}_i$ there are precisely 8 elements for which $\neg\neg x = x$ holds:

$$\mathsf{A}_i^{(r)} := \{\mathbf{0}, \mathsf{a}, \mathsf{b}, \mathsf{a} \vee \mathsf{b}, \neg\mathsf{a}, \neg\mathsf{b}, \neg(\mathsf{a} \vee \mathsf{b}), \mathbf{1}\}.$$

Let us show that only elements a and b can potentially satisfy (2.10) and (2.11).

Indeed, we already know that we cannot use $\mathbf{0}$ and $\mathbf{1}$. In addition, we cannot use elements $\neg\mathsf{a}, \neg\mathsf{b}, \neg(\mathsf{a} \vee \mathsf{b})$, because for each $\mathsf{a}' \in \{\neg\mathsf{a}, \neg\mathsf{b}, \neg(\mathsf{a} \vee \mathsf{b})\}$ and for any $\mathsf{c} \in \mathbf{A}_i$, if $\mathsf{a}' \wedge \mathsf{c} = \mathbf{0}$, then $\mathsf{c} \leq \neg\mathsf{a}'$, that is, $\mathsf{c} \leq \mathsf{a} \vee \mathsf{b}$, while in algebra $\mathbf{A}_i$ all elements smaller then $\mathsf{a} \vee \mathsf{b}$ satisfy condition $\neg\neg\mathsf{c} = \mathsf{c}$.

This leaves us with elements a, b and $\mathsf{a} \vee \mathsf{b}$. But we cannot use element $\mathsf{a} \vee \mathsf{b}$, because neither $(\mathsf{a} \vee \mathsf{b}) \wedge \mathsf{a}$ nor $(\mathsf{a} \vee \mathsf{b}) \wedge \mathsf{b}$ is $\mathbf{0}$.

Next, we observe that in $\mathbf{A}_i$, there are just two elements c and $\neg\mathsf{a}$ whose intersection with a and b gives $\mathbf{0}$, but we cannot select $\neg\mathsf{a}$, because $\neg\neg\neg\mathsf{a} \to \neg\mathsf{a} = \mathbf{1}$, and this element would not satisfy (2.11). Thus, only elements a, b, and c satisfy conditions (2.10) and (2.11), and this observation completes the proof.

Next, we prove that in the homomorphic images of algebras $\mathbf{A}_i$, only images of elements a, b, and c may satisfy conditions (2.10) and (2.11).

Proposition 2.17 *Let algebra* $\mathbf{A}_i$, $i \in [1, \omega]$, *and* $\varphi : \mathbf{A}_i \longrightarrow \mathbf{B}$ *be a homomorphism onto algebra* $\mathbf{B}$. *If for some elements* $\mathsf{a}', \mathsf{b}', \mathsf{c}' \in \mathbf{A}_i$, *their images* $\bar{\mathsf{a}}, \bar{\mathsf{b}}, \bar{\mathsf{c}}$ *satisfy conditions* (2.10) *and* (2.11), *then elements* $\mathsf{a} = \neg\neg\mathsf{a}', \mathsf{b} = \neg\neg\mathsf{b}'$, *and* $\mathsf{c} = \mathsf{c}' \wedge \neg\mathsf{a}' \wedge \neg\mathsf{b}'$ *satisfy* (2.10) *and* (2.11).

Proof First, let us observe that $\varphi^{-1}(\mathbf{0}) = \{\mathbf{0}\}$; that is, $\mathbf{0}$ is the only element of $\mathbf{A}_i$ which φ sends to $\mathbf{0}$.

Indeed, assume for contradiction that there is an element $\mathsf{d}' \in \mathbf{A}_i$ such that $\mathbf{0} < \mathsf{d}'$ and $\varphi(\mathsf{d}') = \mathbf{0}$. Elements a, b, and c are the only atoms of $\mathbf{A}_i$ and therefore,

$\mathsf{a} \leq \mathsf{d}'$, $\mathsf{b} \leq \mathsf{d}'$, or $\mathsf{c} \leq \mathsf{d}'$. Hence, $\varphi(\mathsf{a}) = \mathbf{0}$, $\varphi(\mathsf{b}) = \mathbf{0}$, or $\varphi(\mathsf{c}) = \mathbf{0}$ and therefore, $\varphi(\neg\mathsf{a}) = \mathbf{1}$, $\varphi(\neg\mathsf{b}) = \mathbf{1}$, or $\varphi(\neg\mathsf{c}) = \mathbf{1}$. Recall that $\varphi(\neg\mathsf{a}) = \neg\varphi(\mathsf{a}) = \neg\bar{\mathsf{a}}$ and by (2.11), $\neg\bar{\mathsf{a}} \neq \mathbf{1}$. Likewise, $\neg\bar{\mathsf{b}} = \varphi(\neg\mathsf{b}) \neq \mathbf{1}$. And if $\varphi(\neg\mathsf{c}) = \mathbf{1}$, then $\neg\bar{\mathsf{c}} = \mathbf{1}$ and consequently $\neg\neg\bar{\mathsf{c}} \to \bar{\mathsf{c}} = \mathbf{1}$, which contradicts (2.11). Thus, $\varphi^{-1}(\mathbf{0}) = \{\mathbf{0}\}$.

Next, let us show that $\mathsf{a} \wedge \mathsf{b} = \mathbf{0}$ and hence, $\neg(\mathsf{a} \wedge \mathsf{b}) = \mathbf{1}$. Indeed,

$$\varphi(\mathsf{a} \wedge \mathsf{b}) = \varphi(\neg\neg\mathsf{a}' \wedge \neg\neg\mathsf{b}') = \neg\neg\varphi(\mathsf{a}') \wedge \neg\neg\varphi(\mathsf{b}') = \varphi(\mathsf{a}') \wedge \varphi(\mathsf{b}') = \mathbf{0}.$$

Hence, $\mathsf{a} \wedge \mathsf{b} \in \varphi^{-1}(\mathbf{0})$ and therefore, $\mathsf{a} \wedge \mathsf{b} = \mathbf{0}$.

In addition, $\mathsf{a} \wedge \mathsf{c} = \neg\neg\mathsf{a}' \wedge (\mathsf{c} \wedge \neg\mathsf{a}' \wedge \neg\mathsf{b}') = \mathbf{0}$. Likewise, $\mathsf{b} \wedge \mathsf{c} = \mathbf{0}$.

$$\mathsf{a} \wedge \mathsf{c} = \neg\neg\mathsf{a}' \wedge (\mathsf{c} \wedge \neg\mathsf{a}' \wedge \neg\mathsf{b}') = \mathbf{0},\ \ \mathsf{b} \wedge \mathsf{c} = \mathbf{0},$$
$$\neg\neg\mathsf{a} \to \mathsf{a} = \neg\neg\neg\neg\mathsf{a}' \to \neg\neg\mathsf{a}' = \mathbf{1},\ \ \neg\neg\mathsf{b} \to \mathsf{b} = \mathbf{1}.$$

Thus, elements a, b, and c satisfy (2.10).

Next, we observe that by (2.11), $\neg\bar{\mathsf{a}} \neq \mathbf{1}$ and $\neg\bar{\mathsf{b}} \neq \mathbf{1}$, that is, $\bar{\mathsf{a}} > \mathbf{0}$ and $\bar{\mathsf{b}} > \mathbf{0}$. Hence,

$$\varphi(\mathsf{a}) = \varphi(\neg\neg\mathsf{a}') = \neg\neg\varphi(\mathsf{a}') = \neg\neg\bar{\mathsf{a}} \geq \bar{\mathsf{a}} > \mathbf{0}$$

and by the same reason, $\mathsf{b} \neq \mathbf{0}$. Thus, $\neg\mathsf{a} \neq \mathbf{1}$ and $\neg\mathsf{b} \neq \mathbf{1}$.

Now, let us show that $\neg\neg\mathsf{c} \to \mathsf{c} \neq \mathbf{1}$. That is, we need to demonstrate that

$$\neg\neg(\mathsf{c}' \wedge \neg\mathsf{a}' \wedge \neg\mathsf{b}') \to (\mathsf{c}' \wedge \neg\mathsf{a}' \wedge \neg\mathsf{b}') \neq \mathbf{1}.$$

To that end, we will show that

$$\varphi(\neg\neg(\mathsf{c}' \wedge \neg\mathsf{a}' \wedge \neg\mathsf{b}') \to (\mathsf{c}' \wedge \neg\mathsf{a}' \wedge \neg\bar{\mathsf{b}}) = \neg\neg(\bar{\mathsf{c}} \wedge \neg\bar{\mathsf{a}} \wedge \neg\bar{\mathsf{b}}) \to (\bar{\mathsf{c}} \wedge \neg\bar{\mathsf{a}} \wedge \neg\bar{\mathsf{b}}) \neq \mathbf{1}.$$

Indeed, recall that by (2.11), $\bar{\mathsf{a}} \wedge \bar{\mathsf{c}} = \mathbf{0}$ and hence, $\mathsf{c} \leq \neg\bar{\mathsf{a}}$. Likewise, $\bar{\mathsf{c}} \leq \neg\bar{\mathsf{b}}$ and hence,

$$\bar{\mathsf{c}} \leq \neg\bar{\mathsf{a}} \wedge \neg\bar{\mathsf{b}} \text{ and consequently, } \bar{\mathsf{c}} \wedge \neg\bar{\mathsf{a}} \wedge \neg\bar{\mathsf{b}} = \bar{\mathsf{c}}.$$

Hence,

$$\neg\neg(\bar{\mathsf{c}} \wedge \neg\bar{\mathsf{a}} \wedge \neg\bar{\mathsf{b}}) \to (\bar{\mathsf{c}} \wedge \neg\bar{\mathsf{a}} \wedge \neg\bar{\mathsf{b}}) = \neg\neg\bar{\mathsf{c}} \to \bar{\mathsf{c}},$$

and by (2.11), $\neg\neg\bar{\mathsf{c}} \to \bar{\mathsf{c}} \neq \mathbf{1}$. This observation completes the proof.

Corollary 2.15 *Any homomorphic image of any algebra* $\mathbf{A}_i$, $i \in [1, \omega]$, *contains at most one set of elements satisfying conditions* (2.10) *and* (2.11).

Corollary 2.16 *None of the proper homomorphic images of algebras* $\mathbf{A}_i$, $i \in [1, \omega]$, *has elements satisfying conditions* (2.8) *and* (2.9).

