Null Hypersurfaces and the Rigged Metric

Manuel Gutiérrez and Benjamín Olea

Abstract Given a null hypersurface in a Lorentzian manifold we can construct a Riemannian metric on it called the rigged metric. This is not a canonical construction because it depends on the choice of a rigging, that is, a vector field transverse to the null hypersurface, but it can be used as an auxiliary tool which allows us to apply Riemannian techniques on null hypersurfaces. We show two such applications: in the first one the rigged metric is used to obtain conditions for a totally umbilic null hypersurface to be contained in a null cone. In the second one it is used to ensure that a codimension two spacelike submanifold through a null hypersurface is a leaf of the (integrable) screen distribution.

Keywords Null hypersurface · Rigging technique · Rigged metric · Null cone · Codimension two spacelike submanifold · Maximum principle

1 Introduction

Given a hypersurface L of a n-dimensional Lorentzian manifold (M, g), we call $Rad(T_xL) = T_xL \cap T_xL^{\perp}$ for all $x \in L$. Since the signature of g is one, we have that $0 \leq \dim Rad(T_xL) \leq 1$. If L is a timelike or spacelike hypersurface, then it is clear that $\dim Rad(T_xL) = 0$ for all $x \in L$. In the case $\dim Rad(T_xL) = 1$ for all $x \in L$ it is said that L is a null hypersurface.

A null hypersurface has a unique null direction, given by $Rad(TL)$, which is orthogonal to the whole null hypersurface itself. Moreover, it does not contain timelike directions and it is easy to check that the leaves of the one-dimensional foliation given by $Rad(TL)$ are locally null geodesics. Being foliated by null geodesic is not enough to characterize a null hypersurface, [12, Theorem 1]. For example, a timelike

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plane in the Minkowski space is foliated by null geodesics and obviously it is not a null hypersurface.

The biggest problem that we have to face when dealing with null hypersurfaces is that the induced metric is degenerate. To overcome it we introduced the following idea.

Definition 1 ([8]) A rigging for a null hypersurface L is a vector field ζ defined in some open neighbourhood of L such that $\zeta_x \notin T_x L$ for all $x \in L$. If ζ is defined only on L, then we call it a restricted rigging.

Observe that a timelike vector field in M is a rigging for any null hypersurface. Therefore, locally it always exists a rigging for a null hypersurface, but its global existence is not guaranteed. The reason is that the existence of a rigging implies the existence of a globally defined null section in the null hypersurface $\xi \in \Gamma(Rad(TL))$ normalized by the condition $g(\zeta, \xi) = 1$. It is called the rigged vector field associated to ζ and it allows us to decompose

$$T_x L = \mathcal{S}_x \oplus span(\xi_x) \tag{1}$$

for all $x \in L$, where $\mathcal{S}$ is a spacelike foliation in L called screen distribution and it is defined by $\mathcal{S} = TL \cap \zeta^{\perp}$.

The following example shows that the existence of the rigged vector field is a non-trivial restriction.

Example 1 Consider the Minkowski space $(\mathbb{L}^3, g) = (\mathbb{R}^3, dx^2 + dy^2 + dydz)$ and call M the quotient by the isometry group generated by

$$\Phi(x, y, z) = (x - 1, -y, -z).$$

We can induce a Lorentzian metric on M and the projection of the plane $y = 0$ is a null hypersurface diffeomorphic to a Möbius band. It does not exist a globally defined null section $\xi \in \mathfrak{X}(L)$, so it does not admit a rigging.

On the other hand, the existence of a screen distribution and a null section is equivalent to the existence of a restricted rigging [5, Theorem 1.1, pg. 79] in the sense that there exists a restricted rigging which induces them.

If we call $N = \zeta - \frac{1}{2} g(\zeta, \zeta)\xi$, which is the unique null vector field transverse to L, orthogonal to the screen distribution and normalized by $g(N, \xi) = 1$, then we have the decomposition

$$T_x M = T_x L \oplus span(N_x) \tag{2}$$

for all $x \in L$. The induced connection ∇^L is the projection onto TL according to this decomposition of $\nabla_U V$ for all $U, V \in \mathfrak{X}(L)$. It is a symmetric connection on L which in general is not compatible with the metric g.

The null second fundamental form of L is defined by $B(U, V) = -g(\nabla_U \xi, V)$ for all $U, V \in \mathfrak{X}(L)$ and the local second fundamental form as $C(U, X) = -g(\nabla_U N, X)$ for all $U \in \mathfrak{X}(L)$ and $X \in \Gamma(\mathcal{S})$. B is symmetric, but C is symmetric if and only if the screen distribution is integrable. We define the null mean curvature H and the screen mean curvature Ω as

$$H_x = \sum_{i=3}^{n} B(e_i, e_i),$$
$$\Omega_x = \sum_{i=3}^{n} C(e_i, e_i),$$

being $\{e_3, \ldots, e_n\}$ an orthonormal basis of $\mathcal{S}_x$.

