Bochner-Flat Para-Kähler Surfaces

María Ferreiro-Subrido

Abstract We show that Bochner-flat para-Kähler surfaces are self-dual Walker manifolds and therefore they are locally isometric to the cotangent bundle of an affine surface equipped with a modified Riemannian extension. Explicit examples of constant and non-constant scalar curvature are given.

Keywords Para-Kähler structures · Bochner tensor · Walker structures · Riemannian extension

1 Introduction

A para-Kähler manifold is a symplectic manifold (M^{2n}, Ω) that is locally diffeomorphic to a product of Lagrangian submanifolds. This way its tangent bundle decomposes as a Whitney sum of Lagrangian subbundles $TM = L \oplus L'$. Considering π_L and $\pi_{L'}$ the projections on each subbundle, the $(1, 1)$-tensor field defined by $J = \pi_L - \pi_{L'}$ is an almost paracomplex structure on M. Moreover, since L and L' are Lagrangian subspaces one has that $\Omega(JX, JY) = -\Omega(X, Y)$ for all vector fields X, Y on M and so $g(X, Y) = \Omega(JX, Y)$ defines a neutral signature metric on M such that $g(JX, JY) = -g(X, Y)$ and $\nabla J = 0$, where ∇ denotes the Levi-Civita connection of (M, g).

Para-Kähler structures, which are also called bi-Lagrangian manifolds in the literature, are relevant for both Physics and Geometry. Para-Kähler geometry plays an important role in the study of several geometric problems such as the non-uniqueness of the metric for the Levi-Civita connection [5], the classification of symplectic connections [7], the spaces of oriented geodesics [3], the study of cones over pseudo-Riemannian manifolds [2] or the classical Monge-Kantorovich mass transport [15] (see also [11] for applications to supersymmetry). Paracomplex geometry is also relevant for understanding Weierstrass and Enneper type representations for Lorentzian surfaces in $\mathbb{R}^{2,1}$ [10, 16].

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A. L. Albujer et al. (eds.), *Developments in Lorentzian Geometry*, Springer Proceedings in Mathematics & Statistics 389, https://doi.org/10.1007/978-3-031-05379-5_5

The Bochner curvature tensor was introduced by S. Bochner in 1949 [4]. It is formally defined as an analogue of the Weyl curvature tensor, so that the curvature tensor of a Bochner-flat manifold is completely determined by its Ricci tensor. Let (M^{2n}, g, J) be a para-Kähler manifold. Its Bochner curvature tensor is defined as

$$B(X,Y)Z = R(X,Y)Z + \frac{\tau}{(2n+2)(2n+4)} R_0(X,Y)Z - \frac{1}{2(n+2)} R_1(X,Y)Z$$

for all vector fields X, Y, Z on M, where

$$\begin{aligned} R_0(X,Y)Z &= g(X,Z)Y - g(Y,Z)X \\ &\quad + g(JX,Z)JY - g(JY,Z)JX + 2g(JX,Y)JZ, \\ R_1(X,Y)Z &= g(X,Z)\,\mathrm{Ric}(Y) - g(Y,Z)\,\mathrm{Ric}(X) + g(X,JZ)\,\mathrm{Ric}(JY) \\ &\quad - g(Y,JZ)\,\mathrm{Ric}(JX) + 2g(X,JY)\,\mathrm{Ric}(JZ) + \rho(X,Z)Y \\ &\quad - \rho(Y,Z)X + \rho(X,JZ)JY - \rho(Y,JZ)JX + 2\rho(X,JY)JZ\,. \end{aligned}$$

A para-Kähler manifold is said to be *Bochner-flat* if its Bochner tensor vanishes identically. A para-Kähler manifold has constant paraholomorphic sectional curvature c if and only if its curvature tensor is of the form $R(X,Y)Z = \frac{c}{4} R_0(X,Y)Z$ (see [14]). This way, any para-Kähler manifold of constant paraholomorphic sectional curvature is Bochner-flat. Moreover, a Bochner-flat para-Kähler manifold has constant paraholomorphic sectional curvature if and only if it is Einstein.

Even though the condition of being Bochner-flat is somehow analogous to that of being locally conformally flat, it is more restrictive since a Bochner-flat para-Kähler manifold has constant scalar curvature if and only if it is locally symmetric [17]. Moreover, if its Ricci operator is diagonalizable then the manifold either has constant paraholomorphic sectional curvature or it is locally isometric to a product of two spaces of constant opposite paraholomorphic sectional curvature.

The anti-self-dual Weyl curvature tensor of a four-dimensional para-Kähler manifold is determined by its scalar curvature as $W^- = \frac{\tau}{12}\,\mathrm{diag}[2, -1, -1]$ and the symplectic form Ω is an eigenvector for the distinguished eigenvalue. On the other hand, the self-dual Weyl curvature tensor of a para-Kähler manifold is completely determined by the Bochner tensor, so $W^+ = 0$ if and only if the manifold is Bochner-flat (see [6]). An immediate consequence of these facts is that a four-dimensional para-Kähler manifold is locally conformally flat if and only if it is Bochner-flat and its scalar curvature vanishes identically.

Let (M, g, J) be a para-Kähler manifold and denote $\mathfrak{D}_\pm = \ker(J \mp \mathrm{Id})$ the eigenspaces corresponding to the eigenvalues ± 1 of the paracomplex structure J. $\mathfrak{D}_\pm$ are parallel degenerate distributions and so any para-Kähler surface has an underlying Walker structure. This fact allows us to study para-Kähler structures through Walker manifolds.