Proof Suppose that $\mathbf{A}_i$ is an algebra the diagram of which is depicted in Fig. 2.3 and that $\mathsf{a}, \mathsf{b}, \mathsf{c} \in \mathbf{A}_i$ are elements satisfying conditions (2.8) and (2.9).

For contradiction, assume that $\varphi : \mathbf{A}_i \longrightarrow \mathbf{B}$ is a proper homomorphism of $\mathbf{A}_i$ onto $\mathbf{B}$ and that elements $\overline{\mathsf{a}}, \overline{\mathsf{b}}, \overline{\mathsf{c}} \in \mathbf{B}$ satisfy conditions (2.8) and (2.9). Then, these elements satisfy the weaker conditions (2.10) and (2.11). By Proposition 2.17, elements $\overline{\mathsf{a}}, \overline{\mathsf{b}}, \overline{\mathsf{c}}$ are images of some elements $\mathsf{a}', \mathsf{b}', \mathsf{c}' \in \mathbf{A}_i$ also satisfying condition (2.10) and (2.11). By Proposition 2.16, the set of elements of $\mathbf{A}_i$ satisfying (2.10) and (2.11) is unique; namely, it is $\{\mathsf{a}, \mathsf{b}, \mathsf{c}\}$. By (2.9), $\neg\mathsf{a} \vee \neg\mathsf{b} \vee (\neg\neg\mathsf{c} \to \mathsf{c})$ is a pretop element of $\mathbf{A}$ and hence, because φ is a proper homomorphism,

$$\varphi(\neg\mathsf{a} \vee \neg\mathsf{b} \vee (\neg\neg\mathsf{c} \to \mathsf{c})) = \neg\overline{\mathsf{a}} \vee \neg\overline{\mathsf{b}} \vee (\neg\neg\overline{\mathsf{c}} \to \overline{\mathsf{c}}) = \mathbf{1},$$

and we have arrived at a contradiction: elements $\overline{\mathsf{a}}$, $\overline{\mathsf{b}}$, and $\overline{\mathsf{c}}$ do not satisfy (2.9).

2.5.2 Proofs of Lemmas

2.5.2.1 Proof of Proposition 2.15

To prove Proposition 2.15, we need to show that no algebra $\mathbf{B} \in \mathcal{A}$ can be embedded in any homomorphic image of algebra $\mathbf{A}_i$, $i = 1, 2, \dots$, as long as $\mathbf{B} \not\cong \mathbf{A}_i$.

From Corollary 2.16, we already know that none of the proper homomorphic images of algebras $\mathbf{A}_i$ contains elements satisfying conditions (2.8) and (2.9). Thus, no algebra from $\mathcal{A}$ can be embedded in a proper homomorphic image of any algebra $\mathbf{A}_i$.

Now, assume that $\mathbf{B} \in \mathcal{A}$ and $\varphi : \mathbf{B} \longrightarrow \mathbf{A}_i$ is an embedding. By the definition of $\mathcal{A}$, $\mathbf{B}$ is generated by some elements $\mathsf{a}, \mathsf{b}, \mathsf{c}$ satisfying conditions (2.8) and (2.9). Hence, because φ is an isomorphism, elements $\varphi(\mathsf{a}), \varphi(\mathsf{b}), \varphi(\mathsf{c})$ satisfy (2.8) and (2.9). By Proposition 2.16, there is a unique set of three elements that satisfy (2.10) and (2.11) and therefore, there is a unique set of three elements satisfying (2.8) and (2.9). By the definition of $\mathcal{A}$, this set generates algebra $\mathbf{A}_i$; that is, φ maps $\mathbf{B}$ onto $\mathbf{A}$ and thus φ is an isomorphism between $\mathbf{B}$ and $\mathbf{A}$.

2.5.2.2 Proof of Lemma 2.1

Syntactic proof (cf. Jankov 1968b). For contradiction, assume that $A \in \overline{\boldsymbol{L}}(\mathcal{A})$. Recall that by Proposition 2.13, $A \in \overline{\boldsymbol{L}}(\mathcal{A}^{(m)})$ and hence, for some minimal algebras $\mathbf{B}_i$, $i \in [1, n]$,

$$X_{\mathbf{B}_1}, \dots, X_{\mathbf{B}_n} \vdash A.$$

On the other hand, by Proposition 2.5.2.1, $\{\mathbf{A}_i \mid i = 1, 2 \dots\} \subseteq \overline{\boldsymbol{L}}(\mathcal{A}^{(m)})$. Class $\{\mathbf{A}_i \mid i = 1, 2 \dots\}$ is infinite and thus, there is an $\mathbf{A}_k \notin \{\mathbf{B}_i, i \in [1, n]\}$. Observe that $\mathbf{A}_k \not\models A$: it is not hard to see that valuation $p \mapsto \mathsf{a}$, $q \mapsto \mathsf{b}$, $r \mapsto \mathsf{c}$ refutes A in every $\mathbf{A}_k$.

Hence, by Theorem 2.2, $A \vdash X_{\mathbf{A}_k}$ and therefore,

$$X_{\mathbf{B}_1}, \ldots, X_{\mathbf{B}_n} \vdash X_{\mathbf{A}_k}.$$

This contradicts Proposition 2.14, which states that the characteristic formulas of any antichain form a strongly independent set, and the subclass of all minimal algebras always forms an antichain.

Semantic proof. Observe that formula A is invalid in algebra $\mathbf{A}_\omega$, and let us prove that $\mathbf{A}_\omega$ is a model of $\overline{\boldsymbol{L}}(\mathcal{A})$. To that end, we prove that neither an algebra from $\mathcal{A}$ or its homomorphic image can be embedded into $\mathbf{A}_\omega$ and therefore, by Proposition 2.12, all formulas $X_{\mathbf{A}}$, $\mathbf{A} \in \mathcal{A}$, are valid in $\mathbf{A}_\omega$.

Indeed, by Proposition 2.17, not any algebra from $\mathcal{A}$ can be embedded in a proper homomorphic image of $\mathbf{A}_\omega$. In addition, by Proposition 2.16, $\mathbf{A}_\omega$ contains a unique set of three elements satisfying conditions (2.8) and (2.9), and these elements generate algebra $\mathbf{A}_\omega$. Thus, if algebra $\mathbf{A} \in \mathcal{A}$ was embedded in $\mathbf{A}_\omega$, its embedding would be a map onto $\mathbf{A}_\omega$, which is impossible, because $\mathbf{A}$ is finite, while $\mathbf{A}_\omega$ is infinite.

2.5.2.3 Proof of Lemma 2.2

We need to show that formula A is valid in all finite models of logic $\overline{\boldsymbol{L}}(\mathcal{A})$. To that end, we will show that every finite Heyting algebra $\mathbf{A}$ refuting A is not a model of $\overline{\boldsymbol{L}}(\mathcal{A})$, because there is a homomorphic image $\mathbf{B}$ of $\mathbf{A}$ in which one of the algebras from $\mathcal{A}$ is embedded. Because $\overline{\boldsymbol{L}}(\mathcal{A})$ is defined by characteristic formulas of algebras from $\mathcal{A}$, none of the members of $\mathcal{A}$ is a model of $\overline{\boldsymbol{L}}(\mathcal{A})$. Hence, if $\mathbf{A}' \in \mathcal{A}$ and $\mathbf{A}'$ is embedded in $\mathbf{B}$, algebra $\mathbf{B}$ and, consequently, algebra $\mathbf{A}$ are not models of $\overline{\boldsymbol{L}}(\mathcal{A})$.

Suppose that finite algebra $\mathbf{A}$ refutes formula

$$A = \neg(p \wedge q) \wedge \neg(q \wedge r) \wedge \neg(p \wedge r) \wedge (\neg\neg p \to p) \wedge (\neg\neg q \to q) \wedge \neg\neg(p \vee q \vee r) \to \neg p \vee \neg q \vee (\neg\neg r \to r).$$

Then, by Corollary 2.2, there is a homomorphic image $\mathbf{B}$ of algebra $\mathbf{A}$ and a valuation ν in $\mathbf{B}$ such that

$$\nu(\neg(p \wedge q) \wedge \neg(q \wedge r) \wedge \neg(p \wedge r) \wedge (\neg\neg p \to p) \wedge (\neg\neg q \to q) \wedge \neg\neg(p \vee q \vee r)) = \mathbf{1}_{\mathbf{B}}$$
$$\nu(\neg p \vee \neg q \vee (\neg\neg r \to r)) = \mathsf{m}_{\mathbf{B}}.$$

Let $\mathsf{a} = \nu(p)$, $\mathsf{b} = \nu(q)$, and $\mathsf{c} = \nu(r)$. Then, elements a, b, and c satisfy conditions (2.8) and (2.9) and therefore, these elements generate a subalgebra of $\mathbf{B}$ belonging to $\mathcal{A}$, and this observation completes the proof.

2.6 Calculus of the Weak Law of Excluded Middle

In Jankov (1968a), Yankov studied the logic of calculus $\mathsf{KC} := \mathsf{IPC} + \neg p \vee \neg\neg p$ which nowadays bears his name. Let us denote this logic by Yn.

A formula A is said to be *positive* if it contains only connectives $\wedge$, $\vee$ and $\to$. If $\boldsymbol{L}$ is an si-logic, $\boldsymbol{L}^+$ denotes a positive fragment of $\boldsymbol{L}$—the subset of all positive formulas from $\boldsymbol{L}$. We say that an si-logic $\boldsymbol{L}$ is a *p-conservative extension* of Int when $\boldsymbol{L}^+ = \mathsf{Int}^+$.

An s.i. calculus K *admits the derivable elimination of negation* if for any formula A there is a positive formula A^* such that $A \overset{\mathsf{K}\ \mathsf{K}}{\dashv\vdash} A^*$. If $\boldsymbol{L}$ is a logic of K, we say that $\boldsymbol{L}$ admits derivable elimination of negation. Given an si-logic $\boldsymbol{L}$, its extension $\boldsymbol{L}' \in \mathbf{Ext}\boldsymbol{L}$ is said to be *positively axiomatizable* relative to $\boldsymbol{L}$ just in case $\boldsymbol{L}'$ can be axiomatized relative to $\boldsymbol{L}$ by positive axioms.

The following simple proposition provides some different perspectives on the notion of derivable elimination of negation introduced in Jankov (1968a).

