Spacelike Causal Boundary at Finite Distance and Continuous Extension of the Metric: A Preliminary Report

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Abstract The causal boundary of a spacetime is examined in terms of a foliation by timelike curves—a preferred class of observers. In the present instance, the foliates are assumed to be future-incomplete. Conditions are given on the observers which imply that the Future Causal Boundary is purely spacelike, has the same topology as the foliation space, and fits into the future-completion in a particularly simple manner as boundary of a manifold-with-boundary. However, one of these conditions is not robust, as it fails for Interior Schwarzschild, the intended model. Next, assuming that a spacetime as above has the requisite properties identifying the Future Causal Boundary as desired—and assuming a geometric simplification, that the drift-form vanishes—conditions are given in terms of an integral condition on sectional curvature that allow for the metric to extend in a continuous manner, along each foliation, to the boundary: For Π a 2-plane containing the foliation velocity vector, it's required that the sectional curvature $K(\Pi)$ be monotone along each foliate and that $\sqrt{|K(\Pi)|}$ be integrable. But a further condition is required to have the metric be continuous within the boundary. This is a preliminary report, giving full explanations but very few proofs.

1 Introduction

The causal boundary construction has seen a fair bit of use in elucidating behavior at infinity for a range of classical spacetimes: standard static, especially with spherical symmetry, as in [4, 15]; beginning work in stationary spacetimes, as in [6]; multi-warped products, mentioned in [13] (as well as [1, 2]) and, especially spherically symmetric, in [15]; and some general results on quotients by group actions, as in

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[13]. This note presents preliminary results on causal boundary without any such symmetries or algebraically special situations.

The reason for using the causal boundary, instead of, for instance, conformal boundaries, is that the causal boundary construction is an intrinsic construction depending solely on the causal structure of the spacetime: not only does it have no explicitly ad hoc formulations, but it even has categorically universal properties with respect to causal structure—at least, when one considers solely the Future Causal Boundary, the FCB (see [10]). Imposing a topology is more complex, and there are competing notions for a good topology ([11, 5]—respectively, the Future Chronological Topology and what I like to call the Andalusian topology, introduced originally in [3]). But these proposed topologies agree in the case of a spacelike FCB—and, in that case, the Future Chronological Topology also has categorically universal properties, so that any reasonable candidate for a future boundary must be a topological quotient of the FCB in the Future Chronological Topology (see [11]).

This note presents the current state of a project to characterize FCB in very broad categories of spacetimes, with no assumptions of structure other than the existence of a "reasonable" class of observers, i.e., a foliation by timelike curves. The intent of the project is to find properties of the spacetime, observable by local groups of these observers, that result in identification of the Future Causal Boundary and how it sits within the future completion of the spacetime.

This initial stage of the project examines the case in which the FCB occurs "at finite distance", i.e., the observers are all assumed to have finite lifetimes. The intent is to replicate what happens in Interior Schwarzschild, where the FCB is $\mathbb{R}^1 \times \mathbb{S}^2$ (line cross a sphere), embedded in the future completion as $\{0\} \times \mathbb{R}^1 \times \mathbb{S}^2$ within $[0, 2m) \times \mathbb{R}^1 \times \mathbb{S}^2$; that is to say, with spacetime M having a foliation by unit-speed timelike curves $\{\gamma_q : (A, 0) \to M \mid q \in Q\}$, that the FCB of M, within the future completion, be realized as $\{0\} \times Q$ sitting within $(A, 0] \times Q$. Results will be presented as quasi-local conditions on M, observable by the given class of observers, that guarantee this result, with appropriate causal and topological properties. But these results are preliminary, in that one of them is not robust: it fails for Interior Schwarzschild.

As the causal boundary construction is inherently of low regularity (yielding a topology but not necessarily a differentiable structure, and with the future completion not necessarily being a manifold with boundary), it is also of interest to ask just how regular it might be in certain circumstances. Also appearing in this note are preliminary results on extension of the spacetime metric to the causal boundary in a C^0 manner—again, in terms of properties measurable by the preferred class of observers, particularly sectional curvature. This portion of the project is also still ongoing, and may be strengthened with additional research.

Section 2 presents the detailed definitions and nine conditions that guarantee the appropriate behavior for FCB of the spacetime. Section 3 presents sectional curvature properties that guarantee the continuous extension of the metric to the boundary (clearly not to be obeyed by Interior Schwarzschild).

2 Spacelike Causal Boundary at Finite Distance

The causal boundary of a strongly causal spacetime M was introduced in 1972 by Geroch, Kronheimer, and Penrose, [9], and it was brought to wide attention in [16]. As explored in [10] it consists of a Future Causal Boundary, a Past Causal Boundary, and a melding of those two to produce the complete causal boundary; or, more precisely, there is the future completion $\hat{M}$ of M, in which one identifies M and its Future Causal Boundary $\hat{\partial}(M)$; the past completion $\check{M}$ of M, comprising M and its Past Causal Boundary $\check{\partial}(M)$); and a melding of $\hat{M}$ with $\check{M}$, in which M is preserved, with the remainder being the complete causal boundary. But the current project concerns itself only with $\hat{M}$ and $\hat{\partial}(M)$.

The Future Causal Boundary $\hat{\partial}(M)$ consists of the TIPs (Terminal Indecomposable Past sets) of M: pasts of future-endless timelike curves. Then the future completion $\hat{M}$ of M consists, as a set, of $M \cup \hat{\partial}(M)$; this set must be endowed, first, with an extension of the chronology relation $\ll$ and, second, with a topology that respects that of M. The chronology relation is this: For $x \in M$ and $P, Q \in \hat{\partial}(M)$,

$$\begin{aligned} x \ll Q &\iff x \in Q \\ P \ll x &\iff \text{for some point } y \ll x,\ P \subset I^-(y) \\ P \ll Q &\iff \text{for some point } y \in Q,\ P \subset I^-(y) \end{aligned}$$

This yields a "future-complete" object $\hat{M}$, i.e., for any future chain c given by $x_1 \ll \cdots \ll x_n \ll x_{n+1} \ll \cdots$, there is some $x \in \hat{M}$ with with $I^-(x) = I^-[c]$.