The present work is organized as follows. Section 2 is devoted to the description of Walker structures in dimension four, paying special attention to self-dual Walker structures, in order to pave the way for the understanding of Bochner-flat para-Kähler structures in Sect. 3. Note that the para-Kähler and the Walker structures induce distinguished opposite orientations on the manifold, a fact that plays an important role in the theory. The classification of Bochner-flat para-Kähler surfaces of constant scalar curvature is given in Theorem 2, specifying the different curvature models realized in each situation. Finally, some examples of Bochner-flat para-Kähler surfaces of non-constant scalar curvature are provided in Sect. 3.2.

2 Walker Structures

Let $(M, g, \mathfrak{D})$ be a four-dimensional Walker manifold, i.e. a pseudo-Riemannian manifold (M, g) of neutral signature admitting a parallel degenerate plane field $\mathfrak{D}$ of maximal dimension. Walker showed in [19] the existence of local coordinates $(x^1, x^2, x_{1'}, x_{2'})$ so that $\mathfrak{D} = \operatorname{span}\{\partial_{x_{1'}}, \partial_{x_{2'}}\}$ and the metric expresses as

$$g = dx^i \otimes dx_{i'} + dx_{i'} \otimes dx^i + g_{ij}(x^1, x^2, x_{1'}, x_{2'})dx^i \otimes dx^j. \tag{1}$$

The simplest examples of Walker manifolds are given by the so-called Riemannian extensions. We briefly review their construction as follows. Consider a surface Σ and let $\pi : T^*\Sigma \to \Sigma$ be the projection from its cotangent bundle. Let $(p, \omega) \in T^*\Sigma$ denote a point in $T^*\Sigma$, where $p \in \Sigma$ and $\omega \in T_p^*\Sigma$. For each vector field X on Σ the evaluation map is the function $\iota X \in \mathcal{C}^\infty(T^*\Sigma)$ defined by $\iota X(p, \omega) = \omega(X_p)$. Two vector fields $\bar{X}$ and $\bar{Y}$ on $T^*\Sigma$ satisfy $\bar{X} = \bar{Y}$ if and only if they act on evaluation maps as $\bar{X}(\iota Z) = \bar{Y}(\iota Z)$ for any vector field Z on Σ. Given a vector field X on Σ, its complete lift X^C is the vector field determined by the identity $X^C(\iota Z) = \iota[X, Z]$. In the same way as vector fields on $T^*\Sigma$ are characterized by their action on evaluation maps, $(0, s)$-tensor fields on $T^*\Sigma$ are characterized by their action on complete lifts of vector fields. In particular, any $(1, 1)$-tensor field T on Σ induces a 1-form ιT on $T^*\Sigma$ characterized by $\iota T(X^C) = \iota(TX)$ (see [20] for more details concerning this matter).

Riemannian extensions of torsion-free connections were introduced by Patterson and Walker in [18] as metrics on $T^*\Sigma$ such that $g_D(X^C, Y^C) = -\iota(D_X Y + D_Y X)$, where D is a torsion-free connection on the base manifold Σ. *Deformed Riemannian extensions* are neutral signature metrics on $T^*\Sigma$ such that $g_{D,\Phi} = g_D + \pi^*\Phi$, where Φ is a symmetric $(0, 2)$-tensor field on the affine surface. Afifi showed in [1] that a Walker manifold with parallel null distribution $\mathfrak{D}$ is locally isometric to a deformed Riemannian extension of an affine connection if and only if its curvature tensor satisfies $R(\cdot, \mathfrak{D})\mathfrak{D} = 0$. These metrics were further generalized in [8] as follows. Considering a symmetric $(0, 2)$-tensor field Φ and $(1, 1)$-tensor fields T and S on an affine surface (Σ, D), the *modified Riemannian extension* is the neutral signature metric on $T^*\Sigma$ defined by $g_{D,\Phi,T,S} = \iota T \circ \iota S + g_D + \pi^*\Phi$, where '$\circ$' denotes the symmetric product. Considering local coordinates (x^1, x^2) on a neighbourhood $\mathcal{U}$ in Σ and induced coordinates $(x^1, x^2, x_{1'}, x_{2'})$ on $\pi^{-1}(\mathcal{U})$, one has

$$g_{D,\Phi,T,S} = dx^i \otimes dx_{i'} + dx_{i'} \otimes dx^i + \left\{\tfrac{1}{2}x_{r'}x_{s'}(T_i^r S_j^s + T_j^r S_i^s) - 2x_{k'}{}^D\Gamma_{ij}^k + \Phi_{ij}\right\} dx^i \otimes dx^j,$$

where $T = T_i^j dx^i \otimes \partial_{x^j}$, $S = S_i^j dx^i \otimes \partial_{x^j}$, $\Phi = \Phi_{ij} dx^i \otimes dx^j$ and ${}^D\Gamma_{ij}^k$ are the Christoffel symbols of the affine connection D. Moreover, the Walker distribution is given by $\mathfrak{D} = \ker \pi_*$. Furthermore, a Walker metric corresponds to the modified Riemannian extension of an affine connection if and only if $(\nabla_{\mathfrak{D}} R)(\mathfrak{D}, \cdot)\mathfrak{D} = 0$.

2.1 Self-Dual Walker Manifolds

The existence of a parallel degenerate 2-dimensional distribution $\mathfrak{D}$ on a neutral signature manifold (M, g) of dimension four naturally induces an orientation. We recall the discussion in [12]. Let $p \in M$ and let $\{u, v\}$ be an arbitrary basis of $\mathfrak{D}_p$. Then the Hodge-star operator satisfies $\star(u^* \wedge v^*) = \pm(u^* \wedge v^*)$, where $u^*, v^* \in T_p^*M$ denote the corresponding dual forms. This way, any four-dimensional Walker manifold is naturally oriented by the self-duality of $u^* \wedge v^*$. Let $(x^1, x^2, x_{1'}, x_{2'})$ be local coordinates on a four-dimensional Walker manifold as in (1). Then the Walker orientation determined by $\star(dx_{1'} \wedge dx_{2'}) = dx_{1'} \wedge dx_{2'}$ corresponds to the volume element $\mathrm{vol}_g = dx^1 \wedge dx^2 \wedge dx_{1'} \wedge dx_{2'}$. Self-dual Walker manifolds were described in [8] as follows.