The third fundamental tensor of a null hypersurface is the so-called rotation one-form, given by $\tau(U) = g(\nabla_U \zeta, \xi)$ for all $U \in \mathfrak{X}(L)$. If we call $A^* : TL \to \mathcal{S}$ characterized by $B(U, V) = g(A^*(U), V)$ for all $U, V \in \mathfrak{X}(L)$, then $A^*(\xi) = 0$ and

$$\nabla_U \xi = -\tau(U)\xi - A^*(U),$$

so ξ is a pre-geodesic vector field.

The tensors B, C and τ depend on the chosen rigging, but in a predictable way, [15]. The case of the tensor B is specially favourable. Indeed, if ζ' is another rigging, then the respective rigged vector fields are proportional, $\xi' = f\xi$ where $f = \frac{1}{g(\zeta', \xi)}$, and thus $B' = fB$ and $H' = fH$.

A null hypersurface is totally umbilical if the null second fundamental form is proportional to the metric, $B = \rho g$ for some $\rho \in C^\infty(L)$, and it is totally geodesic if $B = 0$. These definitions are independent of any choice due to the fact commented above.

We say that a rigging is screen conformal if $C = \varphi B$ for some $\varphi \in C^\infty(L)$ and we say that the screen distribution is totally umbilic if $C = \frac{\Omega}{n-2} g$. In both cases, the screen distribution is integrable. On the other hand, we say that the rigging is distinguished if the rotation one-form vanishes. Most of the important examples of null hypersurfaces admit a screen conformal and distinguished rigging.

A rigging also induces a Riemann metric on L given by $\widetilde{g} = g + \omega \otimes \omega$, being $\omega = i^*\alpha$, α the g-metrically equivalent one-form to ζ and $i : L \to M$ the canonical inclusion. With this metric, ξ is unitary and orthogonal to the screen distribution $\mathcal{S}$. The one-form ω is called the rigged one-form.

We have that ω is closed if and only if the screen distribution is integrable and ξ is $\widetilde{g}$-geodesic. Moreover, we have the following.

Lemma 1 ([11]) *Let L be a null hypersurface and ζ a rigging for it.*

1. *If ζ is screen conformal and distinguished, then $d\omega = 0$.*
2. *If the screen is totally umbilical and ζ is distinguished, then $d\omega = 0$.*
3. *If ζ is a conformal vector field and screen conformal, then it is distinguished. Moreover, if the conformal factor of ζ never vanishes, then L is totally umbilical*

If the null hypersurface is totally umbilical and ω is closed, then the rigged metric can be decomposed.

Theorem 1 ([8, 10]) *Let (M, g) be a Lorentzian manifold and L a totally umbilical null hypersurface. Suppose that ζ is a restricted rigging for L such that its rigged one-form ω is closed. Take S a leaf of $\mathcal{S}$ and $\Phi : \mathcal{A} \to L$ the flow of ξ. If Φ restricted to $(a, b) \times \Sigma \subset \mathcal{A}$, where $(a, b) \subset \mathbb{R}$ and $\Sigma \subset S$, is injective, then*

$$\Phi : \left((a, b) \times \Sigma, dt^2 + \lambda(t, x)^2 g|_S\right) \to (L, \widetilde{g}),$$

being $\lambda(t, x) = \exp\left(-\int_0^t \frac{H(\Phi_s(x))}{n-2} ds\right)$, is an isometric embedding.

2 Characterization of a Null Cone

An important family of null hypersurfaces in a Lorentzian manifold are the local null cones. They are defined as

$$C^l_{e_0} = \{\exp_p(u) : u \in T_pM \cap \exp_p^{-1}(\theta) \text{ is null and } g(u, e_0) < 0\},$$

where $e_0 \in T_pM$ is a fixed timelike vector and θ is a normal neighbourhood of p. On the other hand, we define the null cone of e_0 with vertex p as

$$\mathcal{C}_{e_0} = \{\exp_p(u) : u \in T_pM \cap \Theta \text{ is null and } g(u, e_0) < 0\},$$

where Θ is the maximal definition domain of the exponential map at p. Null cones are immersed null hypersurfaces except at conjugate points, see [9] and references therein. In Robertson-Walker spaces (in particular in constant curvature Lorentzian manifolds) the local null cones are totally umbilical null hypersurfaces. The converse holds in some situations.

Theorem 2 ([1, 7]) *A totally umbilical (non-totally geodesic) null hypersurface in a constant curvature Lorentzian manifold is contained in a null cone.*

Theorem 3 ([7]) *Any totally umbilic null hypersurface in a Robertson-Walker space $I \times_f \mathbb{S}^{n-1}$ $(n > 3)$ with*

$$\int_I \frac{1}{f(r)} dr > \pi \qquad (3)$$

is an open set of a null cone. In particular, it does not exist totally geodesic null hypersurfaces.

In a constant curvature space we have $Ric(u, u) = 0$ for all null vector u. This curvature condition is basically enough to ensure that a totally umbilical null hypersurface is contained in a null cone in an arbitrary Lorentzian manifold. Indeed, we have the following.