We also want to specify an extension of the causal-past relationship $\prec$ in M:

$$\begin{aligned} x \prec Q &\iff I^-(x) \subset Q \\ P \prec x &\iff P \subset I^-(x) \\ P \prec Q &\iff P \subsetneq Q \end{aligned}$$

The topology that will be employed here is the Future Chronological Topology, as developed in [4, 11]. It is defined not through a definition of open set, but through the more primitive notion of defining limits of a sequence: Given any set X with a relation $\ll$ much like a spacetime's chronology relation, for any sequence of elements $\sigma = \{x_n\}$ in X, we define the future-limits $\hat{L}(\sigma)$ of that sequence as follows:

$$\begin{aligned} x \in \hat{L}(\sigma) \iff \; & \\ &\text{for all } y \ll x,\ y \ll x_n \text{ for sufficiently large } n,\ \text{and} \\ &I^-(x) \text{ is maximal among all IPs } P \text{ satisfying} \\ &\qquad \text{for all } y \in P,\ y \ll x_n \text{ for infinitely many } n \end{aligned}$$

where an IP in X is the past of any future chain (among the assumptions on $\ll$ in X is that $I^-(x)$ is always an IP). In slightly different words: $x \in \hat{L}(\sigma) \iff I^-(x)$ is contained in $\text{LI}(I^-(x_n))$ and is a maximal IP in $\text{LS}(I^-(x_n))$ ($\text{LI}(A_n)$ denotes points eventually in A_n, $\text{LS}(A_n)$ denotes points in infinitely many A_n). For any set $A \subset X$, let $\hat{L}[A] = \bigcup_{a\in A} \hat{L}(x)$. Then we define $A \subset X$ to be *closed in the Future Chronological Topology* if $\hat{L}[A] = A$ (constructively, the closure of a set A is by means of transfinite induction of the $\hat{L}$ operator: $\text{closure}(A) = \hat{L}^{\Omega}[A]$, where Ω is the first uncountable ordinal; this works because of a separability assumption on X and its chronology relation).

This topology has a number of good points: points are closed, X is dense in $\hat{X}$, and, while X need not be open in $\hat{X}$, the Future Chronological Topology on X coincides with the subspace topology of X in $\hat{X}$; and, most importantly, for M a strongly causal spacetime, the Future Chronological Topology on $\hat{M}$ yields the manifold topology on M (which is open in $\hat{M}$). Furthermore, in the case of spacelike-or-null boundaries (i.e, no chronology relations between boundary elements), the future completion, using the Future Chronological Topology, has a categorically universal property: passing to the future completion is left adjoint to the forgetful functor, forgetting about being future-complete.

The primary example to concentrate on is that of M being Interior Schwarzschild, $\mathbb{S}\text{ch}_{\text{Int}}$: $(0, 2m) \times \mathbb{R}^1 \times \mathbb{S}^2$ with metric $g = -\dfrac{1}{\frac{2m}{r} - 1}\, dr^2 + \left(\frac{2m}{r} - 1\right) dt^2 + r^2\, k_{\mathbb{S}^2}$ ($k_{\mathbb{S}^2}$ being the round metric on the unit 2-sphere). We have $\hat{M} = [0, 2m) \times \mathbb{R}^1 \times \mathbb{S}^2$ with $\hat{\partial}(M)$ occurring as $\{0\} \times \mathbb{R}^1 \times \mathbb{S}^2$, all in the natural topology; and $\hat{\partial}(M)$ is spacelike.

This section will be devoted to explicating a number of assumptions to make about a spacetime M that, together, will ensure that $\hat{M}$, the future completion of M, has (in the Future Chronological Topology) the topology of a manifold with boundary, $(A, 0] \times Q$, with M the interior and $\hat{\partial}(M)$ appearing as $\{0\} \times Q$, spacelike in the induced chronology relation. (Actually, it's slightly more general than that, amounting to an open subset of $(A, 0) \times Q$ for M, but with the same $\hat{\partial}(M)$.) All the assumptions except the last one, the ninth, are obeyed by $\mathbb{S}\text{ch}_{\text{Int}}$; the failure of the last assumption to hold for $\mathbb{S}\text{ch}_{\text{Int}}$ means that this is not yet a robust list of properties.

Assumption 1 Foliation by observers.

We assume that M comes equipped with a foliation $\mathcal{F}$ of future-directed timelike curves, each being a line (i.e., not a closed curve). Physically this means we are looking at a preferred class of observers (one which is at least topologically consistent with a causal spacetime). Let us take these to be unit-speed, as befits a field of observers.

We then let $Q = M/\mathcal{F}$ be the leaf-space with natural projection $\pi : M \to Q$ (i.e., the quotient by the equivalence relation of existing within the same foliate).

In $\mathbb{S}\text{ch}_{\text{Int}}$ this foliation is topologically the set of r-coordinate curves; but we want to parametrize them not by the coordinate r, but by arc-length. So for any $(t, p) \in \mathbb{R}^1 \times \mathbb{S}^2$ we define $\gamma_{t,p} : (-\pi m, 0) \to (0, 2m) \times \mathbb{R}^1 \times \mathbb{S}^2$ via $\gamma_{t,p}(s) =$

$(\rho^{-1}(s), t, p)$, where $\rho : (0, 2m) \to (-\pi m, 0)$ is the orientation-reversing function $\rho(r) = 2m\left(\arctan\sqrt{\frac{2m}{r} - 1} + \frac{r}{2m}\sqrt{\frac{2m}{r} - 1}\right) - \pi m$. For practical use, this is best written as $\gamma_{t,p}(\rho(r)) = (r, t, p)$.

Assumption 2 No ancestral pairs.

In any foliation of $\mathcal{F}$ by timelike curves in a spacetime M, we say two elements of the foliation (γ, γ') form an *ancestral pair* if for all $x \in \gamma$ and all $x' \in \gamma'$, we have $x \ll x'$. It is shown in [12], Theorem 1.2, that if $\mathcal{F}$ has no ancestral pairs, then $Q = M/\mathcal{F}$ is a Hausdorff manifold, and M is diffeomorphic to $\mathbb{R}^1 \times Q$ (we have the projection $\pi : M \to Q$ is a line bundle, and, Q being a manifold, there must be a smooth cross-section).

