Nilpotent Structures of Neutral 4-Manifolds and Light-Like Surfaces

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Abstract Nilpotent structures of neutral 4-manifolds are analogues of complex structures and paracomplex structures. Nilpotent structures give two-dimensional involutive distributions and the integral surfaces are light-like and analogues of complex curves and paracomplex curves. Light-like surfaces in neutral 4-manifolds with local horizontal lifts are characterized in terms of the curvature tensors and such surfaces are analogues of isotropic minimal surfaces in Riemannian 4-manifolds.

Keywords Nilpotent structure · Neutral 4-manifold · Light-like surface

1 Introduction

The purpose of this paper is to study almost nilpotent structures of neutral 4-manifolds and light-like surfaces in neutral 4-manifolds.

Almost nilpotent structures of neutral 4-manifolds are analogues of almost complex structures of Riemannian 4-manifolds. Almost complex structures on an oriented Riemannian 4-manifold (M, h) which are h-preserving and compatible with the orientation of M correspond to sections of a suitable one of the twistor spaces associated with M. Such an almost complex structure I is parallel with respect to the Levi-Civita connection ∇ of h if and only if the corresponding section Θ is horizontal with respect to the connection $\hat{\nabla}$ of the 2-fold exterior power of the tangent bundle TM induced by ∇. It is known that $\nabla I = 0$ just means that (M, h, I) is a Kähler surface and then I is its complex structure. If (M, h, I) is a Kähler surface, then integral surfaces of involutive I-invariant 2-dimensional distributions are complex curves of (M, I). A complex curve of a Kähler surface is just an isotropic minimal surface compatible with the orientation of the space and equipped with at least one complex point and notice that there exist totally geodesic surfaces in $\mathbb{C}P^2$, $\mathbb{C}H^2$, $\mathbb{C}P^1 \times \mathbb{C}P^1$, $\mathbb{C}H^1 \times \mathbb{C}H^1$ with no complex points ([1]). In general, an isotropic minimal surface

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in an oriented Riemannian 4-manifold compatible with the orientation of the space is characterized by horizontality of a suitable one of the twistor lifts ([12]). See [7] for the case where the space is S^4. We can refer to [11] for the twistor spaces and isotropic minimal surfaces.

On oriented neutral 4-manifolds, we can consider not only almost complex structures but also almost paracomplex structures. On such a 4-manifold (M, h), almost complex (resp. paracomplex) structures which are h-preserving (resp. h-reversing) and compatible with the orientation of M correspond to sections of a suitable one of the space-like (resp. time-like) twistor spaces associated with M. See [3, 6] for the space-like twistor spaces and [3, 13, 14] for the time-like twistor spaces. For almost complex structures and almost paracomplex structures, we can find analogues of results on almost complex structures of oriented Riemannian 4-manifolds ([3]). In addition, for complex curves of neutral Kähler surfaces and paracomplex curves of paraKähler surfaces, we can find analogues of results on complex curves of Kähler surfaces; for space-like or time-like surfaces in oriented neutral 4-manifolds with zero mean curvature vector which are isotropic and compatible with the orientations of the spaces, we can find analogues of results on isotropic minimal surfaces in oriented Riemannian 4-manifolds compatible with the orientations of the spaces ([3]).

The space-like (resp. time-like) twistor spaces associated with an oriented neutral 4-manifold (M, h) are fiber bundles such that fibers are hyperboloids of two sheets (resp. one sheet). They are contained in subbundles $\bigwedge^2_{\pm} TM$ of rank 3 in the 2-fold exterior power $\bigwedge^2 TM$ of TM. We can find fiber bundles $U_0(\bigwedge^2_{\pm} TM)$ in $\bigwedge^2_{\pm} TM$ respectively such that fibers are light-like cones. Our main objects of study in the present paper are almost nilpotent structures and they correspond to sections of either $U_0(\bigwedge^2_+ TM)$ or $U_0(\bigwedge^2_- TM)$. We will see that an almost nilpotent structure N is parallel with respect to ∇ if and only if the corresponding section Θ is horizontal with respect to $\hat{\nabla}$. If $\nabla N = 0$, then (h, N) is called a *nilpotent Kähler structure* of M, and M equipped with (h, N) is called a *nilpotent Kähler* 4*-manifold*. Neutral hyperKähler 4-manifolds have almost nilpotent structures parallel with respect to ∇ and we can refer to [10, 15] for neutral hyperKähler 4-manifolds. An almost nilpotent structure N of M gives a light-like 2-plane of the tangent space at each point of M. Therefore we have a light-like two-dimensional distribution $\mathscr{D}$. We will see that $\mathscr{D}$ is involutive if and only if for the section Θ corresponding to N and each tangent vector V of M contained in $\mathscr{D}$, the covariant derivative $\hat{\nabla}_V \Theta$ is given by Θ up to a constant. In particular, if $\nabla N = 0$, then $\mathscr{D}$ is involutive. In the case where $\nabla N = 0$, we can consider integral surfaces of $\mathscr{D}$ to be analogues of complex curves and paracomplex curves. Since $\mathscr{D}$ is light-like, we naturally have interest in light-like surfaces of M. Referring to the discussions on whether $\mathscr{D}$ is involutive, we will study a light-like surface in M with a nonzero horizontal section of a suitable one of the pull-back bundles of $U_0(\bigwedge^2_{\pm} TM)$ on a neighborhood of each point and we will see that a light-like surface in M has such a section if and only if ∇ induces a connection of the surface such that the curvature tensor of $\hat{\nabla}$ vanishes. We can consider light-like surfaces in M with local nonzero horizontal sections as above to be analogues of isotropic minimal surfaces in oriented Riemannian 4-manifolds compatible with the orientations of the spaces.