Theorem 4 ([10]) *Let (M, g) be a null geodesically complete Lorentzian manifold with dimension $n > 3$. Take L a totally umbilic null hypersurface satisfying the following properties.*

1. *It has never vanishing null mean curvature.*
2. $Ric(u, u) = 0$ *for all null vector* $u \in TL$.
3. *It is strongly inextensible.*

Then, L is contained in a null cone.

A null hypersurface is called strongly inextensible if it is not properly contained in any immersed connected null hypersurface. For example, a degenerate hyperplane and a null cone in the Minkowski space are strongly inextensible. This is a topological necessary condition to get the conclusion of the theorem because if a totally umbilical null hypersurface can be extended maybe we loss the umbilicity condition, which is essential to prove that it is contained in a null cone.

In [10] there is an example showing this behavior. To get it we perturb the euclidean metric inside a ball $B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ to obtain a complete Riemann surface $(\mathbb{R}^2, g_0)$ such that the straight lines $y = \pm x$ are pregeodesics. Moreover, we can construct g_0 so that the time to reach the point $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ from the origin when we parametrize $y = x$ by the arc length and the time to reach $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ from the origin when we also parametrize $y = -x$ by the arc length are different.

Now, in the Lorentzian manifold $\left(\mathbb{R} \times \mathbb{R}^2, -dt^2 + g_0\right)$ we have that $L = \{(t, x, y) \in \mathbb{R}^3 : (t + 1)^2 = x^2 + y^2, t > 0\}$ is a totally umbilical null hypersurface with $Ric(u, u) = 0$ for all null vector $u \in TL$, since out of the tube $\mathbb{R} \times B$ the metric coincides with the Minkowski metric. Obviously, L is not strongly inextensible and it is not contained in a null cone because there are null geodesics in L (constructed from $y = \pm x$) which do not intersect themselves at any point.

On the other hand, the null geodesically completeness is also necessary. The easier example is a null cone with the vertex at the origin of the Minkowski space. If we remove the origin, then it is not contained in any null cone since we have removed the vertex.

We can even give an example where the theorem does not holds if we suppose the inextesibility of the ambient space instead of the null geodesically completeness. For this, it is enough to consider the Kruskal space $Q \times_r \mathbb{S}^2$, [16]. The null hypersurface $L = \{(u, v, x) \in Q \times \mathbb{S}^2 : u = u_0\}$ for a fixed $u_0 \in \mathbb{R} - \{0\}$ is totally umbilical with never vanishing null mean curvature but it is not contained in a null cone.

The key to prove the Theorem 4 is the Gauss-Codazzi equation for a null hypersurface given by

$$g(R_{UV}W, \xi) = \left(\nabla^L_U B\right)(V, W) - \left(\nabla^L_V B\right)(U, W) + \tau(U)B(V, W) - \tau(V)B(U, W)$$

for all $U, V, W \in \mathfrak{X}(L)$. If L is totally umbilical ($B = \rho g$), then contracting this equation we get

$$\frac{1}{n-2}Ric(\xi,\xi) = \xi(\rho) - \rho^2 + \tau(\xi)\rho. \tag{4}$$

In order to simplify Eq. (4) it is convenient to deal with a geodesic rigged vector field, i.e. $\tau(\xi) = 0$. Its existence is guaranteed if there exists a conformal rigging and in [12, Theorem 18] its existence is also ensured under a strong topological assumption. On the other hand, we can always construct locally a geodesic rigging. Even more, given an injective null geodesic $\gamma : [0, b) \to L$ there is a open set $V \subset L$ containing $\gamma([0, b))$ and a geodesic rigged $\xi \in \mathfrak{X}(V)$ with $\xi_{\gamma(t)} = \gamma'(t)$.

For our purpose we need a refined version of the above claim, since we want to apply Theorem 1 and for this we need an open set of the form $(0, b) \times \Sigma$ contained in V. To state it, first we need to introduce a function measuring the maximal definition interval of the integral curves of a geodesic rigged. Concretely, suppose that $U \subset L$ is an open neighbourhood and $\xi \in \mathfrak{X}(U)$ is a geodesic rigged. We call

$$b(x) = sup\{t > 0 : \gamma([0, t)) \subset L, \gamma \text{ is the integral curve of } \xi \text{ with } \gamma(0) = x\}.$$

Proposition 1 *Let L be a strongly inextensible totally umbilical null hypersurface of a geodesically null complete Lorentzian manifold with never vanishing null mean curvature. Suppose that $Ric(u, u) = 0$ for all null vector $u \in TL$ and take $\xi \in \mathfrak{X}(U)$ a geodesic rigged with positive associated null mean curvature, being $U \subset L$ an open subset.*

1. *$b(x)$ is finite for all $x \in U$ and it defines a positive smooth function on U.*
2. *If we fix $x_0 \in U$ and we call $\gamma : [0, b(x_0)) \to L$ the null geodesic with $\gamma(0) = x_0$ and $\gamma'(0) = \xi_{x_0}$, then there exists an open set $V \subset L$ such that:*
   - *$\gamma([0, b(x_0)) \subset V$*
   - *ξ can be extended (as a geodesic null section) to V and $\gamma'(t) = \xi_{\gamma(t)}$ for all $t \in [0, b(x_0))$.*
   - *If $\alpha : [0, b(x)) \to L$ is a null geodesic with $\alpha(0) = x \in V$ and $\alpha'(0) = \xi_x$, then $\alpha([0, b(x)) \subset V$.*