(Typically, ancestral pairs result from removing a compact set from a spacetime: If we remove the origin $(0, \mathbf{0})$ from Minkowski n-space $\mathbb{L}^n = \mathbb{L}^1 \times \mathbb{R}^{n-1}$, then the t coordinate curves $\{\gamma_{\mathbf{x}} \mid \mathbf{x} \in \mathbb{R}^{n-1}\}$ for $\mathbb{L}^n$ still form a foliation, except there are two of them, $\gamma_{\mathbf{0}^+}$ and $\gamma_{\mathbf{0}^-}$, where there used to be just $\gamma_{\mathbf{0}}$, and those two are respectively the $t > 0$ and $t < 0$ portion of $\gamma_{\mathbf{0}}$; $(\gamma_{\mathbf{0}^-}, \gamma_{\mathbf{0}^+})$ form an ancestral pair.)

So **we will assume in our spacetime M that the given class of observers has no ancestral pairs,** yielding a line bundle $\pi : M \to Q$ of smooth manifolds.

In $\mathbb{S}\mathrm{ch}_{\mathrm{Int}}$ the foliation $\mathcal{F} = \{\gamma_{t,p} \mid (t, p) \in \mathbb{R}^1 \times \mathbb{S}^2\}$ has no ancestral pairs, yielding $Q = \mathbb{R}^1 \times \mathbb{S}^2$.

Assumption 3 Foliates are future-incomplete.

Thus far, the assumptions made are extremely mild: Any reasonable spacetime will have a foliation by timelike curves, and asserting that there are no ancestral pairs pretty much just rules out "mutilated" spacetimes. But now we start to give some structure to our spacetime: **We assume that all the foliates $\gamma \in \mathcal{F}$ are future-incomplete.** Since we're already parametrizing them by arc-length, we might as well give them all the same finite future-endpoint on their domains: For each $q \in Q$, we will have γ_q defined on $(A_q, 0)$ for some $A_q < 0$, possibly $-\infty$ (we will typically label our foliates by Q). Thus, $\{\gamma_q : (A_q, 0) \to M\}$ are our unit-speed foliates.

Note that this gives us a preferred global time coordinate $\tau : M \to (-\infty, 0)$: For any $x \in M$, there is a unique $t \in (A_q, 0)$, with $q = \pi(x)$, so that $x = \gamma_q(t)$; then define $\tau(x) = t$. This gives us the global function $(\tau, \pi) : M \to \mathbb{R} \times Q$—but without further assumption, this need not be even continuous. Call τ the *observer time-function*.

In $\mathbb{S}\mathrm{ch}_{\mathrm{Int}}$, as noted above, we have $A_{t,p} = -\pi m$ for all (t, p). The observer time-function is given by $\tau(r, t, p) = \rho(r)$.

Assumption 4 τ is differentiable.

For any Riemannian manifold (N, h) and function $f : N \to \mathbb{R}$, we can define the spacetime $N_f = \{(t, x) \in \mathbb{R} \times N \mid t < f(x)\}$ as an open subset in $\mathbb{L}^1 \times N$ (i.e., metric $g = -(dt)^2 + h$). We have the foliation $\{\gamma_x \mid x \in N\}$ given by $\gamma_x : (-\infty, f(x)) \to N_f$ via $\gamma_x(s) = (s, x)$. But this works for any lower-semi-continuous function f; it need not even be continuous (consider $N = \mathbb{R}$ and $f(x) = 0$ for $x \leq 0$, $f(x) = 1$ for $x > 1$). We're not going to get the kind of boundary we want to work with unless we eliminate such eccentricities.

So **we will assume that** $\tau : M \to (-\infty, 0)$ **is differentiable—at least** C^1. This does quite a bit for us in terms of global analysis:

We will use T for the velocity vector field of the foliates, $T = \dot{\gamma}$; this is a future-directed unit-timelike field. Consider the 1-form $\alpha = -T^\flat$, i.e., $\alpha(X) = -\langle X, T\rangle$. For any vector $X \in TM$, let $X^\perp = X - (\alpha(X))T$; then $X^\perp$ lies in the perpendicular space to T; in particular, $X^\perp$ is spacelike. For any t, let $i^t : Q \to M$ be the cross-section of π given by $i^t(q) = \gamma_q(t)$ (actually, this is defined only for $Q^t = \{q \in Q \mid A_q < t\}$). Note that $i^t_*(T_q Q)$ is transverse to T at the point $x = \gamma_q(t)$; also note that i^t_* is an endomorphism. It follows that statements about tensors at x need be checked only on T and vectors of the form $i^t_*(A)$.

For any $q \in Q$ and any $t > A_q$, define the two-covariant tensor h^t at q by $h^t = i^{t*}(g + \alpha^2)$. It then follows that $g + \alpha^2 = \pi^* h^t$, as can be seen by evaluating both sides on (T, T), $(i^t_* A, T)$, and $(i^t_* A, i^t_* B)$ (making use of $\pi_* T = 0$ and $\pi \circ i^t = \text{id}$). Note that h^t is positive-definite: For any non-0 $A \in T_q Q$, $(i^t_* A)^\perp$ is a non-0 spacelike vector, and $|(i^t_* A)^\perp|^2 = \langle i^t_* A + \langle T, i^t_* A\rangle T, i^t_* A + \langle T, i^t_* A\rangle T\rangle = \langle i^t_* A, i^t_* A\rangle + \langle T, i^t_* A\rangle^2 = h^t(A, A)$. Thus, h^t amounts to a "time-dependent" Riemannian metric on Q (actually, h^t exists only on Q^t). We call this the *(time-dependent) observer-space metric*.

Also define the covector η^t at q by $\eta^t = i^{t*}\alpha$. We then have $\alpha - d\tau = \pi^*\eta^t$, again by evaluating both sides on T and on any $i^t_* A$ and using $i^{t*} d\tau = 0$. Then η^t is a time-dependent one-form on Q (again: η^t exists only on Q^t). We call this the *(time-dependent) drift-form*, in analogy with the drift-form on a stationary spacetime, foliated by the stationary observers (see [14]).

We thus have an expression for the spacetime metric as $g = -\alpha^2 + \pi^* h^t = -(d\tau + \pi^*\eta^t)^2 + \pi^* h^t$, valid at points $x \in M^t = \pi^{-1}(Q^t)$. More succinctly, we can write

$$g = -(d\tau + \pi^*\eta^\tau)^2 + \pi^* h^\tau$$

which is quite reminiscent of the way the spacetime metric in stationary spacetimes is expressed in [14]. Also note that (τ, π) provides us a diffeomorphism $M^t \cong (t, 0) \times Q^t$. This also gives us M diffeomorphic to $\mathbb{R} \times Q$ (same differentiability class as τ), but not by a "nice" map.