Proposition 2.18 *Suppose that $\boldsymbol{L}$ is an si-logic. Then, the following are equivalent:*

(a) $\boldsymbol{L}$ admits derivable elimination of negation;
(b) every extension of $\boldsymbol{L}$ is positively axiomatizable relative to to $\boldsymbol{L}$;
(c) any two distinct extensions of $\boldsymbol{L}$ have distinct positive fragments.

Proof (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) is straightforward.

(b) $\Longrightarrow$ (a). Suppose that $\boldsymbol{L}$ is defined by an s.i. calculus K. Then, for every formula A, consider logic $\boldsymbol{L}'$ defined by $\mathsf{K} + A$. If $\boldsymbol{L}' = \boldsymbol{L}$, that is, $A \in \boldsymbol{L}$, we have $A \overset{\mathsf{K}\ \mathsf{K}}{\dashv\vdash} (p \to p)$. If $\boldsymbol{L} \subsetneq \boldsymbol{L}'$, by assumption, there are positive formulas $B_i, i \in I$, such that $\boldsymbol{L}'$ is a logic of $\mathsf{K} + \{B_i, i \in I\}$. Thus, on the one hand, for every $i \in I$, $A \overset{\mathsf{K}}{\vdash} B_i$. On the other hand, $B_i, i \in I \overset{\mathsf{K}\ \mathsf{K}}{\dashv\vdash} A$, and consequently, there is a finite subset of formulas from $\{B_i, i \in I\}$, say, $B_1, \dots, B_n$, such that $B_1, \dots, B_n \overset{\mathsf{K}}{\vdash} A$. It is not hard to see that

$$A \overset{\mathsf{K}\ \mathsf{K}}{\dashv\vdash} \bigwedge_{i=1}^{n} B_i.$$

(c) $\Longrightarrow$ (b). Indeed, if $\boldsymbol{L}_1 \supseteq \boldsymbol{L}$, then $\boldsymbol{L}_1$ is a logic of $\mathsf{K} + \boldsymbol{L}_1^+$: the logics of $\mathsf{K} + \boldsymbol{L}_1^+$ and $\boldsymbol{L}_1$ cannot be distinct, because they have the same positive fragments and by (c) they must coincide.

Remark 2.3 Derivable elimination of negation is not the same as expressibility of negation. For instance, in IPC, $\neg p \dashv\vdash p \to q$, because $\vdash \neg p \to (p \to q)$ and a formula equivalent to $\neg p$ can be derived from $p \to q$ by substituting $p \wedge \neg p$ for q. At the same time, obviously, $\nvdash \neg p \leftrightarrow (p \to q)$. Similarly, $p \vee \neg p \dashv\vdash p \vee (p \to q)$, and Cl can be defined by $\mathsf{IPC} + p \vee (p \to q)$.

The goal of this section is to prove the following theorem.

Theorem 2.5 *The following holds:*

(a) Yn *is the greatest p-conservative extension of* Int*;*
(b) Yn *is a minimal logic admitting derivable elimination of negation.*

Corollary 2.17 *Logic* Yn *is a unique s.i. p-conservative extension of* Int *admitting derivable elimination of negation.*

Remark 2.4 Yn is a minimal logic admitting derivable elimination of negation, but it is not the smallest such logic: it was observed in Hosoi and Ono (1970) that all logics of the second slice are axiomatizable by implicative formulas. Hence, the smallest logic of the second slice has derivable elimination of negation. It is not hard to see that this logic is not an extension of Yn.

In Jankov (1968a), Yankov gave a syntactic proof of Theorem 2.5; we offer an alternative, semantic proof, and we start with studying the algebraic semantics of KC.

2.6.1 Semantics of KC

Let us start with a simple observation that any s.i. Heyting algebra $\mathbf{A}$ is a model for KC (that is, $\mathbf{A} \models (\neg p \vee \neg\neg p)$ if and only if each distinct from $\mathbf{0}$ element $\mathsf{a} \in \mathbf{A}$ is *dense*; that is, $\neg\neg\mathsf{a} = \mathbf{1}$ (or equivalently, $\neg\mathsf{a} = \mathbf{0}$). Thus, a class of all such algebras forms an adequate semantics for the Yankov logic, and we call these algebras the *Yankovean* algebras.

Let us recall some properties of dense elements that we need in the sequel. Suppose that $\mathbf{A}$ is a Heyting algebra and $\mathsf{a}, \mathsf{b} \in \mathbf{A}$. Then, it is clear that if $\mathsf{a} \leq \mathsf{b}$ and a is a dense element, then b is a dense element: $\mathsf{a} \leq \mathsf{b}$ implies $\neg\mathsf{b} \leq \neg\mathsf{a} = \mathbf{0}$. Moreover, if a and b are dense, so is $\mathsf{a} \wedge \mathsf{b}$: by Glivenko's Theorem $\neg\neg(\mathsf{a} \wedge \mathsf{b}) = \neg(\neg\mathsf{a} \vee \neg\mathsf{b}) = \neg(\mathbf{0} \vee \mathbf{0}) = \mathbf{1}$.

Theorem 2.6 (Jankov 1968a) *The class of all finite Yankovean algebras forms an adequate semantics for* KC.

Remark 2.5 In Jankov (1968a), Yankov offered a syntactic proof. We offer a semantic proof based on an idea used in McKinsey (1941).

Proof It is clear that $\mathbf{0} \oplus \mathbf{A} \models \neg p \vee \neg\neg p$ for all Heyting algebras $\mathbf{A}$, and we need to prove that for any formula A such that $\mathsf{KC} \vdash A$, there is a finite Yankovean algebra $\mathbf{B}$ in which A is refuted.

Suppose that $\mathsf{KC} \nvdash A$. Then, there is a Yankovean algebra $\mathbf{A}$ in which A is refuted. Let ν be a refuting valuation and $A_1, \ldots, A_n, \mathbf{1}$ be all the subformulas of A.

Consider a distributive sublattice $\mathbf{B}$ of $\mathbf{A}$ generated (as sublattice) by elements $\mathbf{0}, \nu(A_1), \ldots, \nu(A_n), \mathbf{1}$. Every finitely generated distributive lattice is finite (cf., e.g., Grätzer 2003), and any finite distributive lattice can be regarded as a Heyting algebra. Let us prove that (a) **B** is a Yankovean algebra, and (b) ν is a refuting valuation in **B**.

(a) First, let us note that the meets, the joins, and the partial orders in algebras **A** and **B** are the same. Hence, **B** has a pretop element: the join of all elements from **B** that are distinct from **1**, and therefore, it is an s.i. algebra. In addition, because algebra **A** is Yankovean, all its elements that are distinct from **0** are dense. Hence, as **B** is finite, the meet of all elements from **B** that are distinct from **0** is again a dense element and therefore, it is distinct from **0**. Thus, this meet is the smallest distinct from **0** element of **B** and therefore, all elements that are distinct from **0** are dense and **B** is a Yankovean algebra.

(b) Let us observe that if elements $\mathsf{a}, \mathsf{b} \in \mathbf{B}$, then $\mathsf{a} \wedge \mathsf{b}, \mathsf{a} \vee \mathsf{b} \in \mathbf{B}$, and $\neg\mathsf{a} \in \mathbf{B}$, because $\neg\mathsf{a} = \mathbf{1}$ if $\mathsf{g} = \mathbf{0}$ and $\neg\mathsf{a} = \mathbf{0}$ otherwise. In addition, if $\mathsf{a} \to \mathsf{b} \in \mathbf{B}$ and $\to'$ is an implication defined in **B**, then $\mathsf{a} \to' \mathsf{b} = \mathsf{a} \to \mathsf{b}$: by definition, $\mathsf{a} \to \mathsf{b}$ is the greatest element in $\{\mathsf{c} \in \mathbf{A} \mid \mathsf{a} \wedge \mathsf{c} \leq \mathsf{b}\}$, and because $\mathsf{a} \to \mathsf{b} \in \mathbf{B}$ and **A** and **B** have the same partial order, $\mathsf{a} \to \mathsf{b}$ is the greatest element in $\{\mathsf{c} \in \mathbf{B} \mid \mathsf{a} \wedge \mathsf{c} \leq \mathsf{b}\}$. Thus, because all elements are $\mathbf{0}, \nu(A_1), \ldots, \nu(A_n), \mathbf{1}$, all values of $\nu(A_1), \ldots, \nu(A_n)$ when ν is regarded as a valuation in **B** remain the same and therefore, ν refutes A in **B**.

Given a Heyting algebra **A**, one can adjoin a new bottom element and, in such a way, obtain a new Heyting algebra denoted by $\mathbf{0} \oplus \mathbf{A}$. For instance (cf. Fig. 2.2), $\mathbf{3} = \mathbf{0} \oplus \mathbf{2}$. It is not hard to see that $\mathbf{0} \oplus \mathbf{A}$ is a Yankovean algebra. On the other hand, any finite Yankovean algebra has the form $\mathbf{0} \oplus \mathbf{A}$, where **A** is a finite s.i. Heyting algebra.

Corollary 2.18 *The class of finite Yankovean algebras forms an adequate semantic for* KC.

Let us construct more adequate semantics for KC.

Observe that in any Heyting algebra **A**, the elements $\{\mathbf{0}\} \cup \{\mathsf{a} \in \mathbf{A} \mid \neg\neg\mathsf{a} = \mathbf{1}\}$ form a Heyting subalgebra of **A** denoted by $\mathbf{A}^{(d)}$. It is clear that if **A** is an s.i. algebra, then $\mathbf{A}^d$ is Yankovean. In the sequel, we use the following property of $\mathbf{A}^{(d)}$.

Proposition 2.19 *If $\varphi : \mathbf{A} \longrightarrow \mathbf{B}$ is a homomorphism of Heyting algebra **A** onto Heyting algebra **B**, then the restriction $\widehat{\varphi}$ of φ to $\mathbf{A}^{(d)}$ is a homomorphism of $\mathbf{A}^{(d)}$ onto $\mathbf{B}^{(d)}$.*

Proof It is clear that $\widehat{\varphi}(\mathbf{A}^{(d)})$ is a subalgebra of **B**. Moreover, because for any element of $\mathbf{A}^{(d)}$ that is distinct from **0**, $\neg\neg\mathsf{a} = \mathbf{1}$, it is clear that $\widehat{\varphi}(\neg\neg\mathsf{a}) = \neg\neg\widehat{\varphi}(\mathsf{a}) = \mathbf{1}$ and hence, $\widehat{\varphi}(\mathsf{a}) \in \mathbf{B}^{(d)}$; that is, $\varphi(\mathbf{A}^{(d)}) \subseteq \mathbf{B}^{(d)}$. Thus, we only need to show that $\widehat{\varphi}$ maps $\mathbf{A}^{(d)}$ onto $\mathbf{B}^{(d)}$.