Remark 1 In [5], nilpotent Kähler structures of an oriented vector bundle E of rank 4 over $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ or $T^2 = S^1 \times S^1$ were studied. Let h be a neutral metric of E. Let ∇ be an h-connection of E, which means $\nabla h = 0$. Suppose that E is over S^1. Then we can find a nowhere zero, horizontal section Θ of $\bigwedge^2_+ E$ ([5]). If Θ is light-like, then Θ gives a nilpotent structure N of E and therefore (h, ∇, N) is a nilpotent Kähler structure of E. Suppose that E is over T^2. Then for a light-like, partially horizontal section Θ of $\bigwedge^2_+ E$, there exists an h-connection ∇' related to ∇ such that h, ∇' and Θ give a nilpotent Kähler structure of E ([5]).

2 Complex Structures and Paracomplex Structures of 4-Dimensional Neutral Vector Spaces

Let X be an oriented 4-dimensional vector space and h_X a neutral metric of X, i.e., an indefinite metric of X with signature (2, 2). Let $\bigwedge^2 X$ be the 2-fold exterior power of X and $\hat{h}_X$ the metric of $\bigwedge^2 X$ induced by h_X:

$$\begin{aligned}&\hat{h}_X(x_1 \wedge x_2, x_3 \wedge x_4)\\&= h_X(x_1, x_3)h_X(x_2, x_4) - h_X(x_1, x_4)h_X(x_2, x_3)\end{aligned}$$

$(x_i \in X)$. Let $\mathscr{B}_X$ be the set of ordered pseudo-orthonormal bases of X giving the orientation of X. Then $(e_1, e_2, e_3, e_4) \in \mathscr{B}_X$ satisfies

$$h_X(e_i, e_j) = \begin{cases} 1 & (i = j = 1 \text{ or } 2), \\ -1 & (i = j = 3 \text{ or } 4), \\ 0 & (\text{otherwise}). \end{cases}$$

For $(e_1, e_2, e_3, e_4) \in \mathscr{B}_X$, we set

$$\theta_{ij} := e_i \wedge e_j \quad (i, j \in \{1, 2, 3, 4\},\ i \neq j)$$

and

$$\begin{aligned}\Theta_{\pm,1} &:= \frac{1}{\sqrt{2}}(\theta_{12} \pm \theta_{34}),\\ \Theta_{\pm,2} &:= \frac{1}{\sqrt{2}}(\theta_{13} \pm \theta_{42}),\\ \Theta_{\pm,3} &:= \frac{1}{\sqrt{2}}(\theta_{14} \pm \theta_{23}).\end{aligned}$$

Then $\Theta_{\pm,1}$, $\Theta_{\pm,2}$, $\Theta_{\pm,3}$ form a pseudo-orthonormal basis of $\bigwedge^2 X$ and therefore we see that $\hat{h}_X$ has signature (2, 4). Let $\bigwedge^2_+ X$, $\bigwedge^2_- X$ be subspaces of $\bigwedge^2 X$ generated by $\Theta_{-,1}$, $\Theta_{+,2}$, $\Theta_{+,3}$ and $\Theta_{+,1}$, $\Theta_{-,2}$, $\Theta_{-,3}$, respectively. Then by the definitions of $\bigwedge^2_\pm X$, we have

$$\bigwedge^2 X = \bigwedge^2_+ X \oplus \bigwedge^2_- X$$

and we see that $\bigwedge^2_+ X$, $\bigwedge^2_- X$ are orthogonal to each other and that the restriction of $\hat{h}_X$ on each of them has signature (1, 2). In addition, noticing the double covering

$$SO_0(2,2) \longrightarrow SO_0(1,2) \times SO_0(1,2),$$

we see that $\bigwedge^2_\pm X$ do not depend on the choice of $(e_1, e_2, e_3, e_4) \in \mathscr{B}_X$.

We set

$$U_+\left(\bigwedge^2_\pm X\right) := \left\{ \Theta \in \bigwedge^2_\pm X \;\middle|\; \hat{h}_X(\Theta, \Theta) = 1 \right\}.$$

Then each $\Theta \in U_+\left(\bigwedge^2_+ X\right)$ corresponds to a unique h_X-preserving complex structure I of X satisfying

$$\Theta = \frac{1}{\sqrt{2}}(e \wedge I(e) - e^\perp \wedge I(e^\perp)), \tag{1}$$

where e is a space-like and unit vector of X and $e^\perp$ is a time-like vector of X satisfying

$$h_X(e^\perp, e^\perp) = -1, \quad h_X(e, e^\perp) = h_X(I(e), e^\perp) = 0.$$

Then we have $(e, I(e), e^\perp, I(e^\perp)) \in \mathscr{B}_X$, which means that I is compatible with the orientation of X. Conversely, each h_X-preserving complex structure I of X compatible with the orientation corresponds to a unique element of $U_+\left(\bigwedge^2_+ X\right)$ by (1). Hence we have a one-to-one correspondence between $U_+\left(\bigwedge^2_+ X\right)$ and the set of h_X-preserving complex structures of X compatible with the orientation. Similarly, we have a one-to-one correspondence between $U_+\left(\bigwedge^2_- X\right)$ and the set of h_X-preserving complex structures of X which are not compatible with the orientation.

We set

$$U_-\left(\bigwedge^2_\pm X\right) := \left\{ \Theta \in \bigwedge^2_\pm X \;\middle|\; \hat{h}_X(\Theta, \Theta) = -1 \right\}.$$

Then each $\Theta \in U_-\left(\bigwedge^2_+ X\right)$ corresponds to a unique h_X-reversing paracomplex structure J of X satisfying

$$\Theta = \frac{1}{\sqrt{2}}(e \wedge J(e) - e^\perp \wedge J(e^\perp)), \tag{2}$$

where $e, e^{\perp}$ are as above. Then we have $(e, J(e^{\perp}), J(e), e^{\perp}) \notin \mathscr{B}_X$, which means that J is not compatible with the orientation of X. Conversely, each h_X-reversing para-complex structure J of X which is not compatible with the orientation corresponds to a unique element of $U_-\left(\bigwedge^2_+ X\right)$ by (2). Hence we have a one-to-one correspondence between $U_-\left(\bigwedge^2_+ X\right)$ and the set of h_X-reversing paracomplex structures of X which are not compatible with the orientation. Similarly, we have a one-to-one correspondence between $U_-\left(\bigwedge^2_- X\right)$ and the set of h_X-reversing paracomplex structures of X compatible with the orientation.

