Anisotropic Connections and Parallel Transport in Finsler Spacetimes

Miguel Ángel Javaloyes, Miguel Sánchez, and Fidel F. Villaseñor

Abstract The general notion of anisotropic connections ∇ is revisited, including its precise relations with the standard setting of pseudo-Finsler metrics, i.e., the metric nonlinear connection and the (linear) Finslerian connections. In particular, the vertically trivial Finsler connections are canonically identified with anisotropic connections. So, these connections provide a simple intrinsic interpretation of a part of any Finsler connection closer to the Koszul formulation in M. Moreover, a new covariant derivative and parallel transport along curves is introduced, taking first a self-propagated vector (*instantaneous observer*) so that it serves as a reference for the propagation of the others. The covariant derivative of any anisotropic tensor is given by the natural derivative of a curve of tensors obtained by parallel transport along a curve and, in the case of pseudo-Finsler metrics, this is used to characterize the Levi-Civita–Chern anisotropic connection as the one that preserves the length of parallelly propagated vectors.

Keywords Finsler spaces and spacetimes · Anisotropic connections · Sprays · Nonlinear connections · Finsler connections · Parallel transport · Levi-Civita–Chern connection

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A. L. Albujer et al. (eds.), *Developments in Lorentzian Geometry*, Springer Proceedings in Mathematics & Statistics 389, https://doi.org/10.1007/978-3-031-05379-5_11

1 Introduction

The standard geometric picture for the description of a (pseudo-)Finsler metric $L = F^2 : A\,(\subset TM) \to \mathbb{R}$ comprises two elements: a (nonlinear) connection ν on the fibration $A \to M$ and a linear connection $\nabla^* \equiv (\big(\Gamma^{\mathrm{H}}\big)^k_{ij}, \big(\Gamma^{\mathrm{V}}\big)^k_{ij})$ on the vertical vector bundle $\mathrm{V}A \to A$. The former is canonically associated with the spray determined by L, whose integral curves are the critical points of the energy functional. However, there are quite a few non-equivalent choices for the latter (Berwald, Cartan, Chern, Hashiguchi...). Motivated by the complexity of this and other settings, some researchers have introduced the concept of *anisotropic connection*, a generalization of the (pseudo-)Riemannian setting which incorporates in a natural way the direction dependent geometric structures of Finsler geometry [13, 20, 24, 25].

Recently, one of the authors has developed systematically the *anisotropic calculus* [11, 12], namely, how to make computations with an anisotropic connection, which can be seen as a natural and intuitive generalization of the usual Koszul connections. Some applications have been obtained in [10, 17]. In the present article, we revisit this notion, showing precisely its relations with the other elements of the standard setting and providing a further insight on its associated parallel transport.

More precisely, in Sect. 2 we introduce heuristically the notions of pseudo-Finsler metric, by looking for general ways of measuring the lengths of curves, and Finsler spacetime, by stressing geometric elements related with measurements. In Sect. 3 anisotropic tensor fields on M are introduced and the concept of anisotropic connection ∇ is defined. First, ∇ is regarded as a type of covariant derivative which applies to usual vector fields X, Y on M so that it provides an anisotropic vector field $\nabla_X Y$. Then, the usual rules of derivations are discussed so that ∇_X can be applied to any anisotropic tensor. We emphasize some issues which will become relevant later such as homogeneity (natural invariance by homotheties), the notion of torsion or the affine structure of the space of all the anisotropic connections.

In Sects. 4 and 5 we make a detailed study of the relation between anisotropic connections ∇ and, resp., (nonlinear) connections ν on $A \to M$ and linear connections ∇^* on $\mathrm{V}A \to A$ (ν and ∇^* are not assumed to come from any Finsler function L a priori). A detailed correspondence is given for the former, in particular:

> Any anisotropic connection ∇ with Christoffel symbols Γ^a_{ij} is characterized by a pair composed by a nonlinear connection $N^a_i = \Gamma^a_{ij} y^j$ and a tensor Q satisfying $Q^a_{ij} y^j = 0$. In the homogeneous case, all nonlinear connections can be obtained from anisotropic ones (Theorem 2).

The relation between an anisotropic ∇ and a linear ∇^* becomes subtler. Indeed, if we are given an auxiliary nonlinear connection $\overset{o}{\nu}$, then ∇^* can be determined by specifying the covariant derivatives (of the sections of $\mathrm{V}A \to A$) with respect to the $\overset{o}{\nu}$-horizontal and vertical directions. This is standard in Finsler geometry, and the connections with vanishing vertical derivatives are called *vertically trivial* here; clearly, they are independent of $\overset{o}{\nu}$. Such trivial connections can be put in one to one correspondence with the anisotropic connections by using $\overset{o}{\nu}$. However, this correspondence also becomes independent of $\overset{o}{\nu}$. Summing up:

> There is a natural bijection between vertically trivial connections ∇^* and anisotropic connections ∇. It identifies the homothety invariant ∇^*'s with the homogeneous ∇'s (Proposition 3, Theorem 3).

In Sect. 6 we focus on the pseudo-Finsler case. As explained therein, the last result above becomes essential for the identification of anisotropic connections in the pseudo-Finsler setting. Indeed, the 2-homogeneity of L leads to the homogeneity of the involved linear connection ∇^* (and the canonical nonlinear one $\overset{o}{\nu}$). Some Finsler connections such as Berwald or Chern are vertically trivial and, thus, directly identifiable with anisotropic connections. Moreover, the non vertically trivial ones, as Cartan or Hashiguchi, will project on vertically trivial ones (by using $\overset{o}{\nu}$). So, anisotropic connections provide the non-vertical part of any Finslerian connection, expressed tidily as Koszul-type derivations on M. As already pointed out in [11], the metric L allows one to select a unique Levi-Civita anisotropic connection, which is then identifiable to Chern's.

In Sect. 7, we introduce the covariant derivative D_γ and parallel transport along curves γ for any anisotropic connection ∇. Taking into account the dependence of ∇ on the direction, one can choose a reference W (a vector field on γ which takes values on $A \subset TM$) as in [2, p. 121], to define its associated covariant derivative D_γ^W and W-parallel transport, which will behave as the usual (isotropic) one. This parallel transport is of crucial importance, as it can be used to define in a very natural way the covariant derivative of tensors, but the dependence on the direction inherent to Finslerian geometry introduces additional subtleties. As a first step, one can parallel transport the observer, which in Finsler spacetimes is interpreted as the (timelike) direction on the tangent bundle where we are doing the computations. This is defined using a parallel observer determined by $D_\gamma^V V = 0$, a nonlinear equation whose solutions may not be extended on the whole γ. However, they do extend in the most interesting cases, such as the standard Finsler and the Lorentz-Finsler ones. Once we have a parallel observer along a curve, we can make the parallel transport of any other vector using as a reference this parallel observer. The parallel transport of the observer coincides with the one provided by a nonlinear connection (see for example [19, Chap. VII], [1, Sect. 2.1.6], [25, p. 103], [4, Sect. 2.1], [8, Definition 1.4] and [27, Sect. 7.6]). However, as far as we know, the second parallel transport with respect to an observer has not been considered in literature.

It turns out that the most economical way to codify all the information of the covariant derivatives along curves in a smooth setting with natural assumptions is with an *anisotropic connection*, which allows for covariant derivatives of any kind of tensor (see Theorem 1). These covariant derivatives were introduced in [12] from a rather abstract viewpoint as tensor derivations which satisfy the Leibniz rule of the tensor product and commute with contractions. To enhance the geometric meaning of these covariant derivatives, we will show in Theorem 6 that they coincide with the (usual) derivative in a vector space of the curve of tensors obtained with parallel transport (and therefore using only covariant derivatives along curves). Finally, one can wonder if given a pseudo-Finsler metric, there is an anisotropic connection with a parallel transport which preserves the length of vectors. We show in Sect. 7.3 that this connection exists and it is the Levi-Civita-Chern anisotropic connection, which can be identified (as a vertically trivial linear connection) with the classical Chern connection.

Finally, we would like to emphasize that our approach is useful in the classical Finsler case a well as its (positive definite) variants, such as Kropina, Randers-Kropina, conic and wind Finsler metrics [5, 14, 16, 29]. In this article, we emphasize the case of Finsler spacetimes (introducing notions such as *observer*) not only because these constitute an active topic of research where our setting applies naturally [3, 9, 21], but also because its physical intuitions suggest interesting geometric definitions valid even for the positive definite case.

2 General Background

2.1 Pseudo-Finsler Metrics

Let us pose the following problem. Given a manifold M, we want to define a general smooth structure that allows us to measure the length of curves. It seems quite natural that this length should be defined as

$$\ell(\gamma) := \int_a^b F(\dot{\gamma}(s))ds$$

for some function $F : TM \to \mathbb{R}$ in the tangent bundle TM. From a geometrical viewpoint, this definition should not depend on the parametrization of γ, which can be achieved by requiring that F is a homogeneous function of degree 1 when restricted to any tangent space. Moreover, if one wants to include relativistic measures and to remain in the smooth realm, it is better to consider the square $L = F^2$, because otherwise one would find many examples where F is not smooth on lightlike vectors, namely, vectors $v \in TM$ where $F(v) = 0$. Indeed, this happens when one considers the one-homogeneous function $F : \mathbb{R}^4 \to \mathbb{R}$ given by

$$F(\tau, v^1, v^2, v^3) = \sqrt{\tau^2 - (v^1)^2 - (v^2)^2 - (v^3)^2},$$

which is non-smooth on the lightlike vectors.

We will make two additional assumptions on L.

(i) The first one is that L is not necessarily defined in the whole tangent bundle TM, but only in some directions. Sometimes, there are some forbidden directions because of some constraints of the problem, or as in General Relativity, because only trajectories with directions on a cone (say, the future-directed timelike one) will become relevant. Therefore, we will choose as domain of L a subset A in TM which is conic, to permit arbitrary positive reparametrizations of the curves, and open for the sake of simplicity, even though the boundary of A can be considered in different ways (as in wind Riemannian metrics [5] or Finsler spacetimes [15]).

(ii) The second one is related to the (vertical) Hessian of L,

$$g_v(u, w) := \frac{1}{2}\frac{\partial^2}{\partial t \partial s}L(v + tu + sw)|_{t=s=0},$$

which, as we will see later, can be thought as the best scalar product approximation of L. We will assume that this scalar product is nondegenerate for every $v \in A$ but not necessarily Euclidean (positive definite). Nondegenericity will be essential to obtain the existence and uniqueness of the covariant derivative.

Summing up, the following notion of a *pseudo-Finsler metric* collects all the conditions above for a very general definition of length of curves.

Definition 1 Let M be an n-manifold, $\pi : TM \to M$ the natural projection of TM onto M and $A \subset TM \setminus \mathbf{0}$ an open subset of TM which is conic (namely, for every $v \in A$ and $\lambda > 0$, $\lambda v \in A$) and satisfies $\pi(A) = M$. A smooth function $L : A \to \mathbb{R}$ is a *pseudo-Finsler metric* if

1. L is positive homogeneous of degree 2, that is, $L(\lambda v) = \lambda^2 L(v)$ for every $v \in A$ and $\lambda > 0$.
2. The fundamental tensor of L, namely g_v defined by

$$g_v(u, w) := \frac{1}{2}\frac{\partial^2}{\partial t \partial s}L(v + tu + sw)|_{t=s=0}$$

for any $v \in A$, and any $u, w \in T_{\pi(v)}M$, is nondegenerate.

In this definition we have excluded the zero section from A. As A is open and conic, the only case in which the zero section could be contained in A is when it is the whole tangent bundle. But even in this case, there are problems with the zero section, because L can be C^2 on the zero section only if it comes pointwise from a scalar product.[1]

Given a pseudo-Finsler metric $L : A \to \mathbb{R}$ on a manifold M, for every $p \in M$, we define the *indicatrix* at p as

$$\Sigma_p = \{v \in T_pM \cap A : L(v) = 1\},$$

(sometimes the indicatrix of $-L$ may be of interest too) and the *lightcone* as

$$\mathcal{C}_p = \{v \in T_pM \cap A : L(v) = 0\}.$$

[1] Indeed, if g is one half the Hessian of L at the 0 vector of each tangent space, then $L(v) = g(v, v)$ for every $v \in TM$, see [28, Proposition 4.1].