The time-dependent drift-form controls a lot of structure in the spacetime: If it is constant—$\eta' = 0$, where $\eta' = \frac{d}{dt}\eta^t$—then each foliate γ_q is geodesic (and if $h' = 0$ also, then the spacetime is stationary). If $\eta' = 0$ and $d\eta^t = 0$ for all t, then $T^\perp$ is integrable.

For $\mathbb{S}\text{ch}_{\text{Int}}$, we have $T = \sqrt{\frac{2m}{r} - 1}\,\frac{\partial}{\partial r}$, $\alpha = \frac{1}{\sqrt{\frac{2m}{r} - 1}}\,dr$, $\eta^\tau = 0$, and $h^\tau = \left(\frac{2m}{r} - 1\right) dt^2 + r^2 k_{\mathbb{S}^2}$ (recall $\tau(r, t, p) = \rho(r)$).

Assumption 5 Small drift-form.

In [14] it is shown that in a stationary-complete spacetime, the drift-form must have length no more than 1 for the spacetime to be strongly causal—and, in fact, there are considerable restrictions on the behavior of the drift-form if its length has a supremum of 1, if the spacetime is to be strongly causal. We will need to impose a uniform restriction on the time-dependent drift-form:

We will assume for some $a < 1$, $|\eta^t|_{h^t} \leq a$ **for all points** $q \in Q$ **and all** $t > A_q$; note that measurement of η^t is by means of h^t. (Technically, this is saying $|(\eta^t)^{\sharp^t}|_{h^t} \leq a$, where $\sharp^t : T^*_q Q \to T_q Q$ is the index-raising linear isomorphism that makes use of h^t.)

With this assumption, we can conclude M is strongly causal: For any future-directed causal curve σ, with $c = \pi \circ \sigma$ we have $(\tau \circ \sigma)' + \eta^{\tau\circ\sigma}(\dot{c}) \geq |\dot{c}|_{h^{\tau\circ\sigma}}$, so $\tau' \geq (1-a)|\dot{c}|_{h^\tau}$: We get that τ is strictly increasing along any future-directed causal curve, and that implies strong causality. (Consider a neighborhood U_n of any point x, consisting of $I^-(y_n) \cap I^+(x_n)$ for, say, $y_n = \gamma_q(t_0 + 1/n)$ and $x_n = \gamma_q(t_0 - 1/n)$, where $q = \pi(x)$ and $t_0 = \tau(x)$, and with n large enough that the metric doesn't change much over U_n; then any future-causal curve exiting U_n must do so on the past-null cone from y_n, and, if it re-enters U_n, must do so on the future-null cone from x_n, in violation of increasing τ.) As strong causality is necessary to define the Future Causal Boundary, this was a necessary step to obtain.

In $\mathbb{S}\mathrm{ch}_{\mathrm{Int}}$, $a = 0$ works.

Assumption 6 Bound on shrinkage under h^τ.

We are now in a position to start making assumptions that are specifically related to obtaining the topology and causal structure of $\hat{\partial}(M)$. The plan: Identify $\hat{\partial}(M)$ with Q by means of the IPs $\{P_q = I^-[\gamma_q] \mid q \in Q\}$, show that there are no $\ll$ or $\prec$ relations within this set, and establish the topology to be that of Q, with $\hat{\partial}(M)$ sitting appropriately within $\hat{M}$ to make that the obvious manifold with boundary: $\hat{M} = \{(t, q) \in (-\infty, 0] \times Q \mid t > A_q\}$.

Our first chore is an integral condition that guarantees all the P_q are distinct from one another—and, fortuitously, this also establishes the spacelike nature of $\{P_q \mid q \in Q\}$.

We establish a locally uniform bound on how quickly a given vector $X \in T_q Q$ can shrink in the time-dependent observer-space metric. Specifically: **We assume that for any non-vanishing vector field** X on Q, **each** $q \in Q$ **has a neighborhood** U **such that, for some** $t_0 < 0$,

$$\int_{t_0}^{0} \frac{dt}{\inf_{p\in U} |X_p|_{h^t}} < \infty$$

This guarantees, for all $q \neq q' \in Q$, that $P_q \not\subset P_{q'}$. Thus, all the IPs generated by the foliates are distinct from one another; and, furthermore, we do not have any instances of $P_q \ll P_{q'}$ or $P_q \prec P_{q'}$.

Furthermore: If we know that the $\{P_q\}$ are the only TIPs, then this information leads us to the desired topological conclusion: In the Future Chronological Topology,

$\hat{\partial}(M) \cong Q$ and sits in $\hat{M}$ as $\{0\} \times Q$ in $\{(t, q) \in (-\infty, 0] \times Q \mid t > A_q\}$. And this identification of both $\hat{\partial}(M)$ and $\hat{M}$ is not just by homeomorphism: It is by diffeomorphism of the same class of differentiability as τ.

In $\mathbb{S}\text{ch}_{\text{Int}}$, recall $Q = \mathbb{R}^1 \times \mathbb{S}^2$; let $P : Q \to \mathbb{S}^2$ be projection. For $X \in T_{t,p}Q$, we have $|X|_{h^s} = \sqrt{\left(\frac{2m}{r} - 1\right)((dt)X)^2 + r^2|P_*X|^2}$ (where $s = \tau(r, t, p) = \rho(r)$)—independent of the point (t, p) in Q—so the relevant integral ignores the inf and becomes $\displaystyle\int_{\rho(r_0)}^{0} \frac{ds}{\sqrt{\left(\frac{2m}{r} - 1\right)((dt)X)^2 + r^2|P_*X|^2}}$ with $s = \rho(r)$. Using $ds = \rho'(r)\,dr = -\frac{dr}{\sqrt{\frac{2m}{r}-1}}$, we obtain the following: For $(dt)X \neq 0$, the integral is approximately $\int_0^{r_0} \frac{1}{|(dt)X|} \frac{dr}{\frac{2m}{r}-1} \doteq \frac{1}{|(dt)X|} \frac{r_0^2}{4m}$; and for $(dt)X = 0$, the integral is approximately $\int_0^{r_0} \frac{1}{|P_*X|} \frac{dr}{r\sqrt{\frac{2m}{r}-1}} \doteq \frac{2}{|P_*X|} \sqrt{\frac{r_0}{2m}}$. So in either case, the integral is finite.

