Chapter 5
The Kinematics of Cornering

Cars have to negotiate corners. Everybody knows that. But not all cars do that the same way [4]. This is particularly evident in race cars, where the ability to negotiate a corner is a crucial aspect to minimize lap time.

In this chapter we will exploit the kinematics of a vehicle while taking a corner. At first sight, taking a corner looks quite a trivial task. But designing a vehicle that does it properly is one of the main challenges faced by a vehicle engineer [2]. Therefore, there is the need to investigate what really happens during the cornering process. It will be shown that some very significant kinematical quantities must follow precise patterns for the car to get around corners in a way that makes the driver happy. In some sense, the geometric features of the trajectory must adhere to some pretty neat criteria.

Before digging into the somehow mysterious kinematics of cornering, we will recall some kinematical concepts. Strangely enough, it appears that they have never been employed before in vehicle dynamics, although all of them date back to Euler or so.

5.1 Planar Kinematics of a Rigid Body

As discussed at the beginning of Chap. 3, in many cases a vehicle can be seen as a rigid body in planar motion. Basically, we need a flat road and small roll angles. The congruence (kinematic) equations for this case were given in Sect. 3.2. We will extensively use the symbols defined therein.

Here we recall some fundamental concepts of planar kinematics of a rigid body [1, 3, 6]. They will turn out to be very useful to understand how a car takes a corner.

The original version of this chapter was revised. Legends of figures 5.21 and 5.22 were updated. The correction to this chapter is available at https://doi.org/10.1007/978-3-031-06461-6_13

M. Guiggiani, *The Science of Vehicle Dynamics*,
https://doi.org/10.1007/978-3-031-06461-6_5

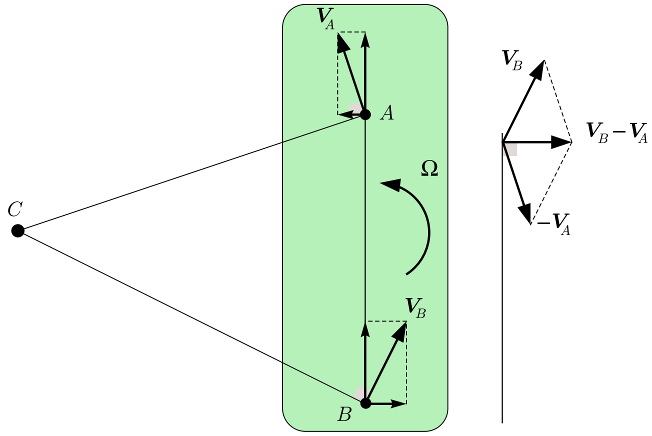

Fig. 5.1 Relationship between the velocities of two points of the same rigid body in planar motion

5.1.1 Velocity Field and Velocity Center

In a rigid body, by definition, the distance between any two points is constant. Accordingly, taken two such points, say A and B, their velocities must have the same component along the direction AB, as shown in Fig. 5.1. More precisely, the two velocities are related by the following equation

$$\mathbf{V}_B = \mathbf{V}_A + \boldsymbol{\Omega} \times AB = \mathbf{V}_A + \mathbf{V}_{BA} \tag{5.1}$$

where $\boldsymbol{\Omega}$ is the angular velocity. This is the fundamental equation of the kinematics of rigid bodies, planar or three-dimensional. It had been already given in (2.1) and (3.3).

It is worth noting that $\boldsymbol{\Omega}$ is the same for all points. It is a kinematic feature of the rigid body as a whole.

Another way to state the fundamental equation (5.1) is saying that the relative velocity $\mathbf{V}_{BA} = \mathbf{V}_B - \mathbf{V}_A$ is orthogonal to the segment AB and proportional to the length of AB, that is $|\mathbf{V}_{BA}| = |\boldsymbol{\Omega}||AB|$ (Fig. 5.1).

It can be shown [1, 3, 6] that in case of planar motion, that is $\boldsymbol{\Omega} = r\,\mathbf{k}$, and with $r \neq 0$, at any instant there is one point C of the (extended) rigid body that has zero velocity. Therefore, applying (5.1) to A and C, and then to B and C we have

$$\mathbf{V}_A = r\,\mathbf{k} \times CA \qquad \text{and} \qquad \mathbf{V}_B = r\,\mathbf{k} \times CB \tag{5.2}$$

as shown in Fig. 5.1.

Several different names are commonly in use to refer to point C:

- instantaneous center of velocity;
- velocity center;

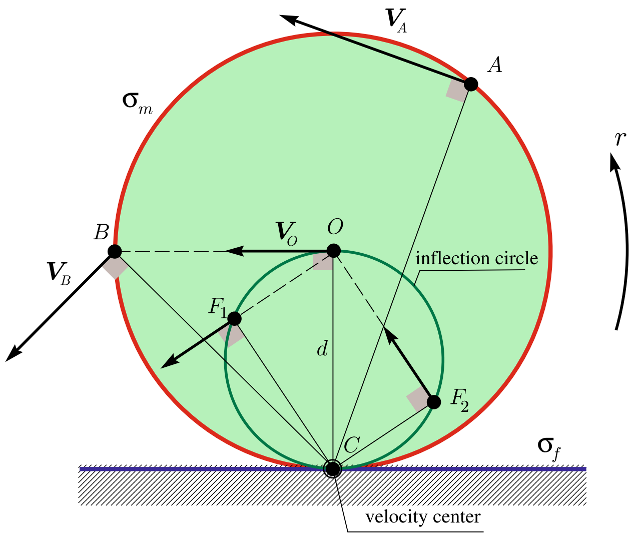

Fig. 5.2 Velocity field of a rigid wheel σ_m rolling on a flat road σ_f

- instantaneous center of zero velocity;
- instantaneous center of rotation.

5.1.2 Fixed and Moving Centrodes

As the body changes its position, the point of the rigid body with zero velocity changes as well. If we follow the positions of this sequence of points we obtain a curve σ_f in the fixed plane, called the *fixed centrode* or space centrode, and another curve σ_m on the moving plane, called the *moving centrode* or the body centrode. It can be shown that the moving centrode rolls without slipping on the fixed centrode, the point of rolling contact being C.

A simple example should help clarify the matter. Just consider a rigid circle rolling without slipping on a straight line, as shown in Fig. 5.2. It is exactly like a rigid wheel rolling on a flat road. The two centrodes are the circle σ_m and the straight line σ_f. Point C as a point of the circle has zero velocity. However, the geometric point[1] $\hat{C}$ that at each instant coincides with C moves on the road with a speed

$$V_{\hat{C}} = rd \tag{5.3}$$

[1] By geometric point we mean a point not belonging to the rigid body.

where d is the diameter of the inflection circle (already defined in Sect. 3.2.10 and to be discussed in detail in Sect. 5.1.4).

The velocity field is like a pure rotation around C (Fig. 5.1). But the acceleration field is not! In fact, the wheel is travelling on the road, not turning around C.

5.1.3 Acceleration Field and Acceleration Center

The counterpart of (5.1) for the accelerations of points of a rigid body is

$$\mathbf{a}_B = \mathbf{a}_A + \dot{\boldsymbol{\Omega}} \times AB + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times AB) = \mathbf{a}_A + \mathbf{a}_{BA} \tag{5.4}$$

In case of planar motion it simplifies into (Fig. 5.3)

$$\mathbf{a}_B = \mathbf{a}_A + \dot{r}\,\mathbf{k} \times AB - r^2 AB \tag{5.5}$$

The relative acceleration $\mathbf{a}_{BA} = \mathbf{a}_B - \mathbf{a}_A$ between any two points is proportional to the length $|AB|$ and forms an angle ξ with the segment AB (Fig. 5.3)

$$\tan \xi = \frac{\dot{r}}{r^2} \tag{5.6}$$

As discussed in Sect. 3.2.9, it can be shown that in case of planar motion, that is $\boldsymbol{\Omega} = r\,\mathbf{k}$, and with $r \neq 0$, at any instant there is one point K of the (extended) rigid body that has zero acceleration. In general, $K \neq C$. The absolute acceleration of any point A forms an angle ξ with the segment KA, as shown in Fig. 5.3. Therefore, the acceleration field is like a pure rotation around K.

