Chapter 4
Braking Performance

Driving a vehicle involves, among other things, braking [1]. Fortunately, most of the times, we brake very softly, far from the braking performance limit. Most drivers, perhaps, never need to experience the limit braking performance of their car in everyday traffic. However, engineers must know very well the mechanics of braking a vehicle, to allow it to stop as soon as possible in case of emergency. Actually, this problem has been somehow mitigated by the advent of ABS systems [2], which now equip every road car. However, many race cars do not have ABS and hence brake design and balance is still a relevant topic in vehicle dynamics.

By *brake balance* or bias, we mean how much to brake the front wheels with respect to the rear wheels. The goal is to stop the vehicle as soon as possible, but avoiding wheel locking. Cars have only one pedal to brake all wheels and brake balance is left to the car. By the way, wheel locking should be avoided because, in order of importance:

1. the steering/directional capability is totally impaired (most important);
2. the grip is lower;
3. energy dissipation switches from the brakes to the contact patches and tires get damaged.

On the other hand, almost all motorcycles and bicycles have independent brake commands for the front wheel and for the rear wheel, thus leaving the duty of brake balance to the rider. Many bicyclists fear using the front brake because they believe it might cause the bicycle to overturn. Actually, overturning a bicycle with the front brake is much harder than it seems. Not using the front brake is a bad habit, since it drastically impairs the braking performance.

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4.1 Pure Braking

As anticipated, we extract tailored models from the fairly general vehicle model developed in Chap. 3.

When braking on a *flat*, *straight* road, with *uniform grip*, we know beforehand that

$$\begin{aligned} Y &= 0 \\ N &= 0 \\ \Delta X_i &= 0 \\ \Delta Z_i &= 0 \end{aligned} \tag{4.1}$$

that is, there are no lateral forces, no yaw moment and no lateral load transfers. Accordingly, the vehicle goes straight, with no lateral acceleration and yaw rate (and also no lateral velocity)

$$\begin{aligned} a_y &= 0 \\ \dot{r} &= 0 \\ v &= 0 \\ r &= 0 \end{aligned} \tag{4.2}$$

Other quantities are usually very small. In particular, if the wheels of the same axle have a bit of convergence (also called toe-in), that means that there are small steering angles and, accordingly, very small lateral slips. Similarly, if the wheels of the same axle have some camber, the tires are subject to a small spin slip:

$$\begin{aligned} \delta_{ij} &\simeq 0 \\ \sigma_{y_{ij}} &\simeq 0 \\ \varphi_{ij} &\simeq 0 \end{aligned} \tag{4.3}$$

At first, all these quantities can be set equal to zero.

4.2 Vehicle Model for Braking Performance

A simple, yet significant, model to study the limit braking performance of a road vehicle is shown in Fig. 4.1. We are dealing here with road vehicles, without significant aerodynamic downforces (however, have a look at Fig. 3.24). Formula cars are dealt with in Sect. 4.12.

We suppose to brake on a flat and straight road, with uniform grip. Therefore, the vehicle goes straight. Moreover, we assume to apply a constant force to the brake pedal. Therefore, pitch oscillations are negligible.

Summing up, we can employ the two-dimensional model shown in Fig. 4.1. The vehicle is just a single rigid body with mass m, moving horizontally with forward

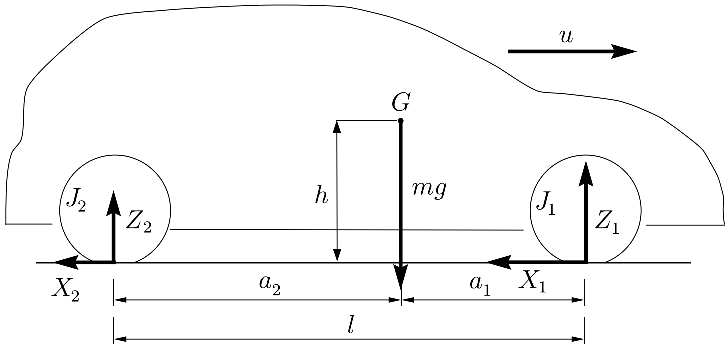

Fig. 4.1 Model for braking performance analysis of road cars

speed u and forward acceleration $\dot{u} < 0$. Beside its own weight mg, it receives two vertical forces Z_1 and Z_2 from the road, one per axle, and two longitudinal (braking) forces X_1 and X_2, again one per axle.

In this chapter only we assume X_1 and X_2 to be positive if directed like in Fig. 4.1. It is more convenient to deal with positive quantities.

4.3 Equilibrium Equations

The three equilibrium equations are readily obtained from Fig. 4.1

$$
\begin{aligned}
m\dot{u} &= -(X_1 + X_2) \\
0 &= Z_1 + Z_2 - mg \\
0 &= (X_1 + X_2)\,h - Z_1\,a_1 + Z_2\,a_2
\end{aligned}
\tag{4.4}
$$

which must be supplemented by the following inequalities

$$
|X_i| \le \mu_p^x Z_i \qquad \text{and} \qquad Z_i \ge 0 \tag{4.5}
$$

where μ_p^x is the global longitudinal friction coefficient defined in (2.87). It is quite obvious that the braking forces cannot exceed the traction limit, nor the vertical forces be negative. For brevity, we will use the symbol μ for μ_p^x in this chapter.

The aerodynamic drag X_a has not been included because in road cars it is really small compared to the braking forces.

The rolling resistance is also very small. The braking forces X_i already include this small contribution.

4.3.1 Rigorous Moment Equation

In the third equation in (4.4) we omitted the rotating inertia J_1 and J_2 of the two axles. The complete equation is

$$(J_1 + J_2)\frac{\dot{u}}{r_r} = (X_1 + X_2)\,h - Z_1\,a_1 + Z_2\,a_2 \tag{4.6}$$

where r_r is the wheel rolling radius. However, the contribution of the rotating inertia is usually negligible. Typically, $(J_1 + J_2)/r_r \simeq 1$ kgm, while, e.g., $mh \simeq 600$ kgm.