In fact, we can not only say that $b(x)$ is finite, but we can give an explicit expression for it. Take $x_0 \in L$ and V a neighbourhood as in Proposition 1 and $\xi \in \mathfrak{X}(V)$ a geodesic rigged. The flow of ξ is given by $\Phi(t, x) = \exp_x(t\xi_x)$ and if we integrate Eq. (4), then we get

$$\rho(\Phi(t, x)) = \frac{1}{\frac{1}{\rho(x)} - t} \tag{5}$$

for all $0 \le t \le b(x)$. Since $\lim_{t \to \frac{1}{\rho(x)}^-} \rho(\Phi(t, x)) = \infty$, then $b(x) \le \frac{1}{\rho(x)}$. We can see that necessarily $b(x) = \frac{1}{\rho(x)}$ using the following lemma.

Lemma 2 *Let (M, g) be a geodesically null complete Lorentzian manifold and L a strongly inextensible and totally umbilic null hypersurface. Suppose that $Ric(u, u) =$*

0 *for all null vector* $u \in TL$, $\gamma : [0, b) \to L$, $0 < b < \infty$, *is a null geodesic such that* $\gamma(b) \notin L$ *and take a geodesic null section* ξ *as above. Then the associated null mean curvature holds*

$$\lim_{t \to b^-} \rho(\gamma(t)) = \infty.$$

Proof Take $\Sigma \subset V$ a spacelike hypersurface in L through $x_0 = \gamma(0)$ and consider $\Phi : \mathbb{R} \times \Sigma \to M$ given by $\Phi(t, x) = \exp_x(t\xi_x)$. We have $\Phi(t, x_0) = \gamma(t)$ for all $t \in [0, b)$. Observe that (b, x_0) is a singular point of Φ, otherwise L would not be strongly inextensible since $\gamma(b) \notin L$. Thus, there is $w \in T_{(b,x_0)}(\mathbb{R} \times \Sigma)$, $w \neq 0$, such that $\Phi_{*(b,x_0)}(w) = 0$. If we take $(t(s), x(s))$ a curve in $\mathbb{R} \times \Sigma$ such that $w = t'(0)\partial_t + x'(0)$ then

$$\Phi_{*(b,x_0)}\big(x'(0)\big) = -t'(0)\gamma'(b).$$

Now, take the geodesic variation $X : [0, \infty) \times (-\varepsilon, \varepsilon) \to M$ given by $X(t, s) = \Phi(t, x(s))$. The Jacobi vector field $J(t) = X_s(t, 0)$ holds $J(0) = x'(0)$, $J(b) = \Phi_{*(b,x_0)}(x'(0)) = -t'(0)\gamma'(b)$ and $J'(t) = \nabla_{J(t)}\xi$ for all $t < b$. If we call $f(t) = g(J, J)$, then $f(0) > 0$, $f(b) = 0$ and

$$f'(t) = 2g(\nabla_{J(t)}\xi, J(t)) = -2B(J(t), J(t)) = -2\rho(\gamma(t))f(t).$$

Therefore,

$$f(t) = f(0)e^{-2\int_0^t \rho(\gamma(t))dt}$$

for $t < b$. Since $f(b) = 0$ we have $\int_0^b \rho(\gamma(t))dt = \infty$, but from Eq. (4) we have that $\rho(\gamma(t))$ is increasing, so $\lim_{t \to b^-} \rho(\gamma(t)) = \infty$.

Now, in order to prove Theorem 4, we only have to choose a suitable screen distribution and apply Theorem 1 to ensure that the null geodesic of the null hypersurface are getting closer.

For this, fix $x_0 \in L$ and $\xi \in \mathfrak{X}(V)$ a rigged as in Proposition 1. Since we are supposing that ρ never vanishes, from Eq. (4) we have that $\xi(\rho) \neq 0$ and so the level sets of ρ form a well-defined screen distribution $\mathcal{S}$ in V.

Take S the leaf of $\mathcal{S}$ through x_0. An integral curve α of ξ with $\alpha(0) = x \in S$ is defined, by construction, in the interval $[0, b(x))$, but from Eq. (5), we have that $b(x)$ is constant through S, thus all the integral curves starting at S are defined in $[0, b)$, where $b = b(x_0)$.

Obviously, the screen distribution defined by the level sets of ρ is integrable and so $d\omega(X, Y) = 0$ for all $X, Y \in \Gamma(\mathcal{S})$. Since $\Phi(t, x) = \exp_x(t\xi_x)$ is the flow of ξ and ρ is given by Eq. (5), the flow of ξ preserves the screen distribution and so $0 = L_\xi\omega = i_\xi d\omega + di_\xi\omega = i_\xi d\omega$. Thus $d\omega(\xi, U) = 0$ for all $U \in \mathfrak{X}(L)$ and we can conclude that the rigged one-form ω is closed.