3 Nilpotent Structures of 4-Dimensional Neutral Vector Spaces

In the present paper, our main objects of study are closely related to the light-like cones of $\bigwedge^2_\pm X$:

$$U_0\left(\bigwedge\nolimits^2_\pm X\right) := \left\{ \Theta \in \bigwedge\nolimits^2_\pm X \setminus \{0\} \;\middle|\; \hat{h}_X(\Theta, \Theta) = 0 \right\}.$$

For each $\Theta \in U_0\left(\bigwedge^2_+ X\right)$, there exists an element (e_1, e_2, e_3, e_4) of $\mathscr{B}_X$ satisfying

$$\Theta = \Theta_{-,1} + \Theta_{+,3}. \tag{3}$$

We call such a basis as (e_1, e_2, e_3, e_4) an *admissible basis* of Θ. Let G be a subgroup of $SO(2, 2)$ defined by

$$G := \left\{ B = \begin{bmatrix} b_1 & -b_2 & b_3 & b_4 \\ b_2 & b_1 & -b_4 & b_3 \\ b_3 & -b_4 & b_1 & b_2 \\ b_4 & b_3 & -b_2 & b_1 \end{bmatrix} \;\middle|\; \begin{array}{l} b_1, b_2, b_3, b_4 \in \mathbb{R}, \\ b_1^2 + b_2^2 - b_3^2 - b_4^2 = 1 \end{array} \right\}.$$

This is isomorphic to $SU(1, 1)$. Let H be a subset of $SO(2, 2)$ defined by

$$H := \left\{ C(h) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{h^2+2}{2} & h & -\frac{h^2}{2} \\ 0 & h & 1 & -h \\ 0 & \frac{h^2}{2} & h & -\frac{h^2-2}{2} \end{bmatrix} \;\middle|\; h \in \mathbb{R} \right\}.$$

We see that H is a subgroup of $SO(2, 2)$. Let (e'_1, e'_2, e'_3, e'_4) be another admissible basis of Θ than (e_1, e_2, e_3, e_4). Then there exist $B \in G$, $h \in \mathbb{R}$ satisfying

$$(e'_1, e'_2, e'_3, e'_4) = (e_1, e_2, e_3, e_4) B C(h). \tag{4}$$

We set

$$\Lambda := \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Then we have $\Lambda B = B\Lambda$ for any $B \in G$ and $\Lambda C(h) = C(h)\Lambda$ for any $h \in \mathbb{R}$. Therefore we see that a linear transformation N of X can be defined by

$$(N(e_1), N(e_2), N(e_3), N(e_4)) = (e_1, e_2, e_3, e_4)\Lambda \tag{5}$$

for an admissible basis (e_1, e_2, e_3, e_4) of Θ and that N is determined by Θ and does not depend on the choice of an admissible basis (e_1, e_2, e_3, e_4) of Θ. We call N a *nilpotent structure* of X corresponding to $\Theta \in U_0\left(\bigwedge_+^2 X\right)$. We denote by $\mathcal{N}_{X,+}$ the set of nilpotent structures of X corresponding to the elements of $U_0\left(\bigwedge_+^2 X\right)$. We have

$$\begin{aligned} \Theta &= \frac{1}{\sqrt{2}}(e_1 \wedge N(e_1) - e_3 \wedge N(e_3)) \\ &= \frac{1}{\sqrt{2}}(e_2 \wedge N(e_2) - e_4 \wedge N(e_4)). \end{aligned} \tag{6}$$

We set

$$V_1 := e_1 - e_3, \quad V_2 := e_2 + e_4.$$

Then we have $\Theta = (1/\sqrt{2})V_1 \wedge V_2$. We see that $\operatorname{Im} N$ is generated by light-like vectors V_1, V_2 and that it coincides with $\operatorname{Ker} N$. We have $h_X(N(x), x) = 0$ for any $x \in X$.

For each $\Theta \in U_0\left(\bigwedge_-^2 X\right)$, there exists an element (e_1, e_2, e_3, e_4) of $\mathcal{B}_X$ satisfying

$$\Theta = \Theta_{+,1} + \Theta_{-,3}.$$

We call such a basis as (e_1, e_2, e_3, e_4) an *admissible basis* of Θ. Let (e_1', e_2', e_3', e_4') be another admissible basis of Θ than (e_1, e_2, e_3, e_4). Then there exist $B \in G$, $h \in \mathbb{R}$ satisfying

$$(e_1', e_2', -e_3', e_4') = (e_1, e_2, -e_3, e_4)BC(h).$$

Therefore we see that a linear transformation N of X can be defined by

$$(N(e_1), N(e_2), -N(e_3), N(e_4)) = (e_1, e_2, -e_3, e_4)\Lambda$$

for an admissible basis (e_1, e_2, e_3, e_4) of Θ and that N is determined by Θ and does not depend on the choice of an admissible basis (e_1, e_2, e_3, e_4) of Θ. We call N a *nilpotent structure* of X corresponding to $\Theta \in U_0\left(\bigwedge_-^2 X\right)$. We denote by $\mathcal{N}_{X,-}$ the set of nilpotent structures of X corresponding to the elements of $U_0\left(\bigwedge_-^2 X\right)$. We have (6). We set

$$V_1 := e_1 + e_3, \quad V_2 := e_2 + e_4.$$

Then we have $\Theta = (1/\sqrt{2})V_1 \wedge V_2$. We see that $\mathrm{Im}\, N$ is generated by light-like vectors V_1, V_2 and that it coincides with $\mathrm{Ker}\, N$. We have $h_X(N(x), x) = 0$ for any $x \in X$.