Given $p \in M$ and $v \in T_pM$, let us discuss why g_v is the best scalar product approximation of L at $v \in A$. Assume for example that $L(v) = 1$, which can be assumed by homogeneity if $L(v) > 0$. Recall that the restriction

$$g_v|_\Sigma : T_v\Sigma_p \times T_v\Sigma_p \to \mathbb{R}$$

coincides with second fundamental form of Σ_p with respect to the opposite of the position vector v computed with the affine connection of T_pM (see for example [14, Eq. (2.3)]). Moreover, one has that v is g_v-orthogonal to $T_v\Sigma_p$ and, by homogeneity (applying Euler's theorem), that $g_v(v, v) = L(v)$. This implies that

$$\Sigma^{g_v} = \{w \in T_pM : g_v(w, w) = 1\}$$

satisfies $T_v\Sigma^{g_v} = T_v\Sigma_p$, and the second fundamental form of Σ^{g_v} at v with respect to the opposite of the position vector v coincides with that of Σ_p.

2.2 Finsler Spacetimes and Its Restspace

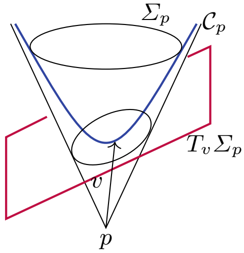
To generalize the definition of spacetime in a certain manifold M, the following observations are in order:

1. We need to measure the length of curves to obtain the elapsed time along the trajectory. By the discussion in the previous section, this leads us to consider a pseudo-Finsler metric $L : A \subset TM \setminus 0 \to \mathbb{R}$.
2. Locally, it must approximate the Lorentz-Minkowski spacetime. This implies that for every $v \in A$, the scalar product g_v must be of Lorentz type since, as argued above, g_v is the best approximation of L around v.
3. There have to be some vectors with zero length, which are the directions of light rays.
4. Moreover, these lightlike directions must be the limit of the timelike directions, therefore, their boundary.

Definition 2 A *Finsler spacetime* is an n-manifold M, $n \geq 2$, endowed with a pseudo-Finsler metric $L : A \to (0, +\infty)$ such that

(i) L is a Lorentz-Finsler metric, i.e., its indicatrix is strongly concave or equivalently the index of g_v is $n - 1$.
(ii) L extends as zero to the closure $\bar{A}$ of A in $TM \setminus 0$ and this extension is smooth with nondegenerate g_v.
(iii) For every $p \in M$, $A_p := A \cap T_pM$ is connected, convex and salient, i.e., if $v \in A_p$ then $-v \notin A_p$. (In fact, the last two conditions follow from the other hypotheses, see [15, Remark 3.6].)

Moreover, the future-directed timelike unit vectors of the indicatrix $\Sigma_p = \{v \in T_pM : L(v) = 1\}$ are used to model the instantaneous observers, while the vectors in the null cone $\mathcal{C}_p = \{v \in T_pM : L(v) = 0\}$ are the lightlike future-directed vectors. The tangent space $T_v\Sigma_p = \{w \in T_pM : g_v(v, w) = 0\}$ is interpreted as the instantaneous restspace of v. Even if we assume that L is defined only in $\bar{A}$, it is possible to extend L (in a non-unique way) to the whole tangent bundle (see [21]). For more details about the interpretation of the restspace see [3] (part (4) after Remark 9).



An *observer* in a Finsler spacetime is a (future-directed) unit timelike curve, namely $\gamma : I = (a, b) \to (M, L)$ such that $\dot\gamma(s) \in A$ and $L(\dot\gamma(s)) = 1$ for all $s \in I$. The *(instantaneous) restspace of the observer* at $s \in (a, b)$ is $T_{\dot\gamma(s)}\Sigma_{\gamma(s)}$. There are two natural metrics in this restspace. The first one is given by

$$g_{\dot\gamma(s)}|_\Sigma : T_{\dot\gamma(s)}\Sigma_{\gamma(s)} \times T_{\dot\gamma(s)}\Sigma_{\gamma(s)} \to \mathbb{R}.$$

It is the fundamental tensor restricted to Σ, which is a definite metric. The direction $\dot\gamma(s)$ is $g_{\dot\gamma(s)}$-orthogonal to $T_{\dot\gamma(s)}\Sigma$. As we have said above, this metric is the best approximation of L with a scalar product in the direction of $\dot\gamma(s)$ as the restriction $g|_\Sigma$ is the second fundamental form of Σ with respect to the opposite to the position vector (using the natural affine connection in T_pM).

The other metric is a Finsler metric with indicatrix $S_{\dot\gamma(s)}$, where $S_v = T_v\Sigma \cap \mathcal{C}_p$ (the set of velocities of light in the restspace of v). It is unclear which one is more suitable to measure spacelike distances, and indeed, the choice of metric could depend on the type of measure.

3 Anisotropic Connections

3.1 Anisotropic Tensor Fields and Their Vertical Derivatives

We will denote by $x = (x^1, \ldots, x^n)$ local coordinates on some open subset U of M and (with a slight abuse of notation) by $(x, y) = (x^1, \ldots, x^n, y^1, \ldots, y^n)$ the natural ones induced on $TU \subset TM$. Let us denote by T^*M the cotangent bundle of the manifold M and by $\overset{r)}{\bigotimes} TM \otimes \overset{s)}{\bigotimes} T^*M$ the classical vector bundle of tensors of type (r, s) over M. Recall that an (r, s)-tensor field on M is a smooth section of this bundle, and let $\mathcal{T}^r_s(M)$ be the space of all such tensor fields. We use the simplified notation for smooth functions and vector fields on M: $\mathcal{T}_0^0(M) = \mathcal{F}(M)$ and $\mathcal{T}_0^1(M) = \mathfrak{X}(M)$ resp. Now, let $\pi_A^*(\overset{r)}{\bigotimes} TM \otimes \overset{s)}{\bigotimes} T^*M)$ be the bundle over A pullbacked by the natural projection $\pi_A : A \to M$. A smooth section $T : A \to \pi_A^*(\overset{r)}{\bigotimes} TM \otimes \overset{s)}{\bigotimes} T^*M)$ of this bundle is called an A-*anisotropic tensor field* on M, and let $\mathcal{T}_s^r(M_A)$ the space of such fields (also by convention, $\mathcal{T}_0^0(M_A) = \mathcal{F}(A)$). In natural coordinates, with summation in repeated indices and $\partial_a \equiv \partial_{x^a}$,

$$T_v = T^{a_1,\ldots,a_r}_{b_1,\ldots,b_s}(x, y)\, \partial_{a_1}\Big|_x \otimes \ldots \otimes \partial_{a_r}\Big|_x \otimes \mathrm{d}x^{b_1}|_x \otimes \ldots \otimes \mathrm{d}x^{b_s}|_x, \qquad v \in A \cap \mathrm{T}U; \tag{1}$$

here, (x, y) and x are, resp., the coordinates of v and $\pi_A(v)$ (the functions $T^{a_1,\ldots,a_r}_{b_1,\ldots,b_s}$ transform tensorially under changes of coordinates). Recall that, naturally, $\mathcal{T}_s^r(M_A)$ becomes a module over the ring $\mathcal{F}(A)$ and the tensor products and contractions induce further operations on sections, as in the case of usual tensor fields on M. In particular, $\mathcal{T}_0^1(M_A)$ and $\mathcal{T}_1^0(M_A)$ will be called resp. the sets of *anisotropic vector fields* and *1-forms* on M.

We emphasize a particularity of anisotropic vector fields. The elements $X \in \mathcal{T}_0^1(M_A)$ are (smooth) sections of the pullback bundle $\pi_A^*(TM) \to A$. This bundle is naturally isomorphic to the *vertical bundle* $\mathrm{V}A \to A$, where

$$\mathrm{V}_v A := \mathrm{Ker}(T_v \pi_A) = \mathrm{Span}\left\{\partial_{y^i}|_v : \ i \in \{1, \ldots, n\}\right\} \subset T_v A.$$

Thus, $X \in \mathcal{T}_0^1(M_A)$ can be identified with a vertical vector field X^{V} on A, called the *vertical lift* of X, such that

$$X_v = X^i(x, y)\partial_{x^i}|_x \in T_{\pi(v)}M \leftrightarrow X_v^{\mathrm{V}} = X^i(x, y)\partial_{y^i}|_{(x,y)} \in \mathrm{V}_v A. \tag{2}$$

Moreover, there is a *canonical anisotropic vector field*:

$$\mathbb{C} = y^i\, \partial_{x^i} \in \mathcal{T}_0^1(M_A), \qquad \mathbb{C}_v := v \in T_{\pi(v)}M. \tag{3}$$

Its vertical lift $\mathbb{C}^{\mathrm{V}}$ is usually called the *Liouville vector field*, and both $\mathbb{C}$ and $\mathbb{C}^{\mathrm{V}}$ are actually smooth on the whole TM. It is also worth pointing out that there is a natural inclusion

$$\mathcal{T}_s^r(M) \hookrightarrow \mathcal{T}_s^r(M_A), \qquad T \mapsto \tilde{T}, \tag{4}$$

just putting the components of $\tilde{T}$ in (1) as independent of directions and equal to those of T. The tensor $\tilde{T}$ will be called *isotropic* and we will not distinguish between T and $\tilde{T}$ when there is no possibility of confusion. Finally, we will say that a local vector field $V \in \mathfrak{X}(U)$ is A-*admissible* (where $U \subset M$ is an open subset) if $V_p \in A$ for all $p \in U$, i.e., V is a local section of the fibered manifold $A \to M$.

Notice that, at each $v \in A$, the fiber of the bundle $\pi_A^*(\overset{r)}{\bigotimes} TM \otimes \overset{s)}{\bigotimes} T^*M)$ becomes the space of all the (r, s)-tensors at $p = \pi_A(v)$. As this is a single vector space, the derivative of any curve in it is well defined. Thus, given an anisotropic tensor $T \in \mathcal{T}^r_s(M_A)$, we can define its *vertical derivative* at $v \in A$ in any direction $w \in T_pM$, $p = \pi(v)$, as follows:

$$(\dot\partial_w T)_v := \frac{d}{dt} T_{v+tw}|_{t=0},$$

which is again a tensor on T_pM. As the map $w \mapsto (\dot\partial_w T)_v$ is linear, we can naturally introduce an $(r, s+1)$ tensor field as follows (we write directly the obvious expression in coordinates).

Definition 3 Given an A-anisotropic tensor $T \in \mathcal{T}^r_s(M_A)$, its *vertical derivative* $\dot\partial T \in \mathcal{T}^r_{s+1}(M_A)$ is given (locally) by

$$(\dot\partial T)_v = \partial_{y^{b_{s+1}}} T^{a_1,\dots,a_r}_{b_1,\dots,b_s}(x, y)\ \partial_{a_1}\Big|_x \otimes \dots \otimes \partial_{a_r}\Big|_x \otimes \mathrm{d}x^{b_1}|_x \otimes \dots \otimes \mathrm{d}x^{b_s}_x \otimes \mathrm{d}x^{b_{s+1}}_x$$

in any natural coordinates as in (1).

3.2 Basic Notion of Anisotropic Connection

As with other kinds of connections, anisotropic connections can be defined in different ways. We introduce them in the spirit of the Koszul formulation of connections, namely as covariant derivatives (on a restricted domain of vector fields first, which is extended later). Thus, we refer to any of their characterizations also as *anisotropic* (or, more properly, *A-anisotropic*) *covariant derivatives*. Anyway, we will prove in Sect. 5 that such derivatives can be identified with certain linear connections on a suitable bundle (which appears naturally in the Finslerian setting).

Definition 4 An *A-anisotropic connection* (or *covariant derivative*) is a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{T}^1_0(M_A), \qquad (X, Y) \mapsto \nabla_X Y,$$

such that

1. $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$ for all $X, Y, Z \in \mathfrak{X}(M)$,
2. $\nabla_X(fY) = (X(f)Y) \circ \pi_A + (f \circ \pi_A)\nabla_X Y$ for all $f \in \mathcal{F}(M)$; $X, Y \in \mathfrak{X}(M)$,
3. $\nabla_{fX+hY} Z = (f \circ \pi_A)\nabla_X Z + (h \circ \pi_A)\nabla_Y Z$ for all $f, h \in \mathcal{F}(M)$ and $X, Y, Z \in \mathfrak{X}(M)$.