Several different names are commonly in use to refer to point K:

- instantaneous center of acceleration;
- acceleration center;
- instantaneous center of zero acceleration.

The velocity and acceleration fields are superimposed in Fig. 5.4.

5.1.4 Inflection Circle and Radii of Curvature

Let us consider again, as an example, the rigid wheel rolling on a flat road. For the moment let us also assume that it rolls at constant speed. The center O of the wheel has zero acceleration, and hence it is the acceleration center K, as shown in Fig. 5.5. The acceleration field is centripetal towards $O = K$. It is worth noting that the acceleration of C is not zero

$$\mathbf{a}_C = \mathbf{n} r^2 d \tag{5.7}$$

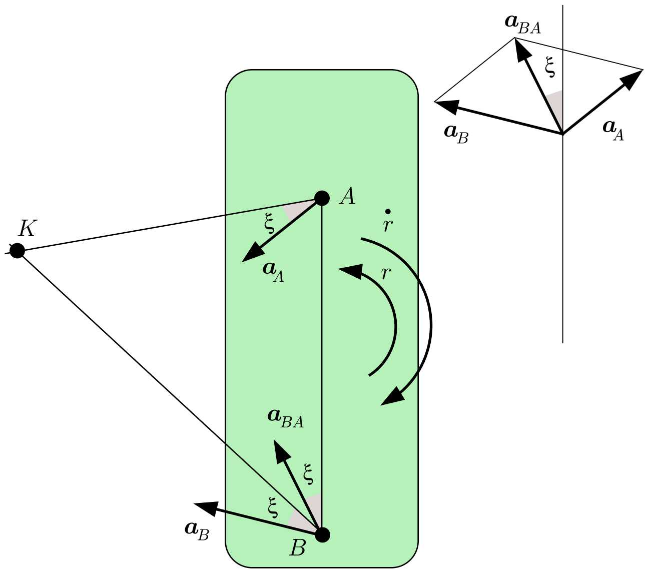

Fig. 5.3 Relationship between the acceleration of two points of the same rigid body in planar motion

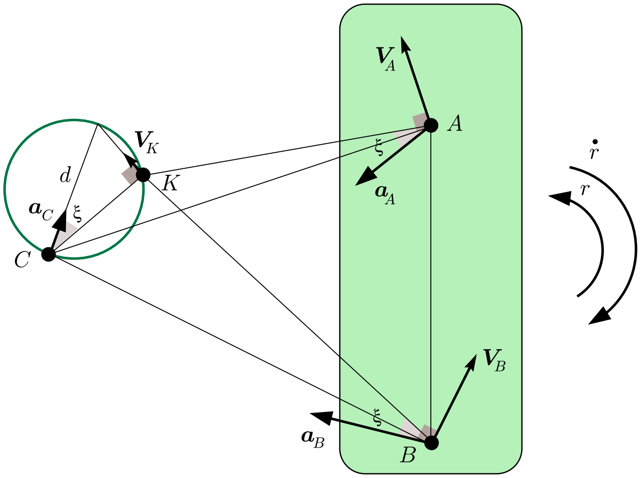

Fig. 5.4 Velocity center, acceleration center, and inflection circle

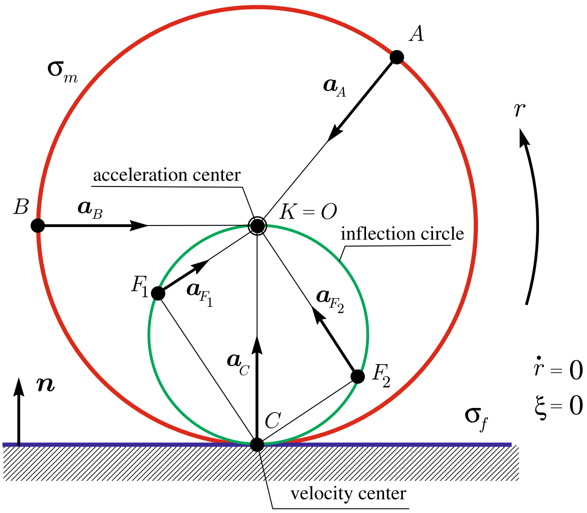

Fig. 5.5 Acceleration field of a rigid wheel rolling at *constant speed* on a flat road

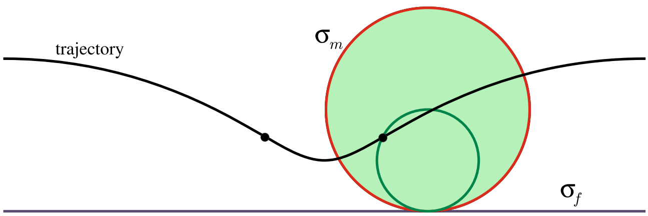

Fig. 5.6 Trajectory with two inflection points

where d is the diameter of the inflection circle (already mentioned in Sect. 3.2.10).

Comparing Figs. 5.2 and 5.5, we see that at a given instant of time there are points, like F_1 and F_2, whose velocities and accelerations have the same direction. They all belong to the inflection circle [5, Sect. 4.5]. Even if we apply an angular acceleration $\dot{r}$, as in Fig. 5.7, the points on the inflection circle still have collinear velocity and acceleration. The points of the rigid body on the inflection circle, as the name implies, have a trajectory with an *inflection point*, that is a point with zero curvature, as shown in Fig. 5.6.

Point C has a nice property: its acceleration is not affected by $\dot{r}$. In other words, Eq. (5.7) holds true even if $\dot{r} \neq 0$. Therefore, it is possible to obtain the diameter d of the inflection circle from the knowledge of $\mathbf{a}_C$ and r.

The inflection circle turns out to be very useful to evaluate the *radius of curvature* of the trajectory of *any* point of the rigid body. The rule is very simple, and it is exemplified in Fig. 5.8. Let us take, for instance, point A. The center of curvature E_A of its trajectory must fulfill the following relationship

$$|AC|^2 = |AE_A||AF_A| \qquad \text{or, more compactly} \qquad a^2 = ef \tag{5.8}$$

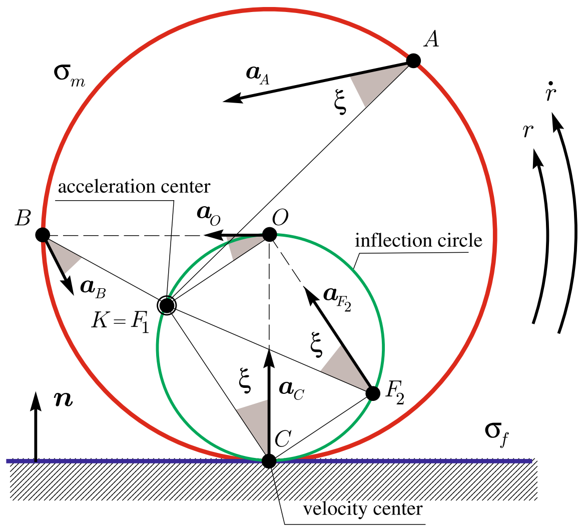

Fig. 5.7 Acceleration field of a rigid wheel rolling at *non-constant speed* on a flat road

where E_A and F_A are always on the same side with respect to A (this is why point A is always first in the three terms in (5.8)). As a consequence, points E_A and F_A are always on opposite sides with respect to C. The distance $R_A = |AE_A|$ is the radius of curvature of the trajectory of A.