Indeed, let us show that for any $\mathsf{b} \in \mathbf{B}^{(d)}$, the preimage $\widehat{\varphi}^{-1}(\mathsf{b})$ contains an element from $\mathbf{A}^{(d)}$.

Suppose that $\mathsf{b} \in \mathbf{B}^{(d)}$. If $\mathsf{b} = \mathbf{0}$, then trivially, $\mathbf{0} \in \widehat{\varphi}^{-1}(\mathsf{b})$. If $\mathsf{b} \neq \mathbf{0}$, then by the definition of $\mathbf{B}^{(d)}$, $\neg\neg\mathsf{b} = \mathbf{1}$, that is, $\neg\mathsf{b} = \mathbf{0}$, and consequently $\mathsf{b} \vee \neg\mathsf{b} = \mathsf{b}$. Hence, for any element $\mathsf{a} \in \widehat{\varphi}^{-1}(\mathsf{b})$, $\mathsf{a} \vee \neg\mathsf{a} \in \widehat{\varphi}^{-1}(\mathsf{b})$, and it is not hard to see that $\mathsf{a} \vee \neg\mathsf{a} \in \mathbf{A}^{(d)}$.

Theorem 2.7 *Suppose that Heyting algebras* $\{\mathbf{A}_i, i \in I\}$ *form an adequate semantics for* IPC. *Then algebras* $\{\mathbf{A}^{(d)}, i \in I\}$ *form an adequate semantics for* KC.

Proof It is clear that algebras $\mathbf{A}_i^{(d)}$ are models for KC, and we only need to prove that for any formula A not derivable in KC, there is an algebra $\mathbf{A}_i^{(d)}$ in which A is refuted. We already know that all finite Yankovean algebras form an adequate semantics for KC. Hence, it suffices to show that each Yankovean algebra can be embedded in a homomorphic image of some algebra $\mathbf{A}_i^{(d)}, i \in I$.

Let $\mathbf{B}$ be a finite Yankovean algebra and $X_\mathbf{B}$ be its characteristic formula. Then, by (2.2), $\mathbf{B} \not\models X_\mathbf{B}$ and consequently, IPC $\nvdash X_\mathbf{B}$, because algebras $\mathbf{A}, i \in I$, form an adequate semantics for IPC. For some $i \in I$, $\mathbf{A}_i \nvdash X_\mathbf{B}$ and by Proposition 2.12, $\mathbf{B}$ is embedded in a homomorphic image $\hat{\mathbf{A}}_i$ of algebra $\mathbf{A}_i$. Recall that $\mathbf{B}$ is Yankovean and all its elements that are distinct from $\mathbf{0}$ are dense. Clearly, embedding preserves density and hence, $\mathbf{B}$ is embedded in $\hat{\mathbf{A}}_i^{(d)}$. The observation that by Proposition 2.19 $\hat{\mathbf{A}}_i^{(d)}$ is a homomorphic image of $\mathbf{A}_i^{(d)}$ completes the proof.

Each Heyting algebra $\mathbf{A}$ can be adjoined with a new top element to obtain a new Heyting algebra that is denoted by $\mathbf{A} \oplus \mathbf{1}$. For instance, $\mathbf{3} = \mathbf{2} \oplus \mathbf{1}$, $\mathbf{4} = \mathbf{3} \oplus \mathbf{1}$, and $\mathbf{Z}_5 = \mathbf{2}^2 \oplus \mathbf{1}$ (cf. Fig. 2.2).

The following Heyting algebras are called *Jaśkowski matrices*, and they form an adequate semantics for IPC:

$$\mathbf{J}_0 = \mathbf{2}, \qquad \mathbf{J}_{k+1} = \mathbf{J}^k \oplus \mathbf{1}.$$

Corollary 2.19 *Algebras* $\mathbf{J}_k^{(d)}, k > 0$, *form an adequate semantics for* KC.

2.6.2 KC *from the Splitting Standpoint*

In what follows, algebra $\mathbf{Z}_5$, the Hasse diagram of which is depicted in Fig. 2.4, plays a very important role.

Proposition 2.20 *Suppose that* $\mathbf{A}$ *is a Heyting algebra. Then* $\mathbf{A} \not\models \neg p \vee \neg\neg p$ *if and only if algebra* $\mathbf{Z}_5$ *is a subalgebra of* $\mathbf{A}$.

Proof It should be clear that $\mathbf{Z}_5 \not\models \neg p \vee \neg\neg p$ (consider valuation $\nu(p) = \mathsf{a}$) and hence, any algebra $\mathbf{A}$ containing a subalgebra isomorphic to $\mathbf{Z}_5$ refutes $\neg p \vee \neg\neg p$.

Conversely, suppose that $\mathbf{A} \not\models \neg p \vee \neg\neg p$. Then, for some $\mathsf{a} \in \mathbf{A}$, $\neg\mathsf{a} \vee \neg\neg\mathsf{a} \neq \mathbf{1}$. It is not hard to verify that subset $\mathbf{0}, \neg\mathsf{a}, \neg\neg\mathsf{a}, \neg\mathsf{a} \vee \neg\neg\mathsf{a}, \mathbf{1}$ is closed under fundamental operations and, therefore, forms a subalgebra of $\mathbf{A}$. In addition, because $\neg\mathsf{a} \vee \neg\neg\mathsf{a} \neq \mathbf{1}$, all five elements of this subalgebra are distinct and thus, the subalgebra is isomorphic to $\mathbf{Z}_5$.

Proposition 2.20 entails that formula $\neg p \vee \neg\neg p$ is interderivable in IPC with characteristic formula $X_{\mathbf{Z}_5}$. Indeed, because $\mathbf{Z}_5 \not\models \neg p \vee \neg\neg p$, by Theorem 2.2,

$\mathbf{A} \vdash X_{\mathbf{Z}_5}$. On the other hand, for any algebra $\mathbf{A}$, if $\mathbf{A} \not\models \neg p \vee \neg\neg p$, then by Proposition 2.20, $\mathbf{A}$ has a subalgebra that is isomorphic to $\mathbf{Z}_5$ and by (2.2), $\mathbf{A} \not\models X_{\mathbf{Z}_5}$.

Corollary 2.20 Yn *coincides with* $\overline{\boldsymbol{L}}(\mathbf{Z}_5)$.

Thus Yn is the greatest logic for which $\mathbf{Z}_5$ is not a model.

2.6.3 Proof of Theorem 2.5

Proof of (a). Because Yn coincides with $\overline{\boldsymbol{L}}(\mathbf{Z}_5)$, for any si-logic $\boldsymbol{L}$, either $\boldsymbol{L} \subseteq \overline{\boldsymbol{L}}(\mathbf{Z}_5)$, or $\boldsymbol{L}(\mathbf{Z}_5) \subseteq \boldsymbol{L}$. Thus, to prove that Yn is the greatest p-conservative extension of Int, it suffices to show (i) that $\overline{\boldsymbol{L}}(\mathbf{Z}_5)$ is a p-conservative of extension of Int and (ii) that $\boldsymbol{L}(\mathbf{Z}_5)$ (and hence all it extensions) is not a p-conservative extension of Int.

(i) It is clear that any formula A and hence, any positive formula derivable in Int is derivable in KC. We need to show the converse: if a positive formula A is not derivable in Int, it is not derivable in KC.

Suppose that positive formula A is not derivable in Int. Then, it is refutable in a finite s.i. Heyting algebra $\mathbf{A}$. Consider algebra $\mathbf{0} \oplus \mathbf{A}$. Observe that operations $\wedge$, $\vee$, and $\rightarrow$ on elements of algebra $\mathbf{0} \oplus \mathbf{A}$ that are distinct from $\mathbf{0}$ coincide with the respective operations on $\mathbf{A}$. Hence, because A is a positive formula, the valuation refuting A in $\mathbf{A}$, refutes A in $\mathbf{0} \oplus \mathbf{A}$. Algebra $\mathbf{0} \oplus \mathbf{A}$ is Yankovean and, thus, it is a model for KC. Hence, formula A is not derivable in KC.

(ii) To prove lack of conservativity, let us observe that the following formula A,

$$((r \rightarrow (p \wedge q)) \wedge (((p \wedge q) \rightarrow r) \rightarrow r) \wedge ((p \rightarrow q) \rightarrow q) \wedge ((q \rightarrow p) \rightarrow p)) \rightarrow (p \vee q),$$

is valid in $\mathbf{Z}_5$ but refuted in $\mathbf{0} + \mathbf{Z}_5$ (cf. Fig. 2.4) by valuation $\nu(p) = \mathsf{a}$, $\nu(q) = \mathsf{b}$, $\nu(r) = \mathsf{c}$.

Remark 2.6 It is observed in Jankov (1968a) that formula A used in the proof of (ii) and formula

$$(((p \rightarrow q) \rightarrow q) \wedge ((q \rightarrow p) \rightarrow p)) \rightarrow (p \vee q)$$

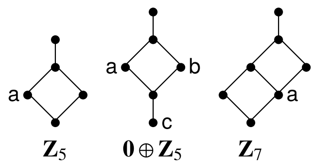


Fig. 2.4 Refuting algebras

are not equivalent in KC, but they are interderivable in KC. That is, even though the logic of KC is a p-conservative extension of Int, the relation of derivability in KC is stronger than that in IPC even for positive formulas.

Proof of (b). To prove (b) we need to show that (i) Yn admits the derivable elimination of negation and that (ii) if $\boldsymbol{L} \subsetneq$ Yn, then $\boldsymbol{L}$ does not admit the derivable elimination of negation. The latter follows immediately from Theorem 2.5 and Proposition 2.18: $\boldsymbol{L}$ and Yn are p-conservative extensions of Int and thus, they have the same positive fragments.