We say that an anisotropic connection ∇ is *homogeneous (of degree zero)*, or *invariant by homotheties*, if for every $v \in A$ and $\lambda > 0$, $(\nabla_X Y)_{\lambda v} = (\nabla_X Y)_v$ (that is, $\nabla_X Y = \nabla_X Y \circ h_\lambda$ where $h_\lambda : A \to A$ is the homothety given by $h_\lambda(v) = \lambda v$).

As in the case of affine connections, ∇ has a local nature. We can eventually use the notation $\nabla^v_X Y := (\nabla_X Y)_v$ and, consistently,

$$\nabla^V_X Y := (\nabla_X Y) \circ V \in \mathfrak{X}(U)$$

for any A-admissible local vector field $V \in \mathfrak{X}(U)$. By using coordinates, we can express ∇ in terms of its *Christoffel symbols* $\Gamma^a_{ij} : TU \cap A \to \mathbb{R}$, which are defined by

$$(\nabla_{\partial_i}\partial_j)_v \; (= \nabla^v_{\partial_i}\partial_j) = \Gamma^a_{ij}(v)\partial_a|_{\pi(v)}, \qquad \text{that is,} \qquad \nabla^V_{\partial_i}\partial_j = \left(\Gamma^a_{ij} \circ V\right)\partial_a. \quad (5)$$

Clearly, the homogeneity of ∇ is then equivalent to the 0-homogeneity of its Christoffel symbols, $\Gamma^a_{ij}(\lambda v) = \Gamma^a_{ij}(v)$, $\lambda > 0$. The following properties of these symbols are proven as in the standard case of affine connections.

Proposition 1 *(1) Under a change of coordinates $(U, x) \rightsquigarrow (\bar{U}, \bar{x})$, the Christoffel symbols Γ^m_{kl} and $\bar{\Gamma}^a_{ij}$ are related by*

$$\bar{\Gamma}^a_{ij}(\bar{x}, \bar{y}) = \frac{\partial \bar{x}^a}{\partial x^m}(x)\left(\frac{\partial^2 x^m}{\partial \bar{x}^i \partial \bar{x}^j}(x) + \frac{\partial x^k}{\partial \bar{x}^i}(x)\,\frac{\partial x^l}{\partial \bar{x}^j}(x)\,\Gamma^m_{kl}(x, y)\right). \quad (6)$$

(2) Conversely, given any local choice of functions Γ^k_{ij} for a coordinate atlas satisfying the cocycle transformation (6), there exists a unique anisotropic connection ∇ whose Christoffel symbols are these functions. Moreover, if the functions are 0-homogeneous in y, then the produced ∇ is homogeneous too.

(3) Any (classical, affine) Koszul connection on M induces naturally an anisotropic one with Christoffel symbols independent of y, for any open conic domain $A \subset TM$ which naturally projects onto the whole M.

(4) Given an anisotropic connection ∇ with Christoffel symbols Γ^a_{ij} for each coordinates (U, x), the choice of functions Γ^a_{ji} for each (U, x) yields a new connection $\hat{\nabla}$, and ∇ is called symmetric *if $\nabla = \hat{\nabla}$.*

3.3 Extension to a Covariant Derivative of Anisotropic Tensors

Note first that the (anisotropic) covariant derivatives of vector fields can be extended to (anisotropic) covariant derivatives of tensor fields on M. That is, using the product and contraction rules of tensor derivations, there is a unique extension of $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{T}^1_0(M_A)$ to a covariant derivative operator

$$\nabla : \mathfrak{X}(M) \times \mathcal{T}^r_s(M) \to \mathcal{T}^r_s(M_A), \qquad (X, T) \mapsto \nabla_X T, \quad (7)$$

such that

$$\nabla_X f = X(f) \circ \pi_A \qquad (\text{i.e.} \nabla_{\partial_i} f = \partial_i f \circ \pi_A) \quad (8)$$

for all $f \in \mathcal{F}(M) = \mathcal{T}_0^0(M)$. For example, if $\omega \in \mathcal{T}_1^0(M)$, then $\nabla_X \omega$ is defined by

$$(\nabla_X \omega)(\tilde{Y}) = X(\omega(Y)) - \tilde{\omega}(\nabla_X Y)$$

(recall (4), in particular $\tilde{\omega}_v = \omega_{\pi_A(v)} \in T^*_{\pi_A(v)}M$). In coordinates,

$$\nabla^V_{\partial_i} dx^k = -(\Gamma^k_{ij} \circ V)\, dx^j.$$

Next, we will go beyond extending the operators ∇_X to an *(anisotropic) covariant derivative of A-anisotropic tensor fields*

$$\nabla : \mathfrak{X}(M) \times \mathcal{T}^r_s(M_A) \to \mathcal{T}^r_s(M_A), \qquad (X, T) \mapsto \nabla_X T, \tag{9}$$

in a natural way (again, as usual, the same symbol ∇ will be used). The key to get this new extension of ∇ is to find a definition of $\nabla_X h$ when $h \in \mathcal{F}(A)$, that is, to find a natural extension of (8). The appropriate choice will be

$$(\nabla_X h)(v) = X_p(h \circ V) - \dot{\partial}_{(\nabla_{X_p} V)} h, \tag{10}$$

where $p = \pi_A(v)$ and $V \in \mathfrak{X}(M)$ is such that $V_p = v$.

Lemma 1 *The definition of $\nabla_X h$ in (10) is independent of the choice of V. Moreover, if $h = f \circ \pi_A$ for some $f \in \mathcal{F}(M)$, then $\nabla_X h$ is equal to $\nabla_X f$ in (8).*

Proof It is enough to check that the expression (10) written in coordinates is independent of V. Let $V = V^j \partial_j$, $X = X^i \partial_i \in \mathfrak{X}(U)$. Then

$$\begin{aligned} X(h \circ V) &= X^i \left(\tfrac{\partial h}{\partial x^i} \circ V\right) + X^i \left(\tfrac{\partial h}{\partial y^j} \circ V\right) \tfrac{\partial V^j}{\partial x^i}, \\ \dot{\partial}_{(\nabla_X V)} h &= \left([X(V^k) + X^i V^j \left(\Gamma^k_{ij} \circ V\right)] \circ \pi_A\right) \dot{\partial}_k h, \end{aligned}$$

in the latter using that $\nabla_X V = X(V^i)\partial_i + X^i V^j \nabla_{\partial_i} \partial_j$. So, (10) reads at each v of coordinates (x, y):

$$\nabla_X h = \left(X^i \circ \pi_A\right) \left(\frac{\partial h}{\partial x^i} - y^j \Gamma^k_{ij} \frac{\partial h}{\partial y^k}\right), \tag{11}$$

which is independent of the chosen V, as required.

Even though this lemma ensures the consistency of the definition of $\nabla_X h$, its meaning is not so evident. Algebraically, it ensures a sort of chain rule for $X(h \circ V)$. Anyway, we will give a further interpretation (see Remark 4). As a summary of this subsection, we obtain

Theorem 1 *Let ∇ be an A-anisotropic connection and $X \in \mathfrak{X}(M)$. The operator $\nabla_X \colon Y \in \mathcal{T}^1_0(M_A) \mapsto \nabla_X Y \in \mathcal{T}^1_0(M_A)$ determines a unique tensor derivation of the* *tensor algebra* $\mathcal{T}(M_A) = \bigoplus_{r,s\geq 0} \mathcal{T}_s^r(M_A)$ *such that* $\nabla_X h$ *is given by* (10) *for* $h \in \mathcal{F}(A)$. *If* $T \in \mathcal{T}_s^r(M_A)$ *has the coordinate expression* (1), *then the components of* $\nabla_k T := \nabla_{\partial_k} T$ *are*

$$T^{a_1,\dots,a_r}_{b_1,\dots,b_s|k} := (\nabla_k T)^{a_1,\dots,a_r}_{b_1,\dots,b_s}$$
$$= \partial_k T^{a_1,\dots,a_r}_{b_1,\dots,b_s} - \Gamma^i_{kj}\, y^j\, \dot{\partial}_i T^{a_1,\dots,a_r}_{b_1,\dots,b_s} + \sum_{l=1}^r \Gamma^{a_l}_{kj_l} T^{a_1,\dots,j_l,\dots,a_r}_{b_1,\dots,b_s} - \sum_{l=1}^s \Gamma^{i_l}_{kb_l} T^{a_1,\dots a_r}_{b_1,\dots,i_l,\dots b_s},$$

where $\partial_k = \partial/\partial x^k$, $\dot{\partial}_k = \partial/\partial y^k$.

The proof can be carried out by following the indications above. Anyway, full computations can be found in [11], where the following intrinsic version of the last displayed formula (regarding T as a $\mathcal{F}(A)$-multilinear map) can also be found in [11, Theorem 11]: for any $v \in A$ and (local) extension $V \in \mathfrak{X}(U)$ of v,

$$\begin{aligned}(\nabla_X T)_v(\theta^1,\dots,\theta^r,X_1,\dots,X_s) =& X_{\pi(v)}(T_V(\theta^1,\dots,\theta^r,X_1,\dots,X_s)) \\ & - (\dot{\partial} T)_v(\theta^1,\dots,\theta^r,X_1,\dots,X_s,\nabla^V_X V), \\ & - \sum_{i=1}^r T_v(\theta^1,\dots,\nabla_X\theta^i,\dots,\theta^r,X_1,\dots,X_s) \\ & - \sum_{j=1}^s T_v(\theta^1,\dots,\theta^r,X_1,\dots,\nabla_X X_j,\dots,X_s),\end{aligned} \tag{12}$$

where $X, X_1,\dots,X_s \in \mathfrak{X}(M)$ and $\theta^1,\dots,\theta^r \in \mathfrak{X}^*(M)$.

Remark 1 One can even extend ∇ to a map

$$\nabla : \mathcal{T}_0^1(M_A) \times \mathcal{T}_s^r(M_A) \to \mathcal{T}_s^r(M_A), \qquad (X,T) \mapsto \nabla_X T,$$

just making it $\mathcal{F}(A)$-linear with respect to the first variable.

Remark 2 We have seen that the domain of an anisotropic connection can be extended from vector fields $X, Y \in \mathfrak{X}(M)$ to anisotropic vector fields in $\mathcal{T}_0^1(M_A)$. Additionally, multilinear maps over anisotropic tensor fields valued on anisotropic vector fields can be regarded as anisotropic tensor fields.

A relevant example appears when two anisotropic connections $\bar{\nabla}, \nabla$ are considered. Their difference $Q = \bar{\nabla} - \nabla$ is naturally an $\mathcal{F}(M)$-multilinear map

$$\mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathcal{T}_0^1(M_A),$$

which can be uniquely extended by $\mathcal{F}(A)$-multilinearity to an anisotropic tensor field $Q \in \mathcal{T}_2^1(M_A)$ (recall the embedding $\mathfrak{X}(M) = \mathcal{T}_0^1(M) \hookrightarrow T_0^1(M_A)$ in (4)). Moreover, Q can also be regarded as an $\mathcal{F}(A)$-multilinear map

$$\mathcal{T}_0^1(M_A) \times \mathcal{T}_0^1(M_A) \to \mathcal{T}_0^1(M_A), \text{ or } \mathcal{T}_0^1(M_A) \times \mathcal{T}_0^1(M_A) \times \mathcal{T}_1^0(M_A) \to \mathcal{F}(A).$$

Applying the previous discussion to the case of the connection $\hat{\nabla}$ obtained from ∇ with Christoffel symbols $\hat{\Gamma}^k_{ji} = \Gamma^k_{ij}$ (see Proposition 1), the following definition becomes consistent.

Definition 5 The *torsion* of an anisotropic connection ∇ is the tensor $\mathrm{Tor} \in \mathcal{T}_2^1(M_A)$ whose components are

$$\mathrm{Tor}^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}.$$

The following consequence is also straightforward now.

Corollary 1 *The space of all anisotropic connections on M has the structure of an affine space with associated vector space $\mathcal{T}_2^1(M_A)$.*

Moreover, the homogeneous anisotropic connections form an affine subspace with associated vector space the subspace of $\mathcal{T}_2^1(M_A)$ composed by its 0-homogeneous tensors (i.e., those which satisfy $Q(\lambda v) = Q(v)$, $\lambda > 0$).

