Mathematics Education in the Digital Era

Alison Clark-Wilson
Ornella Robutti
Nathalie Sinclair *Editors*

The Mathematics Teacher in the Digital Era

International Research on Professional Learning and Practice

Second Edition

MOREMEDIA

Springer

Mathematics Education in the Digital Era

Volume 16

The Mathematics Education in the Digital Era (MEDE) series explores ways in which digital technologies support mathematics teaching and the learning of Net Gen'ers, paying attention also to educational debates. Each volume will address one specific issue in mathematics education (e.g., visual mathematics and cyber-learning; inclusive and community based e-learning; teaching in the digital era), in an attempt to explore fundamental assumptions about teaching and learning mathematics in the presence of digital technologies. This series aims to attract diverse readers including researchers in mathematics education, mathematicians, cognitive scientists and computer scientists, graduate students in education, policy-makers, educational software developers, administrators and teacher-practitioners. Among other things, the high-quality scientific work published in this series will address questions related to the suitability of pedagogies and digital technologies for new generations of mathematics students. The series will also provide readers with deeper insight into how innovative teaching and assessment practices emerge, make their way into the classroom, and shape the learning of young students who have grown up with technology. The series will also look at how to bridge theory and practice to enhance the different learning styles of today's students and turn their motivation and natural interest in technology into an additional support for meaningful mathematics learning. The series provides the opportunity for the dissemination of findings that address the effects of digital technologies on learning outcomes and their integration into effective teaching practices; the potential of mathematics educational software for the transformation of instruction and curricula; and the power of the e-learning of mathematics, as inclusive and community-based, yet personalized and hands-on.

Submit your proposal: Please contact the Series Editors, Dragana Martinovic (dragana@uwindsor.ca) and Viktor Freiman (viktor.freiman@umoncton.ca) as well as the Publishing Editor, Marianna Georgouli (marianna.georgouli@springernature.com).

Forthcoming volume:

- The Evolution of Research on Teaching Mathematics: A. Manizade, N. Buchholtz, K. Beswick (Eds.)

Alison Clark-Wilson • Ornella Robutti
Nathalie Sinclair
Editors

The Mathematics Teacher in the Digital Era

International Research on Professional Learning and Practice

Second Edition

Springer

Editors
Alison Clark-Wilson
UCL Institute of Education
University College London
London, UK

Ornella Robutti
Dipartimento di Matematica
Università di Torino
Torino, Italy

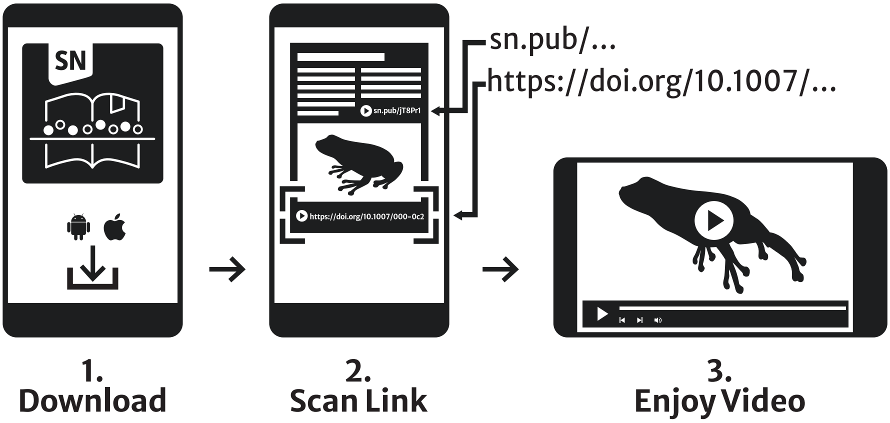
Nathalie Sinclair
Faculty of Education
Simon Fraser University
Burnaby, BC, Canada

This work contains media enhancements, which are displayed with a "play" icon. Material in the print book can be viewed on a mobile device by downloading the Springer Nature "More Media" app available in the major app stores. The media enhancements in the online version of the work can be accessed directly by authorized users.

ISSN 2211-8136 ISSN 2211-8144 (electronic)
Mathematics Education in the Digital Era
ISBN 978-3-031-05253-8 ISBN 978-3-031-05254-5 (eBook)
https://doi.org/10.1007/978-3-031-05254-5

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Springer Nature More Media App



Support: customerservice@springernature.com

Introduction

The eight intervening years between this second edition of *The Mathematics Teacher in the Digital Era* and the first edition have seen increased attention on the role of the teacher within technology-enhanced educational contexts, leading to a more developed understanding of the components of related teacher education programmes and initiatives for both pre- and in-service teachers. The shock to the education system caused by the global coronavirus pandemic simultaneously highlighted the key role that teachers and lecturers play in the nurturing of generations of learners, alongside increased global attention to the role that (educational) technology plays as a mediator of teaching and learning. Studies that have taken place during the pandemic have provided insights into how teachers' practices have had to evolve, whilst also highlighting theoretical and methodological gaps in our understanding of the relatively new phenomena of "hybrid", "at distance" or "remote" teaching in school and university settings (Bretscher et al., 2021; Clark-Wilson et al., 2021; Crisan et al., 2021; Drijvers et al., 2021; Maciejewski, 2021).

As we reflect on the academic impacts of the first edition of the book, the chapters within have offered theoretical constructs and methodological approaches, which have provided other researchers in the field with research tools that are continuing to advance our collective understandings of the field. In this second edition, we invited all of the authors who had contributed to the first edition to submit new research that evidenced advances in their experiences, knowledge and practices. We also invited new authors, whose research had emerged in the intervening years, to offer new critical perspectives that broaden the international commentary, with contributions from Argentina, Australia, Canada, France, Germany, Hong Kong, Iceland, Italy, Mexico, Turkey and the United Kingdom.

A Journey Through the Text

The evolution of the research on technology in mathematics education has enabled a more nuanced understanding of the teacher's perspective to take account of their trajectories of development from pre-service contexts through to in-service practices over time. Hence, we have chosen to loosely organise the text body in accordance with teachers' trajectories of experience with technology use. These experiences concern those within: university undergraduate courses as learners of mathematics; university-based pre-service teacher education courses; university-based teacher education courses and research projects with in-service teachers as participants.

We begin with chapters by Thurm, Ebers and Barzel, and Bozkurt and Koyunkaya that address more practical considerations regarding the provision of support and training for both in-service and pre-service teachers of mathematics.

The growth of large-scale, online professional development initiatives aimed at teachers has resulted in new research that seeks to develop theoretical understanding of the design and impact of such initiatives alongside the development of appropriate methodologies to inform both aspects. The chapter by Thurm, Ebers and Barzel addresses aspects of the design of professional development for mathematics teachers in Germany with a particular focus on the role of the *professional development facilitators* within a regional professional development programme for 30 participants who are all such facilitators. The programme was conducted online (due to the Covid-19 pandemic) and Thurm and colleagues' findings focus on the impact of a module of the programme that supported participants' understanding (and use) of video-based case studies of mathematics teaching that embed multi-representational technology. They use Prediger, Roesken-Winter and Leuders' *Three-Tetrahedron Model* as a framework to highlight the complexities of PD design that has a classroom level, teacher PD level and facilitators' PD level (Prediger et al., 2019). Their findings, which highlight aspects of facilitators' noticing, emphasise the need for carefully structured prompts to support the analysis of video-based activities that serve the dual needs of the facilitators and the teachers with whom they are working.

A pre-service teacher education context in Turkey is the subject of the qualitative action research reported by Bozkurt and Koyunkaya in which they study the impacts of a redesigned practicum course informed by the *Instrumental Orchestration* model (Drijvers et al., 2010; Trouche, 2004). The course design emphasises the pre-service mathematics teachers' (PSTs, n=4) developing use of a dynamic mathematics software (GeoGebra) from the university setting (through micro teaching to their peers) as their practices move to school classrooms. Their study adopts a cyclical research method that draws on data from the PSTs' lesson plans, supported by analyses of their teaching and associated interviews. The research findings offer insights into how the PSTs initially overlooked the exploitation modes for the technology in their planning but became more systematic in their approach through both the processes of micro teaching and during the practicum itself. Given that many pre-service programmes stop short of requiring PSTs to apply their learning about mathematical technologies within authentic teaching situations, this chapter provides valuable insights on the design decisions taken by the teacher educators to develop such an approach.

The majority of the remaining chapters in the book report studies that involve in-service teachers as participants within a range of research settings, each with a different focus. We order these chapters according to teachers' trajectories of development with *novel to them* technologies. We adopt this phrase from Ng and Leung (Chap. 10) as it better reflects our experience and expectation that it is not possible for all teachers to be cognisant of all available (and educationally relevant) technologies at any point in time, irrespective of how mature the wider community considers these technologies to be.

The study by Bakos explores how a novel multi-touch tablet technology, TouchTimes, is used by two primary teachers in British Columbia, Canada, through a lens that considers the teacher, the tool and the mathematical concept as an *ensemble*. Rooted in the instrumental approach, and in particular Haspekian's elaboration of *double instrumental genesis* (2011, 2014), Bakos uses her case studies to reveal three new orchestration types alongside sharing insights on how the agency exerted by the tool extends our existing understandings of the nature of multiplication, and the role of haptic devices within young children's development.

Ng, Liang and Leung's study also focuses on a more novel technology, 3D pens, which enable 3-dimensional models to be drawn as physical objects. The 3D pen warms and extrudes a plastic filament to produce a model that then hardens as it cools. Ng, Liang and Leung's method adopts the use of video-aided reflection with a group of four in-service secondary school teachers in Hong Kong to support their realisations of the affordances of such technologies as a potential teaching tool. In their findings, Ng, Liang and Leung provide evidence for how the videos operate as a *boundary object* between the teachers and researchers in the study (Robutti et al., 2019).

Although the concept of silent animated films to show mathematical concepts dates back to the early twentieth century and was further developed in the 1950s by Nicolet, the design-based research developed by Kristinsdóttir examines aspects of their design and use in her case study in an upper secondary mathematics classroom in Iceland. Kristinsdóttir describes silent videos as short (< 2 min) videos that *do not pose a mathematical problem to be solved* but rather *invite the viewer to wonder, to experience dynamically changing mathematical objects* such that they might *discover something new or consolidate previous thoughts about the mathematics shown in the video*. Each associated *silent video task* invites students to work in pairs to prepare and record a voice-over for the video clip, which is then shared with the class during a whole-class discussion that is led by the teacher. Framed by a lens that focuses on the formative assessment dimension of such discussions, Kristinsdóttir adapted Schoenfeld's *Teaching for Robust Understanding* framework (2018) to identify opportunities and challenges associated with such discussions.

McAlindon, Ball and Chang's study also explores an innovative technology-enhanced pedagogic approach, the *flipped classroom*, through a case study involving an experienced teacher in an Australian secondary school. Defining the flipped classroom as one in which the activities that would normally be conducted in the classroom are flipped with those that would normally be conducted as homework, they explore their case study teacher's experiences and perceptions of a first implementation for the teaching of linear equations. This exploratory study, which involves the teacher making qualitative comparisons with a parallel class that she taught using her traditional approach, concludes positive outcomes such as improved student engagement and improved formative assessment practices. Although the design process for the teacher requires new technology skills and is time consuming, the authors offer some guidelines to inform professional development initiatives that have the goal to support mathematics teachers' flipped classroom pedagogies.

Gueudet, Besnier, Bueno-Ravel and Poisard extend earlier research that featured in the first edition of the book, which shone a theoretical lens on teachers' classroom practices at the kindergarten level from a Documentational Approach to Didactics perspective (Gueudet et al., 2014). In the intervening years, evolutions of this theory and its associated research methods have enabled the authors to consider a kindergarten teacher's development as evidenced by both one of her *documents* (a micro view) and the encompassing *resource system* (a macro view). The authors conclude that both the micro and macro views are necessary to fully appreciate a teacher's design capacity within the context of long-term professional development concerning digital technologies for education.

Staying in France, Abboud-Blanchard and Vanderbrouck report findings from a study in France that explores the implementation of tablet computers in the French primary school setting. Although tablets are no longer widely considered a new technology, the authors' contribution extends ideas reported in the first edition of the book, which concludes three axes (*cognitive, pragmatic* and *temporal*) through which to consider teachers' adoption of new technologies within their mathematics classrooms (Abboud-Blanchard, 2014). Abboud-Blanchard and Vanderbrouck introduce the additional constructs of *tensions* and *proximities*, which they argue align more specifically to classroom uses of tablet computers. In their chapter, the authors articulate how these two new constructs evolve from Activity Theory, and elaborations of Vygotsky's and Valsiner's respective Zone Theories.

Sandoval and Trigueros' chapter is also situated in a primary school setting, this time in Mexico. They offer new perspectives on the teaching of mathematics in primary schools, with an emphasis on how two teachers integrate digital technologies to particularly meet the needs of learners from challenging socio-economic contexts. In common with their contribution to the first edition of the book (Trigueros et al., 2014), they adopt an *enactivist* approach to characterise teachers' actions and the resulting student activities that reveal high levels of participation in immersive environments for learners who are commonly disenfranchised by education systems.

We move from primary school contexts to the secondary phase in the next two chapters, which both follow teachers over a period of time with the aim to identify aspects of their evolving practices. The first, by Simsek, Bretscher, Clark-Wilson and Hoyles, is situated in England and focuses on three in-service teachers' evolving use of a dynamic mathematical technology (*Cornerstone Maths*) for the teaching of geometric similarity to 11–14 year olds over a period of months. The chapter extends the understanding of Ruthven and colleagues' notion of curriculum script, which is one of the five *Structuring Features of Classroom Practice* that was described and critiqued in the first edition of the book (Ruthven, 2014). Simsek and colleagues' chapter contributes a case example of such a curriculum script for the teaching of a specific mathematics topic, highlighting aspects of more productive teaching practices which are often difficult to notice.

Villareal's chapter, in which she describes research in Argentina, follows a secondary school mathematics teacher from her pre-service teacher education programme into her role as a novice in-service teacher. The research dually categorises the teacher's evolving relationships with technology, which adopts Goos' taxonomy of sophistication (*master, servant, partner* and *extension of self* (Goos, 2000), alongside Ruthven's five *Structuring Features of Classroom Practice* (Ruthven et al., 2009). These two frameworks offer an interesting and novel perspective for categorising the evolution of teachers' classroom practices that have implications for the design of teacher education programmes and initiatives.

A university in Canada is the setting for the research reported by Buteau, Muller, Santacruz Rodriguez, Mgombelo, Sacristan and Gueudet, which expands research understanding on the long-term development for a faculty-wide integration of programming technologies within undergraduate-level courses for both mathematics students and future mathematics teachers. Situated in the same context as the earlier study by Buteau and Muller (2014), the instrumental orchestration framework is used to examine the 20-year trajectory of this integration from the perspective of the faculty members. The authors' analysis of the course instructors' and selected students' schemes concludes an *orchestration and genesis alignment model* that highlights the complexities of the instructor's role as both policy maker and teacher with responsibility for orchestrating the students' instrumental geneses.

The Covid-19 pandemic provides the context for the research study that features in the chapter by Sánchez Aguilar, Esparza Puga and Lezama. Set in South America, the authors conducted a survey ($n = 179$) across five Latin American Countries (Argentina, Chile, Colombia, Mexico and Uruguay) that aimed to elicit teachers' perceptions of the abrupt integration of digital technologies into their practices, triggered by widespread and mandatory school closures in the first six months of 2020. This was framed within a methodology that aims to capture the lived experience of teachers by giving them a *voice* to express the obstacles that they faced. The study captures the broad range of technologies in play, extending beyond solely mathematical technologies (i.e., calculators, dynamic geometry software or spreadsheets) to include more general technologies such as videoconferencing software and learning management platforms. The findings revealed six categories of obstacles that capture both what they did and how they felt as they worked to overcome the challenges that they faced.

The penultimate two chapters of the book offer theoretical contributions.

In the first edition of the book, the chapter by Arzarello, Robutti, Sabena, Cusi, Garuti, Malara and Martignone introduced a new theoretical model, *Meta-Didactical Transposition* (MDT), which was developed to respond to the need to consider the complexity of teacher education with respect to the institutions in which teaching operates, alongside the relationships that teachers must have with these institutions

(Arzarello et al., 2014). The original MDT model (now referred to as MDT.1), an extension of Chevallard's *Anthropological Theory of Didactics* (1985, 1992, 1999), describes the evolution of teachers' education over time by analysing the different variables involved: components that change from external to internal (*internalisation*); brokers who support teachers interacting with them; and dialectic interactions between the community of teachers and researchers. The chapter by Cusi, Robutti, Panero, Taranto and Aldon presents an evolution of MDT, namely, *Meta-Didactical Transposition.2* (MDT.2), which offers a deeper insight into the process of *internalisation* that captures the way in which the actors within the teachers education programme develop shared praxeologies over time through the introduction of the external (and, in some cases digital) components.

The final chapter, by Sinclair, Haspekian, Robutti and Clark-Wilson, charts the development of theories that frame research on teaching mathematics with technology from both a historical perspective and an epistemological one. Building directly on Ken Ruthven's chapter in the first edition of this book, it aims to highlight the evolution of the relevant theories since 2014 and highlights trends in the ways that these have been operationalised in recent studies. Furthermore, the authors seek to make explicit the philosophical roots of the commonly adopted theories to provoke the reader to consider what each might reveal—or conceal—concerning aspects of teaching mathematics with digital technologies.

Alison Clark-Wilson
Ornella Robutti
Nathalie Sinclair

References

Abboud-Blanchard, M. (2014). Teachers and technologies: Shared constraints, common responses. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era: An international perspective on technology focused professional development* (pp. 297–317). Springer.

Arzarello, F., Robutti, O., Sabena, C., Cusi, A., Garuti, R., Malara, N., & Martignone, F. (2014). Meta-didactical transposition: A theoretical model for teacher education programmes. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era: An international perspective on technology focused professional development* (pp. 347–372). Springer.

Bretscher, N., Geraniou, E., Clark-Wilson, A., & Crisan, C. (2021, November). *Learning from the pandemic: Capitalising on opportunities and overcoming challenges for mathematics teaching and learning practices with and through technology (Part 3).* Paper presented at the British Society for Research into Learning Mathematics.

Buteau, C., & Muller, E. (2014). Teaching roles in a technology intensive core undergraduate mathematics course. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era: An international perspective on technology focused professional development* (pp. 163–188). Springer.

Chevallard, Y. (1985). *La transposition didactique*. La Pensée Sauvage.

Chevallard, Y. (1992). Concepts fondamentaux de la didactique: perspectives apportées par une approche anthropologique. *Recherches en Didactique des Mathématiques, 12*(1), 73–112.

Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques, 19*(2), 221–266.

Clark-Wilson, A., Bretscher, N., Crisan, C., Geraniou, E., Gono, E., Neate, A., & Shore, C. (2021). *Learning from the pandemic: Capitalising on opportunities and overcoming challenges for mathematics teaching and learning practices with and through technology (Part 2).* Paper presented at the British Society for Research into Learning Mathematics June 2021.

Crisan, C., Bretscher, N., Clark-Wilson, A., & Geraniou, E. (2021). *Learning from the pandemic: Capitalising on opportunities and overcoming challenges for mathematics teaching and learning practices with and through technology.* Paper presented at the British Society for Research into Learning Mathematics March 2021.

Drijvers, P., Doorman, M., Boon, P., Reed, H., & Gravemeijer, K. (2010). The teacher and the tool: Instrumental orchestrations in the technology-rich mathematics classroom. *Educational Studies in Mathematics*, 1–22. https://doi.org/10.1007/s10649-010-9254-5

Drijvers, P., Thurm, D., Vandervieren, E., Klinger, M., Moons, F., van der Ree, H., … Doorman, M. (2021). Distance mathematics teaching in Flanders, Germany, and the Netherlands during COVID-19 lockdown. *Educational Studies in Mathematics, 108*(1), 35–64. https://doi.org/10.1007/s10649-021-10094-5

Gueudet, G., Bueno-Ravel, L., & Poisard, C. (2014). Teaching mathematics with technology at the Kindergarten level: Resources and Orchestrations. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era* (pp. 213–240). Springer.

Haspekian, M. (2011). The co-construction of a mathematical and a didactical instrument. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the seventh congress of the european society for research in mathematics education* (pp. 2298–3007). University of Rzeszów.

Haspekian, M. (2014). Teachers' instrumental geneses when integrating spreadsheet software. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era: An international perspective on technology focused professional development* (pp. 241–276). Springer.

Maciejewski, W. (2021). Teaching math in real time. *Educational Studies in Mathematics, 108*(1), 143–159. https://doi.org/10.1007/s10649-021-10090-9

Prediger, S., Roesken-Winter, B., & Leuders, T. (2019). Which research can support PD facilitators? Strategies for content-related PD research in the Three-Tetrahedron Model. *Journal of Mathematics Teacher Education, 22*(4), 407–425. https://doi.org/10.1007/s10857-019-09434-3

Robutti, O., Aldon, G., Cusi, A., Olsher, S., Panero, M., Cooper, J., … Prodromou, T. (2019). Boundary objects in mathematics education and their role across communities of teachers and researchers in interaction. In G. M. Lloyd & O. Chapman (Eds.), (Vol. *International handbook of mathematics teacher education: Volume 3*, pp. 211–240). Brill.

Ruthven, K. (2014). Frameworks for analysing the expertise that underpins successful integration of digital technologies into everyday teaching practice. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era: An international perspective on technology focused professional development* (pp. 373–394). Springer.

Ruthven, K., Deaney, R., & Hennessy, S. (2009). Using graphing software to teach about algebraic forms: A study of technology-supported practice in secondary-school mathematics. *Educational Studies in Mathematics, 71*(3), 279–297.

Schoenfeld, A. H. (2018). Video analyses for research and professional development: The teaching for robust understanding (TRU) framework. *ZDM, 50*(3), 491–506. https://doi.org/10.1007/s11858-017-0908-y

Trigueros, M., Lozano, M.-D., & Sandoval, I. (2014). Integrating technology in the primary school mathematics classroom: The role of the teacher. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era: An international perspective on technology focused professional development* (pp. 111–138). Springer.

Trouche, L. (2004). Managing the complexity of human/machine interactions in computerized learning environments: Guiding students' command process through instrumental orchestrations. *International Journal of Computers for Mathematical Learning, 9*, 281–307.

Contents

Contributors

Maha Abboud is a professor and researcher in Mathematics Education in France. She is the head of the LDAR lab (Laboratoire de Didactique André Revuz), the largest science and mathematics education laboratory in France. Her research interests focus on teachers' activities with digital technologies. She also studies the practices of teacher educators and their impact on the teaching and learning of mathematics and, more broadly, within STEM education. Her professional interests aim at improving teachers' professional development in the light of technological change. She has supervised eight doctoral theses related to these topics. Her theoretical concerns focus on conceptualising teacher and student activities in technology-based lessons that aim to enhance students' mathematical thinking. As leader of her research team, she initiates and participates in the development and refinement of concepts and methods in educational investigation of the daily life of the mathematics/science classroom. She teaches mathematics and didactics at the Master's degree level at the universities of Cergy and Paris Cité.

Mario Sánchez Aguilar is the Head of the Mathematics Education Program of the National Polytechnic Institute of Mexico. He serves as an Associate Editor of the research journals *Educación Matemática* and *Implementation and Replication Studies in Mathematics Education*. He is a visiting professor at the University of San Carlos of Guatemala. He obtained his PhD in Mathematics Education from Roskilde University in Denmark in 2010. He is interested in the use of digital tools in the teaching and learning of mathematics.

Gilles Aldon is now retired after a career as a mathematics teacher and then as a researcher at the French Institute of Education (Ecole Normale Supérieure de Lyon) where he was head of the EducTice research team until his retirement. His main research topic is the use of technology in mathematics teaching and learning. Particularly, he is interested in the issues of the modifications of teaching and learning in the digital era, the contribution of technology in the experimental part of mathematics, and the problem solving processes. The research methodology that was developed in the EducTice team rests upon design-based research and the collaborative research where teachers and researchers are involved in both the research description and the methodology. He is also the president of the International Commission for the Study and Improvement of Mathematics Teaching (CIEAEM in French), which investigates the conditions and the possibilities for the development of mathematics education, taking into account both teachers' and researchers' experiences.

Sandy Bakos achieved her PhD in mathematics education at Simon Fraser University, Canada. Prior to this, she taught elementary school (K–6) for 15 years in Alberta, Canada, and also spent a year teaching grades 4/5 in Victoria, Australia. Her research examines how elementary school teachers implement digital technologies for teaching and learning mathematics. Sandy's work has primarily focused on teachers' adoption and use of TouchTimes as a pedagogical tool, how this particular digital technology shapes their own understanding of multiplication, and how teachers then use it to build student understanding.

Lynda Ball is a senior lecturer in Mathematics Education at the Melbourne Graduate School of Education at the University of Melbourne. Her doctoral study and subsequent research focuses on teaching and learning secondary mathematics with technology. Her early research focused on the use of Computer Algebra Systems (CAS) for teaching, learning and assessment in senior secondary mathematics, with particular interest in the evolution of written communication of examination solutions in the presence of CAS. Teacher beliefs and experiences in teaching with technology were also a focus. Recent research collaborations, including those with research students, are in the area of STEM education, flipped learning, online diagnostic assessment systems, technology-assisted guided discovery, communication with technology and computer algebra systems in mathematics. The evolution of teacher practices and opportunities to improve student learning with technology, as well as teacher professional development, are ongoing research interests.

Bärbel Barzel is a full professor of Mathematics Education at the University of Duisburg-Essen. Her research focuses on the meaningful use of technology in the mathematics classroom and the professionalisation of teachers to orchestrate and manage this successfully. She is a leading member of the German Centre for Mathematics Teacher Education (DZLM), a nationwide centre that researches and develops professional development (PD) courses and materials for teachers and PD facilitators.

Sylvaine Besnier is a kindergarten teacher in Rennes (France). She has been a member of the Center for Research for Education, Learning and Didactics (CREAD) since 2012. In 2016, she defended a PhD on the documentation work of kindergarten teachers in mathematics, with specific attention to their use of digital resources and its consequences in terms of professional development. Her work focuses on the resources and professional knowledge of teachers. She is particularly interested in the teaching of numbers at kindergarten level and in digital resources. In terms of theory, her research uses the documentational approach to didactics and the notion of orchestration.

Gülay Bozkurt is an assistant professor in the Department of Mathematics Education at İzmir Demokrasi University, Turkey. She received both her MPhil degree in Educational Research (2012) and her doctoral degree in Education (2016) from the University of Cambridge. She also holds an MSc degree in Mathematics Education (2010) from the University of Warwick. Additionally, she has extensive experience as a secondary school teacher of mathematics in Turkey and is thus aware of the practical world of school teaching. Her research interests centre around the use of digital technologies in mathematics education and, in particular, the development of pre-service and in-service mathematics teachers' professional knowledge for teaching with technology.

Nicola Bretscher is a lecturer in Mathematics Education at UCL Institute of Education. Her research interests centre around the use of (digital) technology in mathematics education and, in particular, how mathematics teachers integrate technology into their classroom practice and their mathematical knowledge for teaching. She is also interested in the use of quantitative and mixed methods in mathematics education research. These research interests inform her teaching on initial teacher education, Master's programmes as well as doctoral supervision. Nicola first became interested in the use of technology in the teaching and learning of mathematics as a pre-service teacher during her training year. As a secondary school mathematics teacher, she developed this interest through her Master's dissertation, focusing on the development of teaching techniques for using dynamic geometry software in her own classroom practice. Her doctoral research focused on mathematics teachers' knowledge and how it is involved in interacting with technology, as a means of exploring technology integration further. She is a statistician for the SMART Spaces project, an evaluation of an intervention based on spaced learning in science education funded by the Education Endowment Foundation.

Laetitia Bueno-Ravel is an associate professor of Mathematics Education at the University of Brest and a member of the Center for Research for Education, Learning and Didactics (CREAD). Her PhD (defended in 2003, University of Grenoble, France) concerned the place given to algorithmics by teachers in high school mathematics education. The use of new technologies by teachers has always been a central theme in her research. For several years, she has been reorienting her research work towards the integration of new technologies in the teaching of mathematics and the documentary work of teachers at primary school level, focusing on the teaching of number construction, numeration and calculation. From 2012 to 2018, Laetitia was a member of COPIRELEM, a French association of primary school teacher trainers promoting research in mathematics education to the community of teacher trainers, in particular through the annual organisation of conferences.

Chantal Buteau is a full professor in the Department of Mathematics and Statistics at Brock University (Canada). Since she joined Brock in 2004, Chantal has been progressively involved in education research with a main focus on the integration of digital technology for (university) mathematics learning, including programming, CAS, and epistemic mathematics (computer) games. Over the years, Chantal has taken part in various collaborative research projects funded by the Canadian Social Sciences and Humanities Research Council (SSHRC), such as "Computer Algebra Systems (CAS) in University Instruction: An International Research Study on CAS Usage and Sustainability", and leads the research under which this chapter falls. Chantal recently conducted research work for the Ontario Ministry of Education (Canada) on the teaching and learning of elementary coding and secondary school computer studies. She is Co-director of the Mathematics Knowledge Network (http://mkn-rcm.ca) that brings together diverse mathematics education stakeholders from across Ontario (Canada), and Lead of its Computational Modelling community of practice. In terms of teaching related to this chapter, Chantal has been directly involved in her department in the teaching of the programming-based mathematics MICA courses. In 2018, she also introduced a MICA III course section specifically designed for future teachers.

Shanton Chang is a professor of Information Behavior at the School of Computing and Information Systems, the University of Melbourne. He is an Associate Dean (International) at the Faculty of Engineering and Information Technology. He is also the co-chair of the Digital Access and Equity Program at the Melbourne Social Equity Institute. His research focuses on examining information needs and online behaviours in education, health and business contexts. Shanton has published extensively in this field. He is also the co-author of *Digital Experiences of International Students: Challenging Assumptions and Rethinking Engagement* (2020) as part of the Routledge Series on Internationalization in Higher Education. He is also co-editor of the *Journal of Studies in International Education* Special Issue on "Digitalization of International Education". He was also recognised by the Australian Computer Society as Information Communication Technology (ICT) Educator of the Year in 2017.

Annalisa Cusi graduated in Mathematics from Modena and Reggio Emilia University, where she also obtained a PhD in Mathematics. She worked as a research fellow at Torino University from 2014 to 2016, within the European Project FaSMEd, aimed at investigating the role played by digital technologies in supporting formative assessment processes. She is an associate professor of Mathematics Education at the Department of Mathematics of the Sapienza University of Rome, where she is involved in pre-service and in-service teacher education programmes. Her main research interests are: (1) early algebra and innovative approaches to the teaching of algebra; (2) analysis of teaching/learning processes, with a focus on the role played by the teacher during classroom discussions; (3) methodologies for pre- and in-service teacher education; (4) analysis of the dynamics that characterise teachers' and researchers' interactions within communities of inquiry; (5) formative assessment practices in mathematics within technology-enhanced classrooms; and (6) design and use of digital tools and resources to foster individualisation at university level.

Patrick Ebers is a PhD student in Mathematics Education at the University of Duisburg-Essen. In his PhD thesis he is analysing how video cases can be used to improve teachers' noticing regarding how students use technology. He is an external member of the German Centre for Mathematics Teacher Education (DZLM) and he also works as a secondary school mathematics and physics teacher.

Cristina B. Esteley was a mathematics teacher who worked in several secondary schools in Córdoba, Argentina. She has a Master's degree in mathematics education from the City University of New York (USA). She has a PhD in Education Sciences from the National University of Córdoba (UNC). She has taught mathematics and mathematics education at several Argentine universities. She conducts research focused on professional trajectories of teachers or future teachers of mathematics when they are involved with mathematical modelling activities in contexts that promote collaborative work and the use of technologies. She participates or has participated as a researcher in charge of research projects and as advisor of PhD theses or others graduate works on topics related to her research. She is a member of the Science and Technology Education Group (GECYT) of FAMAF and of the editorial committee of *Revista de Educación Matemática* published at UNC since 1979. She collaborates and has collaborated with colleagues in the framework of the International Commission on Mathematical Instruction. Such collaborations focus on inquiries, analyses and evaluations of research on mathematical modelling and on collaborative work among mathematics teachers.

Ghislaine Gueudet has been a full professor of Mathematics Education at the University Paris-Saclay, France since September 2021 and was previously a professor at the University of Brest (France). Her PhD (defended in 2000, University of Grenoble, France) concerned university Mathematics Education, and she is still researching this theme. She is Co-editor-in-chief of the *International Journal for Research in Undergraduate Mathematics Education*. Since 2006 she has been developing a new research direction on the design and use of educational resources (including digital resources). Concerning these resources, she has introduced in a joint work with Professor Luc Trouche, then with Professor Birgit Pepin, the documentational approach to didactics, analyzing teachers' interactions with resources and the consequences of these interactions in terms of professional development. The documentational approach is now used in studies concerning teaching practices and teachers' professional development at all school levels and in teacher education programmes. Ghislaine has been involved as Co-editor for several collective books on university mathematics education and/or teaching resources and the documentational approach, and is author or co-author of more than fifty articles and book chapters.

Mariam Haspekian is a researcher in Mathematics Education at the EDA laboratory, and a senior lecturer in Didactics of Mathematics at University of Paris Cité, where she is the head of the three-year Bachelor's degree (Licence) in "Educational Sciences" programme. She completed her thesis in 2005 under the direction of

Michèle Artigue, on the integration of spreadsheets for algebra teaching. Since then, her work, within diverse national and international projects, has concerned the teaching of mathematics in digital environments and is oriented along two directions: the networking of theoretical frames in mathematics education, and the analysis of instrumented teaching practices. Contributing to the Instrumental Approach in didactics, her work seeks to develop tools for studying the mathematical practices implemented by teachers in new situations. To analyse these, she introduced and is working on the concepts of distance from practices, didactic reference points and double instrumental genesis of the teacher. She has participated in the organisation of many international conferences (ICME, CERME, EMF). In France, she is a member of the Committee of the ARDM (Association for Research in Didactics of Mathematics, part of the French Commission for Mathematics Teaching-CFEM).

Celia Hoyles taught mathematics in London schools from the late 1960s before moving into higher education, becoming a professor at the Institute of Education, University of London, in 1984. She was inspired by the vision of Seymour Papert to use digital technology to open access to mathematics, and has led many research and development projects to promote this aim with a range of colleagues, notably Richard Noss. Celia worked to change the public face of mathematics by co-presenting a popular TV mathematics quiz show in the UK, *Fun and Games*, which topped the prime-time ratings between 1987 and 1990. She was the first recipient of the International Commission of Mathematics Instruction (ICMI) Hans Freudenthal medal in 2004, and the Royal Society Kavli Education Medal in 2011. She was the UK Government's Chief Adviser for mathematics (2004–07) and the director of the National Centre for Excellence in the Teaching of Mathematics (2007–13). Celia gave a keynote speech at the International Congress on Mathematical Education (ICME, 11), Monterrey, Mexico in 2008. Celia was President of the Institute of Mathematics and its Applications (IMA) (2014–15) and she was made an Officer of the Order of the British Empire in 2004 and a Dame Commander in 2014.

Melike Yiğit Koyunkaya is an associate professor in the Department of Mathematics Education at Dokuz Eylul University, Turkey. She received her doctoral degree in Curriculum and Instruction with a focus on Mathematics Education from Purdue University in 2014. She also holds Bachelor's and Master's degrees in Mathematics from Ege University, Turkey. Her research interests concern geometry education from a constructivist theoretical approach and the professional development of pre-service mathematics teachers in relation to teaching with technology.

Bjarnheiður Kristinsdóttir works as an adjunct lecturer at the University of Iceland. She holds a BSc degree in Mathematics from the University of Iceland, a Dipl. Math. (MSc) degree in Applied Mathematics from the Freiberg University of Mining and Technology in Germany, and a PhD in Mathematics Education from the University of Iceland. Her doctoral project on the definition, development and implementation of silent video tasks was conducted in close collaboration with mathematics teachers in upper secondary schools in Iceland. Before and during her doctoral studies, Bea worked for eight years as a licensed mathematics teacher at upper secondary school level. Her research focus is on mathematics teaching practices and task design, especially involving practices that require students to think, the use of dynamic geometry software and videos, formative assessment, and the orchestration of classroom discussion. Bea has been active within the Nordic-Baltic GeoGebra Network since 2012 and has collaborated on research projects with Professor Zsolt Lavicza and his team at the Johannes Kepler University Linz School of STEM Education in Austria since 2016, most recently on task design for the EU project <colette/> (https://colette-project.eu/), which aims to develop a computational thinking learning environment for teachers and students in Europe.

Allen Leung is a professor of Mathematics Education at Hong Kong Baptist University. He received his PhD in Mathmatics from the University of Toronto, Canada. His research interests include geometric reasoning in dynamic geometry environments, development of mathematics pedagogy using variation, tool-based mathematics task design, and integrated STEM pedagogy. Allen has published in major international mathematics education journals, books and conference proceedings. He was involved in ICME 11, 12 and 13 as a Topic Study Group organising member, a presenter of a Regular Lecture and a member of an ICME survey team. He has contributed to two ICMI Studies and was an IPC member of the 22nd ICMI Study: Task Design in Mathematics Education. He co-edited the Springer book *Digital Technologies in Designing Mathematics Education Tasks – Potential and Pitfalls* (Mathematics Education in the Digital Era Book Series) (2016). Allen is an associate editor of the Springer journal *Digital Experiences in Mathematics Education*.

Javier Lezama has a PhD in Mathematics Education from the research centre CINVESTAV in Mexico City. He is a member of the National System of Researchers of Mexico, Level 1. He is one of the founders of the Mathematics Education Program of the National Polytechnic Institute of Mexico. Javier also created the social network "DocenMat" aimed at mathematics teachers from all over Latin America. He is a visiting professor at the Autonomous University of Guerrero in Mexico. His research interests are connected to the area of mathematics teacher education and development.

Biyao Liang is a postdoctoral fellow sponsored by the Hong Kong Research Grants Committee's (RGC) Research Fellowship Scheme and supervised by Dr Oi-Lam Ng at the Chinese University of Hong Kong. She obtained her BSc in Mathematics from South China Normal University (China) and a PhD in Mathematics Education from the University of Georgia (USA). Her research programme is at the intersections of mathematical cognition, social interactions and teacher education. Specifically, her research draws on radical constructivism and Piagetian theories to characterise students' and teachers' ongoing constructions of mathematical knowledge through social interactions and to design educational opportunities, tools and materials that can support learning through interactions. She has diverse classroom experiences in Mainland

China, Hong Kong, Kansas and Georgia, and has been teaching content and pedagogy courses for pre-service secondary mathematics teachers since 2019.

Andrew McAlindon is a secondary school teacher of Mathematics and integrated Science Technology Engineering and Mathematics (STEM) in Victoria, Australia. His doctoral research centred on the efficacy of the flipped classroom in secondary school mathematics, with a focus on student outcomes and teacher perspectives. Andrew has an ongoing focus on educational improvement in school contexts, with teacher professional development in pedagogical approaches within mathematics and STEM being an ongoing research interest.

Joyce Mgombelo is an associate professor of Mathematics Education at Brock University, Ontario, Canada, where she teaches courses and supervises graduate students in mathematics education and cognition. Her research interests are in the areas of mathematics/STEM education, teacher education (in-service and pre-service) and curriculum studies. Her research programme focuses on mathematics cognition, identity and ethics, based on principles of human cognition. This work is developed from the theoretical perspectives of enactivism, complexity science and psychoanalysis. Joyce's most recent research includes the Canadian Social Sciences and Humanities Research Council (SSHRC)-funded collaborative research projects "Educating for the 21st Century: post-secondary students learning 'progmatics' (computer programming for mathematical investigation, simulation, and real-world modeling)" and "Advancing research methodology in mathematics education for collective learning systems" as well as the Canada Global Affairs collaborative development project "Capacity Development for mathematics teaching in rural and remote communities in Tanzania".

Eric Muller is a professor emeritus in the Department of Mathematics and Statistics at Brock University and a fellow of the Fields Institute for Research in Mathematical Sciences. He has published in the areas of theoretical physics, operations research and mathematics education. He continues to collaborate in research that focuses on the use of programming technology in project-based courses in undergraduate mathematics. Eric completed an MSc in 1963 in the area of Calculus of Variations under Professor Hanno Rund at the University of Natal in Durban. He then briefly taught at Rhodes University in Grahamstown before moving to the University of Sheffield where he completed a PhD in the area of Thermal Conductivity under Professor Norman March. He joined the Department of Mathematics at the fledgling Brock University in 1967 and retired in 2004. Thereafter he spent time with Pacific Resources for Education and Learning located in Honolulu and visited mathematics departments in colleges on isolated islands dispersed over the North Pacific.

Oi-Lam Ng is an assistant professor in the Department of Curriculum and Instruction at the Chinese University of Hong Kong. Her research interests include technology innovations in mathematics education, language and mathematics discourse, mathematics teacher noticing, and STEM education. Particularly, she is interested in advancing a Papert-inspired conception of "learning as making" and the new opportunities it entails for engaging learners in constructionist practices with emergent technologies (3D printing, coding, etc). Oi-Lam's research has been funded by the Research Grant Councils of Canada and Hong Kong respectively, and her funded research is entitled "The effects of implementing a 'learning as Making' pedagogy on school mathematics learning: Primary students' inquiry-based Making with 3D Printing Pens". Her work has been published in *Educational Studies in Mathematics*, *ZDM: International Journal on Mathematics Education*, and the *British Journal of Educational Technology*. Oi-Lam teaches mathematics and STEM education courses. She received her PhD from Simon Fraser University, Canada.

Monica Panero holds a PhD in Mathematics and is a lecturer and researcher in Mathematics Education in the Dipartimento formazione e apprendimento of the Scuola universitaria professionale della Svizzera italiana. Her main research foci are on formative assessment, technology in mathematics education, attitudes towards mathematics and its teaching, and mathematics teacher education. In her recent publications she interrelates such interests by investigating the role of technology formative assessment, during her postdoctoral research within the European project called FaSMEd (2014–16); by studying design principles for fostering and assessing involvement and collaboration in MOOCs for mathematics teachers; and by analyzing the evolution of pre-service primary school teachers' attitudes towards mathematics and its teaching. She is part of the executive board of the International Commission for the Study and Improvement of Mathematics Teaching, and part of the scientific committee of the open access semestral journal *Didattica della matematica. Dalla ricerca alle pratiche d'aula.*

Caroline Poisard is an associate professor of Mathematics Education at the University of Brest and a member of the Center for Research for Education, Learning and Didactics (CREAD). Her research concerns the resources for teaching and learning mathematics at primary school. It has three axes: world languages as a resource for doing mathematics; material and virtual resources for teaching (calculating instruments, the Chinese abacus); and mathematical workshops in the classroom (manipulations and games).

Marisol Santacruz Rodriguez is an assistant professor at the Faculty of Education and Pedagogy of Universidad del Valle (Cali, Colombia). Her specialty is geometric education using digital resources. Her research focuses on the study of the student's activity using digital technologies, the teacher's documentational work and the analysis of the professional knowledge involve in the geometry classroom. For many years, Marisol also taught mathematics at elementary school level. At present, she is more focused on mathematics teacher education and the use of programming for teaching mathematics.

Ana Isabel Sacristán is a full researcher in the Department of Mathematics Education of the Centre for Research and Advanced Studies (Cinvestav) in Mexico City, where she has worked since 1989. Her main area of research is the teaching and learning of mathematics through digital infrastructures. She is particularly fond of the constructionism paradigm as a basis for the design of learning environments where students can explore, and build ideas and concepts through computer programming activities. She has published many academic papers in that area, but has also developed tasks and authored materials for the Mexican Ministry of Education, in particular those for the national "Teaching Mathematics with Technology" programme, on the use of computer programming activities for mathematical learning. She has trained teachers across Mexico and has co-led nationwide research and evaluation on the use of technological tools in Mexican classrooms. She has also been part of many international committees, including the International Programme Committee of the 17th ICMI Study on "Mathematics Education and Technology—Rethinking the Terrain" and has been a visiting professor in several countries, including at the Institute of Education, University of London in England; Université du Québec à Montréal in Canada; and the French Institute of Education at Lyon-ENS in France. More recently she has collaborated with Canada's Brock University.

Ivonne Sandoval has been a teacher and researcher at the National Pedagogical University, Mexico City (Mexico), since 2008. She is a member of the National System of Researchers of Mexico, Level 1. Her research focuses on designing and implementing digital technologies and other resources for mathematics education. Due to this interest, she has co-authored mathematics textbooks for elementary and middle school students in Mexico. She also participated in a Mexican National project dealing with the integration of digital technologies in elementary schools. Ivonne also has an interest in research related to the development of spatial reasoning. She is concerned with elaborating STEM tasks in different cultural contexts through using various resources for students and teachers at elementary school level, specifically in socioeconomically disadvantaged contexts. In this case, she is concerned with studying students, teachers and resources. She also investigates mathematics teaching specialised knowledge, and for this reason, she belongs to the Iberoamerican network for Mathematics Teaching Specialized Knowledge (MTSK) recognised by the Iberoamerican Universities Postgraduates Association (AUIP) since 2019. Her main contributions to mathematics education knowledge have focused on teachers' use of technology and geometry. She has also participated in research with several national and international groups in Mexico, Spain, Colombia, the United States and Canada.

Ali Simsek has primary research interests in secondary school mathematics education in general with a particular focus on the use of dynamic mathematical technologies (DMTs) in the classroom. In his four-year teaching experience in Turkey, Ali developed a keen interest in the use of DMTs to enhance his students' mathematical learning. This led to him pursuing postgraduate studies in the field of Educational Technologies in Mathematics Education. Having been awarded a competitive scholarship from the Turkish government, he completed an MA degree at University College London (UCL)'s prestigious Institute of Education (IOE) in England in 2016. Following this, he then completed a PhD at the same university in 2021 under the supervision of Professor Dame Celia Hoyles, Professor Alison Clark-Wilson and Dr Nicola Bretscher. In his PhD research, he investigated lower secondary mathematics teachers' actual classroom practices as they used DMTs in the classroom to promote their students' understanding of the mathematical domain of geometric similarity (GS). The findings of his PhD research revealed salient differences and some commonalities between the teachers, pointing to key characteristics of classroom practice involving DMTs for teaching GS. Ali now works as a national educational expert at the Ministry of National Education in Turkey.

Danelly Susana Esparza Puga obtained a Bachelor's degree in mathematics from the Autonomous University of Ciudad Juárez (UACJ) in 2011. In 2014 she received a Master's degree in Mathematics Education from the UACJ and obtained her PhD in Mathematics Education from the National Polytechnic Institute of Mexico in 2018. She is a member of the National System of Researchers of Mexico, and her research interests focus on the use of digital tools in the teaching and learning of mathematics.

Eugenia Taranto holds a PhD in Pure and Applied Mathematics and is a postdoctoral researcher at the University of Catania, where she is also a lecturer in Mathematics Education for a graduate-level course. Her research fields include MOOCs (Massive Open Online Courses) for mathematics teacher education, in particular, she collaborated on the design and delivery of five Italian MOOCs and she is the instructional designer of an international MOOC and technologies to mediate the teaching and learning of mathematics (dynamic geometry systems, MathCityMap, learning videos, serious games). She is the author of papers and chapters in various prestigious journals and books.

Daniel Thurm is an assistant professor of Mathematics Education at the University of Siegen in Germany. His research focuses on professional development for teaching mathematics with technology as well as on digital formative assessment. He is a member of the German Centre for Mathematics Teacher Education (DZLM), a nationwide centre that researches and develops professional development (PD) courses and materials for teachers and PD facilitators.

Maria Trigueros is an invited professor in the Mathematics Education Department at the Benemérita Universidad Autónoma de México. She was a professor in the Department of Mathematics at Instituto Tecnológico Autónomo de México for forty years. She received her PhD in Education from Universidad Complutense de Madrid in Spain and her degree and MSc in Physics from the Universidad Nacional Autónoma de México (UNAM). She is a member of the Mexican Academy of Sciences and of the Mexican National Researchers' System. Her research focuses on advanced mathematics teaching and learning, and on the use of technology in the teaching and learning of mathematics at elementary and middle school level. Her main contributions to mathematics education knowledge have focused on teachers' use of technology, algebra, linear algebra and calculus. Maria has received several awards, among them the Luis Elizondo Prize, and has participated in national projects on the use of technology in mathematics teaching. She has developed instructional materials for students at elementary and middle school level and has served as editor for several Mexican and international mathematics education research journals. She has also participated in research with several national and international groups in Mexico and other countries.

Fabrice Vandebrouck is a professor at Université Paris Cité and member of the LDAR lab (Laboratoire de Didactique André Revuz). He is Co-director of the doctoral school Savoirs Sciences Education. He defended his PhD thesis in Mathematics in 1999 before moving into the mathematics education field. He teaches mathematics to undergraduate students at the university and didactics at Master's level. His research concerns the transition from high school to tertiary level alongside the integration of technologies in the teaching of mathematics. Fabrice has supervised eight theses on these topics. He is one of the main contributors to the development of activity theory in didactics of mathematics, alongside Maha Abboud, Aline Robert and Janine Rogalski. He is editor of the book *Mathematics Classrooms: Students' Activities and Teachers' Practices* (2013), and presented an invited lecture at the 13th International Congress on Mathematical Education entitled "Activity Theory in French Didactic Research".

Mónica E. Villarreal has a degree in Mathematics from the Universidad Nacional de Córdoba (UNC) (Argentina). She holds a PhD in Mathematics Education from the Universidade Estadual Paulista (Brazil). She is a professor at the Faculty of Mathematics, Astronomy, Physics and Computer Sciences (FAMAF) of UNC and researcher at the National Council of Scientific and Technical Research (CONICET) of Argentina. Mónica is a member of the Science and Technology Education Group (GECYT) of FAMAF. She is involved in the initial education of mathematics teachers. She has conducted and continues to conduct research on the professional development of pre-service and in-service mathematics teachers, mathematical modelling, and the use of digital technologies in educational contexts, directing research projects and theses on these topics. Mónica has numerous national and international publications in co-authorship, and has participated in many congresses and invited conferences. Since 2017 she has been Associate Editor of the *Revista de Educación Matemática* published at UNC since 1979. She was a member of the International Program Committee for the organisation of ICME 13 in Hamburg (2016). Mónica is the Argentinean representative for the International Commission on Mathematical Instruction (ICMI).

About the Editors

Alison Clark-Wilson is a professorial research fellow at UCL Institute of Education (Faculty of Education and Society), University College London. Alison's research spans aspects of designing, implementing and evaluating educational digital technologies with a particular interest in mathematics education. This includes theoretical contributions (the hiccup theory), alongside design-based research interventions and evaluation studies (Cornerstone Maths, TI-Nspire/TI-Navigator). Between 2017 and 2021 Alison led the European Society for Research in Mathematics Education's working group on "Teaching mathematics with technology and other resources". As a former school mathematics teacher and teacher educator, Alison has particular empathy for the challenges faced by teachers as they seek to stay abreast of technological developments, often with little systemic support and few professional incentives. Alison is the lead editor of the book *Mathematics Education in the Digital Age* (2021) and presented the invited lecture at the 14th International Congress on Mathematical Education in 2021 entitled "*(Re)Assessing Mathematics: Retaining the Integrity of Mathematics as a Human Activity in the Digital Age*".

Ornella Robutti is a full professor of Mathematics Education in the Department of Mathematics "G. Peano" of the University of Torino. Her fields of research are students' cognitive processes in mathematical activities; teaching mathematics within technological environments; teachers' work as individuals and in communities, when teaching mathematics, when learning in professional development programmes, and when designing tasks for students; meanings of mathematical objects in institutional and social contexts; mathematics students' and teachers' identities; and boundary objects and boundary crossing between communities. She is the author of articles/chapters in mathematics education and a team leader/lecturer/participant in many international congresses (ICMI Study, ICME, PME, CERME, ICTMT). In Italy she has been a member of CIIM (the scientific commission for mathematics teaching of the Italian Mathematics Association), and in charge of in-service and pre-service teachers' professional development programmes.

Nathalie Sinclair is a distinguished university professor in the Faculty of Education at Simon Fraser University. She is the founding editor of *Digital Experiences in Mathematics Education* and has written several books, including *Mathematics and the Body: Material Entanglements in the Classroom* (2014). She directs the Tangible Mathematics Project, which has created the multitouch applications TouchCounts and TouchTimes.

Professional Development for Teaching Mathematics with Technology: Fostering Teacher and Facilitator Noticing

Daniel Thurm, Patrick Ebers, and Bärbel Barzel

Abstract Professional development of facilitators has been highlighted as a decisive factor for scaling-up professional development (PD) efforts. However, research on facilitators is still burgeoning and, for many areas like teaching mathematics with technology, little research is available on how to professionalise facilitators. In this paper we question how to prepare facilitators to support teachers for teaching mathematics with multi-representational-tools (MRT). Teaching mathematics with MRT requires a teacher to notice the subtle ways in which technology supports students learning and the challenge for facilitators is to support such a nuanced teacher noticing. Therefore, we focus on the constructs of teacher and facilitator noticing when teaching with MRT, and describe the design and implementation of a video-case-based strategy to professionalise PD facilitators to support teachers noticing of students learning when working with MRT. For this we developed the Content-Activity-Technology-Model (CAT-Model) that helps to capture in graphic form the students' learning processes when working with technology. This provides a more accessible format for teachers and facilitators, which can also be used to reconstruct teacher and facilitator noticing. Our analyses across the three levels of the CAT-Model leads us to identify the potential and challenges for this method and outline how the multi-level video-based design can be further improved.

D. Thurm (✉)
University of Siegen, Siegen, Germany
e-mail: daniel.thurm@uni-siegen.de

P. Ebers · B. Barzel
University of Duisburg-Essen, Essen, Germany

A. Clark-Wilson et al. (eds.), *The Mathematics Teacher in the Digital Era*, Mathematics Education in the Digital Era 16,
https://doi.org/10.1007/978-3-031-05254-5_1

Keywords Professional development · Facilitators · Mathematics · Digital technology · Noticing · Teachers · Video · Cognitive-Activity-Technology Model (CAT-Model)

1 Introduction

Nowadays, in many high-income countries, the use of digital technology for teaching and learning mathematics is well-established in the mathematics curriculum (Clark-Wilson et al., 2020; Thurm et al., 2023). In addition to great progress in the development of teaching ideas and theoretical lenses, research has highlighted the important role of the teacher (Clark-Wilson et al., 2014; Thurm & Barzel, 2020, 2021; Drijvers et al., 2010; Thurm et al., 2023). Teachers contribute decisively to the extent to which the potential of technology is exploited in the classroom. In light of this, PD programs are regarded as important for supporting teachers to integrate technology in their mathematics classrooms in meaningful ways. PD programs can help to equip teachers with the special knowledge they need for teaching with technology (for example, pedagogical technological knowledge; Thomas & Palmer, 2014) and to support them to manage the complex task of orchestrating technology-enhanced mathematics classrooms (Thurm et al., 2023; Drijvers et al., 2010). Focusing on the design of teacher PD programs, research has identified several characteristics that constitute high-quality teacher PD (Ertmer & Ottenbreit-Leftwich, 2010; Grugeon et al., 2010; Ratnayake et al., 2020; Thurm et al., 2023; Thurm & Barzel, 2020). These characteristics (or design principles) include, for example, a focus on teachers' technological pedagogical content knowledge, fostering reflection of technology use, and a focus on helping teachers to understand (and notice) how students might benefit from learning mathematics with technology (Clark-Wilson & Hoyles, 2019; Ertmer & Ottenbreit-Leftwich, 2010; Thurm & Barzel, 2020). Furthermore, professionalising teachers should be case-related, which means relating PD activities to practical aspects such as specific student outcomes, video-cases or other representations of practice (ibid.).

However, designing high-quality PD programs alone is not enough to ensure a high-quality PD experience for teachers. Just as the teacher's role has been found to be critical at the classroom level, so research has pointed out the importance of PD facilitators at the PD level. Facilitators are responsible for the design, adapt, and implement PD programs for teachers: "*Facilitators play a crucial role when scaling up continuous professional development (CPD). They have to design and conduct programs to initiate the process of teachers' professionalization*" (Peters-Dasdemir et al., 2020, p. 457). Yet, despite this important role, little is known about how to professionalise facilitators (Lesseig et al., 2017; Peters-Dasdemir et al., 2020; Prediger et al., 2019; Roesken-Winter et al., 2015; Thurm et al., 2023). In particular more research is needed to identify appropriate design principles that guide the design of PD for facilitators and to investigate associated challenges (ibid). This chapter makes a first step towards addressing this gap by focusing on teacher and facilitator noticing as a key concept of their competencies (Lesseig et al., 2017; Schueler & Roesken-Winter, 2018; Stahnke et al., 2016). Sherin, Russ, and Colestock (2011b) define the concept of noticing as *"professional vision in which teachers selectively attend to events that take place and then draw on their existing knowledge to interpret these noticed events"* (Sherin, Russ, & Colestock, 2011b, p. 80). Clearly, *teacher* noticing is highly relevant for teaching mathematics with technology. Teachers will only use technology if they notice how technology impacts positively on students' learning. In addition, teachers noticing of students' learning is a prerequisite to be able to scaffold students' learning. Similarly, *facilitators* noticing of teacher learning is also important. For example, facilitators need to notice what teachers notice with respect to student learning in order to support teachers in the PD program. In this paper we describe a video-case-based way to foster teacher and facilitator noticing. The methodological basis for our design and research endeavor is based on the Three-Tetrahedron-Model (3 T-Model) for content-related PD research which highlights strategies for connecting classroom level, teacher PD level and facilitator PD level (Prediger et al., 2019).

2 Theory

2.1 Teaching Mathematics with Multi-representational Tools

While the scope of available technologies has increased massively, commonly used technology in the mathematics classroom are multi-representational-tools (MRT) (also called "mathematics analysis software"; Pierce & Stacey, 2010) which combine the capabilities of scientific calculators, function plotters, spreadsheets, statistics and geometry applications, and computer algebra systems. In this chapter, unless stated otherwise, the term "technology" is used to refer to such MRT. MRT can support student learning by providing easy access to different forms of representations, such as numerical and graphical representations and allowing dynamic linking of different forms of representations (Drijvers et al., 2016; Heid & Blume, 2008). In particular, students can work simultaneously with the different mathematical representations and can explore relations between these. This is especially important since research has highlighted that transforming, linking, and carrying out translations between different mathematical representations is crucial for developing an understanding of mathematical concepts (Duval, 2006). In addition, the easy access to different forms of representations can support more student-centered teaching approaches such as discovery learning (Barzel & Möller, 2001; Pierce & Stacey, 2010; Thurm, 2020). In the following we exemplify the affordances of MRT with respect to a particular task, which is shown in Fig. 1. This task was used in the research study as a basis to support teacher and facilitator noticing and we will refer

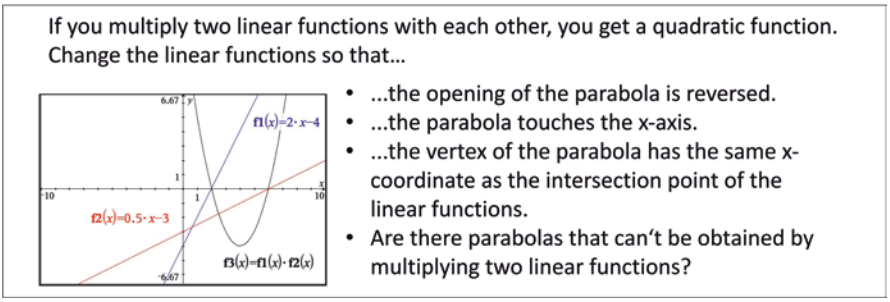


Fig. 1 MRT-task adapted from Drijvers (1994)

to it as the "MRT-task" throughout the chapter. In the MRT-task students are prompted to amend the two given linear functions in such a way that the product function satisfies certain mathematical conditions. In addition, students are asked to write a conjecture about the types of parabolas that cannot be obtained by multiplying two linear functions. In this task, various affordances of MRT become apparent. Firstly, the MRT can support students in generating many pairs of graphs of different linear functions and the respective product functions. Without MRT students would have to engage in the tedious and repetitive work of drawing many functions by hand. This would take much time and would constrain learners from focusing their attention on the relationships between the linear functions and the respective product function. Furthermore, MRT make it possible to *dynamically* change the slope of the linear function, for example, by dragging. At the same time MRT offer simultaneous access to the symbolic and graphical representations, which supports students observing and investigating the links between these two forms of representation. To summarise, using MRT with this task allows students to explore, test and discover mathematical relationships between linear and quadratic functions.

2.2 Facilitators

In the research literature many terms are used to describe the group of people who initiate and lead processes to professionalise teachers, for example, "facilitators", "teacher trainers", "multipliers", "coaches", "didacticians" and "teacher educators" (Peters-Dasdemir et al., 2020). In this chapter we use the term "facilitator" which highlights that the process of facilitating PD for teachers is rather a *"give-and-take than a one-sided teacher-pupil relationship"* (Peters-Dasdemir et al., 2020, p. 457).

Research related to PD facilitators is an emerging field of study (Lesseig et al., 2017; Poehler, 2020; Prediger et al., 2019; Thurm et al., 2023). In the last decade pioneering research studies have focused on identifying the required skills and knowledge for them to be effective (Borko et al., 2014; Elliott et al., 2009; Lesseig et al., 2017; Peters-Dasdemir et al., 2020). Clearly, facilitators require *"competencies about adult learning and the specific knowledge and needs of mathematics' teachers, which are much broader than teachers' competencies."* (Peters-Dasdemir et al., 2020, p. 457). This is, for example, illustrated by the competency model for facilitators developed by Peters-Dasdemir et al. (2020). The model conceptualises facilitators' knowledge as an extension of the knowledge needed for teaching and it adapts the well-established specifications of content knowledge (CK), pedagogical knowledge (PK) and pedagogical content knowledge (PCK) from the classroom level to the PD level. In particular, pedagogical content knowledge on the PD level (PCK-PD) concerns the knowledge needed "*to engage teachers in purposeful activities and conversations about those mathematical concepts, relationships and to help teachers gain a better understanding of how students are likely to approach related tasks"* (Jacobs et al., 2017, p. 3). Moreover, research has generated first insights about the effective design of facilitators' preparation programs (e.g., Kuzle & Biehler, 2015; Lesseig et al., 2017; Roesken-Winter et al., 2015). For example, Lesseig et al. (2017), propose a set of design principles, which include focusing on teacher learning goals, providing opportunities for facilitators to expand their specialised content knowledge, and using video-cases as representations of practice to generate in-depth discussion and reflection of facilitators' practices and beliefs.

Despite these results "*research on preparing and supporting facilitators of mathematics PD is still at a very early stage [...]."* (Jacobs et al., 2017, p. 12), which holds particularly true with respect to facilitators in the context of teaching mathematics with technology (Thurm et al., 2023). This can be illustrated, for example, by the fact, that neither the previous edition of this book (Clark-Wilson et al., 2014) nor the ICMI study of Hoyles and Lagrange (2010), nor the ICME-13 monograph on uses of technology in primary and secondary mathematics education (Ball et al., 2018), nor the ICME-13 topical surveys by Drijvers et al. (2016) and Hegedus et al. (2017), nor the last two proceedings of the International Conference on Technology in Mathematics Teaching (ICTMT, Barzel et al., 2020; Aldon & Trgalová, 2017), nor the recently published ZDM special issue on teaching with technology (Clark-Wilson et al., 2020), have contributions or sections particularly addressing PD for facilitators. However, research activity in this field is slowly burgeoning. For example, Psycharis and Kalogeria (2018) and the recent ICME25 proceedings (Borko & Potari, 2020) provide some elements on this theme. Placing a greater focus on facilitators' professional development is particularly important, since in many countries facilitators are not required to complete any specific PD programme or accreditation to prepare them to offer courses for teachers (Lesseig et al., 2017; Roesken-Winter et al., 2015). Rather *"formalized professional development opportunities for leaders are exceptions rather than the norm"* (Lesseig et al., 2017). In addition, professionalising facilitators in formal ways becomes increasingly important due to the emergence of PD institutions such as the "National Centre for Excellence in the Teaching of Mathematics" (NCETM) in England, the "National Center for Mathematics Education" (NCM) in Sweden, the "Institut für Unterrichts- und Schulentwicklung" (IUS) in Austria or the "German Centre for Mathematics

Teacher Education" (DZLM) in Germany, which aim to provide high-quality PD on a larger scale which brings to the forefront the question of how to professionalise facilitators.

While it is clear that more research with respect to facilitators is needed, conducting such research is not an easy endeavor. For example, Borko (2004) highlighted that facilitators, the PD program, the participating teachers and the context are inevitably intertwined through interactive and reciprocal relationships. Recently Prediger et al. (2019) have started to further unpack this complexity and proposed the Three-Tetrahedron Model (3 T-Model) for PD research and design, which captures the complexity of learning and teaching at the classroom, teacher PD, and facilitator PD level. This model will be explained in detail in the next section.

2.3 *The Three-Tetrahedron Model for Design and Research on PD*

The Three-Tetrahedron Model (3 T-Model) of Prediger et al. (2019) provides a framework for the design of and research on teacher and facilitator PD programs. Its goal is to capture "*the complexity of learning and teaching at the classroom, teacher, and facilitator level that is needed to inform design and research into PD*" (Prediger et al., 2019, p. 407). Extending the idea of the commonplace didactic triangle, which relates teachers, learners and the content to be learned, the 3 T-Model takes the format of a series of tetrahedrons which are considered at the classroom, teacher PD and facilitator PD levels (see Fig. 2). The classroom level tetrahedron comprises

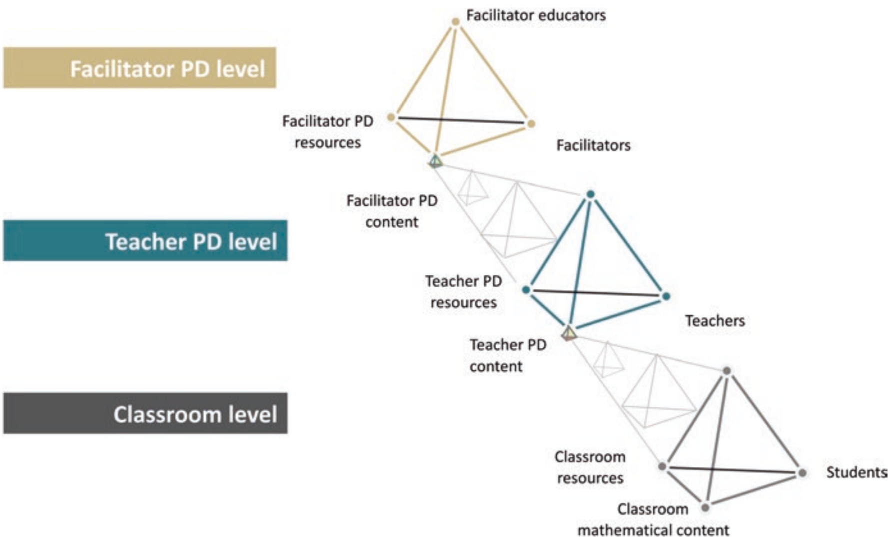


Fig. 2 The three-Tetrahedron Model (3 T-Model) for content-related PD research (Prediger et al., 2019)

relations between students, content, classroom resources and the teacher. This structure can now be transferred to the teacher PD level. Here the teacher takes the position of the learner and the facilitator takes the role of the teacher. At the teacher PD level (TPD), learning is supported by teacher PD resources and the content on the classroom level is replaced by the teacher PD content. Finally, this structure can also be transferred to the facilitator PD level (FPD). This model has been used in a variety of contexts. For example, it has been used to investigate facilitators' practices (Leufer et al., 2019), for describing implementation strategies on different levels (Roesken-Winter et al., 2021) and to gain insights about effective strategies for supporting PD facilitators to incorporate content and skills introduced in facilitator PD sessions into their own practice (Borko et al., 2021).

Prediger et al. (2019) describe three general strategies (lifting, nesting, unpacking) for design and research on PD which take into account the multi-level structure of PD displayed in Fig. 2. In the following we elaborate on the lifting and nesting strategy which we used in the design and research that we report here.

The **lifting strategy** (Fig. 3, left) comprises lifting design and research approaches from one level to the next. For example, lifting a design approach *"means that design principles or design elements developed for the classroom level are implicitly or explicitly transferred (and adapted) to the TPD level (or from the TPD to FPD level)"* (Prediger et al., 2019, p. 412). Similarly, lifting a research practice entails that research questions and/or methods from the classroom level *"are implicitly or explicitly transferred (and adapted) to the TPD level (or from the TPD to FPD level) and applied in an analogous way."* (Prediger et al., 2019, p. 413). For example, design approaches that employ video-case-based learning to support teacher noticing of student learning can be lifted from the teacher PD level to the facilitator PD level, to support facilitator noticing of teacher learning. A further

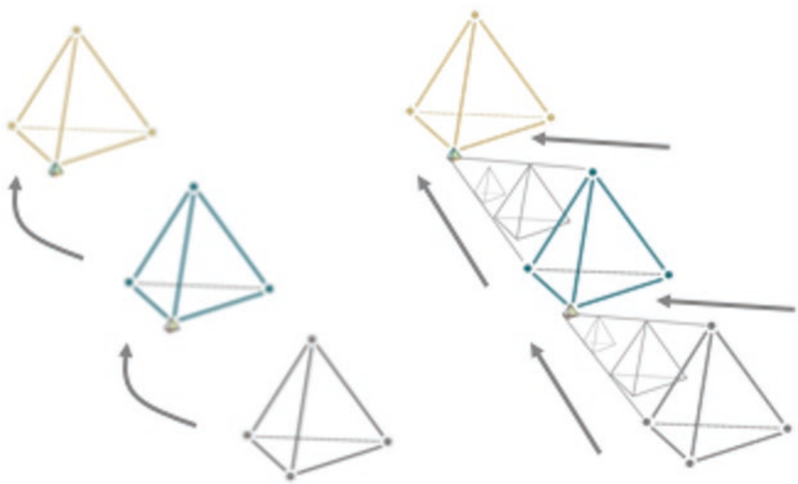


Fig. 3 Lifting strategy (left) and nesting strategy (right) in the Three-Tetrahedron Model (Prediger et al., 2019)

example would be, if research approaches for investigating students' thinking and learning pathways are lifted to the teacher PD level by investigating teachers' thinking and teacher learning pathways.

The **nesting strategy** (Fig. 3, right) accounts for the fact that teacher PD content is usually more complex than classroom content, and that facilitator PD content is usually more complex than teacher PD content (Prediger et al., 2019). Therefore, the nesting strategy considers that aspects of the complete classroom tetrahedron should be nested in the teacher PD content and that aspects of the complete teacher PD tetrahedron should be nested in the facilitator PD content. Hence the nesting strategy *"builds the PD design upon the idea of structuring the TPD/FPD content in a self-similar nested structure, taking into account the complexities of the tetrahedrons below."* (Prediger et al., 2019, p. 413).

In our research project the 3 T-Model was used to guide the design of the PD activities and to situate the different aspects of our project along the different levels of professional learning, while accounting for the complexity resulting from the inherent connections between the different levels. However, while the 3 T-Model is well suited to provide a macro-view on design and research for PD it is often helpful to combine the use of the 3 T-Model with other additional models. For example, Borko et al. (2021) integrate the Learning to Lead Cycle with the 3 T-Model in order to facilitate research about the leadership capacity of experienced teachers.

2.4 Teacher Noticing and Video-Case-Based-Learning

In our research endeavor we used the 3 T-Model to design a PD activity that focused on teacher and facilitator noticing, a construct which we now explain in more detail. Teacher noticing builds on the notion of professional vision which was introduced by Goodwin (1994) as *"socially organized ways of seeing and understanding events that are answerable to the distinctive interests of a particular social group"* (Goodwin, 1994, p. 606). In line with the conceptualisation of Sherin (2007) we understand noticing to be both the perception of aspects in teaching situations that are relevant for teaching quality (selective attention) and the interpretation of these aspects based on appropriate professional knowledge (knowledge-based reasoning). Clearly, if teachers observe a classroom situation they might attend to very different aspects and interpret these in different ways. In particular teacher noticing is important for paying attention to, and interpreting students' mathematical thinking, and to recognise developing mathematical understanding of students (Sherin, Jacobs, & Philipp, 2011a, p. 3). This is particularly true since *"effective instruction requires teachers to notice, pay attention to, and respond to students' ideas"* (Beattie et al., 2017, p. 323; Kilic, 2018). However, noticing is not only crucial for teachers but also for PD facilitators (Lesseig et al., 2017). Facilitator noticing with respect to teachers' learning is important in order to facilitate robust opportunities for teachers' learning, for example, by appropriate facilitation moves (Lesseig et al., 2017;

Schueler & Roesken-Winter, 2018). In literature, different levels of noticing have been described, which capture the depth of noticing starting from general noticing to more specific noticing (Lee & Choy, 2017; van Es, 2011). General noticing focuses, for example, on superficial features that are not directly associated with students learning (classroom level) or teacher learning (teacher PD level) and results in a very general impression of what has occurred. In contrast, specific noticing focuses on relationships between content, teachers, classroom resources and details of student or teacher learning and thinking. Furthermore, specific noticing comprises specificity in recalling details, supporting statements with evidence and providing explanations (van Es, 2011). In this study we use an adapted framework based on the work of van Es (2011), who proposed a model that distinguishes between four levels of noticing, where teachers increasingly attend to more details of students' mathematical thinking (see Table 1).

Given the high importance of teacher and facilitator noticing, the question arises how to support their noticing competencies. In this respect, research has highlighted the potential of the use of video-cases because they can capture the high complexity inherent in classroom teaching or PD without requiring immediate actions, as in a real classroom or PD situation (Koc et al., 2009; Lesseig et al., 2017; Schueler &

Table 1 Framework for levels of noticing adapted from van Es (2011, p. 139)

Level	What teachers notice	How teachers notice
Level 1 Baseline Noticing	Attend to generic aspects of teaching and learning, e.g., seating arrangement, student behavior, etc.	Provide general descriptive or evaluative comments with little or no evidence from observations
Level 2 Mixed Noticing	Begin to attend to particular instances of students' mathematical thinking and behaviors	Form general impressions and highlight noteworthy events or details
		Provide primarily evaluative with some interpretive comments
		Begin to refer to specific events and interactions as evidence
Level 3 Focused Noticing	Attend to particular students' mathematical thinking	Provide interpretive comments
		Refer to specific students' difficulties, events and interactions as evidence
		Elaborate on specific students' difficulties, events and interactions
Level 4 Extended Noticing	Attend to the relations between particular students' mathematical thinking, technology use and mathematical activities.	Provide interpretive comments
		Refer to specific events and interactions as evidence
		Elaborate on specific events, and interactions
		Make connections between events and principles of teaching and learning
		On the basis of interpretations, propose alternative pedagogical solutions

Roesken-Winter, 2018; Sherin, 2007): *"While video captures much of the richness of the classroom environment, it does not require an immediate response from a teacher and can instead promote sustained teacher reflection (*Sherin*,* 2004*). Moreover, because video provides a permanent record of classroom interactions, it can be viewed repeatedly and with different lenses in mind, promoting new ways for teachers to 'see' what is taking place."* (Sherin & Russ, 2015, p. 3). Hence, video-cases allow a case-related approach to PD, where PD activities are centered around authentic representations of practice, which is regarded as an important design principle for PD for teachers as well as for facilitators (Kuzle & Biehler, 2015; Roesken-Winter et al., 2015). However, video-cases are not by themselves sufficient to foster teacher and facilitator noticing: *"[...] using cases alone does not ensure learning, [...] adequate instructional support is needed"* (Goeze et al., 2014, p. 97; Kirschner et al., 2006). Video-cases have to be carefully embedded in PD programs. Research findings suggest that it is helpful if video-cases are combined with appropriate prompts that set a focus in order to guide teacher or facilitator noticing (Lesseig et al., 2017).

Noticing with Respect to Learning Mathematics with Technology

A nuanced and specific type of noticing is important for teaching with technology. Learning mathematics with technology comprises a subtle interplay between the mathematics, the technology and the learner (Trouche & Drijvers, 2010) and teachers will only integrate technology into their teaching in the long term, if they notice this subtle interplay and the potential of learning mathematics with technology. Furthermore, discovery learning tasks, such as the MRT-task introduced earlier (Fig. 1), require teachers to notice different ideas and individual approaches of learners, in order to adequately guide their learning. Moreover, the specific noticing of students' mathematical learning when working with technology is a prerequisite for offering adequate support for students, for example, by providing prompts, hints or questions during the teaching process (Sherman, 2012). For facilitators, specific noticing is not only needed with respect to *student* learning but also with respect to *teacher* learning. In particular, facilitators must support teachers to develop deep noticing of relevant aspects of teaching and learning mathematics with technology. For this, facilitators need to elaborately notice what teachers notice with respect to learning mathematics with technology.

A theoretical approach to describe the subtle processes of learning mathematics with technology is the theory of instrumental genesis (Guin & Trouche, 1999). Instrumental genesis describes the process of an artefact (e.g., a specific technology) becoming an instrument for doing and learning mathematics. However, the theory of instrumental genesis is an explanatory theory. It is not geared towards suggesting how to develop approaches to foster teacher noticing related to teaching mathematics with technology. In particular, the framework does not help to illuminate learning pathways or make the interplay of technology and the learning of mathematics easily accessible for teachers or PD facilitators. Therefore, we developed a framework that builds on the instrumental approach, but is explicitly suitable to analyze, depict and notice learning pathways when learning mathematics with technology within both teacher and facilitator PD programs. Our framework builds on that of Prediger (2019), which was developed in the context of analysing and describing learning pathways with respect to language responsive teaching. The model of Prediger (2019) highlights different categories for teachers' thinking and noticing (content goals, learners discourse practices, lexical means) and their interplay and distinctions. We adapted this framework in order to highlight connections between content goals, mathematical activities and technology use. The resulting Content-Activity-Technology-Model (CAT-Model) is depicted in Fig. 4.

Content goals refer to normative content goals that can be inferred from students' behavior. These content goals are often distinguished as conceptual and procedural knowledge, both regarded as an integral part of mathematical competence: "*Mathematical competence rests on developing both conceptual and procedural knowledge*." (Rittle-Johnson et al., 2015, p. 594). In the CAT-Model we acknowledge that conceptual and procedural knowledge cannot always be separated (Rittle-Johnson & Schneider, 2015) by conceptualising content goals on a continuum ranging from *procedural knowledge* to *conceptual knowledge*. The achieved content goals inferred from student's behavior in a specific situation might not necessarily reflect the anticipated or intended content goal. For example, students working on the MRT-task (Fig. 1) might not display any behavior that is indicative of conceptual knowledge even though the goal of the task is to activate and promote this type of knowledge.

Mathematical activities, refer to observable actions that allow inferences about the cognitive processes. These activities can be located on a continuum reflecting different levels of engagement (Anderson & Krathwohl, 2001; Biggs, 2003). *Procedural activities* relate to lower order activities, for example, if students mainly communicate on a phenomenological level, if students talk about what they are doing, or what they observe. *Conceptual activities* refer to higher order activities like students trying to explain mathematical concepts (where explanations do not necessarily have to be correct), argue about mathematical concepts, or formulate an hypothesis. Again, students might not display procedural or conceptual activities

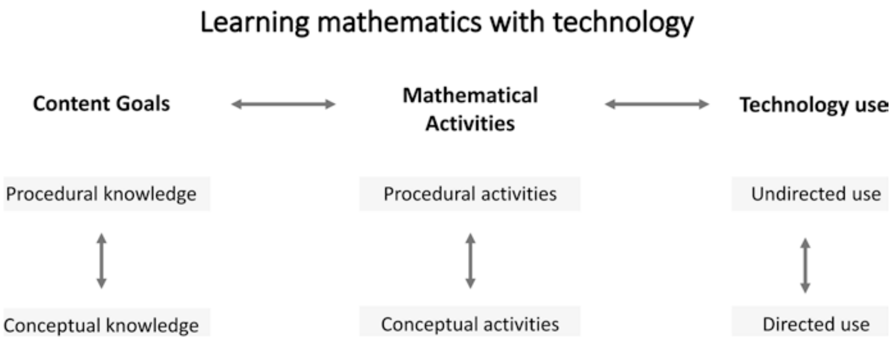


Fig. 4 The categories of the CAT-Model, their interplay (↔) and distinctions (↕)

even though the task or context was intended to do so. In addition, an activity/a task might provoke unintended mathematical activities.

Technology use is differentiated as undirected and directed use. Undirected use refers to students using the technology without a specific purpose, for example, if students apply an undirected trial and error approach or randomly drag a parameter slider. In contrast, *directed use* of technology refers to students deliberately using technology in their learning. This comprises, for example, the use of technology to test hypotheses or to systematically vary parameters to explore mathematical relationships. Hence directed and undirected use echoes a similar use of technology identified by Arzarello et al. (2002) with respect to dragging practices in a dynamical geometry environment.

We want to stress that the categories of the CAT-Model (Fig. 4) reflect a continuum (e.g., from procedural to conceptual knowledge or from procedural to conceptual activities). For example, a mathematical activity might not clearly be either a procedural activity (e.g., calculating a sum) or a conceptual activity (e.g., explaining a concept) but could also be an amalgam of both types of activities located somewhere in-between the two endpoints of the continuum.

The CAT-Model will be used in this paper for two purposes. On the one hand we use the CAT-Model to analyse and describe the learning pathways of a pair of students working with the MRT-task to show how the pair of students' progress from unaimed use through mathematical activities to conceptual understanding (Fig. 7). On the other hand, we use the CAT-Model to analyse what aspects (content goals, mathematical activities, technology use) teachers notice when they observe students working with the MRT-task.

3 Research Questions and Methodology

The outlined theoretical framework enables us to clarify the goals and questions that were informally presented in the introduction. In our study we focus on the constructs of teacher and facilitator noticing with respect to teaching mathematics with technology. The goal of our work was to develop video-case-based activities for teacher PD and facilitator PD that fosters teacher and facilitator noticing and explicitly take into account the connections between the classroom, teacher PD and facilitator PD levels. For design and research across the levels we used the 3 T-Model and the nesting and lifting strategy (see Sect. 2.3). We began the design at the classroom level and subsequently extended it to the teacher PD and facilitator PD level. In the following we detail the considerations regarding design and methodology and outline research questions and goals that were the focus at teacher PD and facilitator PD level. The complete design process and implementation of the teacher PD and facilitator PD program was situated within the German Center for Mathematics Teacher Education (DZLM), which is a nationwide university-based institute for developing and implementing high-quality PD for teachers and facilitators while attending to state-of-the-art design principles identified in the literature (e.g., Goldsmith et al., 2014).

Classroom Level

At the classroom level, we set out to identify a suitable video-case of students working with the MRT-task (Fig. 1). For this, six pairs of students were videod when solving the MRT-task (see Sect. 2.1, Fig. 1). Using the CAT-Model described in Sect. 2.4 (Fig. 4) we identified a student-video-case in which students proceed from unaimed use of technology to conceptual understanding. Details are outlined in Sect. 4.1.

Teacher PD Level

Design: At the teacher PD level our goal was to support teacher noticing with respect to students' learning processes when working with MRT. For this the classroom level tetrahedron consisting of students, classroom content (relationships between linear and quadratic functions) and classroom resources (MRT, MRT-Task) became the content of the teacher PD program (teaching with MRT, see Fig. 2). In the teacher PD program teachers first solved the MRT-task individually. Subsequently teachers watched the student-video-case (recorded at the classroom level, see above) and analysed it with respect to the learning processes of the students. Afterwards, a whole group discussion was held (moderated by a PD facilitator) to discuss what the teachers had noticed in the video-case. Details are outlined in Sect. 4.2. We videographed the whole group discussion and analysed it with respect to the following research questions (Details are outlined in Sect. 4.2):

Research questions:

- What do teachers notice in the student-video-case?
- What challenges can be identified on the teacher PD level?

Facilitator PD Level

Design: At the facilitator PD level the goal was to enable facilitators to support teacher noticing with respect to teaching mathematics with technology. For this the TPD level, which consists of teachers, teacher PD content (teaching with MRT) and teacher PD resources (MRT-task, MRT, student-video-case), became the content of the facilitator PD program (facilitating teaching with PD, see Fig. 2). In the facilitator PD program facilitators first solved the MRT-task individually and subsequently watched the student-video-case and analyzed it with respect to the learning processes that they notice. Afterwards a whole group discussion was held to discuss the what the facilitators had noticed. Hence, up to this point, the facilitators PD activities were identical to the activities of the teachers at the teacher PD level. Facilitators were then split up in groups of 4-5 persons and analyzed the teacher-video-case, which consisted of the teachers whole group discussion recorded at the teacher PD level (see above), with respect to what teachers had noticed. We recorded the facilitators during their small group discussions and analyzed the recordings with respect to the following research questions (Details are outlined in Sect. 4.3):

Research questions:

- What do facilitators notice in the teacher-video-case?
- What challenges can be identified on the facilitator PD level?

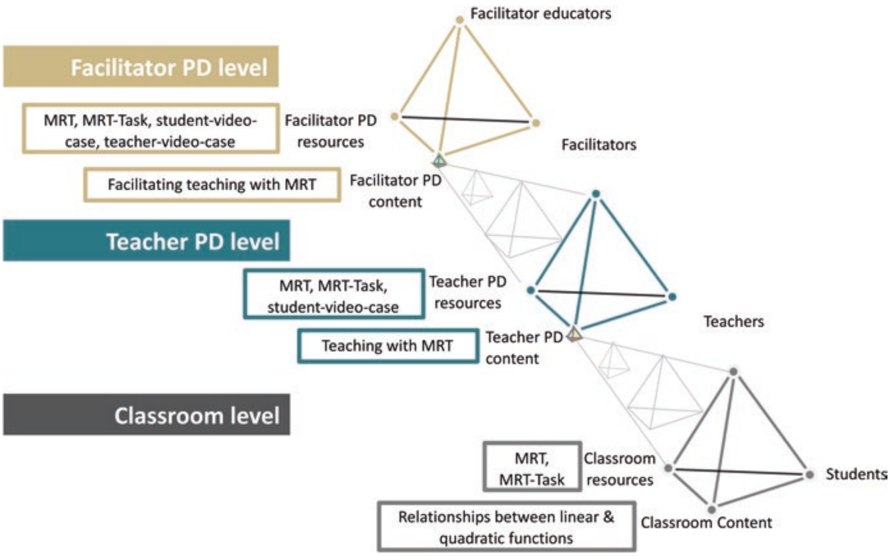


Fig. 5 Nesting of PD content in the 3 T-Model across the different levels

Lifting and Nesting Across Classroom, Teacher PD and Facilitator PD Level
Figure 5 gives an overview of the nested design. Aspects of the lower level tetrahedrons are nested as content within the higher level tetrahedrons. With respect to lifting, several design and research approaches are lifted. On one hand, solving the MRT-task was lifted from classroom level to both the teacher and facilitator PD level. In addition, the use of video-cases as means to support teacher noticing was lifted from the teacher to the facilitator PD level. In addition, the CAT-Model was lifted as a tool to support data analysis from the classroom level to teacher PD level. Moreover, the learning goals and the research question were lifted from the teacher to the facilitator PD level:

Learning Goals:

- Classroom Level: Support students to discover connections between linear and quadratic functions using MRT.
- Teacher PD level: Support teacher noticing with respect to the learning processes of students when working with MRT.
- Facilitator PD level: Support facilitators to be able to notice and support the noticing of teachers.

Participants and the PD Context
The teacher PD program was conducted in the German federal state "Schleswig-Holstein". The PD program comprised of eight one-day modules over a period of 18 months. Four of the modules were taught face-to-face and the remaining four were taught online. The PD program focused on supporting teaching mathematics with technology. Each module addressed a different type of technology (e.g., MRT,

videos, apps, audience response systems, learning management systems). Practical try-outs and reflection phases followed each module (a so-called "sandwich-model", Roesken-Winter et al. (2015)). In total 23 teachers from lower and upper secondary school participated in the PD program. In this paper we draw on data from the first module (taught face-to-face) which focused on teaching mathematics with MRT.

The facilitator PD program took place in the federal state of Hamburg. The program spanned approximately 1 year and was taught entirely online due to the COVID-pandemic. The program comprised three whole-day-sessions and approximately 1 week after each whole-day-session, an additional 2-h-session. In total 30 facilitators participated in the PD program. In this paper we draw on data from the first module that focused on supporting facilitators to conduct high-quality PD sessions for teaching mathematics with MRT. In this session the facilitators were first taught about basic design principles for high-quality PD programs (Roesken-Winter et al., 2015) and fundamentals about basic dimensions of high-quality teaching (Praetorius et al., 2017). Afterwards the facilitators engaged in several activities that centered around the video-case as described in Sect. 4.3.

Both the teacher and facilitator PD program were designed and delivered by the German Center for Mathematics Teacher Education (DZLM).

4 Research-Based Design of the Video-Case-Based Activity and Related Findings

In this section we detail the design and research process outlined in the previous section and provide the results of the analysis conducted at each level. The section first addresses the classroom level in Sect. 4.1 and continues with the teacher and facilitator PD level in Sects. 4.2 and 4.3.

4.1 Classroom Level

Six pairs of students were videorecorded when working on the MRT-task. Based on these recordings we set out to identify an excerpt of the videos to be used in the teacher and facilitator PD program that was not too long (less than 5 min) (Krammer et al., 2008) and had a clear focus on mathematical learning of the students. This meant that the student-video-case should show a noticeable learning process related to the mathematics under focus. In an initial screening of the recordings, a set of potentially suitable video-cases were identified. Expert ratings were then used to evaluate the potential of these video-cases for the use in PD programs. In particular interviews were conducted with experts in relation to what they notice about the interplay of student learning and technology use. Two video-cases that fitted best were subsequently analyzed in depth using the CAT-Model (see Fig. 4). As an often-voiced apprehension of teachers is that technology use in the mathematics classroom does not go beyond undirected use (Mackey, 1999; Thurm, 2017; Thurm & Barzel, 2020), we selected a video-case that showed how students had progressed from undirected use to that which concerned conceptual knowledge. In the following we give a description of the learning pathway of the students that featured in this video-case, which is also depicted in Fig. 7 through the lens of the CAT-Model.