4.4 Longitudinal Load Transfer

When going at constant speed, that is with $\dot{u} = 0$, we have from (4.4) (or, directly, from (3.105)) that the static vertical loads on each axle are

$$Z_1^0 = \frac{mga_2}{l} \qquad Z_2^0 = \frac{mga_1}{l} \tag{4.7}$$

During braking with $\dot{u} < 0$, the two loads change, although their sum must be constantly equal to the vehicle weight mg. We have the so-called longitudinal load transfer ΔZ

$$Z_1 = Z_1^0 + \Delta Z \quad \text{and} \quad Z_2 = Z_2^0 - \Delta Z \tag{4.8}$$

where (cf. (3.104))

$$\Delta Z = -\frac{mh}{l}\dot{u} \tag{4.9}$$

with $\dot{u} < 0$. The front axle is subject to a higher load ($Z_1 > Z_1^0$), while the rear axle to a lower load ($Z_2 < Z_2^0$). It is worth noting that the load transfer does not depend on the type of suspensions.

We have *overturning* of the vehicle if $Z_2 = 0$, that is if

$$|\dot{u}| = a_1 g/h \tag{4.10}$$

This condition is never met in cars, whereas it may limit the brake performance in some motorcycles.

4.5 Maximum Deceleration

The best braking performance $|\dot{u}|_{\max}$ is obtained if both axles brake at their traction limit, that is if

$$X_1 = \mu\,Z_1 \quad \text{and} \quad X_2 = \mu\,Z_2 \tag{4.11}$$

From the equilibrium equations (4.4), it is straightforward to obtain the *limit deceleration*

$$|\dot{u}| = \mu g \tag{4.12}$$

Of course, the maximum deceleration is the minimum between (4.10) and (4.12)

$$|\dot{u}|_{\max} = \min(\mu g,\ a_1 g/h) \tag{4.13}$$

Cars have $\mu < a_1/h$, whereas in some motorcycles it can be the other way around. Here we are mainly dealing with cars, and therefore we have

$$|\dot{u}|_{\max} = \mu g \tag{4.14}$$

4.6 Brake Balance

When braking at the best braking performance, that is with $\dot{u} = -\mu g$, the longitudinal forces are

$$\begin{aligned} X_{1_P} &= \mu\, Z_{1_P} = \mu \left(Z_1^0 + \frac{mh}{l}\mu g \right) = \mu \frac{mg}{l}(a_2 + \mu h) \\ X_{2_P} &= \mu\, Z_{2_P} = \mu \left(Z_2^0 - \frac{mh}{l}\mu g \right) = \mu \frac{mg}{l}(a_1 - \mu h) \end{aligned} \tag{4.15}$$

The *optimal brake balance* (or brake bias) β_P to have the *best* braking performance is promptly obtained as

$$\beta_P = \frac{X_{1_P}}{X_{2_P}} = \frac{Z_{1_P}}{Z_{2_P}} = \frac{a_2 + \mu h}{a_1 - \mu h} \tag{4.16}$$

Typical values in road cars are $\beta_P \simeq 2$ on dry asphalt ($\mu \simeq 0.8$) and $\beta_P \simeq 1.5$ on wet asphalt ($\mu \simeq 0.4$). More commonly, the same concepts would be expressed as front/rear = 66/33 and front/rear = 60/40, respectively.

Let μ_1 be the coefficient of friction of the front axle, and μ_2 be the coefficient of friction of the rear axle. Then, the optimal brake balance β_P is given by

$$\beta_P = \frac{X_{1_P}}{X_{2_P}} = \frac{\mu_1(a_2 + \mu_2 h)}{\mu_2(a_1 - \mu_1 h)} \tag{4.17}$$

which generalizes (4.16) when μ_1 (front) $\neq \mu_2$ (rear).

4.7 All Possible Braking Combinations

If the best braking performance is our ultimate goal, we should also look around to see what happens if we employ a brake balance not equal to β_P. All possible braking combinations can be visualized in a simple, yet very useful, figure.

First solve the equilibrium equations (4.4) with $X_1 = \mu Z_1$, thus getting

$$Z_1 = \frac{X_1}{\mu} = Z_1^0 + \frac{h}{l}(X_1 + X_2) \tag{4.18}$$

and hence

$$X_1 = \mu \left(\frac{Z_1^0 + \dfrac{h}{l} X_2}{1 - \mu \dfrac{h}{l}} \right) \tag{4.19}$$

This is the relationship between X_1 and X_2 to have limit (threshold) braking at the front wheels.

Similarly, solve the equilibrium equations (4.4) with $X_2 = \mu Z_2$, thus getting

$$X_2 = \mu \left(\frac{Z_2^0 - \dfrac{h}{l} X_1}{1 + \mu \dfrac{h}{l}} \right) \tag{4.20}$$

This is the relationship between X_1 and X_2 to have limit (threshold) braking at the rear wheels.

In the plane (X_2, X_1) we can now draw the two straight lines (4.19) and (4.20), as shown in Fig. 4.2. The region inside the two lines contains all possible (admissible) braking combinations. Trying to trespass the upper line means front wheels lock-up. Trying to trespass the right line means rear wheels lock-up. Point P is the condition of best braking performance. It requires the combination of the braking forces X_{1_P} and X_{2_P}, which were obtained in (4.15).

Points with the same level of deceleration all belong to straight lines with slope 45°, that is lines with constant $X_1 + X_2 = -m\dot{u}$. The maximum deceleration corresponds to the line passing through point P. Braking with balance β_P means moving along the line OP.

