Constructing Exact Solutions to Modelling Problems

Mathew O. Aibinu, Surendra C. Thakur, and Sibusiso Moyo

Abstract Modelling of several processes and phenomena, which occur in sciences and engineering, often lead to Nonlinear Partial Differential Equations (NPDEs). Reaction–diffusion systems are members of NPDEs which can be described as mathematical models with applications in diverse physical phenomena. Obtaining the solutions of modelling problems is often a big challenge due to several conditions and parameters which are involved. This study demonstrates how to construct the solutions of modelling problems. A modified method of functional constraints is proposed for constructing exact solutions to nonlinear equations of reaction–diffusion type with delay and which are associated with variable coefficients. Arbitrary functions are present in the solutions, and they also contain free parameters, which make them suitable for usage in solving certain modelling problems, testing numerical, and approximate analytical methods. Specific examples of nonlinear equations of reaction–diffusion type with delay are given and their exact solutions are presented.

Keywords Exact solutions · Generalized traveling-wave · Heat and wave equations · Reaction–diffusion

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M. O. Aibinu (✉)
Institute for Systems Science & KZN E-Skill CoLab, Durban University of Technology, Durban 4000, South Africa
e-mail: moaibinu@yahoo.com; mathewa@dut.ac.za

DSI-NRF Centre of Excellence in Mathematical and Statistical Sciences (CoE-MaSS), Johannesburg, South Africa

National Institute for Theoretical and Computational Sciences (NITheCS), Johannesburg, South Africa

S. C. Thakur
KZN E-Skill CoLab, Durban University of Technology, Durban 4000, South Africa

S. Moyo
Institute for Systems Science & Office of the DVC Research, Innovation & Engagement Milena Court, Durban University of Technology, Durban 4000, South Africa

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Abbreviations

NPDEs	Nonlinear Partial Differential Equations
ODEs	Ordinary Differential Equations
RDEs	Reaction–Diffusion Equations

Introduction

The roles which Nonlinear Partial Differential Equations (NPDEs) play are prominent in the description and analysis of the real-life processes and phenomena. Therefore, it is pivotal to seek the ways of obtaining the exact solutions of NPDEs for a proper and accurate analysis. Several processes and phenomena which occur in sciences and engineering lead to NPDEs as there are several conditions and parameters to be considered in the modeling of such systems. Reaction–Diffusion Equations (RDEs) are members of NPDEs. Reaction–diffusion systems can be described as mathematical models which find applications in diverse physical phenomena. In its simplest form in one spatial dimension, RDE has the form

$$u_t = Du_{xx} + H(u), \tag{1}$$

where $u(x, t)$ denotes the unknown function, H accounts for all local reactions, and D is a diffusion coefficient (which is a constant) (See e.g., [1]). RDEs are pervading in the mathematical modeling of the systems which occur in biology, chemistry, complex physics phenomena, engineering, and mechanics [2]. RDEs delineate the chemical reactions and diffusion processes. Basically, many real-life processes do not only depend on the present state but also on past occurrences. Also, the dynamical systems are constituted by the time delay. The study of nonlinear delay RDEs provides a fundamental tool for the quantitative and qualitative analyses of various dynamical systems such as infections. For RDEs with delay, the kinetic function H which denotes the chemical reactions rates is a function of both $u = u(x, t)$ and $w = u(x, t - \tau)$, which represent the sought concentration function and delayed argument, respectively. Two special cases which can arise are $H(u, w) = H(w)$ and $H(u, w) = H(u)$. A system with local non-equilibrium media is described by $H(u, w) = H(w)$. These are systems which possess inertial properties and reactions will always begin after a time τ. $H(u, w) = H(u)$ represents the classical local equilibrium case [3].

NPDEs are universal in nature and for finding their solutions, several methods have been employed which include spectral collocation and waveform relaxation [4–6], adomian decomposition [7, 8], Tan-Cot [9], residual power series [10], and perturbation [11, 12]. However, there are disadvantages which are commonly associated with these methods. There are conditions which make the universal application of these listed methods and others to be impossible. The objects are different in their geometric shapes. The reaction kinetics and type of fluid flow are erratic. The worthlessness in the presence of singular points is indisputable. Obtaining the exact solutions is imperative for proper analysis of the processes which are under consideration (localization, nonuniqueness, blowup regimes, spatial, etc.).