The remaining assumptions are all addressed to ruling out the existence of any other boundary IPs. To do this, we need to examine any future-directed endless null curve in M; we can accomplish this by looking at any future-directed null lift $\hat{c}$ of any curve c in Q. What we're looking for is

1. $\hat{c}$ approaches some $(0, q)$
2. and then $I^-[\hat{c}] = P_q$

It's that last that is the trickiest.

Assumption 7 Non-timelike past boundary.

For $c : [0, \omega) \to Q$ any curve, we want to make sure the future-null lift $(t(s), c(s))$ (specified by $t' = -\eta^t(\dot{c}) + |\dot{c}|_{h^t}$) isn't future-endless unless $t(\omega) = 0$. That is to say: A future-endless null curve shouldn't "run out of τ" (via $\tau \to A_q$) while approaching a point $q \in Q$. For otherwise, we would have the endpoint of $\hat{c}$ on the past causal boundary of M, instead of on $\{0\} \times Q$. In other words: We need that the past causal boundary $\check{\partial}(M)$ to be spacelike or null, but nowhere timelike; that way no future-directed null curve can meet it.

To that end, **we assume that for any** $q \in Q$, $\bigcap_{\tau<0} I^-(\gamma_q(\tau)) = \emptyset$. This prevents the problem above.

It might be objected, that if it's so simple to express the spacelike nature of a causal boundary component, why not just make a similar assumption for the Future Causal Boundary? The answer is, that we're assuming that while the Future Causal Boundary is somewhat mysterious or tricksy, the Past Causal Boundary of the spacetime at hand is well-understood, and that it's easy to make calculations such as that above.

In $\mathbb{S}\text{ch}_{\text{Int}}$, we have $(r', t', p') \ll (r, t, p)$ implies, among other things, that $r' < r$. Therefore, $\bigcap_{2m<r<0} I^-(r, t, p) = \emptyset$. Indeed, the Past Causal Boundary of $\mathbb{S}\text{ch}_{\text{Int}}$ can be identified with a portion of the Future Causal Boundary of Exterior Schwarzschild, forming the Event Horizon (see [4] or [15]).

Assumption 8 Uniform completeness of h^τ.

We also don't want $\hat{c}(s)$ having the τ coordinate complete its appointed journey—$\tau(\hat{c}(s)) = t(s) = 0$—while $\pi \circ \hat{c}$ fails to do so, i.e., we don't want $\hat{c}$ "running out of q", with $c(s)$ failing to approach some point in Q. So we must be concerned with an assumption concerning curves in Q which escape all compact sets. If we had a fixed metric on Q, we'd just assert that Q must be complete; then no curve $c : [0, \omega) \to Q$ going out to infinity could have a null lift that doesn't hit the $\tau = 0$ boundary somewhere in the interior of $[0, \omega)$, i.e, at some $s_0 < \omega$; $\hat{c}$ would approach $(0, q)$ for $q = c(s_0)$.

But we don't have that luxury of a fixed metric. What's needed is that h^t be complete for each t, but in uniform manner:

We assume that for any curve c in Q exiting all compact sets, there is some parametrization of c on $[0, \infty)$ and a not-necessarily-connected open interval I that is co-extensive with $[0, \infty)$, such that, picking some $s_0 \geq 0$, for all t, for all $s \in I$, $|\dot{c}(s)|_{h^t} \geq |\dot{c}(s_0)|_{h^t}$.

In $\mathbb{S}\text{ch}_{\text{Int}}$ we're looking at $c(s)=(t(s), p(s))$ with, say, $t(s) \to \infty$; we might as well take $t(s)=s$, for our initial parametrization. Then for any r, $|\dot{c}(s)|_{h^\tau} = \sqrt{(\frac{2m}{r} - 1)^2 + r^2|\dot{p}(s)|^2}$. If there is any sequence $\{s_i\}$ increasing monotonically to ∞ with $\{|\dot{p}(s_i)|\}$ non-decreasing, then we form our interval I around those $\{s_i\}$, using our initial parametrization. Failing that, we have $|\dot{p}(s)|$ strictly decreasing after some s_0. We change to a parametrization via $s(\sigma)$ with, say, $s(0) = s_0$; then we're looking for $\left((\frac{2m}{r} - 1)^2 + r^2|\dot{p}(s)|^2\right) s'(\sigma)^2 \geq \left((\frac{2m}{r} - 1)^2 + r^2|\dot{p}(s_0)|^2\right) s'(0)^2$, i.e., $s'(\sigma) \geq s'(0)\sqrt{\dfrac{1 + \left(\frac{r}{\frac{2m}{r}-1}\right)^2 |\dot{p}(s_0)|^2}{1 + \left(\frac{r}{\frac{2m}{r}-1}\right)^2 |\dot{p}(s(\sigma))|^2}}$. But the denominator is never smaller than 1, so it suffices if we just arrange for $s'(\sigma) \geq s'(0)\sqrt{1 + \left(\dfrac{r}{\frac{2m}{r} - 1}\right)^2 |\dot{p}(s_0)|^2}$. Note that the fraction under the radical is decreasing as $r \to 0$. So pick some $r_0 > 0$ and just take $s'(\sigma)$ to be constant at $s'(0)\sqrt{1 + \left(\dfrac{r_0}{\frac{2m}{r_0} - 1}\right)^2 |\dot{p}(s_0)|^2}$ for $\sigma \geq \sigma_0$ for some $\sigma_0 > 0$, and then take $I = [\sigma_0, \infty)$. (What is crucial here is that in the multiply warped product metric, the ratio of the warping factor for the compact factor to that for the non-compact factor, is decreasing.)

Assumption 9 Locally uniform bound on η', h' (not robust).

We now have enough to guarantee that for any endless future-null curve β in M, $\tau \circ \beta \to 0$ and for some $q \in Q, \pi \circ \beta \to q$. But we don't yet know that $I^-[\beta] = P_q$. The problem is that it's entirely possible for multiple null curves to "converge to $(0, q)$" but to have different pasts, properly including P_q. This is the trickiest issue to deal with.

I present here a condition that suffices; but it is clearly not robust, in that it is not obeyed by $\mathbb{S}\mathrm{ch}_{\mathrm{Int}}$.