Some other relevant cases are shown in Fig. 4.3. Region 1 corresponds to low decelerations. So small that they can be obtained with any balance between front and rear braking forces, or even with only a rear braking force X_{2_0}

$$X_{2_0} = \frac{\mu Z_2^0}{1 + \mu \dfrac{h}{l}} \tag{4.21}$$

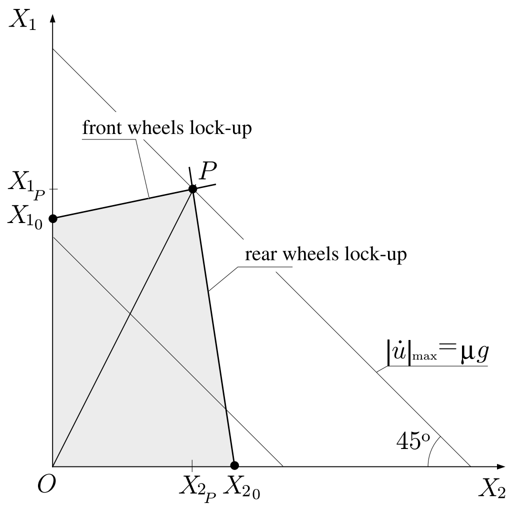

Fig. 4.2 Region of all admissible braking combinations

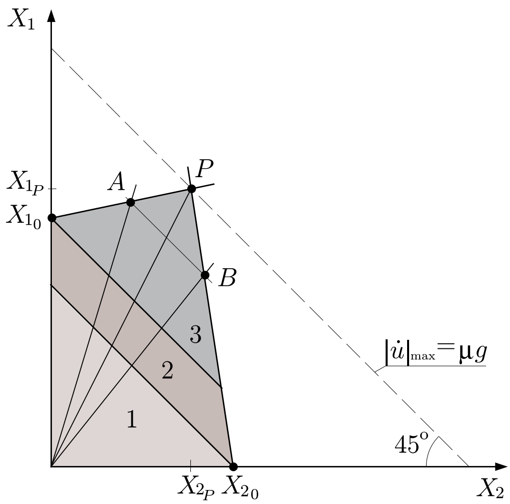

Fig. 4.3 Region of all admissible braking combinations with indication of some particular cases

Region 2 needs necessarily some braking force at the front wheels, but even the front wheels alone, with a braking force X_{1_0}, would do (front/rear = 100/0)

$$X_{1_0} = \frac{\mu Z_1^0}{1 - \mu \dfrac{h}{l}} \tag{4.22}$$

Region 3, that is high decelerations, require intervention of both axles. The higher the deceleration, the narrower the range A–B.

To complete our discussion we have to address the effects of changing the grip coefficient μ and/or the position of G, that is a_1/a_2, and maybe h.

4.8 Changing the Grip

The formulation developed so far includes the grip coefficient as a parameter. Therefore, we have already obtained all formulas to deal with different values of μ. To understand what happens it is helpful to draw the admissible region for, say, three different values $\mu_l < \mu_o < \mu_h$ of the grip coefficient,[1] as shown in Fig. 4.4.

Let us assume that our car has a brake balance that follows line OP_2, that is optimized for $\mu = \mu_o$. If the grip is lower, that is $\mu_l < \mu_o$, there will be less load transfer ΔZ and a lower brake balance would be optimal. If we still follow line OP_2, we exit the admissible region at point A, that is for a deceleration lower than $\mu_l g$ and with the front wheels at lock-up. It can be shown that the deceleration is equal to $\varepsilon_l \mu_l g$, with the braking efficiency $\varepsilon_l < 1$ given by

$$\varepsilon_l = \frac{a_2}{a_2 + h(\mu_o - \mu_l)}, \qquad \text{if } \mu_l < \mu_o \tag{4.23}$$

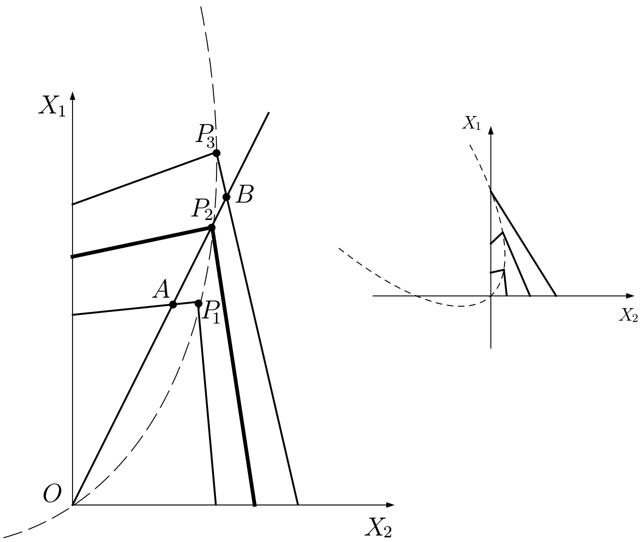

Fig. 4.4 Region of all admissible braking combinations for three different grip coefficients (left) and parabola of limit points (right)

[1] In this section we assume to have the same grip in both axles.

Braking efficiency $\varepsilon_h < 1$ is also obtained when the out-of-balance is due to a higher value $\mu_h > \mu_o$ of the grip coefficient. As shown in Fig. 4.4, we exit the admissible region at point B, which is not optimal. Rear wheels are about to lock up and the deceleration is equal to $\varepsilon_h \mu_h g$, with the braking efficiency $\varepsilon_h < 1$ given by

$$\varepsilon_h = \frac{a_1}{a_1 + h(\mu_h - \mu_o)}, \qquad \text{if } \mu_h > \mu_o \tag{4.24}$$

Also shown in Fig. 4.4 is the parabola that collects all vertices P when varying the coefficient μ. Point P located on the X_1 axis means that maximum deceleration is limited by overturning.

4.9 Changing the Weight Distribution

The longitudinal position of G affects the static load distribution. Therefore, it affects the brake balance, but not the maximum deceleration μg. Accordingly, we get an admissible region like in Fig. 4.5, with a new vertex $\hat{P}$ still on the same line at 45°, and with sides parallel to those of the original region.

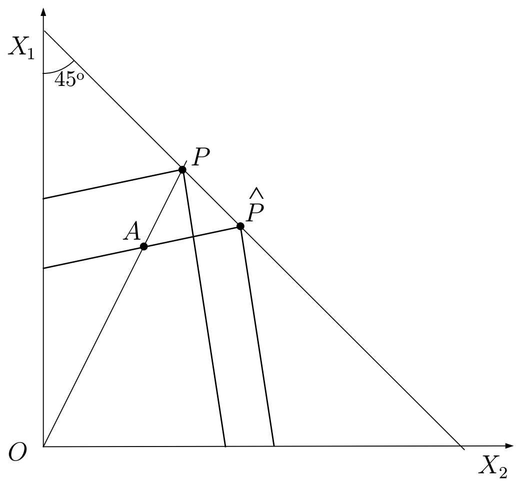

Fig. 4.5 Region of admissible braking combinations for two different weight distributions

4.10 A Numerical Example

A numerical example may be useful to understand better the braking performance of a road car. We take a small car with the following features: mass $m = 1000\,\text{kg}$, wheelbase $l = 2.4\,\text{m}$, $a_1 = a_2 = l/2$, height of the center of mass $h = 0.5\,\text{m}$.