Since ρ is increasing along the integral curves of ξ, the restriction of the flow of ξ to $(0, b) \times S$ is injective and Theorem 1 gives us an isometric embedding $\Phi : (0, b) \times_\lambda S \to (L, \widetilde{g})$, where

$$\lambda(t,x) = \exp\left(-\int_0^t \rho(\Phi(s,x))ds\right) = \exp\left(-\int_0^t \frac{1}{\frac{1}{\rho(x)} - s} ds\right) = 1 - \frac{t}{b}.$$

Now observe that we can extend Φ to $\mathbb{R} \times S$ because M is null geodesically complete, so we can consider the map $\Psi : S \to M$ given by $\Psi(t,x) = \exp_x(b\xi_x) = \Phi(b,x)$. Saying that L is contained in a null cone is equivalent to show that Ψ is a constant map. The key to prove this is to observe that the geodesics $t \to (t,x)$ in $(0,b) \times_\lambda S$ are approaching when t tends to b because $\lim_{t\to b^-} \lambda(t) = 0$.

Lemma 3 *The map $\Psi : S \to M$ is a constant map.*

Proof Take $\alpha : I \to S$ any curve and consider the geodesic variation $X(t,s) = \Phi(t,\alpha(s)) = \exp_{\alpha(s)}(t\xi_{\alpha(s)})$. The curves $s \mapsto X(t,s)$ are in the leaf of $\mathcal{S}$ through $X(t,0)$, so we have

$$g(X_t(t,s), X_s(t,s)) = 0,$$
$$g(X_s(t,s), X_s(t,s)) = \lambda(t)^2 g(\alpha'(s), \alpha'(s))$$

for all $t \in (0,b)$. Taking limit as t approaches b we have that $\Psi_{*_{\alpha(0)}}\left(\alpha'(0)\right) = X_s(b,0)$ is zero or a null vector necessarily proportional to $\frac{d}{dt}\Phi_{\alpha(0)}(t)|_{t=b}$. Thus, for each $x \in S$ we have $\Psi_{*_x} = 0$ or $\dim\ker\Psi_{*_x} = n-3$. Suppose that there is a point $x_0 \in S$ which holds this last case. Since $\dim\ker\Psi_{*_x} < n-2$ is an open condition, there is a neighbourhood $x_0 \in U \subset S$ with $\dim\ker\Psi_{*_x} = n-3$ for all $x \in U$ and thus $\mathcal{H} = \ker\Psi_{*_x}$ defines a codimension one integrable distribution on $U \subset S$.

Take $\alpha : I \to S$ a transverse curve to $\mathcal{H}$ with $\alpha(0) = x_0$ and $\varphi = (z_1, \ldots, z_{n-2})$ an adapted chart to $\mathcal{H}$ in a neighbourhood of x_0 in S, i.e., the leaves of $\mathcal{H}$ are given by $z_{n-2} = c$ for a constant c. Since Ψ is constant through the leaves of $\mathcal{H}$, then we have that the curve

$$\beta(t) = \varphi^{-1}(z_1(\alpha(t)) + \varepsilon, z_2(\alpha(t)), \ldots, z_{n-2}(\alpha(t)))$$

for some $\varepsilon > 0$ holds $\Psi(\alpha(t)) = \Psi(\beta(t))$ and $x_1 = \beta(0) \neq \alpha(0) = x_0$. Therefore, $\Psi_{*_{x_0}}(\alpha'(0)) = \Psi_{*_{x_1}}(\beta'(0))$.

Now, we know that $\frac{d}{dt}\Phi_{x_0}(t)|_{t=b}$ is proportional to $\Psi_{*_{x_0}}(\alpha'(0))$ and $\frac{d}{dt}\Phi_{x_1}(t)|_{t=b}$ to $\Psi_{*_{x_1}}(\beta'(0))$, so there is a constant $a \neq 0$ with $\frac{d}{dt}\Phi_{x_0}(t)|_{t=b} = a\frac{d}{dt}\Phi_{x_1}(t)|_{t=b}$ which means that $\Phi_{x_1}(at+b) = \Phi_{x_0}(t+b)$. If $a > 0$ then we can take a value of t such that $t+b$ and $at+b$ are in $(0,b)$. Evaluating Eq. (5) we get $a = 1$. But this means that the geodesics $\Phi_{x_0}(t)$ and $\Phi_{x_1}(t)$ are the same and in particular $x_0 = x_1$, which is a contradiction.

The case $a < 0$ means that the geodesics $\Phi_{x_0}(t)$ and $\Phi_{x_1}(t)$ meet at $\Phi_{x_0}(b) = \Phi_{x_1}(b)$ but with opposite direction. If we do the same construction as before with a point $x_2 \neq x_1, x_0$ then we can construct two distinct null geodesics which meet with the same direction, but we have seen that this situation is not possible.