Let N be a linear transformation of X satisfying

(i) $\mathrm{Im}\, N = \mathrm{Ker}\, N$,
(ii) $\mathrm{Im}\, N$ is a light-like 2-plane P_N of X,
(iii) $h_X(N(x), x) = 0$ for any $x \in X$.

Let V_2 be a nonzero vector of P_N. Since $P_N = \mathrm{Ker}\, N$, we have $N(V_2) = 0$. Since $P_N = \mathrm{Im}\, N$, there exists a light-like vector U_1 of X satisfying

$$N(U_1) = V_2, \quad h_X(U_1, V_2) = 0.$$

Then there exists a vector V_1 of P_N satisfying

$$h_X(U_1, V_1) = 1.$$

We see that V_1, V_2 form a basis of P_N and that the orientation given by the ordered basis (V_1, V_2) is determined by N. Therefore we define the positive orientation of P_N by (V_1, V_2). There exists a light-like vector U_2 satisfying

$$N(U_2) = V_1, \quad h_X(U_2, V_1) = h_X(U_2, U_1) = 0.$$

We set

$$c := h_X(U_2, V_2).$$

Whether c is equal to -1 is determined by N. We call N a *nilpotent structure* of X if $c = -1$.

Let N be a nilpotent structure of X. We see that U_1, U_2, V_1, V_2 as above form a basis of X and that whether an ordered basis (U_1, U_2, V_1, V_2) is contained in the positive orientation of X is determined by N. Suppose that (U_1, U_2, V_1, V_2) gives the positive orientation of X. We set

$$e_1 := \frac{1}{2}(2U_1 + V_1), \quad e_2 := \frac{1}{2}(-2U_2 + V_2),$$
$$e_3 := \frac{1}{2}(2U_1 - V_1), \quad e_4 := \frac{1}{2}(2U_2 + V_2).$$

Then (e_1, e_2, e_3, e_4) is an element of $\mathscr{B}_X$ satisfying (5). We call such a basis as (e_1, e_2, e_3, e_4) an *admissible basis* of N. For another admissible basis (e_1', e_2', e_3', e_4') of N than (e_1, e_2, e_3, e_4), there exist $B \in G$, $h \in \mathbb{R}$ satisfying (4). Therefore we have

$V_1' \wedge V_2' = V_1 \wedge V_2$, i.e.,

$$(e_1' - e_3') \wedge (e_2' + e_4') = (e_1 - e_3) \wedge (e_2 + e_4).$$

This means that Θ as in (3) does not depend on the choice of an admissible basis (e_1, e_2, e_3, e_4) of N and that it is determined by N. In addition, we see that N is a nilpotent structure corresponding to $\Theta \in U_0\left(\bigwedge_+^2 X\right)$. Hence we have a one-to-one correspondence between $U_0\left(\bigwedge_+^2 X\right)$ and $\mathcal{N}_{X,+}$. Similarly, considering the case where (U_1, U_2, V_1, V_2) does not give the positive orientation of X, we have a one-to-one correspondence between $U_0\left(\bigwedge_-^2 X\right)$ and $\mathcal{N}_{X,-}$.

4 Almost Complex Structures and Almost Paracomplex Structures of Neutral 4-Manifolds

Let M be an oriented neutral 4-manifold and h its neutral metric. An *almost complex structure* I of M is a $(1, 1)$-tensor field of M satisfying $I^2 = -\mathrm{Id}$. There exists a one-to-one correspondence between the set of sections of $U_+\left(\bigwedge_+^2 TM\right)$ (resp. $U_+\left(\bigwedge_-^2 TM\right)$) and the set of almost complex structures which are h-preserving and compatible (resp. not compatible) with the orientation of M.

Let ∇ be the Levi-Civita connection of h and $\hat{\nabla}$ the connection of $\bigwedge^2 TM$ induced by ∇. Then $\hat{\nabla}$ induces connections of $\bigwedge_\pm^2 TM$.

Proposition 1 ([3]) *An almost complex structure I which is h-preserving and compatible with the orientation of M is parallel with respect to ∇ if and only if the corresponding section Θ of $U_+\left(\bigwedge_+^2 TM\right)$ is horizontal with respect to $\hat{\nabla}$.*

If I as in Proposition 1 is parallel with respect to ∇, then (h, I) is a *neutral Kähler structure* of M and M equipped with (h, I) is a *neutral Kähler surface*. Let (h, I) be a neutral Kähler structure of M. Then integral surfaces of involutive I-invariant 2-dimensional distributions are complex curves of (M, I).

Let (M, h) be an oriented neutral 4-manifold. Let S_+ be a Riemann surface and $F : S_+ \longrightarrow M$ a space-like and conformal immersion with zero mean curvature vector. Let Q be a complex quartic differential defined on S_+ by F (see [2, 4]). Then F is isotropic if and only if Q vanishes. Let I_F be the complex structure of the pull-back bundle F^*TM of TM by F corresponding to the lift of F into $U_+\left(\bigwedge_+^2 F^*TM\right)$. Then by definition, F is strictly isotropic, that is, F is isotropic and compatible with the orientation of M if and only if F satisfies $I_F\sigma(T_1, T_1) = \sigma(T_1, T_2)$, where σ is the second fundamental form of F and $T_1 := \partial/\partial u$, $T_2 := \partial/\partial v$ for a local complex coordinate $w = u + \sqrt{-1}v$.