Let **A** and **B** be Heyting algebras. We say that **B** is a *p-subalgebra* of **A** when **B** is an implicative sublattice of **A**. Let us note that if **A** is a model of some logic $\boldsymbol{L}$ and **B** is a p-subalgebra of **A**, then **B** needs not to be a model of $\boldsymbol{L}$. For instance, consider algebras $\mathbf{Z}_5$ and $\mathbf{0} \oplus \mathbf{Z}_5$ from Fig. 2.4: $\mathbf{Z}_5$ is a p-subalgebra of $\mathbf{0} \oplus \mathbf{Z}_5$ (take elements $\mathsf{a}, \mathsf{b}, \mathsf{a} \wedge \mathsf{b}, \mathsf{a} \vee \mathsf{b}, \mathbf{1}$), formula $\neg p \vee \neg\neg p$ is valid in $\mathbf{0} \oplus \mathbf{Z}_5$, while it is not valid in $\mathbf{Z}_5$, because $\neg \mathsf{a} \vee \neg\neg \mathsf{a} < \mathbf{1}$.

Let us demonstrate that every extension $\boldsymbol{L}$ of Yn is positively axiomatizable relative to Yn.

For contradiction, assume that Yn $\subseteq \boldsymbol{L}$ and that $\boldsymbol{L}$ is not positively axiomatizable relative to KC. Then, the logic $\boldsymbol{L}'$ defined relative to Yn by all positive formulas from $\boldsymbol{L}$ is distinct from $\boldsymbol{L}$; that is, Yn $\subseteq \boldsymbol{L}' \subsetneq \boldsymbol{L}$. Let $A \in \boldsymbol{L} \setminus \boldsymbol{L}'$. Then, there is an s.i. Heyting algebra $\mathbf{A} \in \mathrm{Mod}(\boldsymbol{L}') \setminus \mathrm{Mod}(\boldsymbol{L})$ in which A is refuted by some valuation ν. We will construct a positive formula $X(\mathbf{A}, A, \nu)$ similar to a characteristic formula, and we will prove the following lemmas.

Lemma 2.3 *Formula* $X(\mathbf{A}, A, \nu)$ *is refuted in* $\mathbf{A}$.

Lemma 2.4 *Formula* $X(\mathbf{A}, A, \nu)$ *is valid in all algebras from* $\mathrm{Mod}(\boldsymbol{L})$.

Indeed, if $X(\mathbf{A}, A, \nu)$ is refuted in $\mathbf{A}$, and $\mathbf{A} \in \mathrm{Mod}(\boldsymbol{L}')$, then $X(\mathbf{A}, A, \nu) \notin \boldsymbol{L}'$.

On the other hand, if formula $X(\mathbf{A}, A, \nu)$ is valid in all algebras from $\mathrm{Mod}(\boldsymbol{L})$, then $X(\mathbf{A}) \in \boldsymbol{L}$. Recall that by definition, $\boldsymbol{L}$ and $\boldsymbol{L}'$ have the same positive formulas, and formula $X(\mathbf{A}, A, \nu)$ is positive. Hence, $X(\mathbf{A}) \in \boldsymbol{L}'$ and we have arrived at a contradiction and completed the proof.

Let us construct formula $X(\mathbf{A}, A, \nu)$.

Suppose $\mathbf{A}$ is a Heyting algebra, A is a formula, and valuation ν refutes A in $\mathbf{A}$. Then, we take A^ν to denote a formula obtained from A by substituting every variable q occurring in A with variable $p_{\nu(q)}$. It is clear that valuation $\nu' : p_{\nu(q)} \longrightarrow \nu(q)$ refutes A^ν. Let us also observe that because A^ν was obtained from A by substitution, A is refuted in every algebra in which A^ν is refuted.

If $\mathbf{A}$ is a Heyting algebra, and $\mathsf{B} \subseteq \mathbf{A}$ is a finite set of elements, by $\mathbf{A}^+[\mathsf{B}]$ we denote an implicative sublattice of $\mathbf{A}$ generated by elements B. Because B is finite, $\mathbf{A}^+[\mathsf{B}]$ contains the smallest element, namely $\bigwedge \mathsf{B}$ and therefore, $\mathbf{A}^+[\mathsf{B}]$ forms a Heyting algebra, which is denoted by $\mathbf{A}[\mathsf{B}]$. Note that $\mathbf{A}[\mathsf{B}]$ does not need to be a subalgebra of $\mathbf{A}$, because the bottom element of $\mathbf{A}^+[\mathsf{B}]$ may not coincide with the bottom element of $\mathbf{A}$.

Let $\mathbf{A}$ be a Heyting algebra and A be a formula refuted in $\mathbf{A}$ by valuation ν. Suppose that $\{A_1, \dots, A_n\}$ is a set of all subformulas of A and suppose that $\mathsf{A}_{(A,\nu)} := \{\mathbf{0}, \mathbf{1}\} \cup \{\nu(A_i) \mid i \in [1, n]\}$. Let us observe that $\mathsf{A}_{(A,\nu)}$ contains all elements of $\mathbf{A}$ needed to compute the value of $\nu(A)$. It is not hard to see that $\mathsf{A}_{(A,\nu)} = \mathsf{A}_{(A^\nu,\nu')}$. Clearly, $\mathsf{A}_{(A,\nu)}$ does not need to be closed under fundamental operations, but $\mathbf{A}[\mathsf{A}_{(A,\nu)}]$ is a Heyting algebra, and the value of $\nu(A)$, or the value of $\nu'(A^\nu)$ for that matter, can be computed in the very same way as in $\mathbf{A}$. To simplify notation, we write $\mathbf{A}[A, \nu]$ instead of $\mathbf{A}[\mathsf{A}_{(A,\nu)}]$. Thus, ν refutes A in $\mathbf{A}[A, \nu]$ as long as it refutes A in $\mathbf{A}$.

If $\circ \in \{\wedge, \vee, \to\}$, by $\mathsf{A}^\circ_{(A,\nu)}$ we denote a set of all ordered pairs of elements of $\mathsf{A}_{(A,\nu)}$ for which $\circ$ is defined:

$$\mathsf{A}^\circ_{(A,\nu)} = \{(\mathsf{a}, \mathsf{b}) \mid \mathsf{a}, \mathsf{b}, \mathsf{a} \circ \mathsf{b} \in \mathsf{A}_{(A,\nu)}\}.$$

Consider formulas

$$D^+(\mathbf{A}, A, \nu) := \Big(\bigwedge_{\circ\in\{\wedge,\vee\to\}} \bigwedge_{(\mathsf{a},\mathsf{b})\in\mathsf{A}^\circ_{(A,\nu)}} (p_\mathsf{a} \circ p_\mathsf{b} \leftrightarrow p_{\mathsf{a}\circ\mathsf{b}}) \Big),$$

and

$$X^+(\mathbf{A}, A, \nu) := D^+(\mathbf{A}, A, \nu) \to \bigvee_{\mathsf{a},\mathsf{b}\in\mathsf{A}_{(A,\nu)},\mathsf{a}\neq\mathsf{b}} (p_\mathsf{a} \leftrightarrow p_\mathsf{b}).$$

Proof of Lemma 2.3.

Proof We will show that if $\mathbf{A}$ is an s.i. algebra, then $X(\mathbf{A}, A, \nu)$ is refuted in $\mathbf{A}$ by valuation ν'. Indeed,

$$\nu'(p_\mathsf{a} \circ p_\mathsf{b}) = \nu'(p_\mathsf{a}) \circ \nu'(p_\mathsf{b}) = \mathsf{a} \circ \mathsf{b} = \nu'(p_{\mathsf{a}\circ\mathsf{b}})$$

for all $(\mathsf{a}, \mathsf{b}) \in \mathsf{A}^\circ_{(A,\nu)}$ and all $\circ \in \{\wedge\vee, \to\}$ and therefore, $\nu'(D^+(\mathbf{A}, A, \nu)) = \mathbf{1}$.

On the other hand, $\nu'(p_\mathsf{a}) = \mathsf{a} \neq \mathsf{b} = \nu'(p_\mathsf{b})$; that is, ν' refutes every disjunct on the right-hand side of $X(\mathbf{A}, A, \nu)$, and therefore, ν' refutes whole disjunction, because $\mathbf{A}$ is s.i. and disjunction of two elements that are distinct of $\mathbf{1}$ is distinct from $\mathbf{1}$. Thus, ν' refutes $X(\mathbf{A}, A, \nu)$.

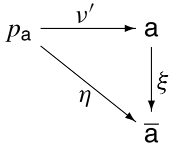
To prove Lemma 2.4, we will need the following property of $X(\mathbf{A}, A, \nu)$.

Proposition 2.21 *Let* $\mathbf{A}$ *be a Heyting algebra, and* ν *be a valuation refuting formula* A *in* $\mathbf{A}$. *Suppose that* $\mathbf{B}$ *is a Heyting algebra and* η *is a valuation refuting formula* $X^+(\mathbf{A}, A, \nu)$ *in* $\mathbf{B}$ *such that*

$$\eta(D^+(\mathbf{A}, A, \nu)) = \mathbf{1}_\mathbf{B}.$$

Then, η *refutes* A^ν *in* $\mathbf{B}[X^+, \eta]$ *and therefore,* A *is refuted in* $\mathbf{B}[X^+, \eta]$.

Proof Indeed, define a map $\xi : \mathsf{A}_{(A,\nu)} \longrightarrow \mathsf{B}_{(X^+,\eta)}$ by letting $\xi(\mathsf{a}) = \eta(p_a)$. Let $\bar{\mathsf{a}} = \eta(p_a)$ for every $\mathsf{a} \in \mathsf{A}_{(A,\nu)}$.



First, let us observe that $\eta(p_\mathbf{1}) = \mathbf{1_B}$. Indeed, by definition, $\mathbf{1} \in \mathbf{A}_{(A,\nu)}$, and $(\mathbf{1}, \mathbf{1}) \in \mathsf{A}^\circ_{(A,\nu)}$, because $\mathbf{1} \to \mathbf{1} = \mathbf{1} \in \mathsf{A}_{(A,\nu)}$. Hence, $(p_\mathbf{1} \to p_\mathbf{1}) \leftrightarrow p_\mathbf{1}$ is one of the conjuncts in $D^+(\mathbf{A}, A, \nu)$ and because $\eta(D^+(\mathbf{A}, A, \nu)) = \mathbf{1_B}$, we have $\eta((p_\mathbf{1} \to p_\mathbf{1}) \leftrightarrow p_\mathbf{1}) = \mathbf{1_B}$; that is, $\eta(p_\mathbf{1}) \to \eta(p_\mathbf{1}) = \eta(p_\mathbf{1})$ and $\eta(p_\mathbf{1}) = \mathbf{1_B}$.