Exactly the same rules apply to any other point, like point B in Fig. 5.8.

Quite interestingly, we can obtain the following formula for the centripetal (normal) component of the acceleration of A

$$a_A^n = \frac{V_A^2}{|AE_A|} = \frac{(ra)^2}{e} = r^2 f = r^2 |AF_A| \tag{5.9}$$

The same kind of formula applies to any other point of the rigid body.

5.2 The Kinematics of a Turning Vehicle

Driving a vehicle to make a turn amounts, roughly speaking, to forcing it to follow a path with *variable radius of curvature*. The traditional approach looks only at the kinematics for a given instant of time, as shown in Fig. 5.9. This is a good starting point, but not the whole story. For instance, from Fig. 5.9 we cannot know the radius of curvature of the trajectory of G (which, of course, is not equal to CG, in general). But let us make the reasoning more precise.

A vehicle has infinitely many points and hence infinitely many trajectories. However, as a rigid body, these trajectories are not independent of each other. It suffices to look at the trajectory (path) of two points. It is perhaps advisable to select the

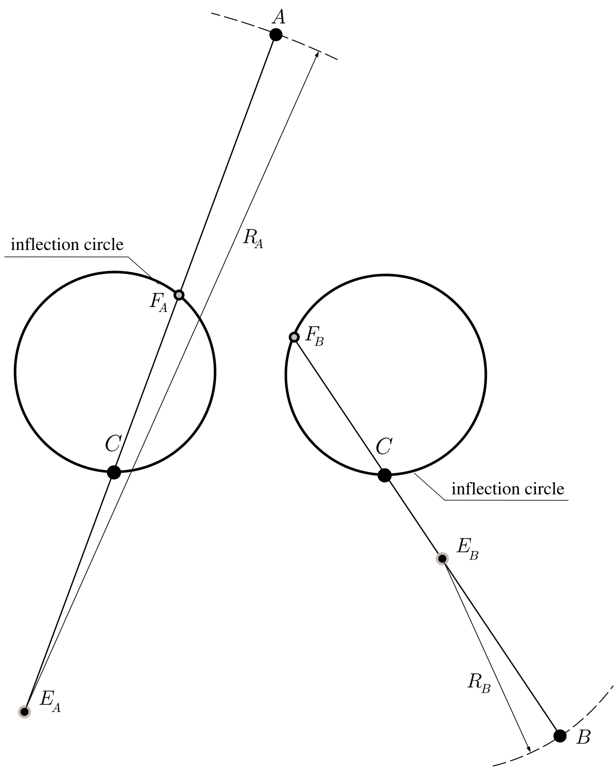

Fig. 5.8 How the inflection circle relates to the centers of curvature of the trajectories of the points of a rigid body

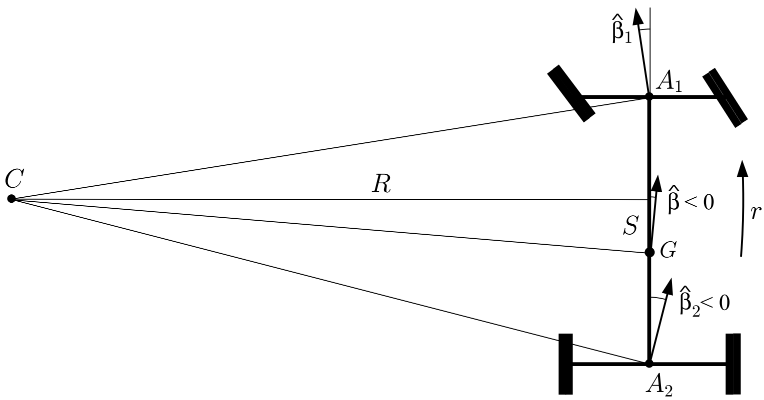

Fig. 5.9 Definition of front slip angle $\hat{\beta}_1$ and rear slip angle $\hat{\beta}_2$ for a turning vehicle

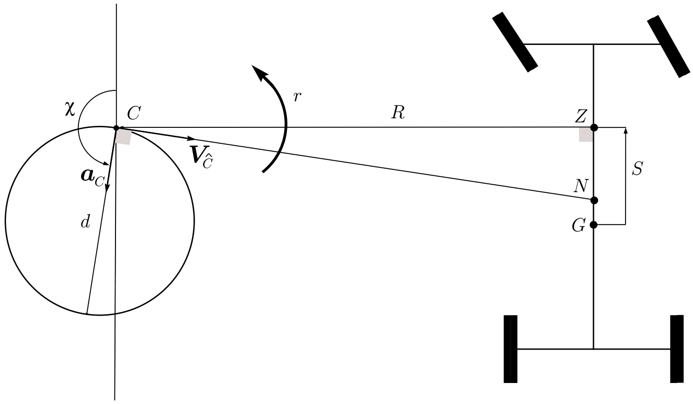

Fig. 5.10 Inflection circle and definition of some relevant quantities

midpoint A_1 of the front axle and the midpoint A_2 of the rear axle (Fig. 5.9). It is not mandatory at all, but maybe convenient.

Looking at the trajectories also implies monitoring the radii of curvature and how they relate to each other.

To monitor whether a vehicle is performing well, or not so well, we can consider also the *fixed* and *moving centrodes*, along with the *inflection circle* (Fig. 5.10). Indeed, we should have clear in mind that the position of the velocity center C changes continuously in time, thus tracing the two centrodes. Therefore, the two centrodes "contain" all the geometric features of the kinematics of the turning vehicle.

However, the centrodes of a vehicle are "built" by the vehicle itself, under the driver actions. In some sense, a vehicle can be seen as a *centrode builder*. It is not like in Fig. 5.2, or in single-degree-of-freedom planar linkages, where the centrodes are completely determined and cannot be modified.

5.2.1 Moving and Fixed Centrodes of a Turning Vehicle

By definition, the centrodes are generated by the successive positions of the velocity center C.

The *moving centrode* σ_m is given by the successive positions of C in the body-fixed reference system $\mathsf{S} = (x, y, z; G)$, that is with respect to the vehicle. As already obtained in (3.11), the position of C with respect to the vehicle is given by (Fig. 5.9)

$$\mathbf{D} = GC = S\,\mathbf{i} + R\,\mathbf{j} \tag{5.10}$$

where, as usual, $S = -v/r$ and $R = u/r$. More explicitly, the parametric equations of the moving centrode in S are

$$
\begin{aligned}
x_m(\hat{t}) &= S(\hat{t}) \\
y_m(\hat{t}) &= R(\hat{t})
\end{aligned}
\tag{5.11}
$$

where we use $\hat{t}$, instead of t, to remark that it is a parameter (like in (3.9)).