Subsequently, the term "exact solution" in relating to NPDEs will refer to where the solution can be expressed in:

(i) terms of elementary functions;
(ii) closed form with definite or/and indefinite integrals;
(iii) terms of solutions to Ordinary Differential Equations (ODEs) or systems of such equations.

Accepted form for exact solutions also includes the combinations of cases listed above (See e.g., [3, 13–17]).

Let $H(u, w)$ denote an arbitrary function which takes two arguments u and w. Consider reaction–diffusion problems of the form

$$c(x)u_t = [a(x)u_x]_x + b(x)H(u, w)u_x, \quad w = u(x, t - \tau), \tag{2}$$

where $a = a(x)$, $b = b(x)$, $c = c(x)$ are appropriate functions with precise roles in the equation. Namely, $a > 0$ casts the diffusion of the second-order divergence form operator, b models the reaction term, c represents a time weight factor (both b and c also casting nonlinearities on the unknown), and $\tau > 0$ is the time delay. We construct the exact solutions of (2) in the form of generalized traveling-wave equations. We apply our results to obtain the solutions of certain essential modelling problems which are peculiar to metal forming processes.

Solutions of Generalized RDEs with Delay

The exact solutions of (2) will be constructed in the form

$$u = U(y), \quad y = t + \int h(x)dx, \tag{3}$$

which is the generalized traveling-wave equations. Substitute (3) into (2) to obtain

$$a(x)h^2U''_{yy} + \left([a(x)h]'_x - c(x)\right) U'_y + b(x)hH\,(U, W)\,U'_y = 0, \tag{4}$$

where $W = U(y - \tau)$ and $h = h(x)$. The coefficients of the equation are chosen such that they conform to the relations

$$b(x) = a(x)h(x), \tag{5}$$

$$[a(x)h]'_x = -ka(x)h^2(x) + c(x), , \tag{6}$$

where k is a constant. The relations reduce (4) to

$$U''_{yy} + [H(U, W) - k]\,U'_y = 0, \quad W = U(y - \tau), \tag{7}$$

which is a delay ODE. Equation (6) can be written in standard form as

$$a(x)h'_x + ka(x)h^2 + a'(x)h - c(x) = 0. \tag{8}$$

Constructing Exact Solutions When the Function $h(x)$ Is not Given

The relation (8) forms a Riccati ODE for $h = h(x)$ when the functions $a(x)$ and $c(x)$ are given. Degenerate and nondegenerate cases will be considered for the Riccati ODE (8).

Degenerate case. For $k = 0$, the general solution for the degenerate form of Riccati equation (8) has the solution which is given by

$$h(x) = \frac{\int a(x)c(x)dx + q}{a(x)}, \tag{9}$$

where q signifies an arbitrary constant.

Example 1 Let $0 < x < \pi$, consider the case where $a(x) = cos(x)$ and $c(x) = 1$. By (8), $h(x) = tan(x)$, where it has been taken that $q = 0$. Equation (5) is applied to obtain that $b(x) = sin(x)$. Thus, for arbitrary functions $H(u, w)$, the nonlinear RDE

$$u_t = [cos(x)u_x]_x + sin(x)H(u, w)\,u_x, \quad w = u(x, t - \tau),$$

admits the generalized traveling-wave equations

$$u = U(y), \quad y = t + tan(x),$$

as its exact solution, where $U(y)$ is determined by

$$U''_{yy} + H(U, W)\,U'_y = 0, \quad W = U(y - \tau). \tag{10}$$

Nondegenerate case. When $k \neq 0$, let

$$h = \frac{1}{k}\frac{\varphi'_x}{\varphi}. \tag{11}$$

Substitution (11) into (9) gives

$$\frac{a(x)}{k}\left(\frac{\varphi''_{xx}}{\varphi}-\left(\frac{\varphi'_x}{\varphi}\right)^2\right)+ka(x)\left(\frac{1}{k}\frac{\varphi'_x}{\varphi}\right)^2+\frac{a'(x)}{k}\frac{\varphi'_x}{\varphi}-c(x)=0. \qquad (12)$$

Simplified form of (12) is

$$a(x)\varphi''_{xx}+a'(x)\varphi'_x-kc(x)\varphi=0, \qquad (13)$$

which is a linear second-order ODE. For the exact solutions of (13) with various functions $a(x)$ and $q(x)$, interested readers are referred to [18, 19].