Next, we observe that if $\mathsf{a}, \mathsf{b} \in \mathsf{A}_{(A,\nu)}$ and $\mathsf{a} \neq \mathsf{b}$, then $\bar{\mathsf{a}} \neq \bar{\mathsf{b}}$. Indeed, because η refutes $X^+(\mathbf{A}, A, \nu)$ and $\eta(D^+(\mathbf{A}, A, \nu)) = \mathbf{1_B}$,

$$\eta(\bigvee_{\mathsf{a},\mathsf{b} \in \mathsf{A}_{(A,\nu)}, \mathsf{a} \neq \mathsf{b}} (p_\mathsf{a} \leftrightarrow p_\mathsf{b})) \neq \mathbf{1_B},$$

and in particular, $\eta(p_\mathsf{a} \leftrightarrow p_\mathsf{b}) \neq \mathbf{1_B}$. Thus, $\eta(p_\mathsf{a}) \neq \eta(p_\mathsf{b})$; that is, $\bar{\mathsf{a}} \neq \bar{\mathsf{b}}$.

Lastly, we observe that if B, C and $B \circ C$ are subformulas of A and $\nu(B) = \mathsf{b}$ and $\nu(C) = \mathsf{c}$, then $\mathsf{b}, \mathsf{c}, \mathsf{b} \circ \mathsf{c} \in \mathsf{A}_{(A,\nu)}$. Moreover, $(\mathsf{b}, \mathsf{c}) \in \mathsf{A}^\circ_{(A,\nu)}$ and consequently, $(p_\mathsf{b} \circ p_\mathsf{c}) \leftrightarrow p_{(\mathsf{b} \circ \mathsf{c})}$ is one of the conjuncts of $D^+(\mathbf{A}, A, \nu)$ and by assumption,

$$\eta((p_\mathsf{b} \circ p_\mathsf{c}) \leftrightarrow p_{(\mathsf{b} \circ \mathsf{c})}) = \mathbf{1_B}.$$

Thus, $\eta(p_\mathsf{b}) \circ \eta(p_\mathsf{c}) = \eta(p_{\mathsf{b} \circ \mathsf{c}})$ and therefore, if $\nu(A) = \mathsf{a}$, then,

$$\eta(A^\nu) = \eta(p_\mathsf{a}) = \bar{\mathsf{a}}.$$

Recall that ν' refutes A^ν; that is $\nu'(A^\nu) \neq \mathbf{1}$ and hence, $\eta(A^\nu) \neq \eta(\mathbf{1}) = \mathbf{1_B}$.

Now, we can prove Lemma 2.4 and complete the proof of the theorem.

Proof of Lemma 2.4

Proof For contradiction, assume that formula $X(\mathbf{A}, A, \nu)$ is refuted in Heyting algebra $\mathbf{B} \in \mathrm{Mod}(\boldsymbol{L})$. By Corollary 2.2, we can assume that $\mathbf{B}$ is s.i. and that the refuting valuation η is such that

$$\eta(D^+(\mathbf{A}, A, \nu)) = \mathbf{1} \text{ and } \eta(\bigvee_{\mathsf{a},\mathsf{b} \in \mathsf{A}_{(A,\nu)}, \mathsf{a} \neq \mathsf{b}} (p_\mathsf{a} \leftrightarrow p_\mathsf{b})) \neq \mathbf{1_B}.$$

Thus, the condition of Proposition 2.21 is satisfied and hence, A is refuted in $\mathbf{B}[X^+, \eta]$. If we show that $\mathbf{B}[X^+, \eta] \in \mathrm{Mod}(\boldsymbol{L})$, we will arrive at a contradiction, because A was selected from $\boldsymbol{L} \setminus \boldsymbol{L}'$ and thus, A is valid in all models of $\boldsymbol{L}$.

Indeed, because $\mathbf{A} \in \mathrm{Mod}(\boldsymbol{L})$ and $\mathsf{KC} \subseteq \boldsymbol{L}$, $\mathbf{A}$ is a Yankovean algebra. Hence, for each element $\mathsf{a} \in \mathbf{A}$ that is distinct from $\mathbf{0}$, $\mathsf{a} \to \mathbf{0} = \mathbf{0}$ and $\mathbf{0} \to \mathsf{a} = \mathbf{1}$. Therefore, for each $\mathsf{a} \in \mathsf{A}_{(A,\nu)}$ that is distinct from $\mathbf{0}$, $\mathsf{a} \to \mathbf{0} = \mathbf{0}$ and $\mathbf{0} \to \mathsf{a} = \mathbf{1}$. Hence, $(p_\mathsf{a} \to p_\mathbf{0}) \leftrightarrow p_\mathbf{0}$ and $(p_\mathbf{0} \to p_\mathsf{a}) \leftrightarrow p_\mathbf{1}$ are conjuncts of $D^+(\mathbf{A}, A, \nu)$. By assumption, $\eta(D^+(\mathbf{A}, A, \nu)) = \mathbf{1}$ and subsequently, $\eta(p_\mathsf{a}) \to \eta(p_\mathbf{0}) = \eta(p_\mathbf{0})$ and $\eta(p_\mathbf{0}) \to \eta(p_\mathsf{a}) = \eta(p_\mathbf{1})$. The latter means that $\eta(p_\mathbf{0})$ is the smallest element of $\mathbf{B}[X^+, \eta]$. Let $\mathsf{c} = \eta(p_\mathbf{0})$. Then, for each distinct from c element $\mathsf{b} \in \mathbf{B}[X^+, \eta]$, $\mathsf{b} \wedge \mathsf{c} = \mathsf{c}$, $\mathsf{b} \vee \mathsf{c} = \mathsf{b}$, $\mathsf{b} \to \mathsf{c} = \mathsf{c}$ and $\mathsf{c} \to \mathsf{b} = \mathbf{1}$.

Recall that $\mathbf{B}$ is also a Yankovean algebra and therefore, for each element $\mathsf{b} \in \mathbf{B}$ that is distinct from $\mathbf{0}$, $\mathsf{b} \to \mathbf{0_B} = \mathbf{0_B}$ and $\mathbf{0_B} \to \mathsf{b} = \mathbf{1_B}$. Hence, the set of all elements of $\mathbf{B}[X^+, \eta]$ that are distinct from c together with $\mathbf{0_B}$ is closed under all fundamental operations and hence, it forms a subalgebra of $\mathbf{B}$. It is not hard to see that this subalgebra is isomorphic to $\mathbf{B}^+[X^+, \eta]$, and this entails that $\mathbf{B}^+[X^+, \eta]$ is isomorphic to a subalgebra of a model of $\boldsymbol{L}$ and therefore, $\mathbf{B}^+[X^+, \eta]$ is a model of $\mathrm{Mod}(\boldsymbol{L})$.

Remark 2.7 Formula $X(\mathbf{A}, A, \nu)$ is a characteristic formula of partial Heyting algebra. The reader can find more details about characteristic formulas of partial algebras in Tomaszewski (2003) and Citkin (2013).

2.7 Some Si-Calculi

Let us consider the following calculi.

(a) $\mathsf{CPC} = \mathsf{IPC} + (\neg\neg p \to p)$—the classical propositional calculus;

(b) $\mathsf{KC} = \mathsf{IPC} + (\neg p \vee \neg\neg p)$—the calculus of the weak law of excluded middle;

(c) $\mathsf{BD}_2 = \mathsf{IPC} + ((\neg\neg p \wedge (p \to q) \wedge ((q \to p) \to p)) \to q)$;

(d) $\mathsf{SmC} = \mathsf{IPC} + (\neg p \vee \neg\neg p) + ((\neg\neg p \wedge (p \to q) \wedge ((q \to p) \to p)) \to q)$.

Let us consider the algebras, whose Hasse diagrams are depicted in Fig. 2.2 and the following series of C-algebras defined inductively:

$$\mathbf{B}_0 = \mathbf{2}, \qquad \mathbf{B}_{k+1} = \mathbf{2}^k \oplus \mathbf{1};$$
$$\mathbf{J}_0 = \mathbf{2}, \qquad \mathbf{J}_{k+1} = \mathbf{J}^k \oplus \mathbf{1}.$$

Algebras $\mathbf{J}_k$ are referred to as Jaśkowski matrices. They were considered by S. Jaśkowski (cf. Jaśkowski 1975) and they form an adequate algebraic semantics for Int in the following sense:

$$\mathsf{Int} \vdash A \iff \mathbf{J}_k \models A \text{ for all } k > 0.$$

In Jankov (1963a), it was observed that algebras $\mathbf{J}_k^{(d)}$, $k > 1$, form an adequate semantics for KC:

$$\mathsf{KC} \vdash A \iff \mathbf{J}_k^{(d)} \models A \text{ for all } k > 0.$$

If C_1 and C_2 are two C-calculi, we write $C_1 = C_2$ to denote that C_1 and C_2 define the same logic. For instance, $\mathsf{Int} + (\neg\neg p \to p) = \mathsf{Int} + (p \vee \neg p)$, because both calculi define Cl.

As usual, if $\mathcal{A}$ is a class of C-algebras and C is a class of C-formulas, $\mathcal{A} \models C$ means that all formulas from C are valid in each algebra from $\mathcal{A}$, and $\mathcal{A} \not\models C$ denotes that at least one formula from C is invalid in some algebra from $\mathcal{A}$.