Assuming a grip coefficient $\mu = 0.8$, the maximum deceleration is vehicle independent and it is equal to $|\dot{u}|_{\max} = \mu g = 7.84\,\text{m/s}^2$, with $g = 9.81\,\text{m/s}^2$.

According to (4.7), the static vertical loads for both axles are $Z_1^0 = Z_2^0 = 4900\,\text{N}$. The load transfer at maximum deceleration is $\Delta Z = 1633\,\text{N}$. Therefore, the vertical loads acting on each axle are $Z_{1_P} = 6533\,\text{N}$ and $Z_{2_P} = 3267\,\text{N}$, which means a brake balance $\beta_P = 2$. This is the optimal value for that car if $\mu = 0.8$.

Should the grip coefficient drop to 0.4 because, e.g., of rain, we would end up with a braking efficiency $\varepsilon_1 = 0.86$. An increase of the grip coefficient up to 1.2 would still bring a reduced braking efficiency $\varepsilon_2 = 0.86$.

4.11 Braking, Stopping, and Safe Distances

The braking distance refers to the distance a vehicle will travel from the point when its brakes are fully applied to when it comes to a complete stop.

The total stopping distance is the sum of the perception-reaction distance and the braking distance. The perception-reaction time ranges from 0.75 to 1.5 s.

In everyday traffic, the driver must keep a safe distance between his/her vehicle and the vehicle in front in order to avoid collision if the car in front brakes or stops. The safe distance corresponds to the distance covered by the vehicles in the perception-reaction time. This is the rationale for the *three-second rule*, by which a driver can easily maintain a safe trailing distance at any speed. The rule is that a driver should ideally stay at least three seconds behind any vehicle that is directly in front. Of course, it can be applied at any speed and with any weather condition.

4.12 Braking Performance of Formula Cars

Formula cars have aerodynamic devices that provide very high downforces at high speed, as briefly explained in Sect. 3.7.2. These loads affect braking pretty much. The first, obvious, effect is that the maximum longitudinal deceleration is speed dependent. In a Formula 1 car it can be up to 5 g at 350 km/h, although the physical grip μ rarely exceeds 1.6. The second, perhaps less obvious, effect is that also the optimal brake balance β_P is speed dependent.

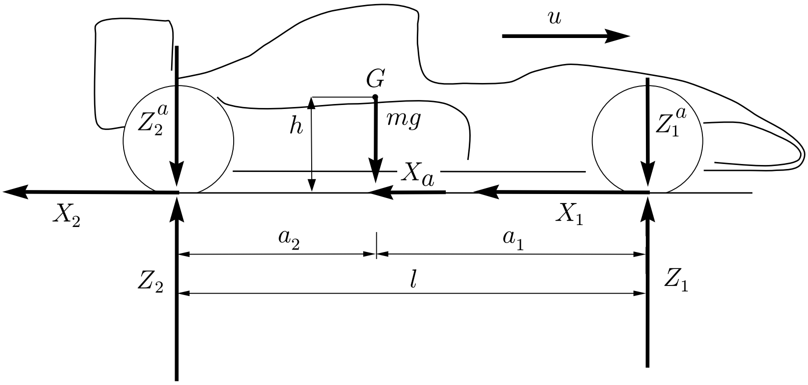

Fig. 4.6 Vehicle model for braking performance of a Formula car (all forces are positive)

4.12.1 Equilibrium Equations

The equilibrium equations (4.4) must be supplemented by the aerodynamic loads. According to Sect. 3.7.2 and as shown in Figs. 3.22 and 4.6, the total aerodynamic force $\mathbf{F}_a$ is equivalent to three forces: a drag force X_a at road level and two vertical forces Z_1^a and Z_2^a acting directly on the front and rear axles, respectively. Therefore, the equilibrium equations become

$$\begin{aligned} m\dot{u} &= -(X_1 + X_2) - X_a \\ 0 &= Z_1 + Z_2 - mg - Z_1^a - Z_2^a \\ 0 &= (X_1 + X_2 + X_a)\,h - (Z_1 - Z_1^a)\,a_1 + (Z_2 - Z_2^a)\,a_2 \end{aligned} \tag{4.25}$$

Unlike in (3.94) and (3.95), here we assume X_1 and X_2 to be positive if directed like in Fig. 4.6, that is to be indeed braking forces. As shown in Fig. 2.43, the tire rolling resistance is part of the braking (grip) forces X_i.

We recall that (cf. (3.81) and (3.82))

$$\begin{aligned} X_a &= \frac{1}{2}\rho_a S_a C_x u^2 = \xi u^2 \\ Z_1^a &= \frac{1}{2}\rho_a S_a C_{z1} u^2 = \zeta_1 u^2 \\ Z_2^a &= \frac{1}{2}\rho_a S_a C_{z2} u^2 = \zeta_2 u^2 \end{aligned} \tag{4.26}$$

where, as it is common practice among race engineers, $C_x > 0$ and $C_{zi} > 0$.

For simplicity, here we assume $S_a C_x$, $S_a C_{z1}$ and $S_a C_{z2}$ to be constant (i.e., speed independent). Actually, this is not strictly true, as shown in Fig. 4.7, because the

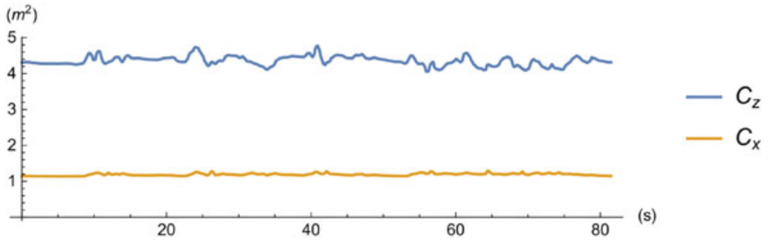

Fig. 4.7 Measured values of $S_a C_z$ and $S_a C_x$ in one lap of an F1 race car

height from the ground of the car is not constant. Therefore, the assumption of constant coefficients should be removed in more advanced analyses.