The parametric equations $(x_f(\hat{t}), y_f(\hat{t}))$ of the *fixed centrode* σ_f in the ground-fixed reference system S_0 can be obtained from the knowledge of the absolute coordinates of G, given in (3.9), and of the yaw angle (3.8)

$$
\begin{aligned}
x_f(\hat{t}) &= x_0^G(\hat{t}) + S(\hat{t}) \cos\psi(\hat{t}) - R(\hat{t}) \sin\psi(\hat{t}) \\
y_f(\hat{t}) &= y_0^G(\hat{t}) + S(\hat{t}) \sin\psi(\hat{t}) + R(\hat{t}) \cos\psi(\hat{t})
\end{aligned}
\tag{5.12}
$$

By definition, the vehicle belongs precisely to the same rigid plane of the moving centrode. They move together. Therefore, the parametric equations of the *moving centrode*, at time t, in the ground-fixed reference system are

$$
\begin{aligned}
x_m^f(t, \hat{t}) &= x_0^G(t) + S(\hat{t}) \cos\psi(t) - R(\hat{t}) \sin\psi(t) \\
y_m^f(t, \hat{t}) &= y_0^G(t) + S(\hat{t}) \sin\psi(t) + R(\hat{t}) \cos\psi(t)
\end{aligned}
\tag{5.13}
$$

Again, the parameter to draw the moving centrode is $\hat{t}$, while t sets the instant of time.

The typical shape of the fixed and moving centrodes of a vehicle making a turn are shown in Figs. 5.11 and 5.12. We see that the moving centrode σ_m is pretty much a straight line, while the fixed centrode σ_f is made of two distinct parts, as is the kinematics of turning: entering the curve and exiting the curve. The velocity center C is the point of rolling contact of the two centrodes.

Actually, the centrodes shown in Figs. 5.11 and 5.12 are typical of a vehicle making a curve the good way. The centrodes changes abruptly if the vehicle does not make the curve properly. This may happen, e.g., if the speed is too high. An example of "bad" centrodes, and hence of bad performance, is shown in Fig. 5.13. We see that the centrodes for the exiting phase (Fig. 5.13c) are totally different from those in Fig. 5.12. The vehicle spins out.

Quite interestingly, as shown in Fig. 5.13b, the two centrodes start having a bad shape although the vehicle still has an apparent good behavior. Therefore, the two centrodes could be used as a warning of handling misbehavior. They depart from the proper shape a little before the vehicle shows unwanted behavior.

To confirm that this is real stuff, we show in Fig. 5.14 the centrodes of a Formula car making Turn 5 of the Barcelona circuit. In this case everything was fine, as confirmed by the "good" shape of both centrodes. Also shown are the trajectory of G and the inflection circle.

But not all laps are the same. Figure 5.15 shows the centrodes for the same curve in a case in which the Formula car did not perform well.

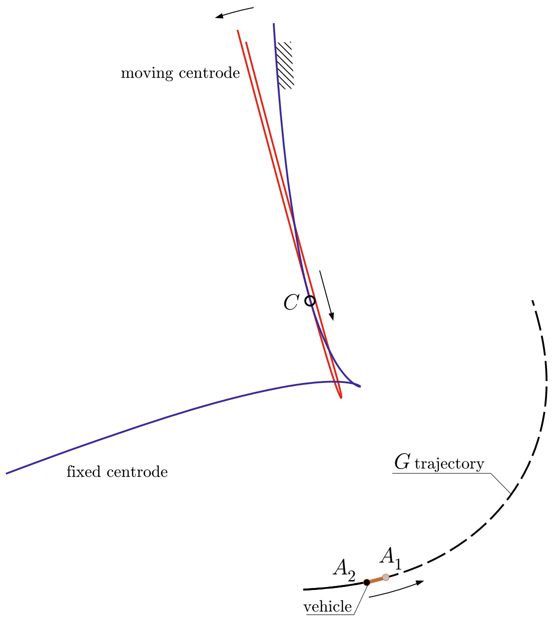

Fig. 5.11 Vehicle *entering a curve*: moving centrode rolling on the fixed centrode

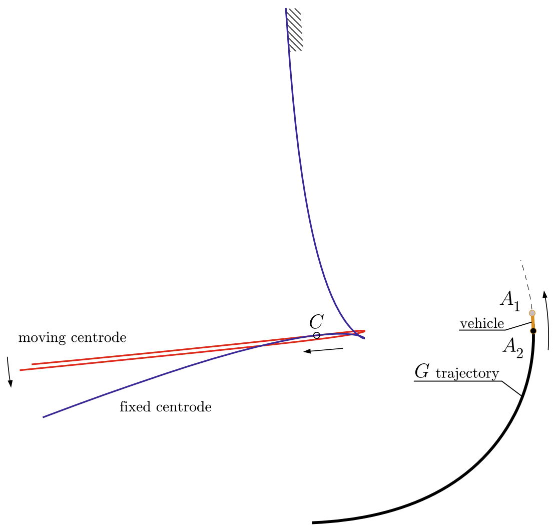

Fig. 5.12 Vehicle *exiting a curve*: moving centrode rolling on the fixed centrode

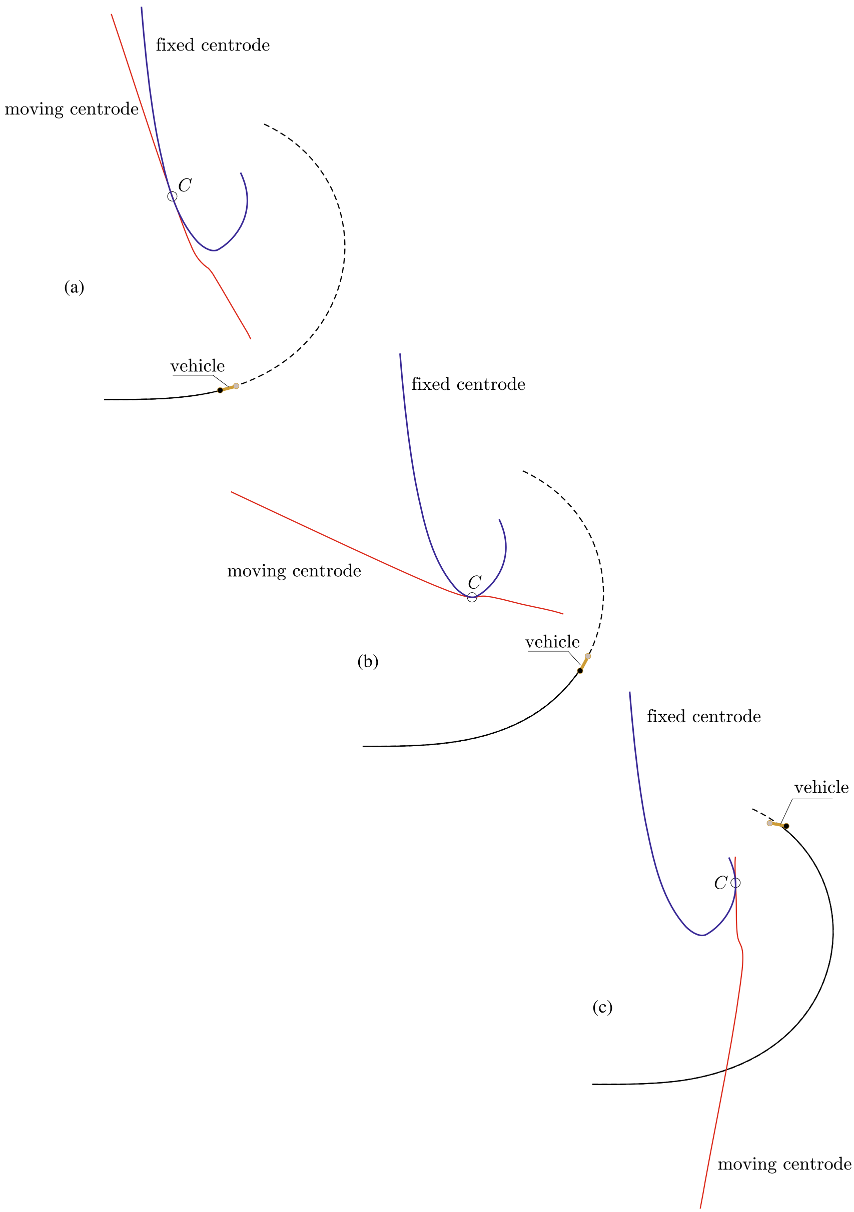

Fig. 5.13 Centrodes of a turning vehicle with handling misbehavior in the final part of the curve (the car spins out)