In the following, η' and h' are to be understood as time-dependent, i.e., more properly written as η'^{τ} and h'^{τ}. We will measure them using h^{τ}. For the former, we'll consider $|\eta'|_{h^{\tau}}$. For the latter, we first raise an index using h^{τ}, i.e., define the (1,1)-tensor $h'^{\sharp}$ via $h^{\tau}(h'^{\sharp}X, Y) = h'(X, Y)$ (so $h'^{\sharp}$ is most accurately written as $h'^{\tau\sharp^{\tau}}$). We then consider the operator-norm on $h'^{\sharp}$: $||h'^{\sharp}||_{\tau} = \sup_{|X|_{h^{\tau}}=1} |h'^{\sharp}(X)|_{h^{\tau}}$.

We assume that for all $q \in Q$, **there is a neighborhood** U **of** q **so that for some** $t_0 > A_q$, $\int_{t_0}^{0} \sup_U \left(|\eta'|_{h^{\tau}}\right) d\tau < \infty$ **and** $\int_{t_0}^{0} \sup_U \left(||h'^{\sharp}||_{\tau}\right) d\tau < \infty$.

I conjecture that it suffices to put such locally uniform integral bounds, instead of on η^{τ} and $h^{\tau\sharp}$, on their h^{τ}-gradients (this is obeyed by $\mathbb{S}\mathrm{ch}_{\mathrm{Int}}$, since $\eta^{\tau} = 0$ anyway and h^{τ} is constant in each τ-slice); but the truth of this conjecture is far from clear.

In $\mathbb{S}\mathrm{ch}_{\mathrm{Int}}$ we have $h^{\tau(r,t,p)} = h^{\rho(r)} = \left(\frac{2m}{r} - 1\right) dt^2 + r^2 k_{\mathbb{S}^2}$, so $h' = \frac{d}{d\tau} h^{\tau} = \frac{dr}{d\tau}\frac{d}{dr} h^{\rho(r)} = -\sqrt{\frac{2m}{r} - 1}\left(-\frac{2m}{r^2} dt^2 + 2r\, k_{\mathbb{S}^2}\right)$, giving us $h'^{\sharp} = \frac{1}{\sqrt{\frac{2m}{r}-1}}\frac{2m}{r^2}\, dt \otimes \frac{\partial}{\partial t} - \frac{2}{r}\sqrt{\frac{2m}{r} - 1}\, \mathrm{id}_{\mathbb{S}^2}$, so $||h'^{\sharp}|| = \max\left\{\frac{1}{\sqrt{\frac{2m}{r}-1}}\frac{2m}{r^2}, \frac{2}{r}\sqrt{\frac{2m}{r} - 1}\right\} \doteq 2\sqrt{2m}\, r^{-\frac{3}{2}}$, which is not integrable on $(0, r_0)$.

Theorem 1 *Let M be a spacetime satisfying Assumptions 1–9, i.e., M is chronological with a foliation by timelike future-incomplete curves having no ancestral pairs, with the observer time-function τ differentiable, the drift-form bounded away from 1 in size, a locally uniform integral bound on shrinkage of vectors in Q (the leaf-space), a simply-behaving (non-timelike) past boundary, a uniform completeness condition on the observer-space metric h^{τ}, and a locally uniform integral bound on the size of both η' and $h'^{\sharp}$. Then*

1. *$\hat{M}$, the future completion of M, is, in the Future Chronological Topology, diffeomorphic to the subset of $(-\infty, 0] \times Q$ consisting of (t, q) with $t > A_q$ (where $-A_q$ is the length of the foliate corresponding to q), and*
2. *$\hat{\partial}(M)$, the Future Causal Boundary, appears there as $\{0\} \times Q$; it is spacelike.*

The diffeomorphism class is that of τ.

Why this is a delicate issue:

The problem is to find a property that can be observed by just a local class of observers, that will prevent a situation such as this: With $M = (-\infty, 0) \times \mathbb{R}^1$, we specify a conformal class of metrics by defining two classes of foliations by curves—the left-directed and right-directed past-null curves. The right-directed are the standard null lines: $\{\beta_a^+ \mid a \in \mathbb{R}\}$, $\beta_a^+ : (0, \infty) \to M$, $\beta_a^+(t) = (-t, a + t)$. The left-directed (all parametrized on $(0, \infty)$) are a portion of the standard null lines, $\{\beta_a^- \mid a \geq 0\}$, $\beta_a^-(t) = (-t, a - t)$; lines of different slope from the origin, $\{\delta_m \mid \frac{1}{2} < m < 1\}$, $\delta_m(t) = (-mt, -t)$; and slope-$\frac{1}{2}$ lines, $\{\hat{\beta}_a^- \mid a \leq 0\}$, $\hat{\beta}_a^-(t) = (-\frac{1}{2}t, a - t)$. The foliation by observers is the set of vertical lines: $\{\gamma_a \mid a \in \mathbb{R}\}$, $\gamma_a : (-\infty, 0) \to M$, $\gamma_a(t) = (t, a)$, so the leaf-space $Q = \mathbb{R}$. The various future-null lifts of the curve $c : (-\infty, 0) \to Q$, $c(s) = s$, (i.e., the various δ_m, parametrized future-wards) all end up at (0, 0), but each $I^-[\delta_m]$ properly contains $I^-[\gamma_0]$; indeed, FCB for M has two spacelike segments (pasts of the $\hat{\beta}_a^-$ and of the β_a^-) separated by a null segment (pasts of the δ_m).

The causal issue here is visible; the problem is expressing it geometrically or analytically, detectable locally, in a robust form. If one nails down this example with an explicit metric, then one finds η' misbehaving in a way that violates Assumption 9. But that is too strong; there ought to be something which prevents this example but allows $\mathbb{S}\text{ch}_{\text{Int}}$.

3 C^0 Extension of the Metric to the Causal Boundary

A number of papers in recent years have addressed the question of possible low-regularity extensions of spacetimes that are not C^2 extendible (see [7, 8]), with some particular focus on Interior Schwarzschild (such as [18, 19], where it is shown that there is no isometric embedding of $\mathbb{S}\text{ch}_{\text{Int}}$ into a spacetime with a C^0 metric, that actually extends $\mathbb{S}\text{ch}_{\text{Int}}$). We focus here on possible extension of the metric in the context of Sect. 2, first in the sense of C^0 extension of the spacetime metric to the Future Causal Boundary, purely along each foliate individually; and then, with somewhat less satisfactory answer, in the sense of C^0 extension of the metric to the Future Causal Boundary in a more complete manifold-with-boundary manner.