4.12.2 Vertical Loads

The vertical loads on each axle are given by the static loads (4.7) (zero speed), plus the aerodynamic (speed dependent) loads (4.26), plus or minus the inertial longitudinal load transfer (cf. (4.8))

$$\begin{aligned} Z_1 &= Z_1^0 + \zeta_1 u^2 + \Delta Z \\ Z_2 &= Z_2^0 + \zeta_2 u^2 - \Delta Z \end{aligned} \tag{4.27}$$

Like in (4.9), the *inertial* longitudinal load transfer ΔZ is given by

$$\Delta Z = -\frac{mh}{l}\dot{u} \tag{4.28}$$

with $\dot{u} < 0$. When braking, the front axle is subject to a higher load, while the rear axle to a lower load, with respect to the static loads. It is a purely inertial effect. However, at high speed the drag force X_a is not negligible and can significantly affect the vertical loads, even if $C_z = 0$ as shown in Fig. 3.24.

4.12.3 Maximum Deceleration

The maximum deceleration is promptly obtained by assuming that both axles are at their limit braking conditions, that is $X_1 = \mu Z_1$ and $X_2 = \mu Z_2$

$$|\dot{u}|_{\max} = \mu \left(g + \frac{\zeta_1 + \zeta_2}{m} u^2 \right) + \frac{\xi}{m} u^2 \tag{4.29}$$

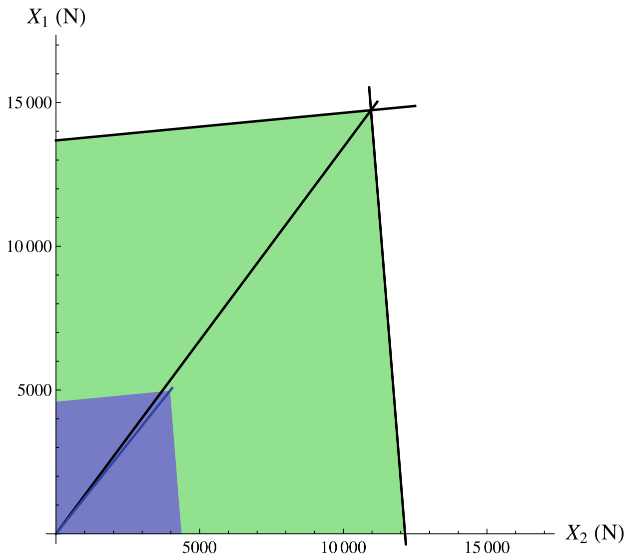

Fig. 4.8 Regions of all admissible braking combinations of a Formula car at 200 km/h, with and without aerodynamic downforces

This formula generalizes (4.14). Of course $|\dot{u}|_{max}$ is very speed dependent, as also shown in Fig. 4.8.

More detailed information can be obtained by looking at (4.29) as a differential equation in the unknown function $u(t)$

$$\dot{u} = -\mu g - \frac{u^2}{k} \tag{4.30}$$

where

$$k = \frac{m}{\mu(\zeta_1 + \zeta_2) + \xi} \tag{4.31}$$

Setting the initial speed $u(0) = u_0$, the analytical solution is as follows

$$u(t) = u_d \tan\left(\arctan\left(\frac{u_0}{u_d}\right) - \frac{t}{t_d}\right) \tag{4.32}$$

where

$$u_d = \sqrt{\mu g k} \qquad \text{and} \qquad t_d = \sqrt{\frac{k}{\mu g}} \tag{4.33}$$

Moreover, to have $u(t) > 0$, it must be $t < t_0$, with

$$t_0 = t_d \arctan\left(\frac{u_0}{u_d}\right) \tag{4.34}$$

Therefore, t_0 is the shortest time to stop the car from speed u_0. Integrating (4.32), we can obtain the distance s_0 travelled by the car to come to a stop

$$s_b = \frac{1}{2} t_d u_d \ln\left(1 + \frac{u_0^2}{u_d^2}\right) = \frac{k}{2} \ln\left(1 + \frac{u_0^2}{\mu g k}\right) \tag{4.35}$$

It is worth comparing this equation with its counterpart (4.52) for cars without aerodynamic devices.

A plot of (4.35) is available in Fig. 4.12 (lower curve), along with a plot of (4.52) (upper curve).

4.12.4 Brake Balance

To brake at the best braking performance, that is with $\dot{u} = -|\dot{u}|_{\max}$, the longitudinal forces must be

$$\begin{aligned} X_{1_P} &= \mu\left(Z_1^0 + \zeta_1 u^2 + \frac{mh}{l}|\dot{u}|_{\max}\right) \\ &= \mu\left(Z_1^0 + \zeta_1 u^2 + \frac{h}{l}[\mu g m + \mu(\zeta_1 + \zeta_2)u^2 + \xi u^2]\right) \\ X_{2_P} &= \mu\left(Z_2^0 + \zeta_2 u^2 - \frac{mh}{l}|\dot{u}|_{\max}\right) \\ &= \mu\left(Z_2^0 + \zeta_2 u^2 - \frac{h}{l}[\mu g m + \mu(\zeta_1 + \zeta_2)u^2 + \xi u^2]\right) \end{aligned} \tag{4.36}$$

Having the right brake balance is very important for lap performance. The *optimal brake balance* (or brake bias) β_P to have the *best* braking performance is promptly obtained as

$$\beta_P(u) = \frac{X_{1_P}}{X_{2_P}} = \frac{(a_2 + h\mu)gm + u^2[(a_1 + a_2)\zeta_1 + h\xi + h(\zeta_1 + \zeta_2)\mu]}{(a_1 - h\mu)gm + u^2[(a_1 + a_2)\zeta_2 - h\xi - h(\zeta_1 + \zeta_2)\mu]} \tag{4.37}$$

which generalizes (4.16). As expected, in general now β_P is explicitly speed dependent.