The hypothesis about the Ricci curvature in Theorem 4 can be relaxed. Indeed, we only need to ensure that the Ricci curvature is a constant through the null geodesics of the null hypersurface, which allows us to solve the differential Eq. (4). An example of a Lorentzian manifolds with this property is the direct product $M = \mathbb{R} \times \mathbb{S}^n$. If $\gamma(t)$ is a null geodesic in M, then $Ric(\gamma'(t), \gamma'(t))$ is a constant, which depends on the geodesic. In a similar way as in Theorem 4 we can prove the following.

Theorem 5 ([10]) *Let (M, g) be a null geodesically complete Lorentzian manifold with dimension $n > 3$. Take L a totally umbilic null hypersurface satisfying the following properties.*

1. *It has never vanishing null mean curvature.*
2. *For each null geodesic $\gamma : I \to L$ it holds $\frac{d}{dt} Ric(\gamma'(t), \gamma'(t)) = 0$.*
3. *It is strongly inextensible.*

Then, L is contained in a null cone.

3 Codimension Two Spacelike Submanifolds Through a Null Hypersurface

There are several papers where codimension two spacelike submanifold through a null hypersurface are studied, but they focus mainly in null cones and constant curvature ambients, [2–4, 13, 14, 17]. In this section we consider a codimension two spacelike submanifold through an arbitrary null hypersurface as a hypersurface of the Riemannian manifold constructed from the rigged metric. This allows us to apply the classical Eschenburg maximum principle to obtain conditions for a codimension two spacelike submanifold through a null hypersurface to coincide with a leaf of an integrable screen.

Take Σ a codimension two spacelike submanifold contained in a null hypersurface L with rigging ζ. Since $(T_x\Sigma)^\perp$ is a timelike plane, we can suppose that

$$(T_x\Sigma)^\perp = span(\xi, \eta),$$

where η is the unique null vector field over Σ with $g(\xi, \eta) = 1$. It is easy to check that the second fundamental form of Σ is given by

$$\mathbb{I}(U, V) = -g(\nabla_U \eta, V)\xi + B(U, V)\eta \tag{6}$$

for all $U, V \in \mathfrak{X}(\Sigma)$ and therefore the mean curvature vector of Σ as a submanifold of M is

$$\vec{H}_\Sigma = tr_\Sigma A_\eta \cdot \xi + H \cdot \eta,$$

where $A_\eta(U) = -(\nabla_U \eta)^{T\Sigma}$.

The null hypersurface L does not need to be totally geodesic or umbilical even if Σ is totally geodesic or umbilical in (M, g), since the Eq. (6) only holds along Σ and not in the whole L.

Now, we consider Σ as a hypersurface of the Riemannian manifold $(L, \widetilde{g})$ and we wonder what the mean curvature of Σ in $(L, \widetilde{g})$ is. First we need a $\widetilde{g}$-unitary and normal vector field to Σ. For this, decompose η according to decompositions (1) and (2) as

$$\eta = X_0 + \alpha\xi + N,$$

where $X_0 \in \Gamma(\mathcal{S})$ and $\alpha = g(\eta, N)$. Observe that $X_0(p) \in \left(T_p\Sigma \cap \mathcal{S}_p\right)^{\perp}$ for all $p \in \Sigma$ and $\alpha \leq 0$ by the construction of N.

The vector field given by

$$E = \frac{1}{\sqrt{1-2\alpha}}\,(X_0 + \xi)$$

is a $\widetilde{g}$-unitary and normal vector field to Σ as a hypersurface of $(L, \widetilde{g})$. Moreover, $\frac{1}{\sqrt{1-2\alpha}}$ can be interpreted as the cosine of the $\widetilde{g}$-angle between $T_p\Sigma$ and $\mathcal{S}_p$ for each $p \in \Sigma$. Therefore, the mean curvature of Σ as a hypersurface of the Riemann manifold $(L, \widetilde{g})$ is given by

$$\widetilde{H}_\Sigma(p) = \sum_{i=3}^{n} -\widetilde{g}(\widetilde{\nabla}_{u_i} E, u_i),$$

where $\{u_3, \ldots, u_n\}$ is a $\widetilde{g}$-orthonormal basis of $T_p\Sigma$. The following proposition gives us an expression for $\widetilde{H}_\Sigma$ in term of the fundamental tensors of L and the second fundamental form and mean curvature vector of Σ.

Proposition 2 ([11]) *Let L be a null hypersurface in a Lorentzian manifold and ζ a rigging vector field for it such that $d\omega = 0$. If Σ is a spacelike codimension two submanifold of M through L, then the mean curvature $\widetilde{H}_\Sigma$ of Σ respect to E holds*

$$\begin{aligned}\frac{\widetilde{H}_\Sigma}{\cos\theta} &= g(\vec{H}_\Sigma, N) - \Omega - B(X_0, X_0) + \frac{1}{\cos^2\theta} H \\ &\quad + \cos^2\theta\left(C(X_0, X_0) - g(\mathbb{I}_\Sigma(V_0, V_0), N) - \tau(X_0 + V_0)\right),\end{aligned}$$

where θ is the $\widetilde{g}$-angle between $T_p\Sigma$ and $\mathcal{S}_p$ and $V_0 = X_0 + 2\alpha\xi$.