Theorem 1 ([8, Theorem 7.1]) *A four-dimensional Walker manifold is self-dual if and only if it is locally isometric to the cotangent bundle $T^*\Sigma$ of an affine surface (Σ, D) with metric*

$$g = \iota X(\iota \,\mathrm{Id} \circ \iota \,\mathrm{Id}) + \iota T \circ \iota \,\mathrm{Id} + g_D + \pi^*\Phi, \tag{2}$$

where g_D denotes de Riemannian extension of the affine connection, X is a vector field on Σ and T and Φ are a $(1, 1)$-tensor field and a symmetric $(0, 2)$-tensor field on Σ, respectively.

Let Σ be a surface with local coordinates (x^1, x^2) and consider $(x^1, x^2, x_{1'}, x_{2'})$ the induced local coordinates on $T^*\Sigma$. The canonical symplectic structure of the cotangent bundle determined by the tautological 1-form $\theta = x_{k'}dx^k$ induces an orientation determined by the volume form $d\theta \wedge d\theta = -dx^1 \wedge dx^2 \wedge dx_{1'} \wedge dx_{2'}$, which is the opposite of the orientation induced by the Walker structure given by $\mathfrak{D} = \ker \pi_*$.

3 Bochner-Flat Para-Kähler Surfaces

Let (M, g, J) be a para-Kähler surface and denote $\mathfrak{D}_\pm = \ker(J \mp \mathrm{Id})$. We consider Walker coordinates $(x^1, x^2, x_{1'}, x_{2'})$ as in (1) and set the Walker distribution to be $\mathfrak{D} = \mathfrak{D}_+$ so that $J|_{\mathfrak{D}} = \mathrm{Id}$. We point out that para-Kähler surfaces are Walker manifolds but the converse is not true, since the parallelizability of $\mathfrak{D} = \mathfrak{D}_+$ does not ensure the integrability of the complementary distribution $\mathfrak{D}_-$. The almost para-Hermitian structures satisfying $J|_{\mathfrak{D}} = \mathrm{Id}$ are locally parametrized by a real-valued function $f(x^1, x^2, x_{1'}, x_{2'})$ so that

$$\begin{array}{ll} J_f\partial_{x^1} = -\partial_{x^1} + g_{11}\partial_{x_{1'}} + f\partial_{x_{2'}}, & J_f\partial_{x_{1'}} = \partial_{x_{1'}}, \\ J_f\partial_{x^2} = -\partial_{x^2} + (2g_{12} - f)\partial_{x_{1'}} + g_{22}\partial_{x_{2'}}, & J_f\partial_{x_{2'}} = \partial_{x_{2'}}. \end{array} \tag{3}$$

Their associated Kähler 2-forms $\Omega_f(X, Y) = g(J_f X, Y)$ are given by $\Omega_f = (f - g_{12})dx^1 \wedge dx^2 + dx_{1'} \wedge dx^1 + dx_{2'} \wedge dx^2$, thus

$$d\Omega_f = \partial_{x_{1'}}(f - g_{12})dx_{1'} \wedge dx^1 \wedge dx^2 + \partial_{x_{2'}}(f - g_{12})dx_{2'} \wedge dx^1 \wedge dx^2.$$

Therefore, $d\Omega_f = 0$ if and only if $f(x^1, x^2, x_{1'}, x_{2'}) = g_{12}(x^1, x^2, x_{1'}, x_{2'}) + h(x^1, x^2)$ for some function $h(x^1, x^2)$ and the almost paracomplex structure becomes

$$\begin{array}{ll} J_h\partial_{x^1} = -\partial_{x^1} + g_{11}\partial_{x_{1'}} + (g_{12} + h)\partial_{x_{2'}}, & J_h\partial_{x_{1'}} = \partial_{x_{1'}}, \\ J_h\partial_{x^2} = -\partial_{x^2} + (g_{12} - h)\partial_{x_{1'}} + g_{22}\partial_{x_{2'}}, & J_h\partial_{x_{2'}} = \partial_{x_{2'}}. \end{array} \tag{4}$$

Considering an almost para-Hermitian structure given by (1) and (4), the associated Kähler 2-form is given by $\Omega_h = h dx^1 \wedge dx^2 + dx_{1'} \wedge dx^1 + dx_{2'} \wedge dx^2$. It is important to emphasize that the para-Kähler and Walker orientations are opposite. Indeed, the Kähler 2-form Ω_h is anti-self-dual for the para-Kähler orientation determined by the paracomplex structure J_h, but it is self-dual for the Walker orientation.