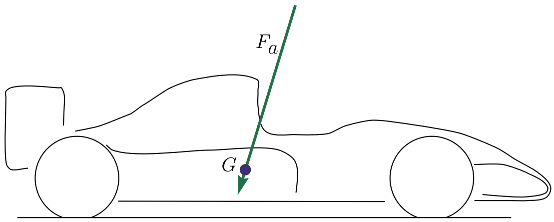

Fig. 4.9 The line of action of the global aerodynamic force must pass through G to have speed independent brake balance

4.12.5 Speed Independent Brake Balance

To avoid explicit speed dependence of β_P, and hence to enhance the lap performance, it must be in (4.37)

$$\beta_P = \frac{a_2 + h\mu}{a_1 - h\mu} = \frac{(a_1 + a_2)\zeta_1 + h\xi + h(\zeta_1 + \zeta_2)\mu}{(a_1 + a_2)\zeta_2 - h\xi - h(\zeta_1 + \zeta_2)\mu} \tag{4.38}$$

that is, with $C_z = C_{z1} + C_{z2}$

$$\beta_P = \frac{a_2 + h\mu}{a_1 - h\mu} = \frac{(a_1 + a_2)C_{z1} + h(C_x + C_z\mu)}{(a_1 + a_2)C_{z2} - h(C_x + C_z\mu)} \tag{4.39}$$

This condition should be taken into account during setup (see also the next two sections).

Interestingly enough, there is a simple physical interpretation of (4.38) and (4.39): the line of action of the global aerodynamic force $\mathbf{F}_a = -\xi u^2\,\mathbf{i} - (\zeta_1 + \zeta_2)u^2\,\mathbf{k}$ must pass through the center of mass G, as shown in Fig. 4.9 and as discussed in Sect. 4.13.4.

4.12.6 Practical Brake Balance

In addition to the brake balance $\beta = X_1/X_2$, we define, as commonly done by race engineers, the *practical brake balance* η

$$\eta = \frac{X_1}{X_1 + X_2} \tag{4.40}$$

along with the *weight distribution* ω

$$\omega = \frac{Z_1^0}{Z_1^0 + Z_2^0} = \frac{a_2}{a_1 + a_2} \tag{4.41}$$

and the *aero balance* α

$$\alpha = \frac{C_{z_1}}{C_{z_1} + C_{z_2}} = \frac{\zeta_1}{\zeta_1 + \zeta_2} = \frac{\zeta_1}{\zeta} \tag{4.42}$$

where

$$\zeta = \zeta_1 + \zeta_2 \tag{4.43}$$

We obtain the following result[2] for the *optimal* practical brake balance η_P

$$\eta_P = \frac{mg(\mu\frac{h}{l} + \omega) + \zeta u^2(\mu\frac{h}{l} + \alpha) + \xi u^2 \frac{h}{l}}{mg + \zeta u^2} \tag{4.44}$$

which is the counterpart of (4.37). Using η_P or β_P is just a matter of taste.

In case of different front to rear grip ($\mu_1 \neq \mu_2$), we have[3]

$$\eta_P = \frac{\mu_1 \left[mg(\mu_2\frac{h}{l} + \omega) + \zeta u^2(\mu_2\frac{h}{l} + \alpha) + \xi u^2 \frac{h}{l}\right]}{mg[\mu_1\omega + \mu_2(1-\omega)] + \zeta u^2[\mu_1\alpha + \mu_2(1-\alpha)] + \xi u^2 \frac{h}{l}(\mu_1 - \mu_2)} \tag{4.45}$$

4.12.7 Speed Independent Practical Brake Balance

Of course, in general, η_P is speed dependent. To avoid speed dependence of η_P, it must be $\partial\eta_p/\partial u = 0$, that means (see also Fig. 4.9)

$$(\omega - \alpha)\zeta - \frac{h}{l}\xi = 0 \tag{4.46}$$

or, equivalently

$$(\omega - \alpha)C_z - \frac{h}{l}C_x = 0 \tag{4.47}$$

which can be rewritten as

$$\omega = \alpha + \frac{C_x}{C_z}\frac{h}{l} \tag{4.48}$$

Quite a compact and interesting formula. We see that, to avoid speed dependence of η_p, it is necessary that $\omega > \alpha$, but just a little. For instance, in a Formula car, it results in $\omega - \alpha \simeq 0.02$.

The last three equations are the counterpart of (4.39). However, they look simpler to be kept in mind.

[2] Ernesto Desiderio, personal communication, 6 May 2020.

[3] Federico Sánchez Motellón, personal communication, 22 June 2020.

4.12.8 Sensitivities

From (4.44) we can easily compute the sensitivities of η_P.

The sensitivity of the optimal practical brake balance η_p with respect to the aero balance α is

$$\frac{\partial \eta_P}{\partial \alpha} = \frac{\zeta u^2}{mg + \zeta u^2} \tag{4.49}$$

Very simple formula. No μ, no h/l, no C_x. Strong speed dependence at high speed.

Similarly, the sensitivity of the optimal practical brake balance η_p with respect to the weight distribution ω is

$$\frac{\partial \eta_P}{\partial \omega} = \frac{mg}{mg + \zeta u^2} \tag{4.50}$$

Again, a very simple formula. No μ, no h/l, no C_x. Strong speed dependence at low speed.

It may be interesting to observe that

$$\frac{\partial \eta_P}{\partial \alpha} + \frac{\partial \eta_P}{\partial \omega} = 1 \tag{4.51}$$

4.12.9 Typical F1 Braking Performance

A typical braking performance of an F1 car is shown in Fig. 4.10. The deceleration grows suddenly up to about 38 m/s^2. Then, as the speed u (m/s) decreases, also the aerodynamic load decreases, thus requiring the driver to gradually release the brake pedal. Meanwhile, the car is already negotiating the curve, as shown by the lateral acceleration and wheel steer angle (deg). Also shown in Fig. 4.10 is the acceleration ($a_x > 0$) when the car exits the curve.