The null second fundamental form B is related to $\widetilde{g}$ by the equation

$$\left(L_\xi \widetilde{g}\right)(X, Y) = -2B(X, Y)$$

for all $X, Y \in \Gamma(\mathcal{S})$, [8]. In particular, the null mean curvature of the null hypersurface is the $\widetilde{g}$-divergence of the rigged vector field,

$$H = -\widetilde{\mathrm{div}}\xi. \tag{7}$$

Suppose that the rigged one-form ω is closed. Then the screen distribution $\mathcal{S}$ is integrable and we can consider a leaf S also as a hypersurface of the Riemannian manifold $(L, \widetilde{g})$. It is clear that ξ is a $\widetilde{g}$-unitary and normal vector field to S and so, taking into account Eq. (7), its mean curvature as a hypersurface of $(L, \widetilde{g})$ coincides with the null mean curvature of the null hypersurface.

Now we can apply the Eschenburg maximum principle to the hypersurfaces Σ and S in $(L, \widetilde{g})$, [6, Theorem 1]. Roughly speaking, it says that if two hypersurfaces are tangent at a point and locally around this point the mean curvatures are suitable bounded by a constant and one of them is "on one side of the other one", then the hypersurfaces are the same. We make more precise this last statement in our context.

Observe that the rigged vector field ξ is always pregeodesics. If $\mathcal{S}$ is integrable and S is the leaf through a fixed point $p \in \Sigma$, then there are neighbourhoods $p \in U \subset S$ and $p \in V \subset L$ such that $\Phi : (-\varepsilon, \varepsilon) \times U \to V$ given by $\Phi(t, x) = \exp_x(t\xi_x)$ is a diffeomorphism. We call $d_S^\zeta = \Pi \circ \Phi^{-1}$, where Π is the projection onto the first factor. If $d_S^\zeta \geq 0$ in a neighbourhood of p in S, then Σ is locally "on one side of S". In this case, Σ and S are tangent at p and $E_p = \xi_p$.

If we suppose that Σ is totally geodesic, from Eq. (6) we have that $B = 0$ and in particular $H = 0$ along Σ. If moreover the rigging is screen conformal we also have $C = 0$ and $\Omega = 0$ along Σ. Therefore, if in addition the rigging is distinguished, then from Proposition 2 we have that the mean curvature of Σ in $(L, \widetilde{g})$ is zero and we obtain the following result ensuring that a codimension two spacelike submanifold through a null hypersurface coincides with a leaf of an integrable screen.

Theorem 6 ([11]) *Let L be a null hypersurface of a Lorentzian manifold, ζ a rigging vector field for it and Σ a spacelike totally geodesic codimension two submanifold of M through L. Take a point $p_0 \in \Sigma$ and let S be the leaf of the screen distribution through p_0. Suppose that*

1. *ζ is distinguished.*
2. *ζ is screen conformal.*
3. *$d_S^\zeta \geq 0$ in a neighborhood of p_0 in Σ.*
4. *$H(p) \geq 0$ for all p in a neighborhood of p_0 in S.*

Then Σ coincides with the leaf S in a neighborhood of p_0.

A more careful application of the Eschenburg maximum principle allows us to obtain other results as the following.

Theorem 7 ([11]) *Let L be a null hypersurface of a Lorentzian manifold, ζ a rigging vector field for it and Σ a spacelike totally umbilical codimension two submanifold of M through L. Take a point $p_0 \in \Sigma$ and let S be the leaf of the screen distribution through p_0. Suppose that*

1. *ζ is distinguished.*
2. *ζ is screen conformal with conformal factor φ.*

3. *$dH = c\omega$ for some non-positive function $c \in C^\infty(L)$.*
4. *$d_S^\zeta \geq 0$ in a neighborhood of p_0 in Σ.*
5. *$H(p_0) \leq 0$.*
6. *$g(\vec{H}_\Sigma, N) \leq \varphi H$ in a neighborhood of p_0 in Σ.*

Then Σ coincides with the leaf S in a neighborhood of p_0.

The Raychaudhuri equation asserts that

$$dH(\xi) = Ric(\xi, \xi) + trace(\left(A^*\right)^2) \geq Ric(\xi, \xi).$$

Thus, if $Ric(\xi, \xi) \geq 0$ and H is not constant, then we can not apply the above theorem, due to point 3. This happens for example in a null cone in a constant curvature Lorentzian manifold.

Theorem 8 ([11]) *Let L be a totally geodesic null hypersurface of a Lorentzian manifold and ζ a rigging for it such that*

1. *ζ is distinguished.*
2. *The screen distribution is totally umbilical.*

Suppose that Σ is a spacelike totally umbilical codimension two submanifold of M through L and there is a point $p_0 \in \Sigma$ such that $d_S^\zeta \geq 0$ in a neighborhood of p_0 in Σ, where S is the leaf of the screen distribution through p_0. If $g(\vec{H}_\Sigma, \zeta) \leq \Omega$ in a neighborhood of p_0, then Σ coincides with the leaf S of the screen distribution in a neighborhood of p_0.