This is reminiscent of a Riemannian result on extension of metric to a putative point-singularity in [20]: Suppose M has a Riemannian metric g on $M - \{p\}$ ($p \in M$), such that the distance function defined by g on $M - \{p\}$ extends to M (i.e., p is at finite distance). Fix some (small) $s_0 > 0$. For any s, define the annulus $A_s = \{x \in M \mid s \leq d(p, x) \leq s_0\}$, and define $K(s) = \sup |K(X, Y)|$ where the supremum is over all non-dependent $X, Y \in T_x M$ for all $x \in A_s$. Then if the ball of radius s_0 around p is simply connected and has no geodesic loops with endpoints going to p, and if $\int_0^{s_0} sK(s)\, ds < \infty$, then the metric extends in a C^1 fashion to p. (Indeed, this theorem was the inspiration for the first conversations on a possible singularity-extension theorem in spacetimes, as referred to in the acknowledgements note.) Note that the simple take-away here is that for a non-extendible (i.e., actual) singularity, sectional curvature must blow up at least inverse-quadratically in distance from the putative singularity.

In light of Assumption 9 from Sect. 2 being non-robust, we will not employ that; instead, we will just assume that each the various $\{P_q\}$ constitute the entire Future Causal Boundary.

The theorems presented here also are weak in that they assume no drift-form: $\eta = 0$. The reason is that this greatly simplifies the calculations for sectional curvature. But this is not quite as strong a restriction as it appears at first glance: Once we know that $\hat{M}$ has the structure (roughly) of $(A, 0] \times Q$, then it is not much of a stretch to assume we can find a foliation of M by timelike geodesics, at least in a neighborhood of $\hat{\partial}(M)$ in $\hat{M}$—and that has $\eta' = 0$ (as $\pi_* \nabla_T T = \eta'^\sharp$ for $T = \dot{\gamma}_q$). Small $|\eta|$ with η constant along foliates may work well for extending these results.

So we will be assuming a foliation of spacetime M along the lines of Sect. 2, Assumptions 1–8, with the additional assumption that there are no "confusing" TIPs masquerading as P_q; we know any future-endless null curve β "approaches" some $(0, q)$, and so we are assuming that $I^-[\beta] = P_q$. Call this suite of assumptions our *Future-Incomplete Spacelike Boundary Ansatz*.

We already know $\mathbb{S}\mathrm{ch}_{\mathrm{Int}}$ satisfies Assumptions 1–8. It fails Assumption 9, but it none the less satisfies the Future-Incomplete Spacelike Boundary Ansatz, as can be seen from its structure as a multiply warped product: As shown in [13], Proposition 3.5, if $M = (a, b) \times_{f_1} N_1 \times \cdots \times_{f_m} N_m$ (i.e., metric is $-(dt)^2 + f_1(t)h_1 + \cdots + f_m(t)h_m$), then $\hat{\partial}(M)$ is spacelike (and all the TIPs are of the form analogous to P_q) iff for all i with N_i non-compact, h_i is complete and $\int_{b^-}^{b} f_i^{-\frac{1}{2}} < \infty$. Employing $\tau = \rho(r)$ as the t-coordinate, we have $\mathbb{S}\mathrm{ch}_{\mathrm{Int}}$ as $(-\pi m, 0) \times_{f_1} \mathbb{R}^1 \times_{f_2} \mathbb{S}^2$, so we just need to check the f_1 integral: $\int_{\tau_0}^{0} \left(\frac{2m}{r} - 1\right)^{-\frac{1}{2}} d\tau = \int_{r_0}^{0} \left(\frac{2m}{r} - 1\right)^{-\frac{1}{2}} \left(-\left(\frac{2m}{r} - 1\right)^{-\frac{1}{2}}\right) dr = 2m \ln \frac{2m}{2m - r_0} - r_0 < \infty$.

The intent here is to find conditions explicable in terms of sectional curvature in the spacetime, that result in C^0 extension of the metric along a foliate—specifically, sectional curvature of planes containing T; this amounts to the tidal accelerations measured by each observer in its rest-space, in various directions, so it is an inherently physical observation, not just a geometric quantity.

The simplest sort of condition to assume is an integrability condition on all planes containing T. But we can weaken this to an assumption on specific planes, if we are able to make a modest assumption on the eigenvectors of h': Note that h' is self-adjoint (with respect to h^τ), so it has a complete set of eigenvectors; let μ be the eigenvalue of h' with maximum absolute value, and let E be some non-0 choice within the eigenspace of μ; call this the "distinguished" eigenvector. Generically, the eigenspaces are 1-dimensional, and it is at least not uncommon for the line-bundle of this particular eigenspace to have a limit as $\tau \to 0$ (though one can certainly create counter-examples). We don't need to assume anything about the dimensionality of the μ eigenspace, but it will be handy if there is a choice of E_q^τ so that E_q^τ converges to some E_q^0 in a manner which is uniform in q. Call this the *Convergent Distinguished Eigenvector Condition*.

In $\mathbb{S}\mathrm{ch}_{\mathrm{Int}}$, we have $h^\tau = \left(\frac{2m}{r} - 1\right)(dt)^2 + r^2 k_{\mathbb{S}^2}$ for $\tau(r, t, p) = \rho(r)$, and $\frac{d\tau}{dr} = -\frac{1}{\sqrt{\frac{2m}{r} - 1}}$; this gives us $h^{\tau\prime} = -\sqrt{\frac{2m}{r} - 1}\left(-\frac{2m}{r^2}(dt)^2 + 2r\, k_{\mathbb{S}^2}\right)$, which yields $h^{\tau\prime\sharp} = \frac{2m}{r^2} \frac{1}{\sqrt{\frac{2m}{r} - 1}}\, dt \otimes \frac{\partial}{\partial t} - \frac{2}{r}\sqrt{\frac{2m}{r} - 1}\, \mathrm{Id}_{\mathbb{S}^2}$. Both terms go to ∞ like $r^{-\frac{3}{2}}$ as $r \to 0$; for $r < \frac{2m}{m+1}$, the second one is larger in absolute value. So we have the eigenspace with the largest absolute value of eigenvalue is (eventually) $T_p\mathbb{S}^2$. We are free to choose any non-0 vector $U_p \in T_p\mathbb{S}^2$ as $E_{t,p}^\tau$ for all τ, doing so in a manner continuous in p, yielding that we have the Convergent Distinguished Eigenvector Condition satisfied.