It is interesting to compare the total acceleration $\sqrt{a_x^2 + a_y^2}$ (lower line in Fig. 4.11) with the potential maximum deceleration (4.29) (upper line in Fig. 4.11). Whenever possible, the driver tries to stay as close as possible to the limit. This can be done in all curves that are grip-limited. Of course, not in those curves that are speed-limited (like, e.g., curve 3 in the Barcelona circuit).

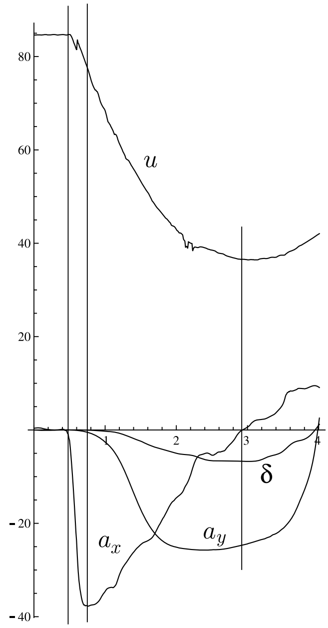

Fig. 4.10 Typical braking performance of an F1 car

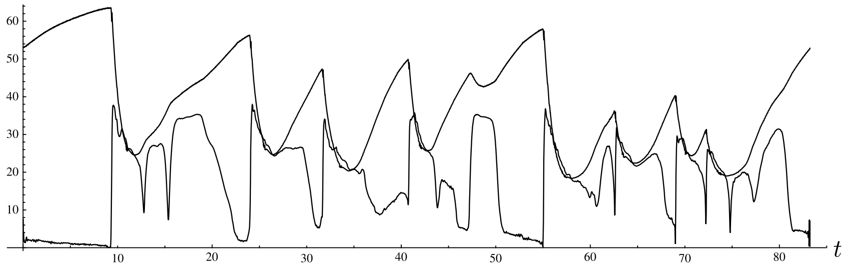

Fig. 4.11 Comparison between the total acceleration (lower line) and the potential maximum acceleration (upper line) of an F1 car

4.13 Exercises

4.13.1 Minimum Braking Distance

For the road car described in Sect. 4.10, *compute the minimum braking distance assuming the following data:*

- *grip coefficient* $\mu = 0.8$ *(dry asphalt)*;
- *initial speed* $u_0 = 100\,\text{km/h}$;
- *braking efficiency* $\varepsilon = 1$.

Then, for the sake of comparison, repeat the same calculation in case the car has only the front brakes, and then in case the car has only the rear brakes.

Solution

First we convert the initial speed in SI units: $u_0 = 100/3.6 = 27.8$ m/s.

In our model, the maximum deceleration is equal to $\mu g = 7.85$ m/s^2. Therefore, it is not affected by the position of G and by the mass m.

We know by elementary physics that the speed decreases from u_0 to zero according to $u(t) = u_0 - \mu g t$. Therefore, we obtain the braking time $t_b = u_0/(\mu g) = 3.54$ s, which is a linear function of the initial speed.

We can now compute the distance covered by the car to come to a stop

$$s_b = \frac{1}{2}\mu g t_b^2 = \frac{u_0^2}{2\mu g} = 49\,\mathrm{m} \tag{4.52}$$

Of course, to get this minimum distance, the brake balance β_P must be set according to (4.16). In this case $\beta_P = 2$, as shown in Sect. 4.10.

It can be of some interest to compare this expression of s_b for road cars with (4.35) for Formula cars.

The braking distance can also be found by determining the work required to dissipate the vehicle kinetic energy, that is $0.5 m u_0^2 = m\mu g s_b$. Of course, the result is the same, but without the byproduct of the braking time t_b.

As well known, the braking distance of a road car is a quadratic function of the initial speed u_0. Doubling the speed makes the braking distance four times longer.

If braking with the front wheels only, we can at most get the braking force X_{1_0} given by (4.22). We see that now the position of G becomes relevant. With a bit of algebra, we obtain that in this case the maximum deceleration is

$$a_f = \mu g \frac{a_2}{l}\left(\frac{1}{1-\mu\dfrac{h}{l}}\right) = \mu g \frac{a_2}{l-\mu h} = 4.71\,\mathrm{m/s^2} \tag{4.53}$$

The braking distance is therefore given by $7.85/4.71 \times 49 = 82$ m, that is

$$s_f = \frac{u_0^2}{2a_f} \tag{4.54}$$

Similarly, but employing (4.21), we obtain that in case of rear braking only the deceleration is

$$a_r = \mu g \frac{a_1}{l}\left(\frac{1}{1+\mu\dfrac{h}{l}}\right) = \mu g \frac{a_1}{l+\mu h} = 3.36\,\mathrm{m/s^2} \tag{4.55}$$

and the braking distance is $7.85/3.36 \times 49 = 114$ m. As expected, front braking only is more efficient than rear braking only. This is particularly true in bicycles and motorcycles. Try to guess why.

4.13.2 Braking with Aerodynamic Downforces

A GP2 race car has the following features (notation as in Sect. 3.7.2):

- $m = 680$ kg;
- $l = a_1 + a_2 = 3.025$ m;
- $a_1/a_2 = 1.27$, *that is weight distribution front/rear* of 0.44/0.56;
- $S_a\, C_{z1} = 1.5\, \text{m}^2$;
- $S_a\, C_{z2} = 2.1\, \text{m}^2$;
- $S_a\, C_x = 1.1\, \text{m}^2$;
- $\mu = 1.35$;
- $h = 0.27$ m;
- *air density* 1.25 kg/m^3.

Compute the minimum braking distance and the minimum braking time when it is running straight at 150 km/h *and at 300* km/h.

Solution

In this race car we have (see (4.26))

- $\zeta_1 = 0.5 \times 1.25 \times 1.5 = 0.9375$ kg/m
- $\zeta_2 = 0.5 \times 1.25 \times 2.1 = 1.3125$ kg/m
- $\xi = 0.5 \times 1.25 \times 1.1 = 0.6875$ kg/m

Therefore, in (4.32) we obtain $u_d = 49.17$ m/s and $t_d = 3.71$ s.