The following proposition gives a characterization of complex curves in terms of isotropicity of space-like surfaces with zero mean curvature vector.

Proposition 2 ([3]) *A surface in a neutral Kähler surface is a complex curve if and only if it is a space-like surface with zero mean curvature vector which is strictly isotropic and equipped with at least one complex point.*

In general, we obtain

Proposition 3 ([3]) *Let* $F : S_+ \longrightarrow M$ *be a space-like and conformal immersion of* S_+ *into an oriented neutral* 4*-manifold* M *with zero mean curvature vector. Then* F *is strictly isotropic if and only if the lift of* F *into* $U_+\left(\bigwedge^2_+ F^*TM\right)$ *is horizontal.*

An *almost paracomplex structure* J of an oriented neutral 4-manifold M is a $(1, 1)$-tensor field of M satisfying $J \neq \mathrm{Id}$ and $J^2 = \mathrm{Id}$. There exists a one-to-one correspondence between the set of sections of $U_-\left(\bigwedge^2_- TM\right)$ (resp. $U_-\left(\bigwedge^2_+ TM\right)$) and the set of almost paracomplex structures which are h-reversing and compatible (resp. not compatible) with the orientation of M.

Proposition 4 ([3]) *An almost paracomplex structure* J *which is* h*-reversing and compatible with the orientation of* M *is parallel with respect to* ∇ *if and only if the corresponding section* Θ *of* $U_-\left(\bigwedge^2_- TM\right)$ *is horizontal with respect to* $\hat{\nabla}$.

If J as in Proposition 4 is parallel with respect to ∇, then (h, J) is a *paraKähler structure* of M and M equipped with (h, J) is a *paraKähler surface*. Let (h, J) be a paraKähler structure of M. Then integral surfaces of involutive J-invariant 2-dimensional distributions are paracomplex curves of (M, J).

Let (M, h) be an oriented neutral 4-manifold. Let S_- be a Lorentz surface and $F : S_- \longrightarrow M$ a time-like and conformal immersion with zero mean curvature vector. Let Q be a paracomplex quartic differential defined on S_- by F (see [3, 4]). Then F is isotropic if and only if Q vanishes. Let J_F be the paracomplex structure of the pull-back bundle F^*TM of TM by F corresponding to the lift of F into $U_-\left(\bigwedge^2_- F^*TM\right)$. Then by definition, F is strictly isotropic if and only if F satisfies $J_F\sigma(T_1, T_1) = \sigma(T_1, T_2)$, where $T_1 := \partial/\partial u$, $T_2 := \partial/\partial v$ for a local paracomplex coordinate $w = u + jv$.

The following proposition gives a characterization of paracomplex curves in terms of isotropicity of time-like surfaces with zero mean curvature vector.

Proposition 5 ([3]) *A surface in a paraKähler surface is a paracomplex curve if and only if it is a time-like surface with zero mean curvature vector which is strictly isotropic and equipped with at least one paracomplex point.*

In general, we obtain

Proposition 6 ([3]) *Let* $F : S_- \longrightarrow M$ *be a time-like and conformal immersion of* S_- *into an oriented neutral* 4*-manifold* M *with zero mean curvature vector. Then* F *is strictly isotropic if and only if the lift of* F *into* $U_-\left(\bigwedge^2_- F^*TM\right)$ *is horizontal.*

5 Almost Nilpotent Structures of Neutral 4-Manifolds

Let M be an oriented neutral 4-manifold and h its metric. Let N be a (1, 1)-tensor field of M. We call N an *almost nilpotent structure* of M if N gives a nilpotent structure of the tangent space of M at each point. Each almost nilpotent structure of M corresponds to a section of either $U_0\left(\bigwedge_+^2 TM\right)$ or $U_0\left(\bigwedge_-^2 TM\right)$.

Theorem 1 *An almost nilpotent structure N of M is parallel with respect to the Levi-Civita connection ∇ of h if and only if the corresponding section Θ of either $U_0\left(\bigwedge_+^2 TM\right)$ or $U_0\left(\bigwedge_-^2 TM\right)$ is horizontal with respect to the connection $\hat{\nabla}$ of $\bigwedge^2 TM$ induced by ∇.*

Proof Let (e_1, e_2, e_3, e_4) be a local ordered pseudo-orthonormal frame field of TM. We set

$$\nabla e_j = \sum_{i=1}^{4} \omega_j^i e_i \quad (j = 1, 2, 3, 4).$$

Then we have

(i) $\omega_i^i = 0$ for $i = 1, 2, 3, 4$,
(ii) $\omega_i^j = -\omega_j^i$ for $\{i, j\} = \{1, 2\}$ or $\{3, 4\}$,
(iii) $\omega_i^j = \omega_j^i$ for $\{i, j\} = \{1, 3\}, \{1, 4\}, \{2, 3\}$ or $\{2, 4\}$.

Let N be an almost nilpotent structure of M corresponding to a section Θ of $U_0\left(\bigwedge_+^2 TM\right)$. Suppose that (e_1, e_2, e_3, e_4) gives an admissible basis of N to the tangent space of M at each point. Then we have (5). Therefore we obtain (3) and

$$\hat{\nabla}\Theta = -(\omega_3^1 - \omega_4^2)\Theta + (\omega_2^1 + \omega_4^1 + \omega_2^3 + \omega_4^3)\Theta_{+,2}. \tag{7}$$

Therefore $\hat{\nabla}\Theta = 0$ if and only if $\{\omega_j^i\}$ satisfies

$$\omega_3^1 = \omega_1^3 = \omega_4^2 = \omega_2^4, \quad \omega_2^1 + \omega_4^1 + \omega_2^3 + \omega_4^3 = 0. \tag{8}$$

We see that $\nabla N = 0$ is equivalent to

$$\nabla(N(e_i)) = N(\nabla e_i) \quad (i = 1, 2, 3, 4).$$

Therefore we see by (5) that $\nabla N = 0$ is equivalent to (8). Hence we see that $\nabla N = 0$ is equivalent to $\hat{\nabla}\Theta = 0$. In the case where Θ is a section of $U_0\left(\bigwedge_-^2 TM\right)$, we can obtain the same result and we have finished the proof of Theorem 1. □

If N is parallel with respect to ∇, then (h, N) is called a *nilpotent Kähler structure* of M, and M equipped with (h, N) is called a *nilpotent Kähler* 4-*manifold*.

Example 1 Let M be a neutral hyperKähler 4-manifold. Then either $\bigwedge^2_+ TM$ or $\bigwedge^2_- TM$ is a product bundle. Suppose that $\bigwedge^2_+ TM$ is a product bundle. Then we can suppose that sections $\Theta_{-,1}$, $\Theta_{+,2}$, $\Theta_{+,3}$ of $\bigwedge^2_+ TM$ are horizontal and that they form a psuedo-orthonormal frame field of $\bigwedge^2_+ TM$. In particular, Θ as in (3) is horizontal. Therefore an almost nilpotent structure N of M corresponding to Θ is parallel with respect to ∇ and (h, N) is a nilpotent Kähler structure of M.

Remark 2 Let M be a manifold and E an oriented vector bundle over M of rank 4 with its neutral metric h. Let N be a section of End (E). We call N a *nilpotent structure* of E if N gives a nilpotent structure of the fiber of E at each point of M. Each nilpotent structure of E corresponds to a section of either $U_0\left(\bigwedge^2_+ E\right)$ or $U_0\left(\bigwedge^2_- E\right)$. Let ∇ be an h-connection of E and $\hat{\nabla}$ the connection of $\bigwedge^2 E$ induced by ∇. Then $\hat{\nabla}$ induces connections of $\bigwedge^2_\pm E$. Referring to the proof of Theorem 1, we can prove that a nilpotent structure N of E is parallel with respect to ∇ if and only if the corresponding section Θ of either $U_0\left(\bigwedge^2_+ E\right)$ or $U_0\left(\bigwedge^2_- E\right)$ is horizontal with respect to $\hat{\nabla}$. We call (h, ∇, N) a *nilpotent Kähler structure* of E if N is parallel with respect to ∇.

Let N be an almost nilpotent structure of M. Then at each point of M, N gives its light-like 2-plane of the tangent space of M. Therefore we have a light-like two-dimensional distribution $\mathscr{D}$.

Theorem 2 *The distribution $\mathscr{D}$ given by N is involutive if and only if for the section Θ of either $U_0\left(\bigwedge^2_+ TM\right)$ or $U_0\left(\bigwedge^2_- TM\right)$ corresponding to N and each tangent vector V of M contained in $\mathscr{D}$, the covariant derivative $\hat{\nabla}_V\Theta$ is given by Θ up to a constant. In particular, if N is parallel with respect to ∇, then $\mathscr{D}$ is involutive.*

Proof Suppose that Θ is a section of $U_0\left(\bigwedge^2_+ TM\right)$. Then $\mathscr{D}$ is locally generated by $e_1 - e_3, e_2 + e_4$. Therefore $\mathscr{D}$ is involutive if and only if the bracket $[e_1 - e_3, e_2 + e_4]$ is contained in $\mathscr{D}$. The latter condition is rewritten into

$$\begin{aligned} h([e_1 - e_3, e_2 + e_4], e_1 - e_3) = 0, \\ h([e_1 - e_3, e_2 + e_4], e_2 + e_4) = 0. \end{aligned} \tag{9}$$

Since ∇ is torsion-free, we have

$$[e_1 - e_3, e_2 + e_4] = \nabla_{e_1 - e_3}(e_2 + e_4) - \nabla_{e_2 + e_4}(e_1 - e_3).$$

Therefore (9) is rewritten into

$$\begin{aligned} h(\nabla_{e_1 - e_3}(e_2 + e_4), e_1 - e_3) = 0, \\ h(\nabla_{e_2 + e_4}(e_1 - e_3), e_2 + e_4) = 0. \end{aligned} \tag{10}$$

Noticing (7) and that (10) is equivalent to

$$h(\nabla_V(e_2+e_4), e_1-e_3)=0 \tag{11}$$

for any tangent vector V in $\mathscr{D}$, we see that $\mathscr{D}$ is involutive if and only if for each $V \in \mathscr{D}$, $\hat{\nabla}_V \Theta$ is given by Θ up to a constant. In the case where Θ is a section of $U_0\left(\bigwedge_-^2 TM\right)$, we obtain the same result. □

Remark 3 We see from the above proof that $\mathscr{D}$ is involutive if and only if (11) holds, that is, the covariant derivatives of the local generators e_1-e_3, e_2+e_4 of $\mathscr{D}$ by $V \in \mathscr{D}$ are contained in $\mathscr{D}$. Therefore $\mathscr{D}$ satisfies this condition if $(M, h, \mathscr{D})$ is a Walker manifold, that is, if the covariant derivatives of the local generators e_1-e_3, e_2+e_4 of $\mathscr{D}$ by any tangent vector of M are contained in $\mathscr{D}$. See [8, 9, 16] for Walker manifolds.

6 Light-Like Surfaces in Neutral 4-Manifolds

Let L be a 2-dimensional manifold and $F : L \longrightarrow M$ an immersion of L into M. We say that F is *light-like* if for any nonzero tangent vector V of L, $dF(V)$ is light-like. Let $F : L \longrightarrow M$ be a light-like immersion of L into M. Let V_1, V_2 be vector fields on a neighborhood O of each point of L which form a local frame field. Then $V_1 \wedge V_2$ gives a local section of either $U_0\left(\bigwedge_+^2 F^*TM\right)$ or $U_0\left(\bigwedge_-^2 F^*TM\right)$. We consider ∇, $\hat{\nabla}$ to be connections of F^*TM, $\bigwedge_\varepsilon^2 F^*TM$, respectively ($\varepsilon \in \{+,-\}$).

Proposition 7 *The Levi-Civita connection* ∇ *of* h *induces a connection of* L *if and only if for* V_1, V_2 *as above and each tangent vector* V *of* O, *there exists a number* c *satisfying*

$$\hat{\nabla}_V V_1 \wedge V_2 = cV_1 \wedge V_2. \tag{12}$$

Proof We see that ∇ induces a connection of L if and only if for any vector field W on L, ∇W gives a section of End (TL). Let (e_1, e_2, e_3, e_4) be a local ordered pseudo-orthonormal frame field of F^*TM satisfying $V_1 = e_1 - e_3$, $V_2 = e_2 + e_4$. Then ∇ induces a connection of L if and only if

$$h(\nabla(e_2+e_4), e_1-e_3)=0,$$

that is, $\omega_2^1+\omega_4^1+\omega_2^3+\omega_4^3=0$. Therefore ∇ induces a connection of L if and only if for each V, there exists a number c satisfying (12). Hence we obtain Proposition 7. □

Remark 4 Let M, E and h be as in the previous remark. Let N be a nilpotent structure of E. Then at each point of M, N gives its light-like 2-plane of the fiber of E. Therefore we have a subbundle E_N of E of rank 2. Referring to the proof of

Proposition 7, we can prove that an h-connection ∇ of E induces a connection of E_N if and only if for local sections ξ_1, ξ_2 of E_N on a neighborhood O of each point of M which form a local frame field and each tangent vector V of O, there exists a number c satisfying $\hat{\nabla}_V \xi_1 \wedge \xi_2 = c\xi_1 \wedge \xi_2$. In particular, if N is parallel with respect to ∇, then ∇ induces a connection of E_N.

Let L be a 2-dimensional manifold and ∇ a connection of L. Then ∇ induces a connection $\hat{\nabla}$ of $\bigwedge^2 TL$. Let (u^1, u^2) be local coordinates of L and Γ^k_{ij} $(i, j, k = 1, 2)$ the Christoffel symbols of ∇ with respect to (u^1, u^2). We set

$$f_k := \Gamma^1_{k1} + \Gamma^2_{k2} \quad (k = 1, 2)$$

and

$$\Omega := d(f_1 du^1 + f_2 du^2).$$

Lemma 1 *The 2-form Ω does not depend on the choice of (u^1, u^2). It is defined on L and determined by ∇.*

Proof We have

$$\hat{\nabla}_{\partial/\partial u^k} \frac{\partial}{\partial u^1} \wedge \frac{\partial}{\partial u^2} = f_k \frac{\partial}{\partial u^1} \wedge \frac{\partial}{\partial u^2} \quad (k = 1, 2). \tag{13}$$

Let $(\tilde{u}^1, \tilde{u}^2)$ be local coordinates of L other than (u^1, u^2). Let $\tilde{\Gamma}^k_{ij}$ $(i, j, k = 1, 2)$ be the Christoffel symbols of ∇ with respect to $(\tilde{u}^1, \tilde{u}^2)$ and set $\tilde{f}_k := \tilde{\Gamma}^1_{k1} + \tilde{\Gamma}^2_{k2}$. Then noticing (13), we obtain

$$f_k = \frac{\partial \log |D|}{\partial u^k} + \frac{\partial \tilde{u}^1}{\partial u^k} \tilde{f}_1 + \frac{\partial \tilde{u}^2}{\partial u^k} \tilde{f}_2 \quad (k = 1, 2),$$

where

$$D := \frac{\partial(\tilde{u}^1, \tilde{u}^2)}{\partial(u^1, u^2)} = \begin{vmatrix} \frac{\partial \tilde{u}^1}{\partial u^1} & \frac{\partial \tilde{u}^1}{\partial u^2} \\ \frac{\partial \tilde{u}^2}{\partial u^1} & \frac{\partial \tilde{u}^2}{\partial u^2} \end{vmatrix}.$$

This yields

$$f_1 du^1 + f_2 du^2 = d \log |D| + \tilde{f}_1 d\tilde{u}^1 + \tilde{f}_2 d\tilde{u}^2.$$

Therefore we obtain

$$d(f_1 du^1 + f_2 du^2) = d(\tilde{f}_1 d\tilde{u}^1 + \tilde{f}_2 d\tilde{u}^2)$$

and we have proved Lemma 1. □

Remark 5 Let L be an l-dimensional manifold and ∇ a connection of L. Let $(u^1, \dots, u^l)$ be local coordinates of L and Γ_{ij}^k $(i, j, k = 1, \dots, l)$ the Christoffel symbols of ∇ with respect to $(u^1, \dots, u^l)$. We set

$$f_k := \sum_{j=1}^{l} \Gamma_{kj}^j \quad (k = 1, \dots, l), \quad \Omega := d\left(\sum_{k=1}^{l} f_k du^k\right).$$

Then Ω does not depend on the choice of $(u^1, \dots, u^l)$.

We will prove

Theorem 3 *Let L be a 2-dimensional manifold and ∇ a connection of L. Then the following are mutually equivalent*:

(a) *on a neighborhood of each point of L, there exists a nonzero horizontal section of $\bigwedge^2 TL$ with respect to $\hat{\nabla}$*;
(b) *the curvature tensor $\hat{R}$ of $\hat{\nabla}$ vanishes*;
(c) $\Omega \equiv 0$.

Proof We obtain

$$\begin{aligned}
&\hat{R}\left(\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}\right)\frac{\partial}{\partial u^1} \wedge \frac{\partial}{\partial u^2} \\
&= \hat{\nabla}_{\partial/\partial u^1}\left(f_2 \frac{\partial}{\partial u^1} \wedge \frac{\partial}{\partial u^2}\right) - \hat{\nabla}_{\partial/\partial u^2}\left(f_1 \frac{\partial}{\partial u^1} \wedge \frac{\partial}{\partial u^2}\right) \\
&= \left(\frac{\partial f_2}{\partial u^1} + f_2 f_1\right)\frac{\partial}{\partial u^1} \wedge \frac{\partial}{\partial u^2} - \left(\frac{\partial f_1}{\partial u^2} + f_1 f_2\right)\frac{\partial}{\partial u^1} \wedge \frac{\partial}{\partial u^2} \\
&= \left(\frac{\partial f_2}{\partial u^1} - \frac{\partial f_1}{\partial u^2}\right)\frac{\partial}{\partial u^1} \wedge \frac{\partial}{\partial u^2}.
\end{aligned} \tag{14}$$

Let Θ be a local nonzero section of $\bigwedge^2 TL$. We locally represent Θ as

$$\Theta = f \frac{\partial}{\partial u^1} \wedge \frac{\partial}{\partial u^2}. \tag{15}$$

If Θ is horizontal with respect to $\hat{\nabla}$, that is, if $\hat{\nabla}\Theta = 0$, then we obtain $\partial f/\partial u^k + f f_k = 0\,(k = 1, 2)$ and therefore we have $\partial f_2/\partial u^1 = \partial f_1/\partial u^2$. Therefore by (14), we obtain (b) from (a). If $\hat{R}$ vanishes, then by (14), we obtain $\partial f_2/\partial u^1 = \partial f_1/\partial u^2$, which means $\Omega \equiv 0$. Therefore we obtain (c) from (b). Suppose $\Omega \equiv 0$. Then there exists a function ϕ defined on a neighborhood of each point of L satisfying $\partial\phi/\partial u^k = f_k$ $(k = 1, 2)$. We set $f := e^{-\phi}$. Then for Θ as in (15), we obtain $\hat{\nabla}_{\partial/\partial u^k}\Theta = 0$. Therefore Θ is horizontal and we obtain (a) from (c). □

Remark 6 Let L, ∇ and Ω be as in the previous remark. Then we can prove that the following are mutually equivalent:

(a) on a neighborhood of each point of L, there exists a nonzero horizontal section of $\bigwedge^l TL$ with respect to $\hat{\nabla}$;
(b) the curvature tensor $\hat{R}$ of $\hat{\nabla}$ vanishes;
(c) $\Omega \equiv 0$.

Corollary 1 *Let M be an oriented neutral 4-manifold. Let h be the neutral metric of M and ∇ the Levi-Civita connection of h. Let L be a 2-dimensional manifold and $F : L \longrightarrow M$ a light-like immersion of L into M. Then the following are mutually equivalent*:

(a) *on a neighborhood of each point of L, there exists a nonzero horizontal section of either $U_0\left(\bigwedge^2_+ F^*TM\right)$ or $U_0\left(\bigwedge^2_- F^*TM\right)$ given by a local section of $\bigwedge^2 TL$;*
(b) *on a neighborhood O of each point of L, there exists a nilpotent structure N of $F^*TM|_O$ with $E_N = F^*dF(TL)|_O$ parallel with respect to ∇;*
(c) *∇ induces a connection of L such that the curvature tensor $\hat{R}$ of $\hat{\nabla}$ vanishes*;
(d) *∇ induces a connection of L satisfying $\Omega \equiv 0$.*

Remark 7 Let M be an m-dimensional manifold and E a vector bundle over M of rank n. Let ∇ be a connection of E. Then ∇ induces a connection $\hat{\nabla}$ of $\bigwedge^n E$. Let $\xi_1, \dots, \xi_n$ form a local frame field of E on a neighborhood O of each point of M. Let $(u^1, \dots, u^m)$ be local coordinates on O. Let $\Gamma^k_{\alpha j}$ $(\alpha = 1, \dots, m,\ j, k = 1, \dots, n)$ be functions on O given by

$$\nabla_{\partial/\partial u^\alpha} \xi_j = \sum_{k=1}^{n} \Gamma^k_{\alpha j} \xi_k.$$

We set

$$f_\alpha := \sum_{k=1}^{n} \Gamma^k_{\alpha k} \quad (\alpha = 1, \dots, m), \quad \Omega := d\left(\sum_{\alpha=1}^{m} f_\alpha du^\alpha\right).$$

Then referring to the proof of Lemma 1, we can prove that Ω depends neither the choice of $(u^1, \dots, u^m)$ nor the choice of $(\xi_1, \dots, \xi_n)$. In addition, referring to the proof of Theorem 3, we see that the following are mutually equivalent:

(a) on a neighborhood of each point of M, there exists a nonzero horizontal section of $\bigwedge^n E$;
(b) the curvature tensor $\hat{R}$ of $\hat{\nabla}$ vanishes;
(c) $\Omega \equiv 0$.

Suppose $n = 4$ and that E is oriented and equipped with a neutral metric h and an h-connection ∇. Let E' be a subbundle of E of rank 2 such that each fiber is light-like. Then the following are mutually equivalent:

(a) on a neighborhood of each point of M, there exists a local nonzero horizontal section of either $U_0\left(\bigwedge^2_+ E\right)$ or $U_0\left(\bigwedge^2_- E\right)$ given by a local section of $\bigwedge^2 E'$;

(b) on a neighborhood O of each point of M, there exists a nilpotent structure N of $E|_O$ with $E_N = E'|_O$ parallel with respect to ∇;
(c) ∇ induces a connection of E' such that the curvature tensor $\hat{R}$ of $\hat{\nabla}$ vanishes;
(d) ∇ induces a connection of E' satisfying $\Omega \equiv 0$.

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