Theorem 2.8 (Jankov 1963a, Theorem 1) *Suppose that C is a set of formulas in the signature $\to, \wedge, \vee, \neg$. Then the following hold:*

(a) $\mathsf{IPC} + C = \boldsymbol{L}(\mathbf{2})$ *if and only if* $\mathbf{2} \models C$ *and* $\mathbf{3} \not\models C$*;*
(b) $\mathsf{IPC} + C = \boldsymbol{L}(\{\mathbf{J}_k^{(d)}, k \geq 0\})$ *if and only if* $\{\mathbf{J}_k^{(d)}, k \geq 0\} \models C$ *and* $\mathbf{Z}_5 \not\models C$*;*
(c) $\mathsf{IPC} + C = \boldsymbol{L}(\{\mathbf{B}_k, k \geq 0\})$ *if and only if* $\{\mathbf{B}_k, k \geq 0\} \models C$ *and* $\mathbf{4} \not\models C$*;*
(d) $\mathsf{IPC} + C = \boldsymbol{L}(\mathbf{3})$ *if and only if* $\mathbf{3} \models C$ *and* $\mathbf{4} \not\models C$ *and* $\mathbf{Z}_5 \not\models C$*.*

Proof In terms of splitting, we need to prove the following:

(a') $\boldsymbol{L}(\mathbf{2}) = \overline{\boldsymbol{L}}(\mathbf{3})$;
(b') $\boldsymbol{L}(\{\mathbf{J}_k^{(d)}, k \geq 0\}) = \overline{\boldsymbol{L}}(\mathbf{Z}_5)$;
(c') $\boldsymbol{L}(\{\mathbf{B}_k, k \geq 0\}) = \overline{\boldsymbol{L}}(\mathbf{4})$;
(d') $\boldsymbol{L}(\mathbf{3}) = \overline{\boldsymbol{L}}(\mathbf{4}, \mathbf{Z}_5))$.

(a') is trivial: the only s.i. Heyting algebra that does not contain $\mathbf{3}$ as a subalgebra is $\mathbf{2}$.

(b') was proven as Corollary 2.19.

(c') It is not hard to see that neither algebra $\mathbf{B}_k$ nor its homomorphic images or subalgebras contain a four-element chain subalgebra. On the other hand, if $\mathbf{B}$ is a finitely generated s.i. Heyting algebra such that $\mathbf{4}$ is not its subalgebra, then $\mathbf{B} \cong \mathbf{B}' \oplus \mathbf{1}$. Elements $\mathbf{0}, \mathbf{1}$, and the pretop element form a three-element chain algebra; hence, $\mathbf{B}'$ contains at most a two-element chain algebra and by (a'), $\mathbf{B}'$ is a Boolean algebra. Clearly, $\mathbf{B}'$ is a homomorphic image of $\mathbf{B}$ and hence, $\mathbf{B}'$ is finitely generated. Every finitely generated Boolean algebra is finite and therefore, $\mathbf{B}$ is isomorphic to one of the algebras $\mathbf{B}_k$.

(d') Let $\mathbf{A}$ be a finitely generated s.i. Heyting algebra that has no subalgebras isomorphic to $\mathbf{Z}_5$ and $\mathbf{4}$. Then by (b'), $\mathbf{A}$ is an s.i. Yankovean algebra and hence, $\mathbf{A} \cong \mathbf{0} \oplus \mathbf{A}' \oplus \mathbf{1}$. By (c'), $\mathbf{0} \oplus \mathbf{A}'$ is a Boolean algebra and therefore, $\mathbf{0} \oplus \mathbf{A}' \cong \mathbf{2}$. Thus, $\mathbf{A} \cong \mathbf{2} \oplus \mathbf{1} \cong \mathbf{3}$.

The above theorem can be rephrased as follows.

Theorem 2.9 (Jankov 1963a, Theorem 3) *Suppose that C is a set of formulas in the signature $\to, \wedge, \vee, \neg$. Then the following hold:*

(a) $\mathsf{IPC} + C = \mathsf{CPC}$ *if and only if* $\mathbf{2} \models C$ *and* $\mathbf{3} \not\models C$*;*
(b) $\mathsf{IPC} + C = \mathsf{KC}$ *if and only if* $\{\mathbf{J}_k^{(d)}, k \geq 0\} \models C$ *and* $\mathbf{Z}_5 \not\models C$*;*

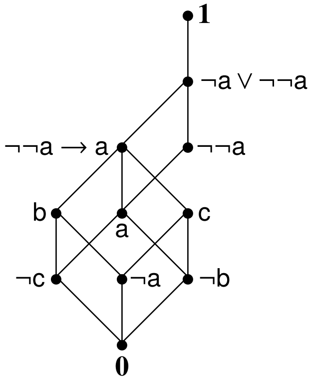


Fig. 2.5 Algebra refuting A

(c) $\mathsf{IPC} + C = \mathsf{BD}_2$ *if and only if* $\{\mathbf{B}_k, k \geq 0\} \models C$ *and* $\mathbf{4} \not\models C$*;*
(d) $\mathsf{IPC} + C = \mathsf{Sm}$ *if and only if* $\mathbf{3} \models C$ *and* $\mathbf{4} \not\models C$ *and* $\mathbf{Z}_5 \not\models C$*.*

Remark 2.8 It is noted in Jankov (1963a) that logic Sm can be defined by $\mathsf{IPC} + ((p \to q) \vee (q \to r) \vee (r \to s))$.

2.8 Realizable Formulas

In 1945, S. Kleene introduced a notion of realizability of intuitionistic formulas (cf. Kleene 1952). Formula A is said to be realizable when there is an algorithm that by each substitution of logical-arithmetical formulas gives a realization of the result. It was observed in Nelson (1947) that all formulas derivable in IPC are realizable; moreover, all formulas derivable in IPC from realizable formulas are realizable, while many classically valid formulas, $\neg\neg p \to p$, for instance, are not realizable. This observation gave the hope that the semantics of realizability is adequate for IPC. It turned out that this is not the case: in Rose (1953), it was proven that formula

$$C = ((\neg\neg A \to A) \to (\neg A \vee \neg\neg A)) \to (\neg A \vee \neg\neg A), \text{ where } A = (\neg p \vee \neg q)$$

is realizable not derivable in IPC. Indeed, formula C is refutable in the Heyting algebra whose Hasse diagram is depicted in Fig. 2.5 by substitution $\nu : p \mapsto \mathsf{b}$, $\nu : q \mapsto \mathsf{c}$, which entails $\nu : A \mapsto \mathsf{a}$.

In Jankov (1963b), Yankov constructed the following sequences of formulas: for each $n \geq 3$ and $i \in [1, n]$, let $\pi_n^i := \neg p_1 \wedge \cdots \wedge \neg p_{i-1} \wedge \neg p_{i+1} \wedge \cdots \wedge \neg p_n$ and

$$A_n := \bigwedge_{1 \leq k < m \leq n} \neg(p_k \wedge p_m) \wedge \bigwedge_{i=1}^{n-1} (\pi_{n-1}^i \to (p_i \vee p_n)) \to (p_n \vee \neg p_n)$$

and

$$B_n := \bigwedge_{1\leq i<j\leq n} \neg(p_i \wedge p_j) \wedge \bigwedge_{i=1}^{n-1}(\pi^i_{n-1} \to (p_i \vee p_j)) \to \bigvee_{i=1}^{n} p_i.$$

In addition, let

$$\rho := ((\neg\neg p \to (p \vee \neg p)) \wedge ((\neg\neg q \to q) \to (q \vee \neg q)) \wedge \neg(p \wedge q)) \to (p \vee \neg q).$$

Theorem 2.10 (Jankov 1963b, Theorem 1) *The following hold:*

(a) formulas A_3 and ρ are realizable and cannot be derived in IPC *from each other; thus, they are not derivable in* IPC*;*
(b) in IPC*, $A_3 \vdash C$ and $C \nvdash A_3$;*
(c) for any $n \geq 3$, formulas A_n and B_n are not derivable in IPC*; nevertheless, $A_3 \vdash A_n$ and $A_n \vdash B_n$ and hence, formulas A_n and B_n are realizable.*

Theorem 2.11 (Jankov 1963b, Theorem 2) *Every realizable formula is valid in algebra $\mathbf{Z}_7$ whose Hasse diagram is depicted in Fig. 2.4.*

Let us observe that formula $C' := ((\neg\neg p \to p) \to (\neg p \vee \neg\neg p)) \to (\neg p \vee \neg\neg p)$ (the skeleton of the Rose formula) is refuted in algebra $\mathbf{Z}_7$ by valuation $\nu : p \mapsto \mathsf{a}$. Hence, Theorem 2.11 entails that the skeleton $C' := ((\neg\neg p \to p) \to (\neg p \vee \neg\neg p)) \to (\neg p \vee \neg\neg p)$ of the Rose formula is not realizable.

On the other hand, formula C' is interderivable in Int with the characteristic formula $X_{\mathbf{Z}_7}$ of algebra $\mathbf{Z}_7$. Hence, if C' is not realizable, all realizable formulas are valid in $\mathbf{Z}_7$. Indeed, assume for contradiction that A is a realizable formula and is invalid in $\mathbf{Z}_7$. Then, by Theorem 2.2, in IPC, $A \vdash C'$ and therefore, C' should be realizable.

More information on the realizability of propositional formulas can be found in Plisko (2009).

2.9 Some Properties of Positive Logic

If A and B are positive formulas, let $A \leq B \overset{\text{def}}{\iff} \vdash A \to B$ in PPC. A set of formulas is *independent* if any two distinct formulas of this set are incomparable relative to $\leq$. In Jankov (1968d), Yankov constructed three infinite sequences of positive formulas on two variables: (a) independent, (b) strongly descending, and (c) strongly ascending. For the duration of this section, $\vdash$ means derivability in PPC (unless otherwise indicated).

2.9.1 Infinite Sequence of Independent Formulas

Consider the following sequence of positive formulas (cf. Jankov 1968d):

$$A_1 := p, \quad B_1 := q, \quad A_{k+1} := B_k \vee (B_k \to A_k), \quad B_{k+1} := A_k \vee (A_k \to B_k). \tag{2.12}$$

Let

$$C_k := (((A_k \to B_k) \to B_k) \wedge ((B_k \to A_k) \to A_k)) \to (A_k \vee B_k). \tag{2.13}$$

In the proofs, we use algebras $\mathbf{A}_i$, the Hasse diagrams of which are depicted in Fig. 2.6, and we use valuation

$$\nu : p \mapsto \mathsf{a}_1 \quad \nu : q \mapsto \mathsf{b}_1. \tag{2.14}$$

Let us observe that for all $1 \leq k \leq i$,

$$\nu(A_k) = \mathsf{a}_k, \quad \nu(B_k) = \mathsf{b}_k, \ \text{ and } \ \nu(C_i) = \mathsf{c}_i.$$

Proposition 2.22 *Formulas $C_k, k > 0$, are independent in* PPC*; that is, for any $i \neq j$, $C_i \nvdash C_j$ and $C_j \nvdash C_i$.*

Proof First, let us observe that $\nvdash C_j \to C_i$ for any $i > j$, because valuation ν defined by (2.14) refutes formula $C_j \to C_i$.