Student-Video-Case of Lara and Rose

The video-case shows two students, named Lara and Rose, working on the first part of the MRT-Task (see Fig. 1, first bullet). They begin by using the MRT's drag mode to change the slope of the linear function f1. They start to randomly change the slope, and by this they manage to invert the opening of the parabola, however they did not notice this as the MRT viewing window only shows a part of the graph. They only see a part of the inverted parabola that looks similar to a slightly curved line (see Fig. 6, left). Next, they start to change the slope of the function f2 using the drag mode. Finally, they try to adjust the viewing window to get a better view, but they do not manage to achieve this. They return to the starting situation by changing the parameters back to the original values. Subsequently they develop the hypothesis that the two linear functions have to be mirrored by the x-axis, so that the resulting parabola is also mirrored at the x-axis. They start to enter two new functions f4 and f5 which are the mirrored functions of f1 and f2. Then they enter a new function f6 = f4·f5. Now they realise that their function f5 is the same as the function f3 = f1·f2 (see Fig. 6, right). They are surprised and suddenly realise the reason for this. They argue algebraically that if they multiply both functions by (−1) the result is positive. The video-case has an open ending in which the students propose a new idea. They hypothesise that mirroring the two linear functions at the first bisector will lead to an inverted parabola.

Although the students do not find a solution to the original problem (to reverse the opening of the parabola) during this video-case, the sequence shows the learning process begins with an undirected use of technology, followed by setting up an

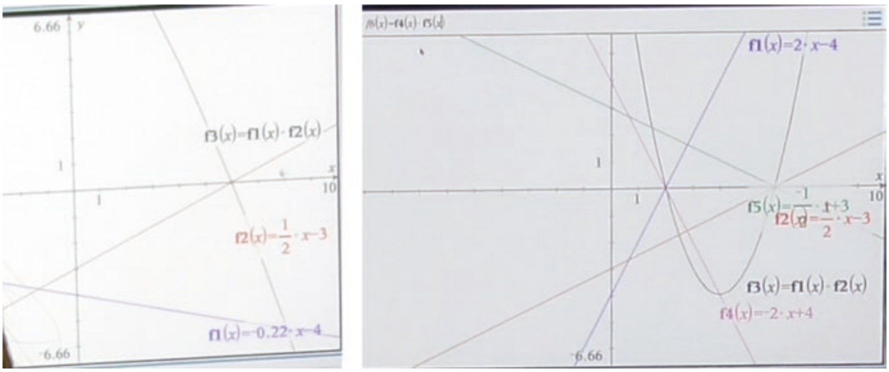


Fig. 6 Screenshots of Lara and Rose's screen when working with the MRT-task

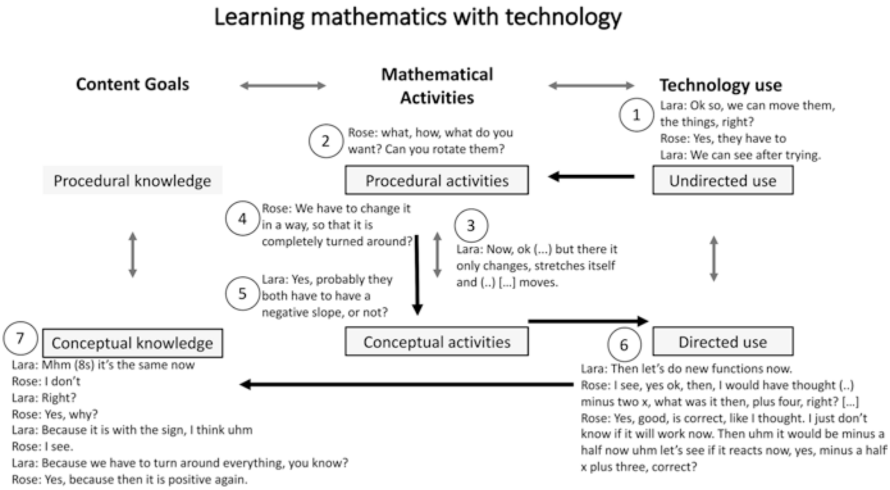


Fig. 7 Learning pathway of Lara and Rose analyzed with the CAT-Model

hypothesis with is subsequently falsified while gaining conceptual insights. This is an example of a process of student instrumental genesis as an interplay between mathematical activities, the use of technology and mathematical thinking. Therefore, it offers the potential to be used in order to specify, capture and reflect on teacher noticing.

4.2 Teacher PD Level

The student-video-case is an authentic case, which shows how even an initially unaimed use of technology can lead to the development of conceptual knowledge. However, to use any student-video-case in teacher PD programs, the design of prompts that can guide teacher noticing is crucial (see Sect. 2.4). Therefore, crafted suitable discussion prompts to support the student-video-case. The preliminary work of Ebers (2020b) has shown that discussion prompt such as "*Discuss the scene with respect to the interaction between cognitive activation of the students and using the MRT.*" is too general and can be further enhanced by more detailed prompts. Therefore, we explicitly focused the discussion prompts in the following way:

> *Discuss the scene with respect to the interaction between cognitive activation of the students and using the MRT. The following question might help you:*
>
> 1. *What are the solution approaches of the students? Do they reflect or alter their thinking?*
> 2. *Which visual prompts do you notice that result from students interacting with the MRT? Which prompts are used and which prompts are not used by the students? What are possible reasons for not using?*
> 3. *Which obstacles do you notice?*

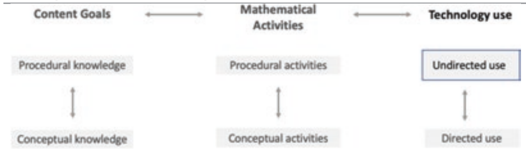
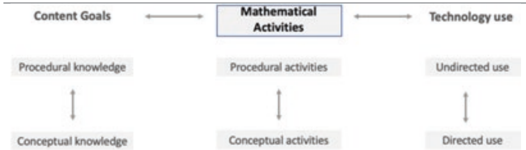
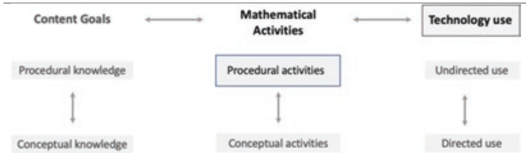
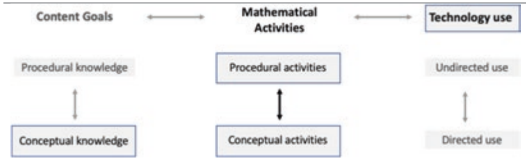
In the teacher PD program, the teachers first solved the MRT task on their own and subsequently analysed the student-video-case in small groups. After finishing the small group work, a whole group discussion, which was moderated by a PD facilitator, was held to summarise the teachers' findings. The whole group discussion had a duration of 6:45 min. In total 7 statements were made by 6 teachers. Two statements that were made by the same teacher were subsequently grouped together as one unit of analysis. We transcribed the group discussion and analyzed the statements of the teachers according to levels and categories of noticing. Levels of noticing were analyzed using the adapted framework based on the work of van Es (2011) (see Sect. 2.4). Categories of noticing (capturing what teachers focused on) were analyzed using the CAT-Model described in Sect. 2.4 (Fig. 4). The analysis of the statements was carried out by 10 mathematics education researchers. At first the researchers individually coded each teacher statement. Afterwards the researchers paired up to discuss differences in their coding. However, there was mostly agreement among the raters. Each pair of researchers then generated a joint coding of the video-case. The resulting coding were then checked for differences and combined to produce a joint coding.

With respect to levels of noticing, the analysis revealed that teachers showed quite different levels of noticing (see Table 2). Three teachers showed baseline noticing (level 1). With respect to categories of noticing two of these teachers focused only on the undirected use of technology while one teacher focused very generally on mathematical activities without paying attention to particular aspects. One teacher showed mixed noticing (level 2). This teacher started to attend to particular aspects of the students' learning but focused strongly on the part where students did not manage to change the viewing window. Finally, two teachers showed focused noticing (level 3). Table 2 gives examples of the results for four of the teachers.

4.3 Facilitator PD Level

At the facilitator PD level, we used the teacher-video-case (i.e., the group discussion that was recorded at the teacher PD level, see Sect. 4.2) as means to support facilitator noticing. Clearly facilitators need to be able to notice and deal with the previously identified heterogenity of levels and categories of teacher noticing. In order to focus facilitator noticing when working with the teacher-video-case we generated specific prompts. An important distinction between the student-video-case and the teacher-video-case is that the student-video-case did not show the teacher, but the teacher-video-case shows the actions of the facilitator that moderates the group discussion of the teachers. Hence the teacher-video-case can be used in two ways. On one hand the teacher-video-case can be used to support the noticing of facilitators with respect to the different levels and categories of teacher noticing in the video. On the other hand, the teacher-video-case can also be used to help facilitators analyse the facilitator moves and how these moves support or hinder teacher learning

Table 2 Analysis of levels and categories of teacher noticing

Level 1 – Baseline noticing	
Categories of noticing	Excerpts from teacher
	*"They have no guidance in the program which pushes them slowly in the right direction, it is just wild – there are two nonmathematicians sitting together who just push around wildly and even if they manage to get the result, they cannot find out how and why they got the result"*
	*"I think a lot of things have been mathematised here, but the question is, was it useful for this task? Well, they talked, but if the goal of the task is that they should be able to deal with parabolas and the parameters of the parabola – they didn't manage to do that […]"*
Level 2 – Mixed noticing	
Categories of noticing	Excerpts from teacher
	*"I think that it is not just a trial and error, they have useful ideas, they remember that a parabola is opened downwards if somehow a minus comes into play. What they can't do is to reflect and question their mistakes. […] they managed to turn the parabola but didn't realise it. They didn't adjust the image section that way, and that was a pity."*
Level 3 – Focused noticing	
Categories of noticing	Excerpts from teacher
	*"Because they found out when they entered the mirrored function equations, oh yes, this is the same parabola again although I have different straight lines now, oh yes, but I put a minus in front of it and I think they say "minus times minus is plus" and therefore we have the same parabola again."* *"[…] what could be better than what the students do, that they test an assumption and interpret and justify […]"*

(Schueler & Roesken-Winter, 2018). Therefore, we deliberately distinguished the prompts for the facilitators in two parts. The first discussion prompt asked facilitators to pay attention to the teachers in the teacher-video-case. The second discussion prompt asked facilitators to pay attention to the facilitator in the teacher-video-case.

In the facilitator PD program, the facilitators first solved the MRT task on their own and then analysed the student-video-case of Lara and Rose in the same way as the teachers within the PD level. In the next PD meeting, which was one week later, the facilitators worked in small groups and analysed the teacher-video-case with respect to the two discussion prompts. Two small group discussions of the facilitators were transcribed. Each of the facilitators' statements was coded by using focus codes and stance codes following Sherin and van Es (2009) and Lesseig et al. (2017).

Focus Codes

Focus codes captured the facilitators' attentions when watching the teacher-video-case. For this we extended the categories of Sherin and van Es (2009) and Lesseig et al. (2017) to account for the nested structure of our PD design. While Sherin and van Es (2009) distinguish between "student", "teacher" and "other" as possible foci with respect to student-video-cases, Lesseig et al. (2017) distinguished between "video-case teachers" "video-case facilitators" and generic codes like "PD in general" or "Self". Since our design has a nested structure, we structured the focus codes according to the TPD and classroom levels. Table 3 gives an overview of the focus codes used to code the teacher-video-case. Each of the facilitators' statements was rated with one *main-focus-code*, which captured what the facilitators mainly focused on in a statement. In addition, each statement could have multiple *sub-focus-codes* which captured which aspects facilitators made connections to in their statement.

Stance Codes

Following Lesseig et al. (2017), stance codes captured whether facilitators described, approved, disapproved, interpreted, speculated (framed comments as wonderings rather than declarative statements) or extended (considered other settings or alternatives).

The coding of the transcripts was carried out by two of the authors. First, the researchers individually coded each transcript. Then, they worked in pairs to discuss differences in their coding. Table 4 shows how often each focus-code was a main-focus, or a sub-focus of the facilitators' discussion. Table 5 shows the distribution of

Table 3 Overview of focus codes

Teacher PD level focus	Classroom level focus	Generic focus
Focus on the teachers	Focus on the task	Students in general
Focus on the facilitator	Focus on the students	Teachers in general
	Focus on the technology	Facilitators in general
		PD in general
		Technology in general
		Self

Table 4 Distribution of main- and sub-focus-codes (VC=video-case)

	Teacher-VC-facilitator	Teacher-VC-teachers	Student-VC-students	Student-VC-task	Student-VC-technology	Teachers in general	Facilitators in general	PD in general	Self
Main-focus	28	5	0	1	0	0	3	2	1
Sub-focus	6	17	3	1	7	3	0	2	3
Total	34	22	3	2	7	3	3	4	4
	56		12			14			

Table 5 Distribution of stance codes

Describe	Approve	Disapprove	Interpret	Speculate	Extend
12	8	8	11	5	9

stance codes across all statements. Clearly, facilitators most of the times focused on the teacher-video-case facilitator, while they rarely focused on the teacher-video-case teachers or on aspects of the classroom level. With respect to the sub-focus, facilitators most often referred to teacher-video-case teachers in their statements. The distribution of stance codes reveals that facilitators often described (29%) and evaluated (28%), while interpreting (19%), speculating (8%) and extending (16%) occurred less often. Since interpreting and imagining alternatives are indicators of more productive noticing compared to simply describing or evaluating (approving, disapproving) (Lesseig et al., 2017) it can be concluded that facilitators' noticing could be further improved.

As mentioned, the facilitators put their main focus on the teacher PD level, in particular on the facilitator who led the teacher-PD-discussion. In order to better understand the topics of the facilitators' discussions, we briefly summarise this for the two most frequently addressed main-focus-codes, namely, teacher-video-case facilitator and teacher-video-case teachers.

Discussion focused on Teacher-Video-Case Facilitator

In the teacher-video-case the facilitator leading the discussion makes two statements during the whole video-case. These statements were both made in response to statements made by the teachers that criticised the unaimed technology use by the students in the student-video-case (level 1 teacher noticing, see Sect. 4.2). The teacher-video-case began with a teacher heavily criticising the unaimed technology use, to which the teacher-video-case facilitator replies: *"I would like to discuss this. To say it carefully that is not completely my opinion. It is a big fear that one can have and we can discuss this a bit"*. This is followed by other teacher-video-case teachers replying with higher-level noticing comments (level 2 and 3, see Sect. 4.2) before another teacher supports the critique concerning the unaimed technology use of the students. This leads to the following reaction of the teacher-video-case facilitator: *"But they didn't just push around, we have heard this from different sides now"*. Facilitators in the PD program heavily discussed whether this intervention of the teacher-video-case facilitator was appropriate or not. Some facilitators argued that a facilitator should generally not judge in this way. Others highlighted explicitly the necessity of such judgements: *"If the first comment is already so devastating, it could then escalate and then it becomes difficult to get it [the discussion] back and she [the facilitator] gets out of this situation right away."*

Discussion focused on Teacher-Video-Case Teachers

During the limited times where the teacher-video-case teachers were the main focus for teacher noticing, facilitators in the PD program mostly described statements of the teachers and superficially related them to the student-video-case-students and student-video-case technology use. Sometimes there was an evaluation of the teacher-video-case teachers' statements, but this did not lead to a deeper discussion among the facilitators.

5 Discussion

In this paper we set out to design a research-based PD activity for teachers and facilitators in a way that accounts for and takes advantage of the connections between the classroom, teacher PD and facilitator PD levels. In the following we discuss the main findings across these three levels.

Classroom and Teacher PD Level

The developed CAT-Model proved very suitable for identifying potential student-video-cases to be used in the teacher and facilitator PD program. In addition, the CAT-Model served as a helpful lens through which teacher noticing could be analysed. The analysis of teacher noticing showed that the student-video-case was well suited to distinguish the different levels and categories. Crucial to this was the student-video-case which showed a complex learning pathway for a pair of students that moved from unaimed use across different levels of mathematical activities to directed use and conceptual understanding. Furthermore we hypothesise that the high heterogeneity of noticing that was found at the teacher PD level is likely to occur frequently in PD programs for teaching mathematics with technology and provides a challenge for PD facilitators. Facilitators must be aware of this possible heterogeneity and the results displayed in Table 2 illustrate the many different levels and categories of teacher noticing. These could be integrated into the design of future facilitator PD programs to inform facilitators about the different starting points of teachers with respect to noticing. In addition, the heterogeneity of teacher noticing shows that even focused discussion prompts do not necessarily trigger all teachers to notice students learning on a deeper level. We doubt that further refinement of the discussion prompts *will* substantially enhance the depth of teachers noticing. Rather we hypothesise that teachers need more specific tools to scaffold their noticing. One such way could be to ask teachers to analyse the student-video-case using the CAT-Model. If the categories of the model are explained beforehand, teachers could be invited to reconstruct the learning pathways of the students using the CAT-Model.

Facilitator PD Level

At the facilitator PD level our analysis revealed that facilitator noticing centered mainly on the facilitation moves of the teacher-video-case facilitator. We did not expect such a strong focus on the teacher-video-case facilitator, as in the teacher-video-case she only makes two very brief statements. The strong focus on the teacher-video-case facilitator limited an in-depth discussion about the teacher noticing in the teacher-video-case. The strong focus on the facilitator in the teacher-video-case may be due to a high identification of the facilitators with their own role. Another explanation is that focusing on their peers is more familiar to facilitators than focusing on teachers' learning. Such a phenomenon of a one-sided focus has been previously described in the literature with respect to student-video-cases used in teacher PD courses (Goeze et al., 2014; Hogan et al., 2003). Hence, our results suggest that this problem is lifted to the facilitator PD level. Facilitators did not explicitly discuss different levels of teacher-video-case teacher noticing and how it is related to student-video-case students' learning, despite the fact that the first discussion prompt for the facilitators was explicitly focusing on this aspect. Spoken with the language of the 3 T-Model, facilitators did not manage to focus on lower level tetrahedrons but rather took the perspective that is nearest to their own role, the PD facilitator. From this we conclude that the design of the facilitator PD video-case might be refined in two ways:

- In the implemented design we administered the discussion prompts focusing on teacher-video-case teacher and teacher-video-case facilitator at the same time. It might be beneficial to solely provide the discussion prompt focusing on the teacher-video-case teachers first while explicitly stressing that facilitators should ignore the teacher-video-case facilitator.
- Another possibility would be to generate a teacher-video-case which does not include a facilitator. A drawback of this approach would be that different teacher-video-cases would be needed to discuss teacher-video-case teacher noticing and teacher-video-case facilitator moves.

In the few instances where the facilitators did actually focus on the teacher-video-case teachers, our findings suggest that the facilitators did not achieve elaborate levels of noticing. Hence facilitators are also likely to need specific tools that scaffold their noticing. One way to scaffold facilitator noticing when working with teacher-video-cases could be to ask facilitators to analyse the teacher-video-case according to the levels and categories of noticing using the CAT-Model and the table that operationalises the different levels of noticing (see Table 1), in the same way that we did.

Finally, our study brings to the fore the role of PCK-PD. Just as teachers' PCK is related to teacher noticing (Schoenfeld, 2011), so facilitator noticing is related to facilitators' PCK-PD (Peters-Dasdemir et al., 2020). Hence, it is necessary to improve PCK-PD, for example, by making different levels of teacher noticing with respect to teaching with technology (Table 2) an explicit content of facilitator PD.

Taking the aforementioned points together, we highlight the following two tentative design-principles for the design of PD programs for facilitators in the context of teaching mathematics with technology:

– Provide information to facilitators about various levels and aspects of teacher noticing. In particular, the CAT-Model and the examples given in Table 2 can be used to inform PD facilitators about different aspects of teacher noticing and help facilitators to analyse such teacher noticing.
– Video-cases for facilitators should stimulate facilitators to notice relevant aspects at different levels (facilitator level, teacher level, classroom level). Tools such as

the CAT-Model should be used in PD programs for facilitators in order to help them understand and notice different levels of teacher noticing.

6 Conclusion

Research on the design and implementation of PD for facilitators is still limited (Thurm et al., 2023). In this chapter we detail an approach for designing a video-case-based strategy to support both teacher and facilitator noticing. The design and research across the different levels of PD (classroom level, teacher PD level, facilitator PD level) highlights the subtle issues that have to be considered at each level, and how research and design decisions on one level impact the other levels. We have identified different levels and categories of teacher noticing with respect to teaching mathematics with technology. We also found that facilitator noticing when working with teacher-video-cases may only be limited to a focus on the facilitator in the teacher-video-case and may not extend to the classroom and teacher PD levels. In addition, we found that even carefully crafted discussion prompts may not be sufficient to focus teacher and facilitator noticing on relevant aspects. Rather we suggest to use the CAT-Model and the taxonomy of levels and categories of noticing not only as research tools (as we exemplify in this paper) but also as “PD-tools” that can help to scaffold teacher and facilitator noticing when working with video-cases. In this sense we argue that facilitators might need to take more of a research stance by adopting similar analytic tools as researchers. A prerequisite for this is that such tools are complex enough to capture the relevant aspects from the research perspective, but at the same time be accessible and usable not only for researchers but also for a wide range of teachers and facilitators.

We are aware that the work reported in this paper is only the first step to extend the knowledge about design and research on adequate facilitator PD programs in the area of teaching with technology. The complexity lies in the fact that classroom level, teacher PD level and facilitator PD level are inevitably intertwined and tools and theories are needed to adequately address this complexity both at the research and the design level.

References

Aldon, G., & Trgalová, J. (2017). Proceedings of the 13th international conference on Technology in Mathematics Teaching. Technology in Mathematics Teaching ICTMT 13 Ecole Normale Supérieure de Lyon/Université Claude Bernard Lyon 1 3 to 6 July, 2017.

Anderson, L. W., & Krathwohl, D. R. (Eds.). (2001). *A taxonomy for learning, teaching, and assessing: A revision of Bloom’s taxonomy of educational objectives*. Allyn & Bacon.

Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A cognitive analysis of dragging practises in Cabri environments. *Zentralblatt für Didaktik der Mathematik, 34*(3), 66–72.

Ball, L., Drijvers, P., Ladel, S., Siller, H. S., Tabach, M., & Vale, C. (2018). *Uses of technology in primary and secondary mathematics education*. Springer.

Barzel, B., & Möller, R. (2001). About the use of the TI-92 for an open learning approach to power functions. *Zentralblatt für Didaktik der Mathematik, 33*(1), 1–5.

Barzel, B., Bebernik, R., Göbel, L., Pohl, M., Ruchniewicz, H., Schacht, F., & Thurm, D. (2020). *Proceedings of the 14th international conference on Technology in Mathematics Teaching – ICTMT 14: Essen, Germany, 22nd to 25th of July 2019*. https://doi.org/10.17185/duepublico/48820.

Beattie, H. L., Ren, L., Smith, W. M., & Heaton, R. M. (2017). Measuring elementary mathematics teachers' noticing: Using child study as a vehicle. In E. O. Schack, M. H. Fisher, & J. Wilhelm (Eds.), *Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks* (pp. 321–338). Springer.

Biggs, J. (2003). *Teaching for quality learning at university* (2nd ed.). The Society for Research into Higher Education and Open University Press.

Borko, H. (2004). Professional development and teacher learning: Mapping the terrain. *Educational Researcher, 33*(8), 3–15.

Borko, H., & Potari, D. (Eds.). (2020). *Teachers of mathematics working and learning in collaborative groups* (ICMI STUDY 25 conference proceedings). University of Lisbon.

Borko, H., Koellner, K., & Jacobs, J. (2014). Examining novice teacher leaders' facilitation of professional development. *Journal of Mathematical Behavior, 33*, 149–167.

Borko, H., Carlson, J., Deutscher, R., Boles, K. L., Delaney, V., Fong, A., … Villa, A. M. (2021). Learning to Lead: An approach to mathematics teacher leader development. *International Journal of Science and Mathematics Education*, 1–23.

Clark-Wilson, A., & Hoyles, C. (2019). A research-informed web-based professional development toolkit to support technology-enhanced mathematics teaching at scale. *Educational Studies in Mathematics, 102*(3), 343–359.

Clark-Wilson, A., Robutti, O., & Sinclair, N. (2014). *The mathematics teacher in the digital era: An international perspective on technology focused professional development*. Springer.

Clark-Wilson, A., Robutti, O., & Thomas, M. (2020). Teaching with digital technology. *ZDM Mathematics Education, 52*, 1223–1242. https://doi.org/10.1007/s11858-020-01196-0

Drijvers, P. (1994). Graphics calculators and computer algebra systems: Differences and similarities. In H. Heugl & B. Kutzler (Eds.), *Derive in education: Opportunities and strategies* (pp. 189–200). Chartwell-Bratt.

Drijvers, P., Doorman, M., Boon, P., Reed, H., & Gravemeijer, K. (2010). The teacher and the tool: Instrumental orchestrations in the technology-rich mathematics classroom. *Educational Studies in Mathematics, 75*(2), 213–234.

Drijvers, P., Ball, L., Barzel, B., Heid, M. K., Cao, Y., & Maschietto, M. (2016). *Uses of Technology in Lower Secondary Mathematics Education: A concise topical survey*. Springer.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*(1–2), 103–131.

Ebers, P. (2020a). Videofälle zur Vorbereitung von Lehrkräften auf das Unterrichten mit digitalen Medien. In H.-S. Siller, W. Weigel, & J. F. Worler (Eds.), *Beiträge zum Mathematikunterricht 2020* (pp. 1297–1300). WTM-Verlag.

Ebers, P. (2020b). Development of video cases regarding technology use for professional development programs. In B. Barzel, R. Bebernik, L. Göbel, M. Pohl, H. Ruchniewicz, F. Schacht, & D. Thurm (Eds.), *Proceedings of the 14th International Conference on Technology in Mathematics Teaching – ICTMT 14: Essen, Germany, 22nd to 25th of July 2019* (pp. 287–288). https://doi.org/10.17185/duepublico/48820

Elliott, R., Kazemi, E., Lesseig, K., Mumme, J., Carroll, C., & Kelley-Petersen, M. (2009). Conceptualizing the work of leading mathematical tasks in professional development. *Journal of Teacher Education, 60*(4), 364–379.

Ertmer, P. A., & Ottenbreit-Leftwich, A. T. (2010). Teacher technology change: How knowledge, confidence, beliefs, and culture intersect. *Journal of Research on Technology in Education, 42*(3), 255–284.

Goeze, A., Zottmann, J. M., Vogel, F., Fischer, F., & Schrader, J. (2014). Getting immersed in teacher and student perspectives? Facilitating analytical competence using video cases in teacher education. *Instructional Science, 42*(1), 91–114. https://doi.org/10.1007/s11251-013-9304-3

Goldsmith, L. T., Doerr, H. M., & Lewis, C. C. (2014). Mathematics teachers' learning: A conceptual framework and synthesis of research. *Journal of Mathematics Teacher Education, 17*(1), 5–36.

Goodwin, C. (1994). Professional vision. *American Anthropologist, 96*(3), 606–633.

Grugeon, B., Lagrange, J.-B., Jarvis, D., Alagic, M., Das, M., & Hunscheidt, D. (2010). Teacher education courses in mathematics and technology: Analyzing views and options. In C. Hoyles & J.-B. Lagrange (Eds.), *Mathematics education and technology—Rethinking the terrain* (pp. 329–345). Springer.

Guin, D., & Trouche, L. (1999). The complex process of converting tools into mathematical instruments: The case of calculators. *The International Journal of Computers for Mathematical Learning, 3*(3), 195–227.

Hegedus, S., Laborde, C., Brady, C., Dalton, S., & Siller, H.-St., Tabach, M., Trgalova, J., & Moreno-Armella, L. (2017). *Uses of Technology in Upper Secondary Mathematics Education*. Springer.

Heid, M. K., & Blume, G. W. (Eds.). (2008). *Research on technology and the teaching and learning of mathematics: Volume 1: Research synthesis*. Information Age Publishing.

Hogan, T., Rabinowitz, M., & Craven, J. A., III. (2003). Representation in teaching: Inferences from research of expert and novice teachers. *Educational Psychologist, 38*(4), 235–247. https://doi.org/10.1207/S15326985EP3804_3

Hoyles, C., & Lagrange, J. B. (Eds.). (2010). *Mathematics education and technology: Rethinking the terrain*. Springer.

Jacobs, J., Seago, N., & Koellner, K. (2017). Preparing facilitators to use and adapt mathematics professional development materials productively. *International Journal of STEM Education, 4*(30), 1–14.

Kilic, H. (2018). Pre-service mathematics teachers' noticing skills and scaffolding practices. *International Journal of Science and Mathematics Education, 16*(2), 377–400.

Kirschner, P. A., Sweller, J., & Clark, R. E. (2006). Why minimal guidance during instruction does not work: An analysis of the failure of constructivist, discovery, problem-based, experiential, and inquiry-based teaching. *Educational Psychologist, 41*(2), 75–86. https://doi.org/10.1207/s15326985ep4102_1

Koc, Y., Peker, D., & Osmanoglu, A. (2009). Supporting teacher professional development through online video case study discussions: An assemblage of preservice and inservice teachers and the case teacher. *Teaching and Teacher Education, 25*, 1158–1168.

Krammer, K., Schnetzler, C. L., Ratzka, N., Reusser, K., Pauli, C., Lipowsky, F., & Klieme, E. (2008). Lernen mit Unterrichtsvideos: Konzeption und Ergebnisse eines netzgestützten Weiterbildungsprojekts mit Mathematiklehrpersonen aus Deutschland und der Schweiz. *Beiträge zur Lehrerbildung, 26*(2), 178–197.

Kuzle, A., & Biehler, R. (2015). Examining mathematics mentor teachers' practices in professional development courses on teaching data analysis: Implications for mentor teachers' programs. *ZDM, 47*(1), 39–51.

Lee, M. Y., & Choy, B. H. (2017). Mathematical teacher noticing: The key to learning from lesson study. In E. Schack, M. Fisher, & J. Wilhelm (Eds.), *Teacher Noticing: Bridging and broadening perspectives, contexts, and frameworks. Research in mathematics education* (pp. 121–140). Springer.

Lesseig, K., Elliott, R., Kazemi, E., Kelley-Petersen, M., Campbell, M., Mumme, J., & Carroll, C. (2017). Leader noticing of facilitation in videocases of mathematics professional development. *Journal of Mathematics Teacher Education, 20*(6), 591–619.

Leufer, N., Prediger, S., Mahns, P., & Kortenkamp, U. (2019). Facilitators' adaptation practices of curriculum material resources for professional development courses. *International Journal of STEM Education, 6*(1), 1–18.

Mackey, K. (1999). Do we need calculators? In Z. Usiskin (Ed.), *Mathematics education dialogues* (p. 3). NCTM.

Peters-Dasdemir, J., Holzäpfel, L., Barzel, B., & Leuders, T. (2020). Professionalization of facilitators in mathematics education: A competency framework. In M. Inprasitha, N. Changsri, & N. Boonsena (Eds.), *Proceedings of the 44th conference of the International Group for the Psychology of mathematics education (interim volume)* (pp. 457–465). PME.

Pierce, R., & Stacey, K. (2010). Mapping pedagogical opportunities provided by mathematics analysis software. *International Journal of Computers for Mathematical Learning, 15*(1), 1–20.

Poehler, B. (2020). Role of facilitators in supporting teacher collaboration during PD courses on language-responsive mathematics teaching. In H. Borko & D. Potari (Eds.), *ICMI study 25: Teachers of mathematics working and learning in collaborative groups* (pp. 516–523). University of Lisbon.

Praetorius, A.-K., McIntyre, N. A., & Klassen, R. M. (2017). Reactivity effects in video-based classroom research: An investigation using teacher and student questionnaires as well as teacher eye-tracking. *Zeitschrift für Erziehungswissenschaft, 20*(1), 49–74.

Prediger, S. (2019). Investigating and promoting teachers' expertise for language-responsive mathematics teaching. *Mathematics Education Research Journal, 31*(4), 367–392.

Prediger, S., Roesken-Winter, B., & Leuders, T. (2019). Which research can support PD facilitators? Strategies for content-related PD research in the three-tetrahedron model. *Journal of Mathematics Teacher Education, 22*(4), 407–425.

Psycharis, G., & Kalogeria, E. (2018). Studying the process of becoming a teacher educator in technology-enhanced mathematics. *Journal of Mathematics Teacher Education, 21*(6), 631–660.

Ratnayake, I., Thomas, M., & Kensington-Miller, B. (2020). Professional development for digital technology task design by secondary mathematics teachers. *ZDM, 52*(7), 1423–1437.

Rittle-Johnson, B., & Schneider, M. (2015). Developing conceptual and procedural knowledge of mathematics. In R. C. Kadosh & A. Dowker (Eds.), *Oxford handbook of numerical cognition* (pp. 1118–1134).

Rittle-Johnson, B., Schneider, M., & Star, J. R. (2015). Not a one-way street: Bidirectional relations between procedural and conceptual knowledge of mathematics. *Educational Psychology Review, 27*(4), 587–597.

Roesken-Winter, B., Schueler, S., Stahnke, R., & Bloemeke, S. (2015). Effective CPD on a large scale: Examining the development of multipliers. *ZDM, 47*(1), 13–25.

Roesken-Winter, B., Stahnke, R., Prediger, S., & Gasteiger, H. (2021). Towards a research base for implementation strategies addressing mathematics teachers and facilitators. ZDM–mathematics. *Education*, 1–13.

Schoenfeld, A. (2011). Noticing matters. A lot. Now what? In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 3–14). Routledge.

Schueler, S., & Roesken-Winter, B. (2018). Compiling video cases to support PD facilitators in noticing productive teacher learning. *International Journal of STEM Education, 2018*(5), 50. https://doi.org/10.1186/s40594-018-0147-y

Sherin, M. G. (2004). New perspectives on the role of video in teacher education. In J. Brophy (Ed.), *Using video in teacher education* (pp. 1–27). Elsevier Science.

Sherin, M. G. (2007). The development of teachers' professional vision in video clubs. In R. Goldman, R. Pea, B. Barron, & S. J. Deny (Eds.), *Video research in the learning sciences* (pp. 383–395). Erlbaum.

Sherin, M. G., & Russ, R. S. (2015). Teacher noticing via video. In B. Calandra & P. J. Rich (Eds.), *Digital video for teacher education* (pp. 3–20). Routledge.

Sherin, M. G., & van Es, E. A. (2009). Effects of video club participation on teachers' professional vision. *Journal of Teacher Education, 60*(1), 20–37.

Sherin, M. G., Jacobs, V., & Philipp, R. (2011a). Situating the study of teacher noticing. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 3–14). Routledge.

Sherin, M. G., Russ, R. S., & Colestock, A. (2011b). Accessing mathematics teachers' in-the-moment noticing. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing; seeing through teachers' eyes* (pp. 79–94). Routledge.

Sherman, M. F. (2012). Supporting students' mathematical thinking during technology-enhanced investigations using DGS. In D. Martinovic, D. McDougall, & Z. Karadag (Eds.), *Technology in mathematics education: Contemporary issues* (pp. 147–182). Informing Science Institute.

Stahnke, R., Schueler, S., & Roesken-Winter, B. (2016). Teachers' perception, interpretation, and decision-making: A systematic review of empirical mathematics education research. *ZDM, 48*(1–2), 1–27.

Thomas, M. O. J., & Palmer, J. (2014). Teaching with digital technology: Obstacles and opportunities. In A. Clark-Wilson, N. Sinclair, & O. Robutti (Eds.), *The mathematics teacher in the digital era* (pp. 71–89). Springer.

Thurm, D. (2017). Psychometric evaluation of a questionnaire measuring teacher beliefs regarding teaching with technology. In B. Kaur, W. K. Ho, T. L. Toh, & B. H. Choy (Eds.), *Proceedings of the 41st Conference of the International Group for the Psy-chology of Mathematics Education (Bd. 4)* (pp. 265–272).

Thurm, D. (2020). *Digitale Werkzeuge im Mathematikunterricht integrieren: Zur Rolle von Lehrerüberzeugungen und der Wirksamkeit von Fortbildungen*. Springer.

Thurm, D., & Barzel, B. (2020). Effects of a professional development program for teaching mathematics with technology on teachers' beliefs, self-efficacy and practices. *ZDM Mathematics Education, 52*, 1411–1422. https://doi.org/10.1007/s11858-020-01158-6

Thurm, D., & Barzel, B. (2021). *Teaching mathematics with technology: A multidimensional analysis of teacher beliefs and practice*. Educational Studies in Mathematics. https://doi.org/10.1007/s10649-021-10072-x

Thurm, D., Bozkurt, G., Barzel, B., Sacristán, A. I. & Ball, L. (2023). A review of research on professional development for teaching mathematics with digital technology. In B. Pepin, G. Gueudet, & J. Choppin (Eds.), *Handbook for Digital Resources in Mathematics Education*. Springer.

Trouche, L., & Drijvers, P. (2010). Handheld technology for mathematics education: Flashback into the future. *ZDM, 42*(7), 667–681.

van Es, E. A. (2011). A framework for learning to notice students' thinking. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 134–151). Routledge.

Using the Instrumental Orchestration Model for Planning and Teaching Technology-Based Mathematical Tasks as Part of a Restructured Practicum Course

Gülay Bozkurt and Melike Yiğit Koyunkaya

Abstract Preparing prospective mathematics teachers (PMTs) to teach with technology has become one of the important concerns facing teacher education programmes. Accordingly, how such programmes can be structured to develop PMTs' skills and knowledge of technology integration into their instruction is arising as a key question. This chapter details a restructured Practicum course at a Turkish University aiming to orient PMTs' technology incorporation in mathematics teaching. Specifically, we integrated the Instrumental Orchestration model as a means to identify and analyse the development of PMTs' teaching practices with the use of the dynamic mathematics software, *GeoGebra*. The participants were enrolled in a 4-year secondary mathematics education programme at a state university in Turkey. In this study, we employed an action research method that involved the PMTs in a cyclical process of designing technology-based lesson plans through planning, implementing and reflecting. The findings indicated that in the planning process the PMTs' focus was on setting their objectives and general structure for a plan of action, in which they overlooked exploitation modes of their classroom orchestrations. Through micro-teaching, they started noticing the complexity of using the features of dynamic technology in line with their objectives, requiring them to organise their tasks in a more systematic way that considered lesson objectives, technological actions, prompts and potential students' responses.

G. Bozkurt (✉)
Faculty of Education, İzmir Demokrasi University, İzmir, Türkiye

M. Y. Koyunkaya
Buca Faculty of Education, Dokuz Eylül University, İzmir, Türkiye

A. Clark-Wilson et al. (eds.), *The Mathematics Teacher in the Digital Era*, Mathematics Education in the Digital Era 16,
https://doi.org/10.1007/978-3-031-05254-5_2

Keywords Instrumental orchestration · Prospective mathematics teachers · Dynamic mathematical task design · Action research · Classroom teaching · Micro-teaching

1 Introduction

Curriculums and standards in many countries have particularly emphasised the significance of the use of digital technologies to improve teaching and learning in mathematics (e.g., Common Core State Standards for Mathematics, 2010; Ministry of National Education, 2018; National Center for Excellence in the Teaching of Mathematics, 2014). For instance, the use of dynamic geometry software/applications is generally suggested to teach geometry in secondary education, however, it does not necessarily provide specific guidance or materials regarding how to do so. Hence, such emphasis for a changing curriculum integrating technology presents significant concern for the preparation of mathematics teachers. In this sense, initial teacher education programmes are expected to promote prospective mathematics teachers' (PMTs) technology adoption (Hofer & Grandgenett, 2012; Niess, 2005; Yeh et al., 2014). Although since the 1990s such programmes included technology-related courses, they have not provided much evidence regarding PMTs' successful use of technology integration in their teaching (McCulloch et al., 2019). Initial criticism is generally made on the grounds that such courses mostly focused on the use of technology considering only the affordances and technical procedures of the technology for teaching school mathematics (e.g., Powers & Blubaugh, 2005). In recent years, therefore, teacher educators/researchers have turned their attention to content and effective instructional practices with technology in initial teacher education courses (e.g., Bowers & Stephens, 2011; Lee & Hollebrands, 2008; McCulloch et al., 2020).

Research has also highlighted the importance of field experiences to support PMTs' incorporation of technology-based tasks into mathematics lessons (e.g., Darling-Hammond et al., 2009; Strutchens et al., 2016). In this sense, practicum courses where PMTs are directly required to teach mathematical content to students through the use of digital technologies play a crucial role in the development of PMTs' knowledge and skills regarding technology adoption (Grugeon et al., 2009; Meagher et al., 2011; Niess, 2005, 2012; Zbiek & Hollebrands, 2008). Hence, researchers suggested that teacher educators should rethink ways in which they can connect the formal training with the field experiences to enhance PMTs' professional knowledge and learning of teaching with appropriate technologies (McCulloch et al., 2020; Niess, 2012; Strutchens et al., 2016). For example, those in which PMTs could "plan, organize, critique, and abstract the ideas for specific content, specific student needs, and specific classroom situations while concurrently considering the affordances and constraints of the digital technologies" (Niess, 2012, p. 332). Although there have been some studies rethinking ways to offer a link between teacher education courses and PMTs field experiences (e.g., McCulloch et al., 2020), this issue is still considered from an international perspective as "an important endeavor and an emerging research area in need of systematic studies and a global effort to develop a cohesive body of literature" (Huang & Zbiek, 2017, p. 23).

In the light of the above discussion, we (as researchers and teacher educators, the authors of this chapter) aimed at preparing PMTs to design and teach technology-based lessons using a cyclical process in the context of a practicum course at a Turkish University. In working with PMTs, our goal was to focus on the development of their professional knowledge and experiences regarding the successful integration of digital technologies. Hence, we restructured the practicum course by systematically focusing on developing PMTs' integration of digital technologies to their teaching practices with the explicit aim of evaluating their technology-based lesson plans, examining their field experiences, and impact of such experiences on their development of the craft of teaching. To orient and conceptualise the PMTs' lesson planning and classroom practices in the course and enable them to make connection between planning and implementing, we employed a theoretical perspective, *Instrumental Orchestration model* (Drijvers et al., 2010; Trouche, 2004), that was particularly developed to address technology integration in classroom teaching and learning. In the first edition of this book (Clark-Wilson et al., 2014), researchers (Drijvers et al., 2014; Gueudet et al., 2014) reflected on different classroom orchestrations of practicing teachers when using digital technology. As a next step, we were interested in how findings from those studies could be valuable when used in teacher education programmes "for purposes of structuring and scaffolding the reflexive appropriation and development by teachers of the expertise that has been identified" (Ruthven, 2014, p. 390). Hence, our focus was on PMTs who were novices in teaching and in using dynamic mathematics software, particularly GeoGebra, in mathematics teaching. We aimed to answer the following questions:

- How do the PMTs engage with the instrumental orchestration model while preparing a technology-based task?
- How do the PMTs orchestrate their tasks in classrooms?
- What changes or development in the PMTs' orchestrations occur while involved in the practicum course?

In the next section, we will discuss the notion of instrumental orchestration and in particular, the identified classroom orchestrations, which we operationalised in this study.

2 Instrumental Orchestration

The instrumental orchestration model (Drijvers et al., 2010; Trouche, 2004) was based on the instrumental approach (Verillon & Rabardel, 1995), which focuses on learning processes involving tools in which the crucial difference between an artefact and an instrument in a psychological sense has been emphasised. As Verillon and Rabardel (1995) stated that "the instrument does not exist in itself, it becomes an instrument when the subject has been able to appropriate it for himself and has integrated it" (p. 84). Following the instrumental approach, Trouche (2004) used the metaphor of orchestration to model the essential role of teachers in directing students' learning processes with the use of tools. As Trouche (2020) stated,

> Designing an orchestration needs to carefully choose a mathematical problem, according to the didactical goals, to anticipate the possible contribution of the artifacts to the problem solving, to anticipate, in this context, the possible instrumentation of students by these artifacts (p. 410).

While defining the instrumental orchestration notion, Trouche (2004) used two constructs: a didactical configuration and an exploitation mode. A didactical configuration is essentially an arrangement of technological tools in the environment. An exploitation mode is concerned with how teachers plan to take advantage of technology in order to achieve their lesson aims. Drijvers et al. (2010) made use of and developed this notion, by adding another construct, didactical performance as a necessary component of teachers' instrumental orchestrations "to highlight that an instrumental orchestration is a living entity rather than something a teacher prepares beforehand" (Drijvers et al., 2020, p. 1457). A didactical performance emphasises teachers' ad hoc strategy when an unexpected aspect of the mathematical task or the technological tool occurs in classroom teaching, with regard to the chosen didactical configuration and exploitation mode.

In this study, we mainly focused on the orchestrations for whole-class teaching due to the configuration of Turkish ordinary classrooms, hence we reviewed the studies on already identified whole-class orchestrations in the existing literature. The instrumental orchestration model was originally used to illuminate observed teaching practices of the function concept involving the use of applets (small applications) in the Netherlands secondary school context (Drijvers, 2012; Drijvers et al., 2010). The researchers identified six orchestrations for whole-class teaching, which were Technical-demo, Link-screen-board, Discuss-the-screen, Explain-the-screen, Spot-and-show and Sherpa-at-work. Such whole-class orchestrations did not create many organisational issues and only required classroom access to the dynamic software and equipment to project the computer screen and a classroom arrangement for students to see the screen for demonstrations. The Technical-demo was used to demonstrate the techniques of a new tool, such as what was possible with the tool and how to use it. The Explain-the-screen orchestration was used to explain what happened on the computer screen to the whole class and sometimes used to provide students with a good starting point for new tasks. The Discuss-the-screen created a classroom interaction where a teacher and students discuss a problem on the screen. All of these orchestrations illustrate technological alternatives of a regular teaching practice, for instance, Discuss-the-screen can be considered as Discuss-the-board in non-technology-based lessons.

Additionally, Link-screen-board was used with the intention of showing students the relationship between the use in a technological environment and a paper and pencil environment. Spot-and-show was used to deliberately bring up student work for a classroom discussion, which was identified by the teacher as relevant for further discussion. Sherpa-at-work was about a so-called Sherpa-student using the technology to present his or her work, or to carry out actions the teacher requests. These orchestrations seem to be more specific to technology-use. Link-screen-board is perhaps the most obvious example of this kind. Drijvers et al.'s (2013) study identified more whole-class orchestrations of Guide-and-explain and Board-instruction. For the former, the teacher provided a closed explanation based on what was on the screen and often asked closed questions for students, resulting in limited interaction, and for the latter, the teacher used the board for writing as a regular teaching for the whole-class without any connections to the use of digital technology. Also, as a whole-class orchestration, we used Predict-and-test (Bozkurt & Ruthven, 2018; Ruthven et al., 2009) that was achieved where the teacher was leading the activity through the whole-class exposition and questioning format and testing students' ideas on the computer rather than teacher validating or invalidating their conjecture.

After the notion of instrumental orchestration was introduced (Trouche, 2004) and developed (Drijvers et al., 2010), it has been used and extended by a number of researchers in different educational contexts (e.g., Bozkurt & Ruthven, 2018; Erfjord, 2011; Ndlovu et al., 2013; Powell et al., 2017; Tabach, 2011; Tabach et al., 2013). These studies have aimed to illustrate teachers' teaching practices with technology and all have concluded the importance of both teachers' preparation before teaching regarding how to exploit technology, followed by their reflections after teaching. In this sense, these studies provided evidence regarding the usability and usefulness of the IO framework as a means for analysing technology-mediated teaching. Additionally, some of the studies (e.g., Tabach, 2011; Tabach et al., 2013) also contributed to extending potential orchestration types. However, as Drijvers et al. (2020) pointed out in their literature review "the focus seems to be on a growing repertoire of didactical configurations and exploitation modes, whereas didactical performance is hardly addressed" (p. 1462). In this study, by attempting to use the notion of instrumental orchestration in the context of the design and implementation of a practicum course in initial teacher education, we particularly aimed to examine and focus on PMTs' didactical performances through both micro-teaching and actual classroom-teaching in school placements. We aimed to encourage the PMTs to reflect on their teaching and improve their planning based on what could happen in classrooms. In this sense, we believe this study has a potential to contribute to the field by operationalising the IO model in initial teacher education and allowing PMTs to consider the repertoire of instrumental orchestrations, which ultimately could help them in developing practical strategies for the organisation of their technology-based lessons.

3 Methods

The study was designed as action research, which is generally considered as a form of systematic educational research by a practitioner into their own practice to understand and improve such practice (Cochran-Smith & Lytle, 1990; Sagor, 1992; Shagoury & Power, 2012). However, in the context of initial teacher education, preservice teachers should be encouraged and supported to become "active learners shaping their professional growth through reflective participation in both programs and practice" (Clarke & Hollingsworth, 2002, p. 984). As research pointed out, preservice teachers have more opportunities for their professional learning and become more critical and reflective regarding their own teaching in classrooms when teacher educators are involved in conducting action research with them (Darling-Hammond, 2006; Shank, 2006; Snow-Gerono, 2005; Stark, 2006). In this sense, we as teacher educators decided to work collaboratively with PMTs in this action research since "collaborative action research processes strengthen the opportunities for the results of research on practice to be fed back into educational systems in a more substantial and critical way" (Burns, 1999, p. 13).

We, as teacher educators, were the facilitators of the process in order to "develop the methodological protocols necessary for the action research process" (O'Leary, 2004, p. 98). We participated in the planning and evaluation of lesson plans and teaching practices, were responsible for analysing their technology-based tasks, and were observers during the implementation of their plans. In particular, we encouraged them to consider the role of the technology and the use of the instrumental orchestration model in designing and teaching mathematical tasks by reflecting our analysis, thoughts and experiences through asking probing questions. PMTs participated in the planning, implementation, and evaluation of their technology-based lesson plans.

3.1 Research Context

The research was conducted in the context of a compulsory university-based practicum course in a 4-year mathematics teacher education programme at a state university located in the west part of Türkiye (Bozkurt & Yigit Koyunkaya, 2020a, b, 2022). In this practice-based course, PMTs were required to observe the assigned mentor teacher in school placements and to teach at least 6 hours of mathematics lessons in classrooms over a 14-week term. However, there was no requirement for PMTs to integrate digital technologies into their lessons. During the regular practicum course, PMTs prepare and practise lesson plans using any teaching materials or methods that they have learned in their programme without any requirement to conduct a micro-teaching session. During the preparation and implementation of their lessons, both the university course instructor and the assigned school placement mentor teacher gave suggestions and feedback to improve their teaching. Based on our aims, we restructured the course in two aspects. Firstly, we specifically focused on preparing and developing technology use in PMTs' teaching practices. Secondly, we conducted the course in three steps including seminars about using digital technology in teaching, micro-teaching to other PMTs in the university setting and actual teaching in school placements. In each step, we conducted individual and group interviews providing opportunities for them to describe and reflect on their planning and practices. Additionally, we incorporated two technology specific models (namely, Dynamic Geometry Task Analysis (Trocki & Hollebrands, 2018) and Instrumental Orchestration) to support PMTs to design and teach technology-based tasks over three distinct cycles (Bozkurt & Yigit Koyunkaya, 2020a, b, 2022). This chapter specifically focuses on the use of the instrumental orchestration model for PMTs' planning and teaching technology-based mathematical tasks.

3.2 Participants

The participants of the study were four PMTs (two female and two male) selected from eight PMTs taking the practicum course based on a purposive sampling method (Merriam & Tisdell, 2016). Among the eight PMTs in the course, we chose the four who were already familiar with several digital technologies in mathematics education as they took several technology courses and whose general academic achievements were relatively higher amongst the eight PMTs in the class. In this sense, the selection criterion was mainly about their existing knowledge and skills relating to the use of digital technologies in mathematics education. The selection process drew on the second author's decision (also the instructor of the course) as she had previously taught both mathematics education and technology-based courses to the PMTs. This selection criterion was particularly important in our research since our focus was on how they planned and implemented the use of digital technologies into mathematics instruction rather than educating them about such technologies from scratch. Additionally, these four PMTs had worked well together in previous group work and had established a good rapport between each other and their instructors. We believe such a rapport between the PMTs as well as between the participants and the instructor was of crucial importance in strengthening the collaboration in the research.

In addition, they had observed the ordinary classroom environments for a term in the context of a School Experience course. While presenting the results, we assigned numbers to the PMTs as PMT1, PMT2, PMT3 and PMT4 considering their confidentiality.

3.3 Procedure

We conducted this research in three different cycles and each cycle consisted of planning, acting, observing, and reflecting (Kemmis & McTaggart, 2005). In the first cycle, the instructor of the course gave seminars about selecting and using various digital technologies when designing mathematical tasks. Particularly, she discussed the affordances of using dynamic geometry software (i.e., GeoGebra and the Geometer's Sketchpad) by pointing out teachers' roles in the design and implementation process of the technology-based tasks. In addition, she introduced interactive applications such as Desmos and NCTM Illuminations, and video-based web sites such as Khan Academy. The following week, she provided a theoretical introduction of the instrumental orchestration model and an examination of different classroom orchestrations through a video discussion on a mathematics teacher's teaching with the use of GeoGebra (the teacher was leading the lesson on angle bisector construction). After the video discussion, the instructor also required the PMTs to write a reflection regarding the model. During the discussion, the instructor intentionally stopped the video in particular moments and asked the PMTs about the teacher's orchestrations, as he integrated the technology into his teaching. In these moments, it became apparent that initially, the PMTs were not able to discuss his didactical performance in detail. Instead, they were indicating his general classroom management skills and they were only able to name the observed orchestration types supported by small descriptions. Hence, the instructor specifically focused on the teacher's particular actions and tried to stimulate their discussion by asking questions such as "If you were the teacher in the video, how did you use the Sherpa student? or how did you guide the discussion?". Introducing the relatively difficult vocabulary of the instrumental orchestration model theoretically and with connection to the classroom practices of a teacher aimed to guide and better support the PMTs' preparation of lesson plans in terms of their planned classroom orchestrations. In particular, we believed that this way of presenting the model could promote their understanding and awareness of the need for deliberate organisation in their didactical configurations and exploitation modes for successful didactical performances.

At the end of this cycle, which lasted 7 weeks of the course, we asked the PMTs to design their technology-based lesson plans on their chosen topics. Then, we conducted individual interviews about their plans and gave feedback to help improve their designing and planning of the technology-based tasks. In the second cycle, the PMTs revised their plans based on these interviews and practised their micro-teaching sessions. Micro-teaching in this study was conducted with the fellow PMTs acting as secondary school students, but conducted in the university setting. In this way, we aimed to provide an opportunity for the PMTs to practise teaching with technology before implementing their plans in their school placement and to collect feedback and suggestions from their peers and instructors to improve their own teaching. We must state that micro-teaching between PMTs in the university setting is typically preferred and used in the local institution since working with real students requires more practical and ethical considerations.

The micro-teaching sessions were done in a classroom at the university, with an interactive whiteboard next to a traditional board and a laptop computer available to each PMT, similar to the classroom in schools. There were 8 participants acting as students during the micro-teaching sessions, including two researchers, three PMTs from the study while one of them performing the micro-teaching and other two PMTs who were not in the study. After each PMT's micro-teaching session, we conducted post-lesson group discussions to provide feedback/suggestions as well as individual interviews with the PMTs. Our main purpose in this 2-week cycle was to provide the PMTs with an opportunity to practise their tasks before teaching in actual classrooms. In the last cycle (lasting 5 weeks), the PMTs again revised their lesson plans based on their experiences and feedback in the second cycle, and accordingly they implemented their revised plans in actual classrooms during their school placements. The Turkish ordinary classrooms included a configuration that was similar to the micro-teaching setting in terms of the place of interactive whiteboard, traditional board and computer. Generally, there were around 25–30 students in each classroom. While the PMTs mainly were using the boards and computer in the whole-class teaching, the students sat on the tables and discussed and responded to the PMTs' questions, without access to any technology. Following the PMTs classroom practices, we conducted post-lesson individual and group interviews. As a final task the PMTs produced a revision of their technology-based lessons plans. Through each cycle, we examined and revised our research plan considering PMTs development and actions, as revealed while planning, designing or implementing their technology-based tasks.

3.4 Data Collection

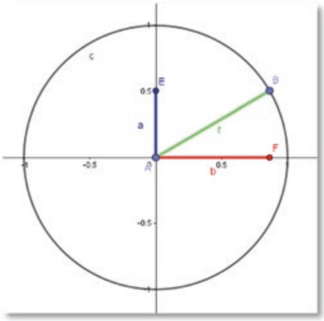
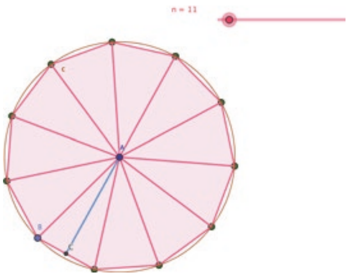
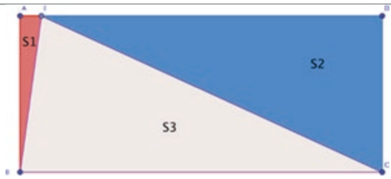
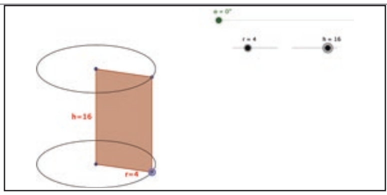
In this chapter, we focus on one main technology-based task from each PMT's lesson plan. In detail, data included in the study consisted of the PMTs' technology-based tasks, video records of their micro-teaching and actual classroom teaching sessions, and post-lesson individual and group interviews/discussions (see Table 1).

Table 1 The data sources of the study

Cycles	The collected data
Cycle 1 (Weeks 1–7)	Video recordings of seminar in which instrumental orchestration model was introduced Video recordings of interviews Lesson plans
Cycle 2 (Weeks 8–9)	Video recordings of micro-teaching sessions Video recordings of post-teaching discussions Revised lesson plans
Cycle 3 (Weeks 9–14)	Video recordings of classroom teaching Video recordings of post-lesson individual interviews Video recordings of post-lesson and final focus group interviews Revised lesson plans

In terms of the topics of the PMTs' lesson plans, they made their decisions based on the syllabus and timetables of the classroom teacher in school placements and designed their tasks accordingly (see Table 2). PMT1 designed a lesson plan to teach trigonometric ratios in the 9th grade and her main technology-based task focused on a unit circle. PMT2 designed a lesson plan to teach the area formula of

Table 2 Summary of the PMTs' technology-based tasks

Name of the task	Image	The goal of the designed task
Unit circle		PMT1 designed a unit circle task to indicate how the changes in one of the angles in a right triangle affects the trigonometric ratios related to this angle. She aimed for students to observe the variants and invariants on the sketch and to reach a generalisation regarding the relationships between angles and trigonometric ratios.
The area formula of the circle		PMT2 designed a task including a regular polygon whose number of sides could be changing with the slider to conclude the area formula of the circle. Particularly, he aimed for students to explore that the polygon tends to a circle when the number of the sides is maximised. With this approach, he planned to use the areas of the n-sided polygons by dividing it into the isosceles triangles to guide students to reach the area formula of the circle.
The area of the triangle		PMT3 designed a task to conclude the fact that the area of the triangle BEC is equal to the area of the triangle ABE and the area of the triangle ECD (S3= S1+S2). He planned to drag the point E on the line segment AD for students to generalise the equality.
Volume and base area of the cylinder		PMT4 designed a task to teach the base area and volume of a shape formed by rotating a rectangle at different angles such as 360°, 180° and 120° using two sliders representing the angle and the radius of the base circle. From this point of view, she aimed for students to reach a general formula of the base area and volume of the shape dependent on the rotated angle.

a circle in the 11th grade and his main technology-based task used inscribed n-sided polygons. PMT3 designed a lesson plan to teach the area formula of a triangle in the 9th grade and his technology-based task involved dividing the rectangle into triangles. PMT4's main technology-based task was related to teaching the volume and base area of a shape in the 11th grade formed by rotating a rectangle through different angles.

3.5 Data Analysis

Qualitative data analysis was guided by instrumental orchestration concepts. In the analysis of the design and teaching process of each task, we compared and contrasted the PMTs' aims and plans of orchestrations and how they orchestrated their tasks in micro-teaching and classroom teaching sessions. Particularly, we identified the changes or development in their orchestrations and considered the reasons for those changes.

We began with a document analysis (Bowen, 2006) to examine the PMTs' objectives and exploitation modes of their tasks in their plans and identify their planned orchestration types. Adopting the video analysis method (Erickson, 2006), we watched all of the video recordings of the teaching sessions for each PMT to analyse their orchestrations in both the micro-teaching and classroom teaching sessions. For each participant, we initially watched the whole video without stopping and took notes. Then, we watched the video and paused to identify notable events that evidenced their orchestration skills and transcribed these video clips.

In the transcribed clips, we particularly focused on the critical events related to the pedagogical purposes of their orchestrations. Additionally, we used the data from individual and group interviews/discussions to support the identified critical events. By triangulating their teaching with post-lesson individual and group interviews/discussions, we compared and contrasted how their orchestrations in the identified critical events changed or developed. To provide a specific example, in PMT2's case, we identified one of the critical events as his plan to employ link-screen-board orchestration. In his plan, he aimed to direct his students to construct a relationship between the area formulas of the inscribed polygons on the traditional board and changes on the screen by animating the slider. In his first micro-teaching, he could not manage to conduct this part of the lesson, as he failed to direct the students to develop a pattern to find the area of inscribed regular polygons and he requested to end the session. As seen in Table 3, we considered the exploitation of his technological actions and didactical performance in terms of the identified event. We used his direct quote from the post-lesson discussion as supportive data to analyse his failure in his didactical performance and from his statements, we deduced that his unplanned discussion of the sketch prevented him from focusing on the inscribed polygon to reach the area formula of the circle. However, in his second micro-teaching, it became evident he had further articulated his exploitation mode and successfully guided the students to link what they saw on the screen by

Table 3 An example of data analysis relating to PMT2's first micro-teaching

	Exploitation of technological action	Didactical performance	Post-micro teaching interview	Orchestration types
Micro-teaching 1	He moved the slider to 3,4 and 5 in order to construct the equilateral triangle, square and pentagon. However, he lost his direction without moving the slider to the n-sided polygon and so on to discuss the relationship.	PMT2 did not reach this step. He could not direct students to construct a relationship between the formulas on the board and changes in the screen.	I think I lost my control at the beginning and then I could not get over the discussion. From a (pointing out one side of the triangle) * h (pointing out the blue segment) / 2, I planned to reach (3*a*h)/2. I mean I planned to construct perimeter multiply by height, perimeter multiply by height, perimeter multiply by height. ... When students represented both line segments using the same value a, I could not get around and I had to stop on pentagon.	Discuss-the-Screen

animating the slider (in which the polygon approached the circle) and the text on the traditional board (the area formula for each inscribed polygon).

Triangulation was used to ensure the trustworthiness of the research findings. We triangulated the PMTs' written objectives and accounts with the video recordings of teaching sessions and post-lesson interviews/discussions to support each part of our analysis. In addition, considering the role of collaboration in the participatory action research, we gave assurance to the participants about the right of withdrawal at any time without any consequences as well as informed them about their confidentiality and anonymity throughout the research.

4 Results

In this section, we present the results of four PMTs' technology-based classroom orchestrations during micro-teaching and actual classroom teaching. Our particular focus is on the evolution of the instrumental orchestrations of their dynamic mathematical tasks. Results are presented in two sections: (1) a general perspective of the four PMTs' orchestrations in micro-teaching and classroom teaching sessions; (2) detailed information about PMT2's orchestration of a dynamic mathematical task.

We specifically extended the PMT2's case since he provided significant evidence of development of particular orchestration types through his teaching sessions. As he stated in the final individual interview:

> My first micro-teaching changed almost everything for me about my orchestration of the class. In particular, it affected how I started my task, what questions I needed to ask while using technology, how to calculate the areas of inscribed polygons etc. I mean it really changed everything.

4.1 General Perspective of the PMTs' Orchestrations in Micro-teaching and Classroom Practices

In the planning process, all PMTs' focus was on setting their objectives and general structure for a plan of action, in which they all seemed to overlook the exploitation modes of their didactical configurations. In their plans, it became evident that the main orchestration type that all PMTs aimed to use was Discuss-the-screen. Additionally, they also stated several other orchestrations such as Explain-the-screen, Predict-and-test, Sherpa-at-work, and Link-screen-board without detailing these exploitation modes. It became apparent during micro-teaching that they had difficulties in orchestrating their tasks in effective ways. In particular they struggled to instrument their tasks with the use of dynamic technology or used the technology limitedly without an apparent intention. Hence, they mostly ended up either with Guide-and-explain or Board-instruction.

For example, considering the use of Discuss-the-screen, PMT1 aimed to teach about the Unit Circle and had indicated the dynamic use of Discuss-the-screen in her plan to enable students to observe the invariants/variants on the screen. Through micro-teaching, she realised that the way she conducted dragging on the screen or how she gave her prompts while coordinating her technological actions were of crucial importance to achieve her goal. One of the aims in her plan was to guide students to explore the relationships between the sine and cosine values of the related angle and the coordinates of point B, by using the line segments in the unit circle. During micro-teaching, PMT1 tried to guide students to observe the trigonometric ratios for different angles by moving the point B on the circle but her use of dragging was random (i.e., dragging the point B on the unit circle back and forth in the first, second, third and fourth quadrants), which is considered as *wandering dragging* in the wider literature (Arzarello et al., 2002). Then, she measured the length of the line segments and showed the coordinates of the point B in the algebra window. During this process, she dragged point B again and asked students to simultaneously observe point B on the circle in the graphic window and its corresponding value in the algebra window. In order to construct the relationship, she proposed the following question "If I try to say something like sin 50 is equal to something connected to point B? What could it be? Can I generalise this, can I conclude something if I try to say something related to B?". However, students stated that they did not understand what she wanted to ask. Then, she continued her teaching completely on the traditional board to explain and show the relationship between the trigonometric ratios and coordinates of the point (see Fig. 1).

Based on her experiences, in the actual classroom, PMT1 added a slider attached to point B in her task and animated it during her teaching, which enabled her to focus on posing questions considering the movement on the screen and improved her Discuss-the-screen orchestration. During the animation of point B around the unit circle, she initially asked students to spot the variants and invariants on the unit circle. Then, she indicated that the red and blue line segments (red line segment refers to the line segment AF and blue line segment refers to the line segment AE) simultaneously change when point B changes and asked students whether the line segments are related to point B. During the discussion, she also measured the line segments to support students to construct the relationship. Based on her didactical performance in micro-teaching, she intentionally added a dynamic text representing point B to make students realise the relationship between the red and blue line segments and the coordinates of point B (see Fig. 2). By animating or dragging the slider, she allowed students to observe the changes in the line segments and point B simultaneously.

Fig. 1 A screenshot from PMT1's micro-teaching on traditional board

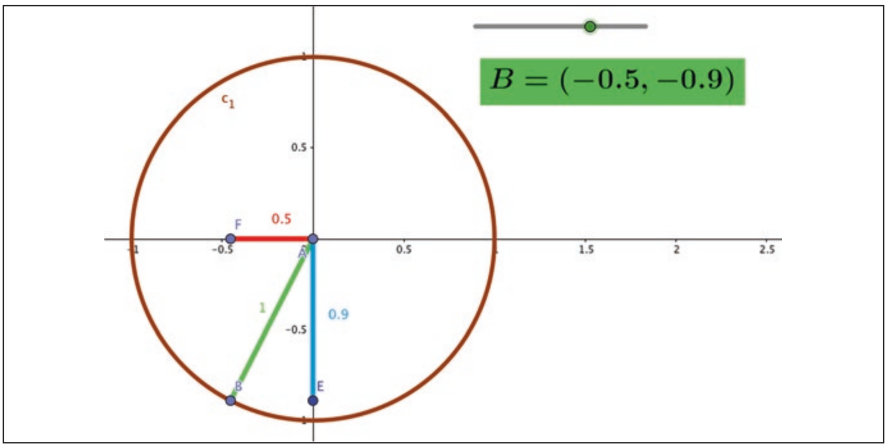


Fig. 2 Unit circle task with the dynamic text representing point B and slider

Similarly, in his first micro-teaching, PMT3 also noticed his lack of use of the software tools to discuss the areas of the triangles in a rectangle as he only dragged the figure once and then measured the area of the triangles. Since he was quite hesitant and nervous in his first teaching experience, he wanted to conduct another micro-teaching of his task before the actual classroom teaching. In his second micro-teaching, he improved his approach by using different technological actions. For instance, having asked students to compare the areas S1, S2 and S3, he guided them to divide the rectangle into triangles by proposing the question "If I want to divide the shape, I mean if I want to represent the triangle S3 by using S1 and S2, how could I do that?". One of the students proposed to draw a perpendicular line from point A to line segment BC. After drawing the perpendicular line and allowing students to compare the areas, he measured the areas of each triangle (see Fig. 3) and verified how the areas S1 and S2 constituted the area S3 by dragging the point E on the rectangle. In this sense, he employed the dragging tool in a more purposeful way and used the tools to verify the equality of areas between the pairs of triangles.

In the actual classroom, PMT3 performed a similar didactical performance as his second micro-teaching experience. Also, he seemed more confident in front of the class both in proposing questions to students and in using the different technological tools.

It also became evident that the PMTs started to comprehend the use of Sherpa-at-work orchestration through micro-teaching. For instance, during the micro-teaching, after PMT4 had discussed the base areas of the shape formed by rotating 360°, 180° and 120°, she asked students how the changes in height affected the ratio between the volumes of the shapes. She initiated the discussion by asking for students' predictions about the changes in ratio in relation to the change in heights. However, during this process, it became evident that she did not include any student input, and instead she mostly explained the answers to her own questions. Even though PMT4 chose a Sherpa student to drive the technology at the front of the

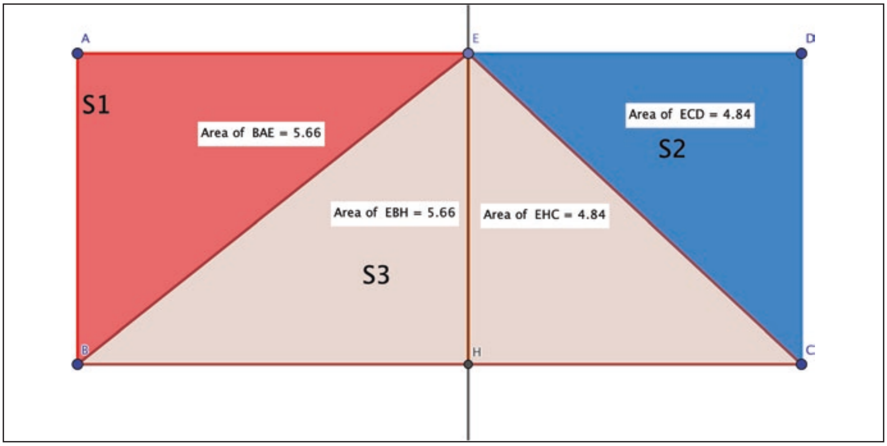


Fig. 3 PMT3's task including the perpendicular line and measured areas

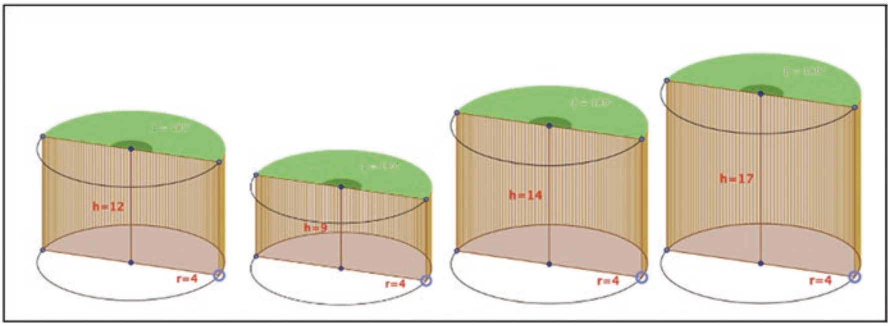


Fig. 4 Different examples of cylinder figures occurred based on the dragging

classroom, she only asked the student to move the height slider without providing any specific directions. The student dragged the slider up and down to change the height for six seconds (see Fig. 4), however PMT4 did not coordinate her question with such movement nor guide the class (or the Sherpa) to make observations nor discuss the changes on the screen. Instead she only explained the ratios using the related formulas on the traditional board.

Some of the PMTs indicated that they understood the real meaning of the Sherpa-at-work orchestration through post-micro-teaching discussion sessions when they reflected on each other's uses of the different orchestrations. As a result, some of them decided not to choose a Sherpa student to use the technology in actual classroom teaching, since students within the school placements would not have previously used GeoGebra. Choosing a Sherpa to help with the use of technology might have ended up focusing on and guiding that student in a technical way, which might hinder the flow of discussion.

Based on the didactical performances and suggestions given in micro-teaching sessions, it became clear that most of the PMTs began to notice the complexity imvolved in using the features of dynamic technology in line with their objectives. There was a realistion that they needed to organise their tasks in a more systematic way that considered lesson objectives, technological actions, questions and potential students' responses. The PMTs also realised the effects of using technology to test students' predictions, organise the discussion and reach their intended goals. Particularly, they realised the meaning, and role, of particular orchestration types such as Discuss-the-screen, Sherpa-at-work and Predict-and-test. To conclude, after introducing and discussing orchestrations of technology-based tasks during the planning stage of their tasks, a general reflection emerged that orchestrating the tasks was not an issue for the PMTs. They tended to believe that they would not need to detail the exploitation modes of their tasks step by step. Then, through the experiences of their teaching practices, they noticed that structuring and planning in advance was of crucial importance for a successful implementation of their tasks that addressed their general lesson objectives. In the final group discussion, PMT3 reflected on this issue by stating:

> At the beginning of this study, I was designing the tasks and then I was looking for the orchestration by considering what orchestration types does this task include? But now,

> I think about what I can do here, does it get better if I manipulate the sketch dynamically? I am considering what happens if I measure this, or should I draw something on it? Is it possible to reach a generalisation if I follow these steps? So, I am thinking about the orchestration of the task by considering all these issues while designing my task. That is what has changed in my mind.

4.2 Extended Results of PMT2

In this section, we share the detailed analysis of PMT2's classroom orchestrations through the processes of planning, micro-teaching and classroom teaching to indicate the evolution of pedagogical purposes of his classroom orchestrations.

4.2.1 What Did He Plan?

After the seminars in which instrumental orchestration model was introduced, PMT2 reflected on this by saying:

> I really like the conductor of the orchestra metaphor for teachers in the classroom. I think orchestration types provide a really useful language for us to describe what we do in the classroom. In my plan, I think I will often use technical-demo, explain-the-screen and link-screen-board. I would also use the other ones depending on the task.

Then, PMT2 prepared a technology-based task to teach the formula for the area of a circle. With the use of a slider attached to the number of sides of inscribed regular polygon, his main purpose was to use more sides of the regular polygon to become closer to the overall shape, which constitutes a circle itself (see Fig. 5).

In his plan, he aimed to use Explain-the-screen, Discuss-the-screen and Link-screen-board orchestrations without detailing the exploitation modes of his orchestration. For instance, his statement for the use of Link-screen-board was the following:

> I planned to use a link-screen-board orchestration to encourage students to find the area formula of the circle by considering the changes in the number of sides of the polygon and constructing a relationship between the formulas on the traditional board and changes on the screen.

The above quotation indicated that he did not consider a particular moment to configure the teaching setting or the tools of GeoGebra he planned to use, whilst selecting the use of the link-screen-board orchestration. The didactical configuration of his whole-class orchestration included access to the dynamic technology and IWB in a whole class setting in which all students were able to see the screen. It became evident that his main orchestration to use was Discuss-the-screen for a whole-class discussion about what is visible on the computer screen. In detail, his objective was to stimulate a discussion in which students could calculate the areas of inscribed regular polygons starting from an equilateral triangle, square... to an n-sided regular polygon and conclude the area formula of the circle. He particularly wanted to guide

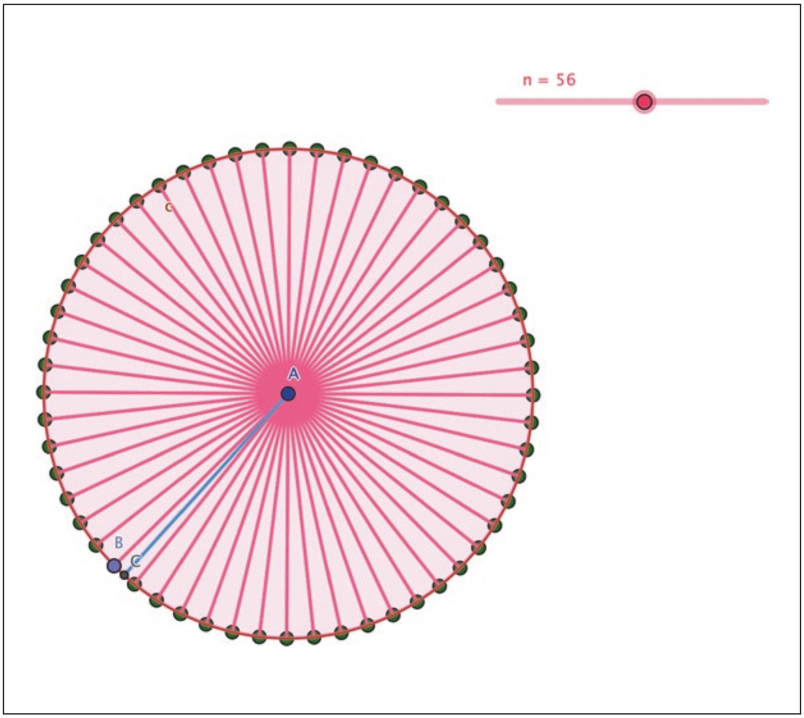


Fig. 5 PMT2's task including the slider

his students through the dynamic file in which each polygon was divided into equal isosceles triangles and by using the height (as blue segment a) the area of each triangle should be calculated by using the formula '$\frac{1}{2}$ **base*height'* (see Fig. 6). By using this formula, he aimed to reach the formula of polygon as '$\frac{1}{2}$ **(the number of the triangles) *base*height'* considering '*(the number of the triangles)*base'* represents the perimeter of the polygon. Based on this, he wanted students to reach the formula of the area of the polygon as '$\frac{1}{2}$ **perimeter of the polygon*height'*.

4.2.2 Micro-teaching 1

In his first micro-teaching, PMT2 only managed to allow students to calculate the areas of inscribed equilateral triangle and square in which the students' way of calculations did not match PMT2's actual plan. The difference in the way of calculations hindered PMT2 to enable students to develop the intended pattern. Then for the inscribed pentagon, he lost his direction and ended the session at his own request before reaching his aims. Table 4 shows how PMT2 guided a discussion with the use of technological actions.

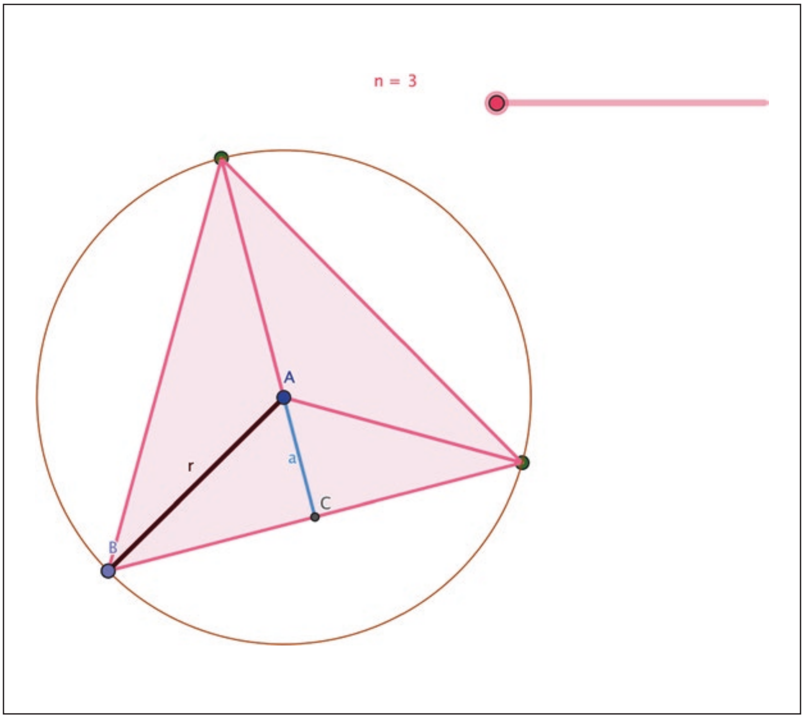


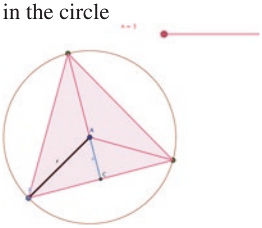
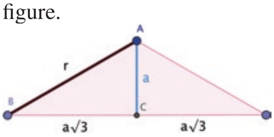
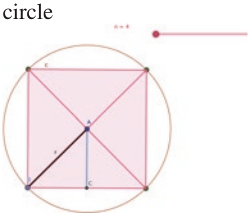
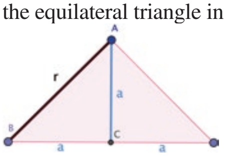
Fig. 6 The case of an equilateral triangle that shows the isosceles triangles, when n=3

The above classroom dialogue shows that the teacher's intentional guiding and the students' actions were different, which created a confusion for him. While PMT2 planned for students to calculate the areas of the regular polygon by using the idea of '*the number of polygon sides * the area of the isosceles triangle'* that was constructed inside the polygon, students used the height of the isosceles triangle representing the same value as *a* (the blue segment in the figure) for different regular polygons. At this point, PMT2 became confused and was not able to direct the students to his planned actions. As he reflected on this during the post micro-teaching interview:

> I think I lost my control at the beginning and then I could not get over the discussion. From a (pointing out one side of the triangle) * h (pointing out the blue segment) / 2, I planned to reach (3*a*h)/2. I mean I planned to construct perimeter multiply by height, perimeter multiply by height, perimeter multiply by height. ... When students represented both line segments using the same value a, I could not get around and I had to stop on pentagon.

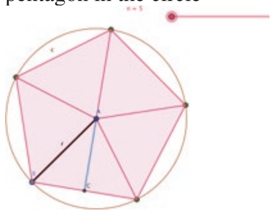
When he stopped his teaching, during the post-lesson discussion, researchers and other PMTs present during the micro-teaching made several suggestions regarding his teaching. One of them was to represent the one side of the polygon as n1, n2, n3 and so on as well as represent the height using different values such as h1, h2, h3 and so on. Additionally, they stated that his aim was not clear within the questions

Table 4 A detailed description of how PMT2 guided the discussion in the first micro-teaching experience

Exploitation of technological action	Didactical performance
Using the slider to move to n=3 to construct an equilateral triangle inscribed in the circle 	Students offered a solution where they used the general formula of area of equilateral triangle $\frac{(one\,side)^2\sqrt{3}}{4}$. Then they calculated one side of the equilateral triangle by using the height (the blue segment a) of the isosceles triangle in the figure.  The height (one side of the isosceles triangle ABC) = a Radius = 2a Other side of the isosceles triangle = a $\sqrt{3}$ One side of equilateral triangle = 2a $\sqrt{3}$ At this point, PMT2 followed their directions and wrote these on the traditional board without considering his plan.
Using the slider to move to 4 to construct a square in the circle 	Students guided PMT2 to represent one side of the square similar to the equilateral triangle in terms of the height (the blue segment a).  The height (one side of the isosceles triangle ABC) = a One side of the square = 2a Radius = a $\sqrt{2}$ During the teaching, PMT2 tried to direct the students to use the isosceles triangles inside the square (See verbatim transcript). T: We calculated the area of the equilateral triangle using the general formula of it. What do you see here (*showing the square on the screen*)? What is in the square? Here, the square is divided into four equal isosceles triangles. S: Can we find the area of one isosceles triangle and multiply it b y 4. T: Yes. We could use the same method for the equilateral triangle. The equilateral triangle consisted of three isosceles triangles (*he moved the slider to 3*). We multiply this (*showing the isosceles triangle*) by 3. T: Now, in the square, what is the radius here (*he was on the traditional board*)? S: r T: How could you write it in terms of a? S. a $\sqrt{2}$ T: What is the height? S: a T: Here, what is the area of one isosceles triangle? S: $\frac{2a^2}{2}$ multiply by 4. T: I am writing like this. Area of one of the triangles is square a, so the area of the square is multiplication of it by 4. [*He was watching the traditional board for a while and then moved the slider to 5*].

(continued)

Table 4 (continued)

Exploitation of technological action	Didactical performance
Using the slider to move to 5 to construct a regular pentagon in the circle 	He was not able to calculate the area of the pentagon; he encountered the situation that he did not manage to guide the students. T: [*by dragging the slider to 5*] How can we calculate this based on the previous one? S: This is a bit difficult to calculate. We should first find the angles. [*while students were discussing the angles, PMT2 drew an inscribed pentagon on the traditional board and seemed puzzled. By moving away from the board, he looked at the figure*] T: Now what do I know? I know the radius is r. S: OK, we found 54° so the other two angles should be 36° *[at this point the PMT was still thinking by looking at the figure from afar*]. The PMT could not continue the discussion and he voluntarily gave up the instruction.

nor his guidance, so they did not really understand the lesson goals. Therefore, one of the suggestions to him was that before the activity students could be prepared regarding the mathematical context of the activity. Also, in order for him to effectively link the screen to the board, they suggested that he should write the formulas on the board in a systematic way and highlight the perimeter in the formulas (i.e., *3* length of one side=perimeter of equilateral triangle*).

4.2.3 Micro-teaching 2

In the second micro-teaching experience, PMT2 orchestrated his class in a more successful way. In particular, he demonstrated an effective use of a slider tool and the zoom tool within GeoGebra. When compared to his first micro-teaching, he had planned his exploitation mode for his Discuss-the-screen orchestration and was better prepared for the questions to ask and how to utilise technology to stimulate a successful classroom discussion in order to fulfil his lesson objectives. Although Discuss-the-screen remained the most apparent orchestration, while discussing the screen, he employed more specific orchestrations such as Predict-and-test and Link-screen-board.

He began the classroom discussion by asking a question “How can you approximate the area of a circle?” and he allowed students to make predictions. By using the dynamic technology, he tested their predictions. For instance, one of the students said that they could find the area by using a square, and the other said an 8-sided polygon. Starting from the students’ ideas, PMT2 opened his prepared GeoGebra file (see Fig. 7) and asked the following question “Let’s start with the smallest polygon that we know. How can I find the area of this triangle?”

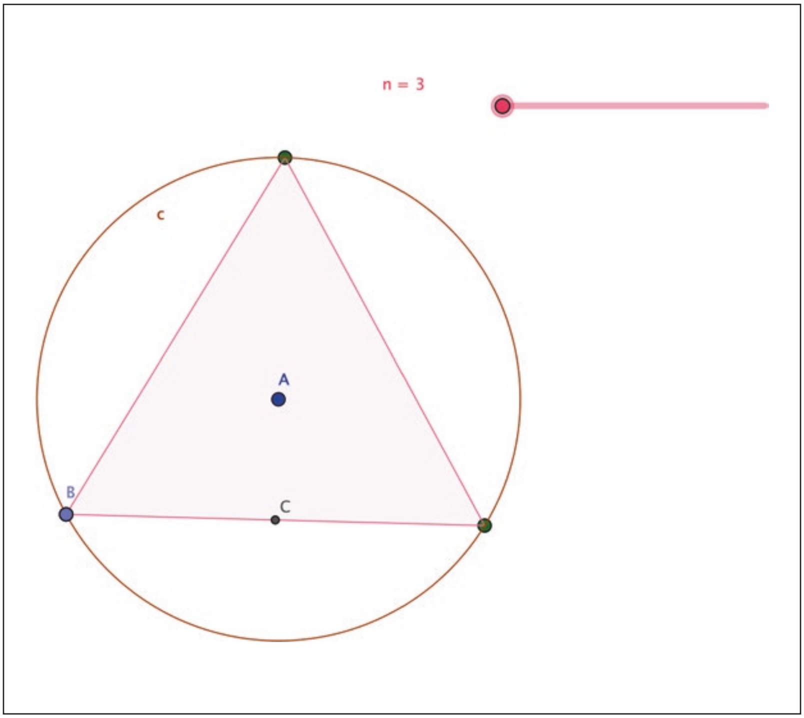


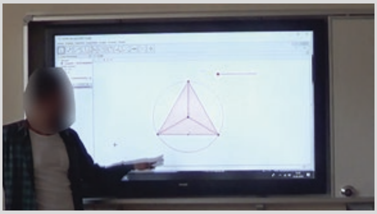
Fig. 7 The GeoGebra file in which he hid the isosceles triangles and the height segment

As seen in the GeoGebra file, he intentionally hid the sides of the isosceles triangles and the blue segment (which showed the height of the isosceles triangle) for the area of regular polygons calculations which led to the failure of his first micro-teaching. In his final post-lesson interview, he said:

> Students could calculate the area of the equilateral triangle by using the isosceles triangles within it. Therefore, I intentionally gave the equilateral triangle without dividing it into the isosceles triangle, and asked students directly how to calculate the area of this equilateral triangle.

Also, he purposefully guided the students to consider different ways to find the area of the equilateral triangle to exploit the steps within his planning. The transcript of this episode was:

S1: If I know one side...
Y: Yes, you can find the area. What else?
...
S2: We can also use $\frac{1}{2}$ **base*height*.
Y: Which height do you mean? The height of the big triangle? *[he shows with his hand on the screen]*
S2: Yes
T: OK, that is another way. What else?
S3: Can we find it by dividing the big triangle into small triangles?
T: Do you mean this? *[he shows the file in the below figure]*



S3: Yes
T: How can I find the area of these triangles? *[he shows one of the isosceles triangles]*
S1: So, the whole area is a small triangle × 3.
T: Ok, how about the area of those small triangles?
S1: base is r $\sqrt{3}$, then, $\frac{r\sqrt{3} * |CA|}{2}$.
T: [*He shows the blue segment in the figure*] OK, now we see the length of AC [*Then he goes up to the normal board and writes what the student says*].

T: Could you repeat it please? Here, let's label one of the sides of the equilateral triangle as n3 and its height as h3 [*He drew the figure on the traditional board*].
...
S1: OK, then $\frac{n3 * h3}{2} * 3$.
[*PMT2 revoice and writes the formula that S1 said on the traditional board*]

Similarly, by increasing the slider step by step, he enabled the students to discuss and calculate the areas of 4 and 5-sided regular polygons to support them to develop a pattern. Then, by dragging the slider to 20, 50 and 100 respectively, he stimulated the discussion on how the area of the polygon changed and also how the area of the inscribed polygon compared to the area of the circle as the number of sides was increased. At this stage, he effectively proposed questions requiring the students to predict what would happen when the number of sides increased and tested their ideas (verify or falsify) with the use of technology and progressed through seeking validation from the dynamic technology. In particular, to develop the approaching idea, he particularly utilised the zoom tool and zoomed in on the screen (see Fig. 8) to discuss the fact that although the overall shape of the polygon resembled the circle, there was still a gap between the polygon and the circle (see Fig. 8).