In order to describe Bochner-flat para-Kähler surfaces we consider the cotangent bundle $T^*\Sigma$ of an affine surface (Σ, D) with metric $g = \iota X(\iota\,\mathrm{Id} \circ \iota\,\mathrm{Id}) + \iota T \circ \iota\,\mathrm{Id} + g_D + \pi^*\Phi$ as in (2) and set the paracomplex structure satisfying the condition $J|_{\ker \pi_*} = \mathrm{Id}$. The almost para-Hermitian structures defined by (1) and (4) are not para-Kähler in general. We use the notation $(\nabla_{\partial_{x^\alpha}} J_h)\partial_{x^\beta} = (\nabla J_h)_{\beta;\alpha}{}^\gamma \partial_{x^\gamma}$ to denote the components of ∇J_h on $T^*\Sigma$ and $(D_{\partial_{x^i}} T)\partial_{x^j} = DT_{j;i}{}^k \partial_{x^k}$, $(D_{\partial_{x^i}} \Phi)(\partial_{x^j}, \partial_{x^k}) = D\Phi_{jk;i}$ to represent the covariant derivatives of the $(1, 1)$-tensor field T and the symmetric $(0, 2)$-tensor field Φ on Σ, respectively. Using the notation in Theorem 1, long but straightforward calculations show that:

Lemma 1 *Let (M, g) be a self-dual Walker manifold of dimension four. Let J_h be an almost paracomplex structure given by $J_h|_{\ker \pi_*} = \mathrm{Id}$ so that (g, J_h) is an almost para-Hermitian structure on M. Then the nonzero components of the covariant derivative ∇J_h are given by*

$$
\begin{aligned}
8\,(\nabla J_h)_{1;1}{}^4 = {} & x_{1'}^3\left\{T_2^1\operatorname{tr}(T)+8\mathcal{S}_2^1\right\}+x_{1'}^2x_{2'}\left\{(T_2^2)^2-(T_1^1)^2+8(\mathcal{S}_2^2-\mathcal{S}_1^1)\right\}\\
&+x_{1'}x_{2'}^2\left\{-T_1^2\operatorname{tr}(T)-8\mathcal{S}_1^2\right\}\\
&+x_{1'}^2\left\{8DT_{1;2}{}^1-4DT_{2;1}{}^1+2X^1(8h-4\Phi_{12})-4X^2\Phi_{22}\right\}\\
&+x_{1'}x_{2'}\left\{8DT_{1;2}{}^2-4DT_{2;1}{}^2-4DT_{1;1}{}^1+16hX^2+8X^1\Phi_{11}\right\}\\
&+x_{2'}^2\left\{-4DT_{1;1}{}^2+4X^2\Phi_{11}\right\}\\
&+x_{1'}\left\{16\rho_{21}^D+10hT_1^1+2hT_2^2-2\operatorname{tr}(T)\Phi_{12}+4(\hat{\Phi}_{21}-\hat{\Phi}_{12})\right\}\\
&+x_{2'}\left\{-16\rho_{11}^D+8hT_1^2+2\operatorname{tr}(T)\Phi_{11}\right\}\\
&+8\left\{\partial_1 h-h({}^D\Gamma_{11}{}^1+{}^D\Gamma_{12}{}^2)+D\Phi_{11;2}-D\Phi_{12;1}\right\},
\end{aligned}
$$

$$
\begin{aligned}
8(\nabla J_h)_{1;2}{}^4 = {} & x_{2'}^3\left\{-T_1^2\operatorname{tr}(T)-8\mathcal{S}_1^2\right\}+x_{1'}^2x_{2'}\left\{T_2^1\operatorname{tr}(T)+8\mathcal{S}_2^1\right\}\\
&+x_{1'}x_{2'}^2\left\{(T_2^2)^2-(T_1^1)^2+8(\mathcal{S}_2^2-\mathcal{S}_1^1)\right\}\\
&+x_{1'}^2\left\{4DT_{2;2}{}^1-4X^1\Phi_{22}\right\}\\
&+x_{1'}x_{2'}\left\{-8DT_{2;1}{}^1+4DT_{1;2}{}^1+4DT_{2;2}{}^2+16hX^1-8X^2\Phi_{22}\right\}\\
&+x_{2'}^2\left\{-8DT_{2;1}{}^2+4DT_{1;2}{}^2+2X^2(8h+4\Phi_{12})+4X^1\Phi_{11}\right\}\\
&+x_{1'}\left\{16\rho_{22}^D+8hT_2^1-2\operatorname{tr}(T)\Phi_{22}\right\}\\
&+x_{2'}\left\{-16\rho_{12}^D+2hT_1^1+10hT_2^2+2\operatorname{tr}(T)\Phi_{12}+4(\hat{\Phi}_{21}-\hat{\Phi}_{12})\right\}\\
&+8\left\{\partial_2 h-h({}^D\Gamma_{22}{}^2+{}^D\Gamma_{12}{}^1)+D\Phi_{12;2}-D\Phi_{22;1}\right\},
\end{aligned}
$$

where $X = X^i\partial_i$, $T = T_j^i\partial_{x^i}\otimes dx^j$ *and* $\Phi = \Phi_{ij}dx^i\otimes dx^j$ *are the vector field, the* $(1,1)$*-tensor field and the symmetric* $(0,2)$*-tensor field on* Σ *given in Theorem 1,* $\mathcal{S}$ *is the* $(1,1)$*-tensor field on* Σ *defined as* $\mathcal{S}(Z) := D_Z X$*,* $\hat{\Phi}(X,Y) := \Phi(TX,Y)$ *and* ${}^D\Gamma_{ij}{}^k$ *are the Christoffel symbols of the affine connection.*

Notice that the expressions in Lemma 1 are polynomials on the fiber coordinates $x_{1'}$ and $x_{2'}$ whose coefficients are functions of the base coordinates x^1 and x^2.

3.1 Bochner-Flat Para-Kähler Surfaces of Constant Scalar Curvature

It follows from Theorem 1 that the scalar curvature of a Bochner-flat para-Kähler surface is given by $\tau = 12\iota X + 3\operatorname{tr}(T)$, where ιX is the evaluation map of the vector field X. Therefore if a Bochner-flat para-Kähler surface has constant scalar curvature then the vector field X vanishes and T must have constant trace. If $\tau \neq 0$ there exist local coordinates in which the $(1,1)$-tensor field $T = c\,\mathrm{Id}$ with $c \in \mathbb{R}$. In this situation, a Bochner-flat para-Kähler surface has constant paraholomorphic sectional curvature and so it is locally isometric to the cotangent bundle of a flat affine surface (Σ, D) endowed with a modified Riemannian extension $g = c\iota\,\mathrm{Id}\circ\iota\,\mathrm{Id} + g_D$ (see [9, Theorem 2.2]).