Next, let us show that $\nvdash C_j \to C_i$ for any $i < j$. To this end, we will show that $\vdash (C_j \to C_i) \leftrightarrow C_i$ and consequently, $\nvdash (C_j \to C_i)$, because $\nvdash C_i$: it is refuted in $\mathbf{A}_i$ by valuation ν.

Let us prove $\vdash (C_j \to C_i) \leftrightarrow C_i$. By definition,

$$C_j \to C_i = C_j \to ((((A_i \to B_i) \to B_i) \wedge ((B_i \to A_i) \to A_i)) \to (A_i \vee B_i)),$$

and hence,

$$\vdash (C_j \to C_i) \leftrightarrow ((C_j \wedge ((A_i \to B_i) \to B_i) \wedge ((B_i \to A_i) \to A_i)) \to (A_i \vee B_i)). \tag{2.15}$$

Thus, by showing that

$$\vdash (((A_i \to B_i) \to B_i) \wedge ((B_i \to A_i) \to A_i)) \to C_j, \tag{2.16}$$

we will prove that (2.15) yields

$$\vdash (C_j \to C_i) \leftrightarrow ((((A_i \to B_i) \to B_i) \wedge ((B_i \to A_i) \to A_i)) \to (A_i \vee B_i)),$$

that is, that $\vdash (C_j \to C_i) \leftrightarrow C_i$.

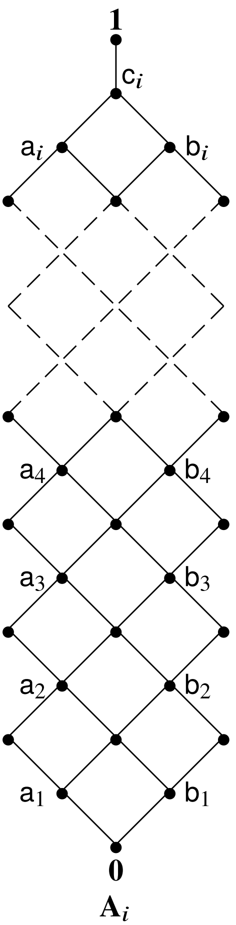


Fig. 2.6 Refuting algebra

To prove (2.16), we consider two cases: (a) $j = i + 1$ and (b) $j \geq i + 2$.
Case (a). Recall that $B_i \vdash A_i \to B_i$ and hence,

$$B_i,\ (A_i \to B_i) \to B_i,\ (B_i \to A_i) \to A_i \ \vdash\ (A_i \vee (A_i \to B_i)). \tag{2.17}$$

In addition, $B_i \to A_i,\ (B_i \to A_i) \to A_i \ \vdash\ A_i$ and hence,

$$B_i \to A_i,\ (A_i \to B_i) \to B_i,\ (B_i \to A_i) \to A_i \ \vdash\ (A_i \vee (A_i \to B_i)). \tag{2.18}$$

From (2.17) and (2.18),

$$B_i \vee (B_i \to A_i),\ (A_i \to B_i) \to B_i,\ (B_i \to A_i) \to A_i \ \vdash\ (A_i \vee (A_i \to B_i)), \tag{2.19}$$

and by the Deduction Theorem,

$$(A_i \to B_i) \to B_i,\ (B_i \to A_i) \to A_i \ \vdash\ (B_i \vee (B_i \to A_i)) \to (A_i \vee (A_i \to B_i)); \tag{2.20}$$

that is,

$$(A_i \to B_i) \to B_i,\ (B_i \to A_i) \to A_i \ \vdash\ A_{i+1} \to B_{i+1}. \tag{2.21}$$

Immediately from (2.21),

$$(A_i \to B_i) \to B_i,\ (B_i \to A_i) \to A_i \ \vdash\ ((A_{i+1} \to B_{i+1}) \to B_{i+1}) \to B_{i+1} \tag{2.22}$$

and consequently,

$$(A_i \to B_i) \to B_i,\ (B_i \to A_i) \to A_i \ \vdash\ ((A_{i+1} \to B_{i+1}) \to B_{i+1}) \to (A_{i+1} \vee B_{i+1}). \tag{2.23}$$

Hence,

$$\begin{aligned}&(((A_i \to B_i) \to B_i) \wedge ((B_i \to A_i) \to A_i)) \ \vdash \\ &((((A_{i+1} \to B_{i+1}) \to B_{i+1}) \wedge ((B_{i+1} \to A_{i+1}) \to A_{i+1})) \to (A_{i+1} \vee B_{i+1}));\end{aligned} \tag{2.24}$$

that is,

$$(((A_i \to B_i) \to B_i) \wedge ((B_i \to A_i) \to A_i)) \ \vdash C_{i+1}. \tag{2.25}$$

Case (b). From (2.21),

$$(((A_i \to B_i) \to B_i) \wedge ((B_i \to A_i) \to A_i)) \vdash A_{i+1} \vee (A_{i+1} \to B_{i+1});$$

that is,

$$(((A_i \to B_i) \to B_i) \wedge ((B_i \to A_i) \to A_i)) \vdash B_{i+2}. \tag{2.26}$$

Using the definition of formulas A_j and B_j, by simple induction one can show that for any $j \geq i + 2$,

$$(((A_i \to B_i) \to B_i) \wedge ((B_i \to A_i) \to A_i)) \vdash B_j \tag{2.27}$$

and subsequently,

$$\begin{aligned}&(((A_i \to B_i) \to B_i) \wedge ((B_i \to A_i) \to A_i)) \ \vdash \\ &\qquad (((A_j \to B_j) \to B_j) \wedge ((B_j \to A_j) \to A_j)) \to (A_j \vee B_j).\end{aligned}$$

Hence, by the definition of formula C_j,

$$(((A_i \to B_i) \to B_i) \wedge ((B_i \to A_i) \to A_i)) \vdash C_j.$$

2.9.2 Strongly Descending Infinite Sequence of Formulas

Consider the following sequence of formulas: for each $k > 0$,

$$D_k := \bigwedge_{i \le k} C_i.$$

Let us prove that formulas D_i form a strongly descending (relative to $\le$) sequence; that is, for any $0 < j < i$, $\vdash D_i \to D_j$, while $\nvdash D_j \to D_i$.

Proposition 2.23 *Formulas D_k, $k = 1, 2, \ldots$ form a strongly descending sequence.*

Proof Let $0 < j < i$. Then $\vdash D_i \to D_j$ trivially follows from the definition of D_k, and we only need to show that $\nvdash D_j \to D_i$.

Indeed, by the definition of formula D_j,

$$\vdash (D_j \to D_i) \leftrightarrow \bigwedge_{k \le i} (D_j \to C_k),$$

and we will demonstrate that $\nvdash D_j \to C_i$ by showing that formula $D_i \to C_j$ is refuted in algebra $\mathbf{A}_i$ whose diagram is depicted in Fig. 2.6.

Let us consider valuation $\nu : p \mapsto \mathsf{a}$ and $\nu : q \mapsto \mathsf{b}$. Then, for all $j < i$, $\nu(C_j) = \mathbf{1}$ and hence, $\nu(\delta_j) = \mathbf{1}$, while $\nu(C_i) = \mathsf{c}_i < \mathbf{1}$ and hence, valuation ν refutes formula $D_j \to C_i$.

2.9.3 Strongly Ascending Infinite Sequence of Formulas

To construct a strongly ascending sequence of formulas of positive logic, one can use an observation from Wajsberg (1931) that a formula A is derivable in IPC if and only if a formula A^+ obtained from A by replacing any subformula of the form $\neg B$ with formula $B \to (p_1 \wedge \cdots \wedge p_n \wedge p_{n+1})$, where $p_1, \ldots, p_n$ is a list of all variables occurring in A, is derivable in PPC. Thus, if one takes any sequence $A_1, A_2, A_3, \ldots$ that is strongly ascending in IPC , the sequence $A_1^+, A_2^+, A_3^+, \ldots$ is strongly ascending in PPC. In particular, one can take sequence of formulas that are a strongly ascending in IPC on one variable constructed in Nishimura (1960) and obtain a desired sequence of formulas that is strongly ascending in PPC on two variables.

A proof that $\mathsf{Int} \vdash A$ if and only if $\mathsf{PPC} \vdash A^+$ can be done by simple induction. It appears that Yankov was not familiar with Wajsberg (1977) and his proof in Jankov (1968d) uses the same argument as the proof from Wajsberg (1931).

Let us also observe that the sequence $A_1, A_2, \ldots$ defined by (2.12) is strongly ascending. Indeed, it is clear that $\vdash A_k \to (B_k \vee (B_k \to A_k))$; that is , $\vdash A_k \to A_{k+1}$. On the other hand, $\nvdash A_{k+1} \to A_k$: the valuation ν refutes this formula.

By the Separation Theorem, all three sequences remain, respectively, independent, strongly ascending and strongly descending in IPC. Moreover, because IPC and KC have the same sets of derivable positive formulas, these three sequences retain their properties. And, if we replace q with $\neg(p \to p)$, we obtain three sequences of formulas that are independent, strongly descending, and strongly ascending in MPC.

2.10 Conclusions

In conclusion, let us point out that Yankov's results in intermediate logics not only changed the views on the lattice of intermediate logics but also instigated further research in this area. In 1971, in Kuznetsov (1971), it was observed that for any intermediate logic $\boldsymbol{L}$ distinct from Int, the segment [Int, $\boldsymbol{L}$] contains a continuum of logics. In the same year, using notion of a pre-true formula, which is a generalization of the notion of characteristic formula, Kuznetsov and Gerčiu presented a finitely axiomatizable intermediate logic without the f.m.p. (Kuznetsov and Gerčiu 1970). Using ideas from Jankov (1968b), Wroński proved that there are continuum many intermediate logics enjoying the disjunction property, among which are the logics lacking the f.m.p. (cf. Wroński 1973).

In Fine (1974), Fine introduced—for modal logics—formulas similar to Yankov's formulas, and he constructed a strongly ascending chain of logics extending S4.

In his Ph.D. Blok (1976), Blok linked the characteristic formulas with splitting, and studied the lattice of varieties of interior algebras. This line of research was continued by Routenberg in Rautenberg (1977, 1980), and his disciples (cf. Wolter 1993; Kracht 1999). Ever since, the splitting technique pioneered by Yankov is one of the main tools in the research of different classes on logics.

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