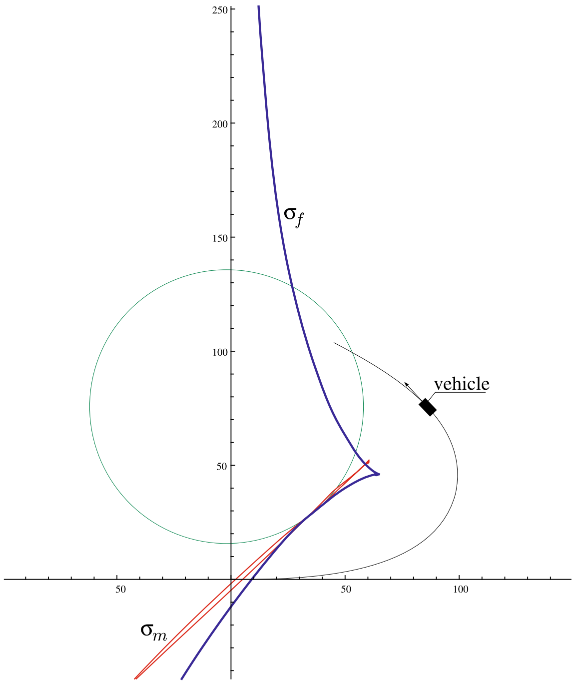

Fig. 5.14 Centrodes of a Formula car making Turn 5 of the Barcelona circuit (the inflection circle is also shown)

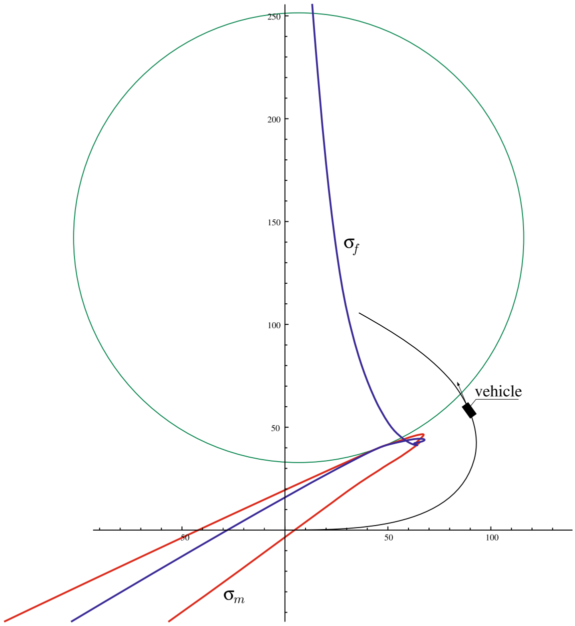

Fig. 5.15 Centrodes of a Formula car *badly* making Turn 5 of the Barcelona circuit (the inflection circle is also shown)

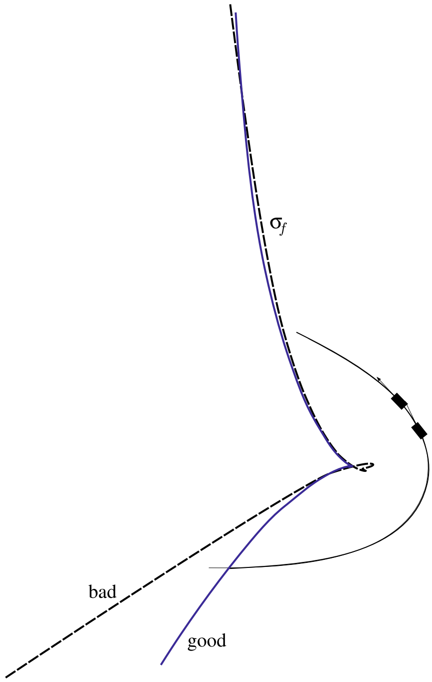

Fig. 5.16 Comparison of the fixed centrodes and of the trajectories of a Formula car making Turn 5 of the Barcelona circuit

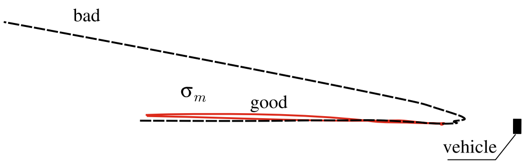

Fig. 5.17 Comparison of the moving centrodes of a Formula car making Turn 5 of the Barcelona circuit

The fixed centrodes for the two cases are compared in Fig. 5.16. The entering part is pretty much the same, whereas the central and the exiting parts are very different. It is worth noting that the trajectories of G are almost the same.

The moving centrodes are compared in Fig. 5.17. Again, they differ markedly in the exiting part.

5.2.2 *Inflection Circle of a Turning Vehicle*

The inflection circle (Fig. 5.10), that is all those points whose trajectory have an inflection point, can be obtained at any instant of time from telemetry data. Perhaps, the main formula is (5.7), that links the diameter d of the inflection circle to the acceleration of the velocity center C. The acceleration $\mathbf{a}_C$ was given in (3.48), which is repeated here for ease of reading (see also (5.19))

$$\begin{aligned}\mathbf{a}_C &= (a_x + vr - \dot{r}u/r)\,\mathbf{i} + (a_y - ur - \dot{r}v/r)\,\mathbf{j}\\ &= \left(\frac{\dot{u}r - u\dot{r}}{r}\right)\mathbf{i} + \left(\frac{\dot{v}r - v\dot{r}}{r}\right)\mathbf{j} \qquad (3.48')\\ &= r(\dot{R}\,\mathbf{i} - \dot{S}\,\mathbf{j})\end{aligned}$$

We see that we also need $\dot{r}$, which is not commonly measured directly, although it should be.

Here we list, with reference to Fig. 5.10, some relevant formulas

$$\sigma = \chi - \pi, \qquad \sin\chi = -\sin\sigma, \qquad \cos\chi = -\cos\sigma \tag{5.14}$$

$$d = \frac{1}{r^2}\sqrt{\left(\frac{\dot{v}r - v\dot{r}}{r}\right)^2 + \left(\frac{\dot{u}r - u\dot{r}}{r}\right)^2} = \sqrt{\frac{\dot{R}^2 + \dot{S}^2}{r^2}} \tag{5.15}$$

$$d\cos\chi = \left(\frac{\dot{u}r - u\dot{r}}{r}\right)\frac{1}{r^2} = \frac{\dot{R}}{r} \tag{5.16}$$

$$d\sin\chi = \left(\frac{\dot{v}r - v\dot{r}}{r}\right)\frac{1}{r^2} = -\frac{\dot{S}}{r} \tag{5.17}$$

$$\mathbf{d} = d\cos\chi\,\mathbf{i} + d\sin\chi\,\mathbf{j} = \frac{\dot{R}\,\mathbf{i} - \dot{S}\,\mathbf{j}}{r} \tag{5.18}$$

$$\mathbf{a}_C = r^2\mathbf{d} = r(\dot{R}\,\mathbf{i} - \dot{S}\,\mathbf{j}) = r^2 d(\cos\chi\,\mathbf{i} + \sin\chi\,\mathbf{j}) \tag{5.19}$$

$$\mathbf{V}_{\hat{C}} = \dot{S}\,\mathbf{i} + \dot{R}\,\mathbf{j} = rd(-\sin\chi\,\mathbf{i} + \cos\chi\,\mathbf{j}) \tag{5.20}$$