We use $K(A, B)$ for the sectional curvature of span$\{A, B\}$. For any $X \in TQ$, let $\bar{X}^t = i^{t*}X$.

First we look at a simple curvature condition, bounding the relevant curvatures by a constant for each foliate:

Theorem 2 *Let M be a spacetime obeying the Future-Incomplete Spacelike Boundary Ansatz (for a foliation $\mathcal{F} = \{\gamma_q \mid q \in Q\}$) with drift-form $\eta = 0$. If either*

1. *for all $q \in Q$, there is some B_q such that for all τ, for any $X \in T_q$, $K(T, \bar{X}^\tau) \leq B_q$, or*
2. *M obeys the Convergent Distinguished Eigenvector Condition and for all q, there is some B_q such that for all τ, $K(T, \bar{E}_q^\tau) \leq B_q$,*

then for all q, the observer-space metric h_q^τ has a limit tensor $h_q^0 = \lim_{\tau \to 0} h_q^\tau$, and h_q^0 is a Riemannian metric on $T_q Q$.

Thus, we have a continuous extension of the spacetime metric $g = -(d\tau)^2 + \pi^* h^\tau$, individually along each foliate τ, to the Future Causal Boundary.

But what's really wanted is a more flexible condition on curvature—an integral condition. This, it turns out, cannot be done without an additional assumption, such as monotonicity (counter-examples exist without this):

Theorem 3 *Let M be a spacetime obeying the Future-Incomplete Spacelike Boundary Ansatz (for a foliation $\mathcal{F} = \{\gamma_q \mid q \in Q\}$) with drift-form $\eta = 0$. If either*

1. *for all $q \in Q$, for any $X \in T_q Q$, for some $\tau_0 > A_q$, $\int_{\tau_0}^0 \sqrt{|K(T, \bar{X}_q^\tau)|}\, d\tau < \infty$, or*
2. *M obeys the Convergent Distinguished Eigenvector Condition and for all q, $K(T, \bar{E}_q^\tau)$ is monotonic and for some $\tau_0 > A_q$, $\int_{\tau_0}^0 \sqrt{|K(T, \bar{E}_q^\tau)|}\, d\tau < \infty$,*

then for all q, the observer-space metric h_q^τ has a limit tensor $h_q^0 = \lim_{\tau \to 0} h_q^\tau$, and h_q^0 is a Riemannian metric on $T_q Q$.

Thus, we have a continuous extension of the spacetime metric $g = -(d\tau)^2 + \pi^* h^\tau$, individually along each foliate τ, to the Future Causal Boundary.

It is worth noting that the main take-away here is that for a non-extendible (i.e., actual) singularity, the sectional curvature of relevant planes—if monotonic—must blow up at least inverse-quadratically in Lorentzian distance from the putative singularity.

In $\mathbb{S}\text{ch}_{\text{Int}}$ we have, for any $U \in T\mathbb{S}^2$, $K(\frac{\partial}{\partial r}, \bar{U}) = -\frac{m}{r^3}$ (see, for instance, [17], Proposition 13.5(2)). This gives us $\int_{\tau_0}^0 \sqrt{|K(T, \bar{E})|}\, d\tau = \int_0^{r_0} \frac{\sqrt{m}}{r^{3/2}} \frac{1}{\sqrt{\frac{2m}{r} - 1}}\, dr = \int_0^{r_0} \frac{\sqrt{m}}{r\sqrt{2m - r}}\, dr = \infty$. In a sense, this is "why" the metric in $\mathbb{S}\text{ch}_{\text{Int}}$ cannot be extended even C^0 to the boundary.

These theorems are weak in that we do not obtain actual continuity of the tensor field h^0 on Q. That requires control on the Q-derivatives of h'. The following condition suffices, though it is less than satisfactory in that it is not expressed in terms of readily observable phenomena.

Let ∇^τ denote the Levi-Civita connection on Q using h^τ. Note that for any vector field X on Q, while ∇^τ_X is a derivative operator and not tensorial, $(\nabla^\tau_X)' = \frac{d}{d\tau}\nabla^\tau_X$ is a (1,1)-tensor field on Q (difference of two connections being tensorial). Let $||(\nabla^\tau_X)'||_\tau$ denote the operator-norm of $(\nabla^\tau_X)'$ with respect to h^τ.

In $\mathbb{S}\mathrm{ch}_{\mathrm{Int}}$, we have $h^\tau = \left(\frac{2m}{r} - 1\right)(dt)^2 + r^2\, k_{\mathbb{S}^2}$, which has very simple covariant derivative: $\nabla^\tau_{\partial_t} = 0$ and for all $U \in T\mathbb{S}^2$, $\nabla^\tau_U = \nabla^{k_{\mathbb{S}^2}}_U \circ P$, where $P : \mathbb{R}^1 \times \mathbb{S}^2 \to \mathbb{S}^2$ is projection. There is no τ-dependence at all, so for any $X \in TQ$, $(\nabla^\tau_X)' = 0$.

For any function f on M, we will say f is *locally uniformly integrable* if for all $q \in Q$, there is a neighborhood U of q and an integrable function $\alpha : [t_0, 0) \to \mathbb{R}$ (with $t_0 > A_p$ for all $p \in U$) such that for all $p \in U$, $f(\gamma_p(t)) \leq \alpha(p)$.

Theorem 4 *Let M be a spacetime obeying the Future-Incomplete Spacelike Boundary Ansatz (for a foliation $\mathcal{F} = \{\gamma_q \mid q \in Q\}$) with drift-form $\eta = 0$. If*

1. either

   (a) (a) for any vector field X on Q, the function $\sqrt{|K(T, \bar{X}^\tau)|}$ is locally uniformly integrable, or

   (b) (b) M obeys the Convergent Distinguished Eigenvector Condition; for all q, $K(T, \bar{E}^\tau_q)$ is monotonic; and the function $\sqrt{|K(T, \bar{E}^\tau)|}$ is locally uniformly integrable,

2. and also for any vector field X on Q, $||(\nabla^\tau_X)'||_\tau$ is locally uniformly integrable,

then h^τ converges in a C^0 manner in Q to a Riemannian metric h^0. Thus, g has a C^0 extension to $\hat{M}$.

Control over $||(\nabla^\tau_X)'||_\tau$ is elusive in terms of quantities readily accessible to observers, such as curvature.

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