Example 2 Taking $a=c=1$ in (13) gives its general solution as

$$\varphi=\begin{cases} A_1\ \cosh(\phi x)+A_2\ \sinh(\phi x), & \text{if } k=\phi^2>0,\\ A_1\ \cos(\phi x)+A_2\ \sin(\phi x), & \text{if } k=-\phi^2<0, \end{cases} \qquad (14)$$

where the arbitrary constants are A_1 and A_2. By using (11), it can be obtained from (14) when $A_1=1$, $A_2=0$, and $k=\phi^2\ (>0)$ that

$$h(x)=\coth(\phi x). \qquad (15)$$

Substitute (15) into (5) to obtain

$$b(x)=\coth(\phi x).$$

Thus, for arbitrary $H(u,w)$,

$$u_t=u_{xx}+\coth(\phi x)G\,(u,w)\,u_x,$$

admits the generalized traveling-wave equations

$$u=U(y),\ y=t+ln\left|\sinh(\phi x)\right|,$$

as its exact solution, where $U(y)$ is determined by the delay ODE

$$U''_{yy}+\left[H\,(U,W)-\phi^2\right]U'_y=0,\quad W=U(y-\tau).$$

Constructing Exact Solutions When the Function $h(x)$ Is Given

Given $h=h(x)$ in (8), the derived generalized traveling-wave equations (3) solves certain RDEs with delay of the form (2) provided (5) and (3) are satisfied. Such derived generalized traveling-wave equations are said to be the exact solutions of the corresponding RDEs. Solving the Riccati ODE (8) is not required as h has been given. An algebraic equation is required to be solved if h in (8) is already given.

Example 3 Degenerate and nondegenerate cases will be considered for arbitrary given functions $a(x)$ and $h = h(x)$.

(I) Degenerate case $k = 0$. Apply (5) to obtain

$$b(x) = a(x)h,$$

and by (8)

$$c(x) = a(x)h'_x + a'(x)h.$$

Thus, for arbitrary function $H(u, w)$,

$$\left[a(x)h'_x + a'(x)h\right] u_t = [a(x)u_x]_x + a(x)hH\,(u, w)\,u_x,$$

admits the generalized traveling-wave equations

$$u = U(y),\ \ y = t + \int h dx, \tag{16}$$

as its exact solution, where $U(y)$ is determined by (10).

(II) Nondegenerate case $k \neq 0$. By (8),

$$c(x) = a(x)h'_x + ka(x)h^2 + a'(x)h.$$

Thus, for arbitrary function $H(u, w)$,

$$\left[a(x)h'_x + ka(x)h^2 + a'(x)hh\right] u_t = [a(x)u_x]_x + a(x)hH\,(u, w)\,u_x,$$

admits the generalized traveling-wave equations (16) as its exact solution, where $U(y)$ is determined by (7).

Application to Metal Forming Processes

The metal forming processes are broad. The results which have been obtained will be applied to an essential metal forming process which is the heating of a uniform metal rod of length L and thermal diffusivity K_0.

Derivation of Heat and Wave Equations in 3D

The derivation of heat and wave equations in 3D is presented in this section to make this study a complete paper. For a full account of the derivation steps, the readers are referred to consult [20, 21]. Let V be an arbitrary 3D subregion of $\mathbb{R}^3$ (i.e. $V \subset \mathbb{R}^3$) and temperature $u = u(x, t)$ be defined for all points $\boldsymbol{x} = (x, y, z) \in V$. The heat energy in the subregion V is given by