$$\mathbf{D} = S\,\mathbf{i} + R\,\mathbf{j} \tag{5.21}$$

$$r\mathbf{d}\cdot\mathbf{D} = \dot{R}S - R\dot{S} \tag{5.22}$$

$$\dot{\mathbf{D}} = \dot{S}\,\mathbf{i} + \dot{R}\,\mathbf{j} - S\dot{\chi}\,\mathbf{j} + R\dot{\chi}\,\mathbf{i} = (\dot{S} + R\dot{\chi})\,\mathbf{i} + (\dot{R} - S\dot{\chi})\,\mathbf{j} \tag{5.23}$$

$$\dot{d} = \frac{1}{r^3 d}\left[r(\dot{R}\ddot{R} + \dot{S}\ddot{S}) - \dot{r}(\dot{R}^2 + \dot{S}^2)\right] \tag{5.24}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{D}}{d}\right) = \frac{\dot{\mathbf{D}}d - \mathbf{D}\dot{d}}{d^2} = \frac{1}{d^2}\left\{[(\dot{S} + R\dot{\chi})d - S\dot{d}]\,\mathbf{i} + [(\dot{R} - S\dot{\chi})d - R\dot{d}]\,\mathbf{j}\right\} \tag{5.25}$$

These equations cover many aspects (Fig. 5.10):

- the diameter d of the inflection circle;
- the orientation χ of $\mathbf{a}_C$, and hence also of the inflection circle, with respect to the vehicle longitudinal axis;
- the acceleration $\mathbf{a}_C$ of the velocity center C;
- the speed $\mathbf{V}_{\hat{C}}$ of the geometric point $\hat{C}$;
- the rate of change of d;
- the rate of change of the vector $\mathbf{D} = GC$.

It is worth noting that almost all quantities depend on r, $\dot{R}$ and $\dot{S}$, that is on u, v, r, $\dot{u}$, $\dot{v}$, and $\dot{r}$

$$\begin{aligned}\dot{R} &= \frac{\dot{u}r - u\dot{r}}{r^2} = \frac{a_x - r^2 S - \dot{r}R}{r} \\ -\dot{S} &= \frac{\dot{v}r - v\dot{r}}{r^2} = \frac{a_y - r^2 R + \dot{r}S}{r}\end{aligned} \tag{5.26}$$

As already discussed in Sect. 3.2.8, mathematically equivalent formulas may not be equivalent at all when dealing with experimental data. Probably, it is better avoiding $\dot{u}$ and $\dot{v}$, and use a_x and a_y instead. It would be also very beneficial to measure directly $\dot{r}$, instead of differentiating the yaw rate r.

It is worth noting that, although S is the longitudinal coordinate of the velocity center C with respect to center of mass G, the quantity $\dot{S}$ is not related to G. It is a global quantity, like u, r, $\dot{r}$, R, $\dot{R}$. Quantities strictly related to G, and hence less general and less reliable, are v, $\dot{v}$, β, $\dot{\beta}$.

As shown in Fig. 5.10, along the axis of the vehicle there are, at any instant of time, some special points. Point Z has zero slip angle, that is, $\beta_Z = 0$, or equivalently $\mathbf{V}_Z = u\,\mathbf{i}$. Point N has $\dot{\beta}_N = 0$. Good handling requires these two points Z and N to be fairly close to each other and not too far from the front axle. Therefore, good handling behavior, like in Fig. 5.14, maybe requires small values of $|\dot{S}|$ or $|r\dot{S}|$. This is a topic that deserves further investigation. See also Sect. 5.2.3.

Also interesting is to observe that

$$|\mathbf{a}_C| = r^2 d \qquad \text{and} \qquad |\mathbf{V}_{\hat{C}}| = |rd| \tag{5.27}$$

They are strictly related.

5.2.3 *Tracking the Curvatures of Front and Rear Midpoints*

To better understand the kinematics of a turning vehicle, we also consider the curvature of the trajectories and how they change in time under the driver action on the steering wheel. In particular, we monitor the trajectories of the midpoints $A_1 = (a_1, 0)$ and $A_2 = (-a_2, 0)$ of both axles (Fig. 5.9), and their centers of curvature E_1 and E_2, respectively. There is a nice interplay between radii of curvature, the velocity center and the inflection circle.

We know from (3.3) that the velocities $\mathbf{V}_1$ and $\mathbf{V}_2$ of A_1 and A_2 are

$$\begin{aligned}\mathbf{V}_1 &= u\,\mathbf{i} + (v + ra_1)\,\mathbf{j} \\ \mathbf{V}_2 &= u\,\mathbf{i} + (v - ra_2)\,\mathbf{j}\end{aligned} \tag{5.28}$$

and hence $V_1 = |\mathbf{V}_1| = \sqrt{u^2 + (v + ra_1)^2}$ and $V_2 = |\mathbf{V}_2| = \sqrt{u^2 + (v - ra_2)^2}$. The corresponding front and rear vehicle slip angles $\hat{\beta}_1$ and $\hat{\beta}_2$ (Fig. 5.9), respectively, are such that

$$\tan(\hat{\beta}_1) = \frac{v + ra_1}{u} = \beta_1$$
$$\tan(\hat{\beta}_2) = \frac{v - ra_2}{u} = \beta_2 \tag{5.29}$$

From (3.39) we obtain the accelerations of A_1 and A_2

$$\mathbf{a}_1 = (a_x - r^2 a_1)\,\mathbf{i} + (a_y + \dot{r} a_1)\,\mathbf{j}$$
$$\mathbf{a}_2 = (a_x + r^2 a_2)\,\mathbf{i} + (a_y - \dot{r} a_2)\,\mathbf{j} \tag{5.30}$$

Through the knowledge of velocity and acceleration we can compute the centripetal (normal) component of the two accelerations, as in (3.40)

$$a_{1n} = \frac{-(a_x - r^2 a_1)(v + ra_1) + (a_y + \dot{r} a_1)u}{V_1}$$
$$a_{2n} = \frac{-(a_x + r^2 a_2)(v - ra_2) + (a_y - \dot{r} a_2)u}{V_2} \tag{5.31}$$

or, more explicitly (but numerically less reliably)

$$a_{1n} = \frac{-(\dot{u} - vr - r^2 a_1)(v + ra_1) + (\dot{v} + ur + \dot{r} a_1)u}{V_1}$$
$$a_{2n} = \frac{-(\dot{u} - vr + r^2 a_2)(v - ra_2) + (\dot{v} + ur - \dot{r} a_2)u}{V_2} \tag{5.32}$$

The curvatures ρ_1 and ρ_2 of the trajectories of A_1 and A_2 are now promptly obtained (cf. (3.37))

$$\rho_1 = \frac{a_{1n}}{V_1^2} = \frac{r}{V_1} + \frac{(\dot{v} + \dot{r} a_1)u - (v + ra_1)\dot{u}}{V_1^3} = \frac{r + \mathrm{d}\hat{\beta}_1/\mathrm{d}t}{V_1} \simeq \frac{r + \dot{\beta}_1}{u}$$
$$\rho_2 = \frac{a_{2n}}{V_2^2} = \frac{r}{V_2} + \frac{(\dot{v} - \dot{r} a_2)u - (v - ra_2)\dot{u}}{V_2^3} = \frac{r + \mathrm{d}\hat{\beta}_2/\mathrm{d}t}{V_2} \simeq \frac{r + \dot{\beta}_2}{u} \tag{5.33}$$

The *entering* phase of making a left turn is characterized by increasing steer angles and diminishing radii of curvature. Moreover, as we have already seen, the velocity center C gets closer and closer to the vehicle. The corresponding transient kinematics is shown in Fig. 5.18. It is worth noting that, according to (5.8), the radius of curvature of point A_1 is equal to E_1A_1, and hence it is shorter than CA_1. On the contrary, the radius of curvature of point A_2 is equal to E_2A_2, which is longer than CA_2. This happens because the vehicle slip angle $\hat{\beta}_1$ at point A_1 is increasing, while the vehicle slip angle $\hat{\beta}_2$ at point A_2 is diminishing (in the sense that it gets bigger, but it is negative), as shown in Fig. 5.18 and according to (5.33).