4 Anisotropic Versus Nonlinear Connections

Any anisotropic connection induces a nonlinear connection on $\pi_A : A \to M$. Let us start recalling the framework of the latter adapted to our case.

The coordinates (x, y) on TU induce naturally coordinates $(x, y, \dot{x}, \dot{y})$ on $TA \subset T(TM)$. Then, the vertical bundle $\mathrm{V}A \subset TA$ is the subbundle of $TA \to A$ composed by the elements with $\dot{x} = 0$, and so $(x, y, \dot{y})$ is a local coordinate system for $\mathrm{V}A$. Clearly, $\mathrm{V}A$ is naturally identifiable with the pullback bundle $\pi_A^*(TM)$, and, consequently, the (smooth) sections of the bundle $\mathrm{V}A \to A$ can be regarded as A-anisotropic vector fields on M (recall (2)).

4.1 Setting for Nonlinear Connections

There are several ways to define a connection on the fibered manifold $A \to M$, commonly called a *nonlinear connection* in the Finsler geometry literature. One way is to provide a vector bundle homomorphism $\nu : TA \to \mathrm{V}A$ such that $\nu|_{\mathrm{V}A}$ is the identity. Then, the *horizontal distribution* $\mathrm{H}A := \mathrm{Ker}\,\nu$ characterizes ν and gives a decomposition $TA = \mathrm{H}A \oplus \mathrm{V}A$, which can also be used as an alternative definition of the nonlinear connection. One has the following representations in coordinates:

$$\nu\left(\dot{x}^i \left.\frac{\partial}{\partial x^i}\right|_{(x,y)} + \dot{y}^a \left.\frac{\partial}{\partial y^a}\right|_{(x,y)}\right) = \left(\dot{y}^a + N^a_i(x,y)\dot{x}^i\right) \left.\frac{\partial}{\partial y^a}\right|_{(x,y)},$$

$$\mathrm{H}_v A = \mathrm{Span}\left\{ \left.\frac{\delta}{\delta x^i}\right|_v := \left.\frac{\partial}{\partial x^i}\right|_v - N_i^a(v) \left.\frac{\partial}{\partial y^a}\right|_v : i \in \{1, ..., n\} \right\}, \tag{13}$$

where the smooth functions N_i^a are defined for $v \in A \cap TU$. When taking a new induced chart $(T\bar{U}, (\bar{x}, \bar{y}))$, the new connection coefficients $\bar{N}_i^a$ are related to the old ones by

$$\bar{N}^a_i(\bar{x}, \bar{y}) = \frac{\partial \bar{x}^a}{\partial x^b}(x) \left(\frac{\partial^2 x^b}{\partial \bar{x}^i \partial \bar{x}^j}(x)\, \bar{y}^j + \frac{\partial x^k}{\partial \bar{x}^i}(x)\, N^b_k(x, y) \right) \tag{14}$$

over $T(U \cap \bar{U})$. One has that $\mathrm{H}_v A$ can be identified with $T_v A/\mathrm{V}_v A = T_{\pi(v)} M$, which allows one to identify any $X \in \mathcal{T}_0^1(M_A)$ with a horizontal vector field X^{H} on A, called the *horizontal lift* of X, determined by

$$X_v = X^i(x, y)\partial_{x^i}|_x \in T_{\pi(v)} M \leftrightarrow X_v^{\mathrm{H}} = X^i(x, y)\delta_{x^i}|_{(x,y)} \in \mathrm{H}_v A, \tag{15}$$

where we have simplified the notation $\delta_{x^i} := \delta/\delta x^i$ in (13).

Remark 3 (a) Conversely, any covering of charts of M endowed with a set of functions satisfying the cocycle transformation (14) determines unequivocally a nonlinear connection of $A \to M$. Incidentally, we recover a standard fact from the theory of fibered manifolds: a nonlinear connection is the same thing as a section of the 1-jet bundle $\mathbf{J}^1 A \to A$; see [18, Sect. 17] for instance. If, by means of such a section, for the selected 1-jet at $v \in A$ one puts

$$N_i^a(v) = -\frac{\partial V^a}{\partial x^i}(\pi(v))$$

(where V is a local extension of v that determines the jet), then it is straightforward to see that these N_i^a's satisfy (14).

(b) It makes sense to assume that the nonlinear connection on $A \to M$ is *positive homogeneous* in the sense that the distribution $\mathrm{H}A$ is invariant under homotheties $h_\lambda : A \to A$, that is, if $Th_\lambda : TA \to TA$ is the tangent map (differential) of h_λ, then for each $\lambda > 0$ and $v \in A$,

$$(Th_\lambda)_v(\mathrm{H}_v) = \mathrm{H}_{\lambda v}. \tag{16}$$

Equivalently, the connection coefficients satisfy

$$N_i^a \circ h_\lambda = \lambda N_i^a \qquad \left(N_i^a(x, \lambda y) = \lambda N_i^a(x, y)\right).$$

As an integral curve $(x(t), y(t))$ of the horizontal vector field δ_{x^i} satisfies $dy^a/dx^i = -N_i^a(x, y)$, the 1-homogeneity of the functions N_i^a characterizes when $(x(t), \lambda y(t))$ $(= h_\lambda(x(t), y(t)))$ is also an integral curve. In this case, the relations

$$\delta_{x^i}|_{\lambda v} = (Th_\lambda)_v\,(\delta_{x^i}|_v), \qquad \partial_{y^i}|_{\lambda v} = \lambda^{-1}\,(Th_\lambda)_v\,(\partial_{y^i}|_v)$$

(0-homogeneity of δ_{x^i} and (-1)-homogeneity of ∂_{y^i}, see [4, Sect. 1.5]) are also satisfied,

$$N_i^a(x, y) = \partial_{y^j} N_i^a(x, y) y^j \tag{17}$$

(by Euler's theorem) and $\partial_{y^j} N_i^a(x, y)$ is 0-homogeneous in y (i.e., invariant under $h_\lambda : (x, y) \mapsto (x, \lambda y)$).

It is worth pointing out that a nonlinear connection ν induces a (nonlinear) covariant derivative of sections of $A \to M$ defined on open subsets $U \subset M$. Namely, let $\mathfrak{X}^A(U)$ be the set of all A-admissible vector fields on U (which behaves in a similar way as a module on the positive functions on U when A is convex at each point) and let $W : U \to A$ be an element of it; in coordinates, $W(x) = (x^i, W^a(x))$. The *ν-covariant derivative* of W is[2] $\nu \circ TW$, so that for any $X \in \mathfrak{X}(U)$ and in natural coordinates $(x, y, \dot{x}, \dot{y})$,

$$\nu \circ TW(X) \equiv (x^i, W^a(x), 0, (D_X W)^a (x)),$$

where

$$(D_X W)^a (x) = X^i \left(\frac{\partial W^a}{\partial x^i} + N_i^a(W(x)) \right).$$

We denote the section $x \mapsto (x^i, (D_X W)^a (x))$ (which is a vector field on U) by $D_X W$. This induces a map

$$D : \mathfrak{X}(U) \times \mathfrak{X}^A(U) \to \mathfrak{X}(U), \qquad (X, W) \to D_X W, \tag{18}$$

which is linear in X but, in general,[3] is not linear in W. Notice that DW characterizes $\nu \circ TW$ (so any of them can be called *ν-covariant derivative*), while D determines the functions N_i^a and, so, the connection ν.

The case of Koszul connections. The nonlinearity of a connection refers to the nonlinearity of its covariant derivative (18). When this derivative is actually linear, the connection is called *linear* too (as usual, the name "nonlinear" must be understood in the sense of "non-necessarily linear"). Anyway, to be more specific, we will use the name *Koszul connection* for the linear connections on $TM \to M$ as in Proposition 1 (3). These (also named *affine connections* in the literature) are locally determined by their Christoffel symbols, which depend only on $x \equiv p \in M$. Finally, to deal with the Koszul case, recall first the following elementary technical result.

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[2] It is worth pointing out that some authors such as Shen, Dahl or Miron-Bucataru, consider an alternative covariant derivative by using the *flip* automorphism of TM, namely $(x, y, \dot{x}, \dot{y}) \mapsto (x, \dot{x}, y, \dot{y})$, see for example [22, Sect. 3.2]. This is avoided here, due to the different role of A and TM in our approach.

[3] Technically, it is *never* linear, as $A \subset TM \setminus \mathbf{0}$ is not a vector bundle, but it could be the restriction to A (i.e., to A-valued vector fields) of a map $\mathfrak{X}(U) \times \mathfrak{X}(U) \to \mathfrak{X}(U)$ linear in the second component, as we will see in Proposition 2.

Lemma 2 *Assume that the functions $N_i^a : A \cap TU \to \mathbb{R}$ are 1-homogeneous and can be smoothly extended to 0. Then there exist smooth functions $\Gamma_{ij}^k : U \to \mathbb{R}$ such that*

$$N_i^a(x, y) = \Gamma_{ij}^a(x) y^j. \tag{19}$$

In particular, $N_{i\cdot j}^a(x) := \partial_{y^j} N_i^a(x, y) = \Gamma_{ij}^a(x)$ and each N_i^a can be naturally extended to TU.

Proof Just recall that for each point $p \in U$ and for any indices a, i, the function f defined on T_pM by $f(v) = N_i^a(v)$ is linear, as $df|_0(v) = \lim_{\lambda \searrow 0} f(\lambda v)/\lambda = f(v)$ for each v by homogeneity. □

Proposition 2 *A nonlinear connection on $A \to M$ induces a Koszul connection (that is, the map (18) is extended to $\mathfrak{X}(U) \times \mathfrak{X}(U)$ and, then, linear in the second component) when its horizontal distribution can be smoothly extended to the zero section (i.e., when its coefficients N_i^a's satisfy (19) in every coordinate chart).*

Most of the framework of linear connections explained here for $A \to M$ can be extended to more general bundles in a standard way and will be used without further comments in Sect. 6.

4.2 Interplay Between Anisotropic Connections and Nonlinear Ones

Consider an anisotropic connection ∇ as in Definition 4. It can be proved that ∇ induces a horizontal distribution i.e., a nonlinear connection by using an intrinsic approach [11, Sect. 3.1]. However, we would like to emphasize the following direct relation with the cocycle transformation associated with the Christoffel symbols Γ_{ij}^a in (5).

Theorem 2 *(1) An anisotropic connection ∇ defines canonically a nonlinear connection ν^∇ whose coefficients N_i^a with respect to a chart are*[4]

$$N_i^a(x, y) = \Gamma_{ij}^a(x, y)\, y^j. \tag{20}$$

If ∇ is homogeneous then ν^∇ is also homogeneous.

(2) A nonlinear connection ν defines canonically an anisotropic connection ∇^ν whose Christoffel symbols Γ_{ij}^a with respect to a chart are

$$\Gamma_{ij}^a(x, y) = \dot{\partial}_j N_i^a(x, y)\, (= N_{i\cdot j}^a(x, y)).$$

---

[4] In fact, there would also be a second nonlinear connection $\Gamma_{ji}^a(x, y)\, y^j$. In our terms, this one would be $\nu^{\tilde{\nabla}}$ for $\tilde{\nabla} = \nabla - \mathrm{Tor}$ (recall Remark 2).

If ν is homogeneous then ∇^{ν} is also homogeneous and $\nu^{(\nabla^{\nu})} = \nu$. In particular, the map $\nabla \mapsto \nu^{\nabla}$ is onto when it is restricted to the sets of homogeneous anisotropic and nonlinear connections.

(3) For any homogeneous nonlinear connection ν, the set $\{\nabla$ anisotropic connection $: \nu^{\nabla} = \nu\}$ is

$$\{\nabla^{\nu} + Q \,:\, Q \in \mathcal{T}_2^1(M_A) \text{ and } Q^k_{ij} y^j = 0\}.$$

(4) On the set of Koszul covariant derivatives (restricted to A), the map $\nabla \mapsto \nu^{\nabla}$ is injective and its image consists precisely of the (restrictions to A of the) linear connections on $TM \to M$.