$$\text{heat energy} = \int \int_V \alpha\rho u \, dV,$$

where ρ is the density of the rod, α is the specific heat, which is the energy required to raise a unit mass of the substance by 1 unit in temperature. In this study, we shall use these basic units: M mass, L length, T time, U temperature, and $[\alpha] = L^2T^{-2}U^{-1}$. Let S denote the boundary of V and $\hat{\boldsymbol{n}}$ be the outward unit normal at the boundary S. We seek the heat flux through S which is the normal component of the heat flux vector ϕ, $\phi.\hat{\boldsymbol{n}}$. Notice that $\phi.\hat{\boldsymbol{n}} < 0$ if ϕ is directed inward and the outward flow of heat is negative. The sum $\phi.\hat{\boldsymbol{n}}$ is taking over the entire closed surface S to get the total heat energy flowing across S. The total heat energy flowing across is denoted by $\int \int_S dS$. It can be recalled from the conservation of energy principle that

rate of change of heat energy	=	heat energy into V from boundaries per unit time	+	heat energy generated in solid per unit time

Applying the conservation of energy principle gives

$$\frac{d}{dt} \int \int \int_V \alpha\rho u \, dV = -\int \int_S \phi.\hat{\boldsymbol{n}} dS + \int \int \int_V Q dV. \tag{17}$$

According to divergence theorem (also known as Gauss's Theorem), for any volume V with closed smooth surface S,

$$\int \int \int_V \nabla.\boldsymbol{A} dV = -\int \int_S \boldsymbol{A}.\hat{\boldsymbol{n}} dS, \tag{18}$$

where $\boldsymbol{A}$ is any function that is smooth (i.e. continuously differentiable) for $\boldsymbol{x} \in V$ and

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

Applying divergence theorem to (17) leads to

$$\int \int \int_V \left(\alpha\rho \frac{\partial u}{\partial t} + \nabla.\phi - Q\right) dV = 0. \tag{19}$$

It is clear that the integrand must be everywhere zero since $V \subset \mathbb{R}^3$ and the integrand is assumed continuous. Thus,

$$\alpha\rho\frac{\partial u}{\partial t} + \nabla.\phi - Q = 0. \tag{20}$$

By Fourier's law of heat conduction (for 3D),

$$\phi = -K_0 \nabla u, \tag{21}$$

where K_0 is called the thermal diffusivity. Substitute (21) into (20) to obtain 3D Heat Equation

$$\frac{\partial u}{\partial t} = a\nabla^2 u + \frac{Q}{\alpha\rho}, \tag{22}$$

where $a = K_0/(\alpha\rho)$ and

$$\nabla^2 = \left(\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2}\right).$$

Normalizing with

$$\tilde{\boldsymbol{x}} = \frac{\boldsymbol{x}}{l}, \quad \tilde{\boldsymbol{t}} = \frac{a}{l^2}t,$$

to obtain a non-dimensional Heat Equation

$$\frac{\partial u}{\partial t} = \nabla^2 u + H, \tag{23}$$

where $H = l^2 Q/K_0$.

Construction of Solution of Heat and Wave Equations

We seek for the exact solution of (23) in the form of the generalized traveling-wave equations (3). Substitute (3) into (23) to obtain

$$h^2 U''_{yy} - U'_y + H = 0, \tag{24}$$

where $H = H(U)$ and $h \equiv 1$. Consequently, it can be deduced that (23) (for 1D Heat Equation) admits the generalized traveling-wave equations

$$u = U(y), \;\; y = t + x, \tag{25}$$

as its exact solution, where $U(y)$ is determined by

$$U''_{yy} - U'_y + H = 0.$$

Conclusion: Recently, exact solutions of RDEs and reaction–diffusion systems have attracted great attention. In this paper, exact solutions are presented for RDEs with delay and variable coefficients. The presence of arbitrary functions and free parameters in the solutions vouches for their feasible application in solving certain modeling problems such as diffusion of pollutants and population models, where the population is spatially distributed. It also makes the obtained solutions to be suitable for usage in testing the numerical and approximate analytical methods. The obtained results also find applications in finding the exact solutions of other forms of partial differential equations which are more complex. Our results are specifically applied as an illustration to obtain the solutions of certain essential modelling problems which are peculiar to metal forming processes.

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