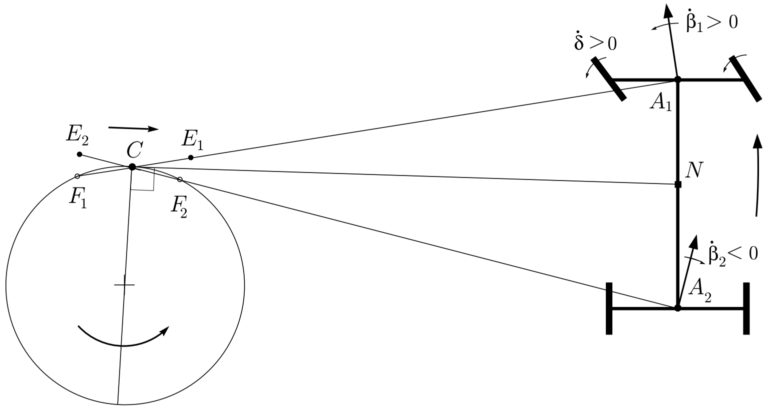

Fig. 5.18 Radii of curvature of a vehicle *entering* a turn properly

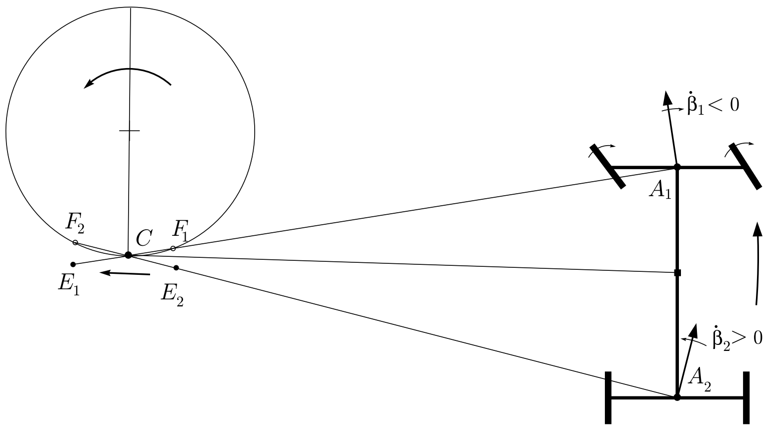

Fig. 5.19 Radii of curvature of a vehicle *exiting* a turn properly

The kinematics of a vehicle *exiting* properly a turn is shown in Fig. 5.19. We see that many things go the other way around with respect to entering.

In both cases, the knowledge of the inflection circle immediately makes clear the relationship between the position of the velocity center C and the centers of curvature E_1 and E_2.

But things may go wrong. Bad kinematic behaviors are shown in Fig. 5.20. We see that the time derivatives of $\hat{\beta}_1$ and $\hat{\beta}_2$ are not as they should be. Indeed, point C is travelling also longitudinally. Again, the positions and orientations of the inflection circle immediately convey the information about the unwanted kinematics of the vehicle.

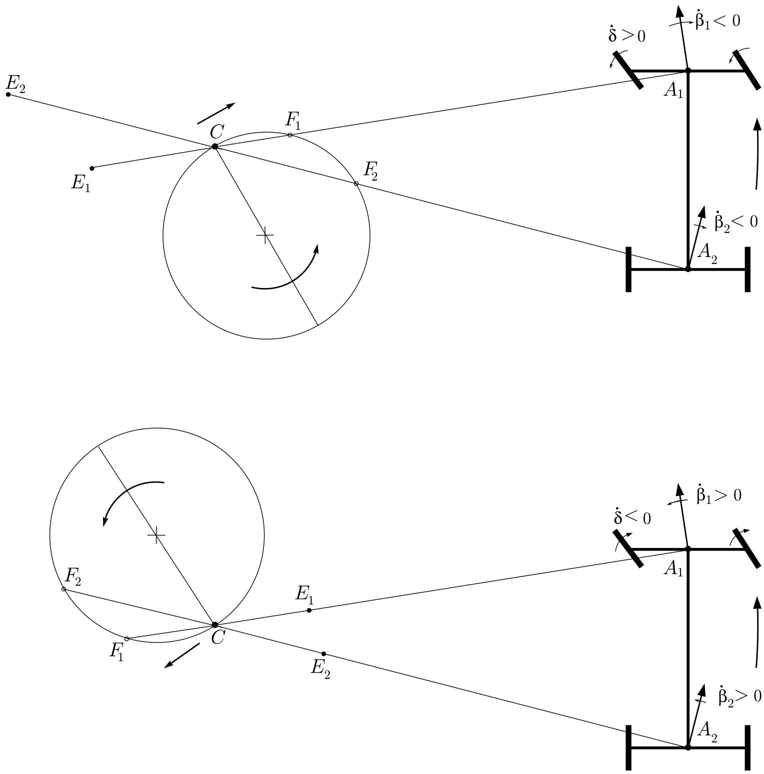

Fig. 5.20 Examples of undesirable kinematics in a turn

5.2.4 Evolutes

Let us go back to good turning behavior. The *evolute* of a curve is the locus of all its centers of curvature. The evolutes of the trajectories of points A_1 and A_2, that is the midpoints of each axle, are shown in the lower part of Figs. 5.22 and 5.21. Also shown are the centers of curvature E_1 and E_2 at a given instant of time, along with the corresponding inflection circle (this one drawn in the upper part with the centrodes). We see that the two evolutes are almost coincident. The relative positions of E_1 and E_2 are consistent with Figs. 5.18 and 5.19.

At the onset of bad turning behavior, the two evolutes depart abruptly from each other. Therefore, monitoring the evolutes of A_1 and A_2 can be another objective way to investigate the handling features of a vehicle.

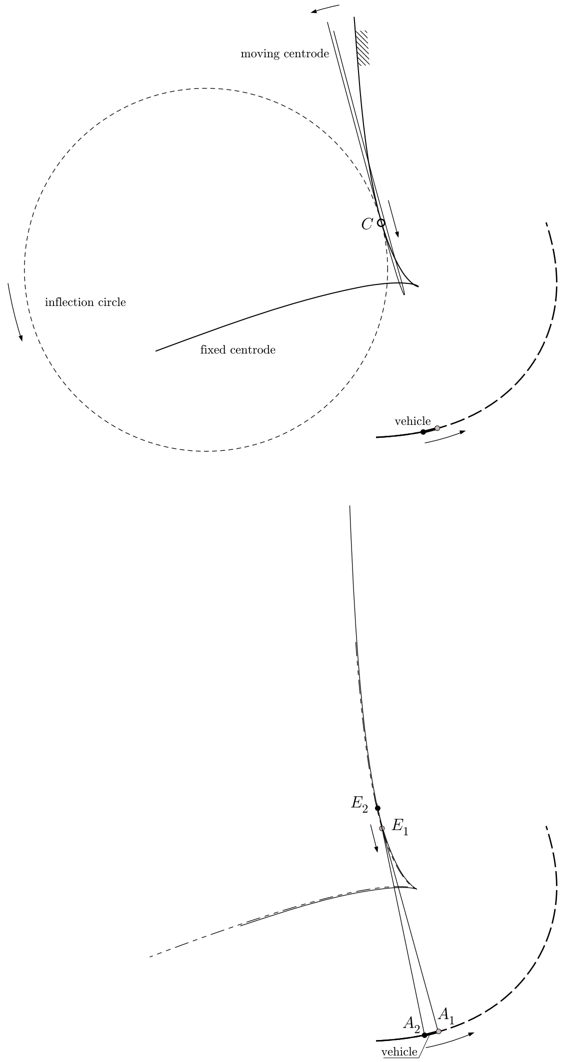

Fig. 5.21 Vehicle *entering* a curve: inflection circle (top) and centers of curvatures with the corresponding evolutes (bottom)