Proof (1) Notice that the functions N^a_i satisfy the cocycle transformation (14). Then, the 0-homogeneity of the Christoffel symbols $\Gamma^a_{ij}(x, y)$ gives the 1-homogeneity of the connection coefficients $N^a_i(x, y)$.

(2) The cocycle (14) for $N^a_i(x, y)$ implies the cocycle (6) for Γ^a_{ij}. Then, the homogeneity of ν implies (17) and thus (20).

(3) Straightforward from part (2) and (20).

(4) For such a ∇, the Christoffel symbols are direction independent and ν^{∇} is linear. Given a second $\bar{\nabla}$ with $\nu^{\bar{\nabla}} = \nu^{\nabla}$, the difference $Q = \bar{\nabla} - \nabla$ is also direction independent and $Q^k_{ij}(x)y^j = 0$, which implies $Q = 0$ by taking derivatives with respect to each y^l. □

In item (2), one sees in coordinates that the torsion of ∇^{ν} (recall Definition 5) coincides with the *torsion of* ν, which can be defined intrinsically [27, (7.8.10)].

Remark 4 Finally, we can give the promised interpretation of the definition of $\nabla_X h$ in (10), (11). Indeed, any $X_p = X^i(x)\partial_i|_x \in T_pM$, $p \equiv x$, gives a horizontal lift $X^{\mathrm{H}}_v = X^i(x)\,\delta/\delta x^i|_{(x,y)} \in \mathrm{H}_v A$ $(\subset T_v A)$ for any $v \equiv (x, y)$ with $\pi_A(v) = p$. So, substituting the expressions in the formulas of $\mathrm{H}_{(x,y)}A$ and (20), we find

$$X^H_v(h) = X^i(\pi(v))\, \left(\partial_i h - \Gamma^k_{ij}\, y^j \dot{\partial}_k h\right)(v) = (\nabla_X h)_v.$$

The last equality is in agreement with (11).

5 Anisotropic Versus Linear Connections

Next, our goal will be to identify anisotropic connections with a class of linear connections on the vector bundle $\mathrm{V}A \to A$.

As shown at the beginning of Sect. 4, $\mathrm{V}A$ $(\subset TA)$ admits as natural coordinates $(x, y, \dot{y})$, which will be relabeled here as (x, y, z). Thus, we can write $z^a \partial_{y^a}|_{(x,y)} \in \mathrm{V}_{(x,y)}A$. The tangent bundle $T(\mathrm{V}A)$ includes the vertical subbundle $\mathrm{V}(\mathrm{V}A)$, whose fiber $\mathrm{V}_w(\mathrm{V}A)$ is generated by the $\partial_{z^a}|_{(x,y,z)}$, where (x, y, z) are the coordinates of $w \in \mathrm{V}A$.

5.1 Linear Connections on $\mathrm{V}A \to A$

In order to define an *(Ehresmann) connection* on $\mathrm{V}A \to A$, we have to provide a smooth horizontal decomposition $T(\mathrm{V}A) = \mathrm{H}(\mathrm{V}A) \oplus \mathrm{V}(\mathrm{V}A)$. Notice first that any positive homothety h_λ on A induces a natural morphism

$$h_{\lambda_*} = (Th_\lambda)|_{\mathrm{V}A} : \mathrm{V}A \to \mathrm{V}A, \quad (x, y, z) \mapsto (x, \lambda y, \lambda z).$$

The new horizontal distribution (and then the Ehresmann connection itself) is called *invariant by positive homotheties* if it is preserved by the tangent map of h_{λ_*}, i.e., if for $w \in \mathrm{V}A$,

$$(Th_{\lambda_*})_w\,(\mathrm{H}_w(\mathrm{V}A)) = \mathrm{H}_{h_{\lambda_*}(w)}(\mathrm{V}A). \tag{21}$$

In what follows, we will focus on the particular case when a *linear connection* ν^* is given on $\mathrm{V}A \to A$. The (linear) covariant derivative operator associated with ν^* will be denoted by ∇^*. As the sections of $\mathrm{V}A \to A$ are identified with the anisotropic vector fields in $\mathcal{T}_0^1(M_A)$ (recall the vertical isomorphism (2)), ∇^* becomes a map

$$\nabla^* : \mathfrak{X}(A) \times \mathcal{T}_0^1(M_A) \to \mathcal{T}_0^1(M_A), \qquad (W, Z) \mapsto \nabla^*_W Z. \tag{22}$$

(The reader may appreciate the similarities and differences between (22) and the anisotropic covariant derivative (9), the latter with $(r, s) = (1, 0)$.) Moreover, it is straightforward but tedious to prove (from the definition of ∇^* in terms of the corresponding horizontal decomposition) that the homothety invariance of ∇^* is characterized as follows. If a section $Z^{\mathrm{V}} : A \to \mathrm{V}A$ is 0-homogeneous (meaning $Z^{\mathrm{V}} \circ h_\lambda = \lambda^{-1} h_{\lambda_*} \circ Z^{\mathrm{V}}$ as in [4, Sect. 1.5]), then

$$\nabla^*_{Th_\lambda(W)} Z^{\mathrm{V}} = \lambda^{-1} Th_\lambda(\nabla^*_W Z^{\mathrm{V}}). \tag{23}$$

To specify ∇^* by means of its Christoffel symbols, one has to choose a basis for $\mathfrak{X}(A)$ and another one for $\mathcal{T}_0^1(M_A)$. A possible choice would be the one associated with coordinates, namely $\{\partial_i|_{(x,y)}, \dot{\partial}_i|_{(x,y)}\} = \{\partial_{x^i}|_{(x,y)}, \partial_{y^i}|_{(x,y)}\}$ for the former and[5] $\{\partial_j|_x\} \equiv \{\dot{\partial}_j|_{(x,y)}\}$ for the latter. However, in case we have a prescribed nonlinear connection $\overset{o}{\nu}$ on $A \to M$, a more convenient choice than $\{\partial_i, \dot{\partial}_i\}$ may be $\{\delta_i, \dot{\partial}_i\}$:

$$\delta_i|_{(x,y)} = \left.\frac{\delta}{\delta x^i}\right|_{(x,y)} = \partial_{x^i}|_{(x,y)} - \overset{o}{N}{}_i^a\,(x, y)\partial_{y^a}|_{(x,y)}, \qquad \dot{\partial}_j|_{(x,y)} = \partial_{y^j}|_{(x,y)}.$$

This happens in the pseudo-Finsler case, where $\overset{o}{\nu}$ is provided by the geodesic spray and, thus, is homogeneous. This last property of $\overset{o}{\nu}$ will be assumed for the sake of simplicity, even though actually it will be important only when the homogene-

[5] Keep in mind that this identification also corresponds exactly to the vertical isomorphism (2).

ity of the Christoffel symbols is involved. From the homogeneity of $\overset{o}{\nu}$, δ_i is 1-homogeneous, namely $\delta_i|_{(x,\lambda y)} = (Th_\lambda)_{(x,y)}\,(\delta_i|_{(x,y)})$, while $\dot\partial_i$ is 0-homogeneous, namely $\dot\partial_i|_{(x,\lambda y)} = \lambda^{-1}\,(Th_\lambda)_{(x,y)}\,(\dot\partial_i|_{(x,y)})$.

Definition 6 The horizontal and vertical Christoffel symbols of ∇^* with respect to a prescribed homogeneous nonlinear connection $\overset{o}{\nu}$ in the coordinates (U, x) are the functions $\left(\Gamma^{\mathrm{H}}\right)^a_{ij}$, $\left(\Gamma^{\mathrm{V}}\right)^a_{ij}$ determined on $A \cap TU$ by

$$\left(\Gamma^{\mathrm{H}}\right)^a_{ij}(x, y)\,\partial_a|_x = \nabla^*_{\delta_i|_{(x,y)}}\partial_j (\equiv \nabla^*_{\delta_i|_{(x,y)}}\dot\partial_j = \left(\Gamma^{\mathrm{H}}\right)^a_{ij}(x, y)\dot\partial_a|_{(x,y)}),$$

$$\left(\Gamma^{\mathrm{V}}\right)^a_{ij}(x, y)\,\partial_a|_x = \nabla^*_{\dot\partial_i|_{(x,y)}}\partial_j (\equiv \nabla^*_{\dot\partial_i|_{(x,y)}}\dot\partial_j = \left(\Gamma^{\mathrm{V}}\right)^a_{ij}(x, y)\dot\partial_a|_{(x,y)}).$$

Proposition 3 *(1) The Christoffel symbols $\left(\Gamma^{\mathrm{H}}\right)^a_{ij}$, $\left(\Gamma^{\mathrm{V}}\right)^a_{ij}$ of ∇^* with respect to $\overset{o}{\nu}$ satisfy:*

(a) The cocycle for $\left(\Gamma^{\mathrm{H}}\right)^a_{ij}$ (resp., $\left(\Gamma^{\mathrm{V}}\right)^a_{ij}$) under a change of coordinates coincides with the one for the Christoffel symbols Γ^a_{ij} of an A-anisotropic connection (6) (resp., the one of an A-anisotropic $(1, 2)$ tensor). In particular, if all the $\left(\Gamma^{\mathrm{V}}\right)^a_{ij}$'s vanish for some coordinates on U, then they vanish for any coordinates therein.

(b) If the linear connection ∇^* is invariant by homotheties then, for all $\lambda > 0$

(b1) $\left(\Gamma^{\mathrm{H}}\right)^a_{ij}(x, \lambda y) = \left(\Gamma^{\mathrm{H}}\right)^a_{ij}(x, y)$ (0-homogeneity), and

(b2) $\left(\Gamma^{\mathrm{V}}\right)^a_{ij}(x, \lambda y) = \lambda^{-1}\left(\Gamma^{\mathrm{V}}\right)^a_{ij}(x, y)$ ((−1)-homogeneity).

(2) Conversely, once a homogeneous nonlinear connection $\overset{o}{\nu}$ is prescribed, any local choice of functions $\left(\Gamma^{\mathrm{H}}\right)^a_{ij}$, $\left(\Gamma^{\mathrm{V}}\right)^a_{ij}$ satisfying (a) for a coordinate atlas determines a unique linear connection ∇^*, whose Christoffel symbols with respect to $\overset{o}{\nu}$ in that atlas coincide with the original $\left(\Gamma^{\mathrm{H}}\right)^a_{ij}$, $\left(\Gamma^{\mathrm{V}}\right)^a_{ij}$. Moreover, if (consistently with (b) above) the functions $\left(\Gamma^{\mathrm{H}}\right)^a_{ij}$, $\left(\Gamma^{\mathrm{V}}\right)^a_{ij}$ are chosen to be, resp., with 0 and (−1) homogeneity in y, then the produced ∇^* is invariant by homotheties.

Proof (1) The cocycles of $\left(\Gamma^{\mathrm{H}}\right)^a_{ij}$ and $\left(\Gamma^{\mathrm{V}}\right)^a_{ij}$ can be checked from their definitions using the transformation laws

$$\bar\delta_i = \frac{\partial x^k}{\partial \bar x^i}\delta_k, \qquad \bar{\dot\partial}_i = \frac{\partial x^k}{\partial \bar x^i}\dot\partial_k, \qquad \partial_j = \frac{\partial \bar x^l}{\partial x^j}\bar\partial_l$$

(recall that the last x^j's are the ones on M and not those on TM). Then, using only the definition of $\left(\Gamma^{\mathrm{H}}\right)^a_{ij}$ and $\left(\Gamma^{\mathrm{V}}\right)^a_{ij}$, the 1-homogeneity of the δ_i's, and the 0-homogeneity of the $\dot\partial_i$'s, the following identities hold true:

$$\left(\Gamma^{\mathrm{H}}\right)^a_{ij}(x, \lambda y)\;\dot\partial_a\Big|_{(x,\lambda y)} = \nabla^*_{\delta_i|_{(x,\lambda y)}}\dot\partial_j = \nabla^*_{(Th_\lambda)_{(x,y)}(\delta_i|_{(x,y)})}\dot\partial_j, \tag{24}$$

$$\left(\Gamma^{\mathrm{H}}\right)_{ij}^{a}(x,y)\,\dot\partial_a\big|_{(x,y)}=\nabla^*_{\delta_i|_{(x,y)}}\dot\partial_j$$
$$=\left(\Gamma^{\mathrm{H}}\right)_{ij}^{a}(x,y)\lambda\,(Th_\lambda)^{-1}_{(x,y)}\,(\dot\partial_a\big|_{(x,\lambda y)})=(Th_\lambda)^{-1}_{(x,y)}\,(\lambda\left(\Gamma^{\mathrm{H}}\right)_{ij}^{a}(x,y)\,\dot\partial_a\big|_{(x,\lambda y)}). \tag{25}$$

Now, in case that ∇^* is invariant by homotheties, one can use (23) with $Z^{\mathrm{V}}=\dot\partial_j$. From (23), (24) and (25) it follows that

$$\begin{aligned}\left(\Gamma^{\mathrm{H}}\right)_{ij}^{a}(x,\lambda y)\,\dot\partial_a\big|_{(x,\lambda y)}=&\lambda^{-1}\,(Th_\lambda)_{(x,y)}\,(\nabla^*_{\delta_i|_{(x,y)}}\dot\partial_j)\\ =&\lambda^{-1}\,(Th_\lambda)_{(x,y)}\,((\Gamma^{\mathrm{H}})_{ij}^{a}(x,y)\,\dot\partial_a\big|_{(x,y)})\\ =&\lambda^{-1}\,(Th_\lambda)_{(x,y)}\,\{(Th_\lambda)^{-1}_{(x,y)}\,(\lambda\left(\Gamma^{\mathrm{H}}\right)_{ij}^{a}(x,y)\,\dot\partial_a\big|_{(x,\lambda y)})\}\\ =&\left(\Gamma^{\mathrm{H}}\right)_{ij}^{a}(x,y)\,\dot\partial_a\big|_{(x,\lambda y)},\end{aligned}$$

thus proving (b1). An analogous calculation proves (b2).

(2) Knowing the cocycles that the Christoffel symbols of such a ∇^* should satisfy, it is possible to define ∇^* by $\left(\Gamma^{\mathrm{H}}\right)_{ij}^{a}$, $\left(\Gamma^{\mathrm{V}}\right)_{ij}^{a}$ on each coordinate chart and, as usual, assert that the local definitions patch together to form a global linear connection on $\mathrm{V}A\to A$. Moreover, (24) and (25) are still valid for this ∇^*, and so are the analogous identities for the $\left(\Gamma^{\mathrm{V}}\right)_{ij}^{a}$'s. So, if the $\left(\Gamma^{\mathrm{H}}\right)_{ij}^{a}$'s are 0-homogeneous and the $\left(\Gamma^{\mathrm{V}}\right)_{ij}^{a}$'s are (-1)-homogeneous, then one can use those identities to show that

$$\nabla^*_{(Th_\lambda)_{(x,y)}(\delta_i|_{(x,y)})}\dot\partial_j=\lambda^{-1}\,(Th_\lambda)_{(x,y)}\,(\nabla^*_{\delta_i|_{(x,y)}}\dot\partial_j), \tag{26}$$

$$\nabla^*_{(Th_\lambda)_{(x,y)}(\dot\partial_i|_{(x,y)})}\dot\partial_j=\lambda^{-1}\,(Th_\lambda)_{(x,y)}\,(\nabla^*_{\dot\partial_i|_{(x,y)}}\dot\partial_j). \tag{27}$$

By using that $\{\delta_i,\dot\partial_i\}$ is a basis for $\mathfrak{X}(A)$, $\{\dot\partial_i\}$ is a basis for the vertical vector fields, and the two expressions $\nabla^*_{(Th_\lambda)_{(x,y)}(W_{(x,y)})}Z^{\mathrm{V}}$ and $\lambda^{-1}\,(Th_\lambda)_{(x,y)}\,(\nabla^*_{W_{(x,y)}}Z^{\mathrm{V}})$ are linear in W and Leibnizian in Z^{V}, the identities (26) and (27) prove that ∇^* satisfies (23), hence that ∇^* is invariant by homotheties. □

5.2 *Anisotropic Connections as Vertically Trivial Linear Connections*

For a linear connection ∇^* on $\mathrm{V}A\to A$, the vanishing of all $\left(\Gamma^{\mathrm{V}}\right)_{ij}^{a}$'s involves only vertical derivatives and, so, it is an intrinsic property (independent also of $\overset{o}{\nu}$ in Proposition 3 item (1)(a)). This makes possible the following definition.

Definition 7 Let ∇^* be a linear connection on $\mathrm{V}A\to A$. We say that ∇^* is *vertically trivial* if its vertical Christoffel symbols vanish ($\left(\Gamma^{\mathrm{V}}\right)_{ij}^{a}=0$ everywhere).

Remark 5 From Proposition 3, it is clear that any homogeneous nonlinear connection $\overset{o}{\nu}$ induces a projection of the set of all ∇^*'s onto the set of vertically trivial connections, such that $(\left(\varGamma^{\rm H}\right)^a_{ij}, \left(\varGamma^{\rm V}\right)^a_{ij}) \mapsto (\left(\varGamma^{\rm H}\right)^a_{ij}, 0)$.

Theorem 3 *Let $\overset{o}{\nu}$ be a homogeneous nonlinear connection on $A \to M$. The map between the sets of the vertically trivial and the A-anisotropic connections, defined locally by*

$$\begin{array}{ccc} \{\textit{vertically trivial } \nabla^*\textit{'s on } \mathrm{V}A \to A\} & \longrightarrow & \{A\textit{-anisotropic}\nabla'\textit{s}\}, \\ (\left(\varGamma^{\rm H}\right)^a_{ij}, \left(\varGamma^{\rm V}\right)^a_{ij} = 0) & \mapsto & \varGamma^a_{ij} = \left(\varGamma^{\rm H}\right)^a_{ij}, \end{array}$$

is well defined and bijective, and it also identifies the homothety invariant ∇^*'s with the homogeneous ∇'s.

Moreover, this map does not depend on the choice of $\overset{o}{\nu}$. So, there exists a natural identification between vertically trivial and anisotropic connections, and it preserves the homothety invariance of the connections.

Proof The first part is straightforward from Proposition 3. For the independence of $\overset{o}{\nu}$, recall that when chosing a second ν', the differences $\delta'_i - \delta_i$ are vertical and thus $\nabla^*_{\delta'_i - \delta_i} \partial_j = 0$ as ∇^* is vertically trivial. For the last assertion, recall that $A \to M$ always admits a homogeneous nonlinear connection (for example, the one determined by a pseudo-Finsler metric on A or, in particular, by a Riemannian metric on M). □

6 Anisotropic Versus Finsler Connections

When a pseudo-Finsler metric L is given on A, the standard approach focuses on two geometric structures.[6] The first one is its geodesic spray on A and, thus, its associated homogeneous nonlinear connection on $A \to M$; these are canonically constructed from L. The second one is a covariant derivative on $\mathrm{V}A \to A$ invariant by homotheties. However, a priori there is not a canonical choice for the latter. Let us explain briefly the interplay of anisotropic connections with these two structures.

Recall that, as L is 2-homogeneous, the nonlinear connections and the covariant derivatives to be considered here will be homogeneous (invariant by homotheties). In particular, all the conclusions of Theorem 2 will be applicable and, for example, the torsion of a nonlinear connection ν can be identified with the torsion of the corresponding anisotropic connection ∇^ν.

[6] We recommend the nice essays by Dahl [7] and Minguzzi [22] for background.

6.1 The Metric Spray

Definition 8 A *spray* on A is a (smooth) section $G\colon A \to TA$ which satisfies: (a) G is a *second order differential equation* (or s. o. vector field), that is, it can be written as[7]:

$$G = y^i \frac{\partial}{\partial x^i} - 2G^a \frac{\partial}{\partial y^a},$$

and (b) The G^a's are 2-homogeneous, i.e., $G^a \circ h_\lambda = \lambda^2 G^a$ for $\lambda > 0$.[8]

We summarize some basic relations between sprays and homogeneous nonlinear connections (analogous to Theorem 2) following [22, Sect. 3.4]. Recall that the integral curves $(x(t), y(t))$ of G, satisfy $dx^i/dt = y^i$ and

$$\frac{dy^a}{dt} + 2G^a(x(t), y(t)) = 0; \tag{28}$$

their projections to M are usually called *geodesics* of G. On the other hand, the *geodesics* of a nonlinear connection ν are its autoparallel curves: in terms of (18), $D_{\dot\gamma}\dot\gamma = 0$, whereas on coordinates, $dx^i/dt = y^i$ and

$$\frac{dy^a}{dt} + N_i^a(x(t), y(t))y^i(t) = 0. \tag{29}$$

Proposition 4 *(1) A homogeneous nonlinear connection ν defines a natural spray $G = \mathbb{C}^{\mathrm{H}}$ (the ν-horizontal lift of the canonical anisotropic field, recall (3) and (15)). In coordinates,*

$$G = y^i \frac{\partial}{\partial x^i} - y^i N_i^a \frac{\partial}{\partial y^a}.$$

The integral curves of G are the geodesics *of ν.*

(2) A spray G in A induces a natural homogeneous nonlinear connection $\overset{o}{\nu}$ with coefficients

$$\overset{o}{N}{}_i^a = \frac{\partial G^a}{\partial y^i} = G^a_{\cdot i}\,.$$

Then, $G = \overset{o}{\mathbb{C}}{}^{\mathrm{H}}$ ($\overset{o}{\nu}$-horizontal lift of $\mathbb{C}$) and $\overset{o}{\nu}$ is torsion-free.

(3) The geodesics of a homogeneous nonlinear connection ν are the integral curves of a spray G iff $G = \mathbb{C}^{\mathrm{H}}$.

(4) Given a homogeneous nonlinear connection ν, consider the natural spray $G = \mathbb{C}^{\mathrm{H}}$ and the nonlinear connection $\overset{o}{\nu}$ associated with this G. Then the difference $\nu - \overset{o}{\nu}$ is in $\mathcal{T}^1_1(M_A)$ with components

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[7] Intrinsically, $T\pi_A \circ G$ is the identity in A.

[8] More intrinsically, $[\mathbb{C}^{\mathrm{V}}, G] = G$, where $\mathbb{C}^{\mathrm{V}}$ is the Liouville vector field on A. In terms of coordinates, $\mathbb{C}^{\mathrm{V}}_v = y^i \partial_{y^i}|_{(x,y)}$ (recall (3) and (2)).

$$N_i^a - G_{\cdot i}^a = \frac{1}{2}\mathrm{Tor}_{ij}^a y^j,$$

where Tor *is the torsion of* ∇^ν *(see part (2) of Theorem 2). Moreover, if this difference vanishes, then actually* Tor *vanishes.*

(5) Any homogeneous nonlinear connection is determined by its geodesics and torsion.

Proof (1) These G^a's satisfy the cocycle transformation required to form a second order equation and their 2-homogeneity comes from the 1-homogeneity of N_i^a.

(2) From the cocycle of a second order vector field, the $\overset{o}{N}{}_i^a$'s satisfy (14). The 1-homogeneity of $\overset{o}{\nu}$ comes from the 2-homogeneity of the G^a's. (Then $G = \overset{o}{\mathbb{C}}{}^{\mathrm{H}}$ is nothing but the Euler relation for the G^a's.) The components of the torsion tensor of $\overset{o}{\nu}$ are $G_{\cdot i \cdot j}^a - G_{\cdot j \cdot i}^a = G_{\cdot i \cdot j}^a - G_{\cdot i \cdot j}^a = 0$.

(3) Recall the geodesic equations (28) and (29). Their solutions coincide if and only if $y^i N_i^a = 2G^a$.

(4) This is a straightforward computation taking into account that $G^a = N_i^a y^i/2$ and $\mathrm{Tor}_{ij}^a = N_{i\cdot j}^a - N_{j\cdot i}^a$. Thus, if $\mathrm{Tor}_{ij}^a y^j = 0$, then $\nu = \overset{o}{\nu}$ and its torsion vanishes due to (3).

(5) This follows directly from (4). □

Remark 6 As a consequence of Theorem 2, the $\overset{o}{\nu}$ associated with G can be always obtained from a canonical homogeneous anisotropic connection $\nabla^{\overset{o}{\nu}}$ (item (2) of the theorem). This $\nabla^{\overset{o}{\nu}}$ is the so-called *Berwald anisotropic connection*. The remaining anisotropic connections that yield G would be controlled by an anisotropic tensor Q satisfying $Q_{ij}^a y^i y^j = 0$.

The spray canonically associated with a pseudo-Finsler *metric* L, called its *metric spray*, can be introduced in the following intrinsic way (see [4, Theorem 5.4.2]). The 1-form

$$\hat{d}L := \frac{\partial L}{\partial y^i}\,\mathrm{d}x^i \in \mathfrak{X}^*(A)$$

is globally well-defined and, due to condition (2) in Definition 1, the 2-form $\mathrm{d}\hat{d}L$ is nondegenerate. Thus, there exists a unique vector field G on A such that

$$\imath_G \mathrm{d}\hat{d}L = -\frac{1}{2}\,\mathrm{d}L,$$

where $\imath$ is the interior product operator. This G is indeed a spray and its geodesics are those of (M, L) (the critical points of the energy functional). It is well-known [25, (4.30)] that its components are $G^a = \gamma_{ij}^a y^i y^j$, where

$$\gamma_{ij}^a = \frac{1}{2}g^{ak}\left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

are the so-called *formal Christoffel symbols*. From them, the *metric nonlinear connection* (that is, the $\overset{o}{\nu}$ in Proposition 4 (2)) is the connection whose coefficients are

$$\overset{o}{N}{}^a_i = \gamma^a_{ij} y^j - g^{aj} C_{ijk} \gamma^k_{lm} y^l y^m,$$

where $C_{ijk} = \partial_{y^k} g_{ij}/2$ is the Cartan tensor of L (which measures how far g is from being pseudo-Riemannian).

Next our aim is to select an anisotropic Levi-Civita connection for a pseudo-Finsler metric L, rethinking the role of the Chern connection.

Theorem 4 *Given a pseudo-Finsler metric L and being its fundamental tensor g, there exists a unique anisotropic connection ∇ that is torsion-free and* metric*, i.e., such that $\nabla g = 0$. It is characterized by the Koszul-type formula*

$$\begin{aligned} 2g_v(\nabla^V_X Y, Z) = {} & (X(g_V(Y, Z)) - Z(g_V(X, Y)) + Y(g_V(X, Z)))\,(\pi(v)) \\ & + g_v([X, Y], Z) + g_v([Z, X], Y) - g_v([Y, Z], X) \\ & - 2C_v(Y, Z, \nabla^V_X V) - 2C_v(Z, X, \nabla^V_Y V) + 2C_v(X, Y, \nabla^V_Z V), \end{aligned} \tag{30}$$

where $v \in A$, $X, Y, Z \in \mathfrak{X}(M)$, $V \in \mathfrak{X}^A(U)$ is any local extension of v and C is the Cartan tensor defined above. Its Christoffel symbols are

$$\Gamma^a_{ij} = \frac{1}{2} g^{ak} \left(\frac{\delta g_{ki}}{\delta x^j} + \frac{\delta g_{kj}}{\delta x^i} - \frac{\delta g_{ij}}{\delta x^k} \right) \tag{31}$$

(the δ_j are the ones of the associated nonlinear connection ν^{∇}). This unique ∇ is called the Levi-Civita anisotropic connection *of L. The corresponding vertically trivial linear connection is the Chern connection.*

Proof Taking into account that $\dot{\partial} g = 2C$, it follows that

$$(\nabla_X g)_v(Y, Z) = X(g_V(Y, Z))(\pi(v)) - g_v(\nabla^V_X Y, Z) - g_v(Y, \nabla^V_X Z) - 2C_v(Y, Z, \nabla^v_X V)$$

and using that $\nabla g = 0$, as well as the above formula permuting X, Y, Z, one gets (30). To get (31), observe that $\nabla g = 0$ in coordinates means

$$\delta_k g_{ij} - \Gamma^l_{ki} g_{lj} - \Gamma^l_{kj} g_{il} = 0,$$

which is equivalent to the structure equations of the Chern connection (see [2]), and therefore its Christoffel symbols coincide with those of Chern's as well as its vertically trivial associated connection. □

Remark 7 We have seen that there are two distinguished anisotropic connections associated with a pseudo-Finsler metric, the Berwald one (see Remark 6) and the

Levi-Civita–Chern one (see the theorem just above). The difference between them is a tensor $\mathcal{L}^\sharp$ metrically equivalent to the Landsberg tensor of L, which satisfies $\mathcal{L}^\sharp_v(u, v) = 0$ for all $v \in A$ and $u \in T_{\pi(v)}M$ (see [25]). This property is essential as it guarantees that both connections have the same associated nonlinear connection $\overset{o}{\nu}$ (the metric nonlinear connection of L). Indeed, the anisotropic connections differing in a symmetric tensor Q with this property from the Chern or Berwald ones are exactly the torsion-free anisotropic connections having $\overset{o}{\nu}$ as their associated nonlinear connection (recall Theorem 2, item (3)). The properties of this family of connections as tools for the study of pseudo-Finsler metrics were collected in [12], where they are referred to as *distinguished connections*.

6.2 The Finslerian Linear Connections

The linear connections associated with a pseudo-Finsler metric L live in the bundle $\mathrm{V}A \to A$. As we have seen, L determines the metric spray G and, thus, the *metric nonlinear connection* $\overset{o}{\nu}$, which plays the role of the prescribed auxiliary connection seen in Sect. 5. Then, Proposition 3 and Theorem 3 are applicable. As a summary, one has:

1. The linear connections ∇^* used in pseudo-Finsler geometry are defined in the vector bundle $\mathrm{V}A \to A$ and they are homothetically invariant.
2. Such a ∇^* can be specified by means of the Christoffel symbols with respect to the metric nonlinear connection $\overset{o}{\nu}$ (Proposition 3), namely: $\left(\Gamma^{\mathrm{H}}\right)^k_{ij}$, which are 0-homogeneous, and $\left(\Gamma^{\mathrm{V}}\right)^k_{ij}$, which are (-1)-homogeneous.
3. The vertically trivial ∇^*'s are in natural correspondence with the homogeneous A-anisotropic connections on M (Theorem 3). Using $\overset{o}{\nu}$, the non-vertically trivial ∇^*'s project onto the vertically trivial ones (Remark 5).
4. The most frequent choices of ∇^* in Finsler Geometry have the following horizontal and vertical parts:

   – Berwald and Hashiguchi: $\left(\Gamma^{\mathrm{H}}\right)^k_{ij} = N^k_{i\cdot j} := \dot{\partial}_j N^k_i$, where the N^j_i's come from $\overset{o}{\nu}$. The Berwald connection is vertically trivial and the Hashiguchi connection has $\left(\Gamma^{\mathrm{V}}\right)^k_{ij} = C^k_{ij} = g^{kl} C_{ijl}$.
   – Chern-Rund and Cartan: $\left(\Gamma^{\mathrm{H}}\right)^k_{ij} = \Gamma^k_{ij}$ as in (31). The Chern-Rund connection is vertically trivial, and in the case of the Cartan connection, $\left(\Gamma^{\mathrm{V}}\right)^k_{ij} = C^k_{ij}$.

7 Parallel Transport and Anisotropic Connections

Next, let us go back to our observers' viewpoint in Sect. 2.2 to introduce parallel transport and show how an anisotropic connection can be recovered from it.

7.1 Observers and Parallel Transport

The most natural way to compare vectors in different tangent spaces of a manifold is by making use of parallel transport along a curve. Depending on what we want to study, this parallel transport should preserve certain properties of vectors. In general, we cannot ensure the preservation of the indicatrix of a pseudo-Finsler metric by a linear map, because the indicatrices at different points may be not linearly equivalent. Perhaps the best idea is to require the preservation of their best approximations by a scalar product, namely, g_v. As there is a dependence on v, we will need different parallel transports for every $v \in A$. Summing up, the Christoffel symbols will depend also on the direction, so the covariant derivative along a curve $\gamma : I \to M$ needs a reference vector field $W \in \mathfrak{X}(\gamma)$:

$$D_\gamma^W : \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma),$$

where $\mathfrak{X}(\gamma)$ denotes the module of smooth vector fields along γ. Moreover, we will assume that the dependence on W is pointwise, in the sense that at the instant t_0, D_γ^W depends only on $W(t_0)$. Thinking about what happens in a Finsler spacetime, where all the computations depend on the observer, we will make first the parallel transport of the observer along γ by searching for a vector field V such that

$$D_\gamma^V V = 0,$$

with the natural requirement that $L \circ V = 1$. Finally, we are ready to parallel transport vectors along γ using a parallel vector field Z, defined by

$$D_\gamma^V Z = 0.$$

As we will see later, if we require this parallel transport to preserve also the metric g_v of the restspace, then the covariant derivative comes from the Levi-Civita–Chern anisotropic connection. Geodesics are recovered as the curves with autoparallel velocity, namely

$$D_\gamma^{\dot\gamma} \dot\gamma = 0.$$

In particular, $L \circ \dot\gamma = const$.

Of course, when we speak about a covariant derivative, we are assuming that it satisfies the natural properties of a derivative. Let us put this on rigorous basis.

In the following, given a smooth curve $\gamma : [a, b] \to M$, $\mathcal{F}(I)$ will denote the ring of smooth real functions defined on $I = [a, b]$. Recall that A denotes a conic open subset of TM, $\pi : TM \to M$ is the natural projection and $\gamma^*(TM)$ is the pullbacked of this bundle by means of the curve $\gamma : [a, b] \to M$.

Definition 9 An A-anisotropic covariant derivative D_γ in A along a curve $\gamma : [a, b] \to M$ is a map

$$D_\gamma : \gamma^*(A) \times \mathfrak{X}(\gamma) \to TM, \qquad (v, X) \mapsto D^v_\gamma X \in T_{\pi(v)}M$$

with a smooth dependence on v, such that if $\pi(v) = \gamma(t_0)$ with $t_0 \in [a, b]$, then

1. $D^v_\gamma(X + Y) = D^v_\gamma X + D^v_\gamma Y$; $X, Y \in \mathfrak{X}(\gamma)$,
2. $D^v_\gamma(fX) = \frac{df}{dt}(t_0)X(t_0) + f(t_0)D^v_\gamma X$; $f \in \mathcal{F}(I)$, $X \in \mathfrak{X}(\gamma)$.

Remark 8 Let $\pi_\gamma : \gamma^*(A) \to [a, b]$ be the pullback fibered manifold obtained from $A \to M$ by $\gamma : [a, b] \to M$. The formal similarity of our definition of D_γ with Definition 4 for anisotropic connections can be stressed by redefining: (a) the domain of D_γ as $\mathfrak{X}(\gamma)$ and (b) its codomain as the sections of the pullback bundle $\pi^*_\gamma(TM) \to \gamma^*(A)$ obtained from $\gamma^*(TM) \to [a, b]$ by $\pi_\gamma : \gamma^*(A) \to [a, b]$.

The notion of anisotropic connection gathers the information of the covariant derivatives along different curves. In fact, there is a unique covariant derivative along curves determined by an anisotropic connection (see [12, Proposition 2.7]).

Proposition 5 *Given a smooth curve $\gamma : [a, b] \to M$, an A-anisotropic connection ∇ determines the unique A-anisotropic covariant derivative along γ with the following property: if $X \in \mathfrak{X}(M)$, then $D^v_\gamma(X_\gamma) = \nabla^v_{\dot\gamma} X$, where $X_\gamma := X \circ \gamma$.*

Proof Indeed, given a local chart (U, x) on M, we can express this covariant derivative in terms of the Christoffel symbols of the A-anisotropic connection ∇, which are defined as the functions $\Gamma^k_{ij} : TU \cap A \to \mathbb{R}$ determined by

$$\nabla^v_{\partial_i} \partial_j = \Gamma^k_{ij}(v)\, \partial_k|_{\pi(v)}.$$

It is easy to check that if $X = X^k \partial_k$, then

$$D^W_\gamma X = (\dot X^i + (\Gamma^i_{jk} \circ W)\dot\gamma^j X^k)\partial_i. \tag{32}$$

This provides coordinate expressions for the covariant derivative.

Moreover, given a curve $\gamma : [a, b] \to M$, if one fixes the reference vector field $W \in \mathfrak{X}(\gamma)$ and $t_1, t_2 \in [a, b]$, then it is possible to define a parallel transport

$$P^W_{t_1,t_2} : T_{\gamma(t_1)}M \to T_{\gamma(t_2)}M$$

in such a way that $P^W_{t_1,t_2}(z) = Z(t_2)$, where $Z \in \mathfrak{X}(\gamma)$ is such that $D^W_\gamma Z = 0$ and $Z(t_1) = z$.

Remark 9 The parallel transport $P^W_{t_1,t_2}$ shares all the natural properties of the parallel transport with respect to an affine connection (it is a well-defined linear isomorphism, invariant under reparametrizations of γ including the reversal of the sign). Indeed, as the value of the Christoffel symbols is determined by W, which is fixed, (32) yields an equation for the transport of the same type as in the affine case.

There is a different type of parallel transport which is not always well-defined, namely, when the goal is to find a vector field V along γ such that $D^V_\gamma V = 0$. The existence of this parallel transport is not guaranteed along the whole curve unless we have some control on the Christoffel symbols.

Definition 10 A smooth curve $\gamma : [a, b] \to M$ is *parallel transport complete* if for every $v \in A \cap T_{\gamma(a)}M$, there exists a (unique) A-admissible vector field $V \in \mathfrak{X}(\gamma)$ such that $D^V_\gamma V = 0$ and $V(a) = v$. Consistently with Sect. 3.1, here A-admissible means that $V(t) \in A$ for all $t \in [a, b]$.

Remark 10 From standard results of ODE's one has that, for every $v \in A \cap T_{\gamma(a)}M$, there exists some $\epsilon > 0$ such that a parallel V as above is well-defined in $[a, a + \epsilon]$. Moreover, all the curves are parallel transport complete in the following two cases: (1) when $A = TM \setminus \mathbf{0}$ and the anisotropic connection is homogeneous, and (2) in the case of a Finsler spacetime (M, L) with a distinguished connection (for the latter, notice that the anisotropic connection is homogeneous and $L(V)$ is constant for any parallelly transported vector V, so that it cannot abandon A). From now on, we will restrict ourselves to work with curves where this parallel transport is defined everywhere.

Let us define the parallel transports which have a geometric meaning to compare what happens in different points of the manifold.

Definition 11 (*Instantaneous observer's parallel transport*) Let ∇ be an A-anisotropic connection and $\gamma : [a, b] \to M$ a parallel transport complete curve. For each $t_1, t_2 \in [a, b]$, the *instantaneous observer's parallel transport* is the map

$$P_{t_1,t_2} : A \cap T_{\gamma(t_1)}M \to A \cap T_{\gamma(t_2)}M$$

given by $P_{t_1,t_2}(v) = V(t_2)$, for $V \in \mathfrak{X}(\gamma)$ satisfying $V(t_1) = v$ and $D^V_\gamma V = 0$.

This parallel transport coincides the one obtained from the nonlinear connection which appears in many classical textbooks devoted to Finsler geometry as [19, Chap. VII], [1, Sect. 2.1.6], [25, p. 103], [4, Sect. 2.1], [8, Definition 1.4] and [27, Sect. 7.6]. In some other textbooks there is an additional notion of parallel transport taking as a reference vector the velocity of the curve (see [25, Definition 7.3.1], [26, Sect. 5.3] and [6, Chap. 4]). Recall that we defined instantaneous observers in the setting of Finsler spacetimes as vectors $v \in A$ of unit length $L(v) = 1$. Of course, the constraint on the length is not relevant for the transport. In the case of general anisotropic connections such a restriction makes no sense but we have maintained the name of observers to stress the relativistic geometric intuitions.

Definition 12 (*Parallel transport with respect to an instantaneous observer*) Let ∇ be A-anisotropic connection and $\gamma : [a, b] \to M$ a parallel transport complete curve. For each $t_1, t_2 \in [a, b]$ and observer $v \in T_{\gamma(t_1)}M \cap A$, the *parallel transport with respect to* v is the map

$$P^v_{t_1,t_2} : T_{\gamma(t_1)}M \to T_{\gamma(t_2)}M$$

obtained as $P^v_{t_1,t_2}(w)=W(t_2)$, where $W \in \mathfrak{X}(\gamma)$ satisfies that $W(t_1)=w$ and $D^V_\gamma W = 0$ with V satisfying $D^V_\gamma V = 0$ and $V(t_1) = v$.

Recall that, as V is fixed by v, this parallel transport satisfies the natural properties of the transport explained in Remark 9. See [23] for a general treatment of parallelism.

7.2 Recovering the Anisotropic Connection from the Transport

First observe that given a smooth curve $\gamma : [a, b] \to M$, we can define the parallel transport of covectors with respect to the instantaneous observer $v \in T_{\gamma(a)}M \cap A$,

$$P^v_{a,b} : T_{\gamma(a)}M^* \to T_{\gamma(b)}M^*,$$

as

$$P^v_{a,b}(\theta)(w) = \theta(P^v_{b,a}(w))$$

for any $\theta \in T_{\gamma(a)}M^*$ and $w \in T_{\gamma(b)}M$, so that a parallel covector field on a parallel vector field will be constant. Imposing commutativity with the tensor product, this parallel transport can be extended to arbitrary (r, s)-tensors.

Next, we write explicitly such a transport regarding the tensors as multilinear maps. Consider an A-anisotropic tensor $T \in \mathcal{T}^r_s(M_A)$ and a curve $\gamma : [a, b] \to M$. Then we can define the parallel transport $P_{t_1,t_2}(T)_v$ for any $t_1, t_2 \in [a, b]$ as the map

$$P_{t_1,t_2}(T)_v : T^*_{\pi(v)}M \times \overbrace{\cdots}^{r} \times T^*_{\pi(v)}M \times T_{\pi(v)}M \times \overbrace{\cdots}^{s} \times T_{\pi(v)}M \to \mathbb{R}$$

given by

$$\begin{aligned} &P_{t_1,t_2}(T)_v(\theta^1, \ldots, \theta^r, v_1, \ldots, v_s) \\ &\qquad = T_{P_{t_2,t_1}(v)}(P^v_{t_2,t_1}(\theta^1), \ldots, P^v_{t_2,t_1}(\theta^r), P^v_{t_2,t_1}(v_1), \ldots, P^v_{t_2,t_1}(v_s)). \end{aligned}$$

In particular, we can define a curve of anisotropic tensors in $T_{\pi(v)}M$:

$$P_t(T) = P_{t,a}(T).$$

Our next goal is to compare this parallel transport with the covariant derivative of any A-anisotropic tensor, as given in Theorem 1 and formula (12). Namely, being $P_t(T)$ a curve in a vector space, let us relate its natural derivative with $\nabla_{\dot\gamma(a)}T$.

Proposition 6 *Given an A-anisotropic tensor $T \in \mathcal{T}^r_s(M_A)$, an A-anisotropic connection ∇ and a regular curve $\gamma : [a, b] \to M$, it holds that*

$$(\nabla_{\dot\gamma(a)}T)_v = \frac{d}{dt}P_t(T)_v|_{t=a},$$

for any $v \in T_{\gamma(a)}M \cap A$.

Proof Assume first that $r = 0$. Recall that we can compute $(\nabla_{\dot\gamma(a)}T)_v(v_1, \ldots, v_s)$ choosing an A-admissible extension V of v and arbitrary extensions $X, X_1, \ldots, X_s$ of $\dot\gamma(a), v_1, \ldots, v_s$, respectively. In particular, these extensions can be chosen in such a way that $\nabla^V_X V = \nabla^V_X X_j = 0$ along γ for all $j = 1, \ldots, s$. With these choices,

$$\begin{aligned}(\nabla_{\dot\gamma(a)}T)_v(v_1, \ldots, v_s) &= X_{\gamma(a)}(T_V(X_1, \ldots, X_s))\\ &= \frac{d}{dt}(T_{V_{\gamma(t)}}((X_1)_{\gamma(t)}, \ldots, (X_s)_{\gamma(t)}))|_{t=a}.\end{aligned} \tag{33}$$

Finally, observe that $V_{\gamma(t)} = P_{a,t}(v)$, since $V(t) := V_{\gamma(t)}$ is a parallel observer along γ (recall that $\nabla^V_X V = 0$) and $V(a) = V_{\gamma(a)} = v$. Moreover, $(X_i)_{\gamma(t)} = P^v_{a,t}(v_i)$ since $\nabla^V_X X_i = 0$ and $(X_i)_{\gamma(a)} = v_i$. Replacing these quantities in (33), we get

$$\begin{aligned}&(\nabla_{\dot\gamma(a)}T)_v(v_1, \ldots, v_s)|_{t=a}\\ &\quad= \frac{d}{dt}T_{P_{a,t}(v)}(P^v_{a,t}(v_1), \ldots, P^v_{a,t}(v_s))|_{t=a} \quad = \frac{d}{dt}P_t(T)_v(v_1, \ldots, v_s)|_{t=a},\end{aligned}$$

as required.

For the general case $r > 0$, observe that given the covector fields $\theta^1, \ldots, \theta^r$, it is possible to choose one-forms ω^i such that $\nabla^V_X \omega^i = 0$ and $(\omega^i)_{\gamma(a)} = \theta^i$. Then, applying the proposition for $r = 0$, it can be shown that $(\omega^i)_{\gamma(t)} = P_{a,t}(\theta^i)$. The result follows analogously to the case $r = 0$ by computing the covariant derivative with V as a reference vector and $\omega^1, \ldots, \omega^r, X_1, \ldots, X_s$ as above. □

It is worth pointing out that, as only the parallel transport close to $t = a$ is needed for each chosen v, the result can be applied even if the curve is not parallel transport complete (see Remark 10).

7.3 Levi-Civita–Chern Connection of a Finsler Spacetime

Let (M, L) be a pseudo-Finsler manifold with $L : A \to \mathbb{R}$. We aim to find an A-anisotropic connection that defines a parallel transport which preserves some metric properties. As we explained in Sect. 7, the parallel transport of an instantaneous observer should preserve the L-length and the parallel transport with respect to an instantaneous observer should preserve the fundamental tensor g_v, namely, for any curve $\gamma : [a, b] \to M$ and $v \in A \cap T_{\gamma(a)}M$ such that the parallel transport of v is well-defined along γ,

$$g_{P_{a,b}(v)}(P^v_{a,b}(u), P^v_{a,b}(w)) = g_v(u, w) \tag{34}$$

for all $u, w \in T_{\pi(v)}M$. Observe that this implies in particular that $L(v) = L(P_{a,b}(v))$, as

$$L(v) = g_v(v, v) = g_{P_{a,b}(v)}(P^v_{a,b}(v), P^v_{a,b}(v)) = L(P^v_{a,b}(v)) = L(P_{a,b}(v)).$$

Proposition 7 *Let (M, L) be a pseudo-Finsler manifold. Then its Levi-Civita–Chern A-anisotropic connection is the only torsion-free connection with a parallel transport that preserves the fundamental tensor of L.*

Proof Observe that by Proposition 6, the fact that the parallel transport of ∇ preserves the fundamental tensor g as in (34) is equivalent to $\nabla g = 0$, and therefore ∇ is the Levi-Civita–Chern connection of (M, L). □

Acknowledgements The thorough revision by Professor Szilasi (Univ. Debrecen) and his worthy and thoughtful suggestions are deeply acknowledged. MAJ was partially supported by the project PGC2018-097046-B-I00 funded by MCIN/ AEI /10.13039/501100011033/ FEDER "Una manera de hacer Europa" and Fundación Séneca project with reference 19901/GERM/15. This work is a result of the activity developed within the framework of the Programme in Support of Excellence Groups of the Región de Murcia, Spain, by Fundación Séneca, Science and Technology Agency of the Región de Murcia.

MS and FFV are partially supported by the project MTM2016-78807-C2-1-P funded by MCIN/ AEI /10.13039/501100011033/ FEDER "Una manera de hacer Europa", by the project A-FQM-494-UGR18 Programa FEDER-Andalucía 2014-202, Junta de Andalucía) and by the framework of IMAG-María de Maeztu grant CEX2020-001105-M funded by MCIN/AEI/ 10.13039/50110001103. FFV is partially supported also by an FPU grant (Formación de Profesorado Universitario) from the Spanish Ministry of Universities.

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