Bochner-flat para-Kähler surfaces with $\tau = 0$ are locally conformally flat. Working at a purely algebraic level, we consider $(V, \langle\cdot,\cdot\rangle, J)$ a para-Hermitian inner product space and a para-Kähler algebraic curvature tensor $\mathcal{A} : V\times V\times V\times V \to \mathbb{R}$

so that $\mathcal{A}(X, Y) \cdot J = J \cdot \mathcal{A}(X, Y)$. There are three non-flat locally conformally flat algebraic curvature models $(V, \langle\cdot, \cdot\rangle, \mathcal{A})$ as follows.

($\mathfrak{M}$): $((V, \langle\,\cdot\,,\,\cdot\,\rangle, \mathcal{A})$ given by

$$\mathcal{A}_{1413} = \mathcal{A}_{3231} = -\frac{1}{2}$$

with respect to pseudo-orthonormal a basis $\{u_1, u_2, u_3, u_4\}$ where the non-zero inner products are $\langle u_1, u_2\rangle = 1 = -\langle u_3, u_4\rangle$.

($\mathfrak{N}_k$): $((V, \langle\,\cdot\,,\,\cdot\,\rangle, \mathcal{A})$ given by

$$\mathcal{A}_{1413} = \mathcal{A}_{1442} = \mathcal{A}_{3224} = \mathcal{A}_{3231} = \frac{k}{2}$$

with respect to an orthonormal basis $\{u_1, u_2, u_3, u_4\}$ where u_1, u_3 are spacelike vectors and u_2, u_4 are timelike vectors.

($\mathfrak{P}_k$): $((V, \langle\,\cdot\,,\,\cdot\,\rangle, \mathcal{A})$ given by

$$\mathcal{A}_{1212} = \mathcal{A}_{4334} = k$$

with respect to an orthonormal basis $\{u_1, u_2, u_3, u_4\}$ where u_1, u_3 are spacelike vectors and u_2, u_4 are timelike vectors.

It follows from Lemma 1 that if $\tau = 0$, then the $(1, 1)$-tensor field T must be parallel and so the classification of Bochner-flat para-Kähler surfaces of constant scalar curvature is summarized as follows.

Theorem 2 ([13, Theorem 4.2]) *Let (M, g, J) be a Bochner-flat para-Kähler surface of constant scalar curvature. Then it is locally isometric to a Riemannian extension of the form $(T^*\Sigma, g = \iota T \circ \iota \operatorname{Id} + g_D)$ with paracomplex structure determined by $J|_{\ker \pi_*} = \operatorname{Id}$, where T is a parallel $(1, 1)$-tensor field on a flat affine surface (Σ, D). Moreover, one of the following holds:*

- *(i)* $T = 0$ *and* (M, g, J) *is flat.*
- *(ii)* $T = c \operatorname{Id}$ *and* (M, g, J) *has constant paraholomorphic sectional curvature* $H = c$.
- *(iii)* $T^2 = \kappa^2 \operatorname{Id}$ *and* (M, g, J) *is isometric to a product of two Lorentzian surfaces of constant opposite curvature, thus modelled on* ($\mathfrak{P}_k$).
- *(iv)* $T^2 = 0$ *and* (M, g, J) *is modelled on* ($\mathfrak{M}$).
- *(v)* $T^2 = -\kappa^2 \operatorname{Id}$ *and* (M, g, J) *is modelled on* ($\mathfrak{N}_k$).

3.2 Some Examples of Bochner-Flat Para-Kähler Structures of Non-constant Scalar Curvature

Consider an affine surface (Σ, D) and let (g, J_h) be an almost para-Hermitian structure on $T^*\Sigma$ given by $J_h|_{\ker \pi_*} = \mathrm{Id}$, where the metric $g = \iota X(\iota\,\mathrm{Id} \circ \iota\,\mathrm{Id}) + \iota T \circ \iota\,\mathrm{Id} + g_D + \pi^*\Phi$ is given as in Theorem 1. Aimed to construct examples of Bochner-flat para-Kähler surfaces of non-constant scalar curvature we analyze the case where the (1, 1)-tensor field T is parallel. In this situation, the nonzero components of the covariant derivative of J_h reduce to

$$\begin{aligned}8\,(\nabla J_h)_{1;1}{}^4 = {} & x_{1'}^3\left\{T_2^1\operatorname{tr}(T) + 8\mathcal{S}_2{}^1\right\} + x_{1'}^2x_{2'}\left\{(T_2^2)^2 - (T_1^1)^2 + 8(\mathcal{S}_2^2 - \mathcal{S}_1{}^1)\right\}\\ & + x_{1'}x_{2'}^2\left\{-T_1^2\operatorname{tr}(T) - 8\mathcal{S}_1{}^2\right\} + x_{1'}^2\left\{2X^1(8h - 4\Phi_{12}) - 4X^2\Phi_{22}\right\}\\ & + x_{1'}x_{2'}\left\{16hX^2 + 8X^1\Phi_{11}\right\} + x_{2'}^2\left\{4X^2\Phi_{11}\right\}\\ & + x_{1'}\left\{16\rho_{21}^D + 10hT_1^1 + 2hT_2^2 - 2\operatorname{tr}(T)\Phi_{12} + 4(\hat\Phi_{21} - \hat\Phi_{12})\right\}\\ & + x_{2'}\left\{-16\rho_{11}^D + 8hT_1^2 + 2\operatorname{tr}(T)\Phi_{11}\right\}\\ & + 8\left\{\partial_1 h - h({}^D\Gamma_{11}{}^1 + {}^D\Gamma_{12}{}^2) + D\Phi_{11;2} - D\Phi_{12;1}\right\},\end{aligned}$$