In the last stage of the discussion, he wanted the students to link the screen and board, in which he animated the slider, and asked students to consider the traditional board, where he wrote the area formula of each polygon (see Fig. 9). He encouraged

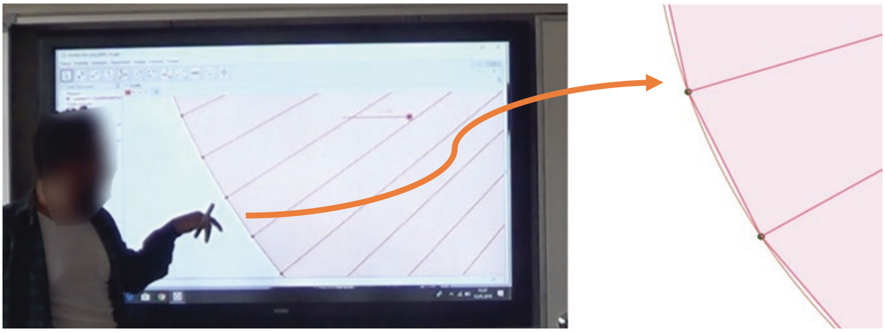


Fig. 8 A screenshot from PMT2's teaching when showing the gap between the circle and polygon

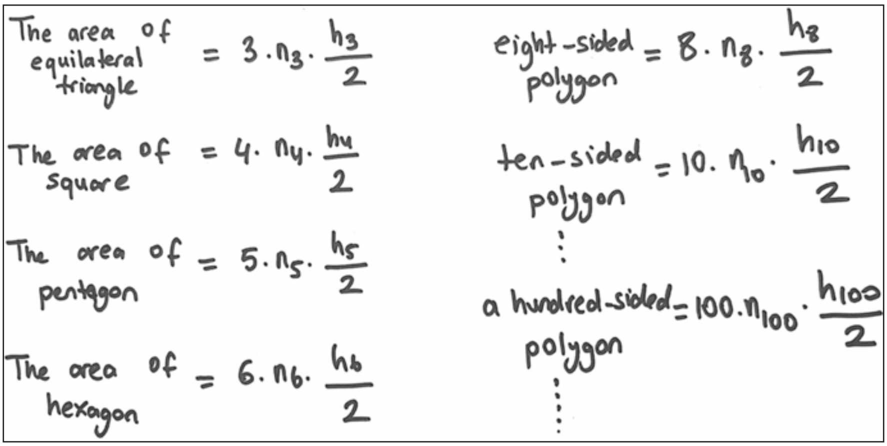


Fig. 9 A re-drawn writing of PMT2 on the traditional board

students to explicitly link what they saw on the screen and what was written on the traditional board.

While on the traditional board, he asked the question "How can I write the area formula for a 100-sided polygon area?". Then he went to the screen and guided students to focus on and interpret the height. By increasing the number of sides with the use of the slider, he asked the students to observe where the sides and height of small triangles inside the polygon were approaching. He was standing in the middle of the IWB and the traditional board and by using his gestures he pointed to the sides and height on the screen (see Fig. 10).

On the traditional board, he asked the students to form the formula of the area of a circle by linking the relationships they had observed on the screen. To ease their observations, he dragged the slider effectively and prompted them with respect to the formula in the following dialogue.

T: *OK, if the polygon approaches the circle, how can we calculate the area of a regular circumscribed n-gon where n is very large? Concerning the congruent triangles inside the n-gon, their sides would approach the radius. What is the circumference formula?*
S: *2∗ pi ∗ r*
T: *Yes, 2 ∗ pi ∗ r. How about the height? Where is it approaching?*
S: *Radius, as well.*
T: *That is also approaching the radius. OK, then, what is the formula?*
S: *2 ∗ pi ∗ r ∗ r /2*
T: *Which means pi multiplied by* r^2.

Concerning the fact that the polygon approaches the circle, one of the students suggested that "It *(the polygon)* goes to infinity, but it never becomes a circle". PMT2 tried to answer the question using the task by indicating that the height of the isosceles triangle (the blue segment) approaches the radius of the circle. Specifically, he stated that "even if it is very tiny, there is still a gap between them", but the students did not seem to be convinced with his response. In the post-lesson discussion, the

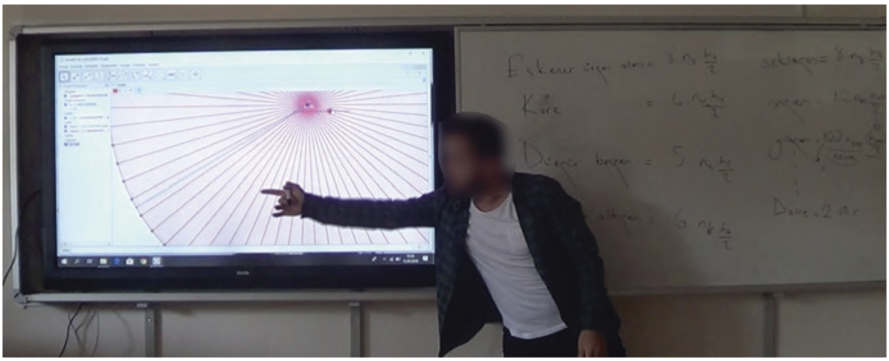


Fig. 10 A screenshot from PMT2's teaching in the position of linking the screen and traditional board

researchers and other PMTs suggested PMT2 to mention the concept of limit to explain this phenomenon.

In the post-lesson discussions, the researchers and other PMTs mentioned the successful points of his teaching. Particularly, they indicated that presenting many examples using the slider and testing students' predictions on the sketch were crucially important to the success of his didactical configuration. However, they suggested moving the slider more slowly to provide an opportunity for students to explore the changes on the screen instead of explaining himself and discussing the screen in a more effective way. Suggestions regarding his second micro-teaching were not related to the mathematical context of the task. Instead, the researchers and PMTs directed him to consider the features and role of the technology in a more detailed way.

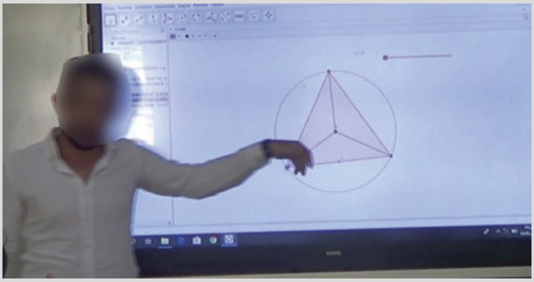
4.2.4 Actual Classroom Teaching

In PMT2's classroom teaching, there was a similar didactical configuration to the micro-teaching, in which the teacher used GeoGebra on one central screen alongside the traditional board. While the teacher controlled the manipulation of the dynamic technology and provided the prompts to stimulate a classroom discussion, students were observing and actively involving in the discussion. As PMT2 planned his exploitation mode of the task, informed by the two previous didactical performances during the micro-teaching, he seemed more confident to orchestrate his activity within the class. He was particularly successful in constructing a link between screen and board by considering the questions to ask alongside how to utilise the technology to stimulate a successful classroom discussion. He also strengthened his orchestration with an effective use of gestures such as his fingers and the movement of his body between the screen and board.

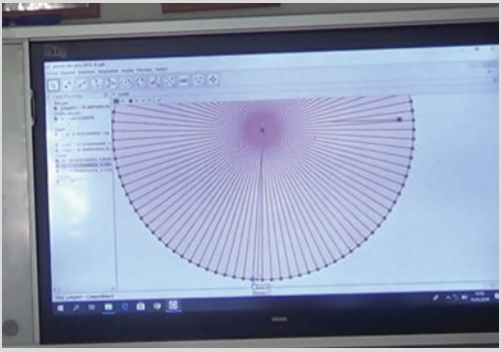
His orchestration of the task showed similarities with that of PMT2 during the second micro-teaching. Additionally, in classroom teaching, it became evident that PMT2 managed to pose more structured questions and he guided his planned discussion with the high school students who were not familiar with the task, technology or context. In this light, during the process of generalising the formula, he conducted an effective discussion by linking the screen to the board, which seemed more structured and indicated more involvement from students.

For example, during the second micro-teaching, the participants were already familiar with PMT2's objectives and the task itself, hence, they seemed to reach intended results relatively easily such as noticing the perimeter in the formulas or the approaching idea of height. However, during the classroom teaching, PMT2 guided the students step-by-step to help them realise the use of perimeter within the formula, and also how the height of the isosceles triangle approaches the radius. In this process, he effectively discussed the context using the task and indicating the concept of limit (see the below text frame including verbatim transcript).

T: OK, please have a look at the screen. What does 3∗n3 indicate?
S: 3 sides.
T: I am asking 3∗n3.
S: Perimeter.
T: Yes, perimeter of the triangle. Perimeter of the equilateral triangle. [*He shows this on the traditional board and then goes to the screen and moves the slider to 3 and continues on the screen*]. OK, we labeled one side as n3 and we have three of it.

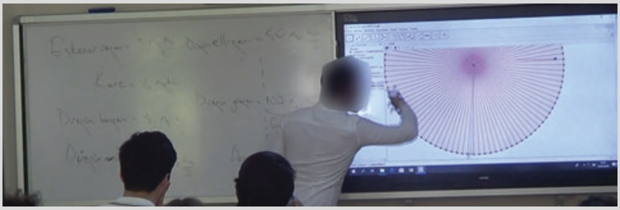


T: How about 4∗n4?
S: Perimeter
T: Whose perimeter is it?
S: Perimeter of the square
[*He discussed the similar operations for the pentagon, 12-sided and 100-sided polygons by dragging the slider and linking the screen and the board*]



T: Where does it go?
S: Infinity
T: Yes, it goes to infinity. How about this $\frac{h3}{2}, \frac{h4}{2}, \frac{h100}{2}\ldots$?
S: Half of the radius.
T: Why is it radius over 2?
S: There is a triangle, its height will be radius.
T: Would it become radius in the end?
S: No, it would not. The shape approaches a circle.

T: Yes, it approaches the circle… But there will always be a gap between these two points, so there is always another point in between. Infinite points construct the circle, and we say it approaches… This is about a concept called limit, which you will learn next year. So, the height will approach r.
S: But it will never be radius...
T: Yes, it will never be radius. Also, the perimeter of the n-sided polygon will never reach the circumference.



T: There will be a gap all the time. I already showed you for 100-gon and now [zooms in] for 1000-gon there is a smaller gap, you see. But, when we learn the limit concept in the twelfth grade, we will ignore this gap. I mean we will not ignore it, but we will take the approximate value of it.

At the end of the lesson, he considered the whole process to reach the area formula of the circle by using the fact about the infinitely-sided polygon.

To conclude, based on his experiences throughout the whole process, his exploitation modes and didactical performances of his instrumental orchestrations evolved (see Fig. 11), PMT2 managed to successfully orchestrate his task to achieve his mathematical goals. In his final individual interview, he reflected across all of the research cycles thus:

> The first micro-teaching changed everything for me. I think it was an indispensable part in my development because I made all the mistakes that I could have done ... Then for the second micro-teaching I revised my plan carefully and considered all the suggestions for a successful organisation of my task. Therefore, the second micro was also quite important for me to think deeply about how to use technology and what kind of questions to ask to guide students. But I think the actual teaching was the most useful experience for me as a teacher. I was more nervous to teach in the real context with the students. … answering their questions, using technology to show the approaching idea, actualising my plan in the classroom… It was a great experience!

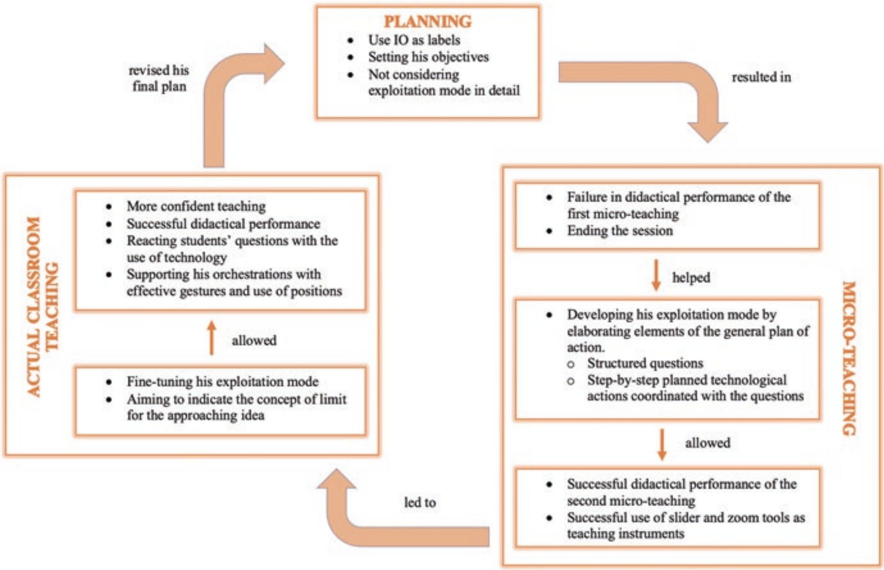


Fig. 11 PMT2's evolution of instrumental orchestration through the cycles

5 Conclusions and Discussion

This chapter focused on a restructured practicum course at a Turkish University in the secondary school context which aimed to develop PMTs' skills to design and teach lessons with the use of dynamic mathematics software focusing on the instrumental orchestration model. Our research questions concerned how the PMTs engaged with the instrumental orchestration model whilst planning their technology-based tasks, and then how they orchestrated these tasks in classrooms. Particularly, we examined what changes or development occurred in the PMTs' instrumental orchestrations through different cycles of the practicum course.

The findings indicated that although the PMTs began to consider orchestration types in their initial plans, they only came to realise the importance of detailing the exploitation modes of their didactical configurations whilst in the process of micro-teaching. They started noticing the complexity of using the features of dynamic technology in line with their objectives and the need to organise their tasks in a more systematic way that considers lesson objectives, technological actions, prompts and potential students' responses. Before practising their tasks, it became apparent they overlooked the pedagogical purposes for instrumental orchestrations. For instance, although most of them designed their tasks around Discuss-the-screen and Link-screen-board orchestrations, they ended up orchestrating mostly with Board-instruction as they either failed to use dynamic features of technology as planned or to coordinate appropriate prompting with their technological actions. For

Link-screen-board, they tended to stay at the traditional board and ignored the technology. As PMT1 reflected, “I realised when I was doing something on the traditional board, I forgot to drag the point B on the screen, which I needed to move simultaneously. So, I continued here on the board, but the screen stayed in the previous position”. In this sense, the insights and results learned from micro-teaching became an intermediate step and served as crucial inputs for the PMTs to reconsider their classroom orchestrations and elaborate their plans for their actual classroom teaching (Agyei & Voogt, 2011; Zbiek & Hollebrands, 2008). As a result, their actual classroom practices and reflections on such practices showed that they were more prepared regarding their teaching trajectory indicating their professional learning regarding technology integration (Goos, 2005; Rocha, 2020; Ruthven, 2014; Trgalova et al., 2018). In particular, they tended to expand and structure their questions as well as their technological actions with an explicit aim to successfully orchestrate their tasks.

This study also provided evidence regarding how a practicum course could be restructured with a specific focus on technology integration (Niess, 2012). This is of essential importance in particular for the education contexts where there is no current requirement for PMTs to incorporate digital technologies into their teaching in school placements. In the restructuring process, two aspects were important: the theoretical model and the methodological approach. For the former, the integration of the instrumental orchestration model into the course in general provided a useful lens both for the PMTs and the researchers. Hence, this study provides evidence for the usability and usefulness of the instrumental orchestration model to help prospective teachers and teacher educators benefit from the practical knowledge about technology integration. However, with the use of already identified whole-class orchestrations, at the first stage the PMTs structured their plans of actions without deeply understanding the concepts of the model. With the classroom practices, they started making sense of the details and systematic orchestrations of their activities in the classroom. In this light, we believe that introducing and discussing a theoretical model only in the planning stage (Bowers & Stephens, 2011) would not result in a successful orchestration in classrooms, in particular for PMTs to comprehend the exploitation modes of their didactical configurations. Hence, we argue that such practicum courses should involve a cyclical process through modification, implementation, and reflection. Also, involvement in participatory action research promoted the process of collaborative learning in two aspects. First, collaboration supported the PMTs’ development relating to the instrumental orchestration model and their teaching practices through our feedback and suggestions. Second, it supported us to revise our plan of actions by considering PMTs’ development of the processes of designing and teaching tasks using the technology. Although conducting a course such intense collaboration in three cycles might not be feasible for a larger size group of PMTs with only the instructor of the course, we believe this study provided and discussed PMTs’ potential hiccups (Clark-Wilson, 2010) and pivotal teaching moments (Stockero & Van Zoest, 2013), which can be used in training PMTs to improve their technology-based teaching in school placements. Also, this study is of value in bridging “acknowledged gap between research and practice" (McIntyre, 2005, p. 357) by making research-based concepts accessible to the future teachers and to encourage them to think about those concepts. McIntyre (2005) argued that members of the academic community must take responsibility to bridge that gap by seeking out and working with the professional knowledge of teachers and should not simply expect practitioners to engage with research knowledge.

Nevertheless, we cannot claim that the professional development arrangement fully developed the PMTs' pedagogical functioning and dynamic use of instrumental orchestrations. In particular, further opportunities to experience learning about the instrumentation of the slider tool as a teaching instrument are necessary for the PMTs to design tasks that aim to allow students to generalise mathematical concepts, processes or relationships and to explore different topics and concepts in their mathematics curriculum. Also, further research could examine PMTs' instrumental orchestrations where students actively work at computers themselves to encourage the PMTs to promote independent learning with the use of digital technologies.

References

Agyei, D. D., & Voogt, J. (2011). ICT use in the teaching of mathematics: Implications for professional development of pre-service teachers in Ghana. *Education and Information Technologies, 16*(4), 423–439.

Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A cognitive analysis of dragging practises in Cabri environments. *ZDM—The International Journal on Mathematics Education, 34*(3), 66–72.

Bowen, G. A. (2006). Document analysis as a qualitative research method. *Qualitative Research Journal, 9*(2), 27–40.

Bowers, J. S., & Stephens, B. (2011). Using technology to explore mathematical relationships: A framework for orienting mathematics courses for prospective teachers. *Journal of Mathematics Teacher Education, 14*(4), 285–304.

Bozkurt, G., & Ruthven, K. (2018). The activity structure of technology-based mathematics lessons: A case study of three teachers in English secondary schools. *Research in Mathematics Education, 20*(3), 254–272.

Bozkurt, G., & Yigit Koyunkaya, M. (2020a). Preparing prospective mathematics teachers to design and teach technology-based lessons. In B. Barbel, B. Ruth, G. Lisa, P. Maximilian, R. Hana, S. Florian, & T. Daniel (Eds.), *Proceedings of the 14th international conference on technology in mathematics teaching – ICTMT 14* (pp. 255–262). University of Duisburg-Essen.

Bozkurt, G., & Yiğit Koyunkaya, M. (2020b). From micro-teaching to classroom teaching: An examination of prospective mathematics teachers' technology-based tasks. *Turkish Journal of Computer and Mathematics Education, 11*(3), 668–705.

Bozkurt, G., & Yigit Koyunkaya, M. (2022). Supporting prospective mathematics teachers' planning and teaching technology-based tasks in the context of a practicum course. *Teaching and Teacher Education, 119, 103830.*

Burns, A. (1999). *Collaborative action research for English language teachers*. Cambridge University Press.

Clarke, D., & Hollingsworth, H. (2002). Elaborating a model of teacher professional growth. *Teaching and Teacher Education, 18*(8), 947–967.

Clark-Wilson, A. (2010). *How does a multi-representational mathematical ICT tool mediate teachers' mathematical and pedagogical knowledge concerning variance and invariance?* Unpublished Ph.D. thesis. Institute of Education.

Clark-Wilson, A., Robutti, O., & Sinclair, N. (2014). *The mathematics teacher in the digital era*. Springer.
Cochran-Smith, M., & Lytle, S. L. (1990). Research on teaching and teacher research: The issues that divide. *Educational Researcher, 19*(2), 2–11.
Common Core State Standards Initiative (2010). *Mathematics standards*. Retrieved April 16, 2021, from http://www.corestandards.org/Math/
Darling-Hammond, L. (2006). Constructing 21st-century teacher education. *Journal of Teacher Education, 57*(3), 300–314.
Darling-Hammond, L., Wei, R. C., Andree, A., Richardson, N., & Orphanos, S. (2009). *Professional learning in the learning profession*. National Staff Development Council.
Drijvers, P. (2012). Teachers transforming resources into orchestrations. In G. Gueudet, B. Pepin, & L. Trouche (Eds.), *From text to 'lived' resources* (pp. 265–281). Springer.
Drijvers, P., Doorman, M., Boon, P., Reed, H., & Gravemeijer, K. (2010). The teacher and the tool: Instrumental orchestrations in the technology-rich mathematics classroom. *Educational Studies in Mathematics, 75*(2), 213–234.
Drijvers, P., Tacoma, S., Besamusca, A., Doorman, M., & Boon, P. (2013). Digital resources inviting changes in mid-adopting teachers' practices and orchestrations. *ZDM—The International Journal on Mathematics Education, 45*(7), 987–1001.
Drijvers, P., Tacoma, S., Besamusca, A., van den Heuvel, C., Doorman, M., & Boon, P. (2014). Digital Technology and Mid-Adopting Teachers' Professional Development: A case study. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era* (pp. 189–212). Springer.
Drijvers, P., Grauwin, S., & Trouche, L. (2020). When bibliometrics met mathematics education research: The case of instrumental orchestration. *ZDM—The International Journal on Mathematics Education, 52*(7), 1455–1469.
Erfjord, I. (2011). Teachers' initial orchestration of students' dynamic geometry software use: Consequences for students' opportunities to learn mathematics. *Technology, Knowledge and Learning, 16*(1), 35–54.
Erickson, F. (2006). Definition and analysis of data from videotape: Some research procedures and their rationales. In J. L. Green, G. Camilli, & P. B. Elmore (Eds.), *Handbook of complementary methods in education research* (pp. 177–191). Erlbaum.
Goos, M. (2005). A sociocultural analysis of the development of pre-service and beginning teachers' pedagogical identities as users of technology. *Journal of Mathematics Teacher Education, 8*(1), 35–59.
Grugeon, B., Lagrange, J. B., Jarvis, D., Alagic, M., Das, M., & Hunscheidt, D. (2009). Teacher education courses in mathematics and technology: Analyzing views and options. In C. Hoyles & J. B. Lagrange (Eds.), *Mathematics education and technology-rethinking the terrain* (pp. 329–345). Springer.
Gueudet, G., Bueno-Ravel, L., & Poisard, C. (2014). Teaching mathematics with technology at the kindergarten level: Resources and orchestrations. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era* (pp. 213–240). Springer.
Hofer, M., & Grandgenett, N. (2012). TPACK development in teacher education: A longitudinal study of preservice teachers in a secondary MA Ed. program. *Journal of Research on Technology in Education, 45*(1), 83–106.
Huang, R., & Zbiek, R. M. (2017). Prospective secondary mathematics teacher preparation and technology. In M. E. Strutchens et al. (Eds.), *The mathematics education of prospective secondary teachers around the world* (pp. 17–24). Springer.
Kemmis, S., & McTaggart, R. (2005). Participatory action research: Communicative action and the public sphere. In N. Denzin & Y. Lincoln (Eds.), *The Sage handbook of qualitative research* (3rd ed., pp. 559–603). Sage.
Lee, H., & Hollebrands, K. (2008). Preparing to teach mathematics with technology: An integrated approach to developing technological pedagogical content knowledge. *Contemporary Issues in Technology and Teacher Education, 8*(4), 326–341.

McCulloch, A. W., Lovett, J. N., Leatham, K. R., Bailey, N., & Reed, S. D. (2019). Preparing secondary mathematics teachers to teach with. technology: Findings from a nationwide survey. In S. Otten, A. G. Candela, Z. de Araujo, C. Haines, & C. Munter (Eds.), *Proceedings of the 41st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1126–1130). University of Missouri.

McCulloch, A. W., Bailey, N., Fye, K., & Scott, G. (2020). Creating a third-space for learning to design technology-based math tasks. *Mathematics Teacher Educator, 9*(1), 7–22.

McIntyre, D. (2005). Bridging the gap between research and practice. *Cambridge Journal of Education, 35*(3), 357–382.

Meagher, M., Ozgun-Koca, A., & Edwards, M. T. (2011). Pre-service teachers' experiences with advanced digital technologies: The interplay between technology in a pre-service classroom and in field placements. *Contemporary Issues in Technology and Teacher Education, 11*(3), 243–270.

Merriam, S. B., & Tisdell, E. J. (2016). *Qualitative research: A guide to design and implementation* (4th ed.). John Wiley & Sons.

Ministry of National Education in Turkey. (2018). *Teaching programs for secondary mathematics lesson (Grades 9, 10, 11, 12)*. MEB.

National Centre for Excellence in the Teaching of Mathematics -NCETM. (2014). *The national curriculum in England.* Retrieved April 16, 2021, from https://assets.publishing.service.gov.uk/government/uploads/system/uploads/attachment_data/file/840002/Secondary_national_curriculum_corrected_PDF.pdf

Ndlovu, M., Wessels, D., & De Villiers, M. (2013). Competencies in using Sketchpad in geometry teaching and learning: Experiences of preservice teachers. *African Journal of Research in Mathematics, Science and Technology Education, 17*(3), 231–243.

Niess, M. L. (2005). Preparing teachers to teach science and mathematics with technology: Developing a technology pedagogical content knowledge. *Teaching and Teacher Education, 21*(5), 509–523.

Niess, M. L. (2012). Rethinking pre-service mathematics teachers' preparation: Technological, pedagogical and content knowledge (TPACK). In D. Polly, C. Mims, & K. Persichitte (Eds.), *Developing technology-rich, teacher education programs: Key issues* (pp. 316–336). IGI Global.

O'Leary, Z. (2004). *The essential guide to doing research*. Sage.

Powell, A. B., Alqahtani, M. M., & Singh, B. (2017). Supporting students' productive collaboration and mathematics learning in online environments. In R. Jorgensen & K. Larkin (Eds.), *STEM education in the junior secondary: The state of play* (pp. 37–56). Springer.

Powers, R., & Blubaugh, W. (2005). Technology in mathematics education: Preparing teachers for the future. *Contemporary Issues in Technology and Teacher Education, 5(3), 254–270.*

Rocha, H. (2020). Using tasks to develop pre-service teachers' knowledge for teaching mathematics with digital technology. *ZDM—The International Journal on Mathematics Education, 52*(7), 1381–1396.

Ruthven, K. (2014). Frameworks for analysing the expertise that underpins successful integration of digital technologies into everyday teaching practice. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era* (pp. 373–393). Springer.

Ruthven, K., Deaney, R., & Hennessy, S. (2009). Using graphing software to teach about algebraic forms: A study of technology-supported practice in secondary-school mathematics. *Educational Studies in Mathematics, 71*(3), 279–297.

Sagor, R. (1992). *How to conduct collaborative action research*. Association for Supervision and Curriculum Development.

Shagoury, R., & Power, B. M. (2012). *Living the questions: A guide for teacher-researchers*. Stenhouse Publishers.

Shank, M. J. (2006). Teacher storytelling: A means for creating and learning within a collaborative space. *Teaching and Teacher Education, 22*(6), 711–721.

Snow-Gerono, J. L. (2005). Professional development in a culture of inquiry: PDS teachers identify the benefits of professional learning communities. *Teaching and Teacher Education, 21*(3), 241–256.

Stark, S. (2006). Using action learning for professional development. *Educational Action Research, 14*(01), 23–43.

Stockero, S. L., & Van Zoest, L. R. (2013). Characterizing pivotal teaching moments in beginning mathematics teachers' practice. *Journal of Mathematics Teacher Education, 16*(2), 125–147.

Strutchens, M., Huang, R., Locano, L., Potari, D., Ponte, J. P., Cyrino, M. C., da Ponte, Z. P., & Zbiek, R. M. (2016). *The mathematics education of prospective secondary teachers around the World*. Springer.

Tabach, M. (2011). A mathematics teacher's practice in a technological environment: A case study analysis using two complementary theories. *Technology, Knowledge and Learning, 16*(3), 247–265.

Tabach, M., Hershkowitz, R., & Dreyfus, T. (2013). Learning beginning algebra in a computer-intensive environment. *ZDM—The International Journal on Mathematics Education, 45*(3), 377–391.

Trgalova, J., Clark-Wilson, A., & Weigand, H.-G. (2018). Technology and resources in mathematics education. In T. Dreyfus, M. Artigue, D. Potari, S. Prediger, & K. Ruthven (Eds.), *Developing research in mathematics education: Twenty years of communication, cooperation and collaboration in Europe* (pp. 142–161). Springer.

Trocki, A., & Hollebrands, K. (2018). The development of a framework for assessing dynamic geometry task quality. *Digital Experiences in Mathematics Education, 4*(2-3), 110–138.

Trouche, L. (2004). Managing the complexity of human/machine interactions in computerized learning environments: Guiding students' command process through instrumental orchestrations. *International Journal of Computers for Mathematical Learning, 9*(3), 281–307.

Trouche, L. (2020). Instrumentalization in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 404–412). Springer.

Verillon, P., & Rabardel, P. (1995). Cognition and artifacts: A contribution to the study of though in relation to instrumented activity. *European Journal of Psychology of Education, 10*(1), 77–101.

Yeh, Y. F., Hsu, Y. S., Wu, H. K., Hwang, F. K., & Lin, T. C. (2014). Developing and validating technological pedagogical content knowledge-practical (TPACK-practical) through the Delphi survey technique. *British Journal of Educational Technology, 45*(4), 707–722.

Zbiek, R. M., & Hollebrands, K. (2008). Incorporating mathematics technology into classroom practice. In K. Heid & G. W. Blume (Eds.), *Research on technology and the teaching and learning of mathematics* (Vol. I, pp. 287–344). Information Age Publishing, Inc.

An *Ensemble* Approach to Studying the Teaching of Multiplication Using *TouchTimes*

Sandy Bakos

Abstract This chapter examines the function of a novel, multi-touch iPad application called *TouchTimes* as it is integrated into the instructional repertoires of two primary school teachers in British Columbia, Canada. The aim of this research was to study the *ensemble* of teacher, tool and mathematical concept – in this case, multiplication – as it played out in the classroom. The *ensemble* views each part in relation to the whole, rather than individually. Using the notion of double instrumental genesis and the construct of instrumental orchestration, I examine case studies in order to identify and highlight specific ways in which the teachers adopted this digital tool into their mathematical pedagogical practice. Three new types of orchestration are identified that emerged from using touchscreen technology for mathematics in the context of primary school classrooms. I also observe ways in which the tool exerted agency in the classroom, especially in relation to new ways of speaking about multiplication and a new attention to fingers as means by which to express and engage with multiplicative relations.

Keywords Touchscreen technology · Double instrumental genesis (personal/professional) · Instrumental orchestration · *TouchTimes* · Multiplication · Primary school mathematics

---

Supplementary Information The online version contains supplementary material available at [https://doi.org/10.1007/978-3-031-05254-5_3]. The videos can be accessed by scanning the related images with the SN More Media App.

S. Bakos (✉)
Faculty of Education, University of Lethbridge, Lethbridge, AB, Canada
e-mail: sandy.bakos@uleth.ca

A. Clark-Wilson et al. (eds.), *The Mathematics Teacher in the Digital Era*, Mathematics Education in the Digital Era 16,
https://doi.org/10.1007/978-3-031-05254-5_3

1 Introduction

With the increasing prevalence of digital technology in schools, mathematics education researchers have a continued interest in how teachers choose to implement these resources for the teaching of mathematics (e.g., Monaghan, 2004; Thomas & Palmer, 2016) and their effect on student learning (e.g., Calder & Murphy, 2018; Sinclair & Baccaglini-Frank, 2015). The emergence of touchscreen devices and their ease of use for younger students are providing new means for primary school teachers to support mathematics learning, though Larkin and Milford (2018) have found that many of the apps downloaded onto these devices for classroom use are chosen, “without a strong conceptual, pedagogical, or methodological underpinning” (p. 12). Integrating technology remains a complex undertaking (Monaghan, 2004) and, despite some aspects of technology becoming more user-friendly, many teachers find it challenging to exploit skillfully the opportunities that technology can offer for learning (Trigueros et al., 2014), and this is also true in particular of *TouchTimes* (Sinclair et al., 2020).

While much of the early research on the use of digital technology in the mathematics classroom focused on student learning (e.g., Behr & Wheeler, 1981; Noss, 1987), often in classrooms where researchers were closely involved (sometimes even as teachers themselves), there has been a shift towards studying the phenomenon of teaching with technology (Artigue, 2010). Several theories that have emerged to account for the impact of digital technology integration on mathematics teaching practice include the notion of double instrumental genesis, which examines teachers’ personal and professional instrumental genesis (Haspekian, 2011), and the construct of instrumental orchestration (Trouche, 2004).

In this chapter, I am specifically interested in the double instrumental genesis of the teacher in relation to *TouchTimes* (TT), as well as the instrumental orchestrations (a notion coined by Trouche and enhanced/expanded by Drijvers et al., 2010, 2013) initiated or led by the teacher. However, while the instrumental orchestration approach has provided important insights into the ways that teachers have organised classroom activity to make use of digital tools, it prioritised the human over the tool (Carlsen et al., 2016). There is also a need to examine the ways in which the functioning of the tool itself impacts the teacher, especially in terms of shaping mathematical concepts. This is relevant to this study because the design of TT promotes an approach to multiplication that is different from—and indeed, sometimes contrary to—existing practices in primary schools in British Columbia (and beyond), a theme that will be explored further in the next section. Therefore, the teachers involved in this study were not only adopting a new digital tool into their instructional repertoire, they were also integrating an entirely different way of thinking about and teaching multiplication based on this application. Clark-Wilson (2010) found that teachers' mathematical ideas shaped how they used digital tools, but that their teaching was also shaped by their increasing familiarity with the tool itself. Indeed, as Noss and Hoyles (1996) point out, "tools wrap up some of the mathematical ontology of the environment and form part of the web of ideas and actions embedded in it" (p. 227).

Prioritising the teacher as the sole intentional agent in the classroom may fail to account for the effects of the tool itself or the mathematics—overly ascribing both responsibility and intentionality to the teacher, while under-appreciating the multiple roles that a teacher performs in a classroom, not all of them strictly didactical. By shifting the subject–object framing towards an *ensemble* approach, in which the teacher–tool–mathematical concept is viewed as a whole, rather than individually, I was better able to notice the emergence of new classroom phenomena that did not fall into intentional teacher choices, but arose from unexpected, spontaneous occurrences.

The next section of this chapter provides some context related to multiplication and, in subsequent sections, I will situate the theoretical foundation and introduce the construct and theory used. This will then be followed by two sections that focus specifically on the project: its methods, which include a brief description of TT, as well as the study context and participants. I will then detail the case studies of two primary teachers who were integrating TT into their classroom practice. Finally, I close with a discussion focused on some of the issues that emerged which are directly linked to the mathematics as it is presented by TT and then on the types of orchestrations used with touchscreen technology in primary classrooms. Though I will articulate my research question after elaborating a theoretical framing, my aim is to study the teacher–tool–mathematics *ensemble* in a primary school classroom (within the context of this chapter, a grade 3 and a 3–4 classroom), as it is perturbed by teacher–*TouchTimes* in concert.

2 Multiplication

In the early primary grades, skip counting, equal grouping and repeated addition are commonly used methods for introducing and working with multiplication (Davis & Renert, 2013; Greer, 1992), even though research indicates that characterising multiplication as repeated addition has limitations (Askew, 2018; Boulet, 1998; Davydov, 1992). Askew (2018) noted that the practice of repeated addition is encouraged by curriculum developers and remains a persistent perception of multiplicative situations for primary teachers and their students. This is also true in British Columbia, where multiplication first appears in the provincially mandated third grade (8–9-year-olds) mathematics curriculum and the examples given are: groups-of, repeated addition and arrays (Province of British Columbia, 2016). Fischbein et al. (1985) claimed that repeated addition is a *primitive* model that, “tacitly affects the meaning and use of multiplication, even in persons with considerable training in mathematics” (p. 6) and that it “reflects the way in which the corresponding concept or operation was initially taught in school” (p. 15).

The benefits of visual representations and the use of different models as mathematical tools for teaching and learning multiplication have been highlighted by researchers (Anghileri, 1989; Kosko, 2018; Maffia & Mariotti, 2018). Though Davis & Renert (2013) outlined multiple models of multiplication, such as number-line hopping, making a grid or rectangular array, area generation, branching, scaling and linear function, they suggested that multiplication as repeated addition is so well rehearsed, that it may eliminate other interpretive possibilities for multiplication. Referring to the work of Bolden et al. (2015) and Davydov (1991), Kosko (2018) noted the importance of how plural aspects of units are conveyed in multiplication and that, “students’ interactions with visual representations may involve attending to the multiplicative nature conveyed by the visual, or may involve counting all units by ones” (pp. 262–263). In the Davydovian approach (Davydov, 1992), the first unitising occurs when the multiplicand is established (e.g., number of tires on a car), followed by the second unitisation, which is the number of units (e.g., how many cars). In order to be considered multiplicative, Steffe (1992) described the necessity of co-ordinating at least two composite units “in such a way that one of the composite units is distributed over the elements of the other composite unit” (p. 264). Jacob and Willis (2003) concluded that it is the “identification or construction of the multiplicand and the multiplier within a situation, and the simultaneous coordination of these factors, that signified a multiplicative response” (p. 460).

When discussing methods for teaching multiplication, while participating in an *a priori* study (see Sinclair et al., 2020), the two primary teachers who are part of this study (Leah and Rachel, pseudonyms) referred to the curriculum requirements and the strategies presented in the textbook as resources that guide their teaching. Leah even pulled out a third-grade textbook, stating, “It’s all groups-of. Groups-of is the first thing, okay repeated addition… Groups-of. Two groups-of five, so this would be two times that [pointing at a picture showing two groups-of five with the associated number sentence, 2×5] whereas this [pointing at the iPad with TT on it], shows it the other way”. The primary resources that these teachers rely on do not include other approaches, such as the use of double number lines (as described in Askew, 2018) or the Davydovian approach based on a double unitisation. This lack of awareness also contributes to these primary teachers’ heavy reliance on equal grouping, skip counting and repeated addition.

3 Situating the Theoretical Foundation

I begin by outlining some theoretical elements of the instrumental approach (Artigue, 2002; Guin et al., 2005) and introduce double instrumental genesis (Haspekian, 2011). I will then detail the construct of instrumental orchestration (Trouche, 2004), highlighting some of its conceptual tools and results, with a particular focus on its relation to the study of technology integration in the context of primary school mathematics.

3.1 The Instrumental Approach

Extending Rabardel's (1995) theory of instrumentation on the human use of tools, the instrumental approach was developed for the analysis of technology-mediated teaching and learning in mathematics (Artigue, 2002; Guin et al., 2005). Two ideas used by the instrumental approach that are part of the theory of instrumentation are the characterisation of artefact/instrument and the acknowledgement that tool use is a two-way process. Vérillon and Rabardel (1995) distinguished an artefact as a physical object or tool that, through human use, becomes an instrument. This interaction between artefact and humans builds an instrument, using a two-way process called *instrumental genesis*, where the user adapts to the tool (*instrumentalisation*), and the tool shapes the user's actions or thinking (*instrumentation*). Before using a digital tool in a classroom context, teachers must first engage in a personal instrumental genesis, similar to all learners, where the artefact becomes an instrument for mathematical activity. In addition to this, teachers must also engage in a professional instrumental genesis, in order to construct and appropriate the artefact into a didactical instrument for teaching mathematics. "The teacher's professional genesis with the tool is much more complicated as it includes the pupils' instrumental genesis" (Haspekian, 2014, p. 254). This dual process has been termed a *double instrumental genesis* by Haspekian (2011). Researchers have also studied how it is that teachers plan for and make decisions within an instrumentalised classroom, which I will discuss in the next sub-section.

3.2 The Construct of Instrumental Orchestration

First used by Trouche (2004), *instrumental orchestration* involves, "the teacher's intentional and systematic organisation and use of the various artefacts available in a—in this case computerised—learning environment in a given mathematical task situation" (Drijvers et al., 2010, pp. 214–215). This may guide the instrumental genesis of individual learners or encourage whole-class, collective instrumental genesis.

Table 1 A summary of whole-class and individual orchestrations

Orchestration type	Didactical intention	Whole-class	Individual
Technical-demo	Techniques for tool use are demonstrated by the teacher	✓	✓
Guide-and-explain	The teacher explains or asks closed-type questions based on what is on the screen	✓	✓
Explain-the-screen	The teacher explains the mathematical content related to the digital technology	✓	
Link-screen-board or link-screen-paper	The teacher connects the mathematical ideas or representations from the technological device to the way this mathematics is commonly recorded	✓	✓
Discuss-the-screen	Class discussion about what is happening on the screen	✓	✓
Spot-and-show	Student work samples are used for class discussion and/or teaching	✓	
Sherpa-at-work	A student uses the technology to present work and/or carries out teacher-directed actions	✓	
Board-instruction	Teaching in front of the board with no real-time reference to, or use of, technology	✓	
Technical-support	Providing technical support		✓

Drijvers et al. (2013)

Eight instrumental orchestration types for teacher-led, whole-class instruction and five further orchestration types used for individual students while working independently (see Table 1) have been identified (Drijvers et al., 2010; Drijvers et al., 2013). The teacher may decide to use teacher-centred orchestrations or ones that explicitly invite student participation. Such decisions are part of the *exploitation modes* used by teachers to benefit their teaching goals. This may include ways that tools or tasks are introduced and engaged with, forms of user interaction (e.g., partner work) or techniques developed by the students. *Didactical performance*, how teachers adapt plans 'on the fly' while teaching, is of particular importance when engaging in student-centred orchestrations. Things such as how to address student input, what questions to ask and when to ask them, and solving unanticipated issues, which Clark-Wilson (2010) terms 'hiccups', related to the technological tool, the mathematical task or the students themselves are all part of the intertwined nature of teacher–tool–mathematics *ensemble*.

Tabach (2011) argued that, if a lesson includes technology that is available, but the teacher intentionally chooses not to use it, then this too is an orchestration type, which she terms a *Not-use-technology* orchestration. Though Tabach (2013) noted that Drijvers (2012) used the term *Work-and-walk-by* for individual orchestrations where the students work independently, while the teacher monitors student progress and assists as needed, she preferred the term *Monitor-and-guide,* an orchestration type where students are working independently while the teacher monitors and guides progress, either in person or through electronic feedback. Also emerging from this research in fifth- and sixth-grade classrooms were instances where the teacher discussed the use of technology, but did so without the technology present. It is termed *Discuss-tech-without-it* orchestration.

When applying the construct of orchestrations in a kindergarten setting in France, Gueudet et al. (2014) identified two new orchestration types: *Autonomous-use,* when children are able to use the technology independently with the teacher monitoring from a distance, and *Supported-use*, when individual children require teacher assistance to engage either with the technology or with the mathematics itself. Both of these orchestrations were used as ways to manage class heterogeneity and differentiate instruction to meet individual learner needs. Also emerging from the French kindergarten context was a variation of the *Link-screen-board* orchestration, which Besnier (2018) termed a *Manipulatives-and-software-duo* orchestration. Here, the teacher creates concrete manipulatives reflective of the digital technology, which are displayed and can be manipulated to demonstrate or explain experiences with the software.

Much of the research focused on instrumental orchestration has been situated either at the secondary-school level (Trouche, 2004; Drijvers, 2012) or at the kindergarten level (Besnier, 2018; Gueudet et al., 2014). In fact, at a digital technology conference in mathematics education with 'orchestrating learning' as the central theme, Joubert (2013) noted that there were only five out of over one hundred submitted papers about technology use related to primary schools. In addition, prior instrumental orchestration research has typically involved either desktop software or interactive whiteboards, rather than touchscreen technology.

I was curious about how the age of the students and the nature of the technology influence the orchestrations used by the teacher. For example, in the two classes I observed, there were multiple iPads available, thus allowing individual or pairs of children to work simultaneously on their own devices, rather than needing to observe a single, shared, digital device, such as a stationary computer connected to a projector or a single interactive whiteboard. Additionally, the dexterity involved in using a computer mouse productively can prove physically difficult for young children, whereas the touchscreen affordances of an iPad make it a more easily accessible form of technology for primary students.

Within the construct of instrumental orchestration, the teacher is usually positioned as the main agent in organising, arranging, adapting and managing the task, the tool and classroom interactions. There is an underlying assumption that the presence of the tool has prompted didactic configurations and exploitations, in addition to shaping didactic performance. However, until recently, little attention has been paid to the agential role of the tool itself. For example, Gueudet et al. (2014) briefly noted that their data clearly demonstrated that several of the software's features did influence the orchestration choices made by the kindergarten teachers, though their research focus stated that, "Orchestrations can be considered as the choices made by the teachers about the use of technology in their classrooms" (p. 215).

In a second example, Carlsen et al. (2016) drew on Pickering (1995) to position the teacher, the learner, the digital tool and the mathematics (as agents) interacting in what they termed "distributed agency". They argued that the phrase *teachers' choices*, used by Drijvers et al. (2010), "obscures the influence/agency of digital tools in understanding teachers' use of digital technology in mathematics classrooms" (p. 15). This prompted me to pay attention to when non-intentional actions on the part of the teacher may be driven or influenced by the tool.

The following three research questions are considered in this chapter:

1. What orchestrations, or sequences of orchestrations, does a primary school teacher employ when using touchscreen technology (TT specifically) in teaching mathematics?
2. How does the way in which TT materialises the mathematical concept of multiplication impact the pedagogical choices of a teacher?
3. How does a teacher's professional instrumental genesis evolve while using TT as a teaching tool?

In order to respond to these questions, I draw on the data gathered during the broader research project described in the method section that follows.

4 On Method

As a member of the initial *TouchTimes* research team, I am aware of some of the intentional choices related to TT's design and its potential to support the teaching and learning of mathematics. I will situate some of the research-informed design choices within the research literature on the teaching and learning of multiplication.

4.1 TouchTimes as a Multiplying Machine

Designed to enable young children's experiences with multiplication that are multiplicative rather than additive, TT (Jackiw & Sinclair, 2019) is a multi-touch, iPad application. Children receive direct visual, symbolic and haptic feedback as they create and transform pictorial representations of multiplicative situations on the TT screen through their fingertips. There are two microworlds in TT, *Grasplify* and *Zaplify*, and I will briefly describe the one relevant to this chapter, namely, Grasplify.[1] Embodying the co-ordination of units in a visually singular form, in this world each of the user's hands becomes either the multiplicand or the multiplier. In order to more easily visualise and better understand how the mathematics in Grasplify functions in response to a user's fingertips, this short, 2-min video demonstration, and brief explanation of some Grasplify basics may be helpful (see video Fig. 1).

Grasplify opens with a blank screen, split in half by a vertical line (Fig. 2a). Designed to be symmetric, coloured dots (called 'pips') appear on whichever side of the screen is touched first, and remain present while the user's fingertips are still

[1] For a more detailed description of Grasplify, see Bakos and Pimm (2020), pp. 148–150.

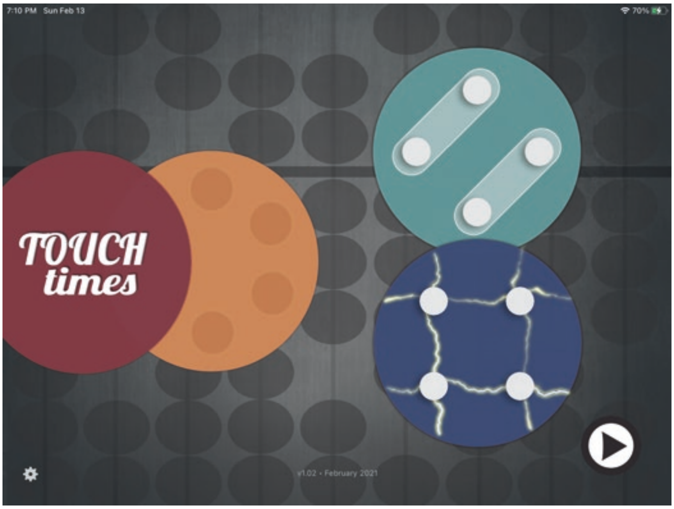


Fig. 1 Short video demonstration of Grasplify (▶ https://doi.org/10.1007/000-8wt)

a b

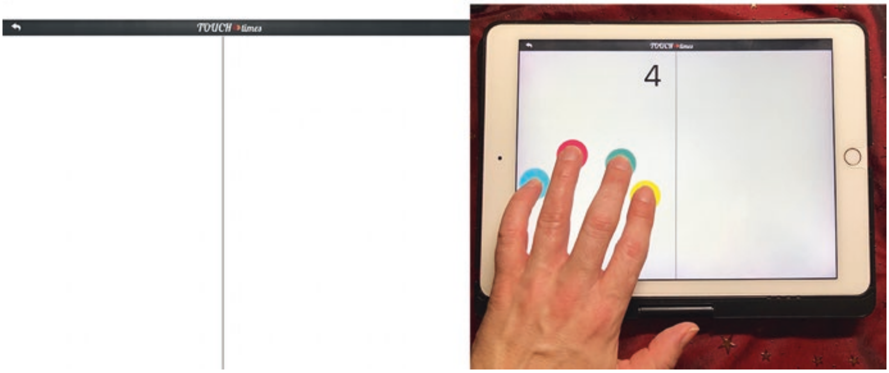


Fig. 2 (**a**) Initial screen; (**b**) pip creation

in contact with the screen (in this instance, the left side in Fig. 2b). Enclosed bundles of pips (named ‘pods’) are then created by finger taps (either singly or collectively) on the opposite side of the screen from where the pips are held (Fig. 3a). Unlike pips, which require continuous finger-screen contact to remain present, pods remain visible on the screen even after finger contact has been removed, and are visual duplicates, both in colour and in spatial orientation, to their corresponding pip configuration. The composition and shape of the pods adjust instantly to the addition or removal of pip-creating fingers from the screen, unless all pip-fingers are removed, which resets the screen (effectively multiplying by zero). The numerical expression at the top of the screen symbolically represents the multiplier, the

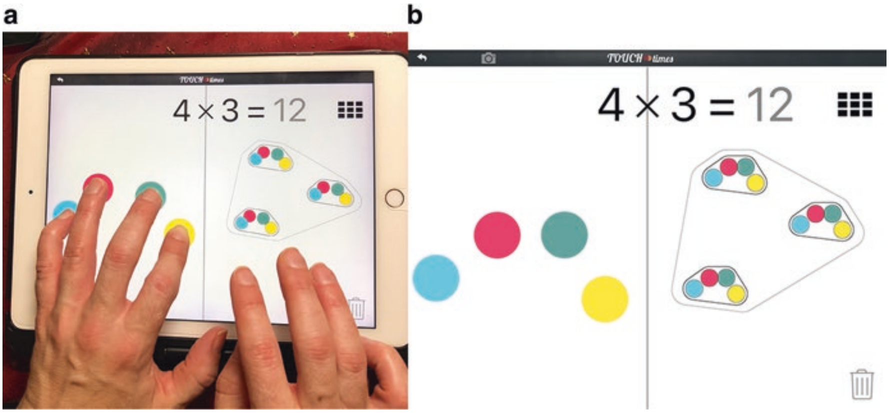


Fig. 3 (**a**) Post-pod creation; (**b**) two composite units showing what would be projected on the screen

multiplicand and the corresponding product (Fig. 3b), and is produced in time, element by element, and adjusts automatically as pips and pods are created or removed.

The functional and relational aspects of TT's design were inspired by Vergnaud's (1983) work on the conceptual field of multiplication, which focuses on doubling, tripling, etc., rather than repeated addition. Grasplify also embodies the co-ordination of Davydov's (1992) double change-in-unit process, as a unit (the multiplicand, represented by the pips) must first be created, before a unit of units (the multiplier, represented by the pods) can be made. This ordering of multiplicand × multiplier is the opposite of what British Columbian teachers (among others) usually encounter in textbooks. However, this ordering is intentional in TT and is grounded in approaches to early mathematics based on measurement and ratio, where the unit quantity is identified prior to the number of units. The order of factors in TT, in which the multiplicand precedes the multiplier, is consistent with the Davydovian approach to multiplication, where one wants to identify the *unit quantity* before asking 'how many units?' This approach is therefore asymmetric, in that the chronological order of the two factors' appearance is important. But this order is the opposite of what is found in most of the textbooks and resources used in Canada, where multiplication is primarily introduced through repeated addition and where the multiplier always precedes the multiplicand in terms of the notation. The TT design embodies alternative models of multiplication, while making the functional and the change-in-unit approaches accessible to young children.

4.2 Study Context and Participants

The episodes described in this chapter took place in two primary classrooms, in different schools in British Columbia, Canada, during the 2019–2020 school year. Both of the teachers involved are experienced Canadian primary teachers, with master's degrees, who teach French Immersion in grade 3 or 3–4. The teachers had volunteered to be part of a larger, multi-phase project involving the integration of TT and the collaborative development of tasks to be used with it.

The data used in this chapter comes from phase two of the project, when the research team (two professors and two doctoral students, including the author) were invited by both teachers into their classrooms to observe mathematics lessons where TT was being used. Members of the team (between two and four depending on availability) observed and video-recorded a total of seven 60- to 90-min mathematics lessons in the two classrooms. These observations began with three visits to Leah's third-grade classroom in October–November 2019. There were then two classroom visits in Rachel's grade 3–4 classroom in December 2019 and another two in March 2020.

There was one video camera set up either in the corner or in the centre of the classroom, in order best to capture the teacher-led, whole-class aspects of the lesson. An additional camera was used by one member of the research team to record what individual students or pairs of students were doing on their iPad while exploring their assigned tasks. Field notes taken during these visits, as well as the digital recordings of the whole-class lessons and partner work, were examined and the episodes shared here have been chosen to illustrate each teacher's orchestrating of TT in their classroom and to highlight how TT has impacted these teachers.

The next sub-section draws on the data outlined above to present case studies of these two primary teachers and the episodes have been chosen to develop a picture of each teacher's orchestrating of TT and to highlight the influence of the digital technology and its presentation of mathematical ideas on the teachers.

4.3 Data Analysis

The data was analysed with two specific aims. The first was related to orchestrations and the second concerned double instrumental genesis. Initially, I identified the orchestration types used by each teacher and then looked for sequences of orchestration that were commonly used by both. Additionally, I wanted to understand how each teacher's professional instrumental genesis evolved while using TT as a teaching tool.

I was conscious of three factors essential to my research: (1) the primary school context; (2) the touchscreen nature of the digital technology; (3) the novelty of the TT model of multiplication (both for teachers and for students). When examining these sequences of orchestrations, I was also looking for lesson 'hiccups' (Clark-Wilson, 2010) arising from the teacher–tool–mathematics *ensemble*: "the perturbations experienced by the teachers during the lesson, triggered by the use of the technology that seemed to illuminate discontinuities in their knowledge and offer opportunities for the teachers' epistemological development within the domain of the study" (p. 138).

Research based on double instrumental genesis (Haspekian, 2011, 2014) has been carried out in the context of teaching with spreadsheets at the secondary-school level. Unlike spreadsheets, TT is a digital tool specifically developed for teaching mathematics with primary students. However, as discussed earlier in this chapter, TT's functional and change-in-unit approaches to multiplication are very different from those with which British Columbian primary teachers are familiar.

5 Case Studies of Instrumental Orchestrations

I begin with some general-level observations about the orchestration types and configurations across both classrooms. As I illustrate them, I also point out instances of the teacher–tool–mathematics *ensemble*, the development of teacher professional genesis and the impact of a lesson 'hiccup' in the creation of a new orchestration.

5.1 Sequences of Orchestrations

Both Leah and Rachel had access to sufficient iPads with TT downloaded onto them for individual or pairs of students to use and were able to plug an iPad into a projector, which presented an enlarged screen image of TT on the wall for all students to see. Rachel sometimes used an Elmo device, so that the projected image also showed the user's fingers maneuvering on the iPad screen.

During our team presence in both classrooms, a similar sequence of orchestrations was used by the teachers when introducing a new task using TT. For example, Leah made 1 × 2 = 2 on an iPad (a *Technical-demo*) that was projected onto the screen for all to see (Fig. 4a) and asked, "What is the product?", thereby introducing unfamiliar mathematical vocabulary (an element of the mathematics register) to her

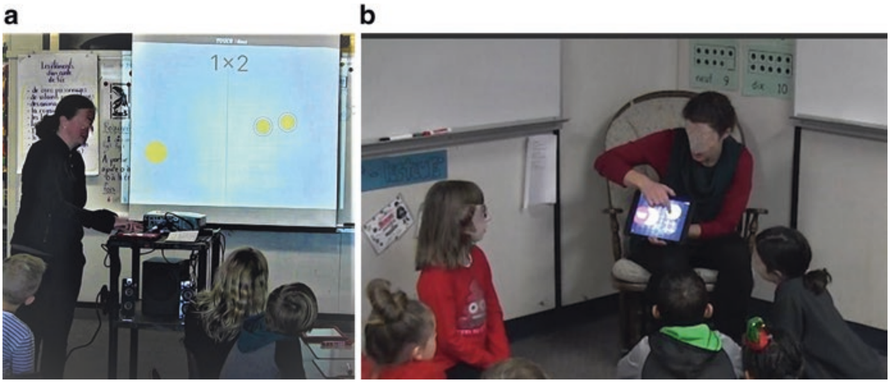


Fig. 4 (**a**) Leah's *Technical-demo* projection; (**b**) Rachel's *Technical-demo*

third-grade students. Invoking the teacher-centred *Explain-the-screen* orchestration, Leah then physically pointed out the product on the screen.

Rachel preferred to begin her lesson by gathering students on the carpet in a group, often holding an iPad up for students to view, while demonstrating (*Technical-demo*) what she wanted students to do (Fig. 4b). On occasion, she also verbally explained the TT task using a *Board-instruction* orchestration where, for example, she wrote the target products that students were to produce using TT (such as multiples of 3).

I now provide a full elaboration of the second lesson observed in Rachel's grade 3–4 classroom, which illustrates the use of follow-up orchestrations. All students were seated on the carpet in front of Rachel, and she began by asking them what colour the product is. When she asked this question, she did not have an iPad in her hands, nor one projected for students to see. Despite this, one of the students immediately answered, "*Blanc* [white]". Though the technology was not physically present, Rachel drew upon student mental images of TT through this *Explain-the-screen* orchestration, linking mathematical vocabulary with the colour of the product symbol displayed by TT. She then explained that students were to use TT to make the product go up by twos. A short (84-second) visual demonstration and explanation of how to skip count using Grasplify by creating additional pods *and* how to skip count by placing more pip-creating fingers on the screen can be viewed in video Fig. 5. A brief description of the mathematical differences in these two ways of skip counting is also included.

As students worked on the task in pairs, Rachel engaged in a *Monitor-and-guide* orchestration, moving throughout the room, monitoring student progress, answering questions and providing differentiated instruction, as necessary. Once most pairs had successfully completed the task, Rachel brought the class back together and, using a *Spot-and-show* orchestration, asked a pair of students to share. The pair chosen came to the front of the group, becoming *Sherpas-at-work* (Fig. 6), holding up an iPad for their classmates to see, while demonstrating and explaining how they had completed the assigned task.

Fig. 5 Video demonstration of skip counting by pips (▶ https://doi.org/10.1007/000-8ws)

Fig. 6 *Sherpas-at-work*

While engaging the pairs in the *Explain-the-screen* orchestration, Rachel also simultaneously interacted with the pair (and sometimes with the whole class) in a *Discuss-the-screen* orchestration, drawing attention to the mathematics. [The transcript below was translated from French.]

Rachel: Show us *one* way to count by two. [Student 1 places two pip-creating fingers on the screen. Then student 2 creates two pods.] Wait, describe what you're doing. [Students remove fingers and start again.]

Student 2: So, go up two. [Student 1 creates two pips.]

Rachel: Those are two pips. [Points to left screen where the pips are being created.]

Student 2: Pips.

Rachel: Pips. [Nodding.]

Student 2: And then we're going to add pods here. [Student 2 adds two pods to create $2 \times 2 = 4$.]

Rachel: Pods. [Nodding.]

Student 2: We just add one and then… [Student 1 places another pip-finger on the screen, creating $3 \times 2 = 6$. Then sequentially adds one, two, three more pip-fingers from her other hand; $4 \times 2 = 8$, $5 \times 2 = 10$, $6 \times 2 = 12$.]

Rachel: You're adding to the pips or the pods? [Student 2 adds one, two, three more pips; $7 \times 2 = 14$, $8 \times 2 = 16$, $9 \times 2 = 18$.]

Student 2: Pips. And it makes it go by two. [Rachel assists by placing two pip-creating fingers on the screen. $10 \times 2 = 20$, $11 \times 2 = 22$.]

Rachel: So, how many pods are there?
Student 2: Two.
Rachel: And how many pips in each pod? How many?
Student 2: Eleven.
Rachel: Have we seen what the girls did?
Students: [Some respond yes, and others respond no].
Rachel: Where do you add? Did you add to the pods or to the pips?
Student 2: The pips.
Rachel: The pips. Okay, thanks. Who found another way to make the product go up by two?

As the pair explained their method of skip counting, Rachel assisted by providing more pip-creating fingers, while simultaneously engaging in a *Discuss-the-screen* orchestration, clarifying how many pips and how many pods there were. She also specifically asked how many pips were in the pods and whether they adjusted the number of pips or pods when making the product go up by two. In so doing, Rachel led the pair beyond the technique used with the technology, towards a more explicit explanation involving mathematical language related to multiplication, while also building collective instrumental genesis by highlighting that the pair had counted up by twos by changing the number of pips each time.

In this moment, the teacher–tool–mathematics *ensemble* was working in concert and the elements were mutually influencing each other. This can be seen as Rachel spoke to the children in terms of pips and pods, directly using the language of TT in her instruction and merging it with the language of skip counting. It can also be seen in the way fingers became part of the multiplicative expression. Later in the lesson, students skip counted backwards from twenty. Rachel asked a pair who had created $5 \times 4 = 20$ on their TT screen, "Can you explain why you can count down by five or four? If you take away one finger right now, will it count down by five or by four?" In this instance of the tool–mathematics merger, it is unclear where the tool stops, and the mathematics begins.

Though Rachel had taught skip counting before, during her implementation of TT, skip counting had become a new mathematical concept, more nuanced and complex. The concept, and therefore her pedagogical strategies (of doing it in different ways), have both changed. Her teacher pedagogy now includes the TT-inflect concept. The design of TT and the manner in which additional finger touches affect the multiplication model displayed on the screen have mathematical implications that influence the didactical performance of the teacher. TT's design allows for the product to increase by two in different ways, thus providing a pedagogical opportunity for Rachel, which will be discussed in more detail in the sub-section that follows. Her ability to recognise and exploit this situation, in order to support students' instrumental genesis and mathematical learning, is indicative of her professional genesis of TT as a teaching tool.

5.2 New Orchestrations

In this sub-section, I highlight three new instrumental orchestrations identified in the two classrooms. The first one can be exemplified by continuing the lesson from Rachel's classroom discussed above. We pick up where a new pair had just shown another way of counting by twos, by adding successive pods.

Rachel	How many pods are there now? [Points toward the right side of the screen where the pods are. 2 × 12 = 24.]
Student 3	Twelve.
Rachel	Yes, twelve. And how many pips are there?
Student 4	Two.
Rachel	Look at this image. [2 × 12 = 24.] How is their image different from the image of the first pair [12 × 2 = 24.]? How is this image different?
Student 5	Because they add the pods and not the pips.

As the two students used their iPad to demonstrate and explain their method for counting up by twos, Rachel once again engaged in a *Discuss-the-screen* orchestration by asking how many pips and how many pods were on the final screen. She then pointed to the screen (Fig. 7), asking the class to look at the current image and identify how it is different from the first one. In this instance, Rachel was engaging in a variation of the *Discuss-the-screen* orchestration by referring to the visible technology *and* to a mental image of the previous screen shared by the first pair, which required students to make a sequential comparison. This orchestration was used by both teachers several times to compare different strategies and I shall refer to it as a *Compare-successive-screens* orchestration.

Fig. 7 *Discuss-the-screen*

This orchestration was also observed and used in whole-class orchestrations in Leah's classroom, though in a slightly different way. Often, with a particular configuration on the screen, Leah would ask the class prediction questions such as, "If I put one more pip-finger down, make a prediction about how the sentence at the top will change. What is that going to look like? What are the numbers at the top going to say?" This involves a *Link-screen-board* orchestration, by having students notice the interaction among the pips and pods and the mathematical sentence displayed by TT. However, it also involves a *Compare-successive-screens* orchestration, where the focus is on comparing the present screen state with an imagined future one.

In drawing students' attention to how the pips (the multiplicand) influence the pods (the multiplier), the *Compare-successive-screens* orchestration is an exploitation mode used by the teacher, that once again exemplifies the emergence of the teacher–tool–mathematics *ensemble*. Rachel was drawing students' attention to the two different ways in which multiplication was occurring in TT. In order to skip count by twos and achieve $2 \times 12 = 24$ in TT, students started with two pips and sequentially created additional pods, a strategy reminiscent of repeated addition, whereas $12 \times 2 = 24$ required students to begin with a single pip and two pods. Students then sequentially placed a second, third, fourth, etc. pip-finger on the screen. By so doing, each additional pip 'spreads' across all pods (two pods results in an increase of two with each new pip). By using *Comparing-successive-screens*, both Rachel and Leah (with her predictive questions) could focus attention on the effects of additional finger placements on the product. Rachel also chose to highlight the different ways of skip counting by twos through the use of this orchestration. The *Comparing-successive-screens* orchestration occurred during both teachers' didactical performances, evidence of the co-implication of TT in their ways of thinking about *and* teaching multiplication. TT had become a mathematics–teaching instrument.

In the following example, Leah used a specific orchestration that is related to *Link-screen-paper,* where students are requested to draw a particular screen configuration. Starting from the initial configuration of one pip and three pods, students were asked to show how they could double three using TT. After students shared their strategies for doing this, they were asked to use their mini-whiteboards to draw what the TT screen would look like after doubling three. I see this as an important and distinct orchestration that I will call *Document-screen-on-paper,* as it requires students to reproduce various elements of the screen, as can be seen in examples Fig. 8a, b, which included the vertical line dividing TT into two, the use of different colours for the pips and the reproduction of those colours in the pods, and the multiplication equations, as well as some of the screen icons.

The dynamic nature of TT allows children an opportunity to create, see and feel through their fingertips an entirely different experience of multiplication from that produced through static images on worksheets. Once their fingers are removed from the TT screen, however, the images vanish. Drawing is commonly used in primary classrooms as a way of capturing and expressing ideas, and the *Document-screen-on-paper* orchestration encourages students to examine more carefully the screen

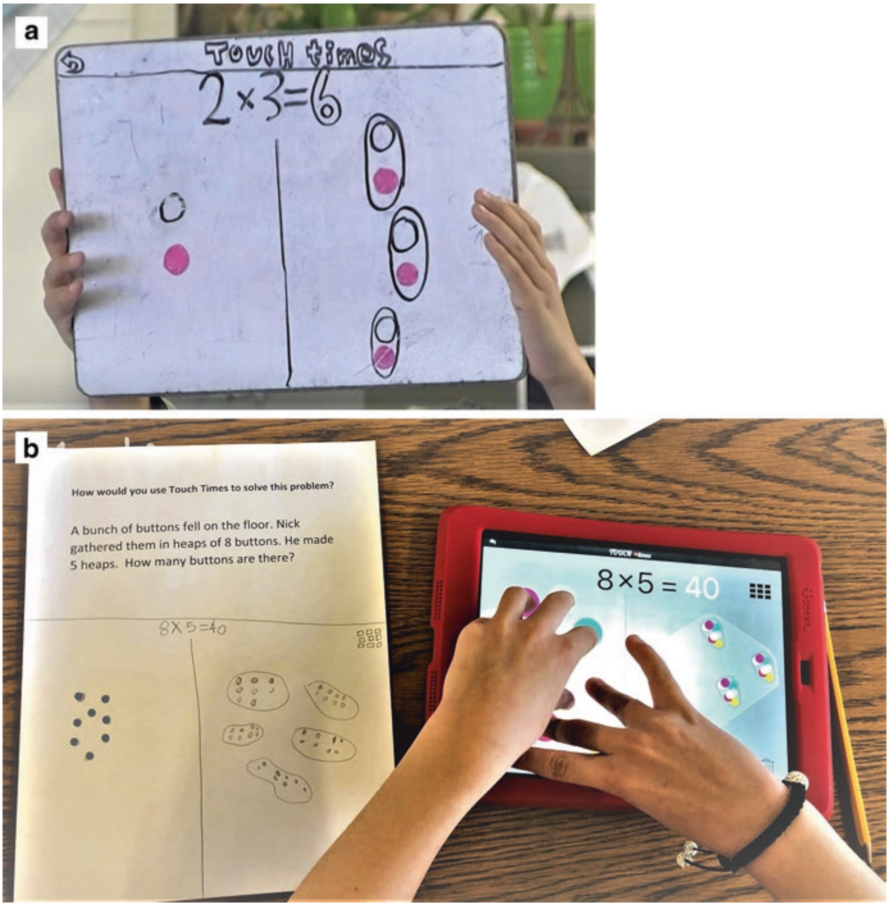


Fig. 8 (**a**) Document-screen-on-paper on a mini-whiteboard; (**b**) Document-screen-on-paper drawing

images that are modeling multiplication, and to replicate these in their drawings. The shape, colour and composition of the pods (multiplier) are all important in relation to the pips (multiplicand).

A final new orchestration observed, which is specific to the touchscreen nature of TT, arose during the whole-class orchestrations and was a result of a lesson 'hiccup'. Although the iPad screen was projected, students were unable to see what the user's fingers were doing on the screen. This was problematic, as students could see the results of the finger manipulations on the projected screen image, but could not see how the fingers themselves were producing this. Leah was in the early stages of her professional genesis, and this was an unforeseen problem. Therefore, she had to address the students' need to see the actions of the user's fingers on the screen in other ways. This sometimes involved pointing to the screen herself (Fig. 9a), asking a child to point at the screen (Figs. 9b and 10a) and occasionally holding up an iPad

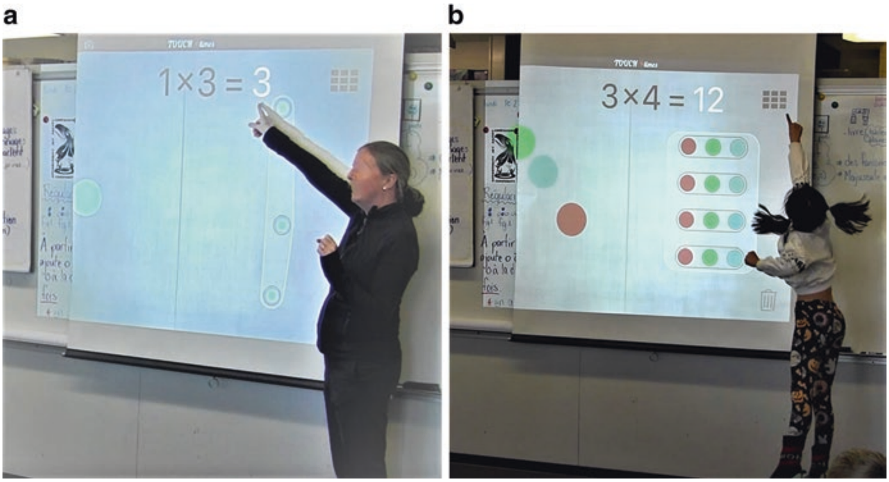


Fig. 9 (**a**) Pointing to the product; (**b**) pointing to the array button

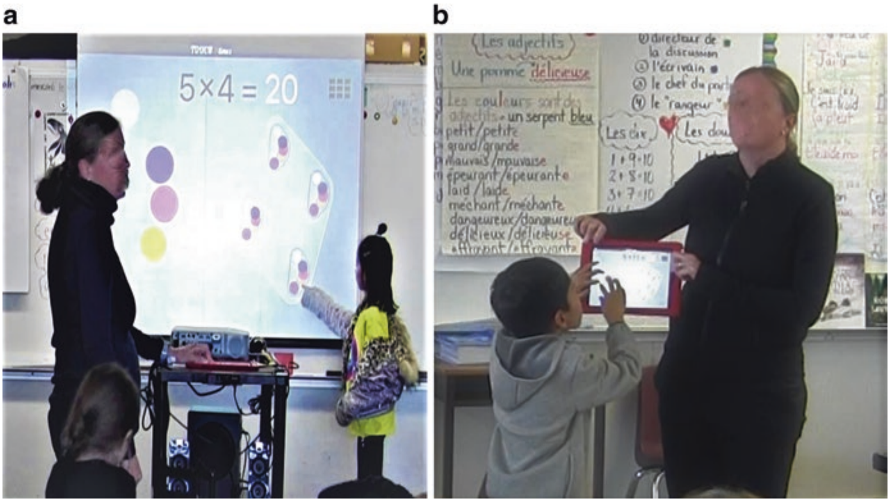


Fig. 10 (**a**) Pointing to a yellow pip within a pod; (**b**) *Sherpa-at-work* on an iPad

vertically for students to explain and demonstrate on the screen for their classmates (Fig. 10b), in what I call a *Discuss-the-finger-and-screen* orchestration.

TT requires finger-screen contact to function and, as a result, there is a natural emergence of an orchestration for *Discuss-the-finger-and-screen*. The mathematics accessed through TT can only be materialised through finger–screen contact. The importance of the fingers and the necessity for seeing them manipulating TT is an instance of the tool's influence on a teacher's actions. Moreover, fingers newly become part of how to express multiplication and how to attend to the process of producing a multiplicative expression, rather than just considering the result (as the static image shows, for example, in Fig. 10a). Again, we can see the emergence of an *ensemble* that performs in a way that is pedagogically, mathematically and technologically very distinct from normal practice.

5.3 Exploring the Tool–Teacher Relation

In the previous sub-sections, I have focused on documenting the sequences of orchestrations used and identifying new ones, while highlighting that *TouchTimes*, the mathematics and the teacher are not independent, but rather are co-implicated. In this sub-section, I focus on episodes where the tool had a strong agential role.

In the first lesson, Leah introduced TT to her students by engaging in a teacher-centred, *Technical-demo* to show students which button to push to enter the Grasplify world. In the video-recording, a student is overheard asking, “the light blue?” in reference to the colour of the Grasplify button. Leah then asked, “What do you notice happens on this side? [Fig. 11a] What do you notice happens on that side? [Fig. 11b]”. She demonstrated few features of the technology, only showing how to create a pip and pods. Though the images on the iPad screen were projected, students were unable to see what Leah’s fingers were doing (Fig. 11c). Her instructions were to play with and explore the technology, and to pay attention to what happens at the top of the screen when lifting, moving or adding a finger.

Students began their individual explorations on their iPads. Early in the exploration, a student asked a question. Leah stopped the class and asked the student to share her question, which was: “I was wondering if you can do… because there’s always like, one times something, but I was wondering if you can do zero times something?” Rather than answer the question, Leah used it as an example of a ‘wondering’ that could emerge while playing with TT, and requested that students record their ‘wonderings’ on their mini-whiteboards.

In this excerpt, I see a coming together of disparate elements that occasioned the taking up of this particular student ‘wondering’. Leah’s purpose for the lesson was clear: “I want you to play with what happens when”. She did not intend to discuss

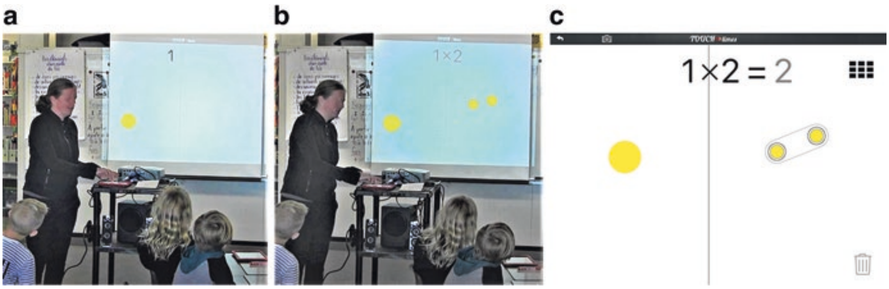


Fig. 11 (**a**) Creating a pip; (**b**) creating a pod; (**c**) screen projection

the issue of multiplication by zero and was likely surprised when a student wondered if it was possible to obtain 0 on the pip-making side of the screen. This instance of a lesson 'hiccup' was prompted by TT.

The multiplication symbols, in which *a* and *b* in the expression *a x b* both can be any numbers, brush up against the physicality of TT, in which there is no possible unit of 0, since one cannot touch the screen zero times. Though this 'hiccup' provided an opportunity to discuss multiplying by zero, which is generally not addressed in teaching approaches that rely on repeated addition, Leah chose to stay with her original lesson purpose, which was to familiarise students with the technology. The question was used as an opportunity to validate the act of 'wondering', positioning her as a teacher who invites open questions. It also concretised TT as a whole-number multiplying machine and provided a clear moment in which the characteristics of the tool destabilised the teacher.

5.4 The Order Matters, So Language Matters Too

During her professional instrumental genesis, Leah was engaging in a process of instrumentalisation where she "instrumentalised the tool in order to service didactic objectives" (Haspekian, 2014, p. 253). As she became more familiar with TT and its design, her thinking about multiplication and the language she used in reference to multiplicative notions began to transform.

When first introduced to TT as part of the research project in 2018, Leah described the app as "backwards", explaining, "because when I'm teaching it, […] always the multiplication is the groups-of, not this five times. For instance, if I am doing three times four, I would expect three groups-of four to show up (Fig. 12a), but four groups-of three is showing up for three times four" (Fig. 12b). Although not experienced during a lesson, this 'hiccup' was clearly triggered by the design of TT and throughout the first year of the project, Leah returned to this idea multiple times. She described showing TT to other teachers, who agreed that, "It's […] the opposite way that the app is looking at it than some of us are used to teaching it" and she referenced the textbook where, "the multiplication is always groups-of. But the app is […] the opposite." Leah's discomfort with the order of the multiplier and the

Fig. 12 (**a**) *TouchTimes* model of 3 × 4; (**b**) 'groups-of' model of 3 × 4

multiplicand in TT was prompted by the technology and, when using TT, she had to think about and understand multiplication differently herself.

A year later, during one of our observational visits in Leah's classroom, this issue spontaneously arose during her teaching while TT was projected onto a screen.

Leah If we think of a pod of whales, is there more than one? It's a group, right? So, the pods, if you think of a pea, with a pod, a pod of peas with all the pods. There's more than one in it, it's a *group of* things. Oh, I don't want to use that word. [Laughs and looks towards the research team.]

Researcher It's okay.

Leah It's a collection of things. It's more than one. Right?

Student Why don't you want to say […] groups-of?

Leah [Leah looks to the research team.] Before Madame started *TouchTimes*, I was stuck on calling multiplication as being groups-of. But we don't always want to be thinking about multiplication that way. And *TouchTimes* helps us start thinking about it in a different way, so I'm trying to avoid that language that I'm using and change the way we look at it. I'm learning that as I go along.

This illustrates how TT influenced her thinking about multiplication and offers an example of Leah's personal genesis interfering with her professional genesis. Initially, it was the multiplicand followed by the multiplier in the expression displayed at the top of the screen by TT that drew her attention towards how she thought about and taught multiplication to her third-grade students. Multiplication was no longer solely about 'groups-of' the way it previously had been for Leah. In reconciling her thinking about this mathematics, it also changed the language that she was using in reference to multiplication with her students.

In interviewing Leah in early March 2020, she shared her curiosity about student thinking in relation to the ordering in TT and whether the order mattered to them.

Leah Yeah, and then I said, "Is seven times one the same thing on *TouchTimes*?" [Fig. 13a] And of course, because of com... how I always say it in French, commutative... commutative property. They all said, well it's the same thing. And I said, "But does it look the same thing?" So, then I put, okay, one times seven on the board. [Fig. 13b] "Make that on your *TouchTimes*. What does that mean on *TouchTimes*? Talk to me in *TouchTimes* language." It was funny how a lot of kids went, "Well, it's one group of seven". Finally, one kid pipes up, "No, it's not. It's one, seven times." So, it was so fascinating because it made them LOOK at that. […] Some of them got confused by that, but we changed it with two. If you change it with that you'll see. There you go. There's two, seven times [Fig. 13c].

During one of our classroom observations, Leah could be overheard in the video-recording, explaining to a research team member that her students were, "okay with the order". Curious to know more about her thoughts relating to this comment, in

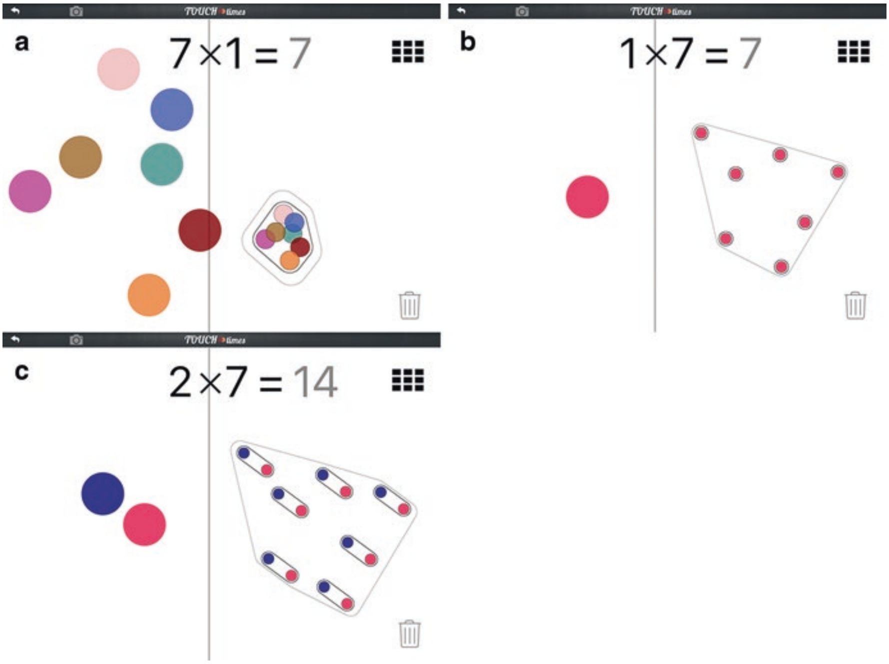


Fig. 13 (**a**) 7 × 1; (**b**) 1 × 7; (**c**) 2, 7 times

my interview with her I shared the video clip containing this comment and asked her to elaborate further.

Leah	That's not the old-fashioned group-of. […] Do you want to know what I meant by that? Was… I'm still trying to go, this whole groups-of thing, the way the order of the sentence is. Which still, people I've showed it to can't get past it. But the kids are okay with it. It's working for them, this five times. So why are we so stuck on it? Right? That's what I meant by it. See the kids are okay with this, they're learning, they're understanding it.
Sandy	Okay and so what prompted the comment?
Leah	Because I was amazed that that child did it exactly how *TouchTimes* would do it, without even a flinch.

Leah's prior way of thinking about and teaching multiplication was primarily based on thinking of 7 × 2, in terms of repeated addition, that is as seven groups-of two. In TT, multiplication involves units of units, and therefore here it is a unit of seven, taken two times. Given the intertwined nature of teacher–tool–mathematics, the multiplicand × multiplier model used by TT created a 'hiccup' for Leah. Not in her lesson *per* se, but in her own understanding of multiplication. This significantly affected her personal genesis of TT, though it did not prove to be a barrier to her professional genesis (nor to the students' explorations themselves). It is striking that

Leah was able to instrumentalise TT to serve her didactic objectives and proceeded to integrate TT into her pedagogical repertoire as a way of teaching multiplication, *in spite* of her own difficulty with the multiplicand × multiplier design of TT being the opposite of repeated addition approaches.

When sharing her continued difficulty with this ordering, Leah brought up commutativity, though repeated addition as a model does not offer a way of understanding commutativity. In trying to reconcile, for herself, that the order of TT should not matter, Leah drew upon her knowledge of the commutative property. Though the product will be the same for $a \times b$ and $b \times a$, in Grasplify the emphasis is not on the product itself: rather, it is on the creation of the pip(s) and the 'spreading' of the pips throughout the pod-units. In this context, the multiplicand is not interchangeable with the multiplier – they have differing roles – and therefore commutativity did not assist Leah's personal genesis of TT. What *was* compelling for her, however, was the successful instrumental orchestration of TT by her students. Unhampered by a deeply entrenched view of multiplication as repeated addition, her students easily used the language of 'three, five times' or 'five groups-of three' for 3 × 5.

It is worth questioning whether or not Leah's personal and professional geneses would have evolved if she had not been part of our research project. When confronted with a tool that did not conform to her accepted model and ways of thinking and speaking about multiplication, would she have persisted with her use of TT? If the tool *and* the mathematical concept *both* create dissonance for the teacher, does this thwart the development of the teacher–tool–mathematics *ensemble*?

6 Discussion

In this section, I draw on the case studies in returning to the purpose of this work, which was to gain a better understanding of the teacher–tool–mathematics *ensemble*. To this aim, the previous sections have introduced some important elements and exemplified the manner in which TT modulated the classroom practice in the case studies examined, which I will discuss here. The scope of my study remains limited, as I observed only two teachers who were both voluntarily part of a research project involving the integration of TT. Nevertheless, some of my comments are not limited to these specific cases. I first present issues that are directly linked to the mathematics as presented by TT and then discuss the types of orchestrations that emerged from these primary classrooms through the use of touchscreen technology.

6.1 The Mathematics, the Teacher and the Tool

It is common practice for many primary school teachers to introduce multiplication through repeated addition, and many teachers, like Leah, become firmly rooted in teaching multiplication using a 'groups-of' approach. The design of TT, which requires the multiplicand to be created by the user before the multiplier, and therefore the numerical expression at the top of the screen which reflects the number of pips and pods created by the user does not match this 'groups-of' idea. In this instance, the digital technology itself, as well as its manner of presenting the mathematics, shapes how a teacher discusses multiplication with her students, as when she refers to 'pips' and 'pods'.

Leah became very conscious of using language such as 'two, seven times', which matched what was visible on the iPad screen. This new way of talking about multiplication is significantly related to the mathematics and exemplifies the mutual influence within the teacher–tool–mathematics *ensemble*. The manner in which Leah talked about multiplication sounded one way when using TT with her students, but it might sound entirely different when teaching multiplication without the presence of TT. This has not been evoked in the literature (e.g., Anghileri, 1989; Kosko, 2018; Davis & Renert, 2013) on using different representations of multiplication, which are shown visually, but are not accompanied by particular language expressions.

Additionally, both teachers used whole-class orchestrations to focus student attention on the effects of adding or removing pip-fingers to the configuration of the pods, and linking these unitisations to the product displayed in the numerical expression. In doing this, new orchestrations such as *Compare-successive-screens, Document-screen-on-paper* and *Discuss-the-finger-and-screen* occurred. Though these orchestrations could also be used at a secondary-school level, they initially emerged from a primary-school context. The *Compare-successive screens* and *Discuss-the-finger-and-screen* orchestrations appeared spontaneously as the teachers interacted with the children and the ideas being shared during whole-class discussion. The *Compare-successive-screens* orchestration was linked to the importance of highlighting mathematical ideas that emerged from TT, such as what happens when the unit is changed, a new idea for teachers when teaching multiplication.

Having children draw mathematical ideas and images in their notebooks is also a common pedagogical practice in many Canadian primary classrooms, and therefore the *Document-screen-on-paper* allows children to re-present the mathematics created with their fingers using TT into their notebooks for future reference. This directs attention to the relationship between the pip-side and the pod-side of the screen, which is helpful for students to appreciate the co-variation that is manifested through the colour and shape of the pips and pods. The *Discuss-the-finger-and-screen* orchestration is very specific to the touchscreen nature of the technology, where the action of the fingers is not solely instrumental (to push buttons), but is conceptual (to express relations).

Leah and Rachel both engaged in a style of teaching where students were given a mathematical task, provided time to explore that task (often in pairs) and then the whole class was brought together for sharing and discussion. This is not an uncommon method of teaching in Canadian primary schools. What is interesting, and appears to be more prevalent in the primary context, is the manner in which whole-class orchestrations occur during whole-class discussion. Rather than being dominated by their teacher, primary students are situated in lead roles, sharing, demonstrating and discussing their findings with their peers, and the teacher assumes a coaching role to assist in drawing out student thinking and highlighting or extending the mathematical ideas. The didactical performance of the teacher is complex and responsive to the ideas presented by the children, the mathematical ideas that are the focus of the lesson and the opportunities that emerge from TT and the task given.

This also differs from the approach that was used by the kindergarten teachers in Carlsen et al.'s (2016) study where the teachers assumed assistant, mediator or teacher roles. In the case study excerpts provided, the situations were very much teacher-led, with the teacher mediating basic child–technology interactions between the interactive whiteboard and the children's eyes and hands, as well as between the technical and mathematical aspects of the digital tool. Though Besnier (2018) mentions the kindergarten teacher's emphasis "on verbalisation in mathematics and the idea that peer-to-peer exchanges promote learning" (p. 261), which is similar to what I observed in third and fourth grades, the focus of Besnier's paper is on the kindergarten teacher's creation of labels used on the board for children to manipulate and mimic the actions of the software, in the *Manipulatives-and-software-duo* orchestration. Besnier does not elaborate on the interactions between the teacher and students during this process.

One way in which the agency of TT was observed occurred through an unexpected student question related to multiplying by zero. This situation unfolded in a non-intentional way, where TT seemed to prompt a novel set of actions. This is noteworthy in that it was clearly an instance in which features of the tool had unintended influence (agency) on the practice of the teacher. There was synergistic movement amongst and between the three components of teacher, technology (TT) and mathematics (multiplication). Although this movement sometimes arises out of teacher choice, it is also at times energised by TT itself. There are the classroom orchestrations that the teacher initiates *because* of the presence of TT. The design of TT occasions certain ways of seeing/feeling multiplication, which also affects how the teacher speaks about multiplication and the body (finger)–TT interaction also prompts new orchestrations.

6.2 Types of Orchestrations in Primary Classrooms

From the outset, I expected there to be some differences between the orchestrations used with touchscreen technology and the orchestrations previously reported using computers or interactive whiteboards (e.g., Carlsen et al., 2016; Drijvers, 2010; Gueudet et al., 2014). Some differences are linked to the availability of iPads, which allows for individuals or pairs to access TT simultaneously, while others are related to the ways in which the TT screen was shared with students during class discussion. When Leah was projecting the image of the TT screen for all to see, students were able to view the digital screen, but were unable to view the movements of the user's fingers on the screen (which was also significant). In order to address this issue, Leah sometimes had students *Document-screen-on-paper*, which could be held up and shared by students during class discussion.