In the above results we have to assume many conditions, but as we said in the introduction, most important examples of null hypersurfaces admit a screen conformal and distinguished rigging. We give some examples.

Example 2 Take the Lorentzian manifold $(M, g) = \left(\mathbb{R} \times \mathbb{H}^{n-1}, -dt^2 + g_0\right)$, where $\mathbb{H}^{n-1}$ is the hyperbolic space $\mathbb{H}^{n-1}$, which in turn can be decomposed as

$$\left(\mathbb{R} \times \mathbb{R}^{n-2}, ds^2 + e^{-2s} h_0\right),$$

being h_0 the Euclidean metric. The hypersurface L given by

$$L = \{(t, t, x) : t \in \mathbb{R}, x \in \mathbb{R}^{n-2}\}$$

is a totally umbilical null hypersurface and its null mean curvature respect to the rigging vector field $\zeta = \partial_t$ is $H = 2 - n$, [7]. Moreover, we can check that

$$C = -\frac{1}{2} g,$$
$$\Omega = -\frac{(n-2)}{2},$$
$$\tau = 0,$$

so ζ is distinguished and screen conformal with conformal factor $\varphi = \frac{1}{2}$.

On the other hand, the leaf of the screen distribution through a point $p_0 = (t_0, s_0, x_0) \in L$ is given by $S = \{(t, s, x) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n-2} : t = t_0, s = s_0\}$. Since $\xi = -\partial_t - \partial_s$, if we fix $p_0 \in L$ and S the leaf through p_0, the condition $d_S^{\zeta}(p) \geq 0$ is equivalent to $t(p) \leq t(p_0)$, where $t : M \to \mathbb{R}$ is the canonical projection onto the first factor. Moreover, the transverse vector field is $N = \frac{1}{2}(\partial_t - \partial_s)$. Therefore, conditions 1, 2 and 3 in Theorem 7 are fulfilled and we can apply it to get the following.

Suppose that Σ is a codimension two totally umbilical spacelike hypersurface in $\mathbb{R} \times \mathbb{H}^{n-1}$ contained in $L = \{(t, t, x) : t \in \mathbb{R}, x \in \mathbb{R}^{n-2}\}$. If there is a point $p_0 = (t_0, t_0, x_0) \in \Sigma$ such that $t(p) \leq t_0$ and $g(\vec{H}_\Sigma, \partial_t - \partial_s) \leq 2 - n$ for all p in a neighborhood of p_0 in Σ, then Σ is locally contained in $\{(t_0, t_0, x) : x \in \mathbb{R}^{n-2}\}$.

Example 3 Let $\varpi > 0$ be a constant and $Q = \{(u, v) \in \mathbb{R}^2 : -\frac{2\varpi}{e} < uv\}$. Take the functions $F(r) = \frac{8\varpi^2}{r} e^{1-\frac{r}{2\varpi}}$, $f(r) = (r - 2\varpi)e^{\frac{r}{2\varpi}-1}$ for $0 \leq r$ and $r(u, v) = f^{-1}(uv)$ for $(u, v) \in Q$. The Kruskal space is the product $Q \times \mathbb{S}^{n-2}$ endowed with the metric

$$2F(r)dudv + r^2 g_0,$$

where g_0 is the standard metric in $\mathbb{S}^{n-2}$. We call $u, v : Q \times \mathbb{S}^{n-2} \to \mathbb{R}$ the canonical projections. The hypersurface

$$L = \{p \in Q \times \mathbb{S}^{n-2} : u(p) = 0\}$$

is a null hypersurface and $\zeta = \partial_u$ is a rigging vector field for it. The rigged vector field is $\xi = \frac{1}{F}\partial_v$ and the null transverse vector field is $N = \zeta$. Observe that $d\omega = 0$, although ζ is not closed, and the leaf of the screen distribution through a point $p_0 = (0, v_0, x_0) \in L$ is $S = \{(0, v_0, x) : x \in \mathbb{S}^{n-2}\}$. Therefore, in this case, the condition $d_S^{\zeta}(p) \geq 0$ is equivalent to $v(p_0) \leq v(p)$.

A direct computation shows that L is totally geodesic. Moreover, using that $r = 2\varpi$ through L we have

$$\begin{aligned} \tau &= 0, \\ C &= -\frac{v}{2\varpi} g, \\ \Omega &= -\frac{(n-2)v}{2\varpi}. \end{aligned}$$

Using Theorem 8, if Σ is a codimension two spacelike totally umbilical submanifold contained in L and $p_0 \in \Sigma$ holds $v(p_0) \le v(p)$ and $g(\vec{H}_\Sigma, \partial_u) \le \frac{2-n}{2\varpi} v$ for all p in a neighborhood of p_0 in Σ, then Σ is locally contained in the sphere $\{p \in Q \times \mathbb{S}^{n-2} : u(p) = 0, v(p) = v(p_0)\}$.

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