$$\begin{aligned}8(\nabla J_h)_{1;2}{}^4 = {} & x_{2'}^3\left\{-T_1^2\operatorname{tr}(T) - 8\mathcal{S}_1{}^2\right\} + x_{1'}^2x_{2'}\left\{T_2^1\operatorname{tr}(T) + 8\mathcal{S}_2{}^1\right\}\\ & + x_{1'}x_{2'}^2\left\{(T_2^2)^2 - (T_1^1)^2 + 8(\mathcal{S}_2^2 - \mathcal{S}_1{}^1)\right\} + x_{1'}^2\left\{-4X^1\Phi_{22}\right\}\\ & + x_{1'}x_{2'}\left\{16hX^1 - 8X^2\Phi_{22}\right\} + x_{2'}^2\left\{2X^2(8h + 4\Phi_{12}) + 4X^1\Phi_{11}\right\}\\ & + x_{1'}\left\{16\rho_{22}^D + 8hT_2^1 - 2\operatorname{tr}(T)\Phi_{22}\right\}\\ & + x_{2'}\left\{-16\rho_{12}^D + 2hT_1^1 + 10hT_2^2 + 2\operatorname{tr}(T)\Phi_{12} + 4(\hat\Phi_{21} - \hat\Phi_{12})\right\}\\ & + 8\left\{\partial_2 h - h({}^D\Gamma_{22}{}^2 + {}^D\Gamma_{12}{}^1) + D\Phi_{12;2} - D\Phi_{22;1}\right\}.\end{aligned}$$

Since the tensor field T is parallel, it has constant trace and X must be nonzero so that $\tau = 12\iota X + 3\operatorname{tr}(T)$ is non-constant.

If $X^1 \neq 0$ it follows immediately from the expression of the coefficient of $x_{1'}^2$ in $(\nabla J_h)_{1;2}{}^4$ above that $\Phi_{22} = 0$. Knowing this, the expression of the coefficient of $x_{1'}x_{2'}$ in $(\nabla J_h)_{1;2}{}^4$ shows that $h = 0$. The same coefficient in $(\nabla J_h)_{1;1}{}^4$ shows that $\Phi_{11} = 0$ and now, focusing on the coefficient of $x_{1'}^2$ in $(\nabla J_h)_{1;1}{}^4$ we see that $\Phi_{12} = 0$. If $X^2 \neq 0$, proceeding analogously it follows that h and Φ vanish identically.

Since both the function h and the symmetric $(0, 2)$-tensor field Φ vanish, the components of the covariant derivative ∇J_h reduce to

$$\begin{aligned}8\,(\nabla J_h)_{1;1}{}^4 = {} & x_{1'}^3\left\{T_2^1\operatorname{tr}(T) + 8\mathcal{S}_2{}^1\right\} + x_{1'}^2x_{2'}\left\{(T_2^2)^2 - (T_1^1)^2 + 8(\mathcal{S}_2^2 - \mathcal{S}_1{}^1)\right\}\\ & + x_{1'}x_{2'}^2\left\{-T_1^2\operatorname{tr}(T) - 8\mathcal{S}_1{}^2\right\} + 16x_{1'}\rho_{21}^D - 16x_{2'}\rho_{11}^D,\\ 8(\nabla J_h)_{1;2}{}^4 = {} & x_{2'}^3\left\{-T_1^2\operatorname{tr}(T) - 8\mathcal{S}_1{}^2\right\} + x_{1'}^2x_{2'}\left\{T_2^1\operatorname{tr}(T) + 8\mathcal{S}_2{}^1\right\} + 16x_{1'}\rho_{22}^D\\ & + x_{1'}x_{2'}^2\left\{(T_2^2)^2 - (T_1^1)^2 + 8(\mathcal{S}_2^2 - \mathcal{S}_1{}^1)\right\} - 16x_{2'}\rho_{12}^D.\end{aligned}$$

The linear terms in these two expressions show that the Ricci tensor of the affine surface must vanish identically. Therefore the affine connection is necessarily flat. Assume that T is trace-free. At this point, the components of ∇J_h take the form

$$8\,(\nabla J_h)_{1;1}{}^4 = 8x_{1'}^3\mathcal{S}_2^1 + x_{1'}^2 x_{2'}\left\{(T_2^2)^2 - (T_1^1)^2 + 8(\mathcal{S}_2^2 - \mathcal{S}_1^1)\right\} - 8x_{1'}x_{2'}^2 S_1^2,$$
$$8(\nabla J_h)_{1;2}{}^4 = 8x_{1'}^2 x_{2'}\mathcal{S}_2^1 + x_{1'}x_{2'}^2\left\{(T_2^2)^2 - (T_1^1)^2 + 8(\mathcal{S}_2^2 - \mathcal{S}_1^1)\right\} - 8x_{2'}^3\mathcal{S}_1^2.$$

The existence of parallel (1, 1)-tensor fields on affine surfaces was studied in [9] showing that (besides the case where $T = 0$) a trace-free parallel (1, 1)-tensor field on an affine surface (Σ, D) corresponds to one of the following.

(a) An *affine para-Kähler structure* ($\det(T) = -\kappa^2 < 0$), which in suitable adapted coordinates becomes $T = \kappa(\partial_{x^1} \otimes dx^1 - \partial_{x^2} \otimes dx^2)$.
(b) An *affine nilpotent Kähler structure* ($T^2 = 0$), which in suitable adapted coordinates becomes $T = \kappa\partial_{x^1} \otimes dx^2$.
(c) An *affine Kähler structure* ($\det(T) = \kappa^2 > 0$), which in suitable adapted coordinates becomes $T = \kappa(\partial_{x^2} \otimes dx^1 - \partial_{x^1} \otimes dx^2)$.

Straightforward calculations now show that, for any case described above, there exist local coordinates in which the (1, 1)-tensor filed $\mathcal{S} = DX$ takes the form $\mathcal{S} = \lambda\,\mathrm{Id}$ for some function $\lambda \in \mathcal{C}^\infty(\Sigma)$ and the scalar curvature is given by $\tau = \iota X$.

We summarize the discussion above in the following

Theorem 3 *Let (Σ, D) be an affine surface and let (g, J_h) be an almost para-Hermitian structure on $T^*\Sigma$ such that*

$$J_h|_{\ker \pi_*} = \mathrm{Id} \quad \textit{and} \quad g = \iota X(\iota\,\mathrm{Id} \circ \iota\,\mathrm{Id}) + \iota T \circ \iota\,\mathrm{Id} + g_D + \pi^*\Phi.$$

If T is parallel and $\mathrm{tr}(T) = 0$ then $(T^*\Sigma, g, J_h)$ is a Bochner-flat para-Kähler surface if and only if $h = 0$, $\Phi = 0$, the affine connection D is flat and $\mathcal{S} = \lambda\,\mathrm{Id}$ for some $\lambda \in \mathcal{C}^\infty(\Sigma)$, being $\mathcal{S}(Z) = D_Z X$.

Remark 1 If the affine connection D is flat, there exist local coordinates on Σ so that all the Christoffel symbols are zero. After a suitable linear transformation on the coordinates one can set T being of one of the forms described above. Straightforward calculations show that there exist suitable adapted coordinates in which Bochner-flat para-Kähler structures determined by a trace-free parallel (1, 1)-tensor field T are given by the vector field $X = ax^i\partial_{x^i}$, where $a \in \mathbb{R}$ and thus $\mathcal{S} = a\,\mathrm{Id}$. We subsequently examine the different possibilities.

(a) Let T be an affine para-Kähler structure on a flat affine surface (Σ, D) and take local coordinates so that ${}^D\Gamma_{ij}{}^k = 0$, $T = \kappa(\partial_{x^1} \otimes dx^1 - \partial_{x^2} \otimes dx^2)$ and $X = ax^i\partial_{x^i}$. Then the Bochner-flat metric induced on $T^*\Sigma$ is given by

$$g = x_{1'}^2(ax_{1'}x^1 + ax_{2'}x^2 + \kappa)dx^1 \otimes dx^1 + x_{2'}^2(ax_{1'}x^1 + ax_{2'}x^2 - \kappa)dx^2 \otimes dx^2$$
$$+ ax_{1'}x_{2'}(x_{1'}x^1 + x_{2'}x^2)(dx^1 \otimes dx^2 + dx^2 \otimes dx^1) + dx^1 \otimes dx_{1'} + dx^2 \otimes dx_{2'}.$$

A straightforward calculation shows that the Ricci curvatures are given by

$$\lambda_\pm = 2\iota X \pm \sqrt{(\iota X)^2 + 2\iota(TX) + \kappa^2},$$

and the Ricci operator diagonalizes with real eigenvalues on the zero section of $T^*\Sigma$. The curvature on the zero section of the cotangent bundle corresponds to that of a locally conformally flat para-Kähler surface determined by an affine para-Kähler structure, thus it is modelled on $(\mathfrak{P}_k)$.

(b) Let T be an affine nilpotent Kähler structure on a flat affine surface (Σ, D) and take local coordinates so that ${}^D\Gamma_{ij}{}^k = 0$, $T = \kappa\partial_{x^1} \otimes dx^2$ and $X = ax^i\partial_{x^i}$. Then the Bochner-flat metric induced on $T^*\Sigma$ is given by

$$\begin{aligned} g = & \, ax_{1'}^2(x_{1'}x^1+x_{2'}x^2)dx^1 \otimes dx^1+x_{2'}(ax_{2'}(x_{1'}x^1+x_{2'}x^2) + \kappa x_{1'})dx^2 \otimes dx^2 \\ & +\tfrac{1}{2}x_{1'}(2ax_{2'}(x_{1'}x^1+x_{2'}x^2)+\kappa x_{1'})(dx^1 \otimes dx^2+dx^2 \otimes dx^1) \\ & +dx^1 \otimes dx_{1'}+dx^2 \otimes dx_{2'}. \end{aligned}$$

A straightforward calculation shows that the Ricci curvatures are given by

$$\lambda_\pm = 2\iota X \pm \sqrt{(\iota X)^2 + 2\iota(TX)},$$

and the Ricci operator is two-step nilpotent on the zero section of $T^*\Sigma$. The curvature on the zero section of the cotangent bundle corresponds to that of a locally conformally flat para-Kähler surface determined by an affine nilpotent Kähler structure, thus it is modelled on $(\mathfrak{M})$.

(c) Let T be an affine Kähler structure on a flat affine surface (Σ, D) and take local coordinates so that ${}^D\Gamma_{ij}{}^k = 0$, $T = \kappa(\partial_{x^2} \otimes dx^1 - \partial_{x^1} \otimes dx^2)$ and $X = ax^i\partial_{x^i}$. Then the Bochner-flat metric induced on $T^*\Sigma$ is given by

$$\begin{aligned} g = & \, x_{1'}(ax_{1'}(x_{1'}x^1 + x_{2'}x^2) + \kappa x_{2'})dx^1 \otimes dx^1 \\ & +x_{2'}(ax_{2'}(x_{1'}x^1 + x_{2'}x^2) - \kappa x_{1'})dx^2 \otimes dx^2 \\ & +\left(ax_{1'}x_{2'}(x_{1'}x^1+x_{2'}x^2)+\tfrac{1}{2}(x_{2'}^2-x_{1'}^2)\kappa\right)(dx^1 \otimes dx^2+dx^2 \otimes dx^1) \\ & +dx^1 \otimes dx_{1'}+dx^2 \otimes dx_{2'}. \end{aligned}$$

A straightforward calculation shows that the Ricci curvatures are given by

$$\lambda_\pm = 2\iota X \pm \sqrt{(\iota X)^2 + 2\iota(TX) - \kappa^2},$$

and the Ricci operator has complex eigenvalues on the zero section of $T^*\Sigma$. The curvature on the zero section of the cotangent bundle corresponds to that of a locally conformally flat para-Kähler surface determined by an affine Kähler structure, thus it is modelled on $(\mathfrak{N}_k)$.

Remark 2 In the case where T is a parallel $(1, 1)$-tensor field with nonzero trace on (Σ, D), we denote $T^0 = T - \frac{\operatorname{tr}(T)}{2}\,\mathrm{Id}$ the traceless part of T so that $T = T^0 + \mu\,\mathrm{Id}$, being $\operatorname{tr}(T) = 2\mu$. Since the affine connection D is flat, there exist local coordinates in which all the Christoffel symbols ${}^D\Gamma_{ij}{}^k$ are zero. After a suitable linear transformation on the coordinates we can set T^0 being of one of the forms described in (a), (b) and (c) above. Straightforward calculations show that there exist suitable adapted coordinates in which Bochner-flat para-Kähler structures determined by a parallel $(1,1)$-tensor field T with $\operatorname{tr}(T) \neq 0$ correspond to one of the following situations.

(a) If T^0 is an affine para-Kähler structure then the $(1,1)$-tensor fields T and $\mathcal{S}$ commute and there exist local coordinates in which $T = (\mu + \kappa)\partial_{x^1} \otimes dx^1 + (\mu - \kappa)\partial_{x^2} \otimes dx^2$ and $X = ax^1\partial_{x^1} + (a + \frac{\kappa\mu}{2})\partial_{x^2}$. The Bochner-flat metric induced on $T^*\Sigma$ has Ricci curvatures given by

$$\lambda_\pm = \tfrac{3}{2}\mu + 3\iota X \pm \sqrt{(\iota X)^2 + \iota(TX) + \iota(KX) + \kappa^2},$$

where $K = T - \operatorname{tr}(T)\,\mathrm{Id}$. The Ricci operator diagonalizes with real eigenvalues $\lambda_\pm = \frac{3}{2}\mu \pm \kappa$ on the zero section of the cotangent bundle $T^*\Sigma$.

(b) If T^0 is an affine nilpotent Kähler structure then the $(1,1)$-tensor fields T and $\mathcal{S}$ commute and there exist local coordinates in which $T = \mu(\partial_{x^1} \otimes dx^1 + \partial_{x^2} \otimes dx^2) + \kappa\partial_{x^1} \otimes dx^2$ and $X = \left(ax^1 - \frac{\kappa\mu}{4}x^2\right)\partial_{x^1} + ax^2\partial_{x^2}$. The Bochner-flat metric induced on $T^*\Sigma$ has Ricci curvatures given by

$$\lambda_\pm = \tfrac{3}{2}\mu + 3\iota X \pm \sqrt{(\iota X)^2 + \iota(TX) + \iota(KX)},$$

where $K = T - \operatorname{tr}(T)\,\mathrm{Id}$. On the zero section of the cotangent bundle the Ricci operator has a single eigenvalue $\lambda = \frac{3}{2}\mu$ that is a double root of its minimal polynomial.

(c) If T^0 is an affine Kähler structure then the $(1,1)$-tensor fields T and $\mathcal{S}$ commute and there exist local coordinates in which $T = \mu(\partial_{x^1} \otimes dx^1 + \partial_{x^2} \otimes dx^2) + \kappa(\partial_{x^2} \otimes dx^1 - \partial_{x^1} \otimes dx^2)$ and $X = \left(ax^1 + \frac{\kappa\mu}{4}x^2\right)\partial_{x^1} + \left(ax^2 - \frac{\kappa\mu}{4}x^1\right)\partial_{x^2}$. The Bochner-flat metric induced on $T^*\Sigma$ has Ricci curvatures given by

$$\lambda_\pm = \tfrac{3}{2}\mu + 3\iota X \pm \sqrt{(\iota X)^2 + \iota(TX) + \iota(KX) - \kappa^2},$$

where $K = T - \operatorname{tr}(T)\,\mathrm{Id}$. The Ricci operator is complex-diagonalizable with eigenvalues $\lambda_\pm = \frac{3}{2}\mu + \sqrt{-\kappa^2}$ on the zero section of $T^*\Sigma$.

(d) If $T^0 = 0$ there exist local coordinates in which the $(1,1)$-tensor field T is a multiple of the identity $T = \mu\,\mathrm{Id}$ and $X = ax^i\partial_{x^i}$. The Bochner-flat metric induced on $T^*\Sigma$ has Ricci curvatures given by

$$\lambda_\pm = \tfrac{3}{2}\mu + 3\iota X \pm \sqrt{(\iota X)^2 + \iota(TX) + \iota(KX)},$$

where $K = T - \operatorname{tr}(T)\,\mathrm{Id}$. On the zero section of the cotangent bundle, the Ricci operator is diagonalizable with a single real eigenvalue $\lambda = \frac{3}{2}\mu$. Therefore the paraholomorphic sectional curvature is constant on the zero section of $T^*\Sigma$.

Acknowledgements This work has been supported by projects PID2019-105138GB-C21(AEI/FEDER, Spain), and ED431C 2019/10, ED431F 2020/04 (Xunta de Galicia, Spain).

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