The *Discuss-the-finger-and-screen* orchestration relates both to the touchscreen aspect of the digital technology and also to the difficulty in projecting the user's fingers and their effects on the TT screen. This orchestration seemed to arise spontaneously in Leah's didactical performance when she specifically wanted to draw attention to the fingers themselves, as well as the effects of adding or removing pip-fingers related to the configuration of the pods. Given the age of the students, both teachers sometimes brought the class together for discussion by having the students seated on the carpet in front of them. In this way, students could hold up their iPads for demonstration, or an iPad could be placed upright along the whiteboard ledge for all to see.

Students both in Leah's and in Rachel's classrooms were observed using mini-whiteboards to document the images on their screens. Both teachers wanted to connect the finger actions on the screen when using TT with written or drawn hand actions that brought attention to the mathematics. The difficulties with unseen fingers on the projection screen and mathematical ideas vanishing with the removal of fingers from TT are both problems that are entangled in the teacher–tool–mathematics *ensemble*. The solutions that emerged for these problems, however, indicate the growing professional genesis of the teachers in their implementation of TT as a teaching tool.

7 Conclusion

This chapter has used an *ensemble* approach to examine the ways in which the teacher–tool–mathematical components mutually influence each other while primary teachers are using TT to teach multiplication. Research on double instrumental genesis and instrumental orchestrations used by primary school teachers and/or the instrumental orchestrations used with touchscreen technology is scarce and this chapter contributes to this area. Many of the orchestrations noted by Drijvers et al. (2010, 2013) were also observed in these primary mathematics lessons.

Though I observed three new orchestrations (*Compare-successive screens, Document-screen-on-paper* and *Discuss-the-finger-and-screen*) that emerged during the use of this particular touchscreen technology in the primary school context, I also observed ways in which the *TouchTimes* tool exerted agency in the classroom, especially in relation to new ways of speaking about multiplication and paying new attention to fingers as means to express and engage physically with multiplicative relations. This therefore underlines the importance of studying the use of different digital technologies, as they too may lead to specific orchestrations.

Acknowledgements I would like to thank Dr. Nathalie Sinclair and Dr. David Pimm for their guidance and support throughout the writing of this chapter. I would also like to acknowledge Dr. Sean Chorney and Victoria Guyevskey for their thoughtful feedback on early drafts.

Additionally, I would like to acknowledge the engagement and effort of the reviewers, whose detailed comments and constructive feedback were instrumental in improving this writing.

This work is funded by Social Sciences and Humanities Research Council of Canada (#435-2018-0433) and has received ethics clearance (REB #20190189).

References

Anghileri, J. (1989). An investigation of young children's understanding of multiplication. *Educational Studies in Mathematics, 20*(4), 367–385.

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematics Learning, 7*(3), 245–274.

Artigue, M. (2010). The future of teaching and learning mathematics with digital technologies. In C. Hoyles & J. Lagrange (Eds.), *Mathematics education and technology: Rethinking the terrain* (pp. 463–475). Springer.

Askew, M. (2018). Multiplicative reasoning: Teaching primary pupils in ways that focus on functional relationships. *The Curriculum Journal, 29*(3), 406–423.

Bakos, S., & Pimm, D. (2020). Beginning to multiply (with) dynamic digits: Fingers as physical–digital hybrids. *Digital Experiences in Mathematics Education, 6*(2), 145–165.

Behr, M., & Wheeler, M. (1981). The calculator for concept formation: A clinical status study. *Journal for Research in Mathematics Education, 12*(5), 323–338.

Besnier, S. (2018). Orchestrations at Kindergarten: Articulation between manipulatives and digital resources. In V. Gitirana, T. Miyakawa, M. Rafalska, S. Soury-Lavergne, & L. Trouche (Eds.), *Proceedings of the Re(s)sources 2018 International Conference* (pp. 259–262). École Normale Supérieure de Lyon (ENS de Lyon).

Bolden, D., Barmby, P., Raine, S., & Gardner, M. (2015). How young children view mathematical representations: A study using eye-tracking technology. *Educational Research, 57*(1), 59–79.

Boulet, G. (1998). On the essence of multiplication. *For the Learning of Mathematics, 18*(3), 12–19.

Calder, N., & Murphy, C. (2018). How might apps reshape the mathematical learning experience? In N. Calder, K. Larkin, & N. Sinclair (Eds.), *Using mobile technologies in the teaching and learning of mathematics* (pp. 31–50). Springer.

Carlsen, M., Erfjord, I., Hundeland, P., & Monaghan, J. (2016). Kindergarten teachers' orchestration of mathematical activities afforded by technology: Agency and mediation. *Educational Studies in Mathematics, 93*(1), 1–17.

Clark-Wilson, A. (2010). *How does a multi-representational mathematical ICT tool mediate teachers' mathematical and pedagogical knowledge concerning variance and invariance?* Unpublished doctoral dissertation. University of London. https://discovery.ucl.ac.uk/id/eprint/10019941.

Davis, B., & Renert, M. (2013). *The math teachers know: Profound understanding of emergent mathematics*. Routledge.

Davydov, V. (1991). The psychological analysis of the operation of multiplication. In V. Davydov (Ed.), *Psychological abilities of primary school children in learning mathematics* (pp. 9–85). National Council of Teachers of Mathematics.

Davydov, V. (1992). The psychological analysis of multiplication procedures. *Focus on Learning Problems in Mathematics, 14*(1), 3–67.

Drijvers, P. (2012). Teachers transforming resources into orchestrations. In G. Gueudet, B. Pepin, & L. Trouche (Eds.), *From text to 'lived' resources: Mathematics curriculum materials and teacher development* (pp. 265–281). Springer.

Drijvers, P., Doorman, M., Boon, P., Reed, H., & Gravemeijer, K. (2010). The teacher and the tool: Instrumental orchestrations in the technology-rich mathematics classroom. *Educational Studies in Mathematics, 75*(2), 213–234.

Drijvers, P., Tacoma, S., Besamusca, A., Doorman, M., & Boon, P. (2013). Digital resources inviting changes in mid-adopting teachers' practices and orchestrations. *ZDM: The International Journal on Mathematics Education, 45*(7), 987–1001.

Fischbein, E., Deri, M., Nello, M., & Marino, M. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathemarics Education, 16*(1), 3–17.

Greer, B. (1992). Multiplication and division as models of situations. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 276–295). Macmillan Publishing Co.

Gueudet, G., Bueno-Ravel, L., & Poisard, C. (2014). Teaching mathematics with technology at Kindergarten level: Resources and orchestrations. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era: An international perspective on technology focused professional development* (pp. 213–240). Springer.

Guin, D., Ruthven, K., & Trouche, L. (Eds.). (2005). *The didactical challenge of symbolic calculators: Turning a computational device into a mathematical instrument*. Springer.

Haspekian, M. (2011). The co-construction of a mathematical and a didactical instrument. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education* (pp. 2298–2307). CERME.

Haspekian, M. (2014). Teachers' instrumental geneses when integrating spreadsheet software. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era: An international perspective on technology focused professional development* (pp. 241–275). Springer.

Jackiw, N., & Sinclair, N. (2019). *TouchTimes [iPad application software]*. Tangible Mathematics Group, Simon Fraser University. https://apps.apple.com/ca/app/touchtimes/id1469862750

Jacob, L., & Willis, S. (2003). The development of multiplicative thinking in young children. In L. Bragg, C. Campbell, G. Herbert, & J. Mousley (Eds.), *Proceedings of the 26th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 460–467). MERGA.

Joubert, M. (2013). Using digital technologies in mathematics teaching: Developing an understanding of the landscape using three 'grand challenge' themes. *Educational Studies in Mathematics, 82*(3), 341–359.

Kosko, K. (2018). Third-grade teachers' self-reported use of multiplication and division models. *School Science and Mathematics, 119*(5), 262–274.

Larkin, K., & Milford, T. (2018). Mathematics apps–stormy with the weather clearing: Using cluster analysis to enhance app use in mathematics classrooms. In N. Calder, K. Larkin, & N. Sinclair (Eds.), *Using mobile technologies in the teaching and learning of mathematics* (pp. 11–30). Springer.

Maffia, A., & Mariotti, M. (2018). Intuitive and formal models of whole number multiplication: Relations and emerging structures. *For the Learning of Mathematics, 38*(3), 30–36.

Monaghan, J. (2004). Teachers' activities in technology-based mathematics lessons. *International Journal of Computers for Mathematical Learning, 9*(3), 327–357.

Noss, R. (1987). Children's learning of geometrical concepts through Logo. *Journal for Research in Mathematics Education, 18*(5), 343–362.

Noss, R., & Hoyles, C. (1996). *Windows on mathematical meanings: Learning cultures and computers*. Kluwer Academic Publishers.

Pickering, A. (1995). *The mangle of practice*. The University of Chicago Press.

Province of British Columbia. (2016). *Building student success – B.C. curriculum – mathematics 3*. https://curriculum.gov.bc.ca/curriculum/mathematics/3/core

Rabardel, P. (1995). *Les hommes et les technologies: Approche cognitive des instruments contemporains*. Armand Colin.

Sinclair, N., & Baccaglini-Frank, A. (2015). Digital technologies in the early primary school classroom. In L. English & D. Kirshner (Eds.), *Handbook of international research in mathematics education* (3rd ed., pp. 662–686). Taylor Francis.

Sinclair, N., Chorney, S., Güneş, C., & Bakos, S. (2020). Disruptions in meanings: Teachers' experiences of multiplication in *TouchTimes*. *ZDM. Mathematics Education, 52*(7), 1471–1482.

Steffe, L. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences, 4*(3), 259–309.

Tabach, M. (2011). A mathematics teacher's practice in a technological environment: A case study analysis using two complementary theories. *Technology, Knowledge and Learning, 16*(3), 247–265.

Tabach, M. (2013). Developing a general framework for instrumental orchestration. In B. Ubuz, C. Haser, & M. Mariotti (Eds.), *Proceedings of the Eighth Congress of the European Society for Research in Mathematics Education* (pp. 2744–2753). Middle-East Technical University.

Thomas, M., & Palmer, J. (2016). Teaching with digital technology: Obstacles and opportunities. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era: An international perspective on technology focused professional development* (pp. 71–89). Springer.

Trigueros, M., Lozano, M., & Sandoval, I. (2014). Integrating technology in the primary school mathematics classroom: The role of the teacher. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era: An international perspective on technology focused professional development* (pp. 111–138). Springer.

Trouche, L. (2004). Managing the complexity of human/machine interactions in computerized learning environments: Guiding students' command process through instrumental orchestrations. *International Journal of Computers for Mathematics Learning, 9*(3), 281–307.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisitions of mathematics concepts and processes* (pp. 127–174). Academic.

Vérillon, P., & Rabardel, P. (1995). Cognition and artifacts: A contribution to the study of though in relation to instrumented activity. *European Journal of Psychology of Education, 10*(1), 77–101.

Using First- and Second-Order Models to Characterise In-Service Teachers' Video-Aided Reflection on Teaching and Learning with 3D Pens

Oi-Lam Ng, Biyao Liang, and Allen Leung

Abstract This study examines the processes through which video-aided reflections may guide teachers to become aware of, and develop, expertise in the use of a novel-to-them technology in mathematics classrooms. Four mathematics teachers participated in this study aimed at exploring teachers' initial experience of using and planning lessons with the technology of "3D Pens", which enables learners to construct 3D models instantly via moving their hands. We analyse the participants' reflections and interpretations of students' actions during semi-structured interview sessions while they were watching video episodes of a lesson integrating 3D Pen for mathematics teaching and learning. Adopting the constructs of first- and second-order models, we provide fine-grained characterisation of the teachers' mathematical and pedagogical learning in the moment of their reflection. The results suggest that the teacher participants not only shifted from operating on their first-order models to constructing second-order models of students' geometrical thinking as supported by 3D Pens, but they also reasoned pedagogically based on their second-order models. Hence, watching videos of authentic technology-rich lessons facilitated a productive noticing experience for mathematics teachers in terms of realising the educational potential for the technology of the 3D Pens.

O.-L. Ng (✉)
Department of Curriculum and Instruction, The Chinese University of Hong Kong, Shatin, Hong Kong
e-mail: oilamn@cuhk.edu.hk

B. Liang
University of Hong Kong, Pokfulam, Hong Kong
e-mail: biyao@hku.hk

A. Leung
Hong Kong Baptist University, Hong Kong, Hong Kong
e-mail: aylleung@hkbu.edu.hk

A. Clark-Wilson et al. (eds.), *The Mathematics Teacher in the Digital Era*, Mathematics Education in the Digital Era 16,
https://doi.org/10.1007/978-3-031-05254-5_4

Keywords Technology-rich teaching · Mathematics teacher · Video-based professional development · Constructivism · First- and second-order modeling · 3D printing

1 Introduction

The effective integration of technology into the mathematics classroom to improve learning poses various challenges—most notably being teachers' participation and expertise (Roschelle, 2006). In other words, what underpins the successful integration of technology is not only the design of the technology, but also the teachers' levels of adaptive expertise (i.e., becoming more adaptive and more expert in helping their students learn). For example, the last decade has seen an increase in teachers' adaptive expertise in using multi-representational and classroom networking technologies for teaching mathematics (e.g., Bellman et al., 2014; Clark-Wilson, 2014); here, teachers were appropriating what were then novel technologies in their practice and transforming their pedagogies to take advantage of multiple representations and connected classrooms to help students learn mathematics. With new forms of technology becoming increasingly adopted in education, more research is needed to address teachers' personal and professional development when adapting a technology—one that is novel to them—in subject teaching (Haspekian, 2014), as well as to explore professional development approaches for supporting teachers' exploitation of technology affordances in classroom practices (Drijvers et al., 2016). Specifically, there is a need to understand how to support teachers' realisation of the opportunities provided by new forms of technology for teaching and learning, which we aim to address in this chapter.

In recent years, teacher noticing has featured prominently in practice and research for developing mathematics teachers' capacity to interpret complex classroom situations and events (e.g., Jacobs et al., 2010; van Es & Sherin, 2002). The term, teacher noticing, has been greatly influenced by Mason (2002)'s work on the discipline of noticing, which encompasses becoming aware of one's practice and keeping such noticing productive through interpretations and enquiry. In particular, video-based approaches to teacher noticing can help capture the complexity of classroom events that could otherwise be easily overlooked or unattended to, such as critical moments of teaching and learning (Stockero & Zoest, 2013). Video-based discussions occasion teachers to observe and interpret what was watched and allow them to re-think their own teaching practices (Coles, 2019). When watching videos of someone else's teaching, teachers may realise they have experienced much of what was happening in the videos themselves (Borko et al., 2008). In addition, the use of videos may help teachers anticipate what they might experience in an unfamiliar situation, such as when teaching in technology-rich environments. As Mason (2014) suggested, certain affordances of technology may or may not be manifested in the classroom; aligning teacher and student attention will improve communication of technology-in-use, and the use of videos can support teachers in anticipating the kinds of classroom communication and activities that may be manifested. Video recording of someone else's lesson can be conceptualised as a "boundary object", which lies at the intersection of different worlds or communities, here with the potential to be interpreted and conveyed in various ways by different viewers (Robutti et al., 2019). For example, Baccaglini-Frank et al. (2018) used videos as the boundary "between the single (real) classroom communities and future, potential classroom communities in which the activities would be realised again" (p. 102). In line with this, one area of future research is to establish empirical evidence of how to effectively engage teachers in video-aided reflections, particularly when it comes to the kinds of noticing tasks that are facilitated in video-based professional development (Coles, 2019; Tripp & Rich, 2012).

This study seeks to inform the processes through which teachers become aware of, and begin to reflect on, the use of a novel-to-them technology in mathematics classrooms. Of significance is that the teacher participants have just learned to use the technology for themselves, but not yet in their classroom teaching; in this way, we are interested in the participating teachers' initial encounters with the target technology and whether they decide to (and how they) adopt it for their future teaching. In the current study, we use video-aided reflection to facilitate a noticing experience for in-service mathematics teachers to consider the potential for the technology of "3D Pens" (Fig. 1), which enables one to draw in three dimensions, thereby enhancing the teaching and learning of geometry topics at the primary and secondary school level (Ng et al., 2018; Ng & Ferrara, 2020). The choice of 3D Pens is suitable for our investigation because it is a novel technology not only for the

Fig. 1 Drawing in the third dimension with a 3D Pen

current study's participating teachers, but also in the mathematics education community at large. Therefore, we are interested in how the teacher participants interpreted what they noticed from watching the video episodes of a mathematics lesson involving the use of 3D Pens. Previous research has shown that video-aided teacher reflection promotes genuine engagement with students' mathematical thinking, thus enriching teachers' pedagogical content knowledge (Jacobs et al., 2010). In the present study, we extend video-aided reflections to the context of technology-rich classrooms by taking videos of authentic 3D Pen-enabled classroom episodes as a tool for providing rich opportunities for teachers to develop their pedagogical knowledge; this can help hypothesise and reflect on student thinking within 3D Pen learning activities.

Our second research goal is to provide a fine-grained characterisation of the teachers' mathematical and pedagogical learning in the moment of their reflection. Extant theoretical framing and empirical work on teacher noticing have focused on the features and actions of teacher noticing (e.g., what teachers notice, how teachers notice) (Sherin & Star, 2011). There is lack of research that unpacks the cognitive processes underlying teachers' noticing activities. In the present study, we aim to address this research gap by gaining insights into how teachers' knowledge interacts during noticing at a mental level. Instead of taking a possessive view of teacher knowledge, we conceptualise teachers' knowledge construction as a dynamic, constructive, and adaptive process (Fennema & Franke, 1992; Liang, 2021). Engaging teachers in viewing and discussing lesson episodes provides a rich opportunity for teachers to construct knowledge of student thinking and adapt their personal mathematical knowledge. In the following section, we discuss the theoretical perspectives and constructs that enable our inquiry into teachers' mental activities during noticing.

2 Theoretical Framework

We adopt an epistemological perspective of constructivism that personal knowledge, as the product of experiential knowing, is *not* a representation of objective truth—rather, it functions and organises viably within a knower's experience and is idiosyncratic to the knower (von Glasersfeld, 1995). Therefore, each individual does not have direct access to other's knowledge but can only construct hypothetical mental models (or interpretations) of the other's knowledge as an observer. Regarding the context of teachers attempting to understand students' mathematical thinking, because teachers do not have access to students' knowledge, they can only construct models of the students' thinking. In what follows, we elaborate on the notions of decentering and first- and second-order models that align with the central tenets of our theoretical framework.

2.1 Decentering

The notion of decentering can be traced back to Piaget's work on children's egocentrism and decentration, including his work on children's socialisation and speech development (Piaget, 1926/1959), children's construction of space (Piaget & Inhelder, 1948/1967), and children's ability to separate themselves from the environment or other objects (Piaget, 1947/2001; 1954/2013). For example, it is nontrivial for an infant to model properties of other objects until they can decenter from the self and conceive of the objects existing independently. Before children's achievement of object permanence, they believe an object does not exist until it is perceptually available.

Building on Piaget's work on decentration, mathematics education researchers have adopted this construct to conceptualise decentering in the context of teaching (Confrey, 1990; Silverman & Thompson, 2008; Steffe & Thompson, 2000a). They defined *teacher decentering* as the mental action of an observer setting aside their own thinking and attempting to understand the perspective of others. Teachers' construction of knowledge of students' mathematical thinking requires teachers to decenter from their personal mathematical knowledge to understand the knowledge of the students. A decentering teacher does not assume that their students would share the same mathematical thinking with the teacher themselves; instead, the teacher actively constructs interpretations of the students' thinking through interactions with students and reflection on the students' mathematical activities (Baş-Ader & Carlson, 2021; Teuscher et al., 2016).

Moreover, engaging in decentering to understand students' mathematical thinking requires the teacher to truly believe that students' mathematical experiences, although potentially distinct from that of the adults', is legitimate, rational, and valuable (Steffe & Thompson, 2000b). As Confrey (1990) stated:

> Decentering, the ability to see a situation as perceived by another human being, is attempted with the assumption that the constructions of others, especially those held most firmly, have integrity and sensibility within another's framework. (p. 108)

In the context of mathematics teaching and learning, this means mathematics teachers need to respect students' mathematical thinking, and here, one of the teachers' tasks is to gain insights into the students' mathematical realities, including how their thinking is rational and internally viable within the students' frameworks.

2.2 First- and Second-Order Models

It may be asked how the notion of decentering can provide analytical power for researchers to investigate teachers' construction of students' mathematics. We find Teuscher et al.'s (2016) framing of decentering through the lenses of first- and second-order models helpful in answering this question. Steffe et al. (1983) defined *first-order models* (FOMs) as "[hypothetical models] the observed subject constructs to order, comprehend, and control his or her experience" (p. xvi). A student's (or a teacher's) first-order mathematics consists of the student's (or the teacher's) mental actions that govern their mathematical activities. Teachers do not have access to their students' first-order mathematics, in the same way that researchers do not have access to the observed teachers' first-order mathematics. However, teachers (or researchers) can make inferences or interpretations of the observed students' (or teachers') knowledge, which can help explain the observable actions of the students (or teachers). Such inferences or interpretations are called *second-order models* (SOMs)—"[the hypothetical models] observers may construct of the subject's knowledge in order to explain their observations (i.e., their experience) of the subject's states and activities" (Steffe et al., 1983, p. xvi).

To recap, a teacher's construction of SOMs requires the teacher to decenter. In other words, a decentering teacher engages in second-order modeling to generate hypotheses of their students' thinking to the extent that if the teacher imagined themselves reasoning with the mathematics in those hypothetical ways, they would act in a similar way as the students. In contrast, a teacher who is constrained to their FOMs operates entirely from their own perspectives (i.e., not decentering) and may assume their students' understandings are identical to their own or do not attempt to discern differences in the students' thinking.

2.3 Pedagogical Consequences of Second-Order Modeling

Upon building the SOMs of students' mathematics, a teacher is poised to compare the mental actions constituting their FOMs and SOMs to reflect on how students' thinking is different, similar, or related to their own thinking. The teacher can also compare the SOMs of different students to discern differences in those students' thinking. These interactions and reorganisations between multiple mathematical meanings have important pedagogical implications (see Silverman & Thompson, 2008; Simon, 1995, 2014), including task design, questioning, and responding. Simon (2014) highlighted "the need for pedagogical theory to connect the work on second-order models with effective pedagogical design and interventions" (p. 350). He stated, "[p]edagogy requires knowing where one is starting, where one is going, and how to get there" (Simon, 2014, p. 349). Knowing where one is starting, and where one is going, necessarily implies simultaneous constructions of SOMs and operations on FOMs. A teacher's ability to discern nuances in the SOMs of different students is also important when considering how to orchestrate meaningful mathematical conversations in their classrooms and provide equitable learning experiences for diverse students. In this book chapter, we provide empirical evidence that teachers' SOMs and FOMs are indeed necessary, and meaningful, to motivate pedagogical considerations sensitive to student thinking.

3 Methods

3.1 Participants and Study Context

The qualitative and in-depth nature of our research necessitated that we only recruit four (three male and one female) teachers to participate in the study. The participants were four in-service secondary school mathematics teachers—Denny, Sam, Denise, and Michael (pseudonyms)—whose teaching experience at the time of the study was one, four, seven, and 10 years, respectively. They were selected due to the range of teaching experience they represented, and they had provided consent to take part in the study upon a call for participation from the first author's networks of in-service mathematics teachers in Hong Kong.

The current study is situated within a larger, design-based research project which explored the technology-rich teaching and learning concerning a specific form of emergent technology: a handheld 3D Pen. It features a collaboration between the first author and the four participants that concerns pedagogical innovations for constructionist learning (Ng, 2020; Papert, 1980), with 3D Pens as the technological media. Given the ability to construct a 3D diagram and manipulate the constructed 3D model, we anticipated that the 3D Pens might be useful in the learning of school mathematics, where many topics involve perception of space and the visualisation and manipulation of shapes. Hence, the larger design-based research intended for the teacher participants, in collaboration with researchers, to create a series of mathematics lesson plans integrating 3D Pens, to be implemented by the participants in their classrooms. We achieved this in four monthly project meetings, during which the participants tried out the use of 3D Pens and discussed suitable topics that complemented their use in mathematics classrooms (Project Meetings 1 and 2); watched videos of a lesson with 3D Pens "in action" and discussed refinements of the watched lesson (Project Meeting 3); and designed 12 secondary mathematics lesson plans that integrated 3D Pens for future implementations (Project Meeting 4). To date, two of the four teachers have implemented their designed lesson plans in their classrooms upon the four project meetings. As aligned with the research goal to analyse the teachers' emerging knowledge of utilising the 3D Pens for their classroom teaching via constructing first- and second- models of students' mathematics, the data source analysed in this chapter was derived from Project Meeting 3, in which the participants engaged in the designed video-based noticing activity.

3.2 The Video

The participants watched a video of a Primary 5 (ages 11–12) mathematics lesson featuring the use of 3D Pens in the learning activities. We considered the video format to be suitable because watching videos of someone else's teaching has been found to be effective for teacher participants to evaluate aspects of the videos (Nickerson et al., 2017). Moreover, the choice of a Primary 5 lesson was to enable the participants to relate to their own context of secondary mathematics, without having explicit impressions of what their own lessons might look like. Indeed, there were numerous secondary mathematics topics that required strong visualisation of 3D shapes and space, such as those captured in the video. The selected video highlights the integration of 3D Pens for teaching and learning the "properties of prisms" (Hong Kong Curriculum Development Council [HKCDC], 2015), namely, the number of bases, lateral faces, and the total number of faces in a prism. In the local context of Hong Kong, this topic has been approached with or without (virtual) manipulatives by teachers and has been emphasised numerically (i.e., how many) rather than relationally (i.e., what are the properties). The video captured a Primary 5 lesson taught by an experienced classroom teacher who had jointly prepared the lesson plan with the first author. Hereafter, we refer to this teacher in the video as the instructor to differentiate him from the participating teachers.

The main learning activity in the lesson was using the 3D Pens to work in pairs to construct two rectangular prisms and two triangular prisms, one of each by each student in the pair (see Fig. 2a). Finally, the classroom teacher led a whole-class discussion about the target properties once the students had constructed the 3D solids (see Fig. 2b,c). Table 1 outlines the nine video episodes compiled into a 13:54 video segment, as well as the camera focus of the episodes. We chose these nine episodes because they chronologically captured key moments of the lesson, namely, the introduction, the learning activity with the 3D Pens, debriefing about the activity and the conclusion. They were taken from different camera foci (whole class and individual students) in order to capture teaching and learning from both general and fine-grained perspectives.

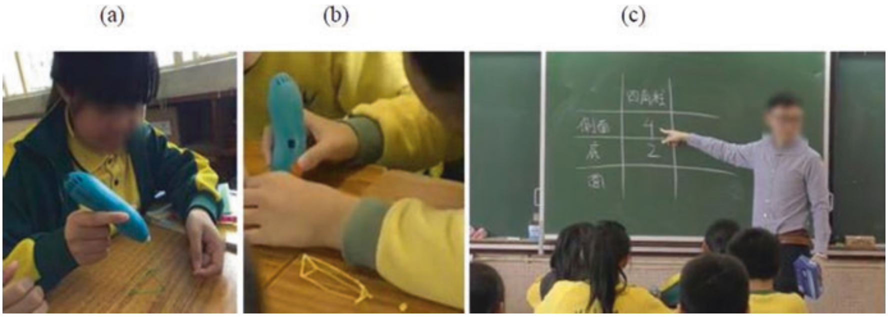


Fig. 2 (**a**) A student drawing a triangular prism with a 3D Pen (captured in Episode 2); (**b**) a final product made by the student (captured in Episode 3); and (**c**) the instructor leading a class discussion on the properties of the prisms (captured in Episode 8)

Table 1 The number and length of video episodes as they relate to different parts of the mathematics lesson with 3D pens and camera focus

Lesson parts in chronological order	Corresponding episode number	Length of selected episodes (in [min:sec])	Camera focus
Whole-class discussion on naming prisms	1	[00:40]	Whole class
Paired students drawing prisms with 3D Pens (see Fig. 2a)	2–6	[01:20], [01:42], [01:43], [01:13], [02:06]	Individual students at their desks
Whole-class discussion on the properties of the solids drawn (see Fig. 2b–c)	7	[01:44]	Whole class
Whole-class discussion on generalising the properties	8, 9	[02:18], [01:08]	Whole class
		Total length: [13:54]	

3.3 Video-Based Semi-structured Interviews

The method of video-based semi-structured interviews is suitable for examining teachers' mathematical and pedagogical learning during noticing because it enables teachers to view and reflect on unique instructional moments. During Project Meeting 3, the four teacher participants and two researchers (the first author and her collaborator, named R1 and R2 in the transcript, respectively) watched the aforementioned nine video episodes. After each episode was played on a large projected screen, the two researchers each conducted a semi-structured interview with their assigned pair of participants. The first author worked with Denise and Michael in one group, and the other researcher worked with Denny and Sam. The prompts used in the semi-structured interviews were drawn from Mason (2002), who argued that two requirements exist for professional noticing. The first requirement is to create an "account of" an event, that is, to reconstruct step by step what the participants saw and briefly but vividly describe some events in the video. The second requirement is to "account for", that is, to offer an interpretation, explanation, value, judgement, justification, or criticism of the accounts. Two video cameras were placed in the room to capture the teachers' verbal and nonverbal communication as they described and reflected on what they noticed from watching the episodes. The video-based semi-structured interviews lasted for 80 min.

Following the video-based semi-structured interviews, the participants spent 15 min individually refining the lesson that they had just watched by annotating or making notes on the printed lesson plans the instructor had prepared. Then, the group of four teacher participants and the two researchers engaged in a 15-min discussion about the specific details of their lesson refinement and the justifications for their decisions. The aims of the study reported in this chapter do not warrant a report on the teachers' lesson-refining activities in detail. However, we did include some statements made during the lesson-refining session to support our claims about the teachers' FOMs and SOMs.

3.4 Method of Analysis

The interviews were videotaped, transcribed in full, and analysed according to the teacher participants' verbal statements and hand movements, that is, we attended to the participants' discourse to infer the areas and processes through which the teachers gained mathematical and pedagogical insights in relation to teaching with 3D Pens. As an example of our analysis, we iteratively watched the data obtained from the video-based interviews and lesson-refining sessions to generate and refine descriptions of the mathematical meanings constituting the participating teachers' FOMs and SOMs and how these models influence their pedagogical thinking.

In our first pass of the data, we focused on identifying the instances where the teachers discussed their own mathematical thinking and the observed students' mathematical thinking. This involved us excluding statements about classroom management, classroom environment, and communication and tasks that were non-mathematical. For example, the statements of "[the instructor] did something related to classroom management…" and "I also think that it is really difficult to engage two people in one task using one Pen" were considered non-mathematical statements. Our second pass focused on constructing SOMs of the teachers' thinking. Regarding each selected instance, we wrote inferential and descriptive memos of the teachers' mathematical thinking and the teachers' interpretations of the students' thinking. We inferred that a teacher was operating on their FOMs when they discussed how they would act or think regarding a mathematical situation, and we inferred a teacher was constructing their SOMs when they discussed their interpretations or explanations of the students' actions. Meanwhile, we inferred and described the interactions of a teacher's SOMs and FOMs when the teacher compared their own mathematical thinking to their inferences of students' thinking, or when the teacher compared different students' thinking. We alternatively tested, refined, and stabilised our inferences as we continued to find supporting or non-supporting evidence of those inferences. During the final pass, we reviewed the transcripts *following* each selected instance and identified additional instances that captured the teachers' discussion on pedagogical decisions that were sensitive to their FOMs and SOMs. These efforts enabled us to re-include some instances that were excluded in the first pass of the data, affording us insights into the consequential pedagogical thinking of the teachers in relation to their construction of and operation on FOMs and SOMs.

4 Results

Our focus on teachers' second-order modeling and the pedagogical potential of their modeling activity led us to characterise the four teacher participants' reflection in relation to three episodes: Episodes 3, 4, and 6 (in which pairs of students were creating prisms with the 3D Pens). Specifically, we discuss the mathematical meanings constituting these teachers' FOMs and SOMs, along with how these meanings and their interactions were situated in their viewing of students' constructions with 3D Pens and were generative to their pedagogical thinking. We illustrate three vignettes: first, we discuss a teacher's pedagogical thinking based on his FOMs; second, we discuss a pair of teachers' constructions and comparison of SOMs with respect to two groups of students and conjecture the pedagogical potential of such constructions; finally, we discuss three teachers' constructions of SOMs (as well as the interactions between SOMs and FOMs) and their pedagogical thinking sensitive to those models. We note that our goal is to illustrate the multiple ways teachers may engage in video-aided reflection on the teaching and learning with 3D Pen construction, but not to make any value judgement on those different approaches.

4.1 Vignette 1: 3D Pen Constructions with Varying Sizes

After watching Episode 3, which offered the participants a video clip of the paired students' geometric construction with the 3D Pens, Denise and Michael discussed the size of the drawings created by the students. In this discussion, Michael noticed that some students' solids were very tiny. He commented that, "it is better if the teachers could ask his students to draw a larger solid" since it would make the construction process easier and the properties of the solid more apparent. Following up on Michael's comment, Denise suggested that maybe the teacher could indicate in the worksheet the *exact* size of the solid the students were supposed to produce. Michael disagreed and said:

> Michael: I don't think [the size] should be exact. It's because in terms of the shapes, if the shapes are different sizes at the end, if you look at them, it shows some kind of mathematical concept […] I think to show them to all students, if they are of different sizes, it can help with the mathematical concept, I think.

In the above quotation, Michael suggested that although it might be helpful to ask students to draw bigger solids, it was unnecessary to require students to produce solids of an identical size. On the contrary, allowing students to use the 3D Pens to produce solids of varying sizes would be helpful for the students to relate to the "mathematical concept". We interpreted that by "mathematical concept", Michael was likely referring to the general structure and property of the solid, namely, the variations in the solids' size could afford students' discussions on the common properties held by multiple solids. We acknowledge that we were constrained by our interactions with Michael and that we could not further confirm our interpretation of his meaning of "mathematical concept". Regardless, we claim that Michael was primarily operating on his FOM instead of SOM when making his statements. Namely, from his own perspective, he could see the general properties and structure of solids of varying sizes, and thus, he considered the pedagogical decision of allowing students to draw in any size as being significant. We do not find evidence that this pedagogical consideration (including the rationale he provided) had its basis in his SOMs of the students' mathematical activities. However, we do not intend to suggest that Michael's reasoning was deficit merely because his pedagogical reasoning was FOM-based. Rather, he demonstrated a nice example of taking into account the use of 3D Pens to consider how to leverage the advantage of this tool to advance students' mathematical reasoning.

4.2 Vignette 2: Straightening the Edges with 3D Pens

Next, we discuss how Sam and Denny, instead of operating on their FOMs, engaged in constructing and comparing the SOMs of two groups of students. Episode 6 depicted two groups of students' activities after completing a triangular prism with 3D Pens. Denny discussed the differences he noticed between the two groups:

> Denny: I noticed some students used scissors to cut the extra material [*see* Fig. 3a]. The first group of students were very happy after they drew the small triangular prism [*see the students' final products in* Fig. 3b]. And then the boy in the second group was persistent in making the lines straight, like making sure that those sides were parallel [*see the students' actions in* Fig. 3c *and their final product in* Fig. 3d]. My interpretation is that maybe different students had different foci. some may be satisfied by just completing the shape of the prism without paying attention to whether the lines were parallel.

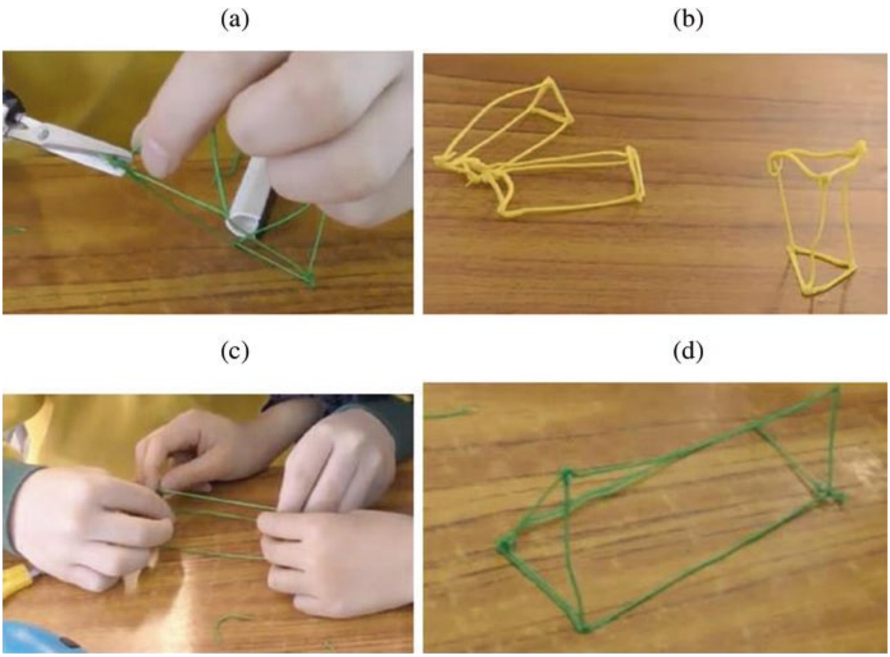


Fig. 3 (**a**) A student cutting the extra materials at the vertex upon completion of a prism, (**b**) the final products of one group of students, (**c**) a group of students straightening the edges of a prism, and (**d**) their final product

Sam: I think the student understood that the sides of a prism should be straight and that they cannot be messy. Otherwise, if the [two bases] were different, it was not a prism. They might not say this explicitly, but at least they had the concept.

R2: So you think they had the concept?

Sam: Yes. That's why he knew he should make the lines straight, and the angles had to be one point, which was a good thing.

In the above transcript, Denny and Sam mentioned that one group of students made efforts to straighten the edges of their prism and cut away the surplus materials at the vertices. They inferred that these students considered parallel sides and uniform bases as the critical features of a prism, while the other group of students might lack such understanding. We consider this to be an evidence of Denny and Sam constructing SOMs of those students; they considered specific student actions associated with their 3D Pen construction (e.g., drawing, straightening, and cutting) as indications of their underlying mathematical thinking. We conjecture that Denny and Sam's interpretations of students' technology-rich learning would have important pedagogical potential in their future teaching. For example, we can envision them reinforcing the ideas of parallel-ness, vertices, and uniform bases when supporting students' 3D construction of prisms.

4.3 *Vignette 3: Two Ways of Constructing and Perceiving a Triangular Prism*

We now turn to providing analyses of how the participating teachers not only shifting from operating on their FOMs to constructing SOMs, but also reasoning pedagogically based on their SOMs. We divide our discussion into three subsections.

4.3.1 "Originally I Thought He Was Drawing Something Wrong"

Upon viewing Episode 3, Denise noticed an unexpected event where she interpreted one group of students drawing a rectangle instead of a triangular prism (see the students' activity in Fig. 4). In this case, she said, "Many of them were drawing triangular prisms, but one group was not." The researcher then asked, "Is it possible to draw a rectangle first?" Denise hesitated to respond. We infer from Denise's utterances that she considered the students could not have been drawing a triangular prism by first drawing a rectangle. In comparison, Michael interpreted that the same pair of students were trying to construct a rectangular prism (as opposed to a rectangle) because he noticed them drawing extra edges on top of the rectangle (also see Fig. 4). Similar to Denise, he concluded that the students were not drawing a triangular prism as instructed by the instructor. We argue that at this point, both Denise

Fig. 4 Two students initially drew a rectangle during the construction of a triangular prism

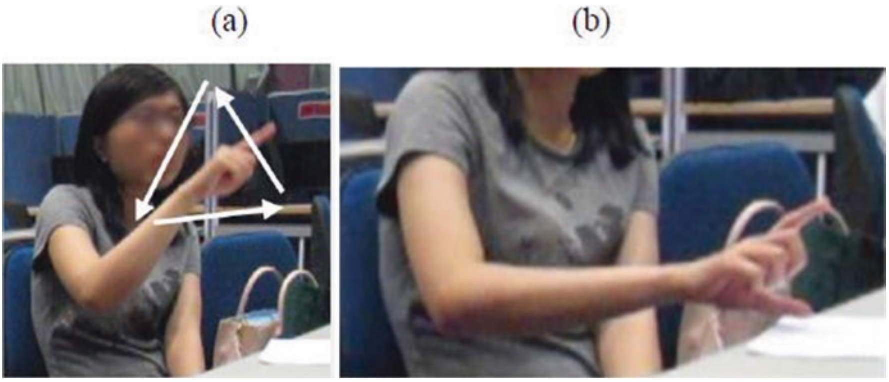


Fig. 5 Denise's gestures of (**a**) drawing a triangle and (**b**) drawing the triangular prism in a vertical orientation

and Michel were primarily operating on their FOMs and were interpreting the students' actions as contra-indications of those FOMs.

As the conversation progressed, Denise conjectured that "perhaps he is drawing a rectangle first and then a triangle". She repeatedly described her imagined process of drawing one rectangle and then drawing two triangles on opposite sides of the rectangle (see her gestures in Fig. 5a). Here, we interpreted that Denise started to construct SOMs of the students' thinking, namely, she was hypothesising a way of thinking that could explain the student's actions of drawing a rectangle. Recall that initially, she considered these actions to be contra-indications of her own thinking without decentering, whereas here, she was able to infer the underlying rationality behind the students' actions. Having re-enacted the drawing with her hands, Denise continued to comment with enthusiasm that this way of drawing with 3D Pen was better because "the triangles would stay sturdy no matter how long the horizontal edges were" and that if drawn otherwise (see her gesture in Fig. 5b), "it's going to fall easily". A few turns later, she said that "she had never thought we could draw it in this way" and that "originally I thought he was drawing something wrong…or something else". Denise's discussion on the affordance of the students' way of drawing suggested that she considered it a viable way of drawing that should be valued despite its novelty. This is evidence of her decentering from her personal mathematics to truly engage with the students' mathematical thinking as she herself would think in their position.

4.3.2 "Why Did He Do That, and What Was He Thinking?"

Following up on Denise's comments, Michael also conveyed his appreciation and curiosity for the students' construction with 3D Pen:

> Michael: [The teacher in the video] didn't teach them how to draw it. [The student] just discovered [it] himself. (…) I mean I was thinking about it just now; if I draw a triangular prism, I'd draw a triangle first and then draw this [*see* Fig. 6a–b]. I'd make sure this and that are the same, so I'd draw a uniform base first, this thing first [*pointing at the diagram; see* Fig. 6c]. So, thinking in the other way, I found the students' way of drawing very interesting. Why did he do that, and what was he thinking? That's what I was actually thinking.

Michael described how his personal way of drawing a triangular prism was different from the students' way of drawing (i.e., drawing a triangle first versus drawing a rectangle first). Here, he was simultaneously operating on his FOM and SOM and comparing them. It is also noteworthy that such a comparison motivated Michael to be inquisitive about the students' intention and thinking (e.g., "Why did he do that, and what was he thinking?").

Later, Michael discussed that the students' unique way of drawing the prism with the 3D Pen might be underpinned by a different way of perceiving the shape:

> Michael: I was thinking that when he was constructing the shape, his concept of the shape governed his way of construction. (…) If we try to trace back what he was thinking from his drawing, he was probably visualising the shape in that way. (…) In terms of cognition, he was thinking differently from other students, but I don't think we can judge it as being good or bad.

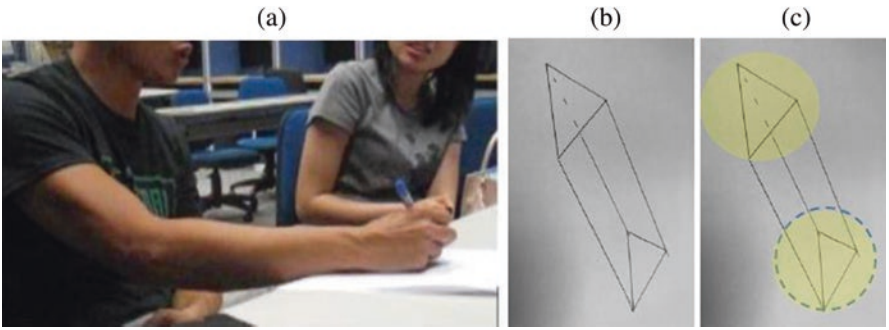


Fig. 6 (**a–b**) Michael sketching his way of drawing a triangular prism on paper; (**c**) Michael gesturing over the shaded region while uttering, "I'd make sure these are the same" and then over the dotted region when uttering "I'd draw a uniform base first, this first"

In the above quote, Michael claimed that the student's actions of constructing the prism implied how he imagined a triangular prism. That he could look beyond the students' physical actions of drawing to infer the students' perceptions of the solids was an important benchmark for engaging in second-order modeling. His last statement also indicated that he valued such a novel way of thinking without judging it as "good or bad".

To sum up, we argue that both Denise and Michael shifted from primarily operating on their FOMs and not decentering to constructing SOMs of the students' mathematics through decentering. This shift had important mathematical outcomes for both teachers. Recall that using 3D Pens to construct geometric shapes was also a novel experience to these participating teachers. It is not surprising that students may use 3D Pens to create drawings in diverse ways, and it is understandable that teachers who lack personal experience of using this tool may not be able to exhaust different construction approaches solely based on their FOMs. The cases of Denise and Michael suggested that second-order modeling had served as an avenue to enrich these teachers' FOMs as they conceived of the students' way of constructing a geometric shape as an additional viable construction approach. In the following sub-section, we move to discussing the teachers' pedagogical reasoning based on their second-order modeling activity.

4.3.3 "They Might Struggle to Understand." "The Instructor Could Have…"

In our interactions with Denise during the lesson-refining phase, we observed that she built on her SOMs to make pedagogical decisions. She commented on the instructor's inability to address different drawing methods and suggested that a classroom discussion should be facilitated:

> Denise: Some people had drawn the lateral faces first and then the base, and [the instructor] completely ignored that group as if he didn't see it. So, I feel that here, the instructor could have talked a little bit about why the students drew in these ways, and so forth.

Denise commented that the instructor was not sensitive to the students' diverse ways of drawing and that he could have built upon the students' different ways of thinking instead of ignoring them. She considered it valuable to have a discussion with, and among students about the different ways of viewing or positioning a triangular prism. Here, we highlight that Denise's emphasis on responding to the students' ways of thinking was contingent upon her SOMs of the students and valuing the students' thinking, as we previously discussed. In contrast, we hypothesise that it is unlikely that the instructor in the video had engaged in second-order modeling as had Denise. This instance suggested the pedagogical potential of teachers' engagement in second-order modeling through the processes of watching, and reflecting on, teaching videos.

Another teacher participant, Sam, also generated hypotheses to explain the same group of students' drawings and commented on the potential consequence of the students' thinking:

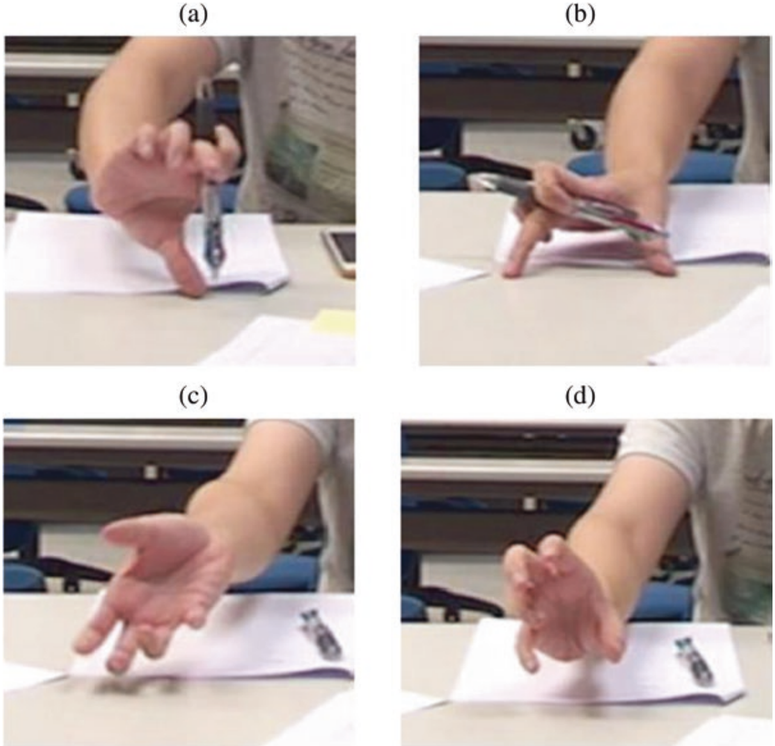


Fig. 7 Sam's gestures of (**a–b**) laying down a solid and (**c–d**) placing a solid upright

Sam: I noticed they lay down the solid [*gesturing from* Fig. 7a *to* Fig. 7b] and drew the lateral face first. This might be due to the lateral face being bigger, and they treated it as the base instead [*gesturing similarly to* Fig. 7b]. (…)

R2: Do you think this method of drawing would influence their conception of the prism?

Sam: I doubt that they had the intention of making the prism stood upright [*gesturing a rotation of 90 degrees with his hand; see* Fig. 7c *and* Fig. 7d]. Consequently, their conception of the prism might be different because if they treated the lateral face as a base, they might struggle to understand, very difficult to understand.

From Sam's words, we interpreted that he was hypothesising how the students who produced the prism by drawing the rectangular face first would struggle to perceive the triangle as the base. Specifically, these students would treat the rectangle as the base of a prism because location-wise, the rectangle was sitting at the bottom from their perspectives. This way of positioning and perceiving the prism was contrasted with his FOM that a prism was constructed upright with a triangular base. Sam was simultaneously constructing and comparing his SOM of the students' technological activities and his FOM. As such, he was able to anticipate the potential mathematical consequences of the observed student actions with 3D Pens given the mismatch between his own thinking and the students'. We consider Sam's reasoning as having important pedagogical potential; if he were to encounter a similar phenomenon in his instruction, he could enact pedagogical moves to facilitate students' discussion on what should be considered a base and lateral face and clarify the conventional definitions of each.

5 Discussion and Conclusion

5.1 Video-Aided Reflections of Teaching and Learning with 3D Pens

To restate, the teachers in the current study had never used 3D Pens in a lesson before; hence, by viewing the videos, they were able to see the authentic teaching and learning episodes that occurred in the lessons that integrated the 3D Pens. Unlike prior studies in which the teachers realised that they had experienced much of what was happening in the videos themselves (Borko et al., 2008), the teachers in the present study used the videos to anticipate what they might experience in an unfamiliar situation. The teachers' discourse has shown to be multimodal, as they often re-enacted their imagined drawing process with 3D Pens, either through gesturing or diagramming on paper. This suggests one affordance of using video-aided reflections. That is, it enables teachers to experientially construct multimodal FOMs associated with novel technology use.

Moreover, video-aided reflections support teachers to go beyond operating on FOMs to become observers of someone else's activities and constructing SOMs of others' thinking (i.e., becoming more decentered from FOMs). As shown in our findings, the teachers attended to students' nuanced actions of 3D drawing, ranging from the hand movements with 3D Pens to the size and orientation of the final product. Importantly, the videos captured the *process* of students' drawing with 3D Pens in addition to merely the final *product*. These dynamic actions of 3D drawing would not have been accessible to the teachers if only students' static written work or final product had been used to facilitate the reflection. Besides, the captured processes are likely not fully accessible to teachers in the moment of their teaching, considering a typical teacher-student ratio in mathematics classrooms. With regard to teaching and learning with 3D Pens, having access to students' construction process is not only beneficial but also essential for teachers' constructions of fine-grained and explanatory SOMs, as students' reasoning is heavily conveyed by, influenced by, and embedded within their physical actions. As suggested by the teachers' discussion on the students' way of drawing a triangular prism horizontally, we argue that the temporality of the 3D drawing process enabled the teachers to construct SOMs as revealed by the students' acts of 3D drawing.

In today's era where multimodal and haptic interfaces are becoming an important design consideration for educational technology, particularly in mathematics education (see, for example, a review in Carreira et al., 2017), we imply that more research is needed in understanding how mathematics teachers may engage in personal and professional development when working with these novel forms of technologies. To this end, we highlight the value of video-based approaches to supporting teachers' professional development in technology-rich mathematics teaching, for the opportunities it occasions for teachers to access and build models of student thinking in technology-rich classroom contexts.

Another affordance of using video-aided reflection is that it engenders teachers' pedagogical reasoning sensitive to their FOMs, SOMs, and technology-rich contexts. The results of this study did not only shed light on what teachers identified as important and noteworthy in a 3D Pen-enabled lesson but also how they realised certain pedagogical affordances of the 3D Pens through the course of the video session. For example, the teachers described that the 3D Pen afforded the students to visualise different orientations of a 3D shape and with varied sizes (e.g., Michael preferred not to restrict the size of the solids drawn and argued that the variation of sizes could serve to support students' constructions of some "mathematical concepts"). From this experience, they realised that certain ways of constructing with 3D Pens (and visualising the product) should not be taken as given (e.g., Sam argued that it might not be trivial to students who constructed a triangular prism horizontally to consider the triangular face as a base). Further, by being open to learning students' diverse ways of construction in a 3D Pen environment, they reflected on their pedagogical moves that responded to individual student thinking for future lesson enactments (e.g., both Denise and Sam valued a deeper discussion on the different ways of constructing and perceiving a triangular prism).

To conclude, we suggest that videos can serve as a boundary object (Robutti et al., 2019), serving purposes from both the teachers' and the researchers' perspectives on pedagogies and curricular design with news forms of technology. Through second-order modeling and associated pedagogical reasoning in the context of video-aided reflections, teachers can learn to be responsive to students' diverse ways of construction with 3D Pens and to facilitate meaningful classroom conversations with students in their classrooms. Given that the participating teachers had already used (or planned to use) 3D Pens in their classroom teaching, it would be worthwhile to examine how they draw on this noticing experience to implement lessons with 3D Pens as a next stage of the study.

Finally, we draw the reader's attention to another aspect of our methodology—the method of interviewing. The interview prompts for the participating teachers' interpretation (*accounting for*) and re-collection (made *accounts*; Mason, 2002) of what they saw in the video were helpful for triggering teachers' second-order modeling as well as guided their reflections of their own FOMs. This allowed the teachers to become conscious of the connections they were drawing between specific classroom episodes and the principles of teaching and learning, as well as to reason about the observed classroom events (Star & Strickland, 2008). We encourage future research to consider using video-aided reflections as a means of developing teachers' expertise in other target areas, particularly those that are unfamiliar and not easily accessible to teachers in their everyday teaching.

5.2 *Researching Teachers' Second-Order Modeling Associated with Technology-Rich Teaching*

Over the past three decades, constructivist researchers have used the notion of second-order modeling and relevant methodologies to generate findings on individuals' mathematical cognition regarding various topical areas and contexts. However, teacher professional programmes have by and large "not been grounded in a similarly extensive research based on the nature of teachers' knowledge and its development" (Doerr & Lesh, 2003, p. 128). Hence, scholars have called for empirical work on *teachers'* second-order modeling to fill in this research gap (see Kastberg, 2014; Liang, 2021; Wilson et al., 2011). To this end, our work echoes Doerr and Lesh's (2003) proposal of a modeling perspective on teachers' development and Simon and Tzur's (1999) *accounts of teachers' practices* (i.e., explaining teachers' perspectives from researchers' perspectives). That is, we characterised these teachers' capabilities of modeling and reflections on student thinking from our interactions with them. In particular, our work is a novel attempt to answer these calls by detailing four in-service teachers' enactment of their first-order mathematics, constructions of SOMs of students' mathematical thinking, and the mathematical affordances of such constructions. As in the cases of Denise and Michael, their constructions of SOMs served as an important avenue for their development of FOMs about triangular prisms associated with 3D Pens construction. Additionally, we identified three different ways that first- and second-order modeling could be generative to teachers' pedagogical thinking: directly operating on FOMs (see Vignette 1), comparing SOMs of different individuals (see Vignette 2), and comparing FOMs and SOMs (Vignette 3). These findings contribute to the literature on second-order modeling and decentering by specifying the types of reasoning involved as well as illuminating the pedagogical potential of this reasoning.

We acknowledge that our use of SOMs is loose when compared to constructivist researchers' common use of second-order modeling to investigate students' mathematical cognition. First, second-order modeling is often conducted through sustained interactions between the observer and the observed (Steffe & Thompson, 2000b; Ulrich et al., 1995). Second, second-order modeling has been considered an ongoing, dynamic, and iterative process (Liang, 2021; Steffe & Thompson, 2000b). An individual's SOMs of another person are always subject to revision, confirmation, and rejection based on additional observations of the observed individual's actions. Therefore, we expect a decentering teacher to continually develop, test, and refine their SOMs so that these models become more stable, coherent, explanatory, and even predictive of their students' mathematical activities.

In the present study, we did not provide our participating teachers with the opportunity to have direct interactions with students. Instead, their construction of SOMs occurred in the context of watching videos that captured students' mathematical activities in a classroom. Because of our limited interactions with the teachers, we did not accomplish the goal of capturing the evolution of the teachers' SOMs either. However, we highlight that our work is a novel attempt to apply the constructivist notion of second-order modeling to analyse teachers' understandings of students' mathematical activities in a technology-rich environment. We focus this initial attempt on the mathematical substance constituting these teachers' SOMs, the interactions between multiple SOMs and between their SOMs and FOMs, and the teachers' consequential technology-rich pedagogical thinking. We call for continued research along this line of inquiry, to include working with other mathematical, technological, and educational contexts to test, refine, and supplement the findings we have reported in this chapter.

Funding Acknowledgements This study is supported by the Research Grants Council (Hong Kong), Early Career Scheme (Reference No. 24615919). The authors are grateful for the teachers who participated in this study and Ms. Ruby Hui for her professional transcription of the video data.

References

Baccaglini-Frank, A., Di Martino, P., & Sinclair, N. (2018). In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.), *Proceedings of the 42nd conference of the international group for the psychology of mathematics education* (Vol. 2, pp. 99–106). PME.

Baş-Ader, S., & Carlson, M. P. (2021). Decentering framework: A characterization of graduate student instructors' actions to understand and act on student thinking. In *Mathematical Thinking and Learning*. Advanced online publication. https://doi.org/10.1080/10986065.2020.1844608

Bellman, A., Foshay, W. R., & Gremillion, D. (2014). A developmental model for adaptive and differentiated instruction using classroom networking technology. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era: An international perspective on technology focused professional development* (pp. 91–110). Springer.

Borko, H., Jacobs, J., Eiteljorg, E., & Pittman, M. E. (2008). Video as a tool for fostering productive discussions in mathematics professional development. *Teaching and Teacher Education, 24*(2), 417–436.

Carreira, S., Clark-Wilson, A., Faggiano, E., & Montone, A. (2017). From acorns to oak trees: Charting innovation within technology in mathematics education. In E. Faggiano, F. Ferrara, & A. Montone (Eds.), *Innovation and technology enhancing mathematics education* (pp. 9–35). Springer.

Clark-Wilson, A. (2014). A methodological approach to researching the development of teachers' knowledge in a multi-representational technological setting. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era: An international perspective on technology focused professional development* (pp. 277–295). Springer.

Coles, A. (2019). Facilitating the use of video with teachers of mathematics: Learning from staying with the detail. *International Journal of STEM Education, 6*(1), 1–13.

Confrey, J. (1990). Chapter 8: What constructivism implies for teaching. *Journal for Research in Mathematics Education Monograph, 4*, 107–210.

Doerr, H. M., & Lesh, R. (2003). A modeling perspective on teacher development. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 125–139). Lawrence Erlbaum Associates.

Drijvers, P., Ball, L., Barzel, B., Kathleen Heid, M., Cao, Y., & Maschietto, M. (2016). *Uses of technology in lower secondary mathematics education: A concise topical survey*. Springer Nature.

Fennema, E., & Franke, M. L. (1992). Teachers' knowledge and its impact. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 147–164). Macmillan.

Haspekian, M. (2014). Teachers' instrumental geneses when integrating spreadsheet software. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era:*

An international perspective on technology focused professional development (pp. 241–276). Springer.
Hong Kong Curriculum Development Council (HKCDC). (2015). *Ongoing renewal of the school curriculum – Focusing, deepening and sustaining: Updating the mathematics education key learning area curriculum (primary 1 to secondary 6)*. The Printing Department.
Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education, 41*(2), 169–202.
Kastberg, S. E. (2014). The power of what we know: Further directions for exploring constructivist model building. *Constructivist Foundations, 9*(3), 352–354.
Liang, B. (2021). *Learning about and learning from students: Two teachers' constructions of students' mathematical meanings through student-teacher interactions*. Unpublished doctoral dissertation. University of Georgia.
Mason, J. (2002). *Researching your own practice: The discipline of noticing*. Psychology Press.
Mason, J. (2014). Interactions between teacher, student, software and mathematics: Getting a purchase on learning with technology. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era: An international perspective on technology focused professional development* (pp. 11–40). Springer.
Ng, O. (2020). How 'tall' is the triangle? Constructionist learning of shape and space with 3D pens. In *International Journal of Mathematical Education in Science and Technology*. Advanced online publication. https://doi.org/10.1080/0020739X.2020.1844910
Ng, O., & Ferrara, F. (2020). Towards a materialist vision of 'learning as making': The case of 3D printing pens in school mathematics. *International Journal of Science and Mathematics Education, 18*, 925–944.
Ng, O., Sinclair, N., & Davis, B. (2018). Drawing off the page: How new 3D technologies provide insight into cognitive and pedagogical assumptions about mathematics. *The Mathematics Enthusiast, 15*(3), 563–578.
Nickerson, S., Lamb, L., & LaRochelle, R. (2017). Challenges in measuring secondary mathematics teachers' professional noticing of students' mathematical thinking. In E. O. Schack, M. H. Fisher, & J. A. Wilhelm (Eds.), *Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks* (pp. 381–398). Springer.
Papert, S. (1980). *Mindstorms—Children, Computers and Powerful Ideas*. New York Basic Books, Inc.
Piaget, J. (1959). *The language and thought of the child* (M. Gabain & R. Gabain, Trans.; 3rd ed.). Routledge. (Original work published 1926).
Piaget, J. (2001). The psychology of intelligence (M. Piercy & D. E. Berlyne, Trans.; 1st ed.). Routledge. (Original work published 1947).
Piaget, J. (2013). *The construction of reality in the child*. Routledge. (Original work published 1954).
Piaget, J., & Inhelder, B. (1967). *The child's conception of space* (F. J. Langdon & J. L. Lunzer, Trans.). The Norton Library. (Original work published 1948).
Robutti, O., Aldon, G., Cusi, A., Olsher, S., Panero, M., Cooper, J., Carante, P., & Prodromou, T. (2019). Boundary objects in mathematics education and their role across communities of teachers and researchers in interaction. In K. Krainer & T. Wood (Eds.), *International handbook of mathematics teacher education* (Vol. 3, pp. 211–240). Brill Sense.
Roschelle, J. (2006). *Effective integration of dynamic representations and collaboration to enhance mathematics and science learning*. Keynote address at Curriculum Corporation 13th National Conference.
Sherin, B. L., & Star, J. R. (2011). Reflections on the study of teacher noticing. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 66–78). Routledge.
Silverman, J., & Thompson, P. W. (2008). Toward a framework for the development of mathematical knowledge for teaching. *Journal of Mathematics Teacher Education, 11*, 499–511.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114–145.

Simon, M. A. (2014). Models of students' mathematics and their relationship to mathematics Pedagogy. *Constructivist Foundations, 9*(3), 348–350.

Simon, M. A., & Tzur, R. (1999). Explicating the teacher's perspective from the researchers' perspectives: Generating accounts of mathematics teachers' practice. *Journal for Research in Mathematics Education, 30*(3), 252–264.

Star, J. R., & Strickland, S. K. (2008). Learning to observe using video to improve preservice mathematics teachers' ability to notice. *Journal of Mathematics Teacher Education, 11*(2), 107–125.

Steffe, L. P., & Thompson, P. W. (2000a). Interaction or intersubjectivity? A reply to Lerman. *Journal for Research in Mathematics Education, 31*(2), 191–209.

Steffe, L. P., & Thompson, P. W. (2000b). Teaching experiment methodology: Underlying principles and essential elements. In R. A. Lesh & A. E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 267–307). Lawrence Erlbaum Associates.

Steffe, L. P., von Glasersfeld, E., Richards, J., & Cobb, P. (1983). *Children's counting types: Philosophy, theory, and application*. Praeger Scientific.

Stockero, S. L., & Van Zoest, L. R. (2013). Characterizing pivotal teaching moments in beginning mathematics teachers' practice. *Journal of Mathematics Teacher Education, 16*(2), 125–147.

Teuscher, D., Moore, K. C., & Carlson, M. P. (2016). Decentering: A construct to analyze and explain teacher actions as they relate to student thinking. *Journal of Mathematics Teacher Education, 19*(5), 433–456.

Tripp, T., & Rich, P. (2012). Using video to analyze one's own teaching. *British Journal of Educational Technology, 43*(4), 678–704.

Ulrich, C., Tillema, E. S., Hackenberg, A. J., & Norton, A. H. (1995). Constructivist model building: Empirical examples from mathematics education. *Constructivist Foundations, 9*(3), 328–339.

van Es, E. A., & Sherin, M. G. (2002). Learning to notice: Scaffolding new teachers' interpretations of classroom interactions. *Journal of Technology and Teacher Education, 10*(4), 571–596.

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Falmer Press.

Wilson, P. H., Lee, H. S., & Hollebrands, K. F. (2011). Understanding prospective mathematics teachers' processes for making sense of students' work with technology. *Journal for Research in Mathematics Education, 42*(1), 39–64.

Opportunities and Challenges That Silent Video Tasks Bring to the Mathematics Classroom

Bjarnheiður Kristinsdóttir

Abstract This chapter provides readers with a new perspective on how short, animated video clips can be used in the mathematics classroom to elicit, attend to, discuss, interpret, and respond to student thinking. It reports on findings from a case study conducted over one school term in collaboration with three Icelandic upper secondary school teachers who implemented *silent video tasks* in their classrooms and took active part in developing the tasks' instructional sequence. By viewing the tasks' potential along the five dimensions of powerful mathematics classrooms defined by the TRU framework (Teaching for Robust Understanding) and comparing them with data from classroom observations and teacher interviews, I aimed to identify opportunities and challenges that silent video tasks bring to the mathematics classroom. Special emphasis was put on the formative assessment dimension. This chapter contributes to the research community's current knowledge of the role that short, animated videos can play in teachers' formative assessment practices. Results of this study confirm previous research indicating that students' responses to silent video tasks can give teachers valuable insights into students' mathematical understanding and enable teachers to refer to students' ideas in a new way in classroom discussion. The biggest challenge created by the silent video tasks was the delicate task of orchestrating meaningful classroom discussions based on students' task responses.

Supplementary Information The online version contains supplementary material available at [https://doi.org/10.1007/978-3-031-05254-5_5]. The videos can be accessed by scanning the related images with the SN More Media App.

B. Kristinsdóttir (✉)
School of Education, University of Iceland, Reykjavík, Iceland

A. Clark-Wilson et al. (eds.), *The Mathematics Teacher in the Digital Era*, Mathematics Education in the Digital Era 16,
https://doi.org/10.1007/978-3-031-05254-5_5

Keywords Task design · Silent video tasks · Teaching for robust understanding · Formative assessment · Video narration technology · Teacher practices · Technology-mediated practices

1 Introduction

The use of short films for the teaching and learning of mathematics has become more common as flipped classroom approaches and resource collections such as Khan Academy gain more popularity (Cargile & Harkness, 2015). Instructional videos used in mathematics classes are usually created by teachers, but as access to technology for video narration and creation becomes more widespread, the roles of teachers and students can change. For example, students can create a narrative to accompany a silent animated video (Kristinsdóttir et al. 2020b) or become video creators (Oechsler & Borba, 2020) and thus take on a more active role in their learning.

This chapter describes and analyses teachers' implementations of silent video tasks (SVTs), in which students are asked to add their own narrative to a silent animated mathematics video, share it with their teacher and peers, and reflect on each other's task responses in a whole class discussion. It draws on data from the final phase of a multi-phased design-based doctoral study that aimed to design, define, develop, and implement silent video tasks in collaboration with Icelandic upper secondary school teachers. From here onwards this particular phase will be referred to as a case study. Previous research indicated that SVTs might give teachers insight into students' current level of understanding, and thus be useful for their formative assessment practices (Kristinsdóttir et al. 2020a, b). Therefore, the case study emphasis was on the further development of instructional sequences for SVTs to support teachers' technology-mediated formative assessment practices, and to gain clarity on how and why teachers could, or would, use SVTs in their classrooms as part of formative assessment practices.

The chapter focuses on the teachers' role in the integration of video narration technology for the purpose of assessment in mathematics. Such technology is an important part of students' social media culture but has hitherto rarely been utilised for mathematics teaching and learning. The chapter aims to contribute to the mathematics education research community's current knowledge (e.g., Aldon & Panero, 2020; Bellman et al., 2014; Olsher et al., 2016; Venturini, 2015; Venturini & Sinclair, 2017) related to the use of technology in assessment, with a specific focus on video narration. The next section gives a short historical overview of the ways in which silent video clips have been used for mathematics teaching and learning in the past. It will be followed by a description of the research context and an introduction to the frameworks used to analyse data.

2 Background: Silent Video Clips for Mathematics Teaching and Learning

Silent videos include no text, music, voice-over, recordings of classroom settings, or human beings. They solely include dynamic representations of mathematical objects; illustrations that change in time but never fail to stay intact with the definition and properties of the object. For example, a silent video could show a triangle inscribed in a circle such that one of its sides is equal to the circle diameter. As the vertex opposite to the diameter is moved along the circle circumference, its angle remains 90°. For a student who is unfamiliar with Thales' theorem, this might be surprising, evoke curiosity and be worth seeking explanation for.

Despite silent videos not being interactive, in a way, animated silent films showing mathematics dynamically can be seen as a predecessor of digital geometry software (DGS), which came about in the 1980s. The use of silent video clips for mathematics teaching and learning dates back to 1910, when the German mathematics teacher Ludwig Münch (1852–1922) produced and screened 30 short, animated films about geometry and astronomy for his students. Twenty of Münch's films are known to exist in archives, on topics such as the Apollonius circle, but they are not accessible to the general public (Kitz, 2013).

Better known are the animated geometry films made by the Swiss teacher Jean Louis Nicolet in the 1930s, as they were widely introduced to teachers by the mathematics educator Caleb Gattegno in the 1950s (Tahta, 1981). Later, Gattegno also introduced films for university teaching made by the UK teacher Trevor Fletcher between 1952–1979 (Tahta and Fletcher 2004). Gattegno, who was a founding member of the Association of Teachers of Mathematics (ATM), reconstructed the Nicolet films in colour with computer animation and underlined that they were not merely illustrations but tools that teachers could use in many ways both in terms of explanations and follow-up work to promote mathematics learning in the classroom (Gattegno, 2007; Tahta, 1981). A recent example of such work is Sinclair's use of the Nicolet-Gattegno film *Circles in the plane* to invoke gestures with her students as they studied the mathematical concept of circle by watching the film a few times in a row, each time with a new task to imitate the video: first by talking, then by moving their hands, and finally by drawing (Sinclair, 2016).

Silent videos differ from the majority of mathematics videos that can be found via YouTube, Vimeo, and similar sources, in that they are not directly instructional. Rather, they are intended to be thought-provoking. Silent videos used in SVTs are usually less than 2 min in length and thus shorter than the Münch, Nicolet, and Fletcher films. Despite differences in length, all these films have in common that they do not pose a mathematical problem to be solved. Rather, they invite viewers to wonder, experience dynamically changing mathematical objects and think about characteristics of mathematical phenomena shown such that they might discover something new or consolidate previous thoughts about the mathematics shown in the video.

3 Silent Video Tasks

Silent video tasks (SVTs) involve the screening of a short (less than 2 min long), silent, animated video clip on a previously studied mathematical topic. The video is designed to invite description, explanation, and/or narrative with possibilities to generalise mathematical ideas. Working in pairs, students are invited to prepare and record their voice-over to the video clip. Students' responses to the task are then listened to, and discussed, in a whole class discussion led by the teacher. During the discussion, teachers can ask and prompt students with the aim to approach common understanding of mathematical concepts and properties. A vignette might help readers visualise the task implementation:

> We enter Anna's classroom. She shows a one-minute video clip (see video Fig. 1) to her 16-year-old students in a remedial class. The video features a topic that they have been working on for the past 2 weeks: Different zones of the Cartesian coordinate system (e.g., $x > 0$) are highlighted successively in distinct colours, and four points appear one after the other. As the class watches the video, a student can be overheard commenting that there is no sound. Anna acknowledges this observation and explains to students that it will be their task to add the narration to the video.
>
> "What are we supposed to talk about?", one student asks, and Anna replies, "Whatever comes to your mind. Imagine a blind person visiting, how would you narrate, describe or explain to them what is going on in the video?". She assigns students randomly into groups of two and gives them twenty minutes to watch the video as often as they want to, whilst they work on their recording. Some students try to fish for what Anna "wants them to say" (without success), but others start recording after a short dialogue. Gathered back in the classroom, Anna plays one student response after the other, stopping the playback every now and then to ask for clarification or point students' attention to something specific: "Did you understand that? What do you think they wanted to say here?" and "Can you explain what you mean by…?"

4 Icelandic Context

Throughout the research project, I worked with upper secondary school teachers in Iceland to develop the instructional sequence of SVTs and determine their value for teaching and learning in the mathematics classroom. There are 38 upper secondary schools in Iceland, out of which 30 offer lines of study that prepare students for further studies in STEM (science, technology, engineering, mathematics) subjects. The majority of upper secondary schools are state run and those privately owned also receive state support. Until 2015, most Icelandic upper secondary school programs were planned for 4 years, but now they are planned for 3 years, during which learners are generally 17–19 years old. Some vocational programs require longer periods of study.

By adding courses to their studies, students in vocational programs have the possibility to complete the matriculation examination in preparation for entering higher education. Although not compulsory, emphasis is placed on providing everyone with the opportunity for upper secondary education, irrespective of their results at the end of compulsory schooling (primary and lower secondary education are compulsory). All three school levels, pre-primary, compulsory and upper secondary, are built on the same values as stated in the national curriculum for each level—values such as respect and care for others, tolerance, and responsibility.

Teaching methods in Icelandic upper secondary mathematics lessons are mainly teacher centred (Sigurgeirsson et al., 2018) and although some schools use DGS, often it is the teacher who uses the DGS for demonstration purposes rather than the learners using it for discovery (Jónsdóttir et al. 2014). Formative assessment and group discussions are rarely practiced in Icelandic upper secondary school mathematics lessons (Jónsdóttir et al., 2014).

In accordance with the Icelandic upper secondary school main curriculum from 2011, mathematics teachers suggest course descriptions to the Icelandic Ministry of Education Science and Culture, which checks them for acceptance. From 2011 onward, the course descriptions are expected to include course objectives and competencies that learners are expected to achieve. At larger upper secondary schools where the same course is taught to many groups of learners (e.g., five groups of 30 learners each), each group having one teacher, the teachers collaborate and usually attempt to follow the same course schedule during each semester. Despite the 2011 National Curriculum Guide's emphasis on competencies, an 'undercover' mathematics course schedule with 'lists of things to cover' exists at the majority of upper secondary schools in Iceland. The same phenomenon was observed after a National Curriculum change in 1999 (Harðarson, 2010).

5 Teaching for Robust Understanding in Mathematics

It is widely accepted that there is no prescribed 'best way to teach'. By analysing mathematically powerful classrooms (classroom environments that support students' mathematical learning) of various kinds with teachers applying a spectrum of different teaching methods, Schoenfeld and his colleagues attempted to distil the characteristics of these classrooms into a small number of dimensions that teachers might be guided towards paying attention to. This was not to claim that one teaching method was best, but to identify what was important to be aware of. Out of this work came five dimensions that constitute the TRU framework (Teaching for Robust Understanding). They are: (i) *Mathematics:* the richness of the mathematical content, (ii) *Cognitive Demand:* the opportunity for students to engage in productive struggle, (iii) *Equitable Access to Content:* that all students are involved in meaningful ways, (iv) *Agency, Ownership, and Identity:* opportunities for students to develop a sense of agency, make mathematics their own, and to develop productive mathematical identities as thinkers and learners, and (v) *Formative Assessment:* the degree to which student ideas are made public and responded to in productive ways (Schoenfeld, 2018). Of course such distillation is problematic in the sense that it puts more emphasis on some critical aspects of the teaching practice over others. Intended to help teachers create classrooms from which students emerge as "knowledgeable, flexible, and resourceful thinkers and problem solvers" (Schoenfeld, 2018, p. 494), the TRU framework, however, does not prescribe any specific practices. It only suggests that teachers become aware of and pay attention to ways (and there are many such ways possible) in which they can improve their current practice along the five TRU framework dimensions.

Since the TRU framework is mostly used to guide teachers' professional development, it was not obvious that it could be useful for the study presented in this chapter. The idea to identify whether silent video tasks offer opportunities to teachers along the dimensions of the TRU framework emerged after the data was collected and in the process of its analysis. This was due to the TRU frameworks' emphasis on conversations between teachers and students and ongoing reflection, i.e., building up awareness of and learning from experience. The intentions of the tasks might align well with the TRU frameworks' dimensions, but theory and practice must grow together. Thus, data on teachers' experiences with using silent video tasks was analysed through the lens of the TRU framework with the aim to identify potential challenges. Such challenges are often connected to tensions that arise in teacher practice when social or sociomathematical norms (in the sense of Yackel & Cobb, 1996) in the classroom are violated. Regarding the fifth dimension of the TRU framework, I will connect to key strategies for formative assessment practices by Wiliam and Thompson (2008) and a list of socio-technical approaches to raising achievement in mathematics education as presented by Wright et al. (2018).

6 Formative Assessment

Malcolm Swan argued that "technology usage must move away from merely rehearsing procedural skills (albeit with feedback) toward a usage that mirrors the outside world; it must become a tool that changes the way we think and reason" (2017, p. 31). After all, mathematics is about generalisations, and what varies or remains invariant. By emphasising that, we might improve student learning of mathematics. Previous research indicates that SVTs might fit the description of being a tool that could be utilised to decide about next steps in instruction (Kristinsdóttir et al. 2020a), as mentioned in Wiliam's (2011) definition of formative assessment:

> *An assessment functions formatively to the extent that evidence about student achievement is elicited, interpreted, and used by teachers, learners, or their peers to make decisions*

about the next steps in instruction that are likely to be better, or better founded, than the decisions they would have made in the absence of that evidence. (p. 43).

For practice, Wiliam and Thompson (2008) further identified the following five key strategies for formative assessment: (1) Clarifying and sharing learning intentions and criteria for success, (2) Engineering effective classroom discussions and learning tasks that elicit evidence of student understanding, (3) Providing feedback that moves learners forward, (4) Activating students as instructional resources for one another, and (5) Activating students as owners of their learning.

The focus of this chapter is on the ways in which technology can be used by teachers to support formative assessment practices. In Table 1, I refer to Wright et al. (2018, p. 219), whose research defined six technology-based formative assessment strategies that have the potential to support teaching and learning (see Table 1). Previous results indicated that SVTs addressed all but the final potential listed in Table 1, because feedback was not immediate but took place later in a follow-up

Table 1 A priori analysis of the ways in which silent video tasks addressed the potential that technology-based formative assessment strategies might have to support teaching and learning, framed by Wright et al. (2018, p. 219)

Technology-based formative assessment strategies that might support teaching and learning	Ways in which silent video tasks addressed these potential before the case study was conducted
Provide immediate feedback	Currently not addressed. Feedback to students is given in a follow-up lesson 1–3 days after students work on the silent video task.
Encourage discussion and developing cooperation	On the basis of some selected or voluntarily played task responses, students are encouraged to discuss and reflect on the ways in which they understand the mathematical concepts that are the topic of the silent video.
Provide an objective and meaningful way to represent problems and misunderstandings	Possible misunderstandings uncovered in student responses can be directly referred to and discussed as useful steps on students' path toward understanding.
Provide opportunities for using preferred strategies in new ways	Teachers who want to build a culture of discussing and collaborating in their classroom surely do not only want students to express their mathematical thoughts in writing. SVTs offer a way to ask for an audio response to a task, thus bringing students' thoughts and ideas to the forefront of the discussion.
Help raise issues that were previously implicit and not transparent for teachers	For example, if students who normally do not speak up in class ('live' format) take the opportunity to speak up via the voice-over (recorded format), there might be a problem of distrust (students not considering the classroom as a safe space for discussion) that needs to be addressed.
Provide different feedback outcomes	Feedback is provided to the whole class via discussion. When needed, teachers can prepare and provide individual feedback after the discussion finds place.

lesson (Kristinsdóttir et al. 2020a). However, as will be reported in more detail in the findings of this chapter, working with teachers who had experience with using formative assessment (referred to as FA hereafter), they emphasised the importance of immediate feedback and changed the SVTs instructional sequence such that discussion based on students' responses would take place immediately after students submitted their responses.

The next section introduces the case study presented in this chapter.

7 Method

In the following subsections, I introduce the participants of the study, describe the data that was collected, and how it was analysed. Challenges in data collection and ethical considerations are also discussed.

7.1 Participants

As I was interested in further developing SVTs as a tool for FA, I purposefully selected and contacted three schools that were known for their emphasis on FA to be involved in the study. Two schools in the urban area agreed, and all teachers who were interested were invited to join. One teacher at Blackbird (school names are pseudonyms), a small, 16–19, urban comprehensive school and two teachers from Mallard, a large, 16–19, urban comprehensive school (see Table 2) took part in the project. Both schools' policies expect students to be "active participants in their studies", Blackbird explicitly states in its school policy that their studies are "characterised by FA",[1] and Mallard emphasises "use of continuous evaluation of students' progress by a variety of assessment methods". It was also considered helpful if teachers at the schools encouraged learners' use of DGS such as GeoGebra and Desmos and had some familiarity with leading group discussions in mathematics lessons.

Table 2 Participating teachers in the second implementation phase of the research project and their teaching experience

Teacher pseudonym (gender)	School pseudonym	Use of DGS	Teaching experience
Andri (m)	Mallard high school	GeoGebra	10 years
Edda (f)	Mallard high school	GeoGebra	20 years
Orri (m)	Blackbird high school	Desmos, GeoGebra	2 years

[1] For participants anonymity, the corresponding school policy paper, which is presented on their website, cannot be referred to. The quotes given here have been translated from Icelandic to English.

Andri, Edda, and Orri (teacher names are pseudonyms, see Table 2) who volunteered to participate in the study were all open to trying out new teaching approaches. Orri was relatively new to teaching and saw both challenge and an opportunity for collaboration in the research project. He had some previous experience with using GeoGebra and Desmos activities in his classroom. Andri and Edda each had around a decade of experience with using GeoGebra at Mallard. Together, they had participated in various Icelandic, Nordic, Baltic, and European collaboration projects for professional development.

Among other teaching duties during fall 2019, Andri, Edda, and Orri all worked with low-achieving 16-year-old students in slow-paced remedial classes of different class sizes. They explained that these courses' schedules accommodated flexibility and thus suggested them as good situations in which to try out SVTs. All three planned to use two or three SVTs over the period of one semester. It turned out that Orri (N = 14–16) implemented three SVTs within one term, whereas Andri (N = 22) and Edda (N = 13) implemented one SVT each. The numbers in brackets denote the number of students, N, who were present in class during task implementation.

As compensation for their participation, I offered the participating teachers support meetings in case they were working on any changes in their practice. Orri accepted the offer and we met six times to discuss ways to build a thinking classroom (e.g., Liljedahl, 2018) at meetings that were recorded but not transcribed or analysed.

7.2 Silent Videos Used in This Study

At initial meetings with Andri, Edda, and Orri, we discussed the teachers' course curricula and ideated to identify topics that might be visualised in a silent video. Based on discussion and ideation with Andri and Edda, I sketched drafts and discussed ideas of three videos on the topics of coordinate geometry and linear equations. Then, based on feedback from Orri, I created the scenarios in GeoGebra and screen recording software to finalise the videos. All three videos show the coordinate system with (0,0) at the centre; the x- and y-axis marked with numbers. The videos were designed intentionally to point students' attention to details in the definitions (characteristics) of the mathematical phenomena in focus. The videos were intended to be used for assessment, but they could also be shown at the start of a lesson sequence to collect students' initial ideas about each mathematical topic within a word cloud.

SVT1 The first video (see Fig. 1) focuses on properties of the coordinate system: First, zones of the coordinate system appear highlighted in light-blue colour one after the other: $x > 0$, $y > 0$, $x < 0$, and $y < 0$. Next, the quartiles appear highlighted one after the other and a point appears in each of them: 1st quartile blue with (4,2), 2nd quartile green with (−3,5), 3rd quartile pink with (−2,−2), and 4th quartile orange with (9,−1).

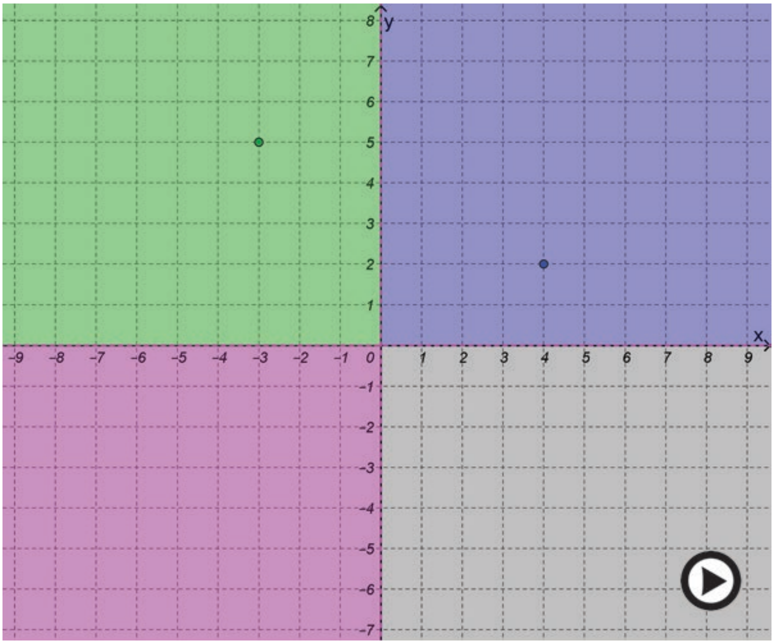


Fig. 1 A still from SVT1, a silent video that focuses on properties of the coordinate system (▶ https://doi.org/10.1007/000-8ww)

SVT2 The second video (see Fig. 2) focuses on the slope of a line. Two points marked A and B, with A = (−3, 1) and B = (−1,2), are shown in blue along with a blue line AB from the start of the video. Point A stays constant while B (and thus the line along with it) moves following a rectangular shaped path, pausing for a short while along the path at the following points: (−1, 3), (−2, 3), (−3,3), (−4,3), (−5,3), (−5,2), and (−5,1). Thus, the movement of point B pauses when the line has the following sequence of slopes: ½, 1, 2, undefined, −2, −1, −½, 0.

SVT3 The third video (see Fig. 3) focuses on the graph of a line as a function of x. One after the other, blue points with integer coordinates along the line $y = x$ show up from (−7,−7) to (8,8) before the line through the set of points is drawn in blue. Next, red points on the line $y = 2x + 4$ show up one after the other from (−5,−6) to (2,8) before the line appears drawn in red. Then, all the red and blue points move along their shortest path to the x-axis and back to their lines again. This movement of the points toward the x-axis and back to their position on the respective lines is repeated once more before the video ends.

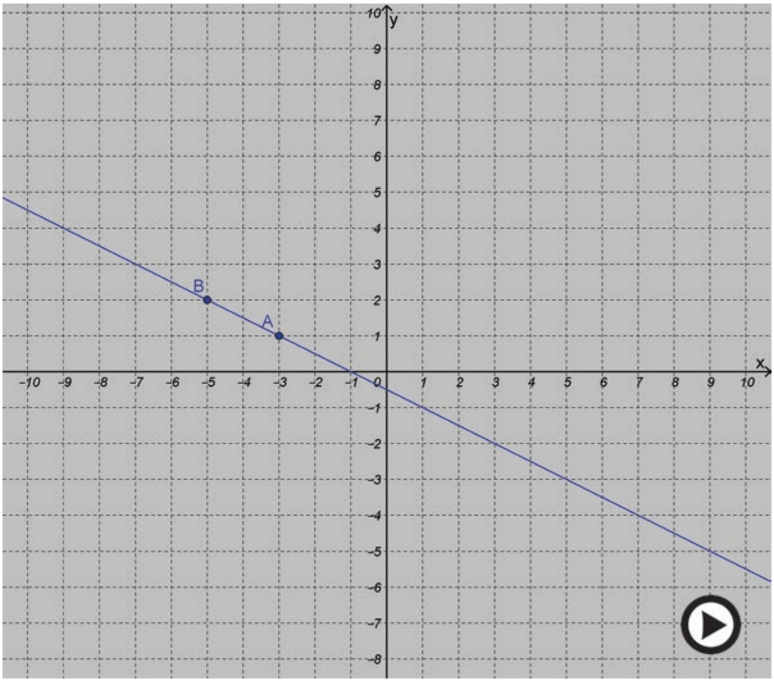


Fig. 2 A still from SVT2, a silent video that focuses on the slope of a line (▶ https://doi.org/10.1007/000-8wv)

These specific SVTs will hereafter be referred to as SVT1, SVT2, and SVT3. The next section will clarify what data was collected within the frame of the presented study.

7.3 Collected Data

Data collected included semi-structured interviews with participating teachers, field notes from classroom observation visits, the students' responses to SVTs, and students' feedback. For the purposes of this case study, my focus was mainly on the interviews and field notes. Prior to the study, I had visited Blackbird and Mallard within the framework of teacher conferences. To get to know the participating teachers and their working places better, I visited them in their schools in August 2019. I also visited Edda once and Orri three times to observe lessons that did not include SVTs. Before and after each SVT implementation, I conducted and audio-recorded semi-structured interviews (Brinkmann & Kvale, 2009) with participating teachers: two with Andri and Edda together, one with Andri, one with Edda, and five with Orri (see Fig. 4). These interviews included questions regarding teachers'

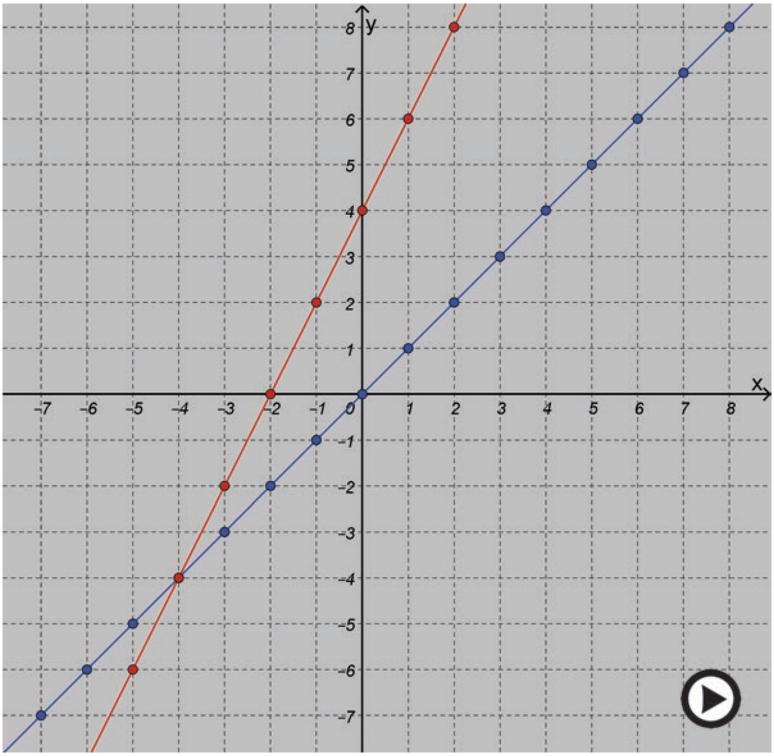


Fig. 3 A still from SVT3, a silent video that focuses on the graph of a line as a function of x (▶ https://doi.org/10.1007/000-8wx)

expectations of, and experiences with, implementing SVTs in their classrooms. All interviews took place either in teachers' classrooms or their schools' meeting rooms. For the purpose of this study, I wrote classroom observation field notes on three visits to Orri's classroom and one visit to each of Andri and Edda's classrooms. The reason for why I collected field notes rather than video recordings from classrooms is given in the next section.

7.4 Challenges in Data Collection

What goes on in classrooms involves speech, gestures, and mimes, many of which happen simultaneously in different corners of the classroom or school building. Even though these actions might seem obvious to the participants involved, they can be less obvious and require more careful examination for visitors. Therefore, to grasp what goes on in classrooms in practice, researchers normally aim to collect video recordings rather than only classroom observation notes.

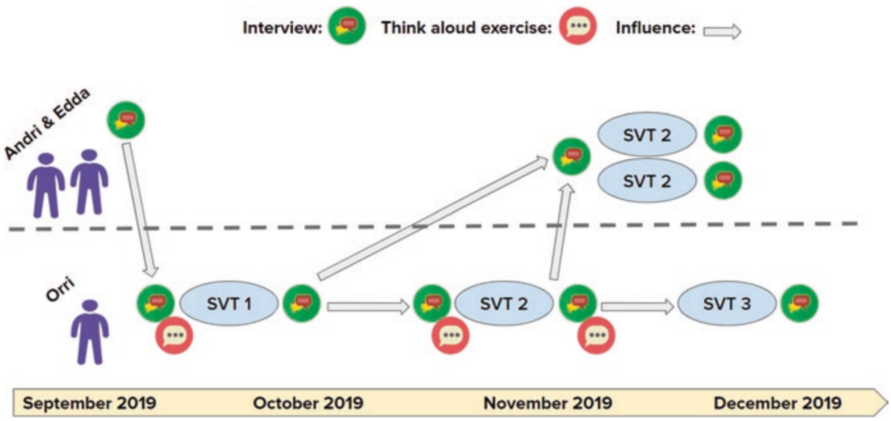


Fig. 4 A timeline of the data collection process. Andri and Edda work closely together at Mallard and thus they were only interviewed separately after the task implementation. Three interviews with Orri at Blackbird included "think-aloud" exercises. When possible, information was transferred by the researcher between teachers, shown by arrows that cross the dotted line between the two schools. The arrows between the three implementations at Blackbird indicate that each implementation informed the next

At the point of data collection—due to the recent introduction of the General Data Protection Regulation (GDPR)—school leaders in Iceland were increasingly aware of complexities regarding data collection in their teachers' classrooms. They preferred field notes over video recordings from teachers' classrooms. To build trust and positive correspondence needed for research that is done in collaboration with teachers, I thus decided to take field notes.

7.5 *Ethical Considerations*

The Icelandic Data Protection Authority was informed about the research project. Teachers signed an informed consent stating their awareness that they could withdraw their participation at any point in time. Principals signed informed consent granting me permission to interview teachers and to visit their classrooms provided that I would not collect identifiable information about students. They trusted me to treat collected data in a respectful manner and anonymise names. Students received written and oral information about the research project and were informed that they could deny participation, meaning that their voice-over recording would only be listened to by their teacher and not the researcher. No student refused participation.

7.6 Research Design and Data Analysis

By working with teachers—asking them to implement SVTs in their classrooms and to reflect on their expectations and experiences—I took a hermeneutic (interpretive) phenomenological stance (van Manen, 2016) towards answering the question of how and why teachers could use SVTs in their mathematics classrooms. I studied teachers' actions and reasons given for their actions, and I transferred between participants (see Fig. 4) all suggestions related to the development of SVTs instructional sequence. In the busy setting of the participating teachers' own classrooms, I observed their work and interviewed them to hear their personal insights on whether, and how, they could use this tool for the teaching and learning of mathematics. Furthermore, I reflected on teachers' insights by writing notes directly after our meetings, and again as I analysed the transcripts from our interviews. Iterative cycles of writing notes and reflections contributed to our collective and evolving understanding of how SVTs could be used in the mathematics classroom.

During my interviews with Orri, I developed a *think-aloud exercise,* asking him to think aloud about how he would implement the SVT next time and why. In the first interview, the purpose of the think-aloud exercise was to hear his ideas and then in later interviews—after each implementation—the purpose was to re-construct his experiences and record his reflection, alongside his expectations for the next round of implementation.

It usually requires training to become aware of, remember and reconstruct our own interpretations; what we were thinking or making sense of. Despite having no training, Orri reflected on what he thought about in-the-moment and related it to planned actions for the next implementation. It was a free-flow and in-the-moment exercise, meaning that Orri could revise his own thinking on the go. During the think-aloud exercise I thus normally did not interrupt unless something needed immediate clarification.

All interviews were transcribed verbatim in Icelandic. When possible, I transcribed directly after the interview took place and thus was able to add some extra notes in parentheses. Analysis started immediately after the first interview and in that first familiarisation phase, I focused on the instructional sequence design and development. After transcribing the last interview, I underwent a second familiarisation phase of the data using open coding in Icelandic on anything that I found interesting in the data. Directly after the second familiarisation phase, I read through the transcripts again, writing detailed notes in English where I summarised and deepened my thoughts. On the basis of the detailed notes, I created a distilled overview of the five interviews with Orri on a large sheet of paper (630x891mm), gaining an overview of how Orri's ideas, experiences, and expectations developed over time.

Regarding opportunities that tasks bring to the mathematics classroom, it comes down to what we consider important. SVTs were developed to be a socio-constructive approach to teaching, a tool that teachers might use to support students in

The Five Dimensions of Mathematically Powerful Classrooms	
The Mathematics	How do mathematical ideas from this unit/course develop in this lesson/lesson sequence? How can we create more meaningful connections?
Cognitive Demand	What opportunities do students have to make their own sense of mathematical ideas? To work through authentic challenges? How can we create more opportunities?
Equitable Access to Content	Who does and does not participate in the mathematical work of the class, and how? How can we create more opportunities for each student to participate meaningfully?
Agency, Ownership, and Identity	What opportunities do students have to see themselves and each other as powerful mathematical thinkers? How can we create more of these opportunities?
Formative Assessment	What do we know about each student's current mathematical thinking? How can we build on it?

Fig. 5 A summary in the form of questions asked within each of five dimensions of mathematically powerful classrooms according to the Teaching for Robust Understanding (TRU) framework (Baldinger et al. 2018, p. 2, reprinted with permission)

developing their own understanding of mathematics. Previous research has indicated that SVTs occasioned for teachers a fundamental shift in perspective from teacher-centred to student-centred instruction (Kristinsdóttir et al. 2020b), and although the TRU framework offers no prescription on how to teach, it also includes such a fundamental perspective shift (Schoenfeld, 2018). Furthermore, the TRU framework seemed to provide a language for talking about instruction along dimensions of interest. Therefore, after the phases of familiarisation and analysis, I once again read through my detailed notes with questions from the TRU framework (Baldinger, Louie, & the ATSMAP, 2018; Schoenfeld, 2018) in mind. I used these questions (see Fig. 5) to determine the opportunities and challenges that SVTs can bring to the mathematics classroom and discussed the findings in a doctoral seminar. The findings are presented in the next section.

8 Findings

This section starts by introducing the teachers' current assessment practices and some ways in which they influenced the SVTs instructional sequence. It continues to describe teachers' implementations of the SVTs and my use of the TRU framework to summarise the opportunities and challenges that SVTs bring to the mathematics classroom.

8.1 Teachers' Existing Assessment Practices and their Influence on SVTs Instructional Sequence

Topic booklets made by Andri and Edda at Mallard and by colleagues of Orri's at Blackbird were the main focus of remedial classes. They were composed of practice problems that were partly exploratory or designed to be worked on using GeoGebra, but mostly they included closed question problems. Work on problems from these booklets counted towards students' final grade. At Mallard, students had access to online self-assessment exercises and completed three exams (each covering part of the material of the course) during the term. A good grade on these three exams could render the final exam as optional. At Blackbird, assessment was based on students' observed work in class, participation in Desmos classroom activities, handed-in work on problems similar to the booklet problems, and participation in final-week group projects that required knowledge of all topics of the term.

Some emphasis was put on group work at Mallard and Blackbird, but no specific emphasis on developing students' verbal communication about mathematics. Despite the teachers' awareness of the importance of whole class discussions, these were seldom practiced and were mostly constrained to teachers asking closed questions, waiting for a response, and either answering the question themselves or evaluating a received response.

All three teachers were familiar with FA practices and what they entailed. This was manifested in the first interview with Andri and Edda as they suggested that feedback via group discussion should take place immediately after students had submitted their task responses. Thus, they changed the initially suggested SVT instructional sequence, which was based on previous research (Kristinsdóttir et al. 2020b), which had placed the group discussion in a follow-up lesson some days after students had submitted their task responses, i.e., to provide time for teachers to prepare themselves by listening to, selecting, and sequencing some task responses on which to base the follow-up discussion. Andri and Edda explained that they expected feedback to be less effective/useful if it was not immediate, similar to Wright et al.'s (2018) suggestions. Also, Andri and Edda wanted all student responses to be listened to in a random order rather than sequencing some selected responses. Otherwise, students might interpret their actions as if they were judging students' responses "from the worst to the best".

Furthermore, Andri asked if self-assessment and peer-assessment practices could be used such that students would provide written feedback to each other's responses. His idea was brought to Orri before the SVT1 implementation. Orri prepared an online reflection sheet for students, where each pair would reflect on three other pairs' responses and at least two pairs would listen to each response. In practice, however, Orri experienced students not taking their role as peer-reviewers seriously. Thus, in the think-aloud exercise before SVT2, he suggested a different approach where the whole group would listen to all task responses in a random order.

8.2 Description of Perceived Classroom Norms

Andri and Edda expressed that they aimed to create a safe and constructive working environment for students, especially those in remedial classes. They set a clear ethos resulting in a calm and engaged atmosphere. Before the SVT implementation, they wanted to make sure that focus would not move from the mathematics to solving technological issues and therefore prepared, by downloading and testing screen recording software (laptops) and sound recording software (smart phones) as well as confirming that the learning management system would suffice to accept students' task responses.

Orri aimed to create a relaxed atmosphere in his classroom. This resulted in a rather loose ethos, with students frequently arriving late and leaving the classroom from time to time. He often put in much time to prepare tasks and create feedback opportunities for students and was repeatedly disappointed by students not putting effort into their work on prepared tasks and not reading/listening to his feedback notes/videos. Orri expressed explicitly and indirectly that he wanted mathematics to be fun, a goal that he expected would lead to students putting in more effort.

8.3 Description of How SVT2 Was Used by Teachers

Orri aimed to collect students' initial ideas about the new mathematical topic and so he showed the video from SVT2 to students at the start of the lesson sequence on the topic of the linear equations. He asked students to think and write words or concepts that connected to the video and he collected these in a word cloud. Orri explained that he never would have thought of some of the words that students wrote (e.g., compass, box, time, and spin) beforehand. Two weeks later, at the end of the lesson sequence, Orri implemented SVT2.

All participating teachers implemented SVT2 in a similar way. At the start of the lesson, they showed the video to the whole class. Its topic (the slope of a line) had been the course focus of the preceding weeks. After explaining that it would be the students' task to add a narrative explanation or description to the video, teachers randomly assigned students into groups of two to work on their voice-over. Despite technical preparations, teachers were observed attending to a few students who had difficulties with either downloading recording software or uploading their voice-over recording.

Some students in each group immediately opened up the link to the video and started to work, but others seemed more confused. This made teachers busy reacting to students who wanted either instructions regarding what to focus on in their response or a confirmation that their contribution was going in a 'right' direction. I had discussed with the teachers the risk of such *stop-thinking* situations (see

Liljedahl, 2018) during preparation and even though they considered it to be challenging not to answer students' questions, they—upon entering such situations—made clear that it was the students' responsibility to make decisions on what to focus on in their voice-over. Nevertheless, there was an apparent tension created by the stop-thinking situations.

Immediately after receiving students' voice-over recordings, the teacher gathered the group together to listen and react to all of them. Similar to what teachers had predicted, students seemed to find it important that their own task response would be played and they were eager to hear their teacher's and peers' reaction. Still, the teachers' effort to involve students in discussions resulted only in some short reflections and surface-level discussion; no student-to-student debate was observed. The questions used by the teachers to facilitate discussion included asking students to recognise differences and similarities among their responses, and clarifying questions regarding whether and how students understood what was being said. At the end of class, Andri and Edda asked students to answer a questionnaire, on what (if anything) they would have liked to change in their voice-over, if there was anything that made them wonder, and if they would like to add any comments or questions regarding the SVT. Orri, on the other hand, asked students to record a new voice-over, wondering what (if anything) they would change.

In the next section, the goals for the SVTs will be identified along the five dimensions of the TRU framework. Then, I will present the opportunities and challenges that were identified based on the classroom observation and interview data from teachers' SVT implementations along the dimensions of this framework.

8.4 In Theory: Opportunities and Challenges that SVTs Might Bring

Within the TRU framework, a Conversation Guide (Baldinger et al. 2018) lists a set of questions intended for teachers' planning and reflection. These questions are organised along the TRU framework's five dimensions (see Fig. 5). Based on the intentions behind the way in which SVTs were designed and developed, the questions are answered in Table 3.

Answering the questions from Fig. 5 offered a way to evaluate whether the SVTs fulfilled their intended role in theory. Then, to connect with practice, I took a new look at the interview and classroom observation data to perform a top-down analysis, collecting instances that evidenced teacher practice along the five dimensions. This top-down analysis was, then, followed up by a bottom-up analysis of the selected data excerpts with a focus on identifying the opportunities and challenges that teachers encountered during task implementation. Results of that analysis are given in the next section, which is organised according to themes that were identified.

Table 3 When intentions behind the design of silent video tasks are viewed along the dimensions of the TRU framework, some alignment can be seen. Here, questions from Fig. 5 are answered based on assumptions about the ways silent video tasks might support teachers in teaching for robust understanding

TRU Dimension	What silent video tasks are intended for
The mathematics	By describing, explaining, or narrating the video, students' mathematical ideas are explicitly put into words that get heard, reflected on and discussed by the whole group. By both recording their voice-over and participating in the whole group discussions, students might develop their mathematical ideas and create meaningful connections. They might also realise whether, and in what ways, they understand the mathematics shown in the video.
Cognitive demand	Students might make their own sense of mathematical ideas that arise when watching the silent video and discuss in pairs what to focus on in their voice-over recording. It can be a challenge for students to decide what to focus on.
Equitable access to content	Students are offered an untraditional way (recorded verbal communication) to participate in the lesson. It might create opportunities for each and every student to participate meaningfully in the mathematical communication of the class.
Agency, ownership, and identity	Students whose response is listened and reacted to might develop a feeling of belonging and gain a new view on their own articulated ideas as these get discussed by the whole group. As participants (not only listeners) in the discussion, students get an opportunity to see themselves and their peers as mathematical thinkers.
Formative assessment	From listening, discussing, and re-listening to students' responses to the SVT, teachers gain insight into what students pay attention to when watching the silent video and thus might gain insight into students' current conceptual understanding. Teachers can build on this insight, probing more deeply into aspects that seem currently unclear to students. Also, it might be possible to lead students' discussion towards more abstraction or to generalise and to lead the group towards some common understanding of the mathematics shown in the video. Teachers' work with the current group of learners can also be helpful the next time teachers work with learners on the same mathematical ideas.

8.5 *In Practice: Opportunities and Challenges That SVTs Brought*

This subsection is based on excerpts from observation and interview data that were identified to be connected with teacher practice along one or more of the five dimensions. Even though each dimension of the TRU framework involves putting on new glasses that highlight that dimension, some overlap is unavoidable because the categories discussed in each dimension are not completely distinct. Still, usually one dimension was identified to be the most prominent one for each excerpt of data. Labels regarding what dimension each quote or data reference belongs to are not provided here. Rather, the focus is on introducing the data organised by themes identified regarding opportunities and challenges that SVTs bring to the mathematics classroom. After the listing of themes, these results will be discussed.

8.5.1 Challenge: It Is Hard to Change a Prevailing Socio-Mathematical Norm (For Example, That There Is a Single Correct Answer)

Andri and Edda were surprised by the students' responses that did not mention slope at all and only focused on coordinate points:

Andri: *This was only*[2] *about the point coordinates.*
Edda: *But they never mention the line slope. And still they say that they would not change their voice-over if they would do a new recording. [referring to students' end-of-class feedback]*[3]

They were happy to see that students mastered how to list the coordinate points:

Andri: *what surprised me was that almost all of them put the x-coordinate before the y-coordinate […] in my experience this is endlessly difficult for some students*
Edda: *At least they figured out the coordinate points completely*

Nevertheless, Andri and Edda expressed that they would like their students to gain understanding about linear functions, not only points, even though they read the coordinates correctly. The teachers found it challenging to change students' ideas of what mathematical practice entails in regard to sharing information that one is not yet sure about. If students decided to avoid mentioning the slope, they suggested it might be a coping strategy due to a socio-mathematical norm that is persistent in the Icelandic school system, which is to assume that mathematics is a practice where only one correct answer exists that matters the most.

8.5.2 Opportunity: Previously Inaccessible Information Revealed by Students' Task Responses

Students' responses made their struggles with the concept of slope graspable and discussable:

Andri: *For example, it becomes painstakingly clear how they have not yet realised, you know,*[4] *[what the concept of] slope [is/means] somehow, the minority of them have, at least.*
Andri: *I actually just feel like I always need to teach them this [slope] anew [..,]. You know, I have shown it to them multiple times, you see, but it does not seem like […] it seems like it does not properly arrive.*

[2] Words in the transcript are underlined if interviewees put special emphasis on them

[3] In a few cases, for clarification purposes, words within square brackets have been added to the excerpts from interviews.

[4] Commas have been added around common hesitations (such as "you know" or "you see") to make the excerpts from interviews clearer to read.

Edda: I noticed that they avoided mentioning [the slope when the line was] horizontal and vertical.

Edda: One sees that it is the slope and negative slope that is something that is confusing them […] they do not realise that when the outcome is negative then the slope is negative well they see it approximately but they do not connect it. I find it very interesting

In the last quote, Edda was referring to students' responses that described the changing slope of the line as ½, 1, 2, 0, 2, 1, ½, and 0 (instead of ½, 1, 2, undefined, −2, −1, −½, and 0). This difficulty hitherto had gone unnoticed but was made visible by students' task response and was received by Edda as valuable information.Similarly, responses to SVT3 revealed a previously unnoticed lack of precision in word use when it came to describing intercepts of a line with the axes of the coordinate system. In retrospect, Orri realised that his own level of precision might be improved:

Orri: One realises when one assigns such tasks and does, you know, something, which one finds so obvious that one forgets that it is not obvious at all, you see, one speaks of intercepts over and over again, you know, it is maybe not always obvious that it is often, you know, that there are two intercepts, you see, that one needs to take, you know, that there is not only an intercept that it is a y-intercept and an x-intercept, you see, you know, we find it completely clear […] but to them it is maybe something that one has never properly covered.

Orri's awareness was thus not only raised regarding students' mathematical discourse, but also his own. Reflecting on how to address students' precision, Orri suggested that he could play a random example task response as an audio file (without viewing the video) and draw on the whiteboard according to what he heard. Then, the students might realise:

Orri: Then one would simply say "ok, cuts the x-axis at negative four and the y-axis at negative four ok then it goes through these two points" and they will just say "no we did not mean it in that way" and then I can say "then how can you say it such that it can be understood?".

This idea was never tested in action because Orri came up with it after the third implementation.

8.5.3 Challenge: It Can Be Tempting to Return to Teacher-Centred Transmission of Knowledge

Upon noticing students' perception of slope as always being positive, Edda felt it was something under her responsibility to clarify:

Edda: […] they have yet to connect that [slope being negative or positive] so it is something one needs to go maybe better through and I did that after showing the last one [student response].

In other words, although Edda wanted to draw upon students' responses when concluding the discussion, she could not resist reacting by giving a lecture about slope. Her return to teacher-centred transmission of knowledge might have been influenced by a part-exam which was planned in the upcoming week. Edda expected her students to connect to her clarification because of their SVT participation:

Edda: *So I think it is good to do such tasks [SVTs] and then you can go through the video and explain it better then maybe they will get it better since they have themselves worded it in their own way already and one understands what is wrong and what not so in that way it is very positive so I think it would be exciting to add this task into the [course] curriculum.*

By developing this idea further to make clearer connections to students' responses, maybe what Swan (2006) called a *conceptual reorganisation* might be facilitated when such inconsistencies or obstacles (based on students' ideas) were identified. In that way, students could develop their ideas or build a bridge over the gap. However, as the next theme shows, this was considered by teachers to be a challenging task.

8.5.4 Challenge: It Is Challenging to Lead Group Discussions Based on Students' Ideas

Leading a group discussion—and especially connecting it to students' words and mathematical ideas—revealed itself to be quite a challenge for teachers:

Andri: *Ahhh, you know, it is a little scary to do this, you see, but I have in a way done similar things before but still not in <u>this</u> way […] it would be amazing to do this again.*

Orri: *I think one needs some training in listening well to this, what they <u>are</u> saying and trying to figure out why they are thinking things or I feel that you [the researcher] often hear such things […] things that I had not figured out myself, you see.*

This was not surprising. After all, the orchestration of classroom discussion in the mathematics classroom is a challenging task that requires much practice (Stein & Smith, 2011). Also, the development of awareness of "what students are saying", that Orri mentions above, is important for reflecting in the moment and reflecting on the moment (Pai, 2018, p. 41). Teachers knew that it would be challenging to lead group discussions based on students' work, and they also knew how important it was:

Orri: *I have to confess that I have not put too much emphasis on that [conversations/discussion] one knows that it is absolutely <u>the</u> thing but somehow one has not dared to dip the toes too much into it.*

After three implementations Orri had a feeling of going in the right direction:

Orri: *But it was like last time we did the silent video task […] I was showing these and […] felt like it did not go well […] because then I tried, you know, then I was through maybe one half [of the responses] and felt like nobody was following and I did not mention that I just felt they did not bother at all […] but now somehow it was easier it got more fun not fun maybe but interesting when we were all watching together.*

Comparing implementations of SVT2 and SVT3, there was less sense of time pressure in the latter one. Orri was observed gradually activating everyone's attention such that the students' participation in commenting on each other's responses slowly increased. Plenty of time had been devoted to reflecting on each students' response and this seemed to have a positive effect on students' participation. There was also more sense of trust, evidenced by one pair of students showing no sign of embarrassment when their peers' initial response to their voice-over was to start laughing (followed by discussion). They told each other "we did well" as the next response was uploaded to play.

The difference between implementations of SVT2 and SVT3 is described further in the next theme.

8.5.5 Opportunity: SVT Practices Might Support Teachers to Institutionalise Knowledge

Orri mainly posed clarifying questions during his moderation of the SVT2 whole group discussion. Even though students did not participate much, he saw potential for improvement and decided to give it another try in the third SVT implementation. In his reflection, he wondered about ways to enhance students' participation in concluding the discussion and finally, he created a plan during the think-aloud exercise:

Orri: *Then toward the end I would like to discuss with them the concepts in general like for instance in this case, you know, slope "What is slope? Can anyone reflect on that?" and yes somehow in this way it would be a summary to tie everything together at the end.*

Then, in group discussion based on students' responses to SVT3, Orri carried out his plan by activating students to participate in summarizing the discussion. Furthermore, he connected students' inputs to the topics that the class had been working on in the preceding weeks. Thus, seemingly attempting to *institutionalise* (Brousseau, 1997) knowledge, which is something that Swan (2006) and Aldon (2014) identify as important but often neglected part of teaching practice.

8.5.6 Challenge: It Is Hard to Change Prevailing Social Norms on the Motivation Role of the Final Grade

At both Blackbird and Mallard, teachers expressed frustration when students did "not show their true potential" or "not put in enough effort":

Orri: *They are not putting in much effort here, […] and they seem not to have bothered to make a new recording.*
Edda: *I could imagine using such tasks again and then letting it count [toward the final grade] then they might put in more effort.*

All three mentioned the motivation role of grades when it came to enhancing effort, i.e., that students generally put more effort into tasks that counted towards a grade. It seemed hard to change this social norm.

8.5.7 Opportunity/Challenge: Providing Access to the Classroom Discussion

Orri: *[…] they might feel uncomfortable that someone is listening to their voice […] I think most of them will be fine […] there just might be some who feel maybe, yeah, uncomfortable.*

Before implementing SVT1, Orri suggested that there might be students who would not feel comfortable with the task. Students can get isolated due to disabilities such as severe anxiety, autism spectrum disorder or language barriers that either cause them to be uncomfortable with group work or have a harder time to communicate. Orri and Andri had one student each on the autism spectrum and Edda had three Icelandic language learners (ILL) in their classes.Orri described how his student often rejected working on unconventional tasks. Still, both his and Andri's students participated in the SVT. Orri got his student to participate by offering him to work individually and hand in a written script instead of a recording. Andri's student surprised him by participating in group work despite some discomfort:

Andri: *[…] this surprised me a bit namely that she took the lead [in the group work] and is the one who speaks.*

Her response included much detail, listing coordinates of points A and B, and intercepts with both axes of the coordinate system as they changed in time. It invited an opportunity to support the student in moving from describing detail towards generalizing about patterns, but that opportunity was not recognised until later.

Edda explained that no extra support was provided for ILLs or their teachers at Mallard. When assigning students into random pairs, she strategically made one exception to make sure that the ILL who also was not fluent in English would be in a group of three:

Edda: But they were actually it was mainly the two of them doing the task.

Still, Edda's intention to create access was observed to have the effect that this student participated in the preparation discussion before the other two recorded the task response.

9 Discussion Along the Five TRU Dimensions

Comparing the two previous subsections, identifying opportunities and challenges that SVTs bring, some trends along the five dimensions of the TRU framework can be seen:

The Mathematics Teachers were observed engaging students in an experience that contrasted with the prevailing sociomathematical norm about one correct answer. They gained insight into previously inaccessible information about students' mathematical ideas (including possible misunderstandings), and by repeated use of SVTs Orri raised his own awareness of the importance of precision when describing mathematical objects. It thus seems fitting to conclude that with SVT practices, teachers might support students' learning by raising their awareness of (a) various explanations/descriptions existing, (b) common misunderstandings related to the mathematical topic presented, and (c) why precision is important in mathematical discourse.

Cognitive Demand The teachers were observed attending to students who had a hard time deciding what to focus on in their task responses. This seemed to be due to the fact that the classroom environment involved mainly tasks with one right answer. We cannot be sure if students participating in SVT2 chose to lower the cognitive demand by only focusing on what they were absolutely sure about or if they simply did not think of line slope when watching the video. However, it is then the teachers' task to gather students' ideas about slope in the group discussion. In the discussion they will be given opportunities to make their own sense of mathematical ideas. Provided that teachers persist in taking on the (clearly identified) challenge of leading group discussions, they might establish that problems and misunderstandings will be discussed in an objective and meaningful way for the benefit of every learner in the classroom. In other words, they might create a safe environment for students to share their thoughts (erroneous or not) with others along the road.

Equitable Access to Content To create opportunities for each and every student to participate in the mathematical communication of the class, teachers were observed adjusting their practices to make clear that everyone was invited to participate in the SVT and that everyone's voice would be both listened and reacted to. With ILLs the adjustments were only partly successful and therefore they remain to be developed further. For example, teachers might invite ILLs to create a task response in any language in which they are fluent.

Agency, Ownership, and Identity Students will not see themselves as powerful mathematical thinkers unless teachers treat them as such. This is connected with why teachers were asked to refrain from answering *stop-thinking* questions during task implementation. Also, it is connected with how problematic (but often tempting, therein lies the challenge) it is to turn back to teacher-centred practices, giving a lecture that does not connect to, or meaningfully build on, students' responses and ideas. For example, if teachers had prepared and played their own version of a task response for students, such an act would confirm students' observed expectation of a 'role model' response.

Formative Assessment By insisting on using immediate feedback, teachers were observed to put extra strain on themselves in terms of orchestrating a meaningful classroom discussion based on their reactions to students' responses in real time. Still, teachers' emphases on immediate feedback made sense theoretically, in terms of Wright et al.'s (2018) list. In practice, one could imagine that lengthening the time between recording a task response and reflecting on peers' responses could make the experience more teacher oriented, as only the teacher would have had time to prepare. In one case, Orri was observed 'tying together' students' ideas at the end of class discussion. This happened during his third implementation and underlines that it takes experience, training, and reflection to develop discussion orchestration skills. By practicing and putting more emphasis on formative assessment and classroom discussions, teachers might prevail over the motivation role of the final grade, because more emphasis would be put on the process of learning than on the final grade.

10 Conclusion

This chapter described how SVTs can be implemented in the mathematics classroom and demonstrated how the TRU framework can be used to identify opportunities and challenges of technology-mediated FA practices for the teaching and learning of mathematics. Based on what was experienced by Andri, Edda, and Orri as they implemented SVTs in their classrooms, three opportunities (one of them also including a challenge) and four challenges were identified by analysing classroom observation and interview data via the lens of the TRU framework. The opportunities can be re-phrased as follows:

- SVTs have the potential to make previously unnoticed inconsistencies, or problems regarding students' mathematical ideas (understanding), or ways in which they express their ideas (precision in word use) visible to teachers, thus, allowing teachers to address them.
- Situations created by SVTs might enable teachers to institutionalise mathematical knowledge.

- By offering students a new way of using technology for communicating (through implementing SVTs), teachers can include more students' voices in the class discussion.

The first of these opportunities is important for FA practices since it helps raise issues that were previously implicit for teachers, which is one of the potential identified by Wright et al. (2018). The second opportunity connects to three of Wiliam and Thompson's (2008) key strategies for FA (regarding providing feedback that moves learners forward, activating students as instructional resources for one another, and as owners of their learning) because in a situation of institutionalisation, students' ideas are discussed and connected to mathematical objects that have previously been discussed in the classroom. In other words, students' ways of describing mathematics are given status by relating them to the ways that had been used by the teacher to describe mathematics. It is important to note, that the act of institutionalising knowledge implies learning as *acquisition* (of knowledge), whereas what was intended with the SVTs was learning as *participation* (Sfard, 1998; Sfard, 2008). However, learning as participation might be achieved by supporting students in the process of *reification* (Sfard, 2008), i.e., in their transition from describing processes towards talking about objects—a process that might be supported via students' participation in the group discussion.

Since the third in the list of opportunities in some cases demanded that teachers adapt the task in ways not necessarily obvious to them, it was also considered to be a challenge. Maybe due to their similarity to students' popular culture (e.g., YouTube, TikTok, SnapChat), the task of adding a voice-over to a silent video was observed to be easily understood by students. Apart from a few students who needed technical support with downloading software or uploading their recordings, the use of technology seemed not to be a great hurdle. What requires practice and support is mainly the facilitation of a meaningful discussion. Two of the challenges had to do with socio-mathematical and social norms. They might be country-specific, although the presented study cannot confirm that. These two challenges can be rephrased as follows:

- Due to the prevalent norm in which students assume 'one correct answer' to exist, teachers can encounter tensions when implementing open tasks like SVTs.
- It can be cumbersome for teachers to enhance students' motivation in formative assessment practices when students are mainly driven by final summative assessment.

The other two challenges concerned the way in which SVTs require teachers to shift to working in a socio-constructive way, basing feedback on students' ideas via discussion:

- It is challenging to lead group discussions based on students' ideas.
- It can be tempting to return to teacher-centred transmission of knowledge

These challenges are significant and important to acknowledge when teachers shift to technology-mediated FA practices. They connect both to key strategies for FA and the potential of technology-based FA strategies (Wiliam & Thompson, 2008; Wright et al., 2018). Teacher-centred transmission of knowledge can take the form of a lecture or a monologue spiced with a few questions such as "What is the slope when the line is vertical?" that reinforce students' perception that questions always having one right answer in mathematics. Considering the fact that most teachers are not used to implementing open tasks that require both the use of technology and the orchestration of classroom discussion, it is an understandable reaction to return to teacher-centred practices. Also, an implication that leading a discussion based on students' ideas might have longer-lasting effect on students' learning could be considered less important than the fact that such discussions often will require more time. However, if teachers are so restricted by tight time schedules that they make no time to get their students to think mathematically, something surely needs to change.

According to Mason (2002, p. 8) it is important for teachers to feel that they have made an informed decision in a moment of choice and responded professionally (based on awareness) rather than just reacting. There were indications in this study that a novice teacher using SVT practices started developing an in-the-moment awareness of possibilities for classroom discussion within one school term, supported by the practice of reflection and think-aloud exercises. For him and the experienced teachers, the orchestration of group discussion was clearly the biggest challenge involved in SVT practices. Kooloos et al. (2020) described how support via a professional development course based on the work of Stein and Smith (2011) can support teachers who want to develop their practice of classroom discussions based on open tasks. Within four lessons, they supported a teacher to establish a discourse community in her mathematics classroom. For teachers who aim to include SVTs and similar practices—developing their ability to elicit, attend to, discuss, interpret, and respond to student thinking in their work—such professional development courses with like-minded teachers building a community of practice would probably be a good next step.

Acknowledgements I would like to thank the editors of this book and the anonymous reviewers for their helpful comments. Also, I would like to thank the participating teachers for their active cooperation and readiness to illustrate complexities that they face in their everyday practice. This work was supported by the University of Iceland Centenary Fund and the Eimskip University Fund.

References

Aldon, G. (2014). Didactic incidents: A way to improve the professional development of mathematics teachers. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era: An international perspective on technology focused professional development* (pp. 319–343). Springer.

Aldon, G., & Panero, M. (2020). Can digital technology change the way mathematics skills are assessed? *ZDM, 52*(7), 1333–1348. https://doi.org/10.1007/s11858-020-01172-8

Baldinger, E. M., Louie, N., & The Algebra Teaching Study and Mathematics Assessment Project. (2018). *The TRU math conversation guide: A tool for teacher learning and growth*. Graduate School of Education, University of California, Berkeley & College of Education, Michigan State University. https://truframework.org/wp-content/uploads/2018/03/TRU-CG-Math-2018-version.pdf

Bellman, A., Foshay, W. R., & Gremillion, D. (2014). A developmental model for adaptive and differentiated instruction using classroom networking technology. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era: An international perspective on technology focused professional development*. Springer.

Brinkmann, S., & Kvale, S. (2009). *Interviews: Learning the craft of qualitative research interviewing* (3rd ed.). Sage.

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Kluwer Academic Publishers.

Cargile, L. A., & Harkness, S. S. (2015). Flip or flop: Are math teachers using khan academy as envisioned by Sal khan? *TechTrends, 59*(6), 21–28. https://doi.org/10.1007/s11528-015-0900-8

Gattegno, C. (2007). The method of Jean Louis Nicolet. *Mathematics Teaching, 205*, 42–43.

Harðarson, A. (2010). Hvaða áhrif hafði Aðalnámskráin frá 1999 á bóknámsbrautir framhaldsskóla? [What effect did the National Curriculum Guide from 1999 have on academic study lines in Icelandic upper secondary schools?] *Netla – Veftímarit um uppeldi og menntun*. 19 pages. http://hdl.handle.net/1946/13799

Jónsdóttir, A. H., Briem, E., Hreinsdóttir, F., Þórarinsson, F., Magnússon, J. I., & Möller, R. (2014). *Úttekt á stærðfræðikennslu í framhaldsskólum* [Assessment of mathematics teaching practices in upper secondary schools]. Reykjavík, Iceland: Mennta- og menningarmálaráðuneytið. Retrieved from https://www.mrn.is/media/frettir2014/Uttekt-a-staerdfraedikennslu-i-framhaldsskolum-2014.pdf

Kitz, S. (2013). Dynamische Geometrie ohne Computer: Die mathematischen Trickfilme des Geheimen Schulrats Münch. *Mathematische Semesterberichte, 60*(2), 139–149. https://doi.org/10.1007/s00591-013-0124-y

Kooloos, C., Oolbekkink-Marchand, H., Kaenders, R., & Heckman, G. (2020). Orchestrating mathematical classroom discourse about various solution methods: Case study of a teacher's development. *Journal für Mathematik-Didaktik, 41*(2), 357–389. https://doi.org/10.1007/s13138-019-00150-2

Kristinsdóttir, B., Hreinsdóttir, F., & Lavicza, Z. (2020a). Using silent video tasks for formative assessment. In B. Barzel, R. Bebernik, L. Göbel, M. Pohl, H. Ruchniewicz, F. Schacht, & D. Thurm (Eds.), *proceedings of the 14th international conference on Technology in Mathematics Teaching—ICTMT 14* (pp. 189–196). University of Duisburg-Essen. https://doi.org/10.17185/duepublico/70763

Kristinsdóttir, B., Hreinsdóttir, F., Lavicza, Z., & Wolff, C. E. (2020b). Teachers' noticing and interpretations of students' responses to silent video tasks. *Research in Mathematics Education.* https://doi.org/10.1080/14794802.2020.1722959

Liljedahl, P. (2018). Building Thinking Classrooms. In A. Kajander, J. Holm, & E. J. Chernoff (Eds.), *Teaching and Learning Secondary School Mathematics: Canadian Perspectives in an International Context* (pp. 307–316). Springer.

Mason, J. (2002). *Researching your own practice: The discipline of noticing*. Routledge.

Oechsler, V., & Borba, M. C. (2020). Mathematical videos, social semiotics and the changing classroom. *ZDM, 52*(5), 989–1001. https://doi.org/10.1007/s11858-020-01131-3

Olsher, S., Yerushalmy, M., & Chazan, D. (2016). How might the use of technology in formative assessment support changes in mathematics teaching? *For the Learning of Mathematics, 36*(3), 11–18.

Pai, J. (2018). Observations and conversations as assessment in secondary mathematics. In D. R. Thompson, M. Burton, A. Cusi, & D. Wright (Eds.), *Classroom assessment in mathematics: Perspectives from around the globe* (pp. 25–44). Springer.

Schoenfeld, A. H. (2018). Video analyses for research and professional development: The teaching for robust understanding (TRU) framework. *ZDM, 50*(3), 491–506. https://doi.org/10.1007/s11858-017-0908-y

Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational Researcher, 27*(2), 4–13. https://doi.org/10.3102/0013189X027002004

Sfard, A. (2008). *Thinking as communicating. Human development, the growth of discourses, and mathematizing*. Cambridge University Press.

Sigurgeirsson, I., Eiríksdóttir, E., & Jóhannesson, I. Á. (2018). Kennsluaðferðir í 130 kennslustundum í framhaldsskólum [Teaching methods in 130 lessons in Icelandic upper secondary schools]. *Netla – Veftímarit um uppeldi og menntun., Special issue 2018: Focus on upper secondary schools,* 27 pages. https://doi.org/10.24270/serritnetla.2019.9

Sinclair, N. (2016). Learning circles: Imitation and imagery. *Mathematics Teaching, 253*, 11–14.

Stein, M. K., & Smith, M. S. (2011). *5 practices for orchestrating productive mathematics discussions* (1st ed.). National Council of Teachers of Mathematics.

Swan, M. (2006). Collaborative learning in mathematics: A challenge to our beliefs and practices. NRDC and NIACE.

Swan, M. (2017). Towards a task-based curriculum: Frameworks for task design and pedagogy. In T. McDougal (Ed.), *Essential mathematics for the next generation: What and how students should learn* (pp. 29–60). Tokyo Gakugei University Press.

Tahta, D. (1981). Some thoughts arising from the new Nicolet films. *Mathematics Teaching, 94*, 25–29.

Tahta, D., & Fletcher, T. (2004). *An account of the first decade of AT(a)M. ATM*. Retrieved from https://www.atm.org.uk/write/MediaUploads/About/History_Gattegno/an_account_of_the_first_decade_of_AT(A)M.pdf

van Manen, M. (2016). *Researching lived experience: Human science for an action sensitive pedagogy* (2nd ed.). Routledge, Taylor & Francis Group.

Venturini, M. (2015). *How teachers think about the role of digital technologies in student assessment in mathematics* [PhD thesis]. University of Bologna.

Venturini, M., & Sinclair, N. (2017). Designing assessment tasks in a dynamic geometry environment. In A. Leung & A. Baccaglini-Frank (Eds.), *Digital technologies in designing mathematics education tasks: Potential and pitfalls* (pp. 77–98). Springer.

Wiliam, D. (2011). *Embedded formative assessment*. Solution Tree Press.

Wiliam, D., & Thompson, M. (2008). Integrating assessment with learning: What will it take to make it work? In C. A. Dwyer (Ed.), *The future of assessment: Shaping teaching and learning* (pp. 53–82). Routledge. https://doi.org/10.4324/9781315086545

Wright, D., Clark, J., & Tiplady, L. (2018). Designing for formative assessment: A toolkit for teachers. In D. R. Thompson, M. Burton, A. Cusi, & D. Wright (Eds.), *Classroom assessment in mathematics: Perspectives from around the globe* (pp. 207–228). Springer.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27*(4), 458–477. https://doi.org/10.2307/749877

Teaching Linear Equations with Technology: A Flipped Perspective

Andrew McAlindon, Lynda Ball, and Shanton Chang

Abstract This chapter discusses the experiences and perceptions of one secondary school teacher's implementation of a technology-enhanced flipped pedagogic approach over a 4-week period whilst teaching the topic linear equations in a Year 9 mathematics class in Victoria, Australia. The study found significant teacher time demands during the initial implementation of the flipped pedagogy, primarily due to the process of establishing teacher technology competence. The use of formative assessment to monitor students' progress was found to be helpful to support the teacher to plan and monitor student participation. Student engagement was increased in the flipped group, as it seemed to allow more time in class for the teacher to help individual students, resulting in reduced time pressure on the teacher in class. We conclude that a number of professional development opportunities should be considered to support teachers' implementations of a flipped approach, to include the development of: teacher technology competence, teacher strategies for monitoring students' expectations for learning mathematics and teachers' abilities to be critical about aspects of teaching and learning, which might be enhanced through a flipped classroom approach.

Keywords Flipped classroom · Professional development · Secondary mathematics · Linear equations

A. McAlindon (✉) · L. Ball
Melbourne Graduate School of Education, The University of Melbourne, Melbourne, Australia
e-mail: ap.mcalindon@gmail.com; lball@unimelb.edu.au

S. Chang
School of Computing and Information Systems, The University of Melbourne , Melbourne, Australia
e-mail: shanton.chang@unimelb.edu.au

A. Clark-Wilson et al. (eds.), *The Mathematics Teacher in the Digital Era*, Mathematics Education in the Digital Era 16,
https://doi.org/10.1007/978-3-031-05254-5_6

1 Introduction

The twenty-first century has ushered in an increased reliance on, and access to, technology for students and teachers alike, both inside and outside physical classroom spaces (Groff & Mouza, 2008). Increased access to technology has presented a myriad of opportunities for teachers that foster “fresh thinking about what is taught, how it is taught and why it is taught” (Moyle, 2010, p. 5). The 2020 COVID-19 pandemic, which has forced remote teaching and learning across much of Australia and many other areas worldwide, has expedited teacher consideration of technology-supported practices for teaching and learning in mathematics.

This chapter reports the teacher component of a larger study, which also investigated student understanding, attitude and perspectives to learning how to solve linear equations in Year 9 mathematics through a flipped approach (McAlindon, 2020). This chapter responds to the following research question by presenting a case study of a particular teacher’s experiences as a flipped pedagogic approach was implemented in an Australian mathematics classroom:

What are the teacher experiences and perspectives of implementing a flipped learning approach for the first time?

Given that most flipped research occurs in higher education settings (Akçayır & Akçayır, 2018), our study adds insights from the perspective of secondary school implementation. In higher education, the flipped approach is reported to improve upon a traditional lecture through increased engagement (Yeung & O’Malley, 2014). However, there are also challenges for implementation ranging from time constraints for teachers in the planning and preparation of a new approach to teaching (Critz & Wright, 2013; Hoffman, 2014) to the inability to adequately support or monitor student participation (Lo et al., 2017). Higher education and secondary classrooms have students at different ages and stages of education, so there are likely to be differences in findings related to implementations of a flipped classroom. This study provides insight into whether identified benefits and challenges associated with implementations of a flipped approach in higher education are also evident in the secondary mathematics classroom setting that is reported here.

The research study that this chapter reports involved two parallel classes, one flipped and one non-flipped. This enabled the teacher to reflect upon the differences between the flipped classroom and the non-flipped classroom. For these two classes:

- the teacher was the same.
- the planned examples were the same.
- the non-flipped group were not given access to the flipped content (i.e., the teacher videos).

1.1 Flipped Classroom: Overview of Components & Implementation

A flipped classroom is defined as an "instructional model, in which some activities traditionally conducted in the classroom (e.g., content presentation by the teacher) become home activities, and activities that normally constitute homework now become classroom activities" (Akçayır & Akçayır, 2018). Content presented by the teacher generally utilises digital technologies through the video recording of instruction, or by identifying appropriate existing online videos, for students to view as homework prior to class (Lo et al., 2017). A flipped classroom where technology is used for content delivery is termed a technology-enhanced flipped classroom (Lo et al., 2017). In the mathematics classroom, content presentation might typically include teacher examples that demonstrate the use of mathematical procedures, explain concepts, demonstrate concrete materials, or model the use of technology, etc.

In general, classroom lessons can be considered as having two main components: an out-of-class and an in-class component. These two components typically differ in the timing and location of the activity, with the out-of-class component (i.e., problems or tasks) occurring asynchronous to the in-class component (typically a face-to-face lesson where there is some content presentation by the teacher). A flipped classroom "flips" this paradigm, with the content presentation by the teacher given as homework as a pre-class activity (asynchronous out-of-class component), and the problems or tasks set by the teacher completed in class.

While appearing a relatively simple concept, the flipped approach is more nuanced than the 'invert-the-process' approach to teaching may suggest, with a number of possible implementations. There is neither global practice for implementation nor a mandate that technology must be used (Lo and Hew, 2017; Love et al., 2014). Lo et al.'s (2017) synthesis of research on flipped pedagogies concluded no standard approach to flipped implementation in the mathematics education literature. The pre-class activities reported in the flipped mathematics classrooms, which included watching videos, reading articles, viewing presentations, and reading a textbook (Lo et al., 2017; Lo & Hew, 2017), highlight the extent to which pre-class activities can vary for both teachers and students. Furthermore, numerous in-class activities have been reported in flipped classrooms, including teachers providing students with time for independent practice, active-learning activities, textbook review work, student presentations, small-group work, group discussions, and targeted focused learning on previously identified problems (Lo et al., 2017).

Multiple approaches to teacher implementations have contributed to differing conclusions about the efficacy of a flipped classroom (Bishop & Verleger, 2013; Kadry & Hami, 2014; Love et al., 2014; Jensen et al., 2015). Differences in pre-class and in-class activities in flipped classroom research make it difficult to determine efficacy or attribute any successes/drawbacks of flipped implementation to any one component (pre-class or in-class). Authors claim a “blurry picture” of the impact of a flipped classroom across multiple subject areas (Låg & Sæle, 2019).

Our research therefore sought to examine the impact of altering the pre-class activity when implementing the flipped approach in a secondary mathematics class and reported the perspectives of the teacher in doing so.

1.2 Technology, Pedagogy and Flipped Implementation

In the flipped classroom it is important to distinguish between the types of technologies used. These include technology for recording the teacher content (e.g., video recording the explanations of examples); technology to support the delivery of flipped content (e.g., students using their computer to access YouTube clips or playing files provided by the teacher); technology used by the teacher to collect formative assessment (e.g., use of online platforms); and technology used by the teacher or students to solve problems, or to assist in explaining a concept or procedure.

Consequently, although the flipped classroom can be supported using technology, there are potential barriers for teachers to overcome to effectively implement technology-enhanced flipped classrooms. Kearney et al. (2018) highlighted challenges faced by teachers in adopting new technology in their teaching practices, including access to timely professional development. The complexities of effective technology integration within pedagogy are highlighted by the Technological Pedagogical Content Knowledge (TPACK) framework (Mishra & Koehler, 2006) which outlines the need for teachers to integrate subject content, pedagogical knowledge, and technological knowledge. The TPACK framework is not directly linked with flipped classroom research, but exemplifies the demands placed on a teacher in the implementation of a technology-enhanced flipped classroom. Teachers must consider how to utilise technology to develop students’ mathematical understanding when students are learning new mathematical skills and concepts outside class, and where this new understanding underpins the subsequent face-to-face class.

An important pedagogical consideration is that students will not be able to ask questions of the teacher while engaging with the flipped content, so teachers must consider the nature and extent of the flipped content in order to engage students, but also to anticipate and ameliorate any expected student difficulties. Mishra and Koehler (2006) suggest that technology may constrain pedagogical decisions. In the flipped mathematics classroom this is evident through flipped content presentation occurring without teacher interaction with students; the teacher cannot monitor or respond to students’ understanding in real time. Teachers must consider the extent to which they choose to include reasoning, mathematical solution steps, technology use (e.g., a CAS calculator to simplify algebraic expressions) or demonstrate concrete materials within the context of an inability to respond to students’ questions in-the-moment. This impacts the pedagogical choices that teachers make when planning the pre-class material.

Alongside these pedagogical issues, there are specific technological challenges that teachers need to overcome, many of which relate to competence with technology and the increased workload to develop the new lesson content (Akçayır & Akçayır, 2018; O'Flaherty & Phillips, 2015).

1.3 Teacher Experiences in Flipped Implementation

A flipped classroom requires teachers to make decisions about both content delivery for pre-class and in-class activities. There are technology considerations in the preparation of videos, or selection of content, for pre-class work. In addition, teachers must decide whether students will use technology (for example, dynamic geometry, computer algebra systems, graphing software, applets, etc) for learning or doing mathematics during in-class activities. Therefore, the teacher must consider how and when to utilise technology for both pre- and in-class activities. The focus in this chapter is on technology use for pre-class activities.

Preparing content for the teaching of mathematical skills and/or concepts in an asynchronous manner requires planning by teachers. Preparing pre-class content increased teacher preparation time, even for the most experienced teachers (Akçayır & Akçayır, 2018; Bergmann & Sams, 2016). Increased preparation time is a challenge for teachers to overcome and a major criticism of the approach (Lo & Hew, 2017). Previous research has reported 70+ working hours to redesign courses for a flipped approach (e.g., Adams & Dove, 2016). The increase in time is predominantly due to time required to create and edit video lectures, in addition to preparing in-class activities (Akçayır & Akçayır, 2018; Lo et al., 2017).

Johnson and Renner (2012) highlighted the perspective of one teacher, who found the workload required for planning the flipped classroom cumbersome, with each lesson requiring two lesson plans (i.e., pre-class and in-class). Similarly, Wanner and Palmer (2015) noted increased workload as the biggest concern of 47 Australian university academics; one academic noted a six-fold increase in preparation time. Despite this, increased job satisfaction has been reported by teachers (e.g., Brunsell & Horejsi, 2013). The flipped classroom has provided opportunities for teachers to differentiate for students' needs (Finkel, 2012; Fulton, 2012; Speller 2015), with Saunders (2014) reporting enhanced opportunities for high school mathematics teachers to enrich learning opportunities for all types of learners.

While some teachers have found a positive trade-off for the increased time commitment (i.e., more targeted in-class activities), others view it as burdensome with few rewards (Johnson & Renner, 2012). This chapter presents benefits and drawbacks of flipped implementation through the experiences and perspectives of one teacher who was able to reflect on the experiences of flipped implementation through teaching a parallel non-flipped class concurrently.

2 Methodology

2.1 Methodological Basis

Given the focus of flipped classroom research in higher education and across a range of subject areas, there was a gap in understanding of the impact of such approaches on secondary mathematics teachers. It is important to understand teachers' experiences when implementing a flipped approach (and also their perspectives on this approach), as teachers play a crucial role in deciding on the teaching approaches in their lessons. To determine the viability of a flipped approach one important aspect is to understand potential opportunities and barriers from the perspective of the teacher.

In this study the teacher's experiences and perspectives were gathered through interviews at three distinct timepoints: before, during and after flipped implementation. Gathering the teacher's experience and perspectives at these three stages was important to ensure adequate understanding of teacher planning (before implementation), teacher considerations (during implementation) and teacher reflection (after implementation).

Two Year 9 classes (flipped and non-flipped) were taught by the same teacher, which focused on the solution of linear equations. The teacher explanations and examples were planned to be the same for each class, with the delivery of instruction the only planned difference. The teacher planned to use explicit instruction for presenting examples to the non-flipped class. Figure 1 provides an example used by the teacher, showing the procedural steps that would be recorded. Analysis of the

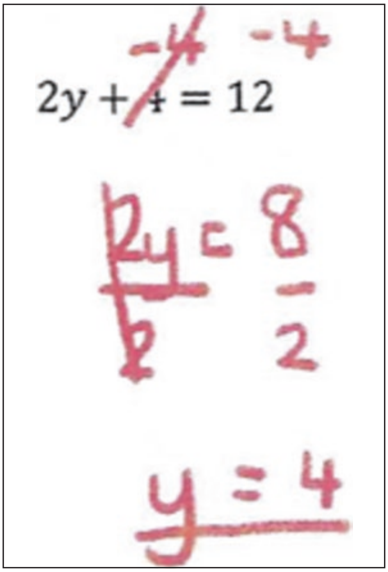


Fig. 1 Teacher procedural example for an algebraic approach for solving linear equations

lesson content and videos is not provided in this paper. The teacher's explicit instructions for the same examples were videoed and provided to the flipped class.

Case-study methodology was used as it allows for rich evidence of teacher practice to inform our understanding (Hamilton & Corbett-Whittier, 2013). In our context, it was used to develop detailed descriptions of the teacher's experiences and perspectives based on her natural observations of her two classes. This approach provided a naturalistic setting for the teacher and the students, with students in their usual mathematics class with their mathematics teacher.

2.2 *Setting of the Study*

The study took place in the first-named researcher's place of employment, a co-educational secondary school in Victoria, Australia (School A). School A has a 1:1 computer implementation strategy, with each student and teacher issued an Apple MacBook for use at school and home. Teachers also have access to a range of technology, including graphics tablets (Wacom Intuos).

School A offers semester-based professional development workshops where teachers choose professional development to suit their interests. One workshop series focused on flipped learning pedagogies and was facilitated by the first-named researcher, who was the professional development coordinator at School A. The 7 × 1.5 h flipped learning workshops covered a general introduction to the flipped classroom, common deliveries of in-class and pre-class activities, how to upload videos and this included time to create and upload a video. Teachers were provided with the following summary of the 10 design principles to support implementation devised by Lo et al. (2017):

1. Manage the transition to the flipped classroom for students.
2. Manage the transition to the flipped classroom for instructors.
3. Consider presenting introductory materials and providing online support in video lectures.
4. Enable effective multimedia learning by using instructor-created short videos.
5. Use online exercises with grades to motivate students' class preparation.
6. Modify in-class teaching plans based on students' out-of-class learning performance.
7. Activate students' pre-class learning by using a structured formative assessment such as a quiz at the start of face-to-face lessons.
8. Require students to solve varied tasks and real-world problems.
9. Meet the needs of students through instructor feedback and differentiated instruction.
10. Facilitate peer-assisted learning through small-group learning activities. (Lo et al., 2017, pp. 62–66)

2.3 Teacher Participant

Teachers from the mathematics faculty who had attended the flipped learning sessions were invited to participate in the study. The first-named researcher explained the study and provided a rationale that having one teacher teaching two parallel classes would help minimise variability in teaching approach and support comparative reflections.

One teacher, Kate (a pseudonym), who taught two classes at Year 9, volunteered to participate in the study. Kate had taught mathematics for over 15 years, and had taught with technology (e.g., MacBook, CAS Calculator) for most of this time. Kate had not taught using a flipped approach before but had previously taught Year 9 linear equations. Kate nominated one of her classes as an experimental group and the other as a control group. The experimental group received their instructional content using a technology-enhanced flipped approach; this class is referred to as the flipped group. The control group received Kate's regular approach to teaching and learning where she presented content and students solved problems during the lesson, with homework involving solution of topic-related problems; this class is referred to as the non-flipped group. The school Principal, Kate, her students and their parents were all provided with a plain language statement outlining the research and signed a consent form to participate in this study. Students who did not return a signed consent form still participated in the classroom activities, however their data was excluded from analysis.

2.4 Research Design

Kate was requested to plan the same explanations and examples for both the flipped and non-flipped groups for the linear equations topic. This enabled comparison between the flipped and non-flipped approaches, as the only planned difference for explanations and examples was the delivery of instruction.

Kate had been provided with the design principles by Lo et al. (2017) during the professional development sessions but she was not directed to apply these in producing the videos. However, as Kate had seen these principles, they may have impacted her lesson design and subsequent teaching.

For each 50- or 100-min lesson over a four-week period (16 lessons) Kate prepared a video for students to watch as a pre-class activity. Kate made her own decisions about the quantity and quality of the explanations and examples for each lesson. As a result, Kate created 73 min and 48 s of online content, across 11 videos.

Kate recorded her examples and explanations (i.e., screen captured audio and video) using a graphics tablet connected to her computer. Recordings were uploaded to an online platform, Edpuzzle (edpuzzle.com), at the first researcher's request. This platform has the functionality to gather student participation data and thus support monitoring. Kate included some formative assessment items, which she expected the flipped group students to answer as they watched each video.

Table 1 Categorisation of similar and different aspects in the flipped and non-flipped groups

Category	Item	Group	
		Flipped	Non-flipped
Pedagogical	Teacher	Same	Same
	Instructional content (examples used)	Same	Same
	Initial delivery of instructional content	Completed for homework via video (not during regular class time)	Explained by teacher during scheduled class time
Student work requirements	Practice questions assigned to students	Same	Same
	Use of scheduled class time	Clarifying concepts from video Solving problems	Listening to teacher explain instructional content Clarifying concepts Solving problems
	Typical homework routine	Watching video for the next lesson	Solving problems
	Overall time commitment	Same	Same

Data on the students' access to each video (i.e., duration watched) and responses to the teacher's formative assessment items enabled Kate to monitor students' participation and understanding. Students accessed the pre-recorded videos for homework (Table 1).

2.5 *Flipped and Non-flipped Lesson Content and Structure*

An overview of the pedagogy and student work expectations for both the flipped and non-flipped group is summarised in Table 1. Students in both groups were expected to have the same time commitment for mathematics.

The mathematical content for the lessons, solving linear equations, included collecting like terms, recognising equivalence, solving arithmetical equations (e.g., equations of the form $Ax + B = C$) and non-arithmetical equations (e.g., equations of the form $Ax + B = Cx + D$). Kate used procedural examples which involved demonstration of step-by-step routine procedures to solve non-contextual problems (Fig. 1). The same explanatory notes, examples and problems were used for both the flipped and non-flipped groups, which enabled her to reflect on the success of the flipped approach.

2.6 Semi-Structured Interviews

Semi-structured interviews provide an appropriate and flexible approach for small-scale research (Drever, 1995), as a set of predetermined questions can have their order, wording, or structure modified based on the interviewer's perception of what seems to be most appropriate at the time (van Teijlingen, 2014). Each semi-structured interview with Kate had pre-planned questions to gain insight into her planning, implementation, technological difficulties, perceptions of student progress and issues with the flipped approach (see Appendices 5.1, 5.2 and 5.3).

Kate participated in three 20-min semi-structured interviews: one before teaching the topic (pre-implementation), a second halfway through teaching the topic (during-implementation), and the third at the end of the topic (post-implementation). Interviews were transcribed verbatim and the full transcript was provided to Kate to check and provide any elaborations or clarifications.

2.7 Data Analysis

Qualitative analysis of each of the three semi-structured interviews was guided by the nine-stage approach outlined by Ball (2011), involving the use of transcripts to determine themes (Fig. 2). This process resulted in subthemes (referred to as clusters by Ball, 2011), which were a collection of similar focused comments made by Kate. Subthemes were grouped into overarching themes that represented similar sets of related subthemes (Table 2).

Data was continually revisited to determine themes that provided the best explanation of "what's going on" (Srivastava & Hopwood, 2009). Transcripts were revisited by all authors to establish a representative set of subthemes, and ultimately, validate a set of themes that captured the content of each interview.

Kate restated or repeated ideas in the same or subsequent interviews, and these similar ideas were grouped into "subthemes" (Stages 4–6). This was achieved by highlighting comments and paraphrasing them in the margin of the transcript, with related paraphrased comments forming a subtheme. An example of the outcomes of Stages 4 and 5 are depicted in Fig. 3.

The subthemes and themes were determined across the three interviews. Using the same paraphrased examples as those from Figs. 3 and 4 exemplifies how subthemes were formed across multiple interviews. The themes relating to the teacher experiences and perspectives of flipped implementation are in the following section.

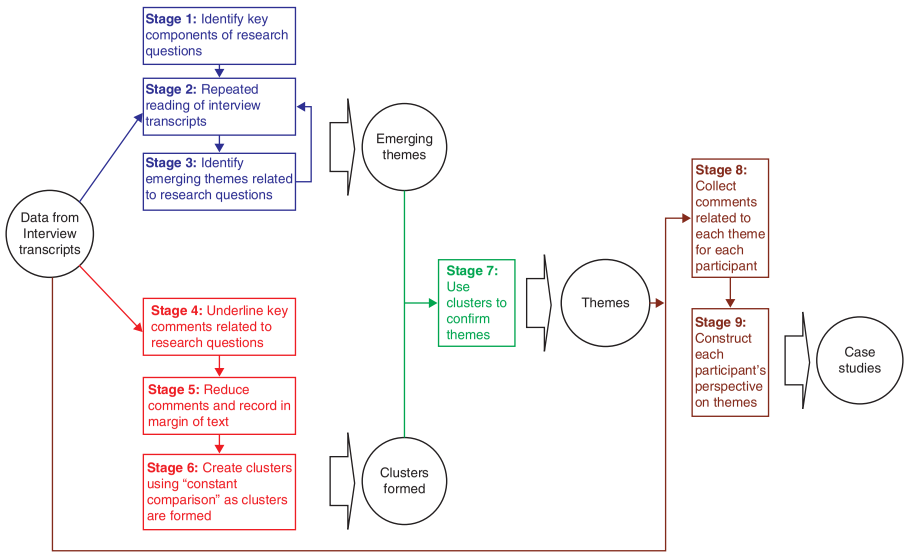


Fig. 2 Nine stage approach for analysing interview transcript data. (Ball, 2011, p. 91)

Table 2 Themes and subthemes from three semi structured teacher interviews and their frequencies

Theme	Subtheme	Frequency	Interview
1. Requirements for the flipped classroom	1.1 Clear expectations are required for student participation in the flipped classroom	15	1, 2, 3
	1.2 The need for teacher technology competence	11	1, 2, 3
2. Understanding the process	2.1 Planning and preparation of lessons is more time consuming for the flipped classroom when compared to the regular classroom	10	1, 3
	2.2 Quality of videos does not need to be perfect	6	1, 2
	2.3 Viewing student progress before a face-to-face lesson can assist with a teacher's future planning and preparation	4	1, 2, 3
3. Key perceived outcomes: Resources	3.1 Flipped tutorials are reusable	6	1, 2, 3
4. Key perceived outcomes: classroom engagement	4.1 More engagement and less behavioural issues in the face-to-face class time for the flipped group when compared to the regular classroom	11	2, 3
	4.2 Flipped classroom provides collaboration opportunities	4	2, 3
5. Key perceived outcomes: Teacher specific	5.1 More opportunity to observe and support students' learning in the face-to-face flipped classroom when compared to the face-to-face regular classroom	7	1, 2, 3
	5.2 Teacher feels less stress in the face-to-face flipped classroom when compared with the face-to-face regular classroom	6	2, 3
6. Key perceived outcomes: Learner specific	6.1 Students have more time to work on questions during face-to-face class time in the flipped classroom when compared to the regular classroom	13	1, 2, 3
	6.2 The flipped classroom supports lower achieving students	8	1, 2, 3
	6.3 Flipped classroom tutorials enable flexible and individualised learning	5	1, 3

Stage 4: Highlighted interview comments relevant to research question	Stage 5: Paraphrased Comments
Umm…Yes, I must say that going into the flipped class I am less stressed with the fact that I'm not time constrained with, you know, oh we need to get through this, this and this. Umm, so I know, sort of I know what is going to happen in the flipped class… umm, and how much time we are going to have to work on it. Whereas with the other class, because I want to allow them time to work in class it's always sort of a struggle, and then you have the behavioural issues because they've had enough of hearing you…. so going into a flipped class…yes, its surprisingly more relaxing… than going into a non-flipped.. with the…you know, with the time stress.	Teacher feels less stressed when entering the flipped classroom. Teacher has a clearer expectation of what will happen in each flipped class and how the lesson will play out. Teacher finds struggles with time and subsequent behaviour in the regular class. Teacher finds more stress in the face-to-face component regular classroom and finds the face-to-face flipped class more relaxing.

Fig. 3 Example of paraphrased comments formed from interview transcript

Subtheme	Paraphrased Comments (Interview 1)	Paraphrased Comments (Interview 2)	Paraphrased Comments (Interview 3)
Teacher feels less stress in the face-to-face flipped classroom when compared with the face-to-face regular classroom	N/A	Teacher feels less stressed when entering the flipped classroom. Teacher finds more stress in the face-to-face component of the regular classroom and finds the face-to-face flipped classroom more relaxing. Teacher has a clearer expectation of what will happen in each flipped class and how the lesson will play out. Teacher finds struggles with time (regular classroom).	Less "energy" needed on the teacher's behalf to run the face-to-face classes during the unit. Teacher was continually less stressed going into the flipped class, with a constant expectation of how the lesson would progress.

Fig. 4 Example of subthemes from multiple paraphrased comments over multiple interviews

3 Results and Discussion

Six themes emerged from the interview analysis with Kate, and these have been organised into the following three broad areas:

- requirements for flipped implementation (Theme 1),
- the processes involved in the implementation (Theme 2), and
- the key perceived outcomes of implementation (Themes 3–6).

The themes, subthemes and frequency of comments that related to each subtheme and the interviews in which they were made in are shown in Table 2.

What follows is an explanation of each of these themes with exemplification from the data.

3.1 Requirements for the Flipped Classroom – Theme 1

This theme categorises the main requirements identified as important for the implementation of flipped lessons. Two subthemes related to expectations of students and teacher technology competence.

3.1.1 Clear Expectations of Students' Participation in the Flipped Classroom

The need for clear student expectations with regard to the watching of the videos for homework was evident in all interviews (subtheme 1.1). Before implementation, Kate raised concerns that students might not watch the videos and may "waste their time in the class". To mitigate these concerns, Kate ensured that the students understood that the time gained in class for solving problems should be used effectively, noting any problems not completed in class "they have to still do, obviously at home, plus watching the new tutorial [i.e., video] and getting the notes". Kate discussed her clear expectations that students were accountable for homework and classwork in Interviews 1 and 2, highlighting student expectation remained a focus. In Interview 2 Kate described the expectations provided to her students to complete the following pre-class activities:

- watch the video tutorial and copy the notes to their workbook; how they watched the tutorial was left up to the students (i.e., pause or rewind as required);
- highlight difficulties prior to entering the next class so these could be clarified in class;
- complete the tutorial quiz questions in their workbook.

Edpuzzle enabled Kate to monitor student work completion. Kate noted some students watched videos at 6 am the morning of class, and subsequently asked all students to complete work the night before, as an additional expectation. By Interview 2, Kate was confident her expectations were being met, and despite “a few students who did not watch the tutorial the first time round”, her continual follow-up ensured all students participated in the flipped classroom as intended, noting that by the mid-point of the topic “all students know what they need to do” and “are quite prepared” for face-to-face lessons.

The three interviews highlighted a progression in Kate’s flipped implementation. In Kate’s experience, teacher intervention was required to maximise student participation. Kate’s communication of expectations assisted appropriate student participation, consistent with Lo et al.’s (2017) guidelines that teachers must manage the transition to the flipped classroom for their students. Kate’s expectation that students watched the videos (before the next lesson) and wrote summary notes once a video had been viewed engaged students with the pre-class videos, which meant they were prepared for the in-class activities where problem solution relied on knowledge from the videos. Furthermore, the need to monitor student participation is highlighted; as without this capacity, teacher intervention is more limited.

3.1.2 Teacher Technology Competence

Teacher technology competence (subtheme 1.2) was a consistent subtheme across the three interviews. In Interview 1, Kate discussed the additional time investment required to become accustomed to working with new forms of technology (i.e., a graphics tablet, screencast software, Microsoft OneNote, Edpuzzle). She noted that it took quite “some time to adjust” to using these technologies. Kate commented that “the first video probably took me … 2 hours, umm, because I couldn’t get everything sort of aligned together, video and pen and everything”, highlighting a substantial time investment to assimilate aspects of technology. For Kate, preparing the first video was the most time-consuming task, but over the interviews her comments indicated the time required to produce videos decreased.

In the second interview, and despite growing competency with technology, Kate noted that aspects of technology continued to remain problematic. These centred around developing adequate technical expertise in using features within the platform, such as “… learning how to get feedback from Edpuzzle of the student’s ongoing progress”. This feedback related to student viewing data and responses to the quiz questions, both crucial aspects in order for Kate to monitor students’ participation.

By Interview 3, no technology issues were evident, however Kate reflected on the initial start-up considerations for teachers implementing the flipped classroom. These included developing technical expertise (i.e., use of graphics tablets, screencast software, upload platforms) and coordinating the use of such tools (sometimes, simultaneously).

Although time consuming, Kate acknowledged, “It’s worth the time spending preparing for the flipped classroom”. Kate cautioned that not taking the time to build this competence would result in teachers “… spend[ing] a lot of time preparing notes and the students still won't get it". Kate's interview suggested that while teachers can understand how to use certain aspects of technology, they needed to be fluent operators and integrate multiple technologies. For example, the graphics tablet needed to be managed alongside a voiceover, with clear handwriting on the tablet to ensure students could make sense of the final product.

McGivney-Burelle and Xue (2013) cautioned that "Those new to flipping should expect many technology glitches especially when creating the first few videos" (p. 484). By Interview 3, Kate suggested her competency with the technology had reached a point where it was no longer an issue, and so this technological competency was built in a matter of weeks. Thus, although there may be an initial increase in time demands in learning to teach with new technologies, the demands may decrease.

This theme highlighted the requirements for a flipped classroom are multifaceted. Teachers should establish student expectations, even if these are the same as for non-flipped classrooms (i.e., the completion of assigned homework), and monitor student participation. Additionally, teachers need to develop technological competence to integrate the use of a number of technologies simultaneously.

3.2 Understanding the Process of Flipping a Classroom – Theme 2

This theme relates to the process of flipping a classroom, such as increased time demands, the required quality of videos and use of student data to inform planning.

3.2.1 Flipping Lessons Is a Time-Consuming Process

This subtheme highlighted Kate's concern about the additional time for implementing flipped lessons. In Interview 1, Kate discussed that planning for her regular mode of teaching would take around 5 min for a 50-min lesson given her experience in teaching (i.e., over 15 years). Kate pointed out that her planning time for flipped lessons was "definitely more lengthy than the 5 minutes". In preparation for recording videos, Kate wrote a plan for her examples and explanations, noting that "preparing the notes … written … took for sure about 2 to 3 hours to prepare for a whole unit". Talbert (2015) described a ratio of "roughly a 6:1 ratio in time spent scripting and producing each video to the running time of the video" (p. 624). Kate noted her first video of approximately 10 min duration took about 2 h to create.

The extended lesson planning time due to the need for a high level of teacher technology competence was discussed in the previous theme; however, the preparation of notes and sequencing for each lesson also required Kate to spend more time than in her regular lesson planning. Kate acknowledged this additional time was a worthy investment, as videos had an immediate impact on students' engagement and her ability to differentiate content (addressed later in themes 4 and 5). Furthermore, Kate acknowledged the benefits in the reusability of the videos (addressed later in theme 3).

3.2.2 Video Quality Does Not Need to Be Perfect

When teachers make videos for students, there could be an expectation that videos should be high quality, with considerable editing and refining. Subtheme 2.2 indicates this was not essential for Kate, and hence need not consume considerable time.

Initially, Kate devoted a large portion of time in her attempts to make perfect videos (Interview 1), with any imperfections in voiceovers or minor imperfections in writing causing her to start recordings over. After approximately five lessons Kate accepted being "less perfect, so if a mistake was done, I would just say no this is the way it needs to be done, rather than just starting from scratch". By Interview 2, Kate had sought feedback from students about the videos and received positive feedback, noting that her students did not expect videos to be perfect:

> They have been "nah we like them Miss", "we can understand them", "there's enough examples" and I can tell that because they can then go straight into their work after we've reviewed the quiz questions, umm—so yeah, the feedback was good.

Acceptance of less than perfect video quality by Kate's students suggests that teachers may not need to allocate significant time for editing videos. In initial videos Kate re-recorded videos in order to eradicate "ums" and "ahs" and fix minor errors, such as use of incorrect notation. While others have suggested self-created videos to be of preference in flipped implementation (Lo et al., 2017), Kate's experience highlights that acceptance of minor errors may be able to save time in video production, without compromising student satisfaction.

3.2.3 Formative Assessment in the Flipped Classroom Assists Planning and Preparation

The preparation for face-to-face lessons was enhanced through Kate's ability to monitor student progress through the platform, which provided a summary of students' answers to the embedded online questions. In both Interviews 1 and 2 Kate discussed use of student progress data from these questions to guide discussion in the next face-to-face class, stating in Interview 2 that she "will show them … how many of you got this one incorrect, let's discuss it on the board".

Given her extensive experience in the teaching of linear equations, Kate mentioned that the student responses to the formative assessment tasks did not reveal anything she had not already anticipated. However, she still used these responses to inform the group focus for the in-class lesson:

> When we did the transposing of equations, the flipped group—I had to go through the tutorial again with them in class, because they got most of the questions, the quiz questions

> incorrect. And so … we discussed extra examples on the board, and once I explained it on the board, then they said "oh now I get it". (Interview 3)

Bhagat et al. (2016) reported the benefit of targeting teaching for specific groups of learners based on progress in pre-class activities. Kate used whole class instruction to clarify any difficulties identified through the quiz questions, rather than design more targeted interventions for specific groups, which may have limited the efficacy of this approach.

3.3 Key Perceived Outcomes: Resources – Theme 3

This theme relates to the reusable nature of the resources created and has only one subtheme.

3.3.1 Flipped Tutorials Are Reusable

Kate highlighted the ability to reuse tutorials (i.e., videos) year-to-year, or for multiple classes in 1 year, as advantages of creating video explanations. While these advantages were discussed in Interview 1, Kate identified an additional advantage during Interview 2, which was to make the videos available to absent students, commenting that the videos "have already been useful for those students who weren't there". This shows a shift in emphasis from thinking about time saving for herself (i.e., to mitigate the extensive start-up time investment from Themes 1 and 2), to considering how videos can be used to enhance opportunities for student learning, opportunities that were not available in a non-flipped approach.

During Interview 3, Kate reemphasised these advantages but returned to the benefits for the teacher from the perspective of being able to use videos with multiple class groups at the same year level:

> When you've got the same class, you know you've got two or three of the same classes as well. It is quite efficient to have the tutorials—because to repeat the same things twice or three times a day—it can be draining.

This subtheme highlights the specific resource outcomes of flipped classroom implementation. The reusable nature of resources serves to benefit both teachers and students. While previous themes have elaborated on the substantial time investment for flipped implementation, this theme begins to acknowledge potential gains. Kate's perspectives aligned with McGivney-Burelle and Xue (2013), who claimed that "Once a polished set of videos and course materials are created the preparation time will be significantly reduced" (p. 484).

However, there is a caveat to any discussion of these advantages as teachers are unable to engage with students during the explanation of new mathematical concepts or procedures, nor can students discuss the mathematics with other students, if necessary. Thus, there is an inability to involve students in building an explanation, questioning aspects of an explanation or taking part in discussions related to the new understanding being developed. This approach of using pre-recorded videos has some advantages in the mathematics classroom, but teachers will need to balance this against recognised benefits that arise from the social nature of learning, for example, through constructing understanding through argumentation (Yackel, 2002) or promoting knowledge development through discussion of technology displays (Ball & Barzel, 2018). This particular caveat was also acknowledged as a drawback of the flipped approach by students in their feedback within the wider study in which this research took place (McAlindon, 2020).

3.4 Key Perceived Outcomes: Student Classroom Engagement – Theme 4

Theme 4 related to aspects of student classroom engagement, viewed from Kate's perspective, that were impacted through the flipped classroom implementation. Two subthemes related to student behaviours and their in-class interactions.

3.4.1 More Engagement and Less Behavioural Issues After Flipping

The emergence of subtheme 4.1 was supported by 11 comments, all occurring during Interviews 2 and 3 (i.e., after implementation of the flipped lessons). Kate noted student behavioural issues in the face-to-face lessons for the flipped group, and despite not discussing this as an anticipated benefit in Interview 1, referred specifically to this group's improvement in their behaviour in Interview 2:

> The behaviour in the flipped group, I think it has improved—you know they work straight through then they ask for a break, they come back in and then they keep working. Whereas the other group, there is still that struggle, once you finish off the explanation, they want a break, because they've had, you know, almost enough, and then they still need to start working. Yeah, so in the flipped group, the behaviour is much better.

The disparity in engagement between the two classes was further reinforced during Interview 3, with Kate commenting that during face-to-face classes, "… the flipped group were more engaged in their work. They were really, you know, heads down and just completing their work". She reported that the flipped group had improved behaviour compared to prior to the study and they seemed more engaged with the work in class when compared to the non-flipped group. Kate suggested that the flipped group were actively engaged with the examples and explanations, while the non-flipped group found the same content tiring, possibly due to the cognitive demands of learning new procedures and/or concepts in the same lesson as solving problems.

The flipped group had a novel format for their classes, with students able to view explanations at their own pace prior to class, with class time devoted to solving problems that involved the application of known procedures and concepts. This resulted in fewer behavioural disruptions and more active participation in the flipped group. Increased student engagement in the flipped group could be attributed to the "flow-on" impact of the pre-class activities, rather than any deliberate change to in-class activities. This contrasts with the findings from Jensen et al. (2015), which attributed the increased engagement of post-graduate students in a flipped classroom to be the product of changes to in-class activities, and not the classroom flip itself.

3.4.2 Increased Collaboration Opportunities in a Flipped Classroom

In Interviews 2 and 3, Kate commented on the impact of the flipped approach on students' inclination to help their peers with mathematics. She noted "students are helping each other more, I noticed, because obviously they've watched the video—they've understood it—and then they might be feeling more comfortable helping out each other". Kate did not report such observations in her non-flipped classroom.

Increased opportunities for students to interact and collaborate during class in a flipped approach have been reported to enhance learning through structured peer-based learning. For example, Lo et al.'s (2017) review of the flipped classroom literature in mathematics highlighted 33 studies with reported benefits that were attributed to peer-assisted learning which occurred in a flipped classroom. While Kate did not explicitly build opportunities for peer-assisted learning into the structure of her face-to-face lessons, the in-class component of the flipped classroom was conducive to peer-based learning as it did not focus on the teacher explaining examples, thus provided more opportunities for interactions.

3.5 Key Perceived Outcomes: Teacher Specific – Theme 5

This theme had two subthemes related to increased teacher support afforded to students and reduced teacher stress in the flipped classroom.

3.5.1 Increased Opportunities to Observe and Support Student Progress

Kate noted an increased ability to observe and support students' learning during in-class activities for the flipped group. In Interview 1, Kate reflected on the potential of the flipped classroom, suggesting the non-flipped group would be disadvantaged as most textbook problems will be completed "when they are at home working by themselves and the teacher is not there". By contrast, her flipped group seemed advantaged by learning at home through video, and "more time to complete class work … with my assistance".

Kate valued the ability for more dedicated teacher time responding to individual student's questions for the flipped group, noting in both Interviews 2 and 3 that the flipped students had an increased ability to just "put their hand up and there's my help". This notion of enhanced support was due to extra time helping students, with Kate commenting she could "get a feel more from the flipped group because I'm seeing them doing their work in front of me". This was not possible in the non-flipped group as content explanations consumed considerable class time.

Bhagat et al. (2016), Clark (2015) and Strayer (2012) also reported that freed-up class time enhancing opportunities to further support students as they actively work through problems. Even in the absence of specific targeted intervention programs, the pre-class component of the flipped classroom provided more time for the teacher to target help to individual students in-class.

3.5.2 Reduced Teacher Stress in a Flipped Classroom

Subtheme 5.2 highlighted reduced teacher stress resulting from the normal time constraints in-class. After implementation, Kate highlighted that she felt less time pressure to teach the curriculum with the flipped approach, compared to the non-flipped approach where finishing the topic was "always sort of a struggle".

Kate identified difficulties with both time and maintaining student engagement in her non-flipped group, both of which were not apparent in her flipped group. A potential of the flipped approach may be reduction of time stress during class, which can be a worthwhile trade-off against the time in initial planning for flipped implementation. This, coupled with the reusability of videos discussed in Theme 3, highlights a potential advantage of the approach in the longer term.

3.6 Key Perceived Outcomes: Learner Specific – Theme 6

This theme focuses on the potential outcomes of the flipped classroom specific to the students, as perceived by Kate. Three subthemes related to enhanced capacity for differentiation within the classroom.

3.6.1 More Time for Student Work in the Face-to-Face Classroom

Kate perceived students in the flipped classroom had more time to work on problems in-class, noting this as a potential advantage during Interview 1:

> So, with the flipped group, I would probably say out of a double period [100 minutes] we're, maximum spending, 20 minutes if I see that they had difficulties with the quiz questions. Um, whereas with the other class having to go through the difficulties from the last lesson and then through the content and then I would say that maybe they end up with 20 minutes of doing work, so that's a big difference.

Kate's description of the "big difference" for students in the flipped group, with 80% lesson time used to complete problems, in comparison to just 20% for the regular group highlights a large disparity in time for student to solve problems in class. This difference is perhaps the explanation for Kate feeling she had more time to support her students (Sect. 3.5.1), engage with their classwork (Sect. 3.4.1) and each other (Sect. 3.4.2). This difference resulted from a change in the pre-class activity, involving students watching a video of approximately 7 min. The ability for students to capitalise on this (i.e., through watching at their own pace—see Sect. 3.6.3), provides natural flow-on opportunities for the teacher (Sect. 3.5.1) and more time for students to solve problems in class (Sect. 3.6.1).

3.6.2 The Flipped Classroom Supports Lower Achieving Students

Kate perceived that the flipped approach had enhanced her ability to support low-achieving students in particular, as it allowed students to better control the pace of explanations (a finding that is supported by the student data in McAlindon, 2020) and created subsequent increased time in the classroom, which could be used to assist those students.

Kate referenced specific students in her class, who would not have otherwise engaged with previous topic content due to perceived difficulty, now actively engaging with content and completing work. This could highlight that the pre-class component of the flipped classroom provided new opportunities for learning for these students that were perhaps not feasible in a non-flipped class.

3.6.3 The Flipped Classroom Enables Flexible and Individualised Learning

Following implementation, Kate felt that students were able to achieve better outcomes "because they could you know, watch the tutorials at their own pace, as many times as they want", which then "allowed me to help them more in class". The feature of on-demand viewing of videos could explain the perceived efficiency of the flipped approach as students were able to re-watch explanations as often as they wished to understand the examples; this contrasted with the non-flipped group where Kate re-explained ideas in class when students were having difficulties.

Kate suggested she was able to explain the same amount of content that could otherwise consume 80 min of her 100-min lessons, in a video with an average duration of 7 min. This was achieved without reduction in content delivered.

What Kate didn't account for in her interviews was the notion that the examples and explanations might need to be further elaborated to help students who were having difficulty or to deepen students' understanding. Kate did not discuss the role of the class discussion in helping students to develop their understanding of solving linear equations. This could be due to the focus on procedures in the examples and might differ when promoting reasoning, problem solving or conceptual understanding.

4 Implications and Conclusions

This chapter reported the experiences and perspectives of one teacher who implemented a flipped approach for the first time when teaching Year 9 students to solve linear equations. Overall, the teacher had a positive experience and believed that the time investment required to implement a flipped classroom was worthwhile. Three broad conclusions can be drawn from the study, namely, using a flipped approach requires an initial time commitment by the teacher, there are potential advantages for both teachers and students in using a flipped approach, and there are considerations for teachers when using a flipped approach in their teaching of mathematics. These findings, which are discussed below, can inform the design of future professional development for teachers.

4.1 Advantages for Teachers

Four advantages of flipped implementation were identified by the teacher: the ability for students to independently view/review content explanations, a reduction in time pressure in class for the teacher, increased student classroom engagement and increased opportunities for the teacher to engage with students.

Students' independent viewing/reviewing of content explanations. This also supported absent students or those who needed to consolidate content knowledge. When a teacher develops a suite of lessons it will be possible to target video explanations to students for revision of previous mathematics concepts or skills where student difficulties have been identified.

Reduced time pressures and teacher stress in-class. The transfer of content explanations to a pre-class activity reduced teacher stress in the face-to-face classroom. Kate noted decreased pressure in the flipped classroom, when compared to her non-flipped classroom, as she did not have to "rush through" explanations.

Increased classroom engagement. Kate noted a more "active" work environment in the flipped group compared to the non-flipped group. Increased engagement resulted in fewer behavioural disruptions, increased collaboration between students, and more focus on tasks. These differences were also apparent when comparing the same flipped group to themselves prior to flipped implementation.

Anderson and Brennan (2015) reported teachers who invested significant time and effort, yet lacked confidence and direction in their implementation, which ultimately led to poor teacher and student outcomes. The professional development recommendations presented in Table 3 attempt to ensure appropriate support and direction in flipped implementation.

Increased Time for the Teacher to Engage with Students. More class time was available for the flipped group to solve problems in Kate's classes (80% of lesson time), compared to the non-flipped group (20%). This afforded extra time in-class

Table 3 Professional development recommendations and associated rationale

Professional Development Recommendation	Rationale
Discuss criteria for choice of content to flip	For content to be appropriate to be flipped, students need to be able to develop an understanding of the key concepts or skills independently, without reliance on classroom discussion or teacher-student interaction. Discussion of criteria for selection of content will support teachers to identify appropriate content for a flipped approach.
Discuss the differing level of time commitment required at various stages of flipped implementation	Teacher time commitments have been shown to be substantial in the initial planning and preparation of flipped implementation (Sect. 3.2). This is due to a range of factors, including increased need for teacher competence in technological pedagogical knowledge Fewer time pressures in class are then observed, with reduced teacher stress in-class and better behavioural engagement outcomes (Sect. 3.4). This along with the reusability of videos (Sect. 3.3) can be acknowledged as trade-off for the initial investment
Discuss and model strategies to capitalise on the additional in-class time to deepen and extend students' understanding	Teachers will have additional time in-class to support students in developing their mathematical understanding (Sects. 3.5 and 3.6) and will need to consider how to best use this time to maximise student learning
Provide video excerpts of flipped tutorials for teachers to discuss and critique	Critique of video of flipped tutorials will provide teachers with the opportunity to identify features that contribute to, or detract from, development of students' understanding. This will help teachers make informed choices about the creation and selection of video content
Discuss and develop students' participation guidelines in a flipped approach	To support students to participate in the flipped approach to develop mathematical understanding, they will need guidance on how they are expected to engage in the flipped classroom (Sect. 3.1)
Discuss and model approaches for monitoring students' participation and understanding	Student data will inform planning of face-to-face teaching and provide the teacher with information regarding the efficacy of the flipped approach

to work with students and observe students as they solved problems enabling Kate to better monitor students' understanding.

4.2 Technology Considerations

Teacher competence with technology is a major requirement for the flipped classroom when videos are created. School-supported professional development will assist in supporting such teacher competence. Discussion and support on the use of a variety of technologies for presentation could be included, for example, helping teachers learn the approach for embedding a dynamic geometry file in a video, or including links to virtual manipulatives for students to engage with. Initially, although Kate attended professional development at the school, the time commitment for technology competence exceeded that provided by the school. Therefore, appropriate professional development and time release is likely to assist teachers, but not eliminate the burden of implementation.

4.3 Student Expectations

The need for teachers to discuss expectations for student engagement with the videos and the ability to use inbuilt features of software to monitor students' use were important in Kate's implementation of the flipped classroom. Teachers will need strategies for achieving buy-in from students in a remote learning environment and also for monitoring students' expectations (i.e., through use of analytics).

4.4 Future Directions for the Flipped Classroom in Secondary School Mathematics

This study focused on exploring the efficacy of a flipped classroom approach where the teacher used their regular examples and explanations to teach students how to solve linear equations. In this case, the teacher demonstrated procedures by writing on a tablet, which mimicked what would be done on a whiteboard in a face-to-face classroom. Having the same examples and explanations enabled the efficacy of flipped class to be contrasted with the teacher's regular approach. Having demonstrated positive benefits through flipped implementation, the next consideration for teachers is investigating the potential for improved student learning through use of a range of explanations and examples, as well as the inclusion of deliberate changes to in-class activities.

Further consideration should be given to whether all explanations should be 'flipped', or if it should be used to target aspects of explanations or particular types of examples. As noted earlier, there is considerable research to support argumentation and communication as essential components of development of students' understanding, so one possibility is that a teacher provides some introductory material for students to engage with as a pre-class activity, and then building on this in the next class, to extend student understanding through discussion, further explanation and activities. More deliberate changes to activities within the in-class component would most certainly contribute to further increases in planning time, however, may offer further enhancement of the benefits observed within this study.

The time duration of this research was less than one school term, and thus Kate's perception could be impacted by the novelty of a new flipped approach. Longer duration research (e.g., Guerrero et al., 2015; one semester) noted that, for students, "the novelty of the videos wore off, fatigue and boredom with the same instructional approach day after day become a factor" (p. 827). Kate reported positive impacts in her classroom which could have been due to the short time frame of the flipped intervention. Further research on the ability of the flipped classroom to maintain student positive attitudes and engagement, across a range of mathematical topics, would be beneficial.