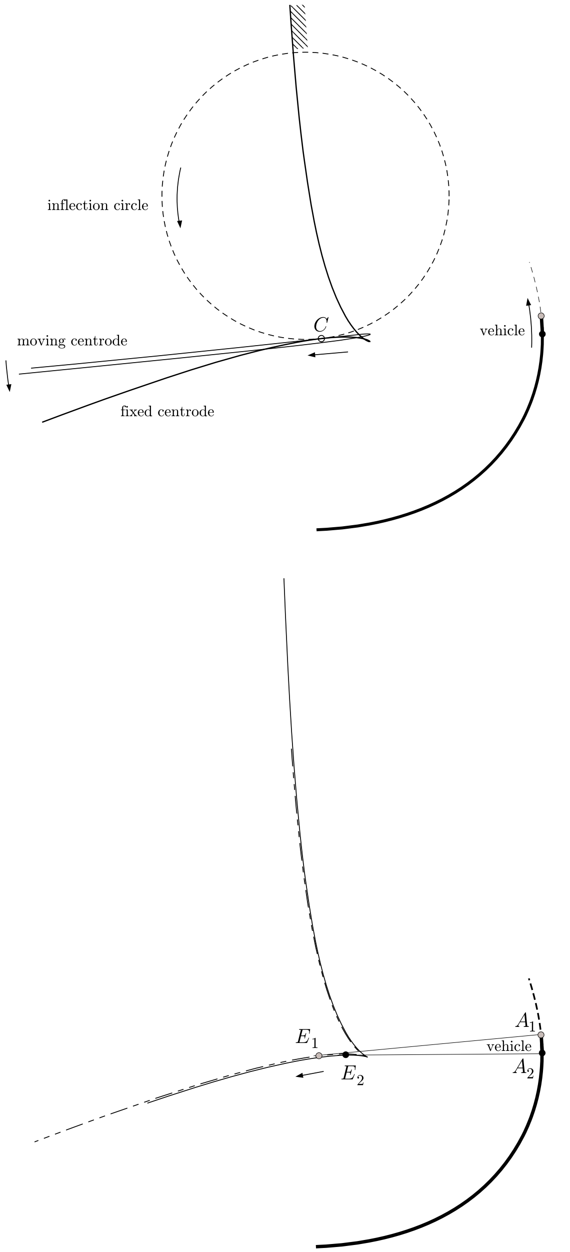

Fig. 5.22 Vehicle *exiting* a curve: inflection circle (top) and centers of curvatures with the corresponding evolutes (bottom)

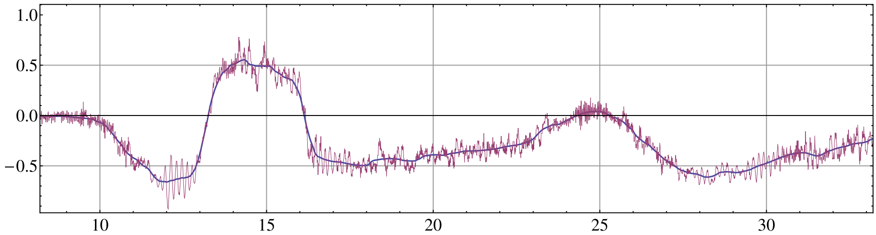

Fig. 5.23 Yaw rate $r(t)$, in rad/s: comparison between raw and filtered data

Table 5.1 Radii of curvature of the trajectories of the axle midpoints

	t (s)	Turn No	R_1 (m)	R_2 (m)
1	10.87	1	−135.01	−148.94
2	12.66	1	−52.54	−50.58
3	30.98	4	−104.20	−102.97
4	61.84	10	75.55	125.94
5	63.87	10	26.19	25.23
6	64.87	10	73.65	59.86

5.3 Exercises

5.3.1 Front and Rear Radii of Curvature

With the data of Tables 8.1 and 8.2 and assuming $a_1 = 1.68$ m *and* $a_2 = 1.32$ m, *compute the radii of curvature* R_1 *and* R_2 *of the trajectories of the axle midpoints* A_1 *and* A_2 *(Fig. 5.18). Discuss the results.*

Solution

According to (5.33), to compute $R_i = 1/\rho_i$ we need the centripetal component a_{in} of the acceleration and the speed V_i. The most reliable formula for a_{in} should be (5.31), because it avoids the computation of $\dot{u}$ and $\dot{v}$. Unfortunately, with the usual telemetry sensors, we cannot avoid the computation of $\dot{r}$. Therefore, the results will strongly depend upon the filter applied to the raw yaw rate before differentiating it, as obvious from Fig. 5.23.

The computed radii of curvature are shown in Table 5.1. The reader is invited to check whether they are consistent with Figs. 5.18 and 5.19, and also to compare them with R_G, already computed in Table 8.2.

5.3.2 Drawing Centrodes

Select the parametric equations to be employed to draw the moving centrode in Fig. 5.17. Then, do the same for Fig. 5.13.

Solution

In Fig. 5.17 the moving centrodes are plotted in the vehicle reference plane. Therefore, we must use (5.10).

On the other hand, in Fig. 5.13 there are three different positions of the moving centrodes. That means that we must use (5.13), for three different instants of time t.

5.4 List of Some Relevant Concepts

Section 5.1.2—the moving centrode rolls without slipping on the fixed centrode, the point of rolling contact being C;
Section 5.1.2—the velocity field is like a pure rotation around C, but the acceleration field is not;
Section 5.1.3—the acceleration field is like a pure rotation around K;
Section 5.1.4—the inflection circle makes it possible to easily evaluate the radius of curvature of the trajectory of any point of a rigid body;
Section 5.2.2—handling misbehavior strongly affects the shape of centrodes;
Section 5.2.4—monitoring the evolutes can be another objective way to investigate the handling features of a vehicle.

5.5 Key Symbols

$\mathbf{a}_C$	acceleration of C
a_x	longitudinal acceleration of G
a_y	lateral acceleration of G
C	velocity center
d	diameter of the inflection circle
G	center of mass;
K	acceleration center
r	yaw rate
R	lateral coordinate of G
S	longitudinal coordinate of G
u	longitudinal velocity
v	lateral velocity of G
χ	orientation of $\mathbf{a}_C$
ψ	yaw angle

References

1. Bottema O (1979) Theoretical kinematics. Dover Publications, New York
2. Heißing B, Ersoy M (eds) (2011) Chassis handbook. Springer, Wiesbaden
3. Pytel A, Kiusalaas J (1999) Engineering mechanics—statics. Brooks/Cole, Pacific Grove
4. Rossa FD, Mastinu G, Piccardi C (2012) Bifurcation analysis of an automobile model negotiating a curve. Veh Syst Dyn 50(10):1539–1562
5. Sandor GN, Erdman AG (1984) Advanced mechanism design: analysis and synthesis, vol 2. Prentice-Hall, Englewood Cliffs
6. Waldron KJ, Kinzel GL (2004) Kinematics, dynamics, and design of machinery. Wiley, New York