Curvature and Killing Vector Fields on Lorentzian 3-Manifolds

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Abstract Although it originated in the study of 4-dimensional spacetimes, the Newman-Penrose formalism is also an effective tool in dimension three, provided that a distinguished vector field is present. Here we show how a 3-dimensional version of the Newman-Penrose formalism can be used to study both the local and global geometry of Lorentzian 3-manifolds. Globally, we find obstructions to Lorentzian metrics generalizing those of constant curvature; locally, we classify Lorentzian 3-manifolds that admit a timelike Killing vector field. These results have appeared in [1] and [3], the latter joint with R. Ream.

Keywords Lorentzian geometry · Ricci curvature · Killing vector fields · Local classification · Global obstructions

1 Introduction

On three-dimensional Lorentzian manifolds, and in the presence of distinguished timelike vector field, the *Newman-Penrose formalism*—when properly adapted to such a setting—is a useful frame technique by which to derive results both global and local. In this paper we give examples of both, to illustrate the method. We start with the global setting first, where we generalize the nonexistence of positive Lorentzian Einstein metrics in dimension 3, by proving the following:

Theorem 1 *Let M be a closed 3-manifold, $\lambda > 0$ a constant, and f a smooth function that never takes the values $0, \lambda$. Then there is no Lorentzian metric g on M whose Ricci tensor satisfies*

$$Ric = fg + (f - \lambda)T^\flat \otimes T^\flat, \tag{1}$$

for any unit timelike vector field T. If (1) holds with $\lambda = 0$, then (M, g) is isometric to $(\mathbb{S}^1 \times N, -dt^2 \oplus h)$ with (N, h) a Riemannian manifold.

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Although Theorem 1 does not cover the case $f = \lambda$, nevertheless this would reduce (1) to the Einstein condition $\text{Ric} = \lambda g$ with positive Einstein constant. In dimension 3, this is equivalent to (M, g) having constant positive sectional curvature—and it is well known that there are *no* such (closed) Lorentzian manifolds. (Its proof came in two stages: [7] showed that there are no geodesically complete such manifolds, in any dimension; [10] then showed that every closed constant curvature Lorentzian manifold must be geodesically complete, with the flat case having already been established in [5]. See [11] for a comprehensive account.) It is in this sense that (1) generalizes the nonexistence of positive Einstein metrics. Notice that the distinguished timelike vector field here is T: it defines the direction of deviation from the Einstein condition.

We now move to the local setting, together with a stronger condition on T, namely, that it be a timelike Killing vector field, $\mathcal{L}_T g_L = 0$. The Newman-Penrose formalism here allows for a complete local classification of Lorentzian 3-manifolds admitting timelike Killing vector fields:

Theorem 2 *Let (M, g_L) be a Lorentzian 3-manifold that admits a unit timelike Killing vector field T. Then there exists local coordinates (t, r, θ) and a smooth function $\varphi(r, \theta)$ such that*

$$T = \partial_t \quad , \quad g_L = -(T^\flat)^2 + dr^2 + \varphi^2 d\theta^2, \tag{2}$$

and where the quotient metric $dr^2 + \varphi^2 d\theta^2$ has Gaussian curvature

$$-\frac{\varphi_{rr}}{\varphi} = \frac{1}{2}\big(S_L - Ric_L(T, T)\big),$$

with S_L and Ric_L the scalar curvature and Ricci tensor of g_L, respectively.

In both Theorems 1 and 2, we use the three-dimensional version of the *Newman-Penrose formalism* [12], which we outline in Sect. 2. This is a frame technique which has by now appeared in many guises in dimensions 3 and 4, both Lorentzian and Riemannian; see, e.g., [2, 4, 9, 13, 20]. The principal virtue of the Newman-Penrose formalism is that *it converts second-order differential equations involving curvature into first-order differential equations involving properties of the flow of our distinguished vector field*, be it the vector field T in (1) or the one in (2). Doing so will, it turns out, simplify the analysis in the proofs of Theorems 1 and 2 considerably.

2 The Newman-Penrose Formalism for Lorentzian 3-Manifolds

Let T be a smooth unit length timelike vector field defined in an open subset of a Lorentzian 3-manifold (M, g) without boundary, so that $\nabla_v T \perp T$ for all vectors v

(∇ is the Levi-Civita connection, and we adopt the index $-++$ for the Lorentzian metric g). Let X and Y be two smooth spacelike vector fields such that $\{T, X, Y\}$ is a local orthonormal frame, where by "timelike" and "spacelike" we mean simply that

$$g(T,T) = -1 \quad , \quad g(X,X) = g(Y,Y) = 1.$$

For such a T, there is an endomorphism D defined on the normal bundle $T^{\perp} \subset TM$:

$$D\colon T^{\perp} \longrightarrow T^{\perp}, \; v \;\mapsto\; \nabla_v T. \tag{3}$$

As is well known, the matrix of D with respect to the frame $\{T, X, Y\}$,

$$D = \begin{pmatrix} g(\nabla_X T, X) & g(\nabla_Y T, X) \\ g(\nabla_X T, Y) & g(\nabla_Y T, Y) \end{pmatrix}$$

carries three crucial pieces of information associated to the flow of T:

1. The divergence of T, denoted div T, is the trace of D.
2. By Frobenius's theorem, $T^{\perp}$ is integrable if and only if the off-diagonal elements of (6) satisfy

$$\omega := g(T,[X,Y]) = g(\nabla_Y T, X) - g(\nabla_X T, Y) = 0. \tag{4}$$

   Since ω^2 equals the determinant of the anti-symmetric part of D (see (6) below), ω is invariant up to sign, called the *twist* of T. We say that the flow of T is *twist-free* if $\omega = 0$.
3. The third piece of information is the *shear* σ of T; it is given by the trace-free symmetric part of D, whose components σ_1, σ_2 we combine here into a complex-valued quantity, for reasons that will become clear below:

$$\sigma := \underbrace{\frac{1}{2}\Big(g(\nabla_Y T, Y) - g(\nabla_X T, X)\Big)}_{\sigma_1} + i \; \underbrace{\frac{1}{2}\Big(g(\nabla_Y T, X) + g(\nabla_X T, Y)\Big)}_{\sigma_2}. \tag{5}$$

   Although σ itself is not invariant, its magnitude $|\sigma^2|$ is: by (6) below, it is minus the determinant of the trace-free symmetric of D. We say that the flow of T is *shear-free* if $\sigma = 0$. As with being twist-free, being shear-free is a frame-independent statement. In terms of div T, ω, and σ, D is

$$D = \begin{pmatrix} \frac{1}{2}\text{div}\, T - \sigma_1 & \sigma_2 + \frac{\omega}{2} \\ \sigma_2 - \frac{\omega}{2} & \frac{1}{2}\text{div}\, T + \sigma_1 \end{pmatrix}. \tag{6}$$

   Note that (3) has been applied to arbitrary n-dimensional Lorentzian manifolds, yielding integral inequalities via Bochner's technique, in particular when D is skew-symmetric, or (in dimension 3) when the vector field of interest is spacelike, in [17] and [18].

We now present the Newman-Penrose formalism for Lorentzian 3-manifolds, which is well known; see [9], and more recently, [13]. Our presentation here parallels the three-dimensional Riemannian treatment to be found in [2], and is meant to fix notation; in what follows, any sign changes that arise due to the Lorentzian index, as compared to the Riemannian case in [2], are indicated by red text—we will in fact need the Riemannian version in Sect. 4. Let $\{T, X, Y\}$ be as above and define the complex-valued quantities

$$\boldsymbol{m} := \frac{1}{\sqrt{2}}(X - iY) \quad , \quad \overline{\boldsymbol{m}} := \frac{1}{\sqrt{2}}(X + iY).$$

Henceforth we work with the complex frame $\{T, \boldsymbol{m}, \overline{\boldsymbol{m}}\}$, for which only $g(T, T) = -1$ and $g(\boldsymbol{m}, \overline{\boldsymbol{m}}) = 1$ are nonzero. The following quantities associated to this complex triad play a central role in all that follows.

Definition 1 The spin coefficients of the complex frame $\{T, \boldsymbol{m}, \overline{\boldsymbol{m}}\}$ are the complex-valued functions

$$\kappa = -g(\nabla_T T, \boldsymbol{m}) \quad , \quad \rho = -g(\nabla_{\overline{\boldsymbol{m}}} T, \boldsymbol{m}) \quad , \quad \sigma = -g(\nabla_{\boldsymbol{m}} T, \boldsymbol{m}),$$
$$\varepsilon = g(\nabla_T \boldsymbol{m}, \overline{\boldsymbol{m}}) \quad , \quad \beta = g(\nabla_{\boldsymbol{m}} \boldsymbol{m}, \overline{\boldsymbol{m}}).$$

Note that the flow of T is geodesic, $\nabla_T T = 0$, if and only if $\kappa = 0$; that

$$\varepsilon = i g(\nabla_T X, Y)$$

is purely imaginary; that σ is actually the complex shear (5); and finally that the spin coefficient ρ is given by

$$-2\rho = \operatorname{div} T + i\,\omega. \tag{7}$$

It is clear that κ, ρ, σ directly represent the three geometric properties of the flow of T discussed above. In terms of the five spin coefficients in Definition 1, the covariant derivatives of $\{T, \boldsymbol{m}, \overline{\boldsymbol{m}}\}$ are given by

$$\begin{aligned}
\nabla_T T &= -\bar{\kappa}\,\boldsymbol{m} - \kappa\,\overline{\boldsymbol{m}},\\
\nabla_{\boldsymbol{m}} T &= -\bar{\rho}\,\boldsymbol{m} - \sigma\,\overline{\boldsymbol{m}},\\
\nabla_T \boldsymbol{m} &= -\kappa\,T + \varepsilon\,\boldsymbol{m},\\
\nabla_{\boldsymbol{m}} \boldsymbol{m} &= -\sigma\,T + \beta\,\boldsymbol{m},\\
\nabla_{\boldsymbol{m}} \overline{\boldsymbol{m}} &= -\bar{\rho}\,T - \beta\,\overline{\boldsymbol{m}},
\end{aligned}$$

while their Lie brackets are

$$[T, \boldsymbol{m}] = -\kappa\, T + (\varepsilon + \bar{\rho})\, \boldsymbol{m} + \sigma\, \overline{\boldsymbol{m}}, \tag{8}$$
$$[\boldsymbol{m}, \overline{\boldsymbol{m}}] = -(\bar{\rho} - \rho)\, T + \bar{\beta}\, \boldsymbol{m} - \beta\, \overline{\boldsymbol{m}}. \tag{9}$$

All other covariant derivatives, as well as the Lie bracket $[T, \overline{\boldsymbol{m}}]$, are obtained by complex conjugation. Now onto curvature; begin by observing that the Riemann and Ricci tensors

$$R(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{z}) = g(\nabla_{\boldsymbol{u}} \nabla_{\boldsymbol{v}} \boldsymbol{w} - \nabla_{\boldsymbol{v}} \nabla_{\boldsymbol{u}} \boldsymbol{w} - \nabla_{[\boldsymbol{u},\boldsymbol{v}]} \boldsymbol{w}, \boldsymbol{z}),$$
$$\mathrm{Ric}(\cdot\,, \cdot) = -R(T, \cdot, \cdot, T) + R(\boldsymbol{m}, \cdot, \cdot, \overline{\boldsymbol{m}}) + R(\overline{\boldsymbol{m}}, \cdot, \cdot, \boldsymbol{m}),$$

satisfy the following relationships in the complex frame $\{T, \boldsymbol{m}, \overline{\boldsymbol{m}}\}$:

$$\begin{cases} \mathrm{Ric}(\boldsymbol{m}, \boldsymbol{m}) = +R(T, \boldsymbol{m}, T, \boldsymbol{m}), \\ \mathrm{Ric}(T, T) = -2R(T, \boldsymbol{m}, T, \overline{\boldsymbol{m}}), \\ \mathrm{Ric}(T, \boldsymbol{m}) = -R(T, \boldsymbol{m}, \boldsymbol{m}, \overline{\boldsymbol{m}}), \\ \mathrm{Ric}(\boldsymbol{m}, \overline{\boldsymbol{m}}) = -\frac{1}{2}\mathrm{Ric}(T, T) - R(\overline{\boldsymbol{m}}, \boldsymbol{m}, \boldsymbol{m}, \overline{\boldsymbol{m}}). \end{cases}$$

Using these, and expressing the Riemann tensor as

$$R(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{z}) = \boldsymbol{u} g(\nabla_{\boldsymbol{v}} \boldsymbol{w}, \boldsymbol{z}) - g(\nabla_{\boldsymbol{v}} \boldsymbol{w}, \nabla_{\boldsymbol{u}} \boldsymbol{z}) - \boldsymbol{v} g(\nabla_{\boldsymbol{u}} \boldsymbol{w}, \boldsymbol{z})$$
$$+ g(\nabla_{\boldsymbol{u}} \boldsymbol{w}, \nabla_{\boldsymbol{v}} \boldsymbol{z}) - g(\nabla_{[\boldsymbol{u},\boldsymbol{v}]} \boldsymbol{w}, \boldsymbol{z}),$$

leads to the following equations; along with (15) and (16) below, play the crucial role in the Newman-Penrose formalism:

$$T(\rho) - \overline{\boldsymbol{m}}(\kappa) = -|\kappa|^2 + |\sigma|^2 + \rho^2 + \kappa\bar{\beta} + \frac{1}{2}\mathrm{Ric}(T, T), \tag{10}$$

$$T(\sigma) - \boldsymbol{m}(\kappa) = -\kappa^2 + 2\sigma\varepsilon + \sigma(\rho + \bar{\rho}) - \kappa\beta - \mathrm{Ric}(\boldsymbol{m}, \boldsymbol{m}), \tag{11}$$

$$\boldsymbol{m}(\rho) - \overline{\boldsymbol{m}}(\sigma) = 2\sigma\bar{\beta} - (\bar{\rho} - \rho)\kappa + \mathrm{Ric}(T, \boldsymbol{m}), \tag{12}$$

$$T(\beta) - \boldsymbol{m}(\varepsilon) = \sigma(\bar{\kappa} - \bar{\beta}) + \kappa(-\varepsilon - \bar{\rho}) + \beta(\varepsilon + \bar{\rho}) - \mathrm{Ric}(T, \boldsymbol{m}), \tag{13}$$

$$\boldsymbol{m}(\bar{\beta}) + \overline{\boldsymbol{m}}(\beta) = -|\sigma|^2 + |\rho|^2 - 2|\beta|^2 - (\rho - \bar{\rho})\varepsilon - \mathrm{Ric}(\boldsymbol{m}, \overline{\boldsymbol{m}}) - \frac{1}{2}\mathrm{Ric}(T, T). \tag{14}$$

Finally, there are two nontrivial differential Bianchi identities,

$$(\nabla_T R)(T, \boldsymbol{m}, \boldsymbol{m}, \overline{\boldsymbol{m}}) + (\nabla_{\boldsymbol{m}} R)(T, \boldsymbol{m}, \overline{\boldsymbol{m}}, T) + (\nabla_{\overline{\boldsymbol{m}}} R)(T, \boldsymbol{m}, T, \boldsymbol{m}) = 0,$$
$$(\nabla_T R)(\overline{\boldsymbol{m}}, \boldsymbol{m}, \boldsymbol{m}, \overline{\boldsymbol{m}}) + (\nabla_{\boldsymbol{m}} R)(\overline{\boldsymbol{m}}, \boldsymbol{m}, \overline{\boldsymbol{m}}, T) + (\nabla_{\overline{\boldsymbol{m}}} R)(\overline{\boldsymbol{m}}, \boldsymbol{m}, T, \boldsymbol{m}) = 0,$$

which, in terms of spin coefficients, take the following forms:

$$T(\mathrm{Ric}(T,\boldsymbol{m})) - \frac{1}{2}\boldsymbol{m}(\mathrm{Ric}(T,T)) - \overline{\boldsymbol{m}}(\mathrm{Ric}(\boldsymbol{m},\boldsymbol{m})) = \tag{15}$$
$$-\kappa\big(+\mathrm{Ric}(T,T)+\mathrm{Ric}(\boldsymbol{m},\overline{\boldsymbol{m}})\big) + \big(\varepsilon+2\rho+\bar{\rho}\big)\mathrm{Ric}(T,\boldsymbol{m})$$
$$+\sigma\,\mathrm{Ric}(T,\overline{\boldsymbol{m}}) - \big(\bar{\kappa}-2\bar{\beta}\big)\mathrm{Ric}(\boldsymbol{m},\boldsymbol{m})$$

and

$$\boldsymbol{m}(\mathrm{Ric}(T,\overline{\boldsymbol{m}})) + \overline{\boldsymbol{m}}(\mathrm{Ric}(T,\boldsymbol{m})) - T\big(\mathrm{Ric}(\boldsymbol{m},\overline{\boldsymbol{m}}) + (1/2)\mathrm{Ric}(T,T)\big) =$$
$$-(\rho+\bar{\rho})\big(\mathrm{Ric}(T,T)+\mathrm{Ric}(\boldsymbol{m},\overline{\boldsymbol{m}})\big) - \bar{\sigma}\mathrm{Ric}(\boldsymbol{m},\boldsymbol{m}) - \sigma\mathrm{Ric}(\overline{\boldsymbol{m}},\overline{\boldsymbol{m}}) \tag{16}$$
$$-\big(-2\bar{\kappa}+\bar{\beta}\big)\mathrm{Ric}(T,\boldsymbol{m}) - \big(-2\kappa+\beta\big)\mathrm{Ric}(T,\overline{\boldsymbol{m}}).$$

E.g., to derive (16), expand the second differential Bianchi identity, beginning with its first term:

$$(\nabla_T R)(\overline{\boldsymbol{m}},\boldsymbol{m},\boldsymbol{m},\overline{\boldsymbol{m}}) = T(R(\overline{\boldsymbol{m}},\boldsymbol{m},\boldsymbol{m},\overline{\boldsymbol{m}})) - R(\nabla_T\overline{\boldsymbol{m}},\boldsymbol{m},\boldsymbol{m},\overline{\boldsymbol{m}})$$
$$- R(\overline{\boldsymbol{m}},\nabla_T\boldsymbol{m},\boldsymbol{m},\overline{\boldsymbol{m}}) - R(\overline{\boldsymbol{m}},\boldsymbol{m},\nabla_T\boldsymbol{m},\overline{\boldsymbol{m}}) - R(\overline{\boldsymbol{m}},\boldsymbol{m},\boldsymbol{m},\nabla_T\overline{\boldsymbol{m}}).$$

In terms of spin coefficients and the Ricci tensor, each term is

$$T(R(\overline{\boldsymbol{m}},\boldsymbol{m},\boldsymbol{m},\overline{\boldsymbol{m}})) = -T(\mathrm{Ric}(\boldsymbol{m},\overline{\boldsymbol{m}}) + \frac{1}{2}\mathrm{Ric}(T,T)),$$
$$R(\nabla_T\overline{\boldsymbol{m}},\boldsymbol{m},\boldsymbol{m},\overline{\boldsymbol{m}}) = -\bar{\kappa}\underbrace{R(T,\boldsymbol{m},\boldsymbol{m},\overline{\boldsymbol{m}})}_{-\mathrm{Ric}(T,\boldsymbol{m})} + \bar{\varepsilon}\underbrace{R(\overline{\boldsymbol{m}},\boldsymbol{m},\boldsymbol{m},\overline{\boldsymbol{m}})}_{-\mathrm{Ric}(\boldsymbol{m},\overline{\boldsymbol{m}})-\frac{1}{2}\mathrm{Ric}(T,T)},$$
$$= R(\overline{\boldsymbol{m}},\boldsymbol{m},\boldsymbol{m},\nabla_T\overline{\boldsymbol{m}}),$$
$$R(\overline{\boldsymbol{m}},\nabla_T\boldsymbol{m},\boldsymbol{m},\overline{\boldsymbol{m}}) = -\kappa\underbrace{R(\overline{\boldsymbol{m}},T,\boldsymbol{m},\overline{\boldsymbol{m}})}_{-\mathrm{Ric}(T,\overline{\boldsymbol{m}})} + \varepsilon\underbrace{R(\overline{\boldsymbol{m}},\boldsymbol{m},\boldsymbol{m},\overline{\boldsymbol{m}})}_{-\mathrm{Ric}(\boldsymbol{m},\overline{\boldsymbol{m}})-\frac{1}{2}\mathrm{Ric}(T,T)},$$
$$= R(\overline{\boldsymbol{m}},\boldsymbol{m},\nabla_T\boldsymbol{m},\overline{\boldsymbol{m}}),$$

Thus the term $(\nabla_T R)(\overline{\boldsymbol{m}},\boldsymbol{m},\boldsymbol{m},\overline{\boldsymbol{m}})$ simplifies to

$$-T\big(\mathrm{Ric}(\boldsymbol{m},\overline{\boldsymbol{m}}) + (1/2)\mathrm{Ric}(T,T)\big) - 2\kappa\mathrm{Ric}(T,\overline{\boldsymbol{m}}) - 2\bar{\kappa}\mathrm{Ric}(T,\boldsymbol{m}),$$

where two terms cancel because $\varepsilon+\bar{\varepsilon}=0$. Repeating this process on the remaining terms in the second differential Bianchi identity yields (16); the first differential Bianchi identity (15) is similarly derived. This concludes the derivation of the Newman-Penrose formalism for Lorentzian 3-manifolds.

3 The Newman-Penrose Formalism and Global Obstructions

3.1 Evolution Equations for Divergence, Twist, and Shear

To prove Theorem 1, we first need to gather some information regarding the flow of T appearing in (1); the first-order differential equations appearing here exemplify "what the Newman-Penrose formalism is good for."

Proposition 1 *Let* (M, g) *be a Lorentzian 3-manifold whose Ricci tensor satisfies*

$$Ric = fg + (f-\mu)T^\flat \otimes T^\flat,$$

for some unit timelike vector field T*, constant* μ*, and smooth function* f *which never takes the value* μ*. Then* T *has geodesic flow, and its divergence, twist, and shear satisfy the following differential equations:*

$$T(\text{div}\,T) = \frac{\omega^2}{2} - 2|\sigma|^2 - \frac{1}{2}(\text{div}\,T)^2 + \mu, \tag{17}$$

$$T(\omega^2) = -2(\text{div}\,T)\,\omega^2, \tag{18}$$

$$T(|\sigma|^2) = -2(\text{div}\,T)\,|\sigma|^2. \tag{19}$$

Furthermore, f *satisfies*

$$T(f-\mu) = -(\text{div}\,T)(f-\mu), \tag{20}$$

and, recalling (6)*, the function* $H := \det D - \frac{\mu}{2}$ *satisfies*

$$T(\text{div}\,T) = 2H - (\text{div}\,T)^2 + 2\mu, \tag{21}$$

$$T(H) = -(\text{div}\,T)H. \tag{22}$$

Proof Let $\{T, X, Y\}$ be an orthonormal frame, with X, Y possibly only locally defined, and let $\{T, \boldsymbol{m}, \overline{\boldsymbol{m}}\}$ be the corresponding complex frame (recall that div T, ω^2, and $|\sigma|^2$ are globally defined, frame-independent quantities). Then the Ricci tensor in this complex frame satisfies

$$\text{Ric}(T,T) = -\mu \quad , \quad \underbrace{\text{Ric}(\boldsymbol{m}, \overline{\boldsymbol{m}})}_{\frac{1}{2}(\text{Ric}(X,X)+\text{Ric}(Y,Y))} = f,$$

with all other components vanishing. That T has geodesic flow, $\nabla_T T = 0$, now follows from the differential Bianchi identity (15), which reduces to

$$\kappa\,\underbrace{(\mathrm{Ric}(\boldsymbol{m},\overline{\boldsymbol{m}})+\mathrm{Ric}(T,T))}_{f-\mu\neq 0}=0,$$

hence $\kappa = 0$. Since $\kappa = 0$ if and only if $\nabla_T T = 0$, this proves the geodesic flow of T. Next, (17) and (18) are the real and imaginary parts of (10), respectively, after setting $\kappa = 0$ therein. With $\kappa = \mathrm{Ric}(\boldsymbol{m},\boldsymbol{m}) = 0$, (11) also simplifies, to

$$T(\sigma)=2\sigma\varepsilon+\sigma\underbrace{(\rho+\bar{\rho})}_{-\mathrm{div}\,T},$$

from which (19) follows because $|\sigma|^2 = \sigma\bar{\sigma}$ and $\varepsilon + \bar{\varepsilon} = 0$. The second differential Bianchi identity (16) yields

$$-T(f-\mu/2)=(\mathrm{div}\,T)(-\mu+f),$$

which is (20), since $T(f-\mu/2) = T(f-\mu)$. Finally, as

$$\det D=\frac{\omega^2}{4}-|\sigma|^2+\frac{(\mathrm{div}\,T)^2}{4},$$

Equation (21) and (22) both follow from (17)–(19).

An immediate consequence of these evolution equations is the following

Corollary 1 *Assume the hypotheses of Proposition 1. If $\mu \geq 0$ and M is closed, then T is also divergence-free.*

Proof By Proposition 1, T has geodesic flow; because M is closed, this flow is complete. We now consider the cases $\mu > 0$ and $\mu = 0$ separately:

1. $\mu > 0$: To show that $\mathrm{div}\,T = 0$, we will use (22), by showing that H is in fact a nonzero constant on M. Indeed, suppose that $H(p) = 0$ at some point $p \in M$, and let $\gamma^{(p)}(t)$ be the (complete) integral curve of T starting at p. By (22), the function $H \circ \gamma^{(p)} \colon \mathbb{R} \longrightarrow \mathbb{R}$ is identically zero; by (21), $\theta(t) := (\mathrm{div}\,T \circ \gamma^{(p)})(t)$ satisfies

$$\frac{d\theta}{dt}=-\theta^2+2\mu. \tag{23}$$

With $\mu > 0$, this has complete solutions $\theta(t) = \sqrt{2\mu}\tanh\big(\sqrt{2\mu}\,t + c\big)$ and $\theta(t) = \pm\sqrt{2\mu}$. But in fact all of these solutions are impermissible, as can be seen by restricting (20) to $\gamma^{(p)}$. Indeed, substituting $\theta(t) = \pm\sqrt{2\mu}$ into (20) yields the solutions

$$f(t)-\mu=(f(0)-\mu)e^{\mp\sqrt{2\mu}\,t}.$$

Likewise, the solution $\theta(t) = \sqrt{2\mu}\tanh\big(\sqrt{2\mu}\,t + c\big)$, when it is inserted into (20), yields the solution

$$f(t) - \mu = \big(f(0) - \mu\big)\operatorname{sech}\big(\sqrt{2\mu}\,t + c\big).$$

But in either case, the right-hand side goes to zero whereas the left-hand side is bounded away from zero (recall that M is closed and f never takes the value μ). Thus H must be nowhere vanishing on M, in which case, consider $1/H$: by (22), we have that $T(1/H) = \frac{\operatorname{div} T}{H}$, and hence

$$T(T(1/H)) \overset{(21)}{=} 2 + \frac{2\mu}{H}. \tag{24}$$

The latter equation has solution

$$(1/H)(t) = -\frac{1}{\mu} + c_1 e^{\sqrt{2\mu}\,t} + c_2 e^{-\sqrt{2\mu}\,t},$$

but unless $c_1 = c_2 = 0$, this solution is unbounded. We therefore conclude that the function H is a nonzero constant on M, which immediately implies that $\operatorname{div} T = 0$ by (22).

2. $\mu = 0$: Once again, suppose that $H(p) = 0$, so that $H \circ \gamma^{(p)} \colon \mathbb{R} \longrightarrow \mathbb{R}$ is identically zero, in which case (21) now becomes

$$\frac{d\theta}{dt} = -\theta^2.$$

This has $\theta(t) = 0$ as its only complete solution. If $H(p) \neq 0$, so that $H \circ \gamma^{(p)} \colon \mathbb{R} \longrightarrow \mathbb{R}$ is nowhere vanishing, then applying (24) along $\gamma^{(p)}$ yields $(1/H)''(t) = 2$, hence $(1/H)(t) = t^2 + c_1 t + c_2$ and thus, by (24),

$$\theta(t) = \frac{2t + c_1}{t^2 + c_1 t + c_2},$$

with $c_1^2 < 4c_2^2$ to ensure that $(1/H)(t)$ is nowhere vanishing. But then (20) would yield

$$f(t) = \frac{c_3}{t^2 + c_1 t + c_2},$$

which is impossible because f, never taking the value $\mu = 0$, must be bounded away from 0 on closed M. We conclude that H must be the zero function, and so $\operatorname{div} T = 0$ once again.

This completes the proof.

(As an aside, it is instructive to consider what happens when $\mu < 0$. In this case, if $H \circ \gamma^{(p)} \colon \mathbb{R} \longrightarrow \mathbb{R}$ is identically zero, then

$$\frac{d\theta}{dt} = -\theta^2 + 2\mu$$

has no complete solutions. Thus H must be nowhere vanishing on M, but this time (24) yields

$$(1/H)(t) = -\frac{1}{\mu} + c_1 \sin(\sqrt{2|\mu|}\, t) + c_2 \cos(\sqrt{2|\mu|}\, t)$$

and (20)

$$f(t) = \mu + \frac{c_3}{-\frac{1}{\mu} + c_1 \sin(\sqrt{2|\mu|}\, t) + c_2 \cos(\sqrt{2|\mu|}\, t)}\cdot$$

But both of these are well behaved for appropriate constants, so there is no contradiction.) In any case, when $\mu > 0$, then a distinguished orthonormal frame appears, which plays a crucial role in the proof of Theorem 1:

Proposition 2 *Assume the hypotheses of Proposition 1. If $\mu > 0$ and M is closed and simply connected, then there exists a global orthonormal frame $\{T, X, Y\}$ with respect to which the spin coefficients κ, ρ, σ take the form*

$$\kappa = \rho + \bar{\rho} = 0 \quad , \quad \sigma = -\sqrt{\frac{\mu}{2}} + i\,\frac{\omega}{2}\cdot \tag{25}$$

Proof By Proposition 1, T has geodesic flow, so that $\kappa = 0$; by its Corollary, T is also divergence-free, so that $\rho = -i\frac{\omega}{2}$ (recall (7)). Now we show the existence of a local orthonormal frame $\{T, X, Y\}$ with respect to which the shear σ takes the form (25). Begin by observing that when $\kappa = \text{div}\, T = 0$, (17) reduces to

$$2|\sigma|^2 - \frac{\omega^2}{2} = \mu.$$

This in turn implies that the endomorphism $D\colon T^{\perp} \longrightarrow T^{\perp}$, whose matrix is given by (6), has the two distinct eigenvalues $\pm\sqrt{\mu/2}$ (note that D is self-adjoint with respect to the induced (Riemannian) metric $g|_{T^{\perp}}$ on $T^{\perp}$ if and only if ω vanishes). Therefore, consider the respective kernels of the two bundle endomorphisms $D \pm \sqrt{\mu/2}I\colon T^{\perp} \longrightarrow T^{\perp}$; if M is simply connected, then these line bundles have smooth nowhere vanishing global sections X, Y_1,

$$D(X) = \sqrt{\mu/2}\, X \quad , \quad D(Y_1) = -\sqrt{\mu/2}\, Y_1,$$

which we can take to have unit length, and which are both spacelike because they are orthogonal to T. If necessary, modify Y_1 so that it is orthogonal to X, by defining

$$Y := -g(X, Y_1)X + Y_1$$

and normalizing Y to have unit length. Then the global orthonormal frame $\{T, X, Y\}$ now has shear σ given by (25); indeed, substituting (6) into $D(X) = \sqrt{\mu/2}\, X$ yields

$$\begin{pmatrix} -\sigma_1 \\ \sigma_2 - \frac{\omega}{2} \end{pmatrix} = \sqrt{\mu/2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \sigma_1 = -\sqrt{\mu/2} \quad , \quad \sigma_2 = \frac{\omega}{2}, \tag{26}$$

completing the proof. □

Armed with Corollary 1 and Proposition 2, we now prove Theorem 1, starting with the $\lambda > 0$ case:

Proof (*Proof of $\lambda > 0$ case of Theorem* 1) Suppose a closed Lorentzian 3-manifold (M, g) exists satisfying (1), with $\lambda > 0$. By the Corollary to Proposition 1, $\text{div}\, T = 0$. Now assume that M is simply connected; then by Proposition 2, there exists a global orthonormal frame $\{T, X, Y\}$ satisfying

$$\kappa = 0 \quad , \quad \rho = -i\frac{\omega}{2} \quad , \quad \sigma = -\sqrt{\frac{\lambda}{2}} + i\frac{\omega}{2}. \tag{27}$$

Using this information, we now show the existence of a vector field Z and a smooth function ψ on M satisfying

$$Z(\psi) = f, \tag{28}$$

which is impossible, as f never takes the value 0 and M is closed. This will be shown by inserting the spin coefficients (27) into (11), (12), and (14). Indeed, (11) simplifies to

$$\underbrace{T(\sigma)}_{0} \overset{(11)}{=} 2\sigma\varepsilon,$$

where $T(\sigma) = 0$ because $\text{div}\, T = 0$, hence $T(\omega^2) = 0$ in (18); in particular, $\varepsilon = 0$ since $\sigma \neq 0$. Next, because $\beta = \frac{1}{\sqrt{2}}\big(g(\nabla_Y X, Y) - ig(\nabla_X Y, X)\big) = \frac{1}{\sqrt{2}}(\text{div}\, X - i\,\text{div}\, Y)$ (since $\nabla_T T = 0$),

$$\underbrace{\boldsymbol{m}(\rho) - \overline{\boldsymbol{m}}(\sigma)}_{-\frac{i}{\sqrt{2}}X(\omega)} \overset{(12)}{=} 2\sigma\bar{\beta},$$

which in turn yields the pair of equations

$$\begin{cases} \sqrt{2\lambda}\,\text{div}\, X + \omega\,\text{div}\, Y = 0, \\ \sqrt{2\lambda}\,\text{div}\, Y - \omega\,\text{div}\, X = X(\omega). \end{cases} \tag{29}$$

Finally, (14), when simplified using $\varepsilon = 0$, yields

$$
\begin{aligned}
\underbrace{\boldsymbol{m}(\bar{\beta}) + \overline{\boldsymbol{m}}(\beta)}_{X(\operatorname{div} X) + Y(\operatorname{div} Y)} &= \underbrace{-|\sigma|^2 + |\rho|^2}_{-\lambda/2} - 2|\beta|^2 - \underbrace{(\rho - \bar{\rho})\varepsilon}_{0} \\
&\qquad - \underbrace{\operatorname{Ric}(\boldsymbol{m}, \overline{\boldsymbol{m}})}_{f} - \frac{1}{2}\underbrace{\operatorname{Ric}(T, T)}_{-\lambda} \\
&= -(\operatorname{div} X)^2 - (\operatorname{div} Y)^2 - f. \qquad (30)
\end{aligned}
$$

This further simplifies,

$$
\begin{aligned}
X(\operatorname{div} X) &\overset{(29)}{=} -\frac{1}{\sqrt{2\lambda}} X(\omega) \operatorname{div} Y - \frac{\omega}{\sqrt{2\lambda}} X(\operatorname{div} Y) \\
&\overset{(29)}{=} -(\operatorname{div} Y)^2 + \underbrace{\frac{\omega}{\sqrt{2\lambda}}(\operatorname{div} X)(\operatorname{div} Y)}_{-(\operatorname{div} X)^2} - \frac{\omega}{\sqrt{2\lambda}} X(\operatorname{div} Y),
\end{aligned}
$$

so that (30) reduces, finally, to

$$
\left(\frac{\omega}{\sqrt{2\lambda}} X - Y\right)(\operatorname{div} Y) = f. \qquad (31)
$$

With $Z := \frac{\omega}{\sqrt{2\lambda}} X - Y$ and $\psi := \operatorname{div} Y$, this is precisely (28). This proves the Theorem in the case when M is simply connected. If M is not simply connected, then pass to its simply connected universal cover $(\widetilde{M}, \tilde{g})$; it is locally isometric to (M, g) via the projection $\pi\colon \widetilde{M} \longrightarrow M$, and therefore its Ricci tensor $\widetilde{\operatorname{Ric}}$ will satisfy (1), with $f \circ \pi$ in place of f, and with $\widetilde{T}$ the (complete) lift of T. Repeating step-by-step our argument on $(\widetilde{M}, \tilde{g})$, we arrive once again at (31). Although $\widetilde{M}$ need not be compact, a contradiction is still obtained because $f \circ \pi$ is bounded away from zero, because $\operatorname{div} Y$ is also bounded (since $d\pi(Y)$ is well defined up to sign, $|\operatorname{div} d\pi(Y)|$ is continuous on M), and because Z is complete on $\widetilde{M}$. This completes the proof. □

Proof (*Proof of* $\lambda = 0$ *case of Theorem* 1) (*This proof parallels that of Theorem 3 in* [2], *but generalizes it: due to our Corollary above, the function* f *is not assumed to be constant, as it was in* [2]; *furthermore*, (35) *below is a necessary step that was not required in* [2].) When $\lambda = 0$,

$$
|\sigma|^2 - \frac{\omega^2}{4} = 0, \qquad (32)
$$

which implies that D has only the eigenvalue 0. As there are no longer two distinct eigenvalues, we cannot call upon Proposition 2; instead, we prove that if (32) holds then T must be parallel. Doing so will then allow us to draw two conclusions: first, that (M, g) must be geodesically complete, by [16]; second, that the universal cover of (M, g) must be isometric to $(\mathbb{R} \times \widetilde{N}, -dt^2 \oplus \tilde{h})$ for some Riemannian 2-manifold $(\widetilde{N}, \tilde{h})$, by the de Rham Decomposition Theorem for Lorentzian manifolds [29]. To begin with, observe that T being parallel is equivalent to the condition

$$\kappa = \rho = \sigma = 0.$$

By Proposition 1 and its Corollary, $\nabla_T T = \operatorname{div} T = 0$, so that we need only show that $\omega^2 = 0$; we'll do this by showing that the open set

$$U = \{p \in M : \omega^2(p) \neq 0\}$$

is empty. Assume for the moment that U is simply connected. Then over U, D has constant rank 1 (recall (6)), so that its kernel is a line bundle over U; as the latter is simply connected, this kernel has a nowhere vanishing section X on U. Now let $\{T, X, Y\}$ be an orthonormal frame, with Y perhaps only locally defined in U. Then the analogue of (26) is now

$$\begin{pmatrix} -\sigma_1 \\ \sigma_2 - \frac{\omega}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so that $\sigma_1 = 0$ and $\sigma_2 = \omega/2$. Thus the analogue of (25) is $\rho = -i\frac{\omega}{2}$ and $\sigma = i\frac{\omega}{2}$. Proceeding as in the proof of the $\lambda > 0$ case above, (29) becomes

$$\omega(\operatorname{div} Y) = 0 \quad , \quad X(\omega) = -\omega\,(\operatorname{div} X). \tag{33}$$

Now $\operatorname{div} Y = 0$ because ω is nowhere vanishing in U, in which case (30) reduces to

$$X(\operatorname{div} X) = -(\operatorname{div} X)^2 - f. \tag{34}$$

If the flow of X was complete in U, then we would obtain a contradiction because (34) has no complete solutions when $f > 0$. Our task is therefore done if we can prove that the flow of X is complete. Thus, let $\gamma\colon [0, b) \longrightarrow U$ be an integral curve of X that is maximally extended to the right, and suppose that $b < \infty$ (the case $(-b, 0]$ will follow from this one by letting $X \to -X$, which leaves (33) and (34) unaltered). To begin with, there is a sequence $t_n \to b$ such that $\{\gamma(t_n)\}$ converges to some $q \in M\backslash U$ (if q were in U, then the integral curve γ would be extendible, contradicting our assumption that it was maximally extended; see, e.g., [15, Lemma 56]). (Though such a sequence is obvious because M is compact, let us give an alternative proof that will also suffice should we need to pass to the universal cover of M below: consider the Riemannian metric h on M defined by

$$h := g + 2T^\flat \otimes T^\flat. \tag{35}$$

Since M is closed, h is complete; as X has h-unit length,

$$d_h(\gamma(0), \gamma(t)) \leq L_h(\gamma|_{[0,t]}) = t,$$

where d_h is the Riemannian distance associated to h. This implies that

$$\gamma([0, b)) \subseteq \{p \in M : d_h(\gamma(0), p) \leq b\}.$$

By the completeness of (M, h), the latter set is compact, hence any sequence $\{\gamma(t_n)\}$ with $t_n \to b$ has a convergent subsequence; cf. [8, Proposition 3.4].) Now set $\theta(t) := (\text{div}\, X \circ \gamma)(t)$ and $\omega^2(t) := (\omega^2 \circ \gamma)(t)$; in particular, observe that $\lim_{n\to\infty} \omega^2(t_n) = 0$ because $q \notin U$. By (33),

$$\omega^2(t) = \omega_0^2\, e^{-2\int_0^t \theta(s)\, ds} \qquad \text{for all } t \in [0, b).$$

By (34), $\theta(t)$ is nonincreasing ($f > 0$), in which case $-2\int_0^t \theta(s)\, ds \geq -2\theta_0 t$ for all $t \in [0, b)$. But then $\omega^2(t) \geq w_0^2 e^{-2\theta_0 t} > 0$, so that

$$\lim_{n\to\infty} w^2(t_n) > 0,$$

a contradiction. Thus we must have $b = \infty$; this proves that U, if simply connected, must be empty. If U is not simply connected, then pass to the universal covers of $(U, g|_U) \subset (M, g)$ and repeat the argument (with $\omega^2 \circ \pi$, and noting that the lift of the Riemannian metric (35) will be complete), noting that any integral curve of T starting in U stays in U, because $T(\omega^2) = 0$ via (18). This completes the proof that the universal cover of (M, g) is isometric to $(\mathbb{R} \times \widetilde{N}, -dt^2 \oplus \tilde{h})$, in which case (M, g) itself is isometric to $(\mathbb{S}^1 \times N, -dt^2 \oplus h)$ with (N, h) a Riemannian 2-manifold. This establishes the $\lambda = 0$ case of Theorem 1. ☐

4 The Newman-Penrose Formalism and Local Classifications

We now move to the local setting. Simply put, the reason why the Newman-Penrose formalism is so effective when T is a unit timelike Killing vector field is because, although divergence and shear are not enough to characterize unit length Killing vector fields in general, they *do* when $\dim M = 3$. (For reasons that will become clear below, we first start in the Riemannian setting.)

Lemma 1 *A unit length vector field T on a Riemannian 3-manifold (M, g) is a Killing vector field if and only if its flow is geodesic, divergence-free, and shear-free.*

Proof The Killing condition, $\mathcal{L}_T g = 0$, is equivalent to

$$g(\nabla_v T, w) + g(\nabla_w T, v) = 0 \qquad \text{for all } v, w \in TM, \tag{36}$$

from which it follows that any Killing vector field T is divergence-free and shear-free, via (5). Finally, (36) also implies that any unit length Killing vector field must have geodesic flow:

$$\nabla_T T = 0.$$

Conversely, suppose that a unit length vector field T is geodesic, divergence-free, and shear-free, and consider (36). Writing v, w with respect to an orthonormal frame $\{T, X, Y\}$ as

$$v = a_0T + a_1X + a_2Y \quad , \quad w = b_0T + b_1X + b_2Y,$$

we have

$$\begin{aligned} g(\nabla_v T, w) + g(\nabla_w T, v) =& a_1 g(\nabla_X T, w) + a_2 g(\nabla_Y T, w) \\ &+ a_1 g(\nabla_w T, X) + a_2 g(\nabla_w T, Y) \\ =& 2a_1b_1 g(\nabla_X T, X) + (a_1b_2 + b_1a_2) g(\nabla_X T, Y) \\ &+ (a_1b_2 + b_1a_2) g(\nabla_Y T, X) + 2a_2b_2 \underbrace{g(\nabla_Y T, Y)}_{g(\nabla_X T, X)} \\ =& 2(a_1b_1 + a_2b_2) \underbrace{g(\nabla_X T, X)}_{\frac{1}{2}\operatorname{div} T} \\ &+ (a_1b_2 + b_1a_2) \underbrace{\big(g(\nabla_X T, Y) + g(\nabla_Y T, X)\big)}_{2\sigma_2} . \end{aligned}$$

This vanishes by our assumptions, completing the proof. □

Our plan of attack is as follows: as we have just seen, if T is a unit length Killing vector field, then $\operatorname{div} T$, σ, and $\nabla_T T$ all vanish, so that only T's twist function ω^2 is unknown. With half of the spin coefficients vanishing, the hope is that this will simplify things enough to allow a full determination of the metric. And it turns out that it will—even in the Riemannian setting, from which the Lorentzian case will then follow easily.

4.1 The Riemannian Case

Staying in the Riemannian setting, we begin with the following Lemma to set the stage:

Lemma 2 *Let (M, g) be a Riemannian 3-manifold admitting a unit length Killing vector field T with twist function ω^2. With respect to any complex frame $\{T, \boldsymbol{m}, \overline{\boldsymbol{m}}\}$, the Ricci tensor* Ric *and scalar curvature S satisfy*

$$T(\omega) = 0 \quad , \quad Ric(T, T) = \frac{\omega^2}{2} \quad , \quad Ric(\boldsymbol{m}, \boldsymbol{m}) = 0,$$
$$\boldsymbol{m}(\bar{\beta}) + \overline{\boldsymbol{m}}(\beta) = -2|\beta|^2 - i\omega\varepsilon - \frac{1}{2}\Big(S - \frac{\omega^2}{2}\Big). \qquad (37)$$

When (37) is written in terms of the underlying orthonormal frame $\{T, X, Y\}$ *of the complex frame, it is*

$$X(\text{div}\, X) + Y(\text{div}\, Y) = -(\text{div}\, X)^2 - (\text{div}\, Y)^2 - i\omega\varepsilon - \frac{1}{2}\Big(S - \frac{\omega^2}{2}\Big)\cdot \qquad (38)$$

Proof By Lemma 1, we know that

$$\kappa = \sigma = \rho + \bar{\rho} = 0;$$

inserting these into (10) and (11) directly yields the first line of equations; e.g.,

$$\text{Ric}(T, T) = \frac{\omega^2}{2} \quad , \quad T(\omega) = 0,$$

are, respectively, the real and imaginary parts of (10). Meanwhile, (37) follows from (14), which has no imaginary part, and the fact that the scalar curvature S in terms of the complex frame $\{T, \boldsymbol{m}, \overline{\boldsymbol{m}}\}$ is

$$S = \text{Ric}(T, T) + 2\text{Ric}(\boldsymbol{m}, \overline{\boldsymbol{m}}) = \frac{\omega^2}{2} + 2\text{Ric}(\boldsymbol{m}, \overline{\boldsymbol{m}}).$$

Finally, (38) follows from the fact that, when $\nabla_T T = 0$,

$$\beta = \frac{1}{\sqrt{2}}\big(g(\nabla_Y X, Y) + ig(\nabla_X X, Y)\big) = \frac{1}{\sqrt{2}}(\text{div}\, X - i\, \text{div}\, Y), \qquad (39)$$

which completes the proof. □

We have not yet considered the differential Bianchi identities; let us do so now. Inserting the contents of Lemma 2 into (15) and (16), as well as $\bar{\rho} = -\rho$, yields

$$T(\boldsymbol{m}(\rho)) = (\varepsilon + \bar{\rho})\boldsymbol{m}(\rho)$$

for (15); but this is precisely the Lie bracket (8) applied to ρ (bearing in mind that $T(\rho) = 0$), and therefore carries no new information. As for (16), it yields

$$-\boldsymbol{m}(\overline{\boldsymbol{m}}(\rho)) + \overline{\boldsymbol{m}}(\boldsymbol{m}(\rho)) = -\bar{\beta}\,\boldsymbol{m}(\rho) + \beta\,\overline{\boldsymbol{m}}(\rho),$$

where $T(S) = 0$ and $\bar{\rho} = -\rho$ have been used. But this is precisely the Lie bracket (9) applied to ρ, so that (16) also yields no new information.

4.2 Local Coordinates

The goal of this section is to establish the "right" local coordinates in which to prove Theorem 3 in the next section. To begin with, recall that because

$$\kappa = \sigma = \rho + \bar{\rho} = 0,$$

the only spin coefficients remaining are ε and β. Observe that the former is in fact purely imaginary,

$$\varepsilon = ig(\nabla_T X, Y), \tag{40}$$

and the latter, when $\nabla_T T = 0$, is given by (39). The following "gauge freedom" simultaneously enjoyed by these two spin coefficients will prove useful in the proof of Theorem 3:

Proposition 3 *Let T be a unit length Killing vector field with twist function ω^2 and $\{T, \boldsymbol{m}, \overline{\boldsymbol{m}}\}$ a complex frame. Then there exists a smooth real-valued function ϑ such that the complex frame $\{T, \boldsymbol{m}_*, \overline{\boldsymbol{m}}_*\}$ defined by the rotation*

$$\boldsymbol{m}_* := e^{i\vartheta}\boldsymbol{m} \quad , \quad \overline{\boldsymbol{m}}_* := e^{-i\vartheta}\overline{\boldsymbol{m}}$$

has spin coefficients $\kappa_* = \sigma_* = 0$, $\rho_* = \rho$,

$$\varepsilon_* = \rho \quad , \quad \text{Re}(\beta_*) = 0 \quad , \quad T(\beta_*) = 0. \tag{41}$$

Proof By definition,

$$\kappa_* = -g(\nabla_T T, \boldsymbol{m}_*) = e^{i\vartheta}\kappa = 0;$$

similarly, $\sigma_* = e^{2i\vartheta}\sigma = 0$, and $\rho_* = \rho$ (in particular, $\omega_*^2 = \omega^2$). Next,

$$\begin{aligned}\varepsilon_* &= g(\nabla_T \boldsymbol{m}_*, \overline{\boldsymbol{m}}_*)\\ &= e^{-i\vartheta} g(\nabla_T (e^{i\vartheta}\boldsymbol{m}), \overline{\boldsymbol{m}})\\ &= \varepsilon + e^{-i\vartheta} T(e^{i\vartheta})\\ &= \varepsilon + iT(\vartheta).\end{aligned} \tag{42}$$

Similarly,

$$\begin{aligned}\beta_* &= g(\nabla_{\boldsymbol{m}_*} \boldsymbol{m}_*, \overline{\boldsymbol{m}}_1)\\ &= g(\nabla_{\boldsymbol{m}} (e^{i\vartheta}\boldsymbol{m}), \overline{\boldsymbol{m}})\\ &= e^{i\vartheta}(\beta + i\boldsymbol{m}(\vartheta)).\end{aligned}$$

By (40) and (42), we may choose a locally defined function ϑ so that

$$\varepsilon_* = \rho_* = \rho.$$

Now, choose any other function ψ satisfying $T(\psi) = 0$ and rotate $\boldsymbol{m}_*, \overline{\boldsymbol{m}}_*$ by ψ; let $\{T, \boldsymbol{m}_o, \overline{\boldsymbol{m}}_o\}$ denote the corresponding frame. Then the analogue of (42) for the frame $\{T, \boldsymbol{m}_o, \overline{\boldsymbol{m}}_o\}$ shows that ε_o remains unchanged,

$$\varepsilon_o = \varepsilon_* = \rho_* = \rho,$$

so that our task would be complete if we can find a ψ satisfying

$$T(\psi) = 0 \quad \text{and} \quad \text{Re}(\beta_o) = 0. \tag{43}$$

To do so, observe that when $\varepsilon_* = \rho_*$, then

$$[T, \boldsymbol{m}_*] \overset{(8)}{=} 0. \tag{44}$$

Let $\{T, X_*, Y_*\}$ denote the underlying orthonormal frame corresponding to the complex frame $\{T, \boldsymbol{m}_*, \overline{\boldsymbol{m}}_*\}$. Since $[T, X_*] = 0$, there exist local coordinates (t, u, v) and functions p, q, r such that

$$T = \partial_t \quad , \quad X_* = \partial_u \quad , \quad Y_* = p\partial_t + q\partial_u + r\partial_v,$$

with p, q, r functions of u, v only, since $[T, Y_*] = 0$, and with r nowhere vanishing. The coframe metrically equivalent to $\{T, X_*, Y_*\}$ is therefore

$$T^\flat = dt - \frac{p}{r}dv \quad , \quad X_*^\flat = du - \frac{q}{r}dv \quad , \quad Y_*^\flat = \frac{1}{r}dv.$$

Next, since $(X_*^\flat)^2 + (Y_*^\flat)^2$ defines a Riemannian metric on the 2-manifold with coordinates $\{(u, v)\}$, and since any Riemannian 2-manifold is locally conformally flat (see, e.g., [6]), it follows that there exist coordinates (x, y) and a smooth function $\lambda(x, y)$ such that

$$(X_*^\flat)^2 + (Y_*^\flat)^2 = e^{2\lambda}(dx^2 + dy^2).$$

By a rotation in x, y if necessary, we may further assume that

$$X_*^\flat = e^\lambda dx \quad , \quad Y_*^\flat = e^\lambda dy.$$

In the new coordinates (t, x, y), we thus have that

$$T = \partial_t \quad , \quad X_* = e^{-\lambda}(\partial_x + a\partial_t) \quad , \quad Y_* = e^{-\lambda}(\partial_y + b\partial_t),$$

for some smooth functions $a(x, y), b(x, y)$. With these coordinates in hand, we now return to the task of satisfying (43), namely, finding a function $\psi(x, y)$ satisfying

$\text{Re}(\beta_o) = \text{Re}\big(e^{i\psi}(\beta_* + i\boldsymbol{m}_*(\psi))\big) = 0$, or

$$e^{i\psi}(\beta_* + i\boldsymbol{m}_*(\psi)) + e^{-i\psi}(\bar{\beta}_* - i\,\overline{\boldsymbol{m}}_*(\psi)) = 0. \tag{45}$$

When expanded, and using the fact that

$$\text{div}\, X_* = \frac{\lambda_x}{e^\lambda} \quad , \quad \text{div}\, Y_* = \frac{\lambda_y}{e^\lambda},$$

(45) is a quasilinear first-order PDE in ψ,

$$(\sin\psi)\psi_x - (\cos\psi)\psi_y = (\cos\psi)\lambda_x + (\sin\psi)\lambda_y,$$

which has a solution by the method of characteristics. Finally, (12) and (13) together yield

$$T(\beta_*) - \underbrace{\boldsymbol{m}(\varepsilon_*)}_{\boldsymbol{m}_*(\rho_*)} \overset{(13)}{=} -\underbrace{\text{Ric}(T, \boldsymbol{m}_*)}_{\boldsymbol{m}_*(\rho_*)\ \text{by}\ (12)},$$

which gives $T(\beta_*) = 0$, completing the proof. □

The following Corollary collects together what we've established so far:

Corollary 2 *Let (M, g) be a Riemannian 3-manifold and T a unit length Killing vector field with twist function ω^2. Then there exists an orthonormal frame $\{T, X, Y\}$ satisfying*

$$\kappa = \sigma = 0 \quad , \quad \rho = \varepsilon = -\frac{i}{2}\omega \quad , \quad \beta = -\frac{i}{\sqrt{2}}\,\text{div}\, Y, \tag{46}$$

and with $T(\omega) = T(\beta) = 0$. In this frame, (38) takes the form

$$Y(\text{div}\, Y) = -(\text{div}\, Y)^2 - \frac{1}{2}\Big(S + \frac{\omega^2}{2}\Big). \tag{47}$$

Notice that (47) implies that such a frame may not always exist *globally*; e.g., if M is compact and S is nonnegative and positive somewhere, then a standard Riccati argument yields that in such a case the only complete solution to (47) is one where $\text{div}\, Y = S + \frac{\omega^2}{2} = 0$, which is impossible. We now proceed to our local classification.

4.3 The Local Classification

One further modification to the orthonormal frame satisfying (46) will, it turns out, give us a complete local classification in the Riemannian setting, from which the Lorentzian analogue will then follow easily:

Theorem 3 *Let (M, g) be a Riemannian 3-manifold that admits a unit length Killing vector field T. Then there exist local coordinates (t, r, θ) and a smooth function $\varphi(r, \theta)$ such that*

$$T = \partial_t \quad , \quad g = (T^\flat)^2 + dr^2 + \varphi^2 d\theta^2, \tag{48}$$

and where the quotient metric $dr^2 + \varphi^2 d\theta^2$ has Gaussian curvature

$$-\frac{\varphi_{rr}}{\varphi} = \frac{1}{2}\big(S + Ric(T, T)\big), \tag{49}$$

with S and Ric the scalar curvature and Ricci tensor of g, respectively.

Proof Let (M, g) be a Riemannian 3-manifold and T a unit length Killing vector field with twist function ω^2. By Corollary 2, there exist a local orthonormal frame $\{T, X, Y\}$ satisfying (46) and coordinates (t, x, y) in which $T = \partial/\partial t$. Let $\{T^\flat, X^\flat, Y^\flat\}$ denote the dual coframe. We now modify the coordinates (t, x, y) while keeping $T = \partial/\partial t$ unchanged. The key is that (8) and (9) satisfy

$$[T, X] = [T, Y] = 0 \quad , \quad [X, Y] = \omega T + (\operatorname{div} Y)X,$$

from which it follows that $Y^\flat$ is closed, $dY^\flat = 0$; hence

$$Y^\flat = dr$$

for some smooth function $r(x, y)$. Similarly,

$$dX^\flat = (\operatorname{div} Y)Y^\flat \wedge X^\flat$$

implies that $X^\flat = \varphi d\theta$ for some smooth functions $\varphi(x, y) > 0$ and $\theta(x, y)$, with the former satisfying

$$Y(\varphi) = (\operatorname{div} Y)\varphi \tag{50}$$

(recall that $T(\beta) = 0$). Since $X(r) = Y(\theta) = 0$, we can define new coordinates (t, r, θ), in terms of which the frame $\{T, X, Y\}$ takes the form

$$T = \partial_t \quad , \quad X = h\partial_t + \frac{1}{\varphi}\partial_\theta \quad , \quad Y = k\partial_t + \partial_r, \tag{51}$$

for some smooth functions h, k; furthermore, $\varphi_t = h_t = k_t = 0$ (recall that $[T, X] = [T, Y] = 0$), so that φ, h, k are all functions of r, θ only. Thus

$$g = (T^\flat)^2 + (X^\flat)^2 + (Y^\flat)^2 = (T^\flat)^2 + dr^2 + \varphi^2 d\theta^2,$$

confirming (48). Now, the quotient metric $dr^2 + \varphi^2 d\theta^2$ has scalar curvature $-2\varphi_{rr}/\varphi$, hence Gaussian curvature $-\varphi_{rr}/\varphi$. To relate this to the curvature of (M, g), we take a Y-derivative of (50), make use of (47), and note that $\partial_t(\operatorname{div} Y) = 0$ by (41), to obtain $\varphi_{rr} = Y(\operatorname{div} Y)\varphi + (\operatorname{div} Y)^2\varphi$, which yields

$$-\frac{\varphi_{rr}}{\varphi} \overset{(47)}{=} \frac{1}{2}\Big(S + \frac{\omega^2}{2}\Big).$$

Since $\mathrm{Ric}(T, T) = \frac{\omega^2}{2}$ by Lemma 2, this confirms (49).

It is now an easy step to the Lorentzian setting and a proof of Theorem 2.

4.4 The Lorentzian Setting

Before proceeding to a proof of the Lorentzian analogue of Theorem 3, first observe that if a timelike T has unit length, $g_L(T, T) = -1$, then the metric

$$g_R := g_L + 2(T^{\flat_L})^2 \tag{52}$$

defines a Riemannian metric on M (here $T^{\flat_L} = g_L(T, \cdot)$). The following properties now hold between g_R and g_L:

1. T is a unit length Killing vector field with respect to g_R if and only if T is a unit timelike Killing vector field with respect to g_L (see, e.g., [14]).
2. If T is a g_R-unit length Killing vector field, then

$$\mathrm{Ric}_R(T, T) = \mathrm{Ric}_L(T, T)$$

(consult [14]; this follows because $\nabla^R_X T = -\nabla^L_X T$ for any unit length X that is g_R- or g_L-orthogonal to T, where ∇^R and ∇^L are, respectively, the Levi-Civita connections of g_R and g_L), while their scalar curvatures S_R and S_L satisfy

$$S_L = S_R + 2\mathrm{Ric}_R(T, T).$$

In particular, $S_R + \mathrm{Ric}_R(T, T) = S_L - \mathrm{Ric}_L(T, T)$.

With these facts established, the Lorentzian analogue of Theorem 3 now follows easily:

Proof (*Proof of Theorem* 2) By our remarks above, T is a unit length Killing vector field with respect to the Riemannian metric g_R, with $S_R + \mathrm{Ric}_R(T, T) = S_L - \mathrm{Ric}_L(T, T)$; thus Theorem 2 follows immediately from Theorem 3. □

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