If the initial speed is 150/3.6 = 41.67 m/s, we obtain from (4.34) the minimum braking time $t_0 = 2.61$ s. According to (4.29), the highest deceleration is 22.75 m/s^2. The braking distance is $s_b = 49.4$ m. It is obtained integrating numerically (7.237) from 0 to 2.61 s.

If the initial speed is 300/3.6 = 83.33 m/s, we obtain from (4.34) the minimum braking time $t_0 = 3.85$ s. According to (4.29), the highest deceleration is 51.28 m/s^2. The braking distance is $s_b = 123.6$ m. It is obtained integrating numerically (4.32) from 0 to 3.85 s.

Just out of curiosity, this car would stop in about 25 m if running at 100 km/h. The time would be less than 2 s.

From Fig. 4.12 we can appreciate how important the aerodynamic loads are in Formula cars. The braking distances with all aerodynamic forces, with drag but no downforces, with no aerodynamics at all, are quite far apart at high speeds.

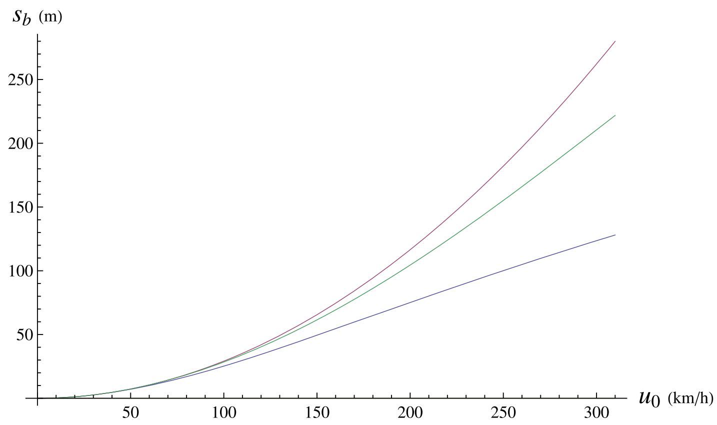

Fig. 4.12 Comparison between the braking distance of a GP2 car (lower curve), the braking distance for the same car, but without any aerodynamic effect (upper curve), the braking distance with drag, but no downforces (intermediate curve)

4.13.3 GP2 Brake Balance

The brake distances computed in the former exercise need a perfect brake balance β_P at any speed. Compute the value of the perfect brake balance for the same GP2 car at 100, 150 and 300 km/h, *and comment on it.*

Solution

First of all, let us test whether this car fulfills (4.39), which would make β_P speed insensitive. The l.h.s. term makes 1.275, while the r.h.s. term makes 1.295. Very good.

Indeed, we have $\beta_P = 1.280$ at 100 km/h, $\beta_P = 1.283$ at 150 km/h, and $\beta_P = 1.290$ at 300 km/h. We see that this car has a brake balance which is almost speed independent.

4.13.4 Speed Independent Brake Balance

Check the physical interpretation of (4.38).

Solution

The physical interpretation of (4.38) requires the global aerodynamic force $\mathbf{F}_a = -(X_a\,\mathbf{i} + Z_a\,\mathbf{k})$ to pass through the center of mass G (Fig. 4.9). Therefore, we have to solve the following system of equations (cf. (4.25))

$$
\begin{aligned}
m\dot{u} &= -(X_1 + X_2) - X_a \\
0 &= Z_1 + Z_2 - mg - Z_a \\
0 &= (X_1 + X_2)\,h - Z_1 a_1 + Z_2 a_2 \\
X_1 &= \mu_1 Z_1 \\
X_2 &= \mu_2 Z_2
\end{aligned} \tag{4.56}
$$

where, for greater generality, the front grip μ_1 is not necessarily equal to the rear grip μ_2.

The resulting brake balance is

$$
\beta_P = \frac{X_1}{X_2} = \frac{\mu_1(a_2 + h\mu_2)}{\mu_2(a_1 - h\mu_1)} \tag{4.57}
$$

which is, indeed, speed independent, and generalizes (4.38). Quite a useful result to optimize the braking performance of a Formula car.

4.14 Summary

The goal of this chapter has been to understand how to stop a vehicle as soon as possible, avoiding wheel locking. This result can be achieved only if the vehicle has the right brake balance. Unfortunately, brake balance is affected by the value of the grip and by the position of the center of mass. This topic has been addressed in detail, both analytically and graphically, through the region of all possible braking conditions. The peculiarity of the braking performance of a Formula car has been also discussed.

4.15 List of Some Relevant Concepts

Section 4.4—the longitudinal load transfer does not depend on the type of suspensions;
Section 4.5—maximum deceleration is limited by either grip or overturning (supposing brakes are powerful enough);
Section 4.6—brake balance depends on grip and weight distribution;
Section 4.7—all possible braking combinations can be represented by a simple figure;
Section 4.12.2—wings do not affect load transfer directly;
Section 4.12.4—brake balance is affected by wings;
Section 4.12.5—the line of action of the global aerodynamic force must pass through G to have speed independent brake balance.

4.16 Key Symbols

a_1	distance of G from the front axle
a_2	distance of G from the rear axle
C_x, C_{z_i}	aerodynamic coefficients
g	gravitational acceleration
G	center of mass
h	height of G
J_y	moment of inertia
l	wheelbase
m	mass
S_a	frontal area
u	longitudinal velocity
$\dot{u}$	longitudinal acceleration
X_i	braking force acting on the i-th axle
Z_i	vertical load on i-th axle
Z_i^0	static vertical load on i-th axle
Z_i^a	aerodynamic vertical load on i-th axle
ΔZ	longitudinal load transfer
α	aero balance
β	brake balance
β_P	optimal brake balance
ε_i	braking efficiency
η	practical brake balance
η_P	optimal practical brake balance
$\mu = \mu_p^x$	coefficient of grip
ρ_a	air density
ω	weight distribution

References

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2. Savaresi SM, Tanelli M (2010) Active braking control systems design for vehicles. Springer, London