This study investigated one teacher's approach to the implementation of the flipped classroom for the teaching of linear equations in Year 9. Although the results from one teacher cannot be generalised, the findings provide a view on some of the potential barriers and opportunities which might be faced by other teachers. For schools or systems that are considering widespread implementation of the flipped classroom, the lived experience of real classroom teachers provides some insight into potential challenges and opportunities that may be encountered in a secondary mathematics setting.

If we consider the principles introduced by Lo et al. (2017) for teacher implementation it appears that Kate utilised the first five of ten listed principles.

Having identified Kate's perspectives and experiences, the next step in implementation could be to consider the remaining five principles that were not evident through interview analysis with Kate, and consider professional development to support flipped classroom implementation. For example, attention could be provided to developing activities for in-classroom differentiation or small-group learning activities that further capitalise on the advantages of the technology-enhanced flipped classroom.

Therefore, to support these suggested future directions, it is also important to provide professional development for teachers to support their pedagogical content knowledge and competence in implementing a flipped approach. Table 3 presents some key considerations for teacher professional development to achieve this support when implementing a flipped approach.

In addition to the requirements for professional development to build teachers pedagogical content knowledge for teaching a flipped approach (Table 3), schools need to provide an environment where such innovations are possible and supported. For schools interested in exploring the flipped classroom, this study highlights the importance of providing assistance and opportunities for teachers to develop their technological competence, alongside a provision for additional planning time in the initial flipped implementation.

Therefore, schools need to acknowledge that teachers may be required to upskill in areas of technology; to create videos, upload to a new platform, monitor progress remotely and engage in the recommendations as outlined above. At the same time, schools should also consult teachers when they are acquiring new technological solutions for flipped classrooms to foster a partnership and identify skill gaps prior to flipped implementation.

All of these factors may require a substantial time investment from the teacher and should be supported by embedded opportunities at the school.

4.5 *Capitalising on Lessons from the COVID-19 Pandemic*

The COVID-19 pandemic, which forced many schools into remote environments provides a unique perspective into the flipped classroom. Many teachers who may not otherwise have had opportunities to create videos found themselves in a position where this commitment was now required to ensure continuity of learning for their students. These teachers are well-placed for flipped implementation, as the time investment already spent in setting up a remote learning environment could place them ahead of the initial barriers of attaining competence with technology and creation of video explanations. These teachers could now be considered prime candidates to capitalise on the learning from the COVID-19 pandemic, and in return to face-to-face learning, look to further utilise these videos to enhance students' learning.

Acknowledgements We would like to thank the teacher for the time she spent creating videos and implementing the flipped approach in her classroom. Thank you also to Catholic Education Melbourne, the school and the students who were participants in this study.

Statement on Research Ethics Human ethics approval was granted from the University of Melbourne. Permission was sought and granted from Catholic Education Melbourne to conduct research in a Catholic School. The Principal, teacher, students and their parents were provided with a plain language statement and consent form to participate in this research.

Appendix: Semi-structured Interview Questions and Rationale

First Interview: Before Starting the Flipped Classroom with Students

Question	Additional prompt (If required)
Describe your usual preparation for a linear equations lesson, in terms of time and resources	Average time planning for a 50 min lesson Typical resources used (i.e., MacBooks – In what way?)
Describe your usual teaching practice in the mathematics classroom, in terms of whiteboard use and questioning	How do you usually 'teach' concepts within linear equations? How do you know when students are understanding what you are teaching?
Discuss your current experiences and comfort with technology	How does this comfort level usually play out in your mathematics classes? How does it usually play out in your preparation of these classes, specifically, previous linear units?

(continued)

Question	Additional prompt (If required)
Describe your preparation for the linear lessons for the flipped classroom group. In doing this, detail your experience in creating the content using technology, including the upload of this to content EdPuzzle	Factors to prompt: did you find anything particularly easy/straight forward? Anything particularly difficult? Anything you ended up abandoning as a result of its difficulty? How do you feel the flipped approach aligns with your usual teaching methodology? Discuss any similarities and differences
In your previous experiences, how have you found students to perform and perceive linear equations in your usual teaching format?	
Do you anticipate any differences with the flipped approach, in terms of student understanding or engagement?	Why/why not?
Was there a reason you selected one particular group to receive the flipped instruction over the other? If so, can you elaborate further on this in terms of your expectations?	

Second Interview: During (Mid-way) Implementation of the Flipped Classroom

Question	Additional prompt (If required)
How did you establish the expectations around the flipped classroom for your flipped students?	
What have you found students have taken well to with the flipped classroom?	
What have the students struggled with?	
Tell me about the differences and similarities between that you have seen in your flipped and non-flipped classrooms.	Have you had to address any of these issues with students?
Have you noticed any obvious differences between engagement or understanding between the flipped and non-flipped groups as a whole?	Describe the difference.
Any differences between groups of students within each group?	Describe the difference.
Did the students have any technical difficulties?	What were these?
How do you think the students are finding the flipped classroom approach to learning linear equations?	
Do you check how often students have completed their work in both classes?	How do you do this for each class?
Do you know when students are accessing the flipped content?	i.e., Home, train, bus?

(continued)

Question	Additional prompt (If required)
Has anything been surprising in the student responses to your EdPuzzle questions?	Describe why this may be of surprise
Have you adjusted/refined any of your face-to-face content based on the flipped student responses to your questions in EdPuzzle?	Why/why not?
Other prompts to ensure reference to any themes that may have arisen from the first interview	Did you have any technical difficulties? Any unexpected benefits or draw backs? Anything different in interactions between student student or student teacher?

Third Interview: After Completion of the Flipped Classroom

Question	Additional prompt (If required)
Can you describe how you found the implementation of the flipped classroom compared to your regular approach?	
Do you feel any groups of students were able to benefit more from any one type of approach (i.e., flipped or non-flipped)?	What makes you think this?
Do you feel any groups of students were at any more of a disadvantage in any of the approaches?	What makes you think this?
You mentioned in the last interview about a student who you thought wouldn't get Algebra, can you explain their journey a little more and how this was turned out different to your expectations?	Mention student name from previous interview
What did the flipped students struggle more or less with when compared to the other class?	Why do you think that?
When you are walking around the classroom, what is your perception of what is going on?	What gave you those impressions?
What would be an outsider's perspective on what is happening if they were to walk into each class?	
Did students in the flipped vs non-flipped have different types of problems?	What were they?
Can you discuss the workload requirements to produce your flipped lessons, and compare this to your regular approach?	
Do you believe the additional workload is worth the effort in the long run?	Why or why not?
On balance, how do you see the future of the flipped classroom in your future mathematics classes?	Consider this in reference to your usual approach Why do you think this?

(continued)

Question	Additional prompt (If required)
Are there factors that you would consider (i.e., student groups, topics taught, classroom setup, year level) as being conducive to a flipped approach?	Discuss these factors and your opinion on why they would influence the success of a flipped approach
What advice would you now offer to anyone wanting to create flipped content for maths?	
What sort of training and resources do you think teachers need to be successful in flipped lessons?	Training, resource, timing.
What advice do you think is pivotal for students to have in order to get the most from flipped learning?	

References

Adams, C., & Dove, A. (2016). Flipping calculus: The potential influence, and the lessons learned. *Electronic Journal of Mathematics & Technology, 10*(3), 155–164.

Akçayır, G., & Akçayır, M. (2018). The flipped classroom: A review of its advantages and challenges. *Computers & Education, 126*, 334–345. https://doi.org/10.1016/j.compedu.2018.07.021

Anderson, L., & Brennan, J. P. (2015). An experiment in "flipped" teaching in freshman calculus. *Primus, 25*(9–10), 861–875. https://doi.org/10.1080/10511970.2015.1059916

Ball, L. (2011). Analysing interview data for clusters and themes. In J. Clark, B. Kissane, J. Mousley, T. Spencer, & S. Thornton (Eds.), *Mathematics: Traditions and (new) practices: Proceedings of the 34th annual conference of the Mathematics Education Research Group of Australasia and the Association of Mathematics Teachers. AAMT–MERGA Conference, Adelaide, 3–7 July 2011* (pp. 89–97). MERGA and AAMT.

Ball, L., & Barzel, B. (2018). Communication when learning and teaching mathematics with technology. In L. Ball, P. Drijvers, S. Ladel, H.-S. Siller, M. Tabach, & C. Vale (Eds.), *Uses of technology in primary and secondary mathematics education: Tools, topics and trends* (pp. 227–244). Springer International Publishing.

Bergmann, J., & Sams, A. (2016). *Flipped learning for maths instruction*. Hawker Brownlow Education.

Bhagat, K. K., Chang, C. N., & Chang, C. Y. (2016). The impact of the flipped classroom on mathematics concept learning in high school. *Journal of Educational Technology & Society, 19*(3), 134–142.

Bishop, J. L., & Verleger, M. A. (2013). The flipped classroom: A survey of the research. In *Proceedings of the 120th American Society for Engineering Education National Conference, Atlanta, 23–26 June 2013* (pp. 1–18).

Brunsell, E., & Horejsi, M. (2013). Flipping your classroom in one 'take'. *The Science Teacher, 80*(3), 8.

Clark, K. R. (2015). The effects of the flipped model of instruction on student engagement and performance in the secondary mathematics classroom. *Journal of Educators Online, 12*(1), 91–115.

Critz, C. M., & Knight, D. (2013). Using the flipped classroom in graduate nursing education. *Nurse Educator, 38*(5), 210–213.

Drever, E. (1995). *Using semi-structured interviews in small-scale research: A teacher's guide*. Scottish Council for Research in Education.

Finkel, E. (2012). Flipping the script in K12. *District Administration, 48*(10), 28.

Fulton, K. (2012). Upside down and inside out: Flip your classroom to improve student learning. *Learning & Leading with Technology, 39*(8), 12–17.

Groff, J., & Mouza, C. (2008). A framework for addressing challenges to classroom technology use. *AACE Journal, 16*(1), 21–46.

Guerrero, S., Beal, M., Lamb, C., Sonderegger, D., & Baumgartel, D. (2015). Flipping undergraduate finite mathematics: Findings and implications. *Primus, 25*(9–10), 814–832. https://doi.org/10.1080/10511970.2015.1046003

Hamilton, L., & Corbett-Whittier, C. (2013). *Using case study in education research* (BERA research methods). SAGE Publications Ltd.

Hoffman, E. S. (2014). Beyond the flipped classroom: Redesigning a research methods course for e3 instruction. *Contemporary Issues in Education Research (CIER), 7*(1), 51–62.

Jensen, J. L., Kummer, T. A., & Godoy, P. D. D. M. (2015). Improvements from a flipped classroom may simply be the fruits of active learning. *CBE—Life Sciences Education, 14*(1), ar5. https://doi.org/10.1187/cbe.14-08-0129

Johnson, L., & Renner, J. (2012). *Effect of the flipped classroom model on secondary computer applications course: Student and teacher perceptions, questions and student achievement.* [Unpublished doctoral dissertation]. University of Louisville.

Kadry, S., & El Hami, A. (2014). Flipped classroom model in calculus II. *Education, 4*(4), 103–107. https://doi.org/10.5923/j.edu.20140404.04

Kearney, M., Schuck, S., Aubusson, P., & Burke, P. F. (2018). Teachers' technology adoption and practices: Lessons learned from the IWB phenomenon. *Teacher Development, 22*(4), 481–496. https://doi.org/10.1080/13664530.2017.1363083

Låg, T., & Sæle, R. G. (2019). Does the flipped classroom improve student learning and satisfaction? A systematic review and meta-analysis. *AERA Open, 5*(3), 1–17. https://doi.org/10.1177/2332858419870489

Lo, C. K., & Hew, K. F. (2017). A critical review of flipped classroom challenges in K–12 education: Possible solutions and recommendations for future research. *Research and Practice in Technology Enhanced Learning, 12*(1), 4. https://doi.org/10.1186/s41039-016-0044-2

Lo, C. K., Hew, K. F., & Chen, G. (2017). Toward a set of design principles for mathematics flipped classrooms: A synthesis of research in mathematics education. *Educational Research Review, 22*, 50–73. https://doi.org/10.1016/j.edurev.2017.08.002

Love, B., Hodge, A., Grandgenett, N., & Swift, A. W. (2014). Student learning and perceptions in a flipped linear algebra course. *International Journal of Mathematical Education in Science and Technology, 45*(3), 317–324. https://doi.org/10.1080/0020739X.2013.822582

McAlindon, A. P. (2020). *Investigating the impact of a flipped classroom approach for a teacher and students in year 9 in the topic of linear equations* [Unpublished doctoral thesis]. The University of Melbourne.

McGivney-Burelle, J., & Xue, F. (2013). Flipping calculus. *Primus, 23*(5), 477–486. https://doi.org/10.1080/10511970.2012.757571

Mishra, P., & Koehler, M. J. (2006). Technological pedagogical content knowledge: A framework for teacher knowledge. *Teachers College Record, 108*(6), 1017–1054.

Moyle, K. (2010). *Building innovation: Learning with technologies*. ACER Press.

O'Flaherty, J., & Phillips, C. (2015). The use of flipped classrooms in higher education: A scoping review. *The Internet and Higher Education, 25*, 85–95.

Saunders, J. M. (2014). *The flipped classroom: Its effect on student academic achievement and critical thinking skills in high school mathematics* [Unpublished doctoral thesis]. Liberty University.

Speller, S. (2015). *Mathematics teacher's experience with flipped learning: A phenomenographic approach* [Unpublished doctoral dissertation]. The University of Toledo.

Srivastava, P., & Hopwood, N. (2009). A practical iterative framework for qualitative data analysis. *International Journal of Qualitative Methods, 8*(1), 76–84. https://doi.org/10.1177/160940690900800107

Strayer, J. F. (2012). How learning in an inverted classroom influences cooperation, innovation and task orientation. *Learning Environments Research, 15*(2), 171–193. https://doi.org/10.1007/s10984-012-9108-4

Talbert, R. (2015). Inverting the transition-to-proof classroom. *Primus, 25*(8), 614–626. https://doi.org/10.1080/10511970.2015.1050616

van Teijlingen, E. (2014). Semi-structured interviews. Retrieved from http://www.fao.org/docrep/x5307e/x5307e08.htm

Wanner, T., & Palmer, E. (2015). Personalising learning: Exploring student and teacher perceptions about flexible learning and assessment in a flipped university course. *Computers & Education, 88*(1), 354–369. https://doi.org/10.1016/j.compedu.2015.07.008

Yackel, E. (2002). What can we learn from analyzing the teacher's role in collective argumentation. *Journal of Mathematical Behavior, 21*, 423–440.

Yeung, K., & O'Malley, P. J. (2014). Making 'the flip' work: Barriers to and implementation strategies for introducing flipped teaching methods into traditional higher education courses. *New Directions in the Teaching of Physical Sciences, 10*, 59–63.

Tensions and Proximities in Teaching and Learning Activities: A Case Study of a Teacher's Implementation of Tablet-Based Lessons

Maha Abboud and Fabrice Vandebrouck

Abstract This chapter presents an example of how mathematics teachers integrate tablets into their classroom activities. It focuses on mediations and actions in the teaching-learning situation from both a cognitive and pragmatic lens and extends our contribution to the first edition of this book by presenting two new theoretical concepts: tensions and proximities. The first is grounded in Activity Theory, as developed in the context of French didactic research focused on teachers' practices and students' activities. The second takes a more Vygotskian perspective. It considers the students' zone of proximal development (ZPD) as well as Valsiner's zones of free movement and promoted action (the ZFM/ZPA complex), which the teacher designs to support learning. These theoretical elements are illustrated within a case study of a sixth-grade mathematics teacher who uses tablet-based dynamic geometry in a problem-solving situation. We highlight several issues related to the evolution of the ZFM/ZPA complex when tablets are introduced. We also identify and characterise the cognitive and pragmatic tensions that emerge from this evolution, and more specifically, the instrumental nature of these tensions. We identify proximities provided by the teacher, which may fall outside the student's ZPD, without the teacher's full awareness. The chapter concludes with a discussion of the insights provided by our theoretical tools, and what remains to be learned for a better understanding of the uses of tablets in day-to-day mathematics teaching practice.

M. Abboud (✉)
CY Cergy Paris Université, Université Paris Cité, Univ Paris Est Créteil, Univ Lille, UNIROUEN, LDAR, Cergy Pontoise, France
e-mail: maha.abboud-blanchard@cyu.fr

F. Vandebrouck
Université Paris Cité, Univ Paris Est Creteil, CY Cergy Paris Université, Univ. Lille, UNIROUEN, LDAR, Paris, France
e-mail: vandebro@univ-paris-diderot.fr

A. Clark-Wilson et al. (eds.), *The Mathematics Teacher in the Digital Era*, Mathematics Education in the Digital Era 16,
https://doi.org/10.1007/978-3-031-05254-5_7

Keywords Tablets · Zone theory · Teachers' activities · Mathematics learning · Proximities · Tensions

1 Introduction

In France, as in many other countries around the world, we can observe that despite the advent of the digital age, the integration of digital technologies into education is still a major challenge for educators and researchers. Much of this challenge relates to teaching practices, notably how to maximise the didactical and pedagogical potential of these new resources. Schools, like other professional environments, regularly see the arrival of new technologies such as laptops and the interactive whiteboard and, more recently, tablets and smartphones. Easy-to-use graphical displays that are, in some ways, an extension of the human body, have given rise to new issues. In mathematics teaching, these are mainly related to the representation, visualisation and manipulation of mathematical objects. Much of the institutional texts and recommendations (at least in France) on the integration of these technologies in mathematics teaching are based on the optimistic premise that their user-friendliness and ease-of access supports their integration into teaching and learning tools used in the classroom.[1]

Several authors (e.g., Depover et al., 2007; Mullet et al., 2019; Tamin et al., 2015) argue that tablet technologies such as iPads have great potential to support learning, and that their use motivates and engages students. They are also claimed to enhance cooperation, dialogue, and negotiation skills (Ingram et al., 2016). However, some authors point out that, although attractive, they need to be used with care for them to have real didactic value in the classroom (Villemonteix & Khaneboubi, 2013). Teachers, therefore, require a better understanding of how to make best use of them (Galligan et al., 2010; Karsenti & Fievez, 2013), and research needs to focus on making sense of the impacts of the resulting learning activities.

Tablets are relatively new to the French classroom. They have been available for less than 10 years, and their full potential remains to be investigated. Their use has led some teachers to develop new forms of mathematical work. Villemonteix and Khaneboubi (2013) indicate some of these forms particularly related to assessment, reactivation of knowledge, or individual work in classroom settings, for example, when tablet applications provide interactive or self-assessment exercises. The multimedia context can be exploited to offer students a richer working environment, while remaining faithful to traditional methods. For instance, students do not need to access a dedicated computer room to carry out individual research. They can stay in the classroom setting, but supplement traditional paper-and-pencil activities using their tablets.

[1] https://www.education.gouv.fr/repenser-la-forme-scolaire-l-heure-du-numerique-vers-de-nouvelles-manieres-d-apprendre-et-d-2678

The wider exploratory project, in which we are involved, aims to study how teachers and students use tablets (provided by the institution) within their day-to-day work in the classroom. More specifically, in this chapter we present a case study of teaching activities focused on learning geometry in a tablet-based lesson. Compared to the use of desktop dynamic geometry environments, tablets can enrich the cognitive and mediating aspects of the activity. For example, the touchscreen can be used to write on, or manipulate geometrical objects directly with the fingers. While tablets are primarily seen as a way to facilitate mediations and enrich students' activities, they might also be used as an instrument for learning new geometrical concepts.

The overall research question for our wider exploratory project is: to what extent can tablets and related technologies support effective mathematics learning in secondary schools? Our work is, therefore, in line with a body of research that seeks to understand how digital and mobile technology can enhance the teaching and learning of mathematics (Hoyles & Lagrange, 2010), and examines the implications for teaching practices and education (Calder et al., 2018; Clark-Wilson et al., 2014).

The research addressed in this chapter aims to observe mediations in the teacher's activity, specifically the instrumental one (Rabardel, 2002) and to examine the impact of these mediations on students' learning. Our specific research questions concern the interactions between teachers, their students, tablet technology, and mathematical knowledge. The detailed research questions are provided at the end of the following section. We first begin by introducing the notions of tensions and proximities in teaching and learning activities to clarify the focus of these research questions.

2 Theoretical Background and Analytical Tools

Our theoretical point of departure is outlined in the first edition of *The Mathematics Teacher in the Digital Era* (Abboud-Blanchard, 2014). This is the general Double Approach (DA) framework introduced in France by Robert and Rogalski (2002). It combines didactical and ergonomic perspectives when analysing teachers' and students' classroom activity and the factors that determine such activity. Rogalski (2008) argues that activity theory provides the frame of reference for the DA. Activity theory was first proposed by Leontiev (1978), building on Vygotsky's sociocultural theory (1934), and adopted in France in the context of ergonomic psychology (Leplat, 1997), before being integrated into the teaching and learning of mathematics (Abboud-Blanchard & Vandebrouck, 2012; Vandebrouck, 2013).

Building on this general framework, our chapter in the first edition (Abboud-Blanchard, 2014) presented an overall theoretical construct that helps to synthesise findings on technology-based practices that arise from different research studies; the original aim was to identify underlying similarities that go beyond factual diversity. The DA construct is structured by three dimensions: cognitive, pragmatic and temporal (CPT). The *cognitive* dimension is related to the teacher's mathematical goals, and the tasks students are asked to undertake in order to achieve these goals. The *pragmatic* dimension emphasises the open character of the classroom environment when digital technology is used. It focuses on class management and the specificities of teacher–student interactions in relation to instrumental issues (Abboud et al., 2018a). The *temporal* dimension concerns how teachers manage two types of time: didactic time and physical (clock) time. It focuses not only on what happens in the classroom, but also includes the time spent away from the classroom, for example, when preparing lessons, searching for resources, collaboration with other teachers, etc.

However, while the CPT construct offers a macro-level analysis of teaching and learning activities, in the context of technology-based lessons, the research presented in this chapter adopts a micro-level analysis, by taking a more fine-grained approach. The aim is to improve our understanding of the teaching activity in environments where tablets are used and to examine the challenges regarding the related students' mathematical activities and learning.

2.1 Defining Tensions

As Abboud and Rogalski (2017b) note, the teacher's conceptions of the mathematical domain to be taught, and their students' relation to it are determinants of their professional activity. These factors condition the didactical path the teacher wants their students to follow, i.e., the planned cognitive route, and the management of processes that unfold during the lesson (Robert & Rogalski, 2005). Although the teacher is likely to be familiar with the didactic format, the diversity of students' activities, and the specific classroom context introduce uncertainty. This uncertainty is exacerbated when students work with a technological tool; the feedback provided by the tool depends on the student's manipulation, and the teacher can struggle to understand their interpretation of this feedback. Thus, in this teaching–learning environment, teachers are faced with tensions that arise not only from the use of the tool and its role in students' activity, but also its interaction with the mathematical knowledge to be learned.

Abboud and Rogalski (2017a) defined *tensions* in the teacher's activity as "*manifestations of struggle between maintaining the intended cognitive route and adapting to phenomena linked to the dynamics of the class situation*" (p. 2336). Some of these tensions can be predicted, and the teacher can plan how to manage them. Others are unexpected. In this case, the teacher must take *in situ* decisions that direct the actions. Tensions relate to various aspects of teacher and students activities, and they take different forms, depending on the three CPT dimensions.

Tensions related to the cognitive dimension are seen in the gap between the mathematical knowledge that the teacher anticipates to be used and developed by students, and what students actually use as they attempt the task. Tensions related to the pragmatic dimension are specifically linked to the instrumental work environment created by the teacher. On the one hand, they can be anticipated by taking into account potential disruptions that affect the autonomy, commitment and motivation of students. On the other hand, they relate to the issue of manipulating the technology and exploiting (individually or collectively) the feedback it provides. Tensions that relate to both pragmatic and cognitive dimensions are mainly linked to the illusion that the mathematical objects and operations implemented in the software closely resemble their counterparts in the paper-and-pencil context. Tensions related to the temporal dimension are linked to the pace of learning, and the discrepancy between the predicted duration of students' activities, and the actual time they need to complete the task.

Teachers can be aware of such tensions while the lesson is in progress. They can often manage the situation, either by giving the expected answer directly, or by manipulating the software themselves. When not identified *in situ*, un-managed or poorly-managed tensions can lead to a significant deviation from the planned cognitive route, or even an exit from it. The teacher can be dissatisfied at not having achieved his or her objectives, or may have to directly supervise students' activity, creating an illusion that the objective has been achieved.

2.2 Defining Proximities

Our new understanding of teaching and learning activities in technology-based lessons builds on Vygotsky's zone of proximal development (Vygotsky, 1986), together with Valsiner's zone of free movement and zone of promoted action (Valsiner, 1987). We use these zones as a framework to conceptualise teaching and learning activities when technologies, in particular tablets, are used in mathematics lessons. Thus, we define *proximities* as teaching actions that support students' activities within these zones. We begin by presenting our understanding of these three zones, and then outline our analytical construct.

From a learning perspective, the zone of proximal development (ZPD) refers to a zone that encompasses the area where a learner is able to complete tasks unaided, and what they are able to learn with assistance from someone who is more knowledgeable than them, e.g., the teacher. It thus represents a set of knowledge development possibilities. From a practical perspective, the teacher creates a working environment that incorporates the use of tablets (and other instruments) to support the students' understanding of the mathematical content to be learned. By setting up the environment, two zones emerge: the zone of free movement (ZFM), which structures how students access different areas of the environment and interact with different instruments in these areas, and the zone of promoted action (ZPA), which seeks to facilitate the acquisition of new learning. The teacher creates an environment where there is direct access to activities and instruments that support students' learning. What the teacher provides (in the ZFM), and promotes (in the ZPA), are interrelated, and usually the two aspects have to be considered simultaneously. Consequently, many authors refer to the ZFM/ZPA complex to describe what the teacher provides and how the teacher organises the environment in order to achieve the task at hand. It captures the notion of the interactive generation of the environment in which the learner develops (Blanton et al., 2005).

Moreover, the ZPA must be consistent with the student's ZPD, and supporting actions that the individual believes to be feasible within the ZFM (as created by the teacher) (Goos, 2020). As Galbraith and Goos (2003, p. 3) state, "*A link between the ZFM and ZPA is provided by the ZPD. For learning to be possible the ZPA must be consistent with an individual's capacity to learn (ZPD), while for the intended approach to learning to have a chance of success the ZPA must lie within the effective ZFM*".

An important observable element relates to teacher–student interactions, which are analyzed in terms of their assumed influence on students' activities. Some of the teacher's interventions are at the cognitive level and relate to the mathematical content (assistance, assessments, review of notions, explanations, presentation of knowledge, etc.), while others are at the pragmatic level and relate to how the student interacts with the environment to achieve the task at hand (the format of classroom work, the available resources, switching from one instrument to another, etc.). An important focus of our study is, therefore, the proximity of these interventions to students' acquired knowledge, i.e., their ZPD.

Our earlier work built on the notion of discursive proximities (Abboud et al., 2018b). *Discursive proximities* are explicit elements or fragments of discourse that the teacher uses to bridge the gap between students' existing knowledge and the mathematical content to be learned during activities that relate—directly or indirectly—to this knowledge. Robert and Vandebrouck (2014) describe discursive cognitive proximities, and show how these elements of the teacher's discourse can influence students' understanding as a function of their existing knowledge, and the activities they undertake. More specifically, the authors develop some of these discursive proximities and use them to study paper-and-pencil environments.

Here, we extend the previous by defining the notion of pragmatic proximities. In particular, we associate them with the teacher's words and actions related to instrumental issues observed in technology-based environments. Investigating the ZFM/ZPA complex in a tablet-based lesson provides an insight into teacher–student–technology interactions. While cognitive proximities within the ZPD direct attention to the new mathematical knowledge students can (potentially) learn, pragmatic proximities tell us about the opportunities teachers give to their students, in the form of the ZFM/ZPA that is set up to engage them in mathematical learning, with the help of technology. Following Galbraith and Goos (2003), we consider that technology may be regarded as a mathematical tool (to increase capacity) or as a transforming tool (to reorganise thinking). In either case, its presence in the work environment changes relationships between teachers, students, and the task to be undertaken.

In the classrooms we observed, two instruments were regularly provided in students' ZFM: paper-and-pencil (and related tools) and tablets (and related software). The corresponding ZPA that was designed by the teacher aimed to facilitate task performance. The teacher's assistance (actions and interventions) takes the form of instrumental and inter-instrumental proximities that support students' development. Other pragmatic proximities linked to this ZFM/ZPA relate to how the teacher uses the tablet as a cultural tool. One example is to draw upon real-life scenarios (games, virtual reality, etc.) that are anchored in the students' adolescent universe, under the assumption that this will enhance learning. However, efficient pragmatic (and cognitive) proximities must be consistent with what can, potentially, be learned (i.e., the ZPD). Hence, the teaching activity should be directed appropriately, and the work environment should support the intended learning activity.

It is important to distinguish between predicted and effective proximities during the data analysis process. The former relates to what is intended by the teacher. These cognitive and pragmatic proximities are (or can be) predicted when setting up the ZFM/ZPA. For example, the aim may be for students to work independently, without any intervention by the teacher during the lesson; alternatively, a discursive or non-discursive intervention can be planned to get as close as possible to students' ZPD. Effective proximities take two forms: (1) planned actions that are actually carried out and (2) improvised actions that are developed *in situ*. The latter depend directly on what the student is doing at the time, and aim to ensure that the actions supported by the ZPA are possible (or reachable) in the ZPD. In case (1), the teacher can assume that there is an effective proximity when, in fact, this is not the case; this contributes to tensions observed in the classroom.

To conclude, we frame our research questions in terms of tensions and proximities. It is clear that the introduction of tablets within teaching–learning leads to a change in the ZFM/ZPA complex. Our first question asks what are the cognitive and pragmatic tensions that result from this change, with a particular focus on instrumental tensions that are directly caused by the introduction of the tool? We examine the extent to which the teacher is aware of these new tensions, and how he or she deals with them. Secondly, we ask what are the proximities that are developed, or could be developed, to support students' learning in this new environment? We seek to evaluate which of these proximities can be predicted, which are effective, and if cognitive and pragmatic proximities fall (or not) within students' ZPD?

3 A Case Study

In this section, we present our theoretical approach and analytical tools through a case study, and use it to highlight the new results these concepts can generate. First, we present the context of our study and the methods associated with our theoretical approach. Secondly, we outline the teacher's profile. Then we develop an *a priori* task analysis of the observed sequence, and students' expected activity. This analysis identifies potential sources of tensions and predicted proximities in the teaching–learning situation. Finally, we analyse the teaching process from the perspective of effective proximities, notably instrumental, and their impact on students' learning.

3.1 Context and Data

We are currently involved in a long-term project[2] that studies how teachers and students use tablets, provided by academic authorities, in their day-to-day classroom work. The project is designed to support the production of tablet-based mathematical resources for secondary school teachers. Ten mathematics teachers, from several regions of France, are participating in experiments that seek to develop, in a collaborative process, tablet-based learning situations and associated resources. These teachers are already members of the Institutes for Research in Mathematics Education (IREM) network, and they already use digital technologies in their classrooms on a regular basis. They have contributed to a book that is aimed at teachers, which offers a range of ways to use the dynamic geometry software GeoGebra in the mathematics classroom.[3] The teachers' goals, in engaging in this new project, are to use tablets in teaching–learning mathematics and to provide other teachers with new resources. Project funding has enabled them to equip their classrooms with tablets and financed regular meetings over the past 3 years. The case study we present here looks at the work of one of these teachers: Roger. He is an experienced teacher who has been involved in the IREM network for about 10 years. He teaches at a secondary school in Lille (northern France) with students from grade 6 to grade 9 (aged from 11 to 14).

A sixth grade geometry lesson was observed. Students had access to tablets equipped with the GeoGebra dynamic geometry software. It should be noted that the tablets did not belong to the students; they were distributed at the beginning of lessons in which they were to be used.

Three types of data were collected. The first was a video recording of the lesson, which was transcribed and the progress of the lesson divided into phases identified in relation to the succession of the teacher's actions. The second data set consisted of responses to a short questionnaire completed by the teacher after the lesson. In this, the teacher was first asked to explain the goals of the lesson and to describe the related work environment. Second, he was asked to report on what actually happened and the extent to which the goals that were initially set was reached. Following our analysis of the video recording and the questionnaire, the third data set was collected. This consisted of responses to a post-lesson interview with the teacher based on elements from the analysis, which served to triangulate some of our interpretations and to remove uncertainty about others. In addition, the post-lesson interview data provided more insights into the teacher's practice and shed light on some of the choices he made during the lesson that had impacted how the session had unfolded.

[2] http://perseverons.inspe-bordeaux.fr/

[3] *Créer avec GéoGébra*: https://tice.univ-irem.fr/lexique/perso/frontLexiqueGGB/

3.2 Introducing Roger

Roger has taught mathematics to lower secondary school (students aged 11–14) for 20 years. He began using technologies in his classroom with the arrival of interactive whiteboards in his school almost 13 years ago. He is very familiar with GeoGebra, and uses it on a regular basis both for lesson preparation work, and to illustrate mathematical concepts and animate geometric figures during classroom sessions.

He teaches at an average suburban school with around 600 students. Students come from heterogeneous socio-economic backgrounds, including a certain number of disadvantaged students (e.g., with no access to technology at home). In his school, other colleagues who teach mathematics use digital technology in the computer room from time-to-time, but not on a regular basis. He is the only member of staff who is interested in the tablet project, and has been using tablets regularly since the school acquired them. He tends to share his experience of their integration into teaching with members of the IREM group, rather than colleagues in his own school.

When interviewed, Roger stated that he sees mathematics teaching as embracing pleasure, creativity and "practising things". He added that this is what made him quickly adopt tablet technology, because it can be manipulated with the fingers (unlike traditional computers), especially geometric objects, and students can have fun simulating real-life situations. Hence, in addition to the use of tablets for predefined "traditional tasks", he also sees it as a tool for creativity and exploration.

The lesson that was observed, and that we analyse below, is with a grade 6 class (students aged 11). It concerns a learning situation that Roger designed. The first objective was to use GeoGebra tablet software to revise the concept of the circle (a topic that was already covered in previous classes). He designed an interaction between the paper-and-pencil environment and the tablet environment that was intended to help students to revise their understanding of the centre and radius of a circle and how to draw circles using a pair of compasses. Second, he aimed to provide students with an opportunity to explore another definition of the circle, which we explain in more detail below. Prior to the observed lesson, he had already introduced his students to GeoGebra. However, he declared in the post-lesson interview that he felt that this initiation to dynamic geometry was too dry. Hence in this lesson he expressed his desire to expose the students to a more engaging situation using the software interface and dragging mode in the tablet environment, in a way that limited access to other GeoGebra menus that might offer alternative ways to accomplish the task.

3.3 Task Analysis

We begin with an analysis of the task, and the expected mathematical activities. This first step is a recurring interest in studies within the DA framework (Vandebrouck, 2013) as it provides in-depth insight into the mathematical knowledge required to complete the task. It sheds light on how the teacher negotiated the inevitable unpredictability of the classroom, when attempting to bring students into contact with mathematical concepts (Abboud & Coles, 2018). In the post-lesson investigation, we take into account the deep links between actual student activities and the task analysis, including its mathematical content (Vandebrouck, 2018). Wherever possible, we make the link with potential pragmatic/cognitive tensions or proximities.

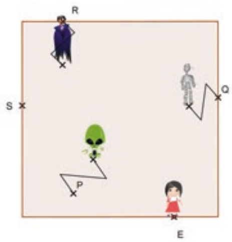
The task is designed to require students to draw upon their existing knowledge of circles. They would have already encountered circles, presented as geometrical forms (a curved line) and investigated circles by observation and with the use of instruments. They were expected to revise the two elements of a circle, its centre and radius, introduced by drawing with pairs of compasses. In the post-lesson questionnaire, Roger stated that the new challenge was to identify the circle as a geometrical set of points at an equal distance from a point that would be identified as the centre of the circle, while this distance would be defined as its radius. This definition requires students to be able to reason and demonstrate their arguments without only using what they could see or their instruments. The paper-and-pencil task they were given is shown below (Fig. 1).

Traversée de la pièce 6^e^
Fiche élève

Élève :

Auteur : PETIT Raphaël © octobre 2015

Une petite fille vient d'entrer dans une pièce sombre par la porte E.
La porte se referme violemment (clac !).
Elle est verrouillée, impossible de l'ouvrir !
Elle se retourne, ses yeux s'habituent à l'obscurité ….
Ah ! Elle devine 3 monstres qui se réveillent !
Vite ! Elle veut rejoindre la porte de sortie qu'elle aperçoit enfin en S ...



A toi de l'aider à traverser la pièce !
Dessine un chemin qu'elle pourra suivre sans se faire attraper.

Quelques informations utiles pour que tu puisses l'aider :

- la pièce est un carré de côté 8 m ;
- elle est représentée à l'échelle avec 1 cm pour 1 m en réalité ;
- un monstre est attaché au point P par une chaîne de 3,8 m de long ;
- un autre monstre est attaché au point Q par une chaîne de 3 m ;
- un dernier monstre est attaché au point R par une chaîne de 2,6 m.

Crossing the room
Worksheet

A little girl enters an unlit room through door E. The door slams shut behind her! It's locked, and there's no way to open it!

As she turns around, her eyes get used to the darkness.... Oh no! She thinks she can see 3 monsters… and they're waking up! Quick, do something! She has to get to the exit, but it's at S...

You have to help her cross the room! Draw a path that she can follow without getting caught by any of the monsters.

Here's some useful information so that you can help her:

- the room is a square, with 8 m sides;
- the scale of the drawing is 1 cm to 1 m;
- the first monster is attached to point P by a 3.8 m chain;
- the second is attached to point Q by a 3 m chain;
- the third is attached to point R by a 2.6 m chain.

Fig. 1 Task statement (French version and English translation)

In the first phase, the task is completed in a paper-and-pencil environment and the students can use artefacts such as a ruler, pairs of compasses and a set square (the first ZFM). They can tackle the problem in two ways. First, they can draw possible paths through the room that avoid being captured by the monsters (the first ZPA). In this case, they have to make measurements, and compare them at several possible positions, by drawing a path with several checkpoints. These activities are accessible to all students as they rely upon acquired knowledge about measuring with a ruler, and do not require specific mathematical knowledge about circles. In this phase, there is no feedback from the environment. Validation can only come from the teacher, who can help to identify points on the path where the girl can be caught (or not) by one of the monsters. There is no possible cognitive proximity with new knowledge.

Nevertheless, the task involves a major adaptation that students must make in order to be able to find a path that prevents the girl from being caught by any of the monsters. They must change their point of view, focusing on the monsters and their areas of action, rather than the girl and her movements around the room. This second approach is only possible for the most advanced students (*a maxima* activity), while the first is accessible to all students (*a minima* activity). If the teacher does not manage this passage between the first and second approaches, we can reasonably hypothesise that a cognitive tension arises and lasts throughout the session.

At this stage, the teacher has set up a ZPA in which students have to explore the monsters' areas of action, and think about the idea of circles that they can draw with a compass. This requires them to recognise existing knowledge about circles, their centre and radius. If this is achieved, they will be able to draw a path that is outside the three areas. Validation is cognitive: the girl has to remain outside the three circles, so that she cannot be caught by the monsters. However in order for the ZPA created by the teacher to be effective, he must provide cognitive proximities to enable students to develop the new knowledge, even if they only have developed *a minima* activity (in other words, even if they did not recognise that the notion of circle is a way to solve the problem). Some students will easily make the leap, while it is outside the ZPD of others.

In the second phase, students undertake the same task in the tablet environment. Here, they have access to GeoGebra dynamic geometry software (the second ZFM).

In this environment (Fig. 2), students can experiment with moving the girl, as in the paper-and-pencil environment, but they can also move the monsters (the second ZPA). To prevent pragmatic tensions due to the change in ZFM/ZPA, the design of the task is the same in both the environments (in fact, the paper-and-pencil version was a printed copy of the tablet's screen). However, there is a significant change to the ZPA. In the tablet environment the monsters can be moved as in real life, with two outcomes: if the girl enters the monster's area, she is immediately caught and the red message "lost" appears (Fig. 3). If she reaches point S without being caught by a monster, the yellow message "win" appears (Fig. 4). Students can make several attempts, with immediate feedback. The adaptation, which represents a change in the viewpoint, is easier than in the paper-and-pencil environment, because the

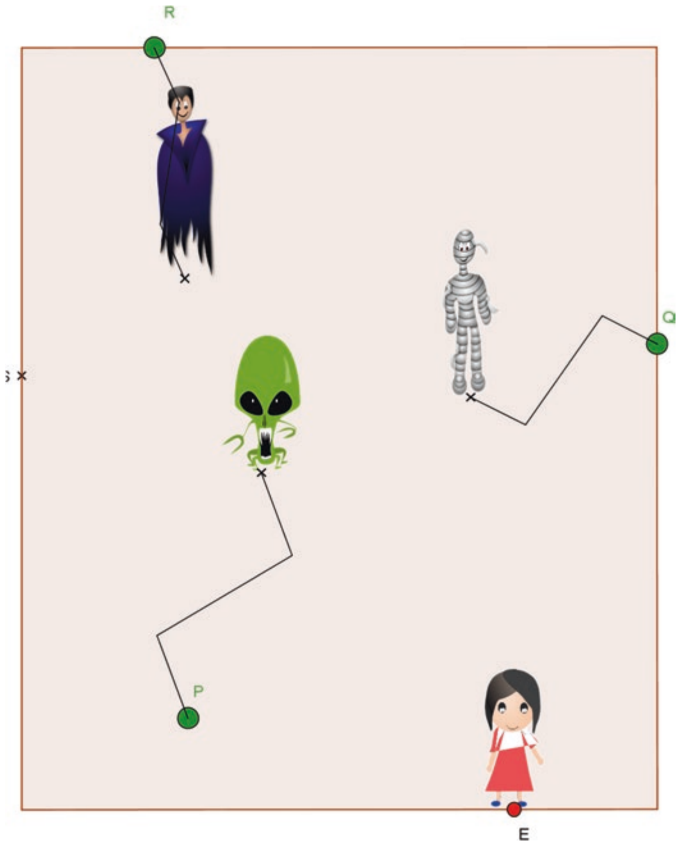


Fig. 2 Tablet environment

monsters can easily be moved. Thus, the tablet environment supports pragmatic and cognitive proximity, along with new knowledge.

All students have access to these activities in the tablet environment. However, the task may not require any mathematical knowledge at all, as the students can move the actors with their fingers and avoid any need to perform measurements and comparisons directly on the screen. Despite a planned pragmatic instrumental proximity between the two ZFM/ZPA, there are new cognitive and pragmatic/instrumental tensions that the teacher must manage to situate the activity within the ZPD of each student. Some students can only develop *a minima* activities, moving actors with their fingers; in this case, even without any mathematical knowledge, they can accomplish the task. The teacher must underline that the task is not just to find *one* way to win, but to find a winning *strategy* using all of the resources provided by the tablet, with reference to previous work in the paper-and-pencil environment (inter-instrumental proximities). Other students are able to visualise circular zones around the monsters, but the teacher may have to intervene to ensure first that these zones are recognised as (pieces) of circles (i.e., as geometric figures) and, second, that the three circles are complete (the ropes are stretched), so that the zones of the monsters

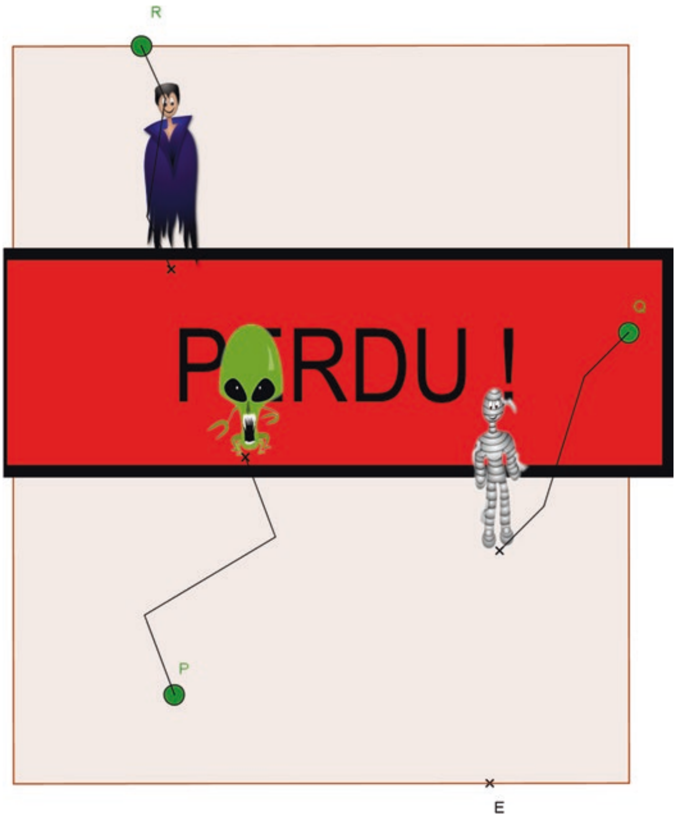


Fig. 3 Lost outcome

and the safe area for the girl are clearly visualised as regions of the plane. GeoGebra's trace (of the monsters) function can help students to visualise disks around the monsters. Let us note here that when asked about this during the post-lesson interview, Roger stated that he was not aware that the use of this function could have led the students to visually perceive the disk as the inner area of the circle. So, recognising circles—with or without the visualisation of the monsters' zones—remains as *a maxima* activity for the best students, and may be even more a more difficult process than in the paper-and-pencil environment.

The ZFM/ZPA created in the second phase can, therefore, reinforce the existing cognitive tension in the teaching activity. This tension lies in the transition from the first (*a minima*) approach, where the expected mathematical knowledge is lacking, and the second (*a maxima*) approach, where the student is able to recognise circles as geometrical objects with a centre and radius. Although the tablet introduces pragmatic and cognitive proximities linked to the change of viewpoint, it can encourage the first approach (*we're not even doing maths anymore, we're just playing*), and the adaptation that brings into play the targeted knowledge (the concept of the circle is even less apparent). Moreover, there is a new pragmatic cognitive tension between the two environments, as recognising circles in the tablet environment can act as an

Fig. 4 Win outcome

obstacle to the use of the compasses in the paper-and-pencil environment. In practice, there is no clear link between moving a monster on the tablet with your fingers, and the centre and radius of a circle, while the latter does not appear as a global set of points. The teacher must be aware of these new cognitive tensions in order to anticipate discursive proximities, while taking into account inter-instrumental aspects.

3.4 Tensions and Proximities in Teaching and Learning Activities

The lesson lasted 55 min, and 11 students were present (Fig. 5). Seven phases were identified. In *Phase 1* the task was presented to students. The teacher explained the situation: the girl is in an unlit room and the door is locked. She sees the monsters, and there is only one way to exit the room – through the door marked S. In *Phase 2,*

Fig. 5 Global view of the classroom

students worked individually on the problem in a paper-and-pencil environment. The teacher gave individual explanations and feedback. In *Phase 3*, students continued to work on the problem with the tablet and on paper. The teacher briefly introduced the tablet environment on the whiteboard, and showed students how to move the monsters and the girl with their fingers. Students who completed the task in the tablet environment were asked to draw their solution on paper. In *Phase 4*, the teacher presented the solution to the whole class. He showed on the whiteboard the monsters' trace (with GeoGebra) (Fig. 11), but students were not able to do the same with their tablets (where the trace function is not available). The monsters' circles were partly drawn but he did not use the word "circle". In *Phase 5*, students once again tackled the problem individually, switching between moving the monsters on their tablets, and the paper-and-pencil environment. The teacher was expecting students to draw the monsters' circles on the paper and then find the path for the girl. In *Phase 6*, the teacher explained the notion of the circle to the whole class. Finally, in *Phase 7* he asked students to write down how they solved the problem and explain their mathematical construction.

In Phase 1, Roger presented the task to students. It was a theatrical performance, with him taking the role of the girl. He turned out the lights, and entered the classroom by the main door; he noticed the three monsters, and then exited a door at the back of the classroom (Fig. 6). This presentation was, thus, far more realistic than the mathematical problem, and was consistent with Roger's aim that his students have fun while learning mathematics. However, this created a pragmatic tension between the simulation and the mathematical problem. This tension would last throughout the lesson—several students and Roger himself mixed realistic vocabulary and geometrical notions. For example, Roger used centimetres or metres

Fig. 6 Presentation of the task

interchangeably, and referred to monsters and the girl even when he used centimetres. During this phase, the tablet window was shown on the whiteboard, and Roger only used it to illustrate possible movements of the monsters. Tablets were provided to the students, but they were told not to use them.

In Phase 2, as underlined by the *a priori* analysis, a cognitive tension arose between the *a minima* activity that was accessible to every student and the *a maxima* activity that was only within the reach of the more able students (involving a change of viewpoint and a recognition of circles). Feedback during the paper-and-pencil environment was only from the teacher, who validated (or not) the paths proposed by students. Figures 7 and 8 show the solutions proposed by some students, and the teacher's corrections.

In this context, to invalidate a path, the teacher showed points on the path where the girl could be caught. The students could, in turn, use measurements and comparisons to see why their solution does not work. Sometimes the teacher highlighted several points: "*So, she can walk between these two monsters, you're right. But here? Does that work?* (Fig. 8: the teacher was referring to points of the path) *You see? So, it's very good at this point, but at this point it's not*".

For most students these interventions were insufficient to develop the expected activity, and the cognitive tension remained. Some students developed alternative strategies, such as running fast, but the teacher was able to show, with the tablet window on the whiteboard, that the monsters ran faster than the girl. As soon as the girl enters the monster's zone, it catches her and the message "lose" is displayed. At the end of Phase 2, none of the students developed the expected *a maxima* activity.

In Phase 3, the teacher enhanced his students' ZPM/ZFA by introducing the tablet environment. Here, the objective was to provide an easy trial-and-error process, based on feedback from the device. However, we observed a crucial switch, as, immediately after telling his students that they could use their tablets, he said: "*now*

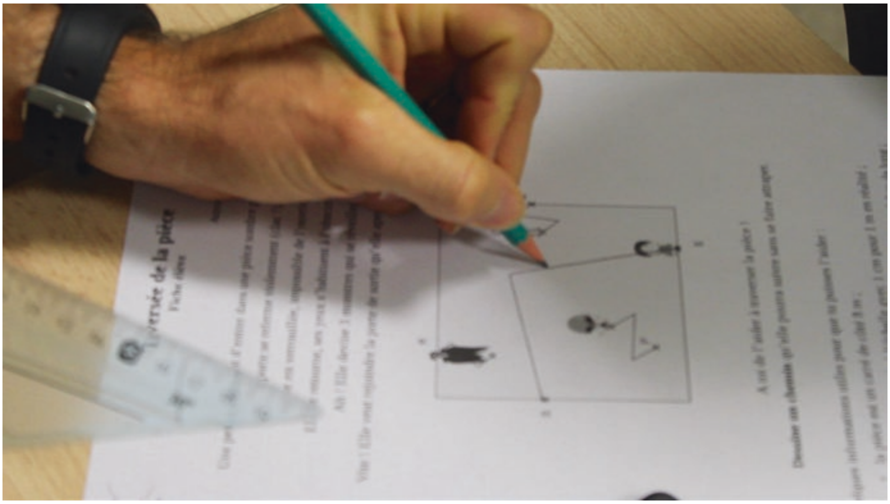


Fig. 7 Example 1 of students' solutions in Phase 2

Fig. 8 Example 2 of students' solutions in Phase 2

we can move the monsters; we can see where the girl will be safe and where she cannot go". This change in viewpoint, which is a major element in the cognitive tension, was initiated by the teacher, and directly supported by the ZPA. Instrumental and cognitive proximities, identified in the *a priori* task analysis, could occur as the teacher himself stated the targeted strategy. Moreover, there was a new pragmatic tension (inter-instrumental) as some students may have thought that the strategy was different because the instruments (within the ZFM) were different. Roger seemed to be unaware of these tensions, and had to repeat the change of viewpoint to most of his students: "*I suggest that you move the monsters*"; "*Lucie, try to find all the places where each monster can go*"; "*the objective is to find all the places they can get to*". A little later, when students were still unable to change their viewpoint, Roger had to justify the reason for this new strategy, "*Fanny, I said that the objective is to try to find out where each of the monsters can go, and once you have understood that, to draw on the sheet of paper all the places the little girl cannot go*"; "*try to draw on the paper all the places that one of the monsters can reach, in other words all the places where the little girl can't go*", "*Try to draw on your sheet of paper all the places where she can't go*", etc.

Students who did manage to switch their viewpoint—and who had understood the reason why—still could not recognise the zones accessed by monsters as circles (disks). As they could not activate the monsters' traces within GeoGebra, it was difficult to visualise circles. The cognitive tension remained, and most students still did not recognise the mathematical knowledge to be learned. The ZPA did not incorporate the action of stretching the rope as the monsters move. At this point, Roger drew upon discursive proximities—he used geometry vocabulary to characterise the monsters' areas and to draw them with instruments on paper—thinking that it was now in most of his students' ZPD. But these efforts failed for most of the class. We see the teacher showing and drawing virtual circles with his finger over most of his students' screens or sheets of paper. At this point, some of the best students can use these proximities, stating that they can see the circles, or use their compass to draw a circle on paper. However, it was a step too far for most of the class. For instance, Roger asked one student, who had drawn several radii around one monster on paper (Fig. 9) what the geometrical shape was. But his attempt fails, as the student has only developed a partial, discrete view of the area, while identifying the area as a circle requires a global and continuous visualisation.

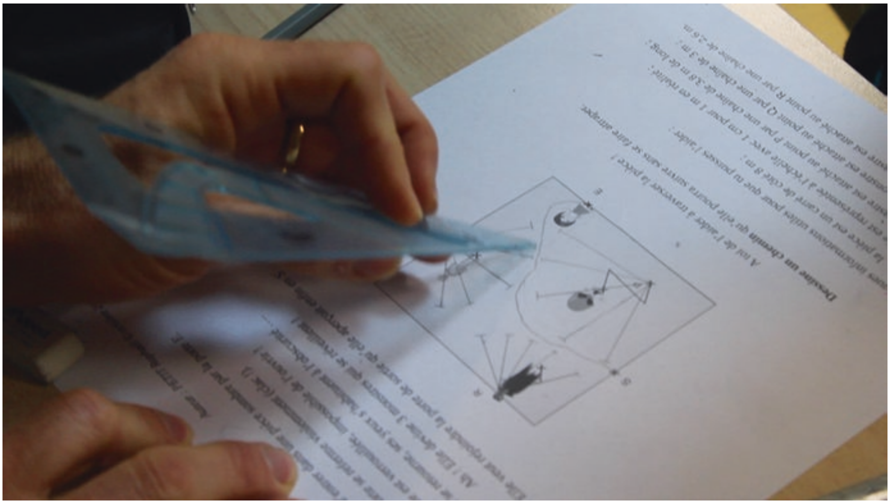


Fig. 9 A student who completed the task by drawing radii, but was unable to visualise the circle

Hence, Roger struggled to manage the classes' cognitive tensions, and most students did not benefit from the available ZPA.

Moreover, as noted in the *a priori* task analysis (and expected activity), there was a new cognitive pragmatic tension that the teacher seemed not to have anticipated. Some students could accomplish the task in the tablet environment without switching their viewpoint, and without any mathematical reasoning (Fig. 10). For instance, Roger had to guide one student towards the expected solution: "*You managed to win, good, the little girl's still there so you won, right? But can you work out all the positions where the green monster can go, the same for the vampire, and for the mummy…*" As predicted in the *a priori* analysis, the initial cognitive tension between *a minima* activity possible for all students and *a maxima* activity achieved by using knowledge about circles seemed to be reinforced. In another example, the teacher tried to stop one student who continued to only move the girl, and always lose, "*I want you to stop playing, I want you to look for the positions that each monster can reach, but that's not what you've done so far. You've been playing and you*

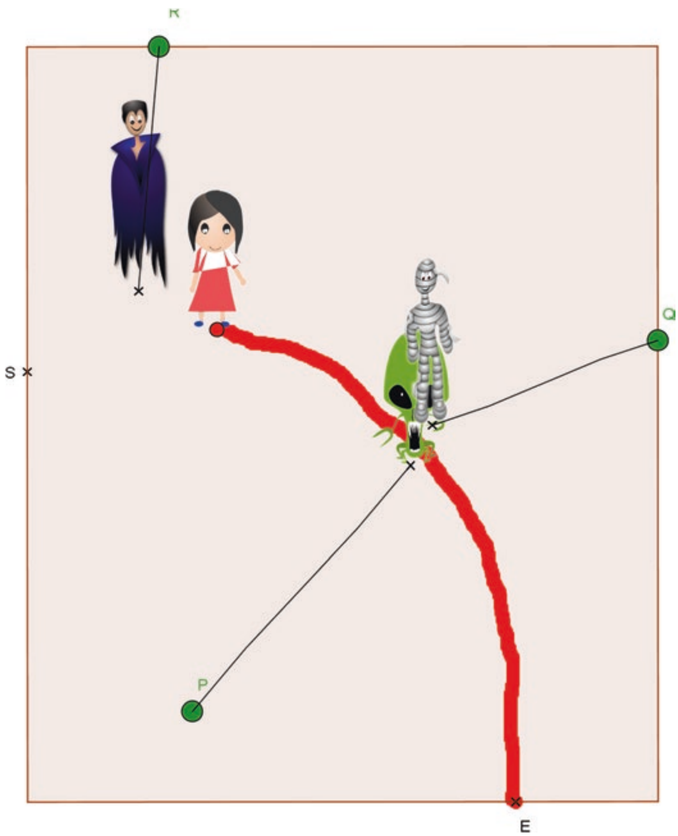


Fig. 10 Winning without visualising any circles

haven't understood what's going on…". Therefore, it appears that the ZPA set up by the teacher in the tablet environment was inappropriate with respect to the ZPD of some students, at this point in the activity.

In phase 4, Roger showed his students the monsters' paths on the whiteboard (Fig. 11). As the students could not do the same on their personal tablets, he provided some new suggestions, "*Take a look. If I move my green monster here, he leaves a slimy trail behind him, like a snail. Okay? (…) I know that these black marks are all places that the little girl can go – or not?* (Students' responses: No!) *If I go as far as I can, then I'll get there, okay*? (Teacher dragging the green monster) *And then if I go a little further… then I can't go any further, you see? And there I can't go any further either. Now I can go there, but then I'm stuck here. And now I'm stuck here. Is that alright? Same for this other monster. Now, he's stuck here, he can go there, he can go there, but there, no, he's stuck. Okay? Does it help you if you know where the limits of all the places they can go are? Does it help you a lot? Does it? So now I want you to take a look at your notebooks, and see if you can find something that tells me exactly where the limits are*". During his explanations, the teacher moved the monsters on the whiteboard, showing several places where they can go

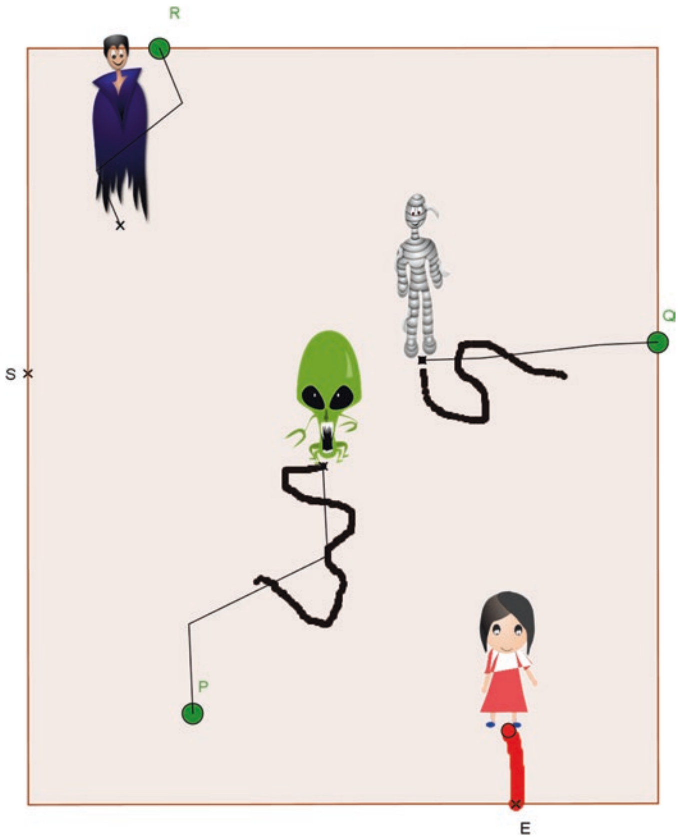


Fig. 11 Monsters' paths shown on the whiteboard

or not. We thus observe that the ZPA incorporated now the action of stretching the rope as the monsters move.

In Phase 5 students' progress and activities became heterogeneous (as is often observed in technological environments). Some continued to play with the tablet, developing *a minima* activity focused on the girl, or having just switched their viewpoint. Some succeeded in moving the monsters from point to point, but failed to grasp the idea of a global circle. Some students asked to have the monsters' traces on their own tablet (as on the whiteboard), but Roger told them it was not possible. By stretching the rope as they moved the monsters, most students eventually completed the task; the little girl exited the room, and they saw the "you have won" message. In these cases, the teacher asked them to go back to their paper-and-pencil, and draw the girl's path with their instruments using a geometrical approach. However, only some seemed to have recognised the monsters' zones as circles (or disks), while most did not. For those who had recognised the circle there was a new task, which was to identify the centre and the radius of each of the three circles, demonstrating that they knew how to use their compasses. For others, this new mathematical task did not make any sense.

In Phase 6, Roger highlighted the idea of a circle. We see cognitive proximity, after one student identified the monsters' zones as circles, "*Well if I go as far as I can each time, if I keep the rope stretched, and I move my monster, something we know appears … what do we see appearing? Use your finger… what do we see appearing when I keep the rope stretched to the maximum and I move my monster? Luna?* (One student says: *a circle*) *A circle. Exactly…*". When Roger asked this student to go to the whiteboard to show the limits of the monsters' zones, we observe (Fig. 12) that even if the rest of the class seemed to see that the monsters' area limits

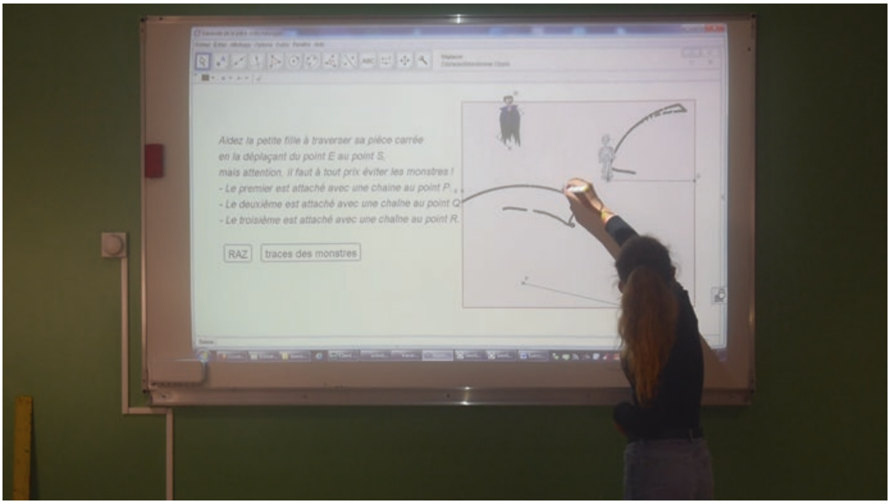


Fig. 12 A student draws a circle by stretching the rope of monster Q and overlapping the zone occupied by monster P

as circles, the movement of the student's arm was not consistent with the idea of a constant radius between the centre and this limit. At this moment, it is far from evident that all students could see circles, and that the cognitive proximity was effective. The fact that, in GeoGebra, it is possible to draw shapes without considering their geometrical properties reinforces the pragmatic (inter-instrumental) tension already present. Roger did not seize the opportunity to manage this tension, and went on to ask every student to write down the mathematical reasoning for their work.

As noted above, the discursive-cognitive proximity between the activity of most of the students, and the mathematical knowledge to be learned, only fell within the ZPD of students who had already identified the monsters' zones as geometrical objects. For instance, one student, Soren, had successfully identified the need to draw circles, and had made the link between circles in the tablet environment and the circles that he was asked to draw with a compass in the paper-and-pencil environment. Roger says, "*Soren, explain to me what you did? (Well, I made circles) What? (I made circles to limit where they can go) Yes. Then how did you make the circles? (Well, I took my ruler with my compass and made the measurements that were written here) Yes…*".

But for other students, the circle and the compasses remained far beyond their ZPD; even those who successfully completed the task in the tablet environment (seeing the winning message) and those who plotted a few points in the monsters' zones. An interaction from Phase 4 illustrates how difficult it was to recognise a circle as a global, continuous set of points: "*Put a mark at 3.8. Put a mark at 3.8 centimetres. He is 3.8 centimetres from the point P. Is that okay? Can you put another point at 3.8 centimetres from P? Go ahead and do another one. From point P. From point P. From point P. Where point P is. Ah, go ahead. Another point 3.8 centimetres from point P (Am I on the same line?) Well no, otherwise you'll be at the same point again. Okay. Here's another one. So, the objective is to plot them all, all the points that are 3.8 centimetres from point P. There are lots of them, aren't there? There's an infinite number of them, okay? It's up to you to find them all. Find me a solution that finds them all*". Another example was observed earlier, at the end of Phase 3 with a different student. Pointing to several of the limit positions of monsters found by the student (Fig. 9), the teacher asked, "*Wouldn't there be a way, isn't there a way? Because this one is OK, this one is OK, this one is OK, but here I don't know where it stops… so can you measure it? Yes, but here? All right, I see. And between those two? And between this one and this one? I don't see where I can stop. How long are we going to do this? (Decades) Decades, yes, but isn't there a quicker way?*". These examples underline the ongoing cognitive tension between what has been done in the tablet environment and recognising circles as a geometrical object defined by a continuous set of points, a centre, and a radius. For these students, the teacher's intervention was ineffective due to the lack of proximity between the work they produced and the mathematical knowledge to be learned. The idea of the circle was not within their ZPD.

In Phase 7 all students had to draw their solution using paper and pencil, and write down their mathematical reasoning. However, despite being able to identify that the monsters' circles were the key to a winning strategy, in this phase, one student still drew three circles around the monsters by hand, again showing that the circle as a geometrical object was not within his ZPD. Roger said, "*Which tool did you use? What did you use as a tool to draw the limits? The pencil and that's all. What can you use to draw a circle? (A pair of compasses) So what? Why don't you use your compasses? Well, if you know you have to use compasses, you have to use it, right?*". Here again, the teacher's intervention could not be called a proximity, despite his intentions. He asks the student a question about how he could draw a circle, whereas the mathematical object associated with the use of the compasses was not yet within the student's ZPD.

The only tension of which the teacher seemed to be aware was the pragmatic tension between the realistic situation and the mathematical problem. He tried to manage this by repeatedly asking his students to use mathematical terms, "*I'll say it again, I want you to use geometric terms. Is that all right? Try to be as precise as possible, as accurate as possible…*". However, there was ongoing confusion in the students' and the teacher's discourses with respect to the two approaches, "*When the monster tries to stretch his rope it's stretched from point P over there; what's the maximum distance? What's the maximum distance for that one – the monster – attached to point P? How long is the rope? (3.8 metres). Okay, so we can actually go up to 3.8 what? (Metres) It's not metres here, it's what? (Centimetres) …*". Even at the end of the lesson he used real-world vocabulary to invalidate some of the geometrical objects produced by the class, "*Is your drawing precise? I want a precise drawing, don't I? Don't forget that it's the little girl's life at stake. We're not taking any risks, right? Do you have your compass? Go ahead and take it out*". The session ended with this confusion between the two approaches. Some students explained their solution without using mathematical terminology. The last interaction between the teacher and a student illustrates the confusion in the student's mind, "*Ah, you've written down your solution, so I'm interested to see, you say: I used the compasses to draw the limits of the monsters and I managed to get the little girl to escape. There's a mathematical term missing, I can't see it. A mathematical word that I don't see (The monsters' metres?). No, no, no, no, no, no, no, no, no, no, no. What have you drawn? You say that you've drawn the limits with a compass. Do you know the name of this limit? What's it called in mathematics? (The circle?) Yes, I would have liked to see that word. With all the vocabulary we already know, circle, centre, radius…*".

4 Findings

Our findings can be summarised according to two aspects.

First, the ZFM/ZPA complex evolves significantly—and in several directions—when the task has to be performed with the tablet instead of paper-and-pencil. The teacher must be aware of all of these evolutions, as they create new tensions, or reinforce existing ones, as the activity progresses, making them increasingly difficult to manage. It seems that the teacher's efforts have only addressed a few predicted pragmatic tensions between the two ZFM/ZPA, for instance by printing out a copy of the tablet's screen and using it as the paper-and-pencil sheet. He was aware that the tablet can provide feedback that is missing in the paper-and-pencil environment, and that it allows the students to experiment more with potential solutions. However, he seemed to be unaware of other, crucial changes in the ZFM/ZPA complex. The new ZPA provided by the tablet (students can move the monsters with their fingers in a natural way, whereas this is not possible using paper-and-pencil) supports a change of viewpoint, and could help students find the correct solution. The tablet, therefore, created a natural link with the targeted knowledge (about circles) and is an instrumental cognitive proximity.

However, in the observed lesson, the teacher himself initiated the change of viewpoint. He redefined the task and, in doing so, reduced the potential of the initial task to support students' learning. Moreover, it caused a new, inter-instrumental tension, as some students may not have understood why they should adopt this new strategy (a new task) in this new environment. We observe that the teacher had to explain this new strategy several times to several students. We argue that although the teacher deliberately set up the new ZFM, he also, unintentionally, changed the ZPA. The instruments themselves changed the ZPA, and an effect of technology that it is important for teachers to be aware. In the tablet environment, some students were able to "win", without learning anything more about the underlying mathematical properties. Freedom of movement was made possible by the feedback provided in the tablet environment, which is not possible in the paper-and-pencil environment. Although students were more motivated in the tablet environment, there was a greater risk of them making less progress than in the paper-and-pencil setting. The cognitive tension between students' *a minima* activity and expected mathematical understanding is reinforced, while the teacher was unaware of the changes engendered by the new ZFM/ZPA complex. Faced with new pragmatic and cognitive tensions, Roger had to manage this new complexity as the lesson progresses, and had to explain more than once the reason for the new strategy in the tablet environment. Another challenge was that some students continued to play with the tablet, instead of engaging in the mathematical task.

The second aspect of our findings concerns the difficulty within the tablet environment for the teacher to exploit discursive cognitive proximities regarding the new knowledge, based on students' activity. As Vandebrouck and Robert (2017) note, the recognition of mathematical objects is different in technological environments. Recognition is cognitive in the paper-and-pencil environment, and precedes students being able to use the compasses to draw the monsters' zones. It is more pragmatic in the tablet environment, and is associated with being able to visualise the monsters' paths (visible or invisible). In the paper-and-pencil environment, this path initially appears as a collection of discrete points, and only becomes a continuous shape with cognitive efforts (associated to the use of the compasses). Whereas the recognition of the shape of the circles in the tablet environment, through the iconic visualisation of the monsters' zones, may not be sufficient to identify these shapes as geometrical figures, nor to draw them using compasses in the paper-and-pencil environment.

Moreover, during phase 6, we observed that on the whiteboard the movement of the student's arm (Fig. 12) was not consistent with a visualisation of the use of a compass. More specifically, it did not allow a cognitive and instrumental proximity with the idea of a constant radius between the centre and the limit of the monster's area. By referring to Duval (1999), who distinguished between iconic and non-iconic visualisation of geometric figures,[4] we argue here that visualisation in the tablet environment was mostly iconic, in other words, it did not associate the observed circular shape with its mathematical attributes (the centre and the radius). Moreover, this iconic recognition can create a pragmatic-cognitive tension with the expected, non-iconic recognition. We observed throughout the lesson how difficult it was for some students to identify circles as global, continuous geometrical objects, based on a collection of discrete points located around the monsters (Fig. 13). Furthermore, they found it difficult to transfer their identification of the circle in the tablet environment to the use of the compass in the paper-and-pencil environment.

As a consequence, it remained challenging for the teacher to exploit discursive and cognitive proximities, based on his students' effective activities in the tablet environment. In Phase 3, we observed that although he did seize opportunities to provide some discursive proximities, his efforts failed for many of students. He had to give procedural help by drawing circles with his fingers over students' screens or over their sheets of paper. We note here that his belief that he was working in, or close to students' ZPD was an illusion, as the recognition of mathematical objects was only iconic.

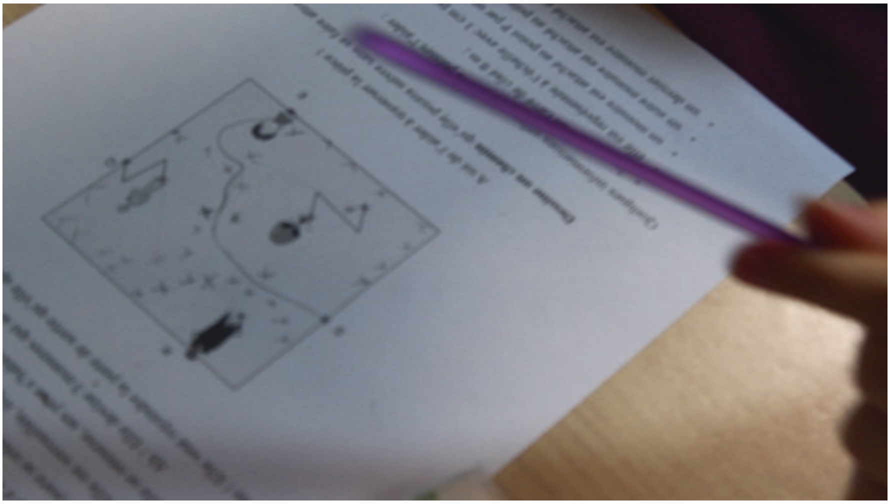


Fig. 13 This student identified several points around monster R, but could not draw a circle by identifying its centre and radius

[4] Duval introduces two types of visualisations that are associated with representations, namely, iconic and non-iconic. "Iconic representation refers to a previous perception of the represented object, from which [we can infer] their concrete character. […] In mathematics, visualisation does not work with such iconic representations: to look at them is not enough to see, that is, to notice and understand what is really represented." (Duval, 1999, p. 14).

5 Conclusion

Policymakers have always advocated for the use of technology in schools, and academic authorities have made financial investments to support its integration into school practices. The assumption is that it is likely that teaching and learning will benefit from the incorporation of an increasing number of technological devices, such as tablets, into classroom activities.[5] In practice, projects that encourage enthusiastic mathematics teachers to engage with emergent technologies may help to address the challenges students face in conceptualising mathematical ideas. Institutional projects that provide technologies to classrooms often rely on a small number of teachers who are recognised for their ability to implement innovative practices. These collaborative projects also create opportunities for teachers to share their experiences, and develop resources that can be disseminated to the wider community. We participated in such a project, as observers and experts in the domain of digital technology integration. Our focus was on how teachers plan their lessons, and try to engage students in a variety of activities, in a classroom environment that is enhanced by the use of tablets.

Our first, global analyses drew upon the analytical tools we had developed in earlier work, namely, the CPT construct, and the concept of tensions in the teacher's activity. Fine-grained, in-class observations led us to develop new tools (still in line with those developed before) that we considered better-adapted to the use of tablets. In particular, we explored Vygotsky's ZPD and Valsiner's ZFM and ZPA as ways to interpret our observations. From this, we developed the concept of *proximities*, namely, teaching actions (discursive or not) that support students' activities within these zones. Proximities provided by the teacher lead to development in the ZPD. There are two kinds: cognitive proximities are directly related to the knowledge at play within the ZPD, while pragmatic proximities are more closely associated with actions within the ZFM/ZPA that are related to the working environment. These analytical tools have helped us gain new insight into teachers' actions, particularly when the class (and the teacher) uses a technological environment to learn (teach) mathematics. In the study presented here, this concerns the tablets that are used in teaching and learning geometry.

A key finding is that in such environments, the ZFM/ZPA complex becomes more difficult for the teacher to grasp. This is because the evolution from the paper-and-pencil environment to the tablet environment brings with it new tensions and opportunities of proximities, certainly cognitive but mostly pragmatic. Even if the teacher conceives these proximities as instrumental or inter-instrumental, they have cognitive aspects that the teacher does not fully realise. This brings us close to findings provided from studies within a semiotic perspective on mathematics teaching/learning that highlighted the potential synergy that may occur between the use of different artefacts linked to the same content (Faggiano et al., 2018). At the same

[5] See the Report of the French Ministry of Education on the use of tablets: https://eduscol.education.fr/numerique/dossier/apprendre/tablette-tactile

time, these studies raise the questions of what happens when a teacher does not recognise semiotic interferences[6] and what is needed to recognise and manage them (Maffia & Maracci, 2019).

Moreover, in our study, discursive proximities became more difficult as the perception of activity (leading to learning) may be an illusion and, thus, there was a failure to identify the real ZPD (for example, related to the iconic visualisation of geometric objects). We argue that this situation is an example of what Blanton et al. (2005) designated as the illusionary zone, in which the teacher believes that opportunities for action are being provided to students but, in reality, this is not the case. Nevertheless the use of Valsiner's zones in Blanton and colleagues' work is rather to explore the ZPD of teachers and gain insights into their potential for development. We plan to examine the illusionary zone concept in more detail in forthcoming work. Our aim is to study its usefulness in understanding the proximities that the teacher seeks to develop in order to reach (or build upon) the ZPD of his or her students.

We are aware that this case study of Roger, and the task he designed, is specific and cannot be generalised. This is not our intention. However, on the one hand, it allowed us to develop theoretical tools that can be generalised to the study of other teachers' practices, and other technological tools. On the other hand, it raises the issue of the extent to which the practices that are developed by teachers who participate in funded projects are supported by the educational institution, and to what extent the learning scenarios that they develop can be disseminated. As experts with an informed critical perspective on these practices, we have the objective to contribute to bridging the gap between policymakers' aspirations and classroom reality regarding the integration of digital technologies. On a local level, our work with teachers, in particular, helping them to reflect on their own practices is one way to bridge this gap. As teacher educators, our research provides resources and methods that seek to improve their uses of technology. On a global level, teachers' professional development can be affected by hoped-for collaborations between the institution and the research community. As Lerman (2014) stated:

> […] the mathematics education research community is largely identical to the mathematics educators' community. This means that as researchers, the tendency is to focus on internal issues of teaching and learning mathematics; an examination of the research field demonstrates the relative lack of attention to policy matters.

Our ambition in participating in the tablet project was to play a constructive part in reflecting on, debating and identifying a realistic vision of how tablets can be integrated into schools. We still have a lot of work ahead of us!

Research Ethics A consent form was signed by the teacher and the students (and their parents) authorising the use of their images for all research purposes and all oral or written communications related to the experimental project.

Both teacher and students were anonymised throughout the current paper.

[6] Maffia and Maracci (2019) define semiotic interference as an enchaining of signs emerging from the contexts of use of different arte facts, referring one to the other (p. 57).

References

Abboud, M., & Coles, A. (2018). The practice theme in mathematics education: Development of a French-English collaboration on the role of theories. *Annales de Didactique et de Sciences Cognitives, SI-2018*, 17–24.

Abboud, M., & Rogalski, J. (2017a). Real uses of ICT in classrooms: Tensions and disturbances in the mathematics teacher's activity. In *Proceedings of the 10th Conference of the European Society for Research in Mathematics Education* (pp. 2334–2341). Dublin City University.

Abboud, M., & Rogalski, J. (2017b). Des outils conceptuels pour analyser l'activité de l'enseignant "ordinaire" utilisant des technologies en classe. *Recherches en Didactique des Mathématiques, 37*(2–3), 161–216.

Abboud, M., Clark-Wilson, A., Jones, K., & Rogalski, J. (2018a). Analyzing teachers' classroom experiences of teaching with dynamic geometry environments: Comparing and contrasting two approaches. *Annales de Didactique et de sciences cognitives, SI-2018*, 93–118.

Abboud, M., Goodchild, S., Jaworski, B., Potari, D., Robert, A., & Rogalski, J. (2018b). Use of activity theory to make sense of mathematics teaching: A dialogue between perspectives. *Annales de Didactique et de Sciences Cognitives, SI-2018*, 61–92.

Abboud-Blanchard, M. (2014). Teachers and technologies: Shared constraints, common responses. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era: An international perspective on technology focused professional development* (pp. 297–318). Springer.

Abboud-Blanchard, M., & Vandebrouck, F. (2012). Analysing teachers' practices in technology environments from an Activity Theoretical approach. *The International Journal for Technology in Mathematics Education, 19*(4), 159–164.

Blanton, M. L., Westbrook, S., & Carter, G. (2005). Using Valsiner's zone theory to interpret teaching practices in mathematics and science classroom. *Journal of Mathematics Teacher Education, 8*, 5–33.

Calder, N., Larkin, K., & Sinclair, N. (Eds.). (2018). *Using mobile technologies in the teaching and learning of mathematics*. Springer.

Clark-Wilson, A., Robutti, O., & Sinclair, N. (Eds.). (2014). *The mathematics teacher in the digital era: An international perspective on technology focused professional development*. Springer.

Depover, C., Karsenti, T., & Komis, V. (2007). *Enseigner avec les technologies. Favoriser les apprentissages, développer des compétences*. Presses Universitaires du Quebec.

Duval, R. (1999). Representation, vision, visualisation: Cognitive functions in mathematical thinking, basic issues for learning. In F. Hitt & M. Santos (Eds.), *Proceedings of the 21st North American PME Conference, Vol. 1* (pp. 3–26). Kluwer Academic Publishers/Morelos.

Faggiano, E., Montone, A., & Mariotti, M. A. (2018). Synergy between manipulative and digital artefacts: A teaching experiment on axial symmetry at primary school. *International Journal of Mathematical Education in Science and Technology, 49*(8), 1165–1180.

Galbraith, P., & Goos, M. (2003). From description to analysis in technology aided teaching and learning: A contribution from zone theory. In L. Bragg, C. Campbell, G. Herbert, & J. Mousley (Eds.), *Proceedings of the 26th Annual Conference of the Mathematics Education Research Group of Australasia*. Deakin University.

Galligan, L., Loch, B., McDonald, C., & Taylor, A. J. (2010). The use of tablets and related technologies in mathematics teaching. *Australian Senior Mathematics Journal, 24*(10), 38–51.

Goos, M. (2020). Theoretical perspectives on learning and development as a mathematics teacher educator. In K. Beswick & O. Chapman (Eds.), *The mathematics teacher educator as a developing professional: Vol. 4 of The International Handbook of Mathematics Teacher Education (Edition 2)* (pp. 53–77). Brill | Sense Publisher.

Hoyles, C., & Lagrange, J. B. (Eds.). (2010). *Digital technologies and Math Education. Rethinking the terrain. The 17th ICMI Study*. Springer.

Ingram, N., Williamson-Leadley, S., & Pratt, K. (2016). Showing and telling: Using tablet technology to engage students in mathematics. *Mathematics Education Research Journal, 28*, 123–147.

Karsenti, T., & Fievez, A. (2013). L'iPad à l'école: usages. In *avantages et défis. Résultats d'une enquête auprès de 6057 élèves et 302 enseignants du Québec*. CRIFPE.

Leontiev, A. N. (1978). *Activity, consciousness and personality*. Prentice Hall.

Leplat, J. (1997). *Regards sur l'activité en situation de travail*. PUF.

Lerman, S. (2014). Mapping the effects of policy on mathematics teacher education. *Educational Studies in Mathematics, 87*, 187–201.

Maffia, A., & Maracci, M. (2019). Multiple artifacts in the mathematics class: A tentative definition of semiotic interference. In M. Graven, H. Venkat, A. Essien, & P. Vale (Eds.), *Proceedings of the 43rd conference of the international group for the psychology of mathematics education* (Vol. 3, pp. 57–64). PME.

Mullet, J., Van de Leemput, C., & Amadieu, F. (2019). A critical literature review of perceptions of tablets for learning in primary and secondary schools. *Educational Psychology Review, 31*, 631–662.

Rabardel, P. (2002). *People and technology. A cognitive approach to contemporary instruments*. https://hal.archives-ouvertes.fr/hal-01020705. Accessed Sept 2020.

Robert, A., & Rogalski, J. (2002). Le système complexe et coherent des pratiques des enseignants de mathématiques: une double approche. *Revue Canadienne de L'enseignement des Sciences, des Mathématiques et des Technologies, 2*(4), 505–528.

Robert, A., & Rogalski, J. (2005). A cross-analysis of the mathematics teacher's activity. An example in a French 10th grade class. *Educational Studies in Mathematics, 59*, 269–298.

Robert, A., & Vandebrouck, F. (2014). Proximités en acte mises en jeu en classe par les enseignants du secondaire et ZPD des élèves: analyses de séances sur des tâches complexes. *Recherche en Didactique des Mathématiques, 34*(2/3), 239–285.

Rogalski, J. (2008). Le cadre général de la théorie de l'activité. Une perspective de psychologie cognitive. In F. Vandebrouck (Ed.), *La classe de mathématiques: activités des élèves et pratiques des enseignants* (pp. 23–30). Eds Octarès.

Tamin, R., Borokhovski, E., Pickup, D., Bernard, R., & El Saadi, L. (2015). *Tablets for teaching and learning: A systematic review and meta-analysis*. Commonwealth of Learning.

Valsiner, J. (1987). *Culture and the development of children's actions: A cultural–historical theory of developmental psychology*. Wiley.

Vandebrouck, F. (Ed.). (2013). *Mathematics classrooms. Student's activities and teachers' practices*. Sense Publishers.

Vandebrouck, F. (2018). Activity theory in French didactic research. In G. Kaiser, H. Forgasz, M. Graven, A. Kuzniak, E. Simmt, & B. Xu (Eds.), *Invited lectures from the 13th international congress on mathematical education* (ICME13) (pp. 679–698). Springer.

Vandebrouck, F., & Robert, A. (2017). Activités mathématiques des élèves avec des technologies. *Recherche en Didactique des Mathématiques, 37*(2-3), 333–382.

Villemonteix, F., & Khaneboubi, M. (2013). Étude exploratoire sur l'utilisations d'iPads en milieu scolaire: entre séduction ergonomique et nécessités pédagogiques. *Revue STICEF, 20*, 1764–7223.

Vygotsky, L. S. (1934/1986). *Thought and language*. MIT Press.

Digital Resources in Kindergarten Teachers' Documents and Resource Systems: A Case Study in France

Ghislaine Gueudet, Sylvaine Besnier, Laetitia Bueno-Ravel, and Caroline Poisard

Abstract This chapter concerns the use of digital resources by kindergarten teachers. It presents a three-year study, using the theoretical perspective and the methods of the documentational approach. We focus on the case of an experienced kindergarten teacher, Mia, and on her teaching of numbers. At a micro-level, we investigate the evolutions of a document she developed around the use of a mathematical software during two years. At a macro-level, we follow the evolutions of her resource system during the three years of our study, in particular concerning the role of digital resources in this system. Drawing on Mia's case, we contend that, at the kindergarten level, digital resources can contribute to teacher professional development. Their use requires at the same time a development of the teacher's design capacity, in particular for designing relevant associations of digital resources with tangible material. The combination of a micro-analysis of documentational geneses and a macro-analysis of the teacher resource system can deepen our understanding of these phenomena.

Keywords Digital resources · Design capacity · Documentational approach to didactics · Kindergarten · Resource system · Tangible material

1 Introduction

In this chapter we focus on teachers at the kindergarten level. In France, where our study takes place, kindergarten welcomes pupils from 3- to 5-years old. It has its own official curriculum, including an objective (to be reached at the end of

G. Gueudet (✉)
Etudes sur les sciences et les techniques, Université Paris-Saclay, Orsay, France
e-mail: Ghislaine.Gueudet@universite-paris-saclay.fr

S. Besnier · L. Bueno-Ravel · C. Poisard
CREAD, University of Brest, Brest, France

A. Clark-Wilson et al. (eds.), *The Mathematics Teacher in the Digital Era*, Mathematics Education in the Digital Era 16,
https://doi.org/10.1007/978-3-031-05254-5_8

kindergarten) labelled as ‘learning numbers and their use’ which incorporates in particular: learn the names of numbers (up to thirty), learn to read numbers written in digits (up to ten), learn the use of numbers to describe quantities, including decomposition and re-composition of small numbers (up to ten).

Using the theoretical and methodological perspective of the documentational approach to didactics (DAD, see e.g., Gueudet, 2019; Trouche et al., 2020a), we study the interactions between teachers and resources in the context of the teaching of numbers at kindergarten, with a particular focus on digital resources. DAD considers that, as part of their professional activity, teachers develop a *resource system*: a structured set of resources that support the different goals of their activity. Moreover, for a given goal, a teacher interacting with resources develops a *document*, associating resources (selected, modified) and professional knowledge (see Sect. 3 for an elaboration of the DAD).

The chapter we authored in the first edition of this book also considered teaching at the kindergarten level in which we used the DAD perspective associated with the concept of orchestration (Gueudet et al., 2014). This chapter introduces two new kinds of evolutions. First, following theoretical and methodological evolutions within the DAD, we focus on the development by a teacher of a document (on the micro-level of one goal of her activity) and on the evolutions of her resource system (on the macro-level of her different goals linked with mathematics). Second, we use new data in our empirical analyses collected in further steps of the same project: MARENE, meaning ‘Mathematical Package of Resources’, a group associating teachers and researchers whose goal is to research and design digital resources for kindergarten and primary school.

In the next section (2) we present a brief synthesis of recent research about the use of technology at the kindergarten level, by teachers in particular. In Sect. 3 we articulate the central concepts of the documentational approach and present our research questions. In Sect. 4 we describe the design of our case study in which we followed a kindergarten teacher, Mia, over a period of three years. In Sect. 5 we present our results that concern a selected document developed by Mia on the one hand, and, on the other hand, the evolutions of her resource system. In Sect. 6 we discuss how our findings answer the research questions.

2 Mathematics at Kindergarten in the Digital Era and Teachers’ Practices

Since the first edition of this book, the number of studies concerning uses of digital tools at the kindergarten level for learning mathematics has increased. This tendency seems to be linked with several factors, in particular the introduction of programming in curricula internationally (especially for enhancing and practising spatial content knowledge through the use of robots such as Bee-Bots) and the development of touchscreen technologies.

Digital tools used to teach numbers at the kindergarten level that have been analysed in these studies can be divided into two categories. Some of these tools are purposefully designed, drawing on research results, to foster the learning of precise mathematical content such as counting (Hundeland et al., 2013; Ginsburg, 2016), cardinality, ordinality and number sense through the use of multimodal representations of numbers (Sinclair & Heyd-Metzuyanim, 2014; Sinclair & Pimm, 2015; Coles & Sinclair, 2017; Sinclair, 2018; Baccaglini-Frank et al., 2020), addition and subtraction (Zaranis et al., 2013; Sinclair & Heyd-Metzuyanim, 2014; Carlsen et al., 2016; Zaranis, 2017) and developing finger gnosis (Sinclair & Pimm, 2015). By contrast, other authors focus on commercial touchscreen apps, which are developed for entertainment, but offer possibilities for young children to interact with mathematical concepts (Byers & Hadley, 2013; Lange & Meaney, 2013; Baccaglini-Frank & Maracci, 2015).

In most of these studies, the analyses concern students' learning and point out that digital tools open up new possibilities for the development of children's number sense, in particular when combining "multi-touch affordances with aural, visual and symbolic ones" (Baccaglini-Frank et al., 2020, p. 2). Using touchscreen technology develops new forms of interactions with mathematics (Byers & Hadley, 2013) and "a new set of tasks" involving finger gnosis in the case of the *TouchCounts* app (Sinclair & Heyd-Metzuyanim, 2014, p. 98).

Whatever the digital tool used at the kindergarten level, its manipulability is always foregrounded. For example, Jung et al. (2014, p. 223) note, "students' achievement in mathematics greatly increased with the presence of mathematics manipulatives". However, the question of articulation between tangible materials and digital tools is scarcely examined, even when both are proposed to students in an experimental intervention (see e.g., Zaranis, 2017). Like Sinclair and Pimm (2015) and Sinclair (2018), we maintain that the use of digital tools cannot be studied without taking into account its relation with tangible materials (and other resources) available in kindergarten classrooms, and its interactions with the mathematical situation.

Considering the combined use of digital tools and other resources requires attention on teachers' choices. Several authors underline that teachers' mediation is crucial to fully exploit the potential of digital technologies (Baccaglini-Frank et al., 2020; Bullock et al., 2017). In particular, without verbalisation, negotiation and discussion guided by teachers about the strategies used by students, students' learning can remain at a surface level and students can keep using the same strategies (Baccaglini-Frank et al., 2020; Baccaglini-Frank & Maracci, 2015). However, research focusing on kindergarten teachers' use of digital technology for their mathematical teaching is still scarce. Hundeland et al. (2013) and Carlsen et al. (2016) have identified that kindergarten teachers adopt three roles in their use of technology: Assistant, Mediator and Teacher. These authors observed that "these roles were used interchangeably and purposefully by kindergarten teachers" (Carlsen et al., 2016, p. 1). Thus, according to Hundeland et al. (2013), kindergarten teachers' mathematical and didactical knowledge has to be taken into account to analyse the quality of digital technology integration in kindergarten mathematics teaching and learning.

More recently, Trgalová and Rousson (2017) designed a digital game for learning numbers at the kindergarten level. Focusing on the concept of 'appropriation', they developed a theoretical model drawing on the instrumental approach (in particular the concept of orchestration) and on the documentational approach. Using this model they analysed the appropriation of the digital game by a kindergarten teacher. They observed that the flexibility of the game (the possibility for the teacher to choose different parameters) was an important factor for its appropriation. They also noted, like Hundeland et al. (2013), that solid mathematical and didactical knowledge was important for this appropriation, because it allowed the teacher to understand the choices of the designers. Their work also evidenced the professional development of this teacher: he reflected on the most suitable orchestrations, and on how to continuously assess his pupils' progress.

In the chapter published in the previous edition of this book (Gueudet et al., 2014), our perspective was very similar to the work of Trgalová and Rousson (2017). We investigated orchestrations developed by kindergarten teachers using mathematical software (a virtual abacus, and a game named 'the passenger train' concerning the ordinal aspect of numbers). Our analyses evidenced strong links between the teacher's knowledge and these orchestrations. In Besnier and Gueudet (2016), we deepened our analyses of orchestrations and initiated a systematic investigation of documents developed during two years by a kindergarten teacher, Mia. In this chapter, we also study the case of Mia (adding a third year of data collection). We do not consider her orchestrations, but instead we study her development of a particular document and the evolutions of her resource system for the teaching of numbers.

3 Teachers' Documents, Teachers' Resource Systems: A Theoretical Framework

The need to understand the impact of digital technologies on learning and teaching processes has led to the development of many theories. In mathematics education research, the instrumental approach (Rabardel, 1995; Guin et al., 2005) is one of these theories, developed in the 1990s and early 2000s. The first studies using the instrumental approach focused on how students learned mathematics using calculators (Guin et al., 2005). The research questions evolved to encompass the use of digital technologies in the classroom by the teacher leading to the introduction of the concept of *orchestration* (Drijvers, 2012; Buteau et al. in this book), which describes the technology-rich environment designed by the teacher and how the teacher exploits this environment with their students for teaching mathematics. The studies about orchestration have often focused on classroom use of a single device or application, for example, the calculator or the spreadsheet. Instrumental orchestration does not provide tools to enlighten the interactions between teachers and the complex sets of resources now available for their activity, which incorporates e-textbooks, video-conference systems, discussions within online communities, etc.

This theoretical gap motivated the development of the documentational approach to didactics (DAD, Trouche et al., 2020a), which focuses on the interactions between teachers and *resources* intervening in their professional activity. DAD considers that the *documentation work* of teachers (selecting resources, modifying them, using them in class) is central in their professional activity and plays a central role for teachers' professional development. The concept of *resource* in DAD corresponds to a definition proposed by Adler (2000), considering as a resource anything likely to 're-source' the teacher's professional activity: a textbook, a software, but also a discussion with a colleague or a student. Referring to activity theory (Vygotsky, 1978), DAD considers the activity of the teacher as goal-directed. For a specific goal of their activity (e.g., 'introducing the correspondence between a set of objects and a number word') teachers interact with different resources (e.g., textbooks, software, discussions with colleagues). Along their activity for this goal they develop a document: resources (selected, transformed, combined) and a scheme of usage of these resources (Vergnaud, 1998). We represent this definition by the simple equation:

$$Resources + Scheme\ of\ usage = Document$$

The development of a document, called a *documentational genesis*, is a twofold process: "the affordances of the resource/s influence teachers' practice (the *instrumentation* process), as the teachers' dispositions and knowledge guide the choices and transformation processes between different resources (the *instrumentalization* process)" (Trouche et al. 2020a, p. 239).

A scheme is a cognitive structure, defined by Vergnaud (1998) as a stable organisation of the activity for a given aim. It associates four components: the *goal* of the activity and sub-goals; *rules of action*; *operational invariants*; possibilities of *inferences*. The operational invariants are the epistemic aspects of the scheme. They are of two kinds: *concepts-in-action* (concepts considered as relevant) and *theorems-in-action* (propositions considered as true). For example, for the goal 'introducing the correspondence between a set of objects and a number word', the operational invariants of a teacher can incorporate 'one-to-one correspondence' as concept-in-action, associated with the theorem-in-action, 'students must learn to establish a one-to-one correspondence between a set of objects and the sequence of number words'. These operational invariants steer the rules of action of the teacher, who can choose, for example, to propose different kinds of sets of objects (e.g., with the possibility to move the objects or not). Within the situation, the teacher can make inferences, leading to adapt their action to the specific features of the situation (e.g., if all students succeed with tangible objects, offer objects represented on a sheet of paper).

The set of all the resources used by a teacher in their professional activity is called their *resource system* (Trouche et al., 2020a). This resource system is organised according to the goals of the activity. Indeed, according to Rabardel and Bourmaud (2003), the activity of a subject is structured in *activity families*, corresponding to similar goals of the activity, for example, 'Planning the teaching of mathematics', 'Designing and implementing an activity about the ordinal aspect of numbers', etc. Resources intervening for two (or more) different activity families are called *pivotal resources*. In this chapter, we focus on the position of digital resources in a kindergarten teacher's resource system and on the evolutions of this resource system linked with the use of digital resources. Our study is framed by the following research questions:

1. What are the characteristics of a document developed by a kindergarten teacher, using digital resources for her teaching of numbers? How do previous resources and professional knowledge influence the resulting document? How do new resources influence the evolution of professional knowledge?
2. What is the role of digital resources in a kindergarten teacher's resource system, and which evolutions of this resource system are linked with the use of digital resources?

We study the case of one teacher, Mia. For question (1) we focus on the development by Mia and on the evolution of a document, at micro-level. For question (2) we consider Mia's resource system for her teaching of numbers, at a macro-level.

4 The Case Study Design

In this section, we first present the principles grounding the general method associated with the DAD, which is called the reflective investigation of teachers' documentation work (Trouche et al., 2020a). Then we present the case study used in this chapter. Finally we introduce our methods for analysis of the data.

4.1 Methodological Principles

The investigation of teachers' documentation work can lead to the collection of different kinds of data. However, it always guided by the principles of *reflective investigation*:

– The organisation and the aims of the data collection are presented to the teacher and discussed with her.
– The material resources used and produced by the teacher are collected.
– The teacher is not followed only for one lesson, but over long periods – several years if possible.
– The teacher is followed in all the different places where her documentation work takes place (in class and out of class).

– The teacher is actively involved in the data collection and analysis.
– The statements of the teacher in interviews are always compared with her actual activity and material resources produced.

The necessity of these principles comes directly from the theoretical perspective of the DAD. The documentation work of the teacher unfolds in multiple moments and places and is only fully accessible to the teacher. The schemes are stable organisations. Consequently, a long-term follow-up is necessary to be able to identify stable patterns in teachers' activity. A large part of these schemes is unconscious, and the declarations of the teachers are not sufficient for identifying the schemes. The observation of the teacher's activity and of the resources designed, compared with the teachers' statements, can provide access to elements of schemes.

4.2 The Case of Mia: Profile and Working Environment

Mia is an experienced kindergarten teacher (of 10 years) who works a medium-sized town in France. She was a member of the MARENE project group. We followed Mia for three years from 2012 to 2015. She worked in a school located in the city centre during the two first years and then moved to a school in an underprivileged area of the same city in 2014–2015. During these three years Mia taught a class called "small section" (3-year-old pupils). In France, 3-year-old pupils take a nap in the afternoon. During this time the other classes of the school are split into small groups who are then taught by the 3-year-old pupils' teacher. This work is organised around a project or a specific learning theme and a rotation of groups is planned to allow each group to meet the different themes over a period of one or two weeks. So, in the afternoons during years 1 and 2, Mia taught specific mathematical activities to small groups of older pupils (4- to 5-year olds). In Table 1, we summarise her working context (school, age of pupils, equipment in terms of digital tools) during the first three years.

During these three years we collected different kinds of data, according to the principles presented above. We summarise the data types in Table 2.

Mia has solid mathematical and didactical knowledge (presentation questionnaire). Indeed Mia was previously a high school economics teacher, and subsequently decided to work with younger students. She mentioned several professional

Table 1 Mia's working context

Year	Working context
Years 1 and 2	A school in the city Centre of R. Mia worked with 3-year-old pupils and with groups of 4-year-old pupils in the afternoon with a focus on mathematics. She had access to three computers in her classroom.
Year 3	Mia changed schools and worked in an underprivileged area of R. with 3-year-old pupils and groups of 4-year-old pupils in the afternoon with a focus on language. She had access to one computer and an IWB in her classroom.

Table 2 Mia's case, data collected

Data collected – Year 1
Presentation questionnaire: Professional history of the teacher, working environment, viewpoint on mathematics teaching, viewpoint on technology, collective work, personal relationship to mathematics. Initial interview about her resources for teaching mathematics and her documentation work. 'Guided tour' of her resources for teaching. Schematic representation of her resource System (SRRS, produced by Mia). Videos and observation of lessons based on 'cars and garages' resources with the 4-year-old pupils, in the afternoon – *6 sessions observed.* Pre-session and post-session discussions with the researcher. Collection of her resources, including her computer files.
Data collected – Year 2
Updated SRRS. Videos and observation of lessons in mathematics with 3-year-old pupils (zoo situation). *3 sessions observed.* Videos and observation of lessons based on "cars and garages" resources with the 4-year-old pupils, in the afternoon. *6 sessions observed*. Pre-session and post-session discussions with the researcher. Collection of her resources, including her computer files.
Data collected – Year 3
'Guided tour' of her resources. Updated SRRS. Videos and observations of lessons in mathematics with 3-year-old pupils (MOOC project). *3 sessions observed.* Pre-session and post-session discussions with the researcher. Collection of her resources, including her computer files.

development courses on the teaching of mathematics in kindergarten that made her aware of the didactic issues in these early learning experiences. According to previous research (Hundeland et al., 2013; Trgalová & Rousson, 2017), this knowledge is a very important condition to guarantee a relevant use of digital resources for her mathematical teaching. This motivated our choice of Mia's case for this chapter, from the different teachers followed in the MARENE project.

4.3 Analysing the Data Collected

In Sect. 4.1 we outlined a central set of principles that guided the analysis of our data, which involved the comparison of teachers' views on their activity and our observations of this actual activity. Here we present more details about how we analysed the data to identify elements of documents on the one hand and to investigate the structure of the resource system on the other hand.

For the analysis of the documents, we introduced a methodological tool called a "document table" (Gueudet, 2019). Table 3 below provides the first example.

Table 3 Example of a "document table"

Goal	Main resources	Rules of action	Operational invariants
Design and implement a synthesis	The black board, and magnetised stickers reproducing elements of the software [...]	Use a vocabulary that approximates that used when working on the software [...]	Verbalisation is important in mathematics.

In such a table, a row represents a document, that is, the resources and an associated scheme of use with its components. The first column describes the goal of the activity. To identify this goal, we searched the teacher's interviews for utterances indicating such goals: "For designing an introductory activity […]"; "When I want to communicate with the parents […]", for example.

Then we noted the resources used to support that achievement of this goal (across all the data collected). The next step is to search for operational invariants and associated rules of action. The operational invariants can be inferred from the interviews, alongside the content of the resources designed by the teacher. For example, the resources designed can provide evidence that "It is important that the children associate different representations of numbers" is probably a theorem-in-action for a teacher, associated with a rule-of-action such as "I always propose several representations of the numbers to my students". If the researchers observed such operational invariants from the analysis of the resources, they would record it in the table in a particular colour and then present the table to the teacher, inviting a reflective view on the elements that had been included in the document table.

Moving to the resource system, according to the DAD it is structured by the different goals of the activity. Several similar goals can be gathered to identify "activity families" (Rabardel & Bourmaud, 2003). As described above for schemes, our starting point was again the teachers' interviews to support the identification of goals. We gathered similar goals, also drawing on previous literature (Gueudet et al., 2012) where activity families have been identified for secondary school mathematics teachers. We then made an inventory of the resources associated with these goals, starting with the interviews, and compared them with the observed activity. We produced a representation of the resource system (see Sect. 5.3) and again submitted it to the teacher who amended it as she felt appropriate.

5 Analysis of Mia's Case

In this section we first describe some digital resources present in Mia's resource system for her teaching of mathematics. Then we analyse a particular document developed by Mia during the two first years. Finally, we discuss the structure of her resource system and the evolutions of this system across these three years.

5.1 Selected Digital Resources Used by Mia

Educational Software

When we started our data collection, Mia was already using software called 'GCompris'.[1] 'GCompris' is an educational software that offers a variety of interactive exercises (maths, science, reading, etc.) for pupils from 2 to 10 years old. Mia used 'GCompris' for computer discovery activities (keyboard, mouse) and mathematical activities (numbers and forms). 'GCompris' allows pupils to practise some techniques and it is not problem-solving oriented.

By contrast, the "Cars and Garages" (CG in what follows) resources are inspired by a previously existing problem-solving situation (not digital, Charnay et al., 2005). This situation was designed for enhancing children's number sense, and more precisely the 'cardinality' aspect (Baccaglini-Frank et al., 2020) by using number as the memory of a quantity.

Pupils have at their disposal several garages (boxes). They are required to make a single journey in a remote place such that they have exactly the number of cars so that each garage has exactly one car. That is, there are no cars without a garage. For pupils, the challenge is to notice that they can use number-based procedures, counting garages, then cars to keep track of their quantity. The MARENE group designed a digital version of this situation and associated tangible material (Fig. 1).

In the software, garages appear in an orange area (the teacher chooses the numeric field). Then they disappear, and cars appear on the right of the screen. Pupils must move the correct number of cars to the grey area at the bottom of the screen and click on the arrow symbolising the storage of cars. The garages reappear; the cars chosen are stored in these garages, which validates or invalidates their choice.

The resources designed by the group also include classroom scenarios. For more details, see Besnier (2016, 2019), or the MARENE group's website.[2]

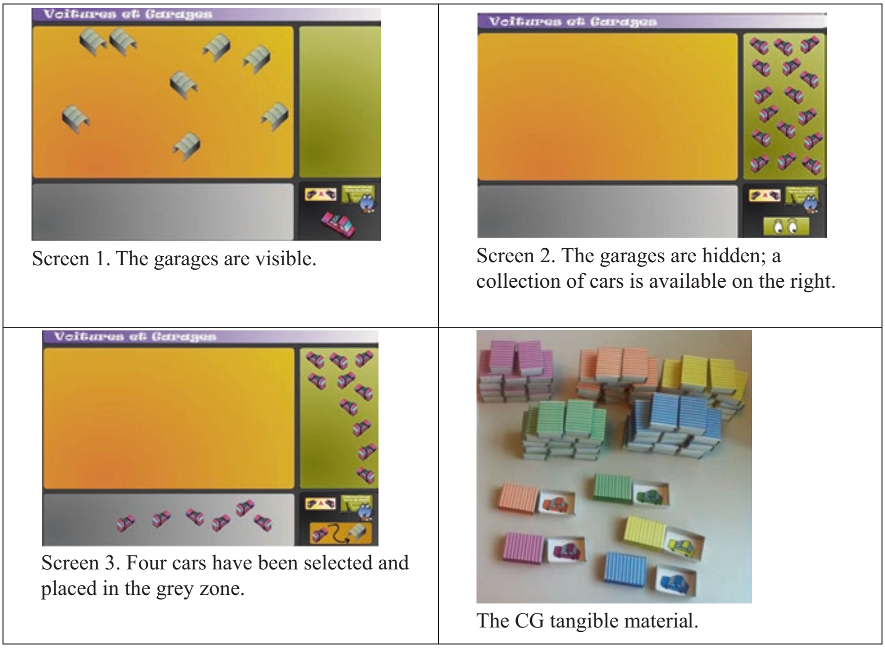
MOOC 'Teach and Learn with Digital Tools in Mathematics' and Nursery Rhymes

During year 3, Mia decided to participate in a Massive Open Online Course (MOOC) called 'Teaching and Learning in Mathematics with Technology' (Panero et al., 2017) to learn more about the use of digital resources for her mathematics teaching. The MOOC offered videos, quizzes and different kinds of resources. During the MOOC, the participants designed and shared teaching projects that used digital technologies. Mia formed a team with a colleague in her school and the second author of this chapter, and they proposed a project entitled 'Develop enumeration strategies and build the number 3'. This project used the IWB and a nursery rhyme called 'The magpies'. This rhyme, traditionally sung in French schools, tells the story of three magpies who land successively on a tree and then leave. We present this project further in Sect. 5.3.

---

[1] « GCompris » means « I got it » in French.

[2] http://seminaire-education.espe-bretagne.fr/wp-content/uploads/marene_main.pdf



Screen 1. The garages are visible.

Screen 2. The garages are hidden; a collection of cars is available on the right.

Screen 3. Four cars have been selected and placed in the grey zone.

The CG tangible material.

Fig. 1 CG software and tangible material

5.2 Focus on the Development of a Document by Mia

Our analysis in this section is based on data collected during years 1 and 2 (Mia did not work on the CG situation in year 3).

In year 1, Mia taught mathematics once a week in her class to a group of 6 pupils (4-year-olds) during the afternoon. The work around the CG situation comprised 6 sessions (Table 4).

Mia developed several documents incorporating the CG software. For the sake of brevity, we focus here on the goal 'Design and implement practice and consolidation activities around the CG situation' (sessions 3 and 4 in year 1).

The six pupils were organised in pairs to work with the CG software, and Mia supported them very closely (Fig. 2).

This choice reflects professional knowledge: the importance that Mia gives to verbalisation in mathematics and the idea that exchanges between peers promote learning. Indeed, in several interviews, Mia emphasised the importance of asking questions of the students, to enable them to verbalise and justify their answers. Moreover, she considered that for such young students the verbalisation has to be supported by the teacher.

At the same time, she observed (final interview year 1) that her presence might have hindered some pupils in their search for a solution. Thus she decided to give more autonomy to the pupils for their work on the software. This required prior

Table 4 Work around the CG situation, year 1

S1	Initial diagnostic, computer (mouse) and numbers
S2	Discovery of the software
S3, S4	Practice and training in pairs on the software
S5	Synthesis on the board
S6	Reinvestment activity: On a worksheet, exercises about number as memory of a quantity

Theo: We will take all the cars!
Mia: You want to take all the cars? Do you agree, Ana?
Ana: No!
Mia: Why don't you agree, Ana?
Theo: I agree! We will take all the cars!
Ana: Not at all, or we are going to lose!

Fig. 2 Practice on the CG software, year 1 – Mia supports the work of one pair (Theo and Ana)

Table 5 Work around the CG situation, year 2

S1	Initial diagnostic, computer (mouse) and numbers
S2	Discovery of the situation with tangible material
S3, S4	Practice in pairs on the software/practice in pairs with tangible material
S5	Synthesis on the board
S6	Practice with software and material

understanding of the situation by the pupils. Thus she decided for the following year to use the tangible material first, and then to associate the software and the tangible material for the practice and consolidation.

In year 2, Mia worked with several groups of 6 to 7 pupils daily in the afternoon. The work around the CG situation still comprised 6 sessions (Table 5).

Mia organised the work according to her intentions presented above. Unlike the previous year, the different groups all became familiar with the situation by handling the boxes. Then they practised in pairs on the software, or using both the software and the tangible material. Halfway through, a synthesis was carried out with each group. After this synthesis, pupils continued to practise with the software and the tangible material (Fig. 3).

We observed here, as in year 1, the importance that Mia gave to verbalisation in mathematics. We also note evolutions of the document developed over the two years (see Table 6).

Fig. 3 Year 2. Practice on the computer and with the tangible material

Table 6 Document for the goal 'Design and implement practice and training activities around the CG situation'

Goal	Main resources	Rules of action	Operational invariants
Design and implement practice and training activities around the CG situation	The CG software *The tangible CG material* Discussions, strategies with the software proposed by the students	*The students have to practise on the computer only after the appropriation of the situation.* *The students practise on the computer and with the tangible material.* *A synthesis is organised between two phases of training and practice.* Have the students express themselves and have them try the different procedures they offer.	*The tangible material supports the understanding of the situation.* *Organising a second practice moments after the synthesis supports learning.* Verbalisation is important in mathematics. Peer-to-peer exchanges promote learning.

The evolutions between Y1 and Y2 are presented in italics

Through her interactions with the resources, Mia developed professional knowledge. Mia had previously developed operational invariants, for example, 'verbalisation is important in mathematics', 'exchanges with peers promote learning' which intervene in the document 'Design and implement practice and training activities around the CG situation'. She also developed new operational invariants, about the association of tangible material and software. Indeed she observed during year 1 that it was difficult for the students to discover both a new situation and a new software at the same time. The tangible material was used for discovery and also for consolidation, while the software was limited to consolidation.

The document developed in year 2 connects the CG software and the tangible material. Both kinds of resources belong to Mia's resource system. In the next section we consider this system more broadly and discuss its evolutions.

5.3 *Mia's Resource System and Its Evolutions*

In this section, we present Mia's resource system for her teaching of numbers in year 1 (Y1) and year 2 (Y2) (Fig. 4) and for year 3 (Y3) (Fig. 6). Our data analyses led us to identify several activity families. Some activity families are linked with mathematics and others are general families whose scope exceeds mathematics but whose resources inform this teaching and probably the teaching of all the other subjects.

For each year, we analysed the structure of these systems, identifying pivotal resources (defined as resources that intervened in several activity families) and discussing the role of digital resources in these systems. We also analysed the evolutions, first between Y1 and Y2, then between Y2 and Y3.

Structure of Mia's Resource System (Years 1 and 2): Professional Activity Families

As presented above, the representation of this system is constructed by the researcher, drawing on Mia's SRRS (for the list of resources) and on her general interview (for the aims of the activity). We identified from her interview seven activity families concerning her teaching of numbers. Four of them can also concern other subjects: planning her teaching (this includes planning for the whole year, but also detailed planning of a lesson); communicating (with colleagues and with the families); following the students; professional development. Three of them concern only her teaching of numbers: design and implement introductory activities; design and implement practice and training activities; design and implement syntheses. This last activity family only concerns the groups of 4-year-old students that she teaches in the afternoon, as Mia considers that 3-year-old students are too young for participating in syntheses.

Pivotal Resources in Mia's Resource System, Y1

– *Mia's computer and the internet.* Mia uses her computer for searching the internet for resources for different aims, and for designing a large part of her teaching resources. She does not directly use it for syntheses, since she has no video projector, but also uses it to design stickers for the syntheses on the board.
– *Detailed planning tables.* During year 1, Mia decided to design with a colleague a common sequencing of topics for the whole year. They designed common planning tables, which were very detailed. These detailed planning tables are pivotal resources, since they were used with a goal of communication with her colleague, and with the goal of planning the learning.
– *Life Notebooks.* Students in Mia's class have individual Life Notebooks, gathering traces of all the activities done in class. Mia also designed a Life Notebook of the class, and she used it the next year as a record of what was done during the previous year. So these Life Notebooks are designed and used to follow the students and to communicate with the families.

– *Mathematical software*. The two mathematical software: GCompris and CG are integrated in two mathematical activity families: for introductory activities and for practice.
– *Stickers*. Mia designed stickers, linked with many different activities (Fig. 5—in particular, in mathematics. When Mia integrated the CG software, she designed stickers associated with this software, in an instrumentalisation process. These stickers were used both for introductory activities and for syntheses, thus they are also pivotal resources.

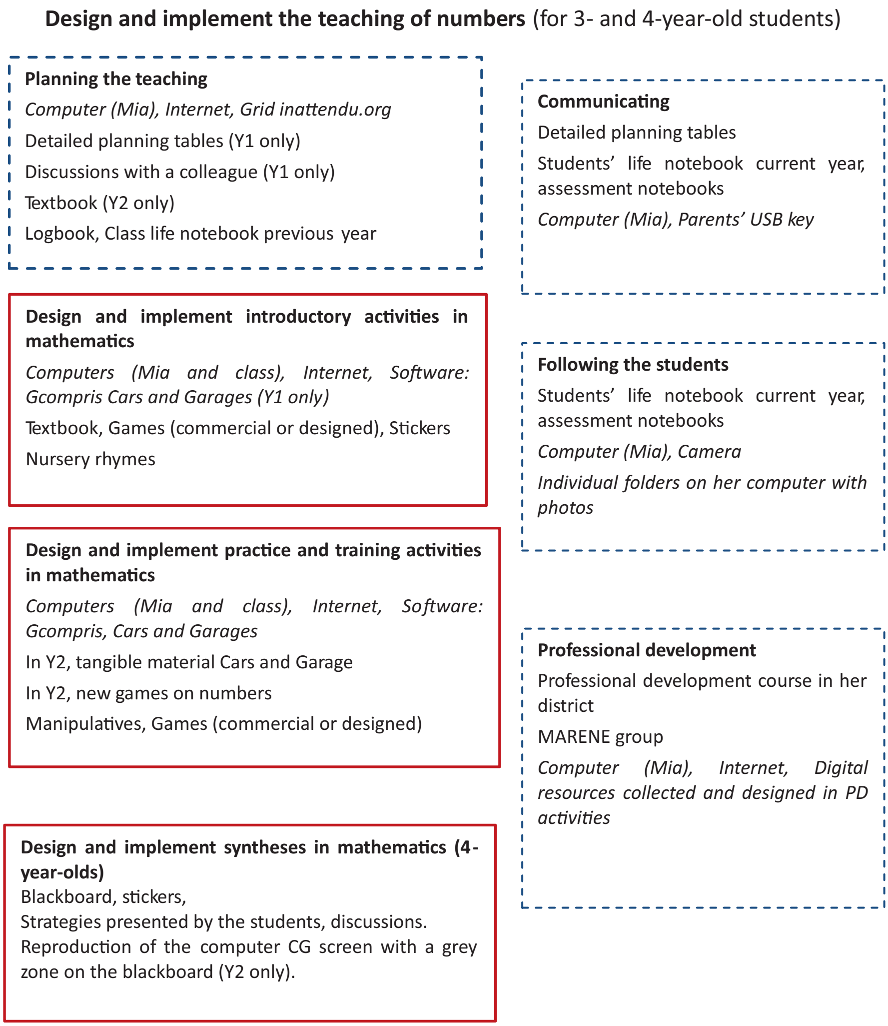


Fig. 4 Mia's resource system for Y1 and Y2 as represented by the researcher. In italics, digital resources. In solid frames, activity families specific to mathematics. In dotted frames, general activity families

Fig. 5 Stickers designed by Mia

Digital Resources in Mia's Resource System, Year 1

Concerning digital resources, we note that:

- Mia's computer was the most important pivotal resource she cites, often in association with the internet. Her computer, resources found on the internet and other digital resources (photos in particular) were essential for designing her other pivotal resources, in particular the Life Notebooks.
- Several digital resources were associated with collective aspects in Mia's work: her planning activity was done with a colleague, using the grids from a website (inattendu.org). Communication with the families involved a lot of photos of the activities. These photos (in a digital format) were copied onto USB keys belonging to the parents.
- Concerning mathematics, Mia was already used to working with 'GCompris'. This facilitated the integration of the CG software in her system: she knew that some software can be useful for the learning of mathematics, and she was already used to organising group work on the few available computers in her class.

Stabilities and Evolutions in Mia's Resource System Between Year 1 and Year 2

We did not notice major changes in Mia's resource system during these two years. We note below the most important modifications.

- *Use of the textbook and planning*

During year 1, Mia used the textbook (for mathematics) only for designing introductory activities. In year 2, the role of the textbook extended to planning her teaching of mathematics. She no longer used the grid from the inattendu.org website, but used the textbook for her planning activity. The textbook hence became a pivotal resource.

– *CG software and associated material*

Evolutions concerning the use of the CG resources are linked with the analyses presented in 5.2. During year 2, Mia integrated the tangible material in her work about the CG situation. She considered that this material was needed for the appropriation of the situation, and also decided to associate the software and the tangible material for training. In year 2, she also designed for the synthesis of the CG situation a grey zone to be displayed on the board, to faithfully reproduce the appearance of the software screen on the board. She noticed that it is very important to provide 4-year-old pupils with similar representations of the CG situation, whatever the medium used (software, tangible material, stickers on the board).

– *Introduction of new games on the topic of numbers*

During year 2 Mia used new games on numbers with her 3-year-old students. She designed, for example, material for a game entitled "the zoo", inspired from the textbook. The introduction of this new game was motivated by her feeling that she had to strengthen her teaching of numbers. According to Mia's interview (Y2), this was a consequence of her work in the MARENE group and with the CG software in particular. Mia's knowledge about the way to introduce numbers to 4-year-old students has been developed by her participation in the MARENE group.

We note that in year 2, no new digital resource was integrated into Mia's resource system. On the contrary, she no longer used the CG software for the introduction of the situation or the inattendu.org website to design her planning.

We now consider year 3. We remind the reader that Mia changed schools between year 2 and year 3. This significant change impacted what Rocha has introduced as the teacher's "*documentational trajectory*" (Trouche et al., 2020b), defined as "a path (with continuities and ruptures) linking professional events (individual and/or collectives) lived by the teacher" (ibid., p. 1245). In this new school, she met a new team of colleagues and she no longer taught 4-year-old students and so no longer used the CG resources, and she had an IWB in her class. We observed significant changes in her resource system (Fig. 6).

We focus here on the evolutions linked with digital resources and with her teaching of numbers.

– *The IWB, a new pivotal resource*: the IWB was very quickly integrated in Mia's resource system, and became immediately a pivotal resource. Indeed the compatibility of the IWB with her previous resource system was very high. The IWB enabled games to be organised, and the associated IWB software provided a record of classroom activities that Mia transformed into booklets contributing to the communication with the families.

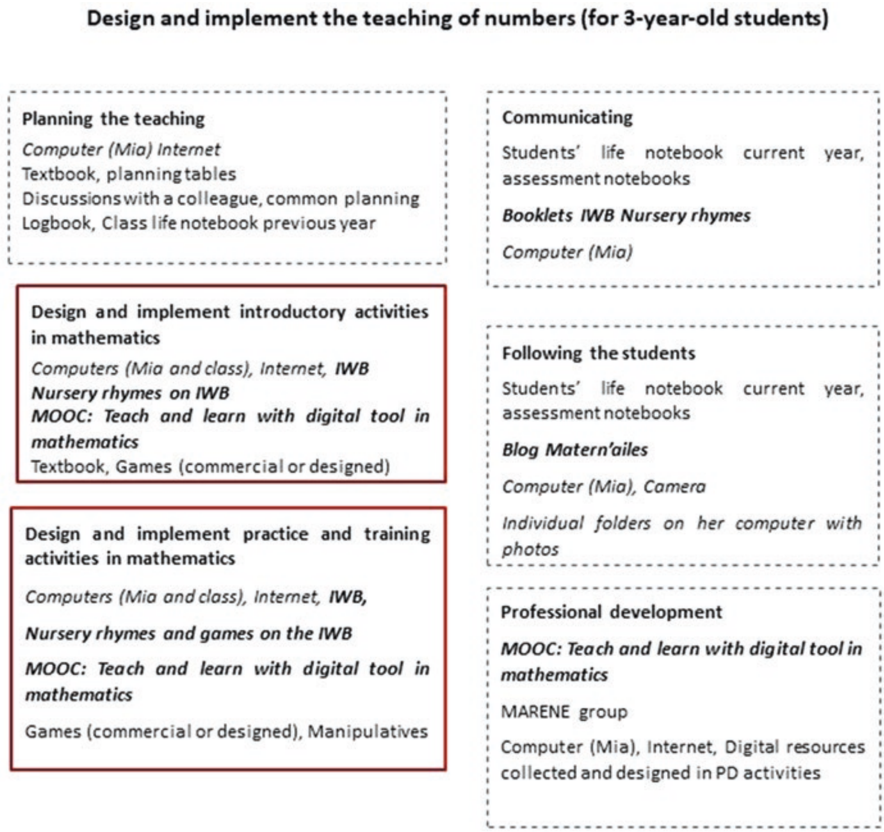


Fig. 6 Mia's resource system in Y3, as represented by the researcher. In italics, digital resources. Changes in the resources (compared with Y2) are in bold. In solid frames, activity families specific to mathematics. In dotted frames, general activity families

– *The MOOC project, another new pivotal resource:* In Y1 and Y2, working around the situation of CG, Mia was challenged by the question of the link between tangible material and digital resources. So she proposed designing a mathematical project articulating tangible material and digital resources for her 3-year-old students. The MOOC project led to the design of a lesson with three sessions. Session 1 was a work on a rhyme ('The three magpies') to be learned by the students. Session 2 was a guided workshop based on the use of a digital version of the rhyme on the IWB (Fig. 7). This digital version was designed by Mia for the MOOC project at the beginning of year 3. Session 3 was a workshop around a situation from the textbook, the 'Bears' birthday cake'. In the workshop the students worked firstly with tangible material (birthday candles) then on a digital version of the 'Bears' birthday cake' situation on the IWB.

Fig. 7 Digital version of "The magpies rhyme"

After this first experience in the context of the MOOC, Mia designed digital versions of other mathematical rhymes on the IWB.

We note between Y2 and Y3 an increase in the frequency of use and importance of digital resources in Mia's resource system. These changes are linked with the presence of the IWB in her class, but they are also supported by Mia's operational invariants, which we discuss in more detail in the section that follows.

6 Discussion and Conclusion

The study presented in this chapter only concerns the case of Mia, and this naturally limits the conclusions we can draw. Mia was an expert teacher; she worked within the MARENE group, so her use of the CG resources does not inform us about the integration of this software by less expert teachers, not involved in its design. Moreover she worked in France, where kindergarten is considered part of primary school with its own mathematics curriculum. This institutional and cultural context certainly influenced our observations.

Nevertheless our analyses provide elements of answers to the research questions, and also suggest hypotheses about kindergarten teachers beyond this single case. Our first research question concerned the micro-level of a document:

1. *What are the characteristics of a document developed by a kindergarten teacher, using digital resources for her teaching of numbers?*

In the case of Mia, we followed the development of a document for the aim 'Design and implement practice and training activities around the CG situation'. We observed that her previous knowledge of the general aim 'Design and implement practice and training activities' influenced the document developed, in an instrumentalisation process. She considered verbalisation very important for learning, while it is simultaneously difficult for these young children. Thus in year 1 she strongly supported pairs of students working on the software. These instrumentalisation processes were associated with instrumentation processes, within Mia's documentational genesis. She noticed indeed during year 1 that it was difficult for the pupils to handle the software and investigate the CG situation at the same time, and that her presence could hinder the search for solutions. Thus in year 2 she decided to introduce the CG situation with the tangible material, and to associate the material and the software. She also decided to conduct a synthesis and then to organise a new phase of practice and training, thus reinforcing the links between tangible material and CG software.

We claim that this analysis of Mia's case provides general insights about the integration by kindergarten teachers of specific software in mathematics. In year 1, Mia had already created links between the software and the work without computers, with stickers for the synthesis that reproduced the software. During year 1 she also developed further knowledge about associating the software and the tangible material. Previous research (e.g., Maschietto & Soury-Lavergne, 2013) has evidenced the potential of a duo of material and digital artefacts for the learning of numbers. At the kindergarten level, the tangible material is especially important, but combining it productively with the use of software is complex. Mia developed new knowledge about this issue through her documentation work during Year 1.

The CG software and its associated resources can contribute to kindergarten teachers' professional development if they are appropriated by these teachers (Trgalová & Rousson, 2017), which means, for us, integrated into their resource systems. The MARENE project's research results concerning the association of virtual and tangible materials and the importance of verbalisation were the common thread of a 3-h training for all kindergarten teachers in Brittany (approximately 1200) on the subject 'Playing to learn numbers'. Moreover, following the design of the CG software and its first implementations, the MARENE group created a hybrid professional development course (4.5 h of remote training, including implementation time, plus 4.5 h of face-to-face training) entitled 'Mathematical Games for Kindergarten: Materials and Software for Number Sense'. One of the challenges of this training is to get teachers to think about the integration of new technologies in relation to manipulatives for their mathematics teaching. This course is part of a national catalogue of 95 hybrid training courses for kindergarten teachers. It has been implemented 52 times since 2016 and there are 20 sessions currently open.[3]

Research question 2 was:

2. *What is the place of digital resources in a kindergarten teacher's resource system, and which evolutions of this resource system are linked with the use of digital resources?*

Mia's computer was, from the beginning, a pivotal resource in her system and the same holds for the internet. This corresponds to a use of digital resources for her preparation work, but not in class. We note nevertheless that she used digital means for communications with the parents, a very important aspect with students of this

[3] https://magistere.education.fr/local/magistere_offers/index.php?v=course#offer=84 (accessed 30 March 2021).

age. Moreover she had already experienced the use of the mathematical software GCompris in class with her students. We contend that the presence of this software in her resource system facilitated the integration of the CG software, which was integrated in year 1 exactly in the same activity families ('introduction' and 'practice'). During the second year the CG software was not used for introduction. It was replaced by an introduction to the situation using tangible material; moreover, the software and the tangible material were associated for practice and training as discussed above.

In year 3, we observed significant changes in Mia's resource system, primarily because of the presence of an IWB in her classroom in her new school. We want to foreground here that these changes are not discontinuities, in particular because her professional knowledge connects her practice in year 3 with her practice in years 1 and 2 (and the same holds for her resource system).

Using the IWB and in the context of a MOOC, Mia designed (with a colleague and a researcher) a project where students worked on mathematical nursery rhymes with the IWB, and simultaneously with tangible material. This practice was new and linked with an evolution in her resource system. Nevertheless, Mia's choices evidence the intervention of her professional knowledge, developed in years 1 and 2, about the association of tangible material and digital resources. Moreover the choice of nursery rhymes is also linked with other professional knowledge, already observed in year 1: the importance of verbalisation for the learning of mathematics for these young students. We also consider that the design, with the IWB, of booklets for communication with the families evidences a continuous evolution between the resource system in years 1–2 and the resource system in year 3. Indeed the communication with the families was always important for Mia; she used the IWB to develop it further (in particular about mathematical classroom activities).

What do we learn from the case of Mia concerning the resource systems of kindergarten teachers for their teaching of numbers and the role of digital resources in these systems? The appropriation of the IWB by Mia was very quick. IWBs afford a common visual focus for a group of students, and this can be useful, especially at the kindergarten level (Carlsen et al., 2016). Nevertheless, Mia did not only present ready-made animated pictures: she also designed illustrated nursery rhymes with the Openboard software. We note that the integration of new digital resources (here the CG software, then the IWB) enriches the teacher's resource system; as stated in previous work, this is linked with the development of the teacher's *design capacity* (Pepin et al., 2017). The digital resources create new opportunities for teacher design, and at the same time require new expertise which can be developed through collective documentation work.

Beyond these answers to our research questions concerning the case of Mia and the integration of digital resources by kindergarten teachers, we want to emphasise some contributions of this chapter in terms of both theory and methodology.

Mia was followed for three years; this long period gave us access to her documentational geneses and to her resource system. An increasing number of studies using the DAD consider the work of teachers over several years (see e.g., Trouche et al., 2020b). This chapter confirms the value of such studies to understand the evolutions (and the stabilities) of teachers' resource systems.

We developed the analyses at the micro-level of a document and at the macro-level of the resource system simultaneously. Analyses at the micro-level evidence precise professional knowledge developed along the use of a particular resource or that which influences this use. Analyses at the macro-level permitted us to follow how the teacher's knowledge can lead to associate several resources, and how the introduction of a new resource in this system can impact the role of other resources. Such analyses also evidenced the importance of the pivotal resources, which we have shown to be very influential in terms of the teacher's activity. When digital resources become pivotal, the teacher's resource system is transformed (Trouche et al., 2020b).

Further research is needed about teachers' resource systems, their structure, and the role of pivotal resources in teachers' professional development (in particular, in the development of their design capacity). The 2020 Covid-19 pandemic and associated lockdowns have disrupted the work of kindergarten teachers in many countries and opened the path to new roles for digital resources in their resource systems. Thus this direction of research is crucial to inform the design and implementation of relevant professional development programmes for kindergarten teachers.

Ethics Statement The teacher Mia was a member and active participant of the research project; she gave written consent for the use of all the data collected for research purposes, including publication and communication. The parents of the children in Mia's classes for the three years of data collection gave a written consent for the classroom observation and the videos, and for the use of all the data collected for research purposes including publication and communication.

References

Adler, J. (2000). Conceptualising resources as a theme for teacher education. *Journal of Mathematics Teacher Education, 3*, 205–224.

Baccaglini-Frank, A., & Maracci, M. (2015). Multi-touch technology and preschoolers' development of number-sense. *Digital Experiences in Mathematics Education, 1*, 7–27.

Baccaglini-Frank, A., Carotenuto, G., & Sinclair, N. (2020). Eliciting preschoolers' number abilities using open, multi-touch environments. *ZDM, 52*(4), 779–791.

Besnier, S. (2016). *Le travail documentaire des professeurs à l'épreuve des ressources technologiques: le cas de l'enseignement du nombre à l'école maternelle [Thèse de doctorat]*. Université de Bretagne Occidentale.

Besnier, S. (2019). Travail documentaire des professeurs et ressources technologiques: le cas de l'enseignement du nombre à l'école maternelle. *Education & Didactique, 13*(2), 119–153.

Besnier, S., & Gueudet, G. (2016). Usages de ressources numériques pour l'enseignement des mathématiques en maternelle: orchestrations et documents. *Perspectivas em Educação Matemática, 9*(21), 978–1003. http://seer.ufms.br/index.php/pedmat/article/view/2215/2279

Bullock, E. P., Shumway, J. F., Watts, C. M., & Moyer-Packenham, P. S. (2017). Affordance access matters: Preschool children's learning progressions while interacting with touch-screen mathematics apps. *Technology, Knowledge and Learning, 22*, 485–511.

Byers, P., & Hadley, J. (2013). Traditional and novel modes of activity in touch screen math apps. In J. P. Hourcade, N. Sawhney, & E. Reardon (Eds.), *Proceedings of the 12th international conference on interaction design and children*. ACM.

Carlsen, M., Erfjord, I., Hunderland, P. S., & Monaghan, J. (2016). Kindergarten teachers' orchestration of mathematical activities afforded by technology: Agency and mediation. *Educational Studies in Mathematics, 93*, 1–17.

Charnay, R., Bouculat, N., Colomb, J., Drouaire, J., & Guillaume, J.-C. (2005). *ERMEL, apprentissages numériques et résolution de problèmes: grande section*. Hatier.

Coles, A., & Sinclair, N. (2017). Re-thinking place value: From metaphor to metonomy. *For the Learning of Mathematics, 37*(1), 3–8.

Drijvers, P. (2012). Teachers transforming resources into orchestrations. In G. Gueudet, B. Pepin, & L. Trouche (Eds.), *From text to 'lived' resources: Mathematics curriculum materials and teacher development* (pp. 265–281). Springer.

Ginsburg, H. P. (2016). *Interactive mathematics books and their friends*. 13th International Congress on Mathematical Education, Hamburg, 24–31 July 2016.

Gueudet, G. (2019). Studying teachers' documentation work: Emergence of a theoretical approach. In L. Trouche, G. Gueudet, & B. Pepin (Eds.), *The "resource" approach to mathematics education* (pp. 17–42). Springer.

Gueudet, G., Pepin, B., & Trouche, L. (Eds.). (2012). *From textbooks to 'lived' resources: Mathematics curriculum materials and teacher documentation*. Springer.

Gueudet, G., Bueno-Ravel, L., & Poisard, C. (2014). Teaching mathematics with technology at kindergarten: Resources and orchestrations. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era* (pp. 213–240). Springer.

Guin, D., Ruthven, K., & Trouche, L. (Eds.). (2005). *The didactical challenge of symbolic calculators: Turning a computational device into a mathematical instrument*. Springer.

Hundeland, P. S., Erfjord, I., & Carlsen, M. (2013). Use of digital tools in mathematical learning activities in the kindergarten: Teachers' approaches. In B. Ubuz et al. (Eds.), *Proceedings of the eighth congress of European research in mathematics education* (pp. 2108–2117). Middle East Technical University.

Jung, E., Brown, E., & Karp, K. (2014). Role of teacher characteristics and school resources in early mathematics learning. *Learning Environments Research, 17*(2), 209–228.

Lange, T., & Meaney, T. (2013). iPads and mathematical play: A new kind of sandpit for young children? In *Eight Congress of European Research in Mathematics Education*. Retrieved from http://cerme8.metu.edu.tr/wgpapers/WG13/WG13_Lange_Meaney%20.pdf

Maschietto, M., & Soury-Lavergne, S. (2013). Designing a duo of material and digital artefacts: The pascaline and Cabri Elem e-books in primary school mathematics. *ZDM, 45*(7), 959–971.

Panero, M., Aldon, G., Trgalova, J., & Trouche, L. (2017). Analysing MOOCs in terms of teacher collaboration potential and issues: The French experience. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the tenth congress of the European mathematical society for research in mathematics education* (pp. 2446–2453). DCU Institute of Education and ERME.

Pepin, B., Gueudet, G., & Trouche, L. (2017). Refining teacher design capacity: Mathematics teachers' interactions with digital curriculum resources. *ZDM, The International Journal on Mathematics Education, 49*(5), 799–812.

Rabardel, P. (2002/1995). *People and technology, a cognitive approach to comtemporary instruments/Les hommes et les technologies: Approche cognitive des instruments contemporains*. Armand Colin. Retreived on http://ergoserv.psy.univ-paris8.fr/Site/default.asp?Act_group=1

Rabardel, P., & Bourmaud, G. (2003). From computer to instrument system: A developmental perspective. In P. Rabardel & Y. Waern (eds.), Special Issue "From Computer Artefact to Mediated Activity", Part 1: Organizational Issues, Interacting With Computers 15(5), 665–691.

Sinclair, N. (2018). Time, immersion and articulation: Digital technology for early childhood mathematics. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, & C. Benz (Eds.), *Contemporary research and perspectives on early childhood mathematics education* (pp. 205–221). Springer Nature.

Sinclair, N., & Heyd-Metzuyanim, E. (2014). Learning number with *touch counts*: The role of emotions and the body in mathematical communication. *Technology, Knowledge and Learning, 19*, 81–99.

Sinclair, N., & Pimm, D. (2015). Mathematics using multiple senses: Developing finger gnosis with three- and four-year-olds in an era of multi-touch technologies. *Asia-Pacific Journal of Research in Early Childhood Education, 9*(3), 99–110.

Trgalová, J., & Rousson, L. (2017). Model of appropriation of a curricular resource: A case of a digital game for the teaching of enumeration skills in kindergarten. *ZDM, 49*(5), 769–784.

Trouche, L., Gueudet, G., & Pepin, B. (2020a). Documentational approach to didactics. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (2nd ed., pp. 239–247). Springer. [An updated version and translations in different languages in the contetxt on the DAD-Multilingual project is available at: https://hal.archives-ouvertes.fr/DAD-MULTILINGUAL/]

Trouche, L., Rocha, K., Gueudet, G., & Pepin, B. (2020b). Transition to digital resources as a critical process in teachers' trajectories: The case of Anna's documentation work. *ZDM, 52*(7), 1243–1257.

Vergnaud, G. (1998). Toward a cognitive theory of practice. In A. Sierpinska & J. Kilpatrick (Eds.), *Mathematics education as a research domain: A search for identity* (pp. 227–241). Kluwer.

Vygotsky, L. S. (1978). *Thought and language*. MIT Press. (Original work published 1934).

Zaranis, N. (2017). Does the use of information and communication technology through the use of realistic mathematics education help kindergarten students to enhance their effectiveness in addition and subtraction? *Preschool and Primary Education, 5*(1), 46–62.

Zaranis, N., Kalogiannakis, M., & Papadakis, S. (2013). Using mobile devices for teaching realistic mathematics in kindergarten education. *Creative Education, 4*(7A1), 1–10.

Analysis of Primary School Teachers' Roles in the Dynamics of Mathematics Lessons That Integrate Technology Resources in Challenging Socio-economic Contexts

Ivonne Sandoval and María Trigueros

Abstract This chapter reports research on teachers using technology to teach mathematics in schools located in low socioeconomic neighborhoods in Mexico. The aim of the study is to research how aspects of the roles of two teachers using technology in their lessons supports the development of rich environments promoting students' demanding mathematical activity and learning. We focus on how their actions and use of digital resources contribute to the creation of a classroom culture that allows these students to work with important mathematical ideas. Using an enactivist approach, the teachers' lessons were examined to characterise their actions, and the resulting students' activity, by focusing on how students living in unfavorable contexts can be motivated to fully participate in lessons where technology is used. Our results highlight how teachers can develop immersive environments to help students who usually lack motivation to develop their mathematical thinking and learn by taking advantage of available technological resources.

Keywords Digital resources · Role of the teacher · Under-resourced schools · Elementary school · Enactivism

I. Sandoval
Universidad Pedagógica Nacional, Cd. de México, Mexico

M. Trigueros (✉)
Instituto Tecnológico Autónomo de México, México City, Mexico
e-mail: trigue@itam.mx

A. Clark-Wilson et al. (eds.), *The Mathematics Teacher in the Digital Era*, Mathematics Education in the Digital Era 16,
https://doi.org/10.1007/978-3-031-05254-5_9

1 Introduction

Research on how teachers' uses of technology in their classrooms can create an environment where it is possible to promote students' learning is important. This outcome depends on teachers' planning, the selection of digital resources and their abilities to adapt their roles according to students' needs throughout their lessons. This is more important when they teach in schools that are located in low socioeconomic neighborhoods, as it is well known that the context where learning takes place affects students' learning outcomes. Such research can offer clues about students' opportunities to learn, about teachers' professional development needs and inform the direction for future teachers' training.

We are particularly interested in studying how teachers work with students in unfavorable socioeconomic conditions, where advanced technology is not widely available and students do not have access to technology at home. There are many underserved schools in under resourced countries, such as México, where teaching conditions are difficult, and teachers have specific needs due to limited access to technological resources and the particular needs of their students. Knowledge of teachers' roles in these conditions can be valuable, particularly with respect to those teachers who have overcome school circumstances and, by using technology in their lessons, created environments where students learn productively. In this chapter we intend to contribute to the literature by analysing the aspects of such teachers' roles that can be related to their creation of rich learning environments in underserved schools. Results obtained can help to gain understanding about how teachers use technological resources to introduce these students to key mathematical ideas and to better understand the constraints that teachers face.

2 Literature Review and Background

Many studies have shown that socioeconomic conditions at home and at the environment surrounding schools have a strong influence on students' achievement (i.e., Ferguson et al., 2007). In Mexico there is a large number of schools where students can be considered at risk of dropping out because of the prevailing conditions in the school neighborhood, in their families and in the school itself. Mexico's government has developed a marginalisation index, a multidimensional measure that includes socioeconomic indicators that imply exclusion from school: illiteracy; housing without basic services such as water, electricity, drainage; overcrowding and household income lower that ten U.S. dollars (Conapo, 2013). In those underserved schools, even where technology is available, the equipment and software are old. They were distributed as part of a former large-scale national project (170,000 classrooms) "Enciclomedia", which was created in 2004 with the intention of complementing already existing materials in primary school classrooms—such as the mandatory textbooks—with digital resources (SEP, 2004). The project aimed to support teaching and learning of all subjects in grades 5 and 6 in the elementary school through the use of one computer and an interactive whiteboard. Although this project was abandoned in 2008 for political reasons, many teachers around the country continue to use its interactive software, particularly in mathematics teaching, for its support in motivating students to engage in mathematical problems through games and interactive activities and by providing them with interesting learning contexts (Trigueros et al., 2014, 2020).

Students attending underserved schools are prone to drop out and most of them show a low attainment in standardised tests (Solís, 2010; Tapia & Valenti, 2016). This was confirmed by the 2018 national exams results where 90.5% of students attending highly-marginalised schools obtained a low grade in mathematics, in contrast with 77.9% of those attending mid-marginalised schools (INEE, 2018). There are, however, studies where authors claim that schools can help students overcome those difficulties (Mc Neil et al., 2009; UNESCO, 2017). Academic goals, teachers' experience, and care for students together with resources available, institutional climate and students' sense of belonging are among aspects that can play a role in retaining students and in overturning their performance (Schmalenbach, 2017; Treviño Villarreal et al., 2019). Other studies about social mobility (i.e., Adelantado, 2000) assert that education is the most important factor predicting students' future success.

In studies related to mathematics achievement, researchers have examined its relation to students' socioeconomic status. Findings with respect to mathematics teachers' behaviours underline the importance of the creation of conditions where students can enjoy mathematics and develop a good relationship with it as a discipline (Lerman, 2000). Other studies point out the importance of promoting students' autonomy by using motivating tasks as a factor to change undeserved students' knowledge (Jorgensen et al., 2014). A respectful and "equitable" relation has been found in other studies to contribute in the creation of environments favourable to mathematics learning (Boaler & Staples, 2008), while other researchers emphasise the role of communication and the importance of students' opinions being valued by their teacher and peers (Civil & Planas, 2004) and the importance of students' reasoning in problem solving and respectful relations between teacher, students and parents (Civil, 2006). Regarding the use of technologies in mathematics teaching at elementary school in México, it has been found that even though teachers' access to technology is limited and long-term projects to make them available face many difficulties to be sustained (Sacristán & Rojano, 2009), some teachers working in underserved schools continue to use technology stimulated by observed improvement in students' achievements (Sacristán, 2017).

For many years, studies have shown that the use of digital resources that address higher order thinking skills and introduce students to powerful mathematical ideas can improve students' mathematical learning (e.g., Papert, 1980; Noss & Hoyles, 1996). These resources change the learning contexts, the communication and the interactions among the actors involved (Niess, 2005; Drijvers et al., 2018). However, access to digital resources is not enough to improve teaching, their learning potential depends on teachers' didactic planning, selection of digital resources and on their actions in the classroom. Moreover, the possibility to foster students' and teachers' learning is closely linked to activities contributing to create rich environments where participants are engaged (Hoyles, 2018; Ruthven, 2018; Trigueros et al., 2020).

Recently, studies on teachers' use of digital resources in mathematics lessons conclude that there is a growing need to develop learning opportunities for teachers to enable them to encounter new innovations, and how they can be effectively integrated in their mathematics lessons (Urbina & Poly, 2017; Ruthven, 2018). Studies focusing on mathematics classrooms point out the diversity of uses those teachers can create for technology (Goos, 2010; Trigueros & Lozano, 2012; Trigueros, Lozano and Sandoval, 2014; Urbina & Poly, 2017; Loong & Herbert, 2018) and the variety of possible interrelations between mathematical, pedagogical, and technological knowledge (e.g., Niess, 2005; Urbina & Poly, 2017).

Literature on mathematics teaching and learning with technology in elementary schools has increased in the past decade (e.g., Sinclair & Yerushalmy, 2016; Sinclair & Baccaglini-Frank, 2016; Spiteri & Chang-Rundgren, 2020). Most studies focus on students' learning (e.g., Moyer-Packenham et al., 2018) and there is still a need for studies addressing its potential in everyday teaching (Drijvers et al., 2018). Research results on elementary school teachers use of technology in their lessons concur that digital resources are not always used to promote deep mathematical thinking and that doing so requires specialised mathematics knowledge, alongside the selection and innovative uses of specific technologies (Goos, 2010). Some researchers point out that the rapid changes of platforms and the creation and deletion of digital resources, together with the growing supply of interactive mobile technologies, increases the need for elementary school teachers to have clear selection criteria to enable them to choose appropriate technology to reach the program goals and foster students' learning (Cuban, 2001; Pierce & Stacey, 2013; Larkin, 2015).

Research on the practices of successful mathematics teachers working in underserved elementary school students in technology enriched classrooms has shown that relationships, verbal interactions and trust are central for students to attain self-esteem and mathematical achievement (Page, 2002). Other studies show that these students' mathematical learning, improved engagement, self-efficacy, attitude toward school, and skill development benefit from interaction with technology particularly when teachers provide a supportive environment in which higher order thinking skills are developed, discovery is promoted and opportunities for interaction among them are provided, (Zielezinski & Darling-Hammond, 2016). The study of the elementary school classroom conditions that influence teachers' opportunities to interact both with students and with technology still needs attention, even more so in the case of teachers working in underserved schools. This study intends to make a contribution by addressing this research gap. In this chapter, we analyse how aspects of elementary school teachers' roles using digital resources make it possible to promote students' mathematical activity. We focus on how the use they make of digital resources and their actions contribute to creating a culture in the classrooms that introduces students in socioeconomically disadvantaged contexts to important mathematical ideas.

3 Theoretical Framework

Enactivist theory developed by Maturana and Varela (1992) considers the body, mind and world as inseparable. It focuses on the importance of embodiment and action to cognition. Consequently, it takes into account how individuals change and make sense of new experiences and challenges during interactions between themselves and with their environment. Learning is considered as the production of meaning through interaction with the world and past personal experiences (Lozano, 2015).

Enactivism backs up our ideas about learning and acting in the classroom. It considers knowing as *effective action*, which refers to actions which allow an individual to continue existing in a given context. To act effectively in a given environment means performing actions that are acceptable in that environment; in our perspective, effective actions mean that teachers and students behave in a way consistent with interactions between them in a context where technology is used. Effective behavior includes immediate coping (Varela, 1999) which consists of acting when a situation needs a reaction, in our case, the teacher acts even though he or she does not have time to think; these actions only come to mind when the person reflects about it later.

Enactivism proposes teaching as a collaborative process and considers learning as noticing particular features through individuals' actions, thus any given learning situation must encompass the teacher, the student, the content and the context in order for interactions to take place (Davis, 1996). From an enactivist perspective, the use of technologies is part of human experiences, since they comprise human practices and their cultural experience (Davis et al., 2000). When digital resources are used in the classroom, they become part of the environment as a tool which shapes teachers and student actions and promotes interaction among teachers, learners, and resources. The relations that emerge from these interactions are dynamic and complex. In an environment where digital resources are present, learning occurs through a dynamic interaction. The technology generates a context where actions can take place and a set of restrictions on possible actions. When technology is used in the classroom, teachers as learners, according to their history, can modify their actions. Actions undertaken by teachers as learners define their different roles in the classroom and their use of the technological tools (Trigueros et al., 2014) (Table 1). In our study, we look at these actions in detail to find out how the teachers' actions throughout the lessons (involving their use of technology as described in Hughes (2005)) enable them to

Table 1 Aspects of the role of teachers who use technology

Role in terms of	Description
Communication of mathematics	Technology might influence the teachers' role regarding mathematics by providing complementary information which teachers and students can comment on and work with. Effective behaviors include several forms of interaction with the mathematical content included in the programs.
Interaction with students	The inclusion of technology can influence the way in which the teacher regulates interactions by presenting unexpected situations that might have not occurred without the use of particular programs. Effective behaviors might include students' exploration, the use of the program and discussing the mathematics that arises while using technology.
Validation of mathematical knowledge	Technology may give feedback to students. Teachers might discuss answers with them before the program validates them or might allow students to use the program as a means for validating their answers. Teachers' actions might include encouraging students to solve those problems, even when unexpected uses of technology appear.
The source of mathematical problems	Technological devices lead to mathematical problems that had not been addressed before and that might not be included in the lesson plan or in the curriculum. Effective actions might include addressing these problems, taking into account students' needs.
Actions and autonomy of students	Actions on mathematical objects and tools can be carried out both by teachers and students. Technology may change the dynamics in the classroom. Teachers' actions might include allowing the students to work with the program and the mathematical problems and to explore with it.

Trigueros et al. (2014, pp. 114–115)

develop rich environments that promote underserved students' motivation and learning. We are aware that the different aspects of the teacher's role overlap and cannot be clearly differentiated and we use this classification only for the purpose of data analysis.

From an enactivist perspective, by considering all of these aspects together, we can have a clearer picture of how teachers integrate technology in their mathematics lesson. These aspects also inform us about teachers' work with groups of students who live in an unfavorable environment. Also, through teachers' actions with technology, we can appreciate how and when they immediately cope with students' needs to create rich opportunities for learning.

4 Research Questions

We are interested in gathering information to address the following questions:

- What aspects of the teachers' role promote underserved students' mathematical activity when digital resources are used in classrooms?

- To what extent do aspects of the teachers' role account for the emergence of specific mathematical activity, for example, visualisation, argumentation and the construction of relationships between concepts?
- How does the use of digital resources and the teacher's actions contribute to creating a culture in the classrooms that introduces students in socioeconomically disadvantaged contexts to important mathematical ideas?

5 Methodology

Five underserved schools in Mexico City and in cities located in surrounding states participated in this study. Teachers were selected to take part according the following criteria: they promoted students' mathematical activity in their lessons; they had experience using digital resources in their mathematics class, and their actions contributed to creating a culture of participation in the classrooms. All of the participants agreed to participate in the research through the National Pedagogical University research partnerships with those schools where our research was conducted. Fifteen teachers, who used technology while teaching at least two lessons related to a specific mathematical topic, were observed and their lessons video recorded. We then reviewed and analysed all of the video recordings and considered different teaching experiences, personal history and, from our perspective, if teachers created a rich environment for learning during their lessons.

We then selected as case studies two of those teachers (Carla and Yasmin[1]) who were additionally recognised by their peers or administrators as good teachers, who had different teaching experience with technology and who worked in schools with different marginalisation indices. We selected two lessons that addressed the same mathematical topic and where they also used digital resources, as representing their teaching behaviour and conducted a new more specific interview with each of them. The interviews aimed to gather further information about their teaching experience, their interest in the use of digital resources in their mathematics lessons at underserved schools and the criteria they used to select and use these resources in their class plans. Carla used *Cycle Track*, an Enciclomedia interactive environment designed to teach proportionality in a movement related context that had been developed to be used by teachers on the interactive whiteboard and one computer, Yasmin used *Lego-Logo Digital Designer (LDD)* software to teach geometrical and spatial thinking using the 14 computers available in her classroom.

[1] We use pseudonyms for both teachers and for all students.

5.1 The Teachers, Students and Schools

Carla had 20 years of teaching experience in a suburban elementary school in Cuernavaca. She frequently used digital resources in her classes. She liked participating in training workshops; one of them on the use of "Enciclomedia's resources" when this project started and others on the use of different programming software such as "GeoGebra" and "Scratch". Over a period of 4 years, she had met informally and regularly with other teachers to discuss and share experiences on using digital resources in their classrooms. She was concerned with her students' living conditions and tried to create motivating environments for them, to motivate their interest in attending school and participating in class. During the interview she said: "*Since my first use of technologies in the classroom, I noticed children liked it. There are always some children that are not interested in school … Some of them have problems at home or like it better to be with friends on the streets… but when I use technology, they are more interested and involved, … I feel they can think differently and learn"*. Carla taught a group of 33 fifth grade students (10–12 years old) who had some earlier experience with technology in previous school years.

Yasmin is a new teacher-researcher who had been teaching for 2 years in an urban elementary school in Mexico City. Her initial training was in mathematics teaching and it included the use of digital resources. She had been involved in mathematics education for 4 years and was interested in the design of activities to develop children's spatial reasoning abilities through the use of easily accessible resources. We selected her because she showed interest in learning from her experiences and wanted to use innovative projects directed to vulnerable populations that enable the integration of STEM disciplines. When we looked at her videos, we considered that she offered rich mathematics learning opportunities to her students while integrating digital resources in her classes. During the interview, Yasmin commented "*digital technology enables students living in these contexts to have access to valuable mathematical ideas and to use them to do mathematics with representations that provide a different experience*". Yasmin taught a group of 28 3rd grade students who had little experience in using technology such as Microsoft Word and Paint softwares.

Both schools are free and public, they are located on the outskirts of each city and both teachers work in the afternoon shift (2 pm to 6:30 pm). The school where Carla works serves a population of 700 students distributed in two shifts in a moderately marginalised neighborhood, while the school where Yasmin works serves a population of 1000 students also distributed in two shifts in a highly marginalised neighborhood.

5.2 Research Tools

In order to study teachers' actions when investigating their role in the classrooms, we analysed classroom observation notes, video-recordings of teachers' lessons and audio-recordings of the interviews. Observers took notes during each lesson and they registered important events in terms of teachers and students' actions and the use of technology (Hughes, 2005) during lessons. During the interviews, which were carried out as part of a larger project, teachers were asked about their background, their training and about how they worked with technology during their mathematics lessons. For this particular study attention was also paid to students' behaviours, their use of language, and the relations between participants during the lessons.

Each researcher analysed the information obtained from different sources using both enactivist theory and the codes in Table 1. Then, we discussed our independent findings and negotiated until we reached consensus on our interpretations, to guarantee the analysis' reliability.

6 Results

6.1 Carla's Case

Carla selected the Cycle track program because she liked to teach the relation between physics and mathematics and, in her experience, it was appropriate to teach proportionality. Most teachers in this school interacted with students during lessons, however, as Carla explained "*most of them focus on having children work on the textbook activities and verify if they are correct, and on explaining correct answers; some use technology, frequently Enciclomedia sometimes PowerPoint. I prefer to let students discuss and listen to their ideas...*". During the interview, she told us that she was kind but exigent with students but emphasised, from their first day, some behavior rules (respect, tolerance and participation) as "*indispensable for learning*". She also described her school context:

> conditions outside the school are rough, so the principal and all of us teachers try to create a different context inside the school where students learn to be responsible and tolerant, where all students are valued. This is how we help them not to drop out and to feel self-assured and value knowledge usefulness in their lives. In my case, I believe that students interactions and their being able to talk about their ideas knowing that they will be listened to is fundamental for developing self-confidence and thus their feeling that they can learn and begin to enjoy learning.

6.1.1 Proportionality and Movement with the Cycle Track Program

The Cycle track can be considered as an interactive tool that offers new infrastructures for teaching specific mathematical concepts (Hoyles, 2018). Its purpose is to introduce uniform movement as an example of a proportional relation. This program includes three movement representations that can be presented simultaneously, by pairs or individually (see Fig. 1). It dynamically simulates the movement of cyclists going from home to school at three different constant speed. It can show

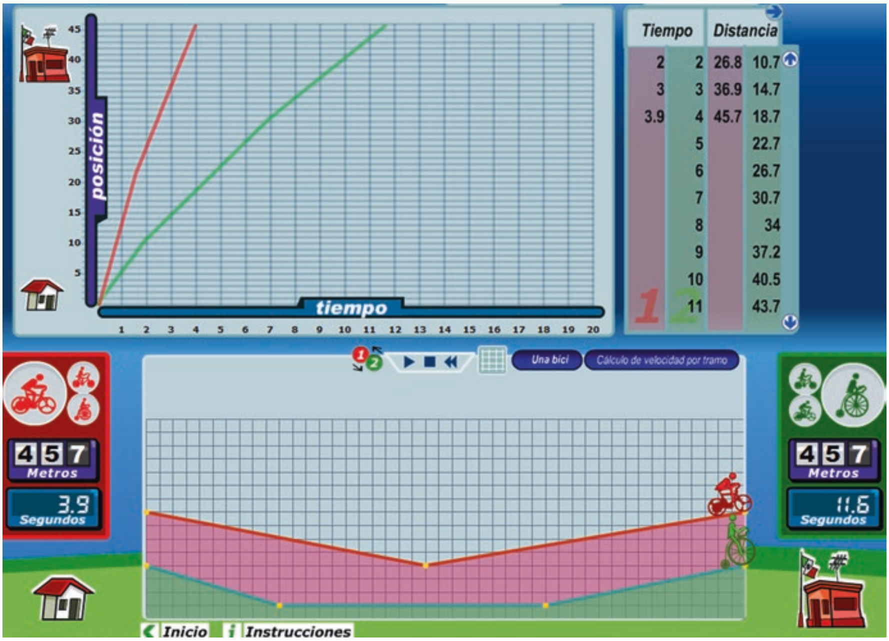


Fig. 1 The Cycle Track program

one or two cyclists moving simultaneously with different speeds on tracks that can be shaped in terms of their incline or by dividing them in different straight segments by touching the screen or electronic whiteboard (Trigueros et al., 2020).

6.1.2 Carla's Case: First Session

Carla was working on proportionality and, as she was also interested in science, chose constant speed motion as an interesting example connected to students' experience. She designed a two-session teaching plan where different possibilities offered by the resource were used.

Carla introduced students to proportionality during a previous lesson through team work on proportional and non-proportional examples using different representations, which she concluded by providing the definition. She used the Cycle track program in the subsequent two sessions.

During the first of these sessions, she asked students to discuss how they would describe a cyclist's movement. Important ideas related to the description of movement emerged and Carla listened to students' ideas and arguments and interacted with them to validate their proposals. Carla selected an activity related to students' experiences which is particularly important for underserved students. Through her actions the teacher motivated students' participation and focused their attention on what she would later do with the technology.

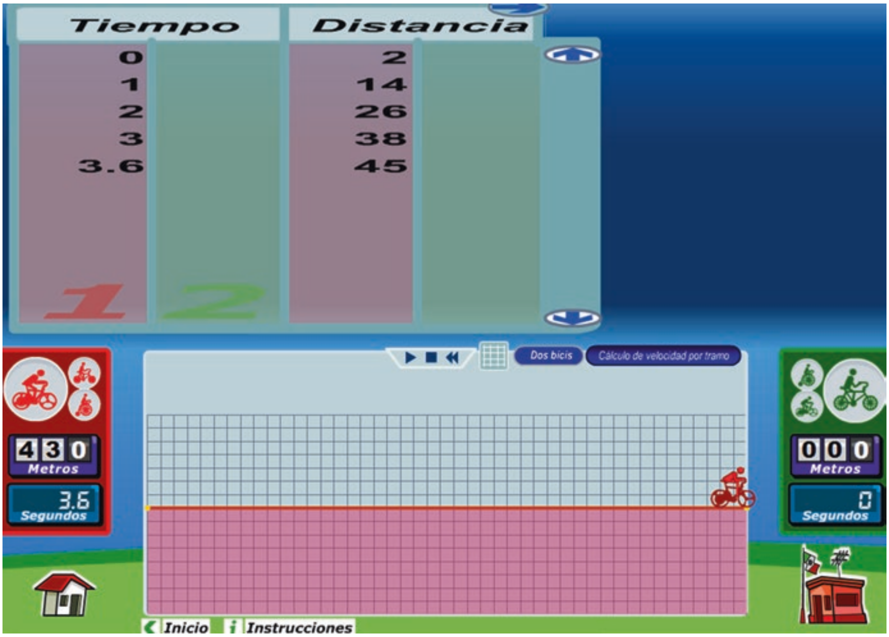


Fig. 2 Two representations of movement

Then, using the Cycle track program and the electronic whiteboard, she let them continue their discussion within this new context. As they had mentioned the relation between velocity and time, Carla showed the cyclist's movement together with its table, and asked "*What do you see?*" (Fig. 2). Students discussed using arguments based on their experience but when looking at the table they focused on it and assertions such as "*As he said. He goes all the time with the same velocity … well, if you think that each time a minute passes, he moves the same stretch, always, that is it goes even*" were proposed. Then Elena suggested. "*Teacher, the table is like what we saw before… that about being proportional*". Carla asked the class what they thought and students discussed the relation between the table and the concept of proportionality.

Carla listened and let students communicate the mathematics. Although they had not been introduced to the study of motion before, they used their experience to identify and relate the variables involved: time, distance and velocity. She promoted students' interactions and they were able not only to relate the two representations but also to recognise the relationship to proportionality.

Carla was the source of the mathematical problem but students communicated the mathematical ideas. They considered the variables involved and looked for a relation between them. Carla's actions, showing the cyclist movement on the animation, asking appropriate questions, listening to students' proposals, and letting them communicate the mathematics, promoted students' interactions and the use of their own language; she used technology as amplifier to help students focus on the two representations and the proportional relation. Carla thus created a rich environment where students' contributions were valued, which is not generally the case in the living context of these students.

Carla posed effective questions at appropriate moments. She continued the discussion by asking: *Can you say something about the graph of the movement?* Elena answered "*It will be inclined, a line*", other students added "*Yes, as the cyclist goes slowly, the velocity… the graph will grow slowly"* or *"if we use another bike, it will be more slanted. Let's check it!*". Carla agreed and suggested to use two cyclists on the Cycle track together with their tables and graphs. Students exclaimed: "*We were right!*" and Carla asked "*What does the slant of the line depend on*?" Students replied "*On the velocity!*" She closed this episode by formalising the proportional relation regarding movement.

Carla's actions created a rich environment where her students had autonomy to interact and offer interesting arguments, thus promoting learning. Students' answers supported the adequateness of her actions combining the use of technology (Hughes, 2005) and interaction with students to promote their reflection on the mathematics involved in the situation. Her effective actions involved insisting in students' predicting and justifying their ideas. Then, using technology as a transformational tool (Ibid.), she helped students explore, explain, and validate their thoughts. Her decision to use two cyclists as a response to students' proposals was not in her plan since "*I decided that would help assess their arguments*" (after class interview). Carla's action to show three different representations of movement, using technology as transformation, promoted a rich environment for students' reflection and argumentation, thus motivating them and fostering their learning about movement and proportionality.

Throughout the session Carla adapted her role and the use of resources from being the source of mathematical problems and communication to promoting students' interaction and communication of mathematical ideas and giving them autonomy to pose mathematical problems She adapted aspects of her role and her use of technology from amplifier to transformation (Ibid.) to create a respectful environment where these students could feel safe to speak out and be listened and where they could be confident to pose questions and share their mathematical ideas.

6.1.3 Carla's Case: Second Session

Carla showed students how the path could be changed on Cycle track. She selected a new path (Fig. 3) and asked: "*How do you expect the velocity to be?*" This dialogue followed:

> S[2]: *First, the velocity is small… then it will be larger.*
> Carla: *Let's see it with the table and graph.*
> E: *…The two lines in the graph go up! One should go down.*
> J: *Why? It is fine. He is always going to be farther and farther as the time passes… velocity changes, it is larger in the second part.*

[2] In the dialogue, the initials represent the different students involved.

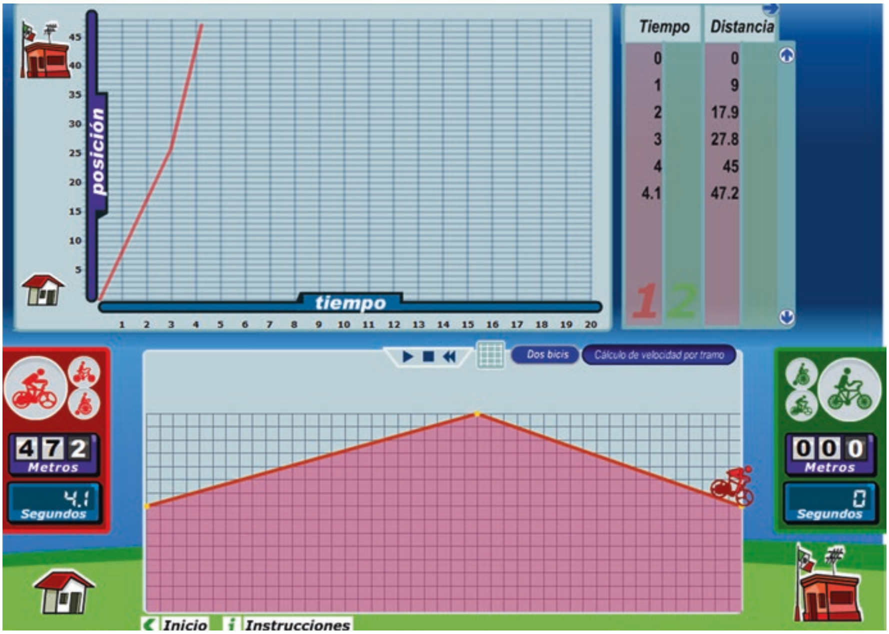


Fig. 3 A different path

L: *Yes, but when he goes up his velocity should be less and less, not the same as it seems in the table, then it would grow.*
S: *I am not sure, it should go down, because the path goes down.*
Carla: *Let's see it again but pay attention to the table this time.*
M: […] *the distance grows all the time, he is farther! The graph goes up.*
Carla: *What would a line going down in the graph mean?*
M: *He goes back to his house?*

Carla's actions involved the use of technology as amplifier and transformational tool and as the source of the problem to promote discussion to assess students' understanding. She gave students the autonomy to predict and to argue their ideas. Her immediate coping strategies when faced with different and conflicting student opinions was apparent, which included adapting her role and her use of technology to direct the students' attention to the information given in different representations and to let students compare their arguments. By opening a collaborative argumentation space, she promoted students' interactions and fostered their noticing of the important relations between the representations and to "see" the difference between trajectory and mathematical representations of movement. She thus opened their possibility to learn. Then, new mathematics arose. Ismael said: "*…the graph is… not proportional… the first little part is, but the other … it doesn't start from zero*" and Regina asked: "*Why if he is going up, the velocity is the same… if I go up … velocity diminishes… but when I go down it should grow*". Other students agreed. Carla's use of technology as amplifier helped a student to focus on the movement and its representations, technology became the source of the mathematical problems when they confronted different representations of movement and their own experience. Carla's role changed to cope with the situation, she communicated the mathematics. Her effective actions consisted in discussing with students the limitations of the technology used and referring to their experience to explain average velocity through an everyday example and as a way to simplify the problem.

Ismael insisted: "*It is not proportional! … that graph does not start at zero"*. Carla's immediate coping strategy included an explanation of how the graph could represent two consecutive movements in a way that was convincing for the student: "*If for the second movement you start counting the time from zero that graph would represent proportionality…. Here to describe the whole movement and compare …, the two graphs are together."* Ismael agreed.

Carla's effective actions were focused on her students' needs: Guiding their interactions and letting students listen to each other's ideas. She then became the source of a new related mathematical problem by illustrating the distance-time variation for an accelerated movement by constructing, together with students, a table and a graph. Students visualised, discussed and reasoned on differences between proportional and non-proportional relations. Also, resorting to students' everyday experience, she took responsibility for the communication of mathematics and explained the role of average velocity as a simplification of accelerated movement. Again, Carla's actions and the dynamics of aspects of her role fostered students' interactions and her willingness to listen to the students' opinions. Students felt free to express their concerns even if they were incorrect. Through her creation of a respectful environment her actions promoted students' trust in her, their own self-esteem and interactions, thus learning. Throughout this episode Carla's varying roles and alternating use of technology and other resources created a culture in the classroom where interactions among participants fostered knowledge exchange.

Carla then proposed a team-game for the students. They had to draw a piece-wise motion graph for a cyclist that comprised three parts and challenge other teams to draw the cyclist's trajectory. Students were given autonomy and became the source of new mathematical problems. Their actions involved discussing, justifying and arguing mathematical ideas. Carla invited them to use the technology to draw the proposed path and the corresponding distance-time graph to validate their mathematical thinking. Carla thus created a motivating environment where students were involved in doing mathematics and in interacting with the technology. For example, a team proposed a graph. Adri described: "*The cyclist moves quickly, this large distance, a short time. Then… it is horizontal… time passes but he is in the same place… maybe he stopped, and then continued slowly… the path would be something like this* (Fig. 4)".

Other students complained: "*It is not right! Going up like that, he would go slowly!"* or "*He moves in the second part*". Adri corrected: *"I was wrong. It does not stop; he moves very slowly and then a little quicker."* Adri introduced the idea of *"not moving"*, although it was not correct for this situation, she was able to describe it corresponding to a horizontal graph.

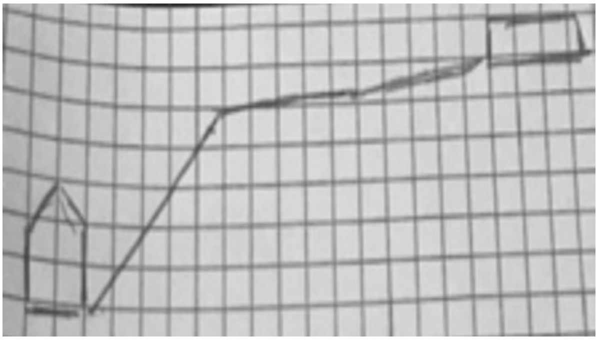


Fig. 4 A team's drawing

Carla's actions created a rich context where students were immersed in mathematical activity. When Adri seemed to confound the path with the distance-time graphs, other students were able to correct her mistake showing their understanding. Students were able to work on an inverse problem, that is, they reconstructed the path of the cyclist from the distance- time graph demonstrating they understood the relations between variables involved in movement. Students were able, in general, to draw the cyclist path on the interactive whiteboard, using technology as a transformational tool. They sometimes struggled when they wanted to show a very steep path or a very sharp angle between two parts of a piece-wise function. The program does not allow a trajectory where the cyclist would stop for a while. When students had problems, Carla's role changed from fostering students' exploration and argumentation to explaining the technological restrictions to the whole class. These restrictions made it possible for students to find new strategies to represent cyclists' trajectories, and the corresponding position-time graph and/or tables that were not allowed by the resource.

To summarise, through the two sessions, Carla's aspects of her roles and the use of technology created spaces for students' interaction where their arguments and contributions were valued. Her actions demonstrated she looked for opportunities for students to perceive similarities and differences in movements and noticing relations among the variables involved, thus promoting their learning. The exploration space she created when students could interact with the technology, allowed students' problem posing, communication and interaction. Students felt free to speak, to use mathematics to explain different problems and to validate their ideas: An encouraging environment which propitiated both learning and students' autonomy and confidence in their mathematical knowledge. Students gave evidence of being motivated and interested in participating. Again, Carla's actions, the aspects of her roles and use of technology contributed to develop underserved students' interest in learning and self-confidence in their possibility to work with and enjoy mathematics.

6.2 Yasmin's Case

6.2.1 Developing Spatial Reasoning and Collaborative Skills with Lego: 2D–3D Dimensional Change

Yasmin selected an accessible technology for 8-year-old students which enabled them to identify relations between 2D and 3D representations by showing different perspectives together with the possibility to manipulate 3D objects.

Lego Digital Designer (LDD) is free software to explore 3D objects' digital representations. It provides a set of digital modular virtual blocks with different shapes, sizes and functions that can be selected from a list. Students can initiate different actions such as inserting, rotating, moving, attaching or detaching the virtual bricks. With LDD, students can experiment with different representations to analyse, design, build simple assemblies, or follow step-by-step instructions to construct objects. LDD was considered a tool that "*offers connections between school mathematics and learners' agendas and culture*" (Hoyles, 2018, p. 3).

Yasmin previously designed constructions on LDD to be reproduced with Lego blocks by each student. She created an activity for two lessons where individual, team and whole group work was combined as the teams constructed a park containing all of the objects they had constructed.

In the previous session students had analysed 3D objects by looking at different views and the correspondence to their 2D representations. Yasmin organised the students into teams, and the children selected responsible peers for picking up and delivering materials for each lesson. The activity's goal was the interpretation of LDD instructions to construct objects with Lego blocks. Students were motivated and interested to play with Lego blocks, which were not available at their homes. This geometrical activity promoted actions to visualise, to interpret the representations of a solid figure's views and perspectives to support students' spatial reasoning. Students were asked to predict the result of rotations or other movements conducted using either the Lego or the virtual blocks before moving them. These activities developed both students' mathematical thinking and their collaborative work skills.

Due to the number of available computers, the group was divided into two subgroups. During the first lesson, one group worked with Lego blocks (Fig. 5) to create castles for the park while the other used them individually to assemble objects shown on LDD. For the second lesson, the subgroups interchanged activities. Students had not used this software before, so they performed actions to get acquainted with rotation, move forward and backward arrows for each construction move. The use of LDD changed the lesson dynamics by promoting students' autonomy to follow step by step instructions in constructing a given object and validating their results.

While using LDD to assemble their objects, students worked at their own pace and self-assessed their results. Yasmin's actions promoted the articulation between the physical and virtual Lego blocks by focusing students' attention on the figure's

Fig. 5 Classroom's activities

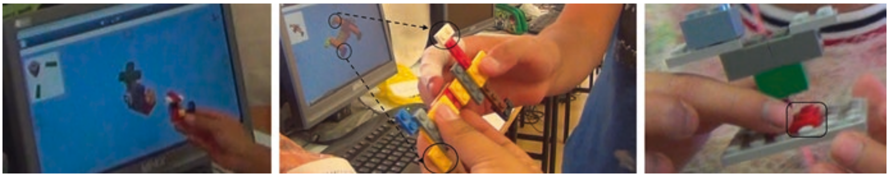


Fig. 6 Superposition of constructions and comparison using rotation and translation

characteristics instead of their size or colour. She encouraged comparison and visualisation of forms by using different views of the object to be constructed, particularly rotation which was a useful approach for a successful outcome. Through her actions, Yasmin strengthened and made connections between students' geometric knowledge on similarity and scaling, geometric transformations, relations between representations (2D-3D) and perspective views. Her actions included allowing the students to work and explore with the program alongside solving mathematical problems. Yasmin used the technology as an amplifier and as a transformational tool to enable students to assemble and disassemble a construction and to use the available tools, such as rotation and interpretation of representations.

The LDD task design favored students' autonomy to validate their construction. They compared actions shown on LDD to those made on Lego blocks, since LDD does not give feedback about the correctness of their process. Nevertheless, students' actions, such as attaching or detaching blocks, opened possibilities to contrast their results and helped students explore through interactive processes; something impossible to achieve with instructions on paper and pencil. Yasmin's roles during the lesson promoted students' autonomy to take decisions while constructing and correcting their mistakes. She created an environment where communication with her and other students to share procedures, pose questions and ask for help together with the construction of respectful and trust relations between students and with her were fostered and could be observed through the verbal interactions and collaborative work in teams.

Students' strategies to validate their constructions were to superimpose their construction onto LDD (Fig. 6) and to change and check if different views coincided. When they struggled, Yasmin invited them to focus on similarities and differences or asked them to reconstruct their process "*in their mind*". Her teacher's role was questioning: "*how did you do it?*", "*is this block the same as that you used?*" or "*can you explore again?*" For example, when discussing Samuel's mistake:

Y. Which block? Where should it go?
S. This. [Showing a new block and changing the construction].
Y. Isn't a block missing?
S. That's it!
Y. Are you sure? Check it. I think there is something missing. Is something in a different place? You can compare if you want (Fig. 6b).

Samuel compared and used instructions on LDD until he could solve the problem by detaching and attaching four blocks and exclaimed: *"I did it"*. In this process, visualisation and interpretation of 2D and 3D representations actions were fundamental.

The use of arrows and exploring each step at a time became verification tools for students. As Hoyles (2018) pointed out, the interplay between the dynamic and the static was a key factor as students could pause, reflect, go back and test through feedback from what was shown on the screen.

When students faced difficulties relating to their interpretation of symmetry (Fig. 6b, c) Yasmin alerted them: "*check the duck's legs", "the plane wings' directions are different than those in the construction", "look! count how many studs on LDD are free"*. During the interview Yasmin confirmed that she promoted children exploration to help them identify their mistakes and rely on their capability to solve them. Yasmin' actions included allowing students autonomy while working with the geometrical problems. She considered this strategy *"promoted students' confidence to compare their actions with others and to participate in classroom activities"*. Through her and her students' actions it could be observed that Yasmin valued the communication of mathematical ideas and students considered their opinions as valuable independently of their being or not correct. Students felt confident to share their difficulties and to show their accomplishments.

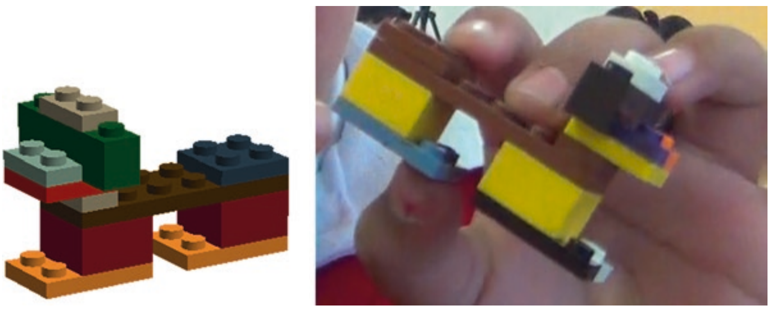


Fig. 7 Pedro's mistake

When Pedro had troubles related to the observer's position and symmetry, he shared with Yasmin the actions he was planning to do, and while doing this he reflected on the correct actions required (Fig. 7). This was an achievement for him. Pedro, as common to most of these students, did not participate much at the beginning of this experience. Initially, he did not recognise his mistakes and expected the teacher to tell him what to do, or if what he was doing was fine; now he demonstrated more confidence in his own ideas:

> *Y. What was your mistake?*
> *P. This* [showing the dogs' back legs]. *It was like this, and it should be backwards. And these little blocks* [referring to a plate de 2 × 3] *had only one stud line outside, while there should be two lines.*

This episode shows the importance of exploration with digital resources. Pedro identified the changes needed and explained them using geometrical terms: *"this was backwards"* (symmetrical). Actions on digital resources together with manipulation of blocks, helped students to develop abilities to construct, describe, and explain the resources' effects.

Unexpectedly, Ana and Juan found out they were missing blocks. They told Yasmin and she took advantage of this situation by asking them to find the missing block from those on a table. Students compared the blocks. Ana selected two similar blocks. Yasmin invited her to compare them with the LDD's construction.

> *Y. What does the block look like?*
> *A. This one. Because this one here [she had it in her hand] doesn't have this* [pointing to the screen].
> *Y. Just for that?*
> *A. No. Because this is larger* [referring to its thickness].

Ana identified the missing block and finished her construction, but she placed it in the opposite position (Fig. 6c). Yasmin said "*Look closely. Turn it around* [referring to rotating it to see other views]. *Look at it from above, it looks like an L. What can you do for it to look like this?* [referring to the LDD construction]." Ana noticed that her L was upside down (symmetrical), explored the digital construction and reconstructed it step-by-step with her blocks. At each step, Yasmin asked her reflection questions: *"which one is below here? which one is next? how do you know? Let's turn it a little bit, look how it should be"*.

Through her interaction, Yasmin promoted Ana's reflection on the process to be followed and helped her compare each result to its virtual representation. In this interaction, Yasmin used spatial location vocabulary, since, as she informed us during the interview, "*it is necessary in communication activities about location and in relation to blocks to use relations such as being on or, under, how many studs are left in a plate, to the right of, rotate it, observe the inferior view*". Asking "*how you identified which blocks should go there* [indicating with her finger the place]*? how did you move the block so that it is in the correct position?*" promoted students' reflection and consciousness of their actions. It was observed how Yasmin effectively responded to students' needs and created opportunities to learn geometric transformations. Several students demonstrated learning as they compared the constructed objects to those on LDD by discriminating symmetrical positions (Fig. 8a), the blocks' form, counting studs, and relations between them.

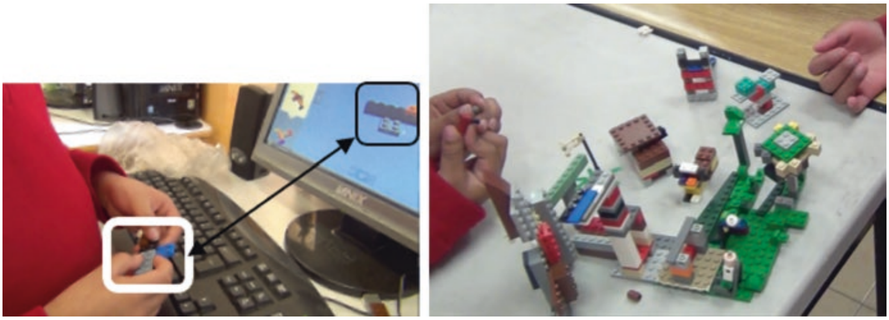


Fig. 8 (**a**) Identifying symmetrical figures, (**b**) Final construction

All students were able to replicate the LDD constructions and all teams assembled their parks (Fig. 8b). Yasmin created a motivating environment where students could imagine and visualise the forms, they needed to construct so that scale and proportion among components were maintained. While doing so, students learned to work collaboratively and to communicate their understanding and conceptualisation of spaces, objects and descriptions. LDD's use enhanced students' learning experience by providing meaningful hands-on activities, developing their construction description and explaining the effects of moving blocks around abilities to communicate mathematical ideas.

The teacher's effective actions promoted the construction of rich learning environments which fostered mathematics learning. Yasmin made appropriate decisions throughout the lesson adapting to the students' prior needs. She attentively listened and responded to students while they worked independently.

In the interview, she told us that students had difficulties to work in teams and they had little tolerance to work with disabled students, or with those who were not their friends. Students seldomly participated in class because of lack of self-confidence and were not responsible in using and taking care of class materials. Yasmin worked with them to create a new classroom culture based on respect and recognition of other's opinions where teamwork and individual work were both valued. She considered it a way to "*offer opportunities to develop social and emotional abilities so children learn and counteract the adverse environment in which they live*".

7 Discussion

On Aspects of the Role of the Teachers

During both teachers' lessons we observed that they adapted aspects of their role, which involved their use of technology and taking account of the mathematical and social needs of their students. They both continuously invited students to *predict* the results of their actions before using the technology to *validate* their proposals. Carla created a class atmosphere where students were immersed in mathematical situations related to movement. She stimulated students' participation and focus on the use of different representations through their analysis. She insisted on students' engagement with data, by seeking their *involvement*, their *autonomy*, the *communication* of mathematical ideas and, through interactions with the technology, she encouraged teamwork and students' interest in working with demanding mathematical situations to *validate* their assumptions. In the environment created by Carla's actions, the students' self- confidence and learning were stimulated. In Yasmin's case, working with real objects together with one-to-one technology enabled her to create an environment where *autonomy*, *participation* and *engagement* were favored. The problem, and the activities she created, stimulated students' actions that would not be possible without the technology and the blocks. For instance, her actions focused students' attention on important transformations, such as rotation, to obtain hidden information about the object, but the students had to explore further using the technology to accomplish their goal. Throughout the sessions, she stimulated students' *interactions* both with the technology and with each other. As a result, they became more open, showed trust in their teacher, were able to communicate their difficulties and their progress with both Yasmin and their peers.

In both cases the teachers' effective actions created rich environments where students felt free to explore, interact and discover higher order mathematical ideas, which promoted their learning. The students' consistent actions throughout the lessons contributed not only to their motivation and self-esteem, but to their development of trust in adults, teamwork abilities, responsibility, and mathematical communication skills. Moreover, Carla and Yasmin's effective actions also created supportive environments for their underserved students through practices that can be considered central for their mathematics learning and the development of higher order thinking skills (Page, 2002; Zielezinski & Darling-Hammond, 2016).

7.1 On the Emergence of Mathematical Activity

The digital technologies used by the teachers in this experience afforded different opportunities to use their knowledge and experience to create rich learning environments (Hoyles, 2018; Ruthven, 2018).

Both teachers' actions introduced their students to complex and age-relevant mathematical ideas: For Yasmin, spatial geometry, projection and transformations and for Carla, proportionality, the study of movement and the use of different representations (Noss and Hoyles, 1996). They used each technology's affordances by designing tasks that gave students opportunities to act and reflect upon their actions to make sense of the related variables—concerning the movements, in Carla's case, or in the search of appropriate transformations to assemble objects, in Yasmin's case. Even though both teachers' students were introduced to different mathematical topics and technological tools, Carla and Yasmin continuously used prediction to foster students' use of their everyday experiences and their imaginations to promote the emergence of new mathematical ideas.

It was observed that Carla created a space where technology was used as an amplifying or transformational tool such that students could use and contrast their experience through visualisation and reasoning about movement. The Cycle track technology offers a range of different motion possibilities that she used to offer her students opportunities to make sense and reason about how variables (and their relation) change in different situations and what remains constant.

Yasmin's students showed that they were able to use technology as an amplifier or transformer, as needed, to accomplish their goal. They also developed appropriate vocabulary to communicate position and orientation spatial relations between Lego blocks and LDD. Yasmin's effective actions helped students to consolidate the meaning of those terms needed to describe spatial relations, and to incorporate them in some of their procedures' descriptions or in their communication with others. In particular, assembling Lego blocks by following bidimensional representations helped students to "see" how to assemble the blocks and how to move forward or backwards and rotate their construction. These actions were relevant in recognising different views of the same object and in using them to assemble their 3D object, thus progressing towards identification of hidden elements in a construction.

These two teachers promoted a justification and argumentation space in their lessons where students' suggestions of new mathematical ideas were valued (Lerman, 2000; Civil, 2006). They always listened to their students when the students were working and guided them by using age-appropriate questioning. Together with the technology Yasmin and Carla promoted a respectful atmosphere (Civil & Planas, 2004) where identification of possible misunderstandings was possible. They also encouraged refutations to develop students' awareness of the validity of their arguments. By encouraging students' argumentation and discussion both teachers promoted the development of students' confidence in themselves, and in their mathematical thinking, which is uncommon in the contexts where they live. Carla, for example, fostered the emergence of the notion of the slope of the line while discussing the graphical representation of different movements. Yasmin' actions to insist on students responding to questions such as "*why?*" or "*what do you need to do?*" promoted students' use of rotations and of different views with the technology hence developing awareness of the relation between objects on the screen and those in their hands. Both teachers' effective actions contributed to developing an enjoyable environment and a rich mathematical context where, according to enactivism, they and students can learn.

7.2 Creation of a Classroom Culture That Introduces Students in Socio-Economically Disadvantaged Context to Important Mathematical Ideas

The Cycle track technology is a unique simulator in that it offers students the possibility to explore the variables involved in constant speed movement. Students in this school do not have technology at home and it is difficult for them to pay to play digital games in commercial establishments. Carla's selection of the Cycle track task, the dynamics of her role and her own, and students' uses of technology provided important experiences that are uncommon in marginal contexts. The students were able to notice that when the cyclist path was changed, the distance-time graph and table simultaneously changed. They compared two simultaneous movements through different paths while contrasting them with their corresponding tabular and graphical representations. The technology–teacher–students interaction patterns opened interesting opportunities for students to engage with data and different representations. As a result, the students could abstract mathematical relationships by perceiving what remained constant and what changed, while differentiating movement representations and path, which is difficult even for more advanced students.

For Yasmin's students, the LDD tool opened the possibility to interact dynamically with the technology in order to build a specific three-dimensional object. By using it, students had the opportunity to familiarise themselves with symbolic referents that helped them to move, and particularly to rotate virtual objects to see them from different perspectives and thus to be able to construct a real object correctly, step by step. In their lives, these students neither had access to Lego blocks nor to technology. During the experience, they discovered important 3D geometric properties of objects and were able to interpret projections, symmetry and transformations while enjoying playing with Lego blocks.

The two teachers in this study had different life histories and experiences, however, our analysis of their role while teaching with technology shows how they share important traits in terms of the classroom culture they created through their lessons.

Carla's history, as evidenced by her interview, showed her to be a diligent and responsible teacher. She cared about her students and about her own preparation, as shown by her continuous participation in courses and collaboration with other teachers to share ideas about the use of Enciclomedia resources to support students' learning. Despite the poor technological conditions at the school where she works, Carla was able to maximise its impacts on the students' learning outcomes through her experience, her knowledge about the context where the school is located, her students' needs and the school administrators' support.

Yasmin was a young teacher, with less teaching experience but was interested to learn through her involvement in research on the use of technology in mathematics education. She was aware of the possibilities offered by different digital resources and was capable of using LDD to design tasks for all her students to work on the Lego project. She knew that her students did not have access to construction toys or technology at home so she gave them an opportunity to work with both during the class through an activity she knew they would enjoy.

Both teachers promoted student involvement by flexibly adapting their roles to students' needs and by designing enriching and motivating activities to enable them to learn. The teachers' actions created opportunities for their students to share their experiences. They were given autonomy throughout the session (Jorgensen et al., 2014), they were able to share their mathematical ideas, to feel free to ask questions and to interact respectfully with teachers and other students and to use interactive technology designed for them to propose, use, validate and value mathematics.

Yasmin did it through the construction of the park. Her students were responsible for the construction blocks and other materials they used in their work; they helped other students when needed; negotiated within their teams and with the whole group to design a park where all the students' constructions could be compatible in terms of size and location. Carla gave students multiple opportunities to reflect on the relations about different representations with and without technology; she gave students opportunities to interact directly with the Cycle track technology and used games to maintain students' interest on the discussed ideas, to assess their mathematical knowledge and to friendly discuss their difficulties.

Through these multiple opportunities, these teachers created a class culture where students living in unfavorable conditions were involved and felt confident in their own mathematical learning (Schmalenbach, 2017; Treviño Villarreal et al., 2019). Carla and Yasmin's effective actions with the technology created a rich classroom culture where students thought and talked about important mathematical topics and became aware of their capability to think mathematically and to enjoy their activity and thus learn (Civil & Planas, 2004).

Both teachers demonstrated high expectations on their students' learning capabilities. They used stimulating technology to help children in vulnerable conditions to learn demanding mathematical ideas. Through the use of the interactive technology, and its immediate feedback and the possibility for students to select (and contrast the results) of interesting scenarios, students who live in contexts where expectations of them are low, had opportunities to engage with data, compare results obtained and abstract mathematical concepts. Such opportunities could be invaluable to inform their decisions to remain in school, and for their future (Boaler & Staples, 2008). The teachers also demonstrated that the opportunity to use the technology to do mathematics can stimulate students' self-esteem together with their interaction and communication skills (Sacristán, 2017).

Although it might be perceived that all the above-mentioned features are common to many other classes, and that results obtained can also be achieved in class contexts where technology is not available, this may not be the case. The use of technology by these teachers played a particularly important role for students living in unfavorable conditions who, in most cases, are limited to work on lower demand and unmotivating exercises. For these students, access to technology as an learning tool that can have uses that go beyond communication or entertainment, is in itself an advantage. Moreover, the use of technology in the studied cases made it possible for students to value themselves, value mathematics and develop specific competences such as engagement with data, abstraction and problem-solving strategies that can help them revalue the role of school and can play an important role in their future.

8 Conclusions

This chapter's contribution is its analysis of how elementary school teachers who teach mathematics using technology in the context of underserved elementary schools, a topic that has not received much attention from research in mathematics education. The findings provide insight into how these teachers can help students, who normally feel undervalued, to approach mathematical thinking by taking advantage of available technological resources in participatory and motivating ways. Framed by an enactivist approach, the teachers' lessons were examined to characterise their actions in terms of the development of environments promoting both students' learning.

Both teachers designed activities that in spite of limited technology propitiated a balanced development of different forms of thought. Both were able to create a space where students felt free to express their ideas and to actively participate, thus producing contexts rich in constructive relationships (Goos, 2010). The use of games in both classrooms contributed to students' motivation and all of them enjoyed doing mathematics (Lerman, 2000; Civil, 2006).

Some specific aspects of technology use by the studied teachers played an important role in creating rich environments that fostered the emergence of positive behaviors and interesting ideas for these underserved students. The dynamics expected by these students in a mathematics classroom were completely changed due to the amount of autonomy they were given. They profited from the possibility to explore different scenarios with the technology, and through interaction, the possibility it offered for them to pose mathematical problems independently. This developed their confidence to work actively with mathematics, have their own mathematical ideas, and to openly communicate with the teacher and other students while using the technology. The teachers' encouragements supported the students to seek validation of their mathematical ideas, to solve the original and newly arisen problems and by fostering their self- confidence. The teachers' actions fostered the emergence of key mathematical ideas involved the use of prediction of results at different moments in the activities; collaborative and respectful discussion of students' ideas; justification and argumentation of students' proposals and students' possibility to use the technology to explore the mathematics involved in the proposed tasks or to validate their own conceptions. All these actions promoted student autonomy and contributed towards the creation of a learning environment in which respectful relations and communication were possible (Boaler & Staples, 2008), and students' opinions were valued both by teachers and their peers (Planas & Civil, 2008). In other words, both teachers fostered conditions considered crucial for impactful disadvantaged students' mathematics learning.

Students in both groups showed good attitudes towards mathematics learning through the use of the selected technological resources, their teachers' collaborative approach to mathematics teaching together with institutional conditions supporting a stable environment. As underlined by Schmalenbach (2017) and Treviño Villarreal et al. (2019) these conditions are factors contributing to retaining disadvantaged students at school and in overturning their generally poor performance.

This study also shows the importance of teachers' knowledge and abilities to adapt to their students' particular circumstances. Their history indicates how the positive impacts of working collaboratively with a support group of teachers or researchers to discuss their experiences when using technology in these school conditions played an important role in enriching their classroom practices.

The case studies reported in this study provide important insights for the design of tasks that might be included in pre- or in-service teacher education programs, particularly those for teachers who serve in schools in low socioeconomic neighborhoods. For example, during seminars promoting reflection on actions, which can create conditions fostering students' learning and workshops to discuss the learning results of collaborative work and the development of imaginative ways to use existing technological resources to promote the emergence of key mathematical ideas in their classrooms. Such initiatives can have a major impact on teachers. They can discover that students who face difficult conditions in their daily life can learn mathematics through enriching experiences that may help them change their future through the appreciation of powerful mathematical knowledge.

Acknowledgements We thank Andrea Ortiz for her contribution to this research study. This project was funded by Conacyt Grant No. 145735 and the program "Aprendizaje de las matemáticas en contextos diversos" from UPN-Escuela Bonfil and supported by Asociación Mexicana de Cultura A.C. and ITAM.

References

Adelantado, J. (coord.). (2000). *Cambios en el Estado del Bienestar. Políticas sociales y desigualdades en España*. Icaria.

Boaler, J., & Staples, M. (2008). Creating mathematical futures through an equitable teaching approach: The case of Railside School. *Teachers College Record, 110*, 608–645.

Civil, M. (2006). Working toward equity in mathematics education: A focus on learners, teachers and parents. In S. Alatorre, J. L. Cortina, M. Sáiz, & A. Méndez (Eds.), *Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 30–50). Universidad Pedagógica Nacional.

Civil, M., & Planas, N. (2004). Participation in the mathematics classroom: Does every student have a voice? *For the Learning of Mathematics, 24*, 7–12.

Conapo. (2013). *Índice absoluto de marginación 2000–2010*. http://www.conapo.gob.mx/es/CONAPO/Indice_Absoluto_de_Marginacion_2000_2010

Cuban, L. (2001). *Oversold and underused: Computers in the classroom*. Harvard University Press.

Davis, B. (1996). *Teaching mathematics. Toward a sound alternative*. Routledge.

Davis, B., Sumara, D., & Luce-Kapler, R. (2000). *Engaging minds: Learning and teaching in a complex world*. Lawrence Erlbaum.

Drijvers, P., Tabach, M., & Vale, C. (2018). Uses of technology in K–12 mathematics education: Concluding remarks. In *Uses of technology in primary and secondary mathematics education* (pp. 421–435). Springer. https://doi.org/10.1007/978-3-319-76575-4_26

Ferguson, H. B., Bovaird, S., & Mueller, M. P. (2007). The impact of poverty on educational outcomes for children. *Pediatrics & child health, 12*(8), 701–706.

Goos, M. (2010). Using technology to support effective mathematics teaching and learning: What counts? Proceedings of the Teaching Mathematics? *Make it count: What research tells us about effective teaching and learning of mathematics Research Conference*. Retrieved from http://research.acer.edu.au/cgi/viewcontent.cgi?article=1067&context=research_conference

Hoyles, C. (2018). Transforming the mathematical practices of learners and teachers through digital technology. *Research in Mathematics Education, 20*(3), 209–228.

Hughes, J. (2005). The role of teacher knowledge and learning experiences in forming technology-integrated pedagogy. *Journal of Technology and Teacher Education, 13*(2), 277–302.

INEE. (2018). *Evaluaciones de logro referidas al Sistema Educativo Nacional. Sexto grado de primaria. Ciclo escolar 2017–2018*. https://historico.mejoredu.gob.mx/evaluaciones/planea/sexto-primaria-ciclo-2017-2018/

Jorgensen, R., Gates, P., & Roper, V. (2014). Structural exclusion through school mathematics: Using Bourdieu to understand mathematics as a social practice. *Educational Studies in Mathematics, 87*, 221–239. https://doi.org/10.1007/s10649-013-9468-4

Larkin, K. (2015). BAn app! An app! My kingdom for an app^: An 18-month quest to determine whether apps support mathematical knowledge building. In T. Lowrie & R. Jorgensen (Eds.), *Digital games and mathematics learning: Potential, promises and pitfalls* (Vol. 4, pp. 251–276). Springer.

Lerman, S. (2000). The social turn in mathematics education research. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning*. Ablex Publishing.

Loong, E., & Herbert, S. (2018). Primary school teachers' use of digital technology in mathematics: The complexities. *Mathematics Education Research Journal, 30*(4), 475–498.

Lozano, M. D. (2015). Using enactivism as a methodology to characterise algebraic learning. *ZDM, 47*(2), 223–234.

MacNeil, A., Prater, D., & Busch, S. (2009). The effects of school culture and climate on student achievement. *International Journal of Leadership in Education, 12*, 73–84.

Moyer-Packenham, P. S., Litster, K., Bullock, E. P., & Shumway, J. F. (2018). Using video analysis to explain how virtual manipulative app alignment affects Children's mathematics learning. In *Uses of technology in primary and secondary mathematics education* (pp. 9–34). Springer.

Niess, M. L. (2005). Preparing teachers to teach science and mathematics with technology: Developing a technology pedagogical content knowledge. *Teaching and Teacher Education, 21*(5), 509–523. https://doi.org/10.1016/j.tate.2005.03.006

Noss, R., & Hoyles, C. (1996). *Windows on mathematical meanings: Learning cultures and computers*. Kluwer.

Page, M. S. (2002). Technology-enriched classrooms. *Journal of Research on Technology in Education, 34*(4), 389–409. https://doi.org/10.1080/15391523.2002.10782358

Papert, S. (1980). *Mindstorms: Children, computers, and powerful ideas*. Basic Books.

Pierce, R., & Stacey, K. (2013). Teaching with new technology: Four 'early majority' teachers. *Journal of Mathematics Teacher Education, 16*(5), 323–347.

Ruthven, K. (2018). Constructing dynamic geometry: Insights from a study of teaching practices in English schools. In G. Kaiser, H. Forgasz, M. Graven, A. Kuzniak, E. Simmt, & B. Xu (Eds.), *Invited lectures from the 13th international congress on mathematical education* (pp. 521–540). Springer. https://doi.org/10.1007/978-3-319-72170-5_29

Sacristán, A. I., & Rojano, T. (2009). The Mexican national programs on teaching mathematics and science with technology: The legacy of a decade of experiences of transformation of school practices and interactions. In A. Tatnall & A. Jones (Eds.), *Education and technology for a better world. IFIP advances in information and communication technology* (Vol. 302, pp. 207–215). Springer. https://doi.org/10.1007/978-3-642-03115-1_22

Sactristán, A. I. (2017). Digital technologies in mathematics classrooms: Barriers, lessons and focus on teachers. In E. Galindo & J. Newton (Eds.), *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Hoosier Association of Mathematics Teacher Educators.

Schmalenbach, C. (2017). Cooperación y participación como respuesta a situaciones desafiantes en las vidas de estudiantes. *Revista de Investigación Educativa-UANL, 4*, 46–59.

SEP. (2004). *Programa enciclomedia: Documento base. Subsecretaría de Educación Básica y normal*. Secretaría de Educación Pública.

Sinclair, N., & Baccaglini-Frank, A. (2016). Digital technologies in the early primary school classroom. In L. D. English & D. Kirshner (Eds.), *Handbook of international research in mathematics education* (pp. 662–686). Taylor & Francis.

Sinclair, N., & Yerushalmy, M. (2016). Digital technology in mathematics teaching and learning: A decade focused on theorising and teaching. In *The second handbook of research on the psychology of mathematics education* (pp. 235–274). Brill Sense.
Solís, P. (2010). La desigualdad de oportunidades y las brechas de escolaridad. In A. Arnaut y S. Giorguli (Coords.), *Educación*. El Colegio de México.
Spiteri, M., & Chang-Rundgren, N. (2020). Literature review on the factors affecting primary teachers' use of digital technology. *Technology, Knowledge and Learning, 25*, 115–128. https://doi.org/10.1007/s10758-018-9376-x
Tapia, L. A., & Valienti, G. (2016). Desigualdad educativa y desigualdad social en México. Nuevas evidencias desde las primaria generales en los estados. *Perfiles educativos, 38*, 32–54.
Treviño Villarreal, M. A., & González Medina, M. A. (2019). Resultados en matemática y su asociación con algunas prácticas de los docentes en un estado mexicano. *Dilemas Contemporáneos Educación Política y Valores, 2*, 1–25.
Trigueros, M., & Lozano, M. D. (2012). Teachers teaching mathematics with Enciclomedia: A study of documentational genesis. In G. Gueudet, B. Pepin, & L. Trouche (Eds.), *From Text to "Lived" Resources* (Vol. 7, pp. 247–263). Springer. https://doi.org/10.1007/978-94-007-1966-8_13
Trigueros, M., Lozano, M. D., & Sandoval, I. (2014). Integrating technology in the primary school mathematics classroom: The role of the teacher. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era* (Vol. 2, pp. 111–138). Springer. https://doi.org/10.1007/978-94-007-4638-1
Trigueros, M., Sandoval, I., & Lozano, M. D. (2020). Ways of acting when using technology in the primary school classroom: Contingencies and possibilities for learning. *ZDM Mathematics Education, 52*, 1–13. https://doi.org/10.1007/s11858-020-01171-9
UNESCO. (2017). *Guía para asegurar la inclusión y la equidad en la educación*. http://unesdoc.unesco.org/ima-ges/0025/002595/259592S.pdf. Accessed 5 Jan 2021.
Urbina, A., & Polly, D. (2017). Examining elementary school teachers' integration of technology and enactment of TPACK in mathematics. *The International Journal of Information and Learning Technology, 34*(5), 439–451.
Zielezinski, M. B., & Darling-Hammond, L. (2016). *Promising practices: A literature review of technology use by underserved students*. Stanford Center for Opportunity Policy in Education.

Characterising Features of Secondary Teachers' Curriculum Scripts for Geometric Similarity with Dynamic Mathematical Technology

Ali Simsek, Nicola Bretscher, Alison Clark-Wilson, and Celia Hoyles

Abstract We report part of a larger research study that explores secondary teachers' integration of dynamic mathematical technology (DMT) to their classroom practice with a particular focus on the mathematical domain of geometric similarity (GS). The study adopted a multiple case study approach and was situated in an English lower secondary school setting. The participants were three teachers with different levels of experience and expertise both in teaching and in using digital technology. Data collection involved video-recorded classroom observations, audio-recorded post-observation teacher interviews, and teachers' resources and students' work. The Structuring Features of Classroom Practice (SFCP) framework guided the data collection and analysis. The findings presented in this chapter focus specifically on the SFCP construct of 'curriculum script' and revealed salient differences between the teachers' practices that concern several key characteristics of their curriculum scripts for GS with DMT. The contributions of this paper are two-fold: (1) bringing together the research on GS and the SFCP framework to specify a fine-grained mathematics-specific version of curriculum script for GS with DMT, and (2) showing the value of this fine-grained framework for characterising features of teachers' classroom use of DMT for teaching GS.

Keywords Classroom practice · Dynamic mathematical technology · Geometric similarity · Secondary mathematics teacher · Structuring features of classroom practice · Curriculum script

A. Simsek (✉) · N. Bretscher · A. Clark-Wilson · C. Hoyles
UCL Institute of Education, University College London, London, UK
e-mail: ali.simsek.15@ucl.ac.uk; n.bretscher@ucl.ac.uk; a.clark-wilson@ucl.ac.uk; c.hoyles@ucl.ac.uk

A. Clark-Wilson et al. (eds.), *The Mathematics Teacher in the Digital Era*, Mathematics Education in the Digital Era 16,
https://doi.org/10.1007/978-3-031-05254-5_10

1 Introduction

Since the early 2000s, researchers have sought to understand how teachers can fully exploit the potential of dynamic mathematical technologies (DMTs), by focussing their research lenses on classroom practice incorporating DMT(s) (e.g., Bozkurt & Ruthven, 2017; Monaghan, 2004; Ruthven et al., 2008; Vahey et al., 2020). Researchers have aimed to characterise and develop an improved, holistic understanding of the features of classroom practice with DMTs (Ball & Stacey, 2019; Trgalová et al., 2018). There is a motivational assumption that the development of such characteristics, and the underlying mechanisms, promotes the complex process of teachers' technology integration in ways that might lead to improvements in students' learning (Clark-Wilson et al., 2014).

Further research is needed to fully address the complexities of teachers' integration of DMTs into the teaching and learning of mathematics in the classroom (Clark-Wilson et al., 2020; Drijvers, 2019). Specifically, researchers have argued that there is a need for *systematic* case studies guided by different theoretical lenses (Goos, 2014; Ruthven, 2014; Trgalová et al., 2018).

Moreover, research into DMT-enriched practice has drawn attention to the need to conduct studies within specific mathematical domains (e.g., Vahey et al., 2020). Geometric similarity (GS) is one such domain: an investigation into practices of teachers using DMTs for the teaching of GS is important both in terms of the significance of GS for school mathematics and the difficulties that it presents to students (Denton, 2017; Edwards & Cox, 2011; Lamon, 2008; Noss & Hoyles, 1996). This is because the dynamically linked multi-representational potential of such technologies offers opportunities (e.g., dragging, dynamic measuring, animation) for students to explore this topic in more tangible ways (Adelabu et al., 2019; Denton, 2017; Noss & Hoyles, 1996). Students' engagement with DMTs might promote their understanding of both the underlying concepts and the embedded variant and invariant relationships. However, to the best of our knowledge, there is no research investigating *deeply* how teachers use DMTs in their practice to teach about GS.[1]

The research reported in this chapter forms part of a larger study that aimed to address this gap. Guided by the Structuring Features of Classroom Practice (SFCP) framework (Ruthven, 2009), the larger study aimed to investigate teachers' practices as they used a particular DMT, the Cornerstone Maths (CM) software, to teach GS to lower secondary-aged learners (13–14 years old) (Simsek, 2021). This chapter focuses on one of the five features of the SFCP framework, 'curriculum script' (which we describe in more detail in the section that follows)

[1] One exception is Clark-Wilson and Hoyles's (2017) research. However, their research did not focus specifically on GS, hence their data provides useful but limited insights.

as our analysis suggests this construct is more closely aligned with both mathematical and technological aspects of our study. We address the following research questions:

> What differences can be identified in the characteristics of secondary teachers' curriculum scripts when they use DMT in the classroom to promote students' understanding of GS?
> What effect might these differences have on the ways in which teachers enact their curriculum scripts when using DMT in the classroom to promote students' understanding of GS?

In the following sections, we argue for the need to specify the SFCP framework for mathematics and then describe our operationalisation of curriculum script as a construct to deeply explore the teaching of GS with DMT.

2 Specifying the SFCP Framework for Mathematics: The Case of GS

The SFCP framework was chosen for this research because of its systematic and holistic approach to understanding the complex and dynamic nature of DMT-enriched practices (Bozkurt & Ruthven, 2017; Ruthven, 2009). With the specific issue of technology integration in mind, Ruthven (2009) identified the features of classroom practice based on the previous research into classroom organisation and interaction along with teacher professional knowledge (e.g., Anderson, 1981; Burns & Anderson, 1987; Cohen et al., 2002; Putnam, 1987; Rivlin & Weinstein, 1984). The resulting five features of classroom practice are *curriculum script*, *resource system*, *activity structure*, *working environment*, and *time economy*. We first define the focal construct of *curriculum script* and then briefly outline how these five features pertain to the integration of new technologies in the classroom.

The concept of *curriculum script*, in the psychological sense, refers to an event-structured organisation of knowledge incorporating potential emergent issues and alternative courses of action that establishes a loosely (but well-defined) ordered model of relevant goals, resources, and actions for teaching a topic (Leinhardt et al., 1991; Ruthven, 2009). When teachers implement new technologies in their classroom practices, they need to develop and/or modify their *curriculum script* for teaching a particular mathematical domain. This script informs the ways in which teachers originate the overall structure for the lesson agenda and enact it in a flexible and responsive way. Likewise, the exploitation of new technologies, together with the tools and resources already in use, entails that teachers establish a coherent *resource system* and develop appropriate pedagogical approaches to use such a system in a complementary and coordinated way. Teachers also need to make adaptations to the settled patterns of their *activity structures* so that they can create new classroom routines to encourage interaction between themselves, students, and technologies during the different phases of a lesson. Moreover, the incorporation of new technologies makes several demands on teachers in their *working environment* of lessons depending on their teaching and pedagogical goals, involving change of room location or physical layout in addition to change in classroom organisation. Finally, the use of new technologies might have an impact on the *time economy* of the classroom, inviting teachers to reconsider how to manage the use of allocated lesson time efficiently and economically so that they can maximise students' learning time with technologies.

It is important to note that this framework is not originally specific to mathematics nor in particular to the concept of GS. On its own, it would not provide the basis for a sufficiently fine-grained analysis to gain a more holistic understanding of the research phenomenon under scrutiny. Thus, it seemed essential to draw on ideas from research on GS (e.g., Chazan, 1988; Clark-Wilson & Hoyles, 2017; De Bock et al., 2002; Denton, 2017; Edwards & Cox, 2011; Noss & Hoyles, 1996; Seago et al., 2014; Son, 2013) to specify the SFCP framework by building a theoretical model more appropriate for the context of classroom teaching of GS with DMT.

In the next sub-section, we present our operationalisation of *curriculum script* for teaching GS with DMT (see Table 1). A full description of the theoretical model, operationalising all five components of the framework, is available in Simsek (2021).

Table 1 A theoretical model for operationalising the 'curriculum script' construct of the Structuring Features of Classroom Practice (SFCP) framework for teaching geometric similarity (GS) with dynamic mathematical technology (DMT) in the classroom

Operational definitions of the accompanying professional knowledge	Exemplification
(C1) Setting and specifying teaching goals for GS and supporting students to achieve these goals by exploiting the affordances of DMT	The teacher identifies the variant and invariant properties of mathematically similar shapes by making use of the dynamic scale factor slider, angle slider, and the ratio checker in DMT, such as: for a set of mathematically similar shapes, the overall appearance of shapes and their corresponding angles are invariant; for mathematically similar shapes, while the *between* ratios of corresponding sides are variant, the *within* ratios of corresponding sides are invariant

(continued)

Table 1 (continued)

Operational definitions of the accompanying professional knowledge	Exemplification
(C2) Using the full range of vocabulary necessary to connect the mathematical and technological aspects of DMT-enriched tasks in relation to GS and promoting students to make use of precise mathematical and technological language to support their oral and written explanations	By using the technological word 'scale factor slider' and the mathematical words 'corresponding sides' and 'corresponding angles', the teacher encourages students to explore and explain that when moving the scale factor slider, for three or more mathematically similar shapes, while the corresponding angles are staying the same, the scale factor and the ratios of corresponding sides vary together. The teacher also pays attention to how precisely students are able to use the words 'scale factor slider', 'corresponding sides' and 'corresponding angles' in their explanations and justifications
(C3) Focusing more on the mathematical aspects of tasks rather than the technical aspects, which results in making the mathematics that underpins each task explicit to students	The teacher focuses students' attention to the variant and invariant properties of mathematically similar shapes when moving the scale factor slider in DMT rather than predominantly to the function of the scale factor slider with no or little reference to the underlying mathematics
(C4) Developing efficient questions (along with follow-up questions) and posing them to the class during the different phases of the lesson, which may be connected with the mathematical and technological aspects of tasks and implementing 'think-pair-share' pedagogic routine into the lesson to encourage students to think about the questions posed and articulate their associated thoughts with justifications	The teacher develops and poses questions to the class during whole-class discussion involving DMT, such as "When playing the animation in DMT, how do the values in the two statements in the ratio checker change dynamically and what do such changes in the values tell us about *within* ratios across mathematically similar shapes?". The teacher then invites students to think and make conjectures in their pairs and then to share their conjectures with the class
(C5) Anticipating students' likely misconceptions about GS when planning lessons and probing for additional misconceptions while teaching in the DMT-enriched class and then identifying ways involving DMT to help students to confront, reflect upon and therefore address their misconceptions	The teacher anticipates the most common misconceptions about GS such as that adding the same length measure to the sides of a geometric figure (except for rhombus, square) always results in a mathematically similar figure. The teacher assists students to confront, reflect upon and therefore address this misconception by enabling them to increase the sides of a geometric figure by a certain amount using the length sliders in the dynamic environment, and then to observe and explore visually and numerically if the obtained figure is mathematically similar to the original

2.1 Operationalising Curriculum Script in the Context of GS with DMT

A curriculum script includes: the mathematical ideas that teachers aim to develop through DMT; the mathematical and technological vocabularies that they seek to highlight; the questions referencing DMT that they intend to pose; and the misconceptions that they want students to confront and reflect upon through DMT (Bozkurt & Ruthven, 2017; Ruthven, 2009, 2014; Simsek, 2021). In the context of our research, teachers might be expected to set the appropriate teaching goals for teaching GS and to exploit the DMT to achieve these goals. They might, for example, set a teaching goal that concerns exploring the variant and invariant properties of mathematically similar shapes (e.g., angle and length properties) through the DMT (Clark-Wilson & Hoyles, 2017; Edwards & Cox, 2011). When developing a curriculum script, teachers might also be expected to identify a range of vocabulary to use in the lesson to connect the mathematical and technological aspects of tasks. For example, teachers might aim to use explicitly the words 'scale factor slider', 'corresponding sides', 'corresponding angles' while moving the scale factor slider to draw students' attention to the fact that for mathematically similar shapes, while the corresponding angles stay the same, the scale factor and the ratios of corresponding sides vary together.

Furthermore, it is important that teachers *foreground* the mathematical ideas being taught rather than the technical features of DMT (Clark-Wilson & Hoyles, 2017). This implies teachers might be expected to use the DMT to improve students' mathematical understanding of GS rather than making its technical characteristics the main discourse in the lesson. For example, after enabling students to make sense of how to use the scale factor slider, teachers might focus students' attention on the variant and invariant properties of mathematically similar shapes (e.g., invariant ratio property) *while dragging* the scale factor slider. Moreover, teachers might need to form more open-ended questions focusing *both* on the mathematical and technical aspects of the tasks (Hollebrands & Lee, 2016). Teachers might, for example, develop and pose questions such as "When playing the animation, how do the values in the two statements in the ratio checker change dynamically and what do such changes tell us about *within* ratios across mathematically similar shapes?". Finally, when planning their curriculum scripts, teachers might typically foresee the likely misconceptions that students may encounter when engaging with mathematical ideas (De Bock et al., 2002). They might also be expected to identify misconceptions that students may confront and/or develop during classroom teaching, and to respond to these misconceptions *using* DMT. For example, teachers might anticipate that students are likely to confront a misconception regarding the incorrect use of additive strategies within GS tasks (Chazan, 1988; Son, 2013) and then identify ways in which they use the DMT to enable students to encounter and reflect upon this misconception (Noss & Hoyles, 1996).

3 Research Context

In this section, we describe the research context by setting the scene and outlining the research design, participants, data collection and analysis procedures.

3.1 Setting the Scene

In this study, we investigated the classroom practices of teachers approximately one year after they had undergone the professional development (PD) provided as part of the Cornerstone Maths (CM) project (Clark-Wilson & Hoyles, 2017). This research was undertaken independently from the original CM project, recruiting teachers from the community of the project to explore their practices where they used the DMT to teach GS. We considered these teachers to be suitable participants for our research because, after their involvement in the project, they were still committed to using the CM software to teach the CM curriculum unit on GS for lower secondary mathematics.

3.1.1 Introducing the CM Project

CM was a large-scale multi-year project[2] (2011–17) conducted at the UCL Institute of Education, University College London in England. The central aim of the project was to address underuse of DMTs by secondary teachers in the classroom. To do so, the researchers supported teachers' integration of the DMT to enhance the teaching and learning of a selection of three mathematical ideas central to the English National Curriculum for lower secondary mathematics (i.e., GS, algebraic patterns and expressions, linear functions). Each of these three CM curriculum units comprises carefully designed DMT (the CM software), student workbooks and teacher guides, along with teacher PD activities. The researchers assumed, with the resources and PD opportunities, teachers rethink the mathematical ideas as they employ new pedagogies underpinned by the DMT (Clark-Wilson & Hoyles, 2017).

3.1.2 Defining the CM Software as DMT

The CM software[3] is a web-based DMT embedded within the three aforementioned CM curriculum units. It was specially designed to exploit the dynamic, visual, and multi-representational potential of digital technology. It contains several carefully

[2] The website of the CM project is: https://www.ucl.ac.uk/ioe/research/projects/cornerstone-maths

[3] The website of the software is: https://www.cornerstonemaths.com/modules/2/investigations

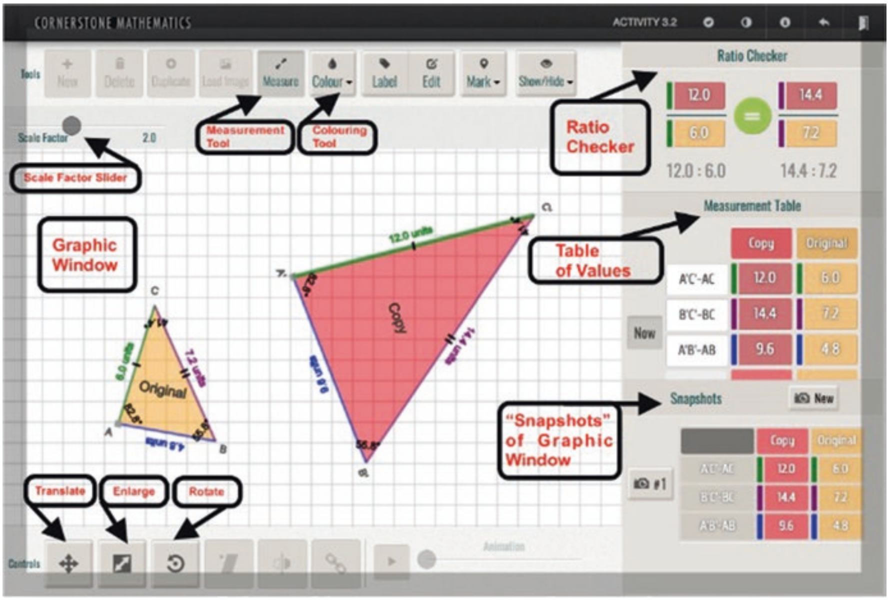


Fig. 1 The key affordances of the CM software available in a task

pre-designed dynamic tasks associated with each of the three units, including GS. These tasks were created on the basis of realistic contexts in a technological learning environment in which there are visual, dynamically linked multiple mathematical representations (e.g., geometric shapes, measurement tables, graphs, and algebraic expressions). The software was designed to offer the potential for students to make and test conjectures by manipulating the dynamic representations, leading them to engage with the underlying mathematical concepts and relationships in a realistic context (Clark-Wilson & Hoyles, 2019).

In relation to GS, the CM software was designed to offer students 12 different dynamic investigations and 19 sub-tasks. The tasks allow students to engage with geometric shapes using several key features encompassing *dragging*, *translating*, *enlarging*, *rotating*, *measuring*, *colouring*, and *labelling*, together with *length*, *angle* and *scale factor sliders*, *ratio checker*, *measurement table*, and *snapshot* (see Fig. 1 for some of the key affordances of the DMT). Therefore, students can potentially recognise and identify the variant and invariant properties of mathematically similar shapes. In its visual, dynamic and interactive learning environment, students and teachers, for example, can translate, enlarge and rotate geometric shapes to determine whether they are mathematically similar, or produce a family of similar shapes to the original shape. Additionally, while moving the various scale factor, length and angle sliders, they can also examine and identify the variant and invariant properties of mathematically similar shapes, by focusing on the appearance and orientations of shapes, corresponding angles and sides, and the ratios of corresponding sides and lengths *within* a shape and *between* the shapes.

3.2 Design for the Study

We chose a qualitative approach, a *multiple case study research design*, as our research seeks to examine and develop a deep insight into a contemporary and complex phenomenon, *teachers' classroom practices involving DMT with a focus of GS*, in a naturalistic way (Ruthven, 2009), by taking classroom realities or conditions into consideration (Yin, 2014).

3.3 Participants

Participants were selected to have contrasting levels of involvement in the original CM project, their use of the CM resources and different levels of experience in teaching since we expected such variation to reveal differences in the characteristics of teachers' DMT-integrated classroom practices with a focus on GS. We expected teachers with high levels of these attributes to exemplify characteristics of expert teachers and those with relatively lower levels to exemplify those of advanced beginners (Berliner, 2004). We assumed that working with these teachers would provide exemplary contrasting performances, which may become a source of case study information from which teachers, particularly novice teachers, could gain insight and understanding into the complex and dynamic nature of classroom practices with DMT.

Through the help of the principal investigators of the CM project (the third and fourth named authors of this chapter), three teachers were identified who were willing to participate in the research and who had contrasting levels of involvement in the original CM project, use of the CM resources and experience in teaching. They were from two London-based, co-educational secondary schools. Below, the brief distinctive characteristics of the three participant teachers who were selected to exemplify expert and advanced beginner teachers are summarised in Table 2 and described as follows:

Table 2 Profiles of the case study teachers

Teacher	Experience in teaching	Involvement in the original CM project and use of CM resources
Jack	10 years	High-level involvement; consistent use
Lara	3 years	Mid-level involvement; intermittent use
Alex	4 years	

- **Jack** (pseudonym) was expected to exemplify characteristics of an *expert teacher* in the use of DMTs within mathematics teaching and learning as he had considerable experience and expertise in the exploitation of the dynamic affordances of such technologies in the classroom. He was well respected and regarded as a successful teacher in his school, especially in terms of his skills and confidence in the use of technology. In his school, he had the responsibility to support his colleagues in planning and conducting lessons with technologies. He had high-level involvement in the original CM project by voluntarily participating fully in all sessions of the PD provided by the project. He also led his colleagues to take part in the CM project and his school supported their attendance in the CM project. Jack was committed to continuing use of the CM resources in the classroom.
- **Alex** and **Lara** (pseudonyms) were expected to exemplify characteristics of *advanced beginner teachers* in the use of DMTs within mathematics teaching and learning as they gained experience and expertise to some extent in the use of such technology in the classroom. They were colleagues who taught in the same school. They had mid-level involvement in the sessions of the PD provided by the original CM project. Their school encouraged them to take part in the CM project. Although they both only used the CM resources intermittently in the classroom, they agreed to use them in their classroom for this research as they wanted to participate in the research.

Although Lara and Alex had similar profiles, any differences in their observed practices would provide insight for the research in terms of analytic generalisation. The differences between the three case study teachers presented an advantage for the research to make the characteristics of classroom practices with DMT across the cases more visible and analysable (Bellman et al., 2014; Bozkurt, 2016; Bozkurt & Ruthven, 2018; Thomas & Hong, 2013).

During the process of data collection, which took place in April to November 2018, all three teachers taught the first eight CM investigations from the CM curriculum unit on GS to a Year 9 class (13–14 years old) approximately over a period of a month. Before planning and teaching their lessons, they were all provided with the same DMT-enriched teaching resources for the teaching of GS, that was the CM curriculum unit on GS. As outlined previously, this unit involves the use of carefully designed DMT (along with the student and teacher booklet) containing a set of the learning objectives in relation to GS and a sequence of tasks. Therefore, in their lessons, the teachers had the same opportunity to adopt the identical learning objectives for GS and offer the related tasks to students.

3.4 Methods of Data Collection

The research reported in this chapter is taken from a larger doctoral study conducted by the first author and co-supervised by the second, third, and fourth authors. The first author collected and analysed all data during the study.

Each of the three case studies in the research drew on multiple data sources including non-participant, video-recorded classroom observations (involving recordings of the teacher's eye view); semi-structured, audio-recorded post-observation teacher interviews; and scrutiny of teachers' lesson resources involving students' work in both digital and paper environments.

3.4.1 Classroom Observation

For each case study, the first author observed eight each of both Lara's and Jack's lessons and seven of Alex's lessons. In the cases of Lara and Alex, they both taught their lessons either in a pre-booked computer room or in an ordinary traditional classroom, depending on the availability of the computer rooms. In the computer rooms, they provided students with desktop computers to interact with the DMT to accomplish the tasks. However, in the ordinary classrooms, they both allowed students to use iPads. Jack conducted his lessons in his normal classroom where he taught all his lessons with laptop computers.

To gather good evidence on what was actually occurring in the teachers' classrooms, with their permission, we video-recorded their practices with two different types of video technology. We used two conventional digital video cameras located on tripods. While one of these cameras (located at the back of the classroom) focused on the whole-class and was stationary, the other camera was directed on the teacher and therefore the observing researcher operated the camera to turn it to follow the teacher's movements around the classroom. Being able to rotate the direction of the camera was vital to capture the teachers' movements and speech, especially their actions on the DMT when they used it 'live'[4] on the Interactive Whiteboard (IWB) (or from their desktop computers) and their accompanying discourse.

Besides these two video cameras, for rich and reliable data collection especially during students' independent work with the DMT, we also used an ear-mounted wearable mini digital video camera as a data collection instrument. All three teachers agreed to use this camera which was mounted on the user's ear and pointed forward roughly at eye level, allowing to record person-centred point-of-view scenes and the associated dialogue (see Fig. 2 for the wearable mini video camera used).

We decided to use such a camera as a supplementary observational instrument so that we could seek to overcome the obstacle that researchers have faced (e.g., Bozkurt, 2016). This obstacle concerns the difficulty of capturing both the teacher's eye view of the classroom and visual and audible interactions between the teacher, students and technology. The two conventional digital cameras allowed us to capture the teachers' activities and engagements and their actions from the researcher's

[4] Throughout the chapter, the term 'live' is used to refer to that the teacher operates DMT for themselves for different purposes.

Fig. 2 One of the case study teachers, Jack, wearing the ear mounted wearable mini digital camera in his teaching

point of view. However, this wearable camera offered us an access to what was 'seen' by the teachers undertaking their activities with the DMT in a natural way, especially during students' independent work with the DMT. It therefore helped explore the teacher's point of visual attention on the screen of the computer or the IWB when using the DMT 'live'.

During the observations, the observing researcher positioned himself at the side of the classroom next to the camera that focused on the teacher so that he could observe the teachers' actions without disturbing the teaching process.

3.4.2 Teacher Interview

The first author conducted six post-lesson interviews with Alex and Jack and eight with Lara. The interviews occurred face-to-face in an empty classroom and lasted between 35–45 min. With the permission of the teachers, all interviews were audio-recorded to obtain more accurate data by concentrating more on the content discussed and capturing the non-verbal communication. During the interviews, two digital recording tools were used, including Apple's QuickTime multimedia software and a mobile smartphone to avoid any possible problem with recording. Furthermore, a MacBook laptop was used during the interviews to enable the teachers to use the DMT 'live' to show their actions on the DMT or to articulate better their thoughts by referring to the DMT-enriched CM tasks displayed on the screen. Using Apple's QuickTime software, the screen of the laptop was also recorded during the interviews.

3.4.3 Lesson Resources

In addition to the collection of data from classroom observations and teacher interviews, the teachers' lesson resources were also collected, in the forms of IWB or PowerPoint slides, their worksheets created for use by students, photographs of students' DMT screens and their written work in the workbook in response to the DMT-enriched tasks.

3.5 Data Analysis

The following general analytic approach was adopted for the qualitative data analysis that led to the identification of the salient differences (and some similarities) across the cases and revealed some common themes. This analysis was led by the first author.

First, a within-case analysis (Mills et al., 2010) was used initially to examine each case in-depth by creating an individual description of the case in written form. This required the first author to re-watch all of the video recordings of each of the observed practices and then produce the rich and thick narrative descriptions to summarise what happened in the lessons. The descriptions of the lessons were then coded and categorised based on our theoretical model (see Table 1 for the theoretical model for operationalising the construct of curriculum script) using NVivo 12, a computer-assisted qualitative data analysis software and identified and modified the associated concepts and themes. Likewise, the first author re-listened to the audio recordings of the interviews and transcribed the interviews verbatim on the computer. He then reviewed the transcripts to gain insights and understandings of the contained aspects as relevant to the theoretical model and developed an overall picture. Having imported the interview transcripts into NVivo 12, the first author coded and categorised the interview data according to its relevance to the theoretical model and identified and modified the associated themes, if helpful. Following these steps, he triangulated all the data available by categorising the teachers' observed actions using the DMT and their rationale for doing so, as articulated during their post-observation reflections. Secondly, a cross-case analysis (Mills et al., 2010) was then carried out, which used spreadsheets to compare and contrast the cases of the teachers according to the themes for each individual case. This was to identify the key differences and similarities in their practices and therefore make the practices more visible and analysable in a holistic sense.

To address the reliability of the analysis, the first author also discussed regularly his interpretations and analysis of the data and the accompanying results with the three co-authors during each step of the data analysis process. We all reached an agreement on the data coded, the different themes and the associated interpretations and conceptualisation of the results.

Lastly, it is noteworthy that our analysis process is not a linear, but more a recursive process, implying that we usually moved back and forth between the different analysis steps outlined above.

4 Results

In this chapter, we present only a cross-case analysis of teachers' curriculum scripts enacted in the classroom for teaching GS with a particular DMT. From the cross-case comparison, we were able to identify characteristic features of expert and advanced beginner teachers' curriculum scripts, which served to distinguish their expertise in teaching GS with the DMT.

Our expectations about the teachers' expertise were borne out through the cross-case comparison. Specifically, Jack, as an expert teacher, had a more comprehensive curriculum script for teaching GS with the DMT than Alex and Lara with respect to the following characteristics:

– the diversity of GS-related teaching goals enacted in the lessons with the DMT;
– the range of mathematical and technological discourse in the classroom;
– the depth and variety of questioning referencing the DMT; and
– the variety of students' misconceptions about GS anticipated, identified, and addressed using the DMT in the lessons.

We now discuss in detail each of these characteristics in relation to the three case study teachers.

4.1 The Diversity of GS-Related Teaching Goals Enacted in the Lessons with the DMT

Although all three teachers broadly taught all of the first eight CM investigations in their lessons, there were differences between them in terms of both the diversity of the teaching goals they focused upon and the ways these were addressed using the DMT in their enacted scripts. In comparison to Alex and Lara, Jack included three additional mathematical goals in his scripts and exploited the dynamic potential of the DMT more effectively and frequently to accomplish these goals by:

– introducing the use of decimal scale factors along with integer scale factors in the creation of mathematically similar shapes;
– introducing the idea of 'congruency' as a special case of similarity; and
– introducing the use of a transformations-based approach to GS in determining whether geometric shapes are mathematically similar.

For example, through the use of the DMT, Jack introduced the class to the use of decimal scale factors (i.e., between 0 and 1) in addition to integer scale factors (including 1) when creating mathematically similar shapes to the original shape. In his post-lesson interview, he reported that his "emergent goal with that [the introduction of decimal scale factors] was to get them [students] to play with different decimal scale factors because [he] noticed that kids do not always get it and [he] just thought that it [the use of the DMT] was an opportunity to kind of focus on that a little bit for some students". Specifically, in Task 3.2 (see Fig. 3) Jack invited

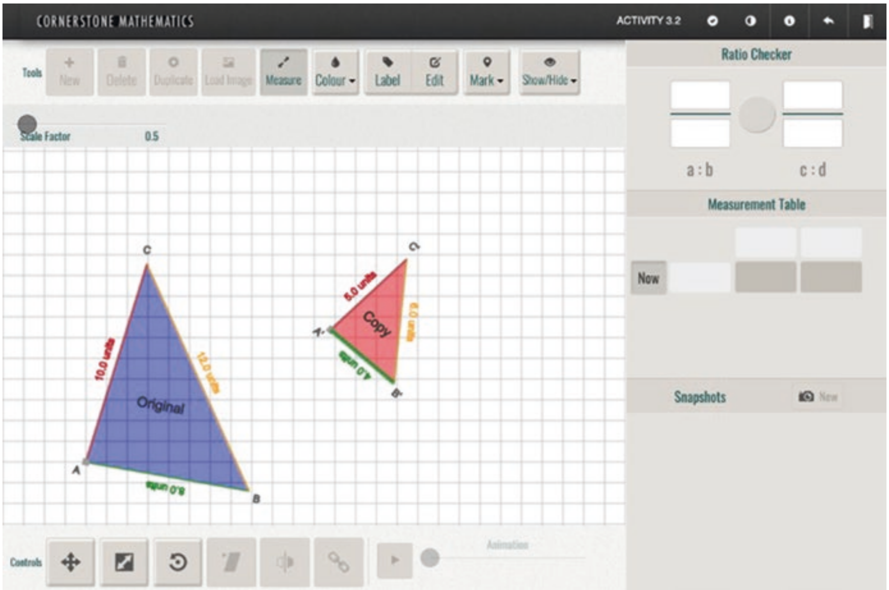


Fig. 3 A task, Task 3.2, in the DMT in which Jack used (and allowed students to use) to exploit the dynamic scale factor slider to create mathematically similar triangles to the original

students to first predict the effect of various scale factor values (including 1, 2, 0.5, respectively) on the side lengths of shapes and then validate or refute their accompanying conjectures *using* the DMT. Following this, Jack used the IWB to drag the scale factor slider to create many different cases of similar triangles. For example, by manipulating the scale factor slider, he set the slider to several different values, including 0.2 and 0.5, to generate particular cases of similar triangles. However, Alex and Lara introduced their classes only to integer scale factors. Also, Jack was the only teacher who used the DMT dynamically by dragging the scale factor slider to create several similar shapes, enabling students to examine and explore the multiplicative relationship between the measurements of the corresponding sides of the similar triangles.

Furthermore, unlike Lara, both Jack and Alex mentioned the idea of 'congruency' as a special case of similarity in their lessons. However, although the idea of congruency was prominent within the relevant DMT-enriched CM task, while Jack created and used opportunities for students to examine and discuss the set of properties of congruent shapes using the DMT, Alex only verbally explained it to students and did not make (either dynamic or static) use of the DMT. For example, Jack used a paper-based CM task incorporating the DMT in the lesson (see Fig. 4). In the task, he invited students to originate and write down their conjectures about the set of four statements related to congruent shapes and then tested their conjectures using the DMT 'live'. In Task 3.2 (see Fig. 3), Jack set the scale factor slider to several numbers including 1 and asked the whole class to investigate if the triangles were mathematically similar by examining their variant and invariant properties. This

What is true about congruent shapes? Use what you learnt in Activity 3.2 (Question 8).

	This is true Sometimes / Always / Never
The lengths of corresponding sides in congruent shapes are equal.	
The scale factor between congruent shapes is 1.	
Congruent shapes are similar.	
Similar shapes are congruent.	

Fig. 4 A paper-based task involving the use of the DMT, used by Jack in his lesson

prompted students to check their conjectures and then explain their justifications for their answers. For example, they were observed saying that "While congruent shapes are *always* similar, similar shapes are *sometimes* congruent".

Lastly, all three teachers introduced students to a transformations-based approach to GS in the classroom to some degree. For example, Jack and Alex introduced students to this approach to facilitate their comparisons of geometric shapes in the environment of the DMT when determining if they were congruent or mathematically similar. During whole-class discussions, they both used the DMT 'live' to apply a sequence of transformations (e.g., translations, rotations, and enlargements) to superimpose the shapes on each other, aiming to support students to examine and compare the properties such as corresponding sides and angles (see Fig. 5 for an example from Jack's case). To apply a transformation-based approach to GS in the dynamic environment, Jack made more frequent use of the DMT (eight times over six lessons) than Alex (two times over seven lessons). When making use of the DMT for this purpose, in each of the eight times he used the DMT, Jack exploited the dynamic features of the software more efficiently than Alex. An example, which represents or typifies Jack's more efficient use, is when he made dynamic links between the different mathematical representations available in the environment of the DMT while using the software 'live' (see Fig. 5). In Lara's case, although she seemed to invite students to use a transformations-based approach to GS, this only happened in one of the eight observed lessons and she did not use the DMT in this context in any way. However, in her interview, when asked to reflect on this, it became clear that she did not intentionally promote students to use the types of transformations to accomplish the GS-related task. She seemed to be unaware of such an approach to GS.

Despite the importance of these three ideas related to GS, which are also acknowledged in the broader literature (e.g., Seago et al., 2014; Son, 2013), Alex and Lara did not appear to pay attention to them in their enacted curriculum scripts by either not introducing them to the class or by not using the DMT 'live' to support students to engage with them.

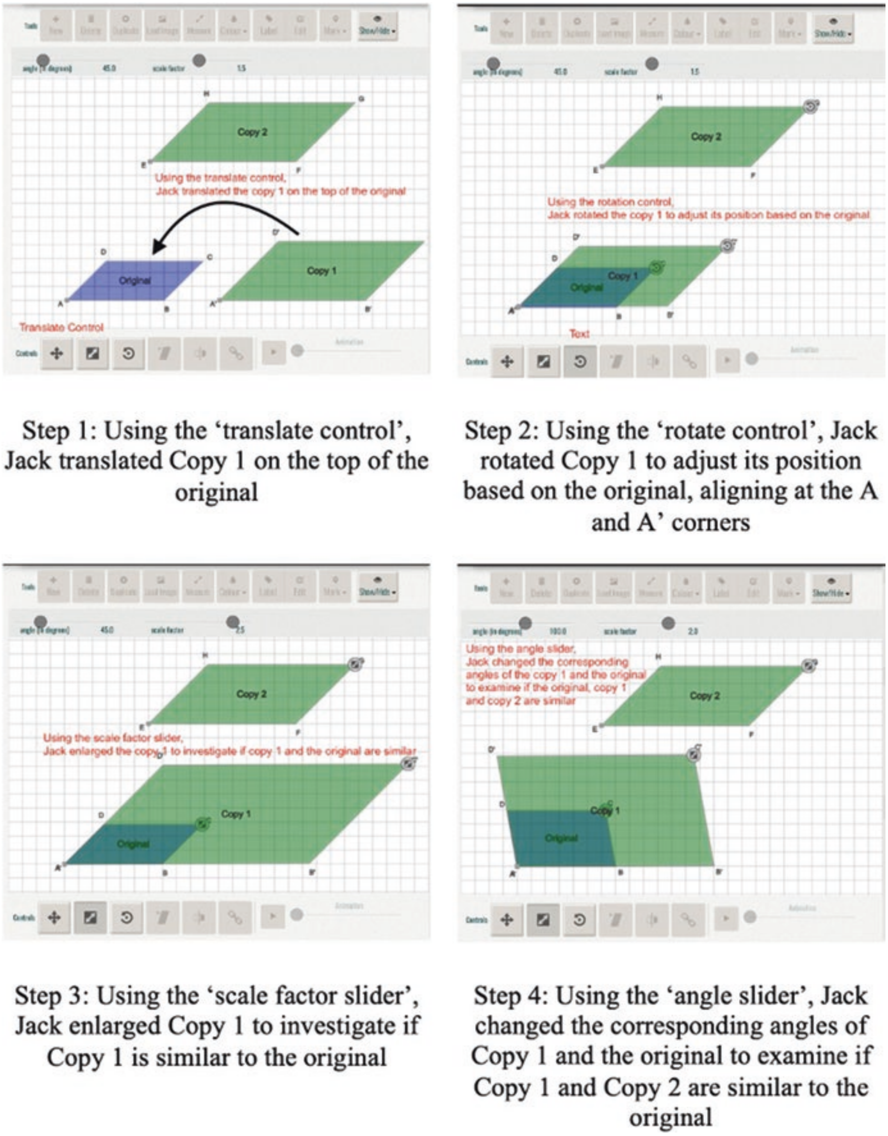


Fig. 5 The steps that Jack followed when using a transformations-based approach to GS

4.2 *The Range of Mathematical and Technological Discourse in the Classroom*

Our analysis showed differences in how the three teachers paid attention to, and promoted, students' mathematical and technological language (e.g., corresponding sides and angles, scale factor, slider, ratio checker). Jack's overall awareness of, and attention to, the development of students' mathematical and technological language was notably more precise when compared to that of Alex and Lara, in terms of the:

– frequency and length of his verbal interactions with students;
– nature of the language focused on;
– extent to which students' articulations were valued.

Compared to Alex and Lara, Jack more frequently interacted with students when he had discussions with each pair of students about their work. Jack's interactions with students were for much longer periods of time than Alex's and Lara's. Such interactions gave Jack more chances to use (and encourage his students to use) a wide range of mathematical and technological vocabularies, leading to a rich associated mathematical discourse. On the contrary, in the case of Alex, while students were working on the DMT-enriched tasks using the DMT, there was a low level of student–teacher interaction as he very rarely interacted with them to create discussions (once or twice in a lesson). When his interaction(s) with a (pair of) student(s) occurred in small-group format, it lasted only for a short period of time (less than 1 min). Similarly, in the case of Lara, although she more often interacted with students during their independent work at computers than Alex and created some discussions on their work with the DMT, her interactions and discussions were much shorter compared to Jack's. The following dialogue provides one example that typifies the brief interactions that Lara had with students:

> **Lara:** Can you observe what is happening [on the screen]?
> **Student:** When I change the scale factor, it [the shape] is getting bigger or smaller. (The student kept dragging the scale factor slider to enlarge the side lengths of the shape).
> **Lara:** Okay, so, is it getting bigger on just one side or both of sides? Okay, make your observation and write it down [in the workbook].

Additionally, during both whole-class discussions and student independent work at computers, Jack more often, regularly and effectively, used the DMT 'live' than Alex and Lara in each lesson. Jack therefore created and used more opportunities to use and highlight a wide range of mathematical and technological language (e.g., scale factor, scale factor slider, ratio checker) when describing and explaining his (and his students') actions on the DMT and the associated outcomes. This, in turn, enabled him to support the development of students' skills to use more appropriate correct vocabulary. However, as Alex and Lara used the DMT less frequently and effectively, compared to Jack, they had fewer opportunities to support students' use of precise mathematical and technological language.

Second, during the periods of whole-class teaching, there were also differences in the focus of the language that the teachers used, mainly depending on whether they made reference to the DMT in their explanations and discussions. For example, Lara's language during whole-class teaching focused far more on the mathematical aspects of the tasks than the technological aspects. This was because she tended not to make reference to the DMT (displayed statically on the IWB) nor did she make dynamic use of it. Compared to Lara, Alex and Jack had more opportunities to promote students' use of both mathematical and technological language as they did use the DMT dynamically, leading them (and their students) to use both types of language explicitly. However, the more detailed data analysis revealed some differences between the cases of Alex and Jack. For example, the questions Alex posed to the class provided clear evidence that he sometimes prioritised the language of technology over that of mathematics and thus the discussions he created did not often go beyond the technical aspects of the tasks in the DMT, particularly during his first few lessons in which he did not make use of the DMT 'live' when convening whole-class plenaries. For example, typical questions that he posed were: "What functions do we have here which are slightly different from the last activity [task] which we carried out?" or "What did you realise when playing with the functionality stuff [available in the DMT]?". However, over time, he began to foreground the mathematical rather than the technological content, asking more open-ended questions to the class to provide their explanations and justifications about the mathematics at stake.

Third, there were also differences in the degree to which, and the way in which, the three teachers appreciated students' use of correct vocabulary. In the case of Jack, during both whole-class discussions and student independent work, he frequently encouraged and valued students' oral and written explanations in response to the questions asked by him or presented in the workbook. Jack did this by sharing students' language with the whole class through explanation and/or demonstration on the IWB (see Fig. 6).

In Lara's case, her questions did not necessarily provoke students to give full explanations in their responses, nor probe and seek to understand their underlying thinking and associated justifications. Her primary goal seemed to be to examine if students' responses were correct or not. For Alex, he sometimes encouraged students to articulate their oral and written responses using correct vocabulary and appreciated precise language. Alex did this more frequently than Lara, but less frequently than Jack. However, Jack's practice was most developed in this respect. In his post-lesson reflection, Jack justified his practice by saying:

> If I am asking them [students] to write down [their explanations and justifications], I often say to them I want you to use the best mathematical language you can. Also, when I put other students' work up on the [interactive white] board, they can see it done well and they can see the good mathematical language used [in the work].

4.3 The Depth and Variety of Questioning Referencing the DMT

Our analysis elucidated differences between the three teachers that concerned, in particular; the extent to which they asked open-ended questions followed by further probing questions (particularly referencing the DMT) during both whole-class teaching and student independent work with the DMT; and the foci and aims of such questions.

The difference in the extent to which the teachers posed open-ended questions in their teaching practice can be explained as follows. First, Jack tended to pose a sequence of follow-up open-ended questions to his students that required them to clarify and elaborate their thinking. He was using this approach to probe their

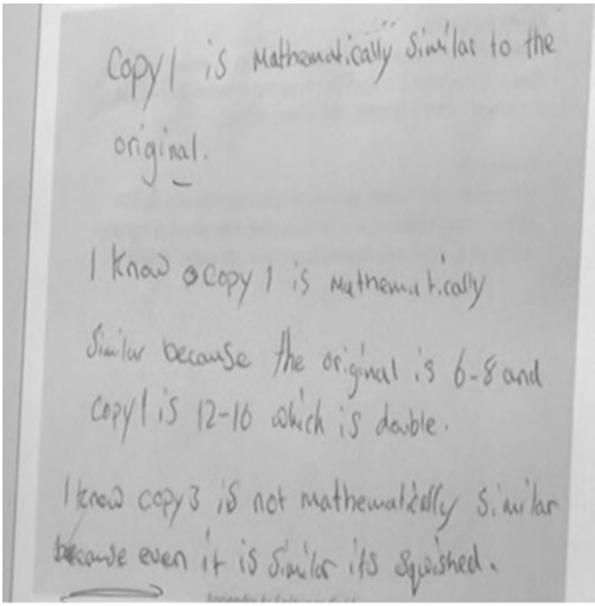


Fig. 6 A student's explanations that Jack selected to display on the IWB

understandings. He invited students to explain and justify their work with (and without) the DMT and supported and guided them to originate conjectures, predict the associated outcomes, and eventually justify their answers. However, unlike Jack, Alex and Lara were inclined not to pose follow-up open-ended questions to their students. Although they sometimes asked this type of questions, these did not usually create meaningful, rich discussions, as Jack did, which might lead students to examine and explore the variant and invariant properties of GS. Both Alex and Lara were inclined not to pursue their discussions with students. Second, the way Jack exploited the DMT in his lessons enabled and triggered his questioning approach. He regularly used the DMT dynamically during both whole-class discussions and student independent work at computers. He exploited the dynamic and visual nature of the DMT by using its dynamic affordances, such as the scale factor slider, and by making the (dynamic) links between the main mathematical representations explicit. Therefore, his questions concerned what was happening in the different representations, the links between them, and their possible meanings, while using the DMT 'live'. This contributed to stimulating interactions between Jack, the students and the dynamic mathematical environment and therefore brought to light the underlying mathematical concepts and relationships. However, Alex and Lara rarely asked such probing questions referencing the DMT in their lessons. Nevertheless, it is notable that Alex began to integrate his dynamic use of the DMT more frequently into his whole-class teaching in his latter lessons, which led to him having more opportunities than Lara in this respect.

Furthermore, in terms of the foci and aims for the open-ended questions, Jack's questions incorporated both mathematical and technological aspects. For example, when interacting with a pair of students, his questioning implied that they should drag the scale factor slider to 1 and explore the properties of the resulting shapes.

> If the scale factor is 2, the sides [of the original shape] are doubled, right? Every sides are multiplied by 2. Then, what do you think what happens when the scale factor is 1? […] With the scale factor of 1, what do you think may happen?

It appears that Jack's aim was to focus students' attention to the underlying key mathematical concepts, both in the DMT and on paper, to probe and support the development of their understandings in both contexts. However, as mentioned in the previous section, Alex's questions focused predominantly on the technological aspects, especially in his initial lessons. After students had offered their responses, Alex was inclined not to pose follow-up questions to investigate and probe students' thinking about the mathematics. Nonetheless, as he began to make more dynamic use of the DMT during whole-class discussions in his latter lessons, the focus of his questions shifted more towards the mathematical content. In this new context, the aim of Alex's questions seemed to probe mostly what students noticed and explored in terms of the mathematics during their interaction with the DMT. However, unlike Jack's case, Alex's open-ended questions did not necessarily result in students noticing and exploring the underlying mathematical concepts and the relationships between them. His questions did not tend to prompt students to make explicit the mathematical links between the different representations within the DMT, such as that how manipulating the scale factor slider changes the respective pairs of corresponding angles in the two mathematically similar parallelograms. This is similar to the findings of Hollebrands and Lee (2016) who reported that although teachers posed students questions focusing on both mathematics and technology, they did not invite or encourage students to explain and justify what a relationship they realised might be true. Finally, Lara's open-ended questions concentrated generally on the mathematical aspects of the tasks due to the lack of her use of the DMT during whole-class teaching. However, her questions were usually broad (e.g., what did you notice?) and were used to obtain information from students further evidence that she seemed to aim to evaluate the correctness of their answers rather than understanding their way of thinking.

4.4 The Variety of Students' Misconceptions About GS Anticipated, Identified, and Addressed in the Lessons Using the DMT

The data evidenced that when compared to Alex and Lara, Jack showed more awareness and understanding of the likely misconceptions and the potential teaching strategies incorporating the dynamic use of the DMT to tackle them. While Jack anticipated and identified a total of six different key misconceptions (e.g., the angles of a geometric figure are multiplied by a scale factor along with its side lengths) and made dynamic use of the DMT (e.g., manipulating the dynamic angle slider alongside the scale factor slider) to provide opportunities for students to encounter and reflect upon them, Alex and Lara predicted only three and two different misconceptions, respectively, and did not necessarily use the DMT in the same way as Jack used.

Additionally, our analysis identified more prominent differences between the teachers in terms of their spontaneous identification of misconceptions and their exploitation of the DMT to tackle them. For example, Jack tended to engage the whole-class or a group of students to investigate and diagnose the causes of such misconceptions and sought to enable them to see the correct solutions through the dynamic use of the DMT. This was evident, for example, by the misconception related to the areas of mathematically similar shapes that Jack identified in a lesson when interacting with a pair of students during their independent work with the DMT. It is noteworthy that even though Jack spontaneously spotted this misconception during his circulation around the class, he exploited (and encouraged his students to exploit) the affordances of the DMT 'live' most effectively, particularly the animation and the gridlines. Consequently, the students could realise and correct their misconception and understand the underlying reason why it had occurred. This echoes a skill that an expert teacher would be expected to have as part of his/her professional knowledge. According to Bellman et al. (2014), an expert teacher is inclined to take responsibility to 'make real-time decisions' about how to use DMT to address the misconceptions they identify in the classroom (p. 98). Unlike Jack, both Alex and Lara tended not to recognise and identify misconceptions students developed and/or encountered during their lessons. Although both teachers drew students' attention to some misconceptions that they might hold (e.g., a scaled shape is *always* larger than the original), they did not tend to use the DMT to support students to notice and reflect upon them.

5 Conclusion

In this study, we adopted a multiple case study approach to investigate the classroom practices of three English lower secondary mathematics teachers with different levels of experience and expertise in the use of digital technology for the teaching and learning of school mathematics. One of the case study teachers was termed an *expert teacher* who had experience of teaching and of using DMTs in the classroom, while the other two were called *advanced beginner teachers* as they were neither novice nor expert in teaching and in using DMTs in the classroom. Our findings indicated that the teachers, particularly the advanced beginner ones, experienced difficulties with the incorporation of the dynamic affordances of the DMT into their classroom teaching of GS (e.g., developing (and extending) their curriculum script) as they needed to "draw on a matrix of [their] professional knowledge" to (re)think and (re)formulate their associated curriculum scripts (Ruthven, 2009, p. 138).

Across the cases of the three teachers, our findings also suggested several important differences in relation to one of the features of classroom practice identified within the SFCP framework, *curriculum script*. In what follows, we conceptualise these differences across classroom practices of expert and advanced beginner teachers, leading to the development of an understanding of characteristics of teachers' curriculum scripts for teaching GS with DMT. It is important to acknowledge that since we worked with only three teachers from two different schools in this case study, the findings from this study cannot be generalised in a simplistic way to a wider population of teachers without further empirical validation. Further research involving different teachers would be needed to address the issue of generalisability of our conceptualisation that follows.

1. Expert teachers develop more comprehensive curriculum scripts compared to advanced beginner teachers for teaching GS with DMT in terms of:
   - the variety of GS-linked teaching goals enacted in the lessons incorporating the use of DMT;
   - the range of mathematical and technological vocabulary used, and paid attention to, during students' use of DMT in the classroom;
   - the depth and variety of questions referencing the DMT, particularly encompassing more open-ended questions and follow-up questions; and
   - the range of GS-related student misconceptions anticipated, identified and addressed in the lessons that exploit the dynamic features of DMT.
2. Expert teachers show a deeper understanding of the dynamic nature of GS than advanced beginner teachers. This enables them to use DMT dynamically, for example, to:
   - introduce both decimal and integer scale factors to create mathematically similar shapes;
   - introduce the idea of 'congruency' to the class and promote students' understanding of the properties of congruent shapes;
   - anticipate and identify common student misconceptions related to GS and to allow students to confront and reflect upon them;
   - make links between GS and geometric transformations by applying the types of transformations (e.g., translations, rotations, enlargements) to geometric shapes in the dynamic mathematical environment to facilitate determining mathematical similarity or congruency;
   - make multiple mathematical connections between the representations of GS in DMT which promotes meaningful whole-class and individual discussions with students.
3. Expert teachers predominantly foreground the mathematics rather than the technology while advanced beginner teachers tend to do vice versa. Expert teachers tend to use DMT 'live' to enhance students' understanding of GS focusing on the *mathematical* content rather than on *techniques* for using DMT. However, in advanced beginner teachers' lessons, the mathematics tends to remain implicit,

especially in their initial lessons. But, over time, they begin to foreground the mathematics.

4. Expert teachers' 'live' use of DMT tends to create opportunities to encourage the use of precise and correct mathematical and technological vocabulary in students' oral and written explanations and justifications. By contrast, advanced beginner teachers tend not to use DMT 'live', which in turn restricts such occasions. Expert teachers tend to use a broad range of mathematical and technological vocabulary while advanced beginner teachers display a more limited range when describing actions on DMT and to articulate the mathematical outcomes of such actions.
5. Expert teachers develop a wide range of open-ended questions referencing DMT to provoke students to make and test mathematical conjectures by exploiting the dynamic features of DMT, and then to articulate and justify their reasoning and results. However, advanced beginner teachers develop a more limited range of open-ended questions that do not necessarily refer to the use of DMT and their questioning requires low-level convergent reasoning, mainly leading to students finding and articulating the right answers, without giving detailed explanations and justifications.

Our research contributes to the literature by articulating in finer detail the important characteristics of teachers' classroom practice with a particular emphasis on their curriculum scripts. More particularly, our study reveals that the further articulations of curriculum script are helpful in identifying the more efficient practice of an expert teacher. Additionally, in the first edition of this book series, Ruthven (2014) invited researchers to use the SFCP framework both in the process of data collection and analysis so that they test, elaborate and refine this novel framework. In this regard, our study offers a further test of how the SFCP framework is useful and functional as a tool for investigating teachers' everyday classroom practices with (new) technologies. It also shows how the SFCP can be specified for a particular mathematical domain by operationalising the framework to analyse the data and to identify differences (and similarities) in teachers' classroom practices, particularly in their curriculum scripts. Finally, our research has also made a methodological contribution to the literature. We used three different digital video cameras in our lesson observations: one focusing on the teacher, the other capturing the whole-class, and the third to capture the 'teacher's eye view'. To the best of our knowledge, this study has been the first research in the field of educational technology within mathematics education that employed an ear-mounted wearable video technology as a data collection instrument to capture teachers' unique view of the classroom.

Acknowledgements The first author expresses his warm thanks to the Ministry of National Education (MoNE) of Türkiye, from which he received financial support for his doctoral education. We would like also to thank all three participating teachers in our study for their committed contribution to this research project.

References

Adelabu, F. M., Makgato, M., & Ramaligela, M. S. (2019). The importance of dynamic geometry computer software on learners' performance in geometry. *The Electronic Journal of E-Learning, 17*(1), 52–63.

Anderson, L. W. (1981). Instruction and time on task: A review. *Journal of Curriculum Studies, 13*(4), 289–303.

Ball, L., & Stacey, K. (2019). Technology-supported classrooms: New opportunities for communication and development of mathematical understanding. In A. Büchter, M. Glade, R. Herold-Blasius, M. Klinger, F. Schacht, & P. Scherer (Eds.), *Vielfältige zugänge zum mathematikunterricht konzepte und beispiele aus forschung und praxis* (pp. 121–129). Springer Spektrum.

Bellman, A., Foshay, W. R., & Gremillion, D. (2014). A developmental model for adaptive and differentiated instruction using classroom networking technology. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era: An international perspective of technology focused professional development* (pp. 91–110). Springer.

Berliner, D. C. (2004). Describing the behaviors and documenting the accomplishments of expert teachers. *Bulletin of Science Technology and Society, 24*(3), 200–212.

Bozkurt, G. (2016). *Teaching with technology: A multiple-case study of secondary teachers' practices of GeoGebra use in mathematics teaching*. Doctoral thesis (PhD), University of Cambridge, UK.

Bozkurt, G., & Ruthven, K. (2017). Classroom-based professional expertise: A mathematics teacher's practice with technology. *Educational Studies in Mathematics, 94*(3), 309–328.

Bozkurt, G., & Ruthven, K. (2018). The activity structure of technology-based mathematics lessons: A case study of three teachers in English secondary schools. *Research in Mathematics Education, 20*(3), 254–272.

Burns, R. B., & Anderson, L. W. (1987). The activity structure of lesson segments. *Curriculum Inquiry, 17*(1), 31–53.

Chazan, D. (1988). Similarity: Exploring the understanding of a geometric concept. *Technical Report*, 88–15.

Clark-Wilson, A., & Hoyles, C. (2017). *Dynamic digital technologies for dynamic mathematics: Implications for teachers' knowledge and practice*. London.

Clark-Wilson, A., & Hoyles, C. (2019). A research-informed web-based professional development toolkit to support technology-enhanced mathematics teaching at scale. *Educational Studies in Mathematics, 102*, 343–359.

Clark-Wilson, A., Robutti, O., & Sinclair, N. (2014). Introduction. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era: An international perspective of technology focused professional development* (pp. 1–10). Springer.

Clark-Wilson, A., Robutti, O., & Thomas, M. O. J. (2020). Teaching with technology. *ZDM Mathematics Education, 52*(7), 1223–1242.

Cohen, D. K., Raudenbush, S. W., & Ball, D. L. (2002). Resources, instruction, and research. In R. Boruch & F. Mosteller (Eds.), *Evidence matters: Randomized trials in education research* (pp. 80–119). Brookings Institution Press.

De Bock, D., Van Dooren, W., Janssens, D., & Verschaffel, L. (2002). Improper use of linear reasoning: An in-depth study of the nature and the irresistibility of secondary school students' errors. *Educational Studies in Mathematics, 50*(3), 311–334.

Denton, J. (2017). Transforming mathematics: Using dynamic geometry software to strengthen understanding of enlargement and similarity. *Warwick Journal of Education, 1*, 69–84.

Drijvers, P. (2019). Head in the clouds, feet on the ground: A realistic view on using digital tools in mathematics education. In A. Büchter, M. Glade, R. Herold-Blasius, M. Klinger, F. Schacht, & P. Scherer (Eds.), *Vielfältige zugänge zum mathematikunterricht: Konzepte und beispiele aus forschung und praxis* (pp. 163–176). Springer Spektrum.

Edwards, M. T., & Cox, D. C. (2011). The frame game. *Journal of Mathematics Education at Teachers College, 2*, 18–27.
Goos, M. (2014). Technology integration in secondary school mathematics: The development of teachers' professional identities. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era: An international perspective of technology focused professional development* (pp. 139–162). Springer.
Hollebrands, K. F., & Lee, H. S. (2016). Characterizing questions and their focus when pre-service teachers implement dynamic geometry tasks. *The Journal of Mathematical Behavior, 43*, 148–164.
Lamon, S. J. (2008). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers* (2nd ed.). Lawrence Erlbaum Associates.
Leinhardt, G., Putnam, R. T., Stein, M. K., & Baxter, J. (1991). Where subject knowledge matters. In J. Brophy (Ed.), *Advances in research on teaching* (*Teachers' knowledge of subject matter as it relates to their teaching practice*) (Vol. 2, pp. 87–113). JAI Press.
Mills, A., Durepos, G., & Wiebe, E. (2010). *Encyclopedia of case study research*. Sage.
Monaghan, J. (2004). Teachers' activities in technology-based mathematics lessons. *International Journal of Computers for Mathematical Learning, 9*, 327–357.
Noss, R., & Hoyles, C. (1996). *Windows on mathematical meanings: Learning cultures and computers*. Kluwer Academic Publishers.
Putnam, R. T. (1987). Structuring and adjusting content for students: A study of live and simulated tutoring of addition. *American Educational Research Journal, 24*(1), 13–48.
Rivlin, L. G., & Weinstein, C. S. (1984). Educational issues, school settings, and environmental psychology. *Journal of Environmental Psychology, 4*(4), 347–364.
Ruthven, K. (2009). Towards a naturalistic conceptualisation of technology integration in classroom practice: The example of school mathematics. *Education & Didactique, 3*(1), 131–159.
Ruthven, K. (2014). Frameworks for analysing the expertise that underpins successful integration of digital technologies into everyday teaching practice. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era: An international perspective of technology focused professional development* (pp. 373–394). Springer.
Ruthven, K., Hennessy, S., & Deaney, R. (2008). Constructions of dynamic geometry: A study of the interpretative flexibility of educational software in classroom practice. *Computers and Education, 51*(1), 297–317.
Seago, N. M., Jacobs, J. K., Heck, D. J., Nelson, C. L., & Malzahn, K. A. (2014). Impacting teachers' understanding of geometric similarity: Results from field testing of the learning and teaching geometry professional development materials. *Professional Development in Education, 40*(4), 627–653.
Simsek, A. (2021). *Characterising features of secondary mathematics teachers' classroom practices with dynamic digital technology: The case of geometric similarity.* Doctoral thesis (PhD), UCL Institute of Education, University College London, London, UK.
Son, J.-W. (2013). How preservice teachers interpret and respond to student errors: Ratio and proportion in similar rectangles. *Educational Studies in Mathematics, 84*, 49–70.
Thomas, M. O. J., & Hong, Y. Y. (2013). Teacher integration of technology into mathematics learning. *International Journal for Technology in Mathematics Education, 20*(2), 69–84.
Trgalová, J., Clark-Wilson, A., & Weigand, H.-G. (2018). Technology and resources in mathematics education. In T. Dreyfus, M. Artigue, D. Potari, S. Prediger, & K. Ruthven (Eds.), *Developing research in mathematics education: Twenty years of communication, cooperation and collaboration in Europe* (pp. 142–161). Routledge.
Vahey, P., Kim, H. J., Jackiw, N., Sela, H., & Knudsen, J. (2020). From the static to the dynamic: Teachers' varying use of digital technology to support conceptual learning in a curricular activity system. *ZDM Mathematics Education, 52*, 1275–1290.
Yin, R. K. (2014). *Case study research: Design and methods* (5th ed.). Sage.

Instrumental Orchestration of the Use of Programming Technology for Authentic Mathematics Investigation Projects

Chantal Buteau, Eric Muller, Joyce Mgombelo, Marisol Santacruz Rodriguez, Ana Isabel Sacristán, and Ghislaine Gueudet

Abstract This chapter focuses on teaching the use of programming technology for pure or applied mathematics investigation projects to university mathematics students and future mathematics teachers. We investigate how the theoretical frame of instrumental orchestration contributes to our understanding of this teaching. Our case study is situated within the implementation of three undergraduate mathematics courses offered at Brock University (Canada) over the past 20 years, whose main activities include investigation projects. The study examines an instructor's actions and decision-making in relation to potential students' schemes that might have been promoted, implicitly or explicitly, by the instructor. The analysis also focuses on two student schemes, namely, the scheme of articulating a mathematics concept within the programming language and the scheme of validating the programmed mathematics. The case study led us to develop an orchestration and genesis alignment (OGA) model that associates different elements of the instructor's orchestration with the intended student development of specific schemes. Our findings highlight the instructors' dual role as policy makers and as teachers responsible for orchestrating students'

C. Buteau (✉) · E. Muller
Department of Mathematics and Statistics, Brock University, St. Catharines, Ontario, Canada
e-mail: cbuteau@brocku.ca; emuller@brocku.ca

J. Mgombelo
Faculty of Education, Brock University, St. Catharines, Ontario, Canada
e-mail: jmgombelo@brocku.ca

M. S. Rodriguez
Faculty of Education and Pedagogy, Universidad del Valle, Cali, Valle del Cauca, Colombia
e-mail: marisol.santacruz@correounivalle.edu.co

A. I. Sacristán
Department of Mathematics Education, Cinvestav (Unidad Zacatenco), Mexico City, Mexico
e-mail: asacrist@cinvestav.mx

G. Gueudet
Etudes sur les sciences et les techniques, Université Paris-Saclay, Orsay, France
e-mail: ghislaine.gueudet.1@univ-rennes1.fr

A. Clark-Wilson et al. (eds.), *The Mathematics Teacher in the Digital Era*, Mathematics Education in the Digital Era 16,
https://doi.org/10.1007/978-3-031-05254-5_11

instrumental geneses (i.e., their web of schemes development). Findings also highlight the integration of projects as a key element of the exploitation mode.

Keywords Instrumental approach · Instrumental orchestration · Schemes · Programming · University mathematics · Investigation projects

1 Introduction

Cassie walks into the computer lab. Today, she was told that the class will start programming an environment for the investigation of the bifurcation diagram of the logistic function. She is pretty excited and looks forward to what her instructor has promised: She will "see and experience chaos" with her own program! We ask: What kind of teaching prompts such interest in students and supports their skill development when programming mathematics concepts and designing their investigation—or in other words, to use programming technology as an "object-to-think-with" (Papert, 1980) in mathematics?

In this chapter, we examine a case study of an instructor in a university setting to investigate how the frame of instrumental orchestration can enhance our understanding of this kind of teaching. In Sect. 2 we give an overview of programming integration in the mathematics classroom and illustrate the activity of using programming for mathematical investigation projects by continuing our story of Cassie. We present in Sect. 3 the frame of the instrumental approach and outline how we operationalise this theory when the artefact is programming and the activity is a university-level mathematics investigation project. We describe in Sects. 4 and 5 our case study methods and findings involving one instructor, and propose in Sect. 6 a model of this kind of teaching. In Sect. 7 we discuss insights gained about this teaching through the lens of the instrumental approach and conclude in Sect. 8 with some recommendations and wider perspectives.

In their recent survey paper about mathematics teaching with technology, Clark-Wilson et al. (2020) stress that an area "that would benefit from closer study…is research that looks at the outcomes of longer-term use of technology in teachers' own mathematics classrooms" (p. 1237). This chapter contributes to this area. Indeed, our case study involves a sustained integration of programming as part of a sequence of three university mathematics courses called *Mathematics Integrated with Computers and Applications* (MICA I−II−III/III*) taught at Brock University in Canada. The courses were first implemented in 2001 and in the intervening period, 10 instructors have since taught them. In these project-based courses, mathematics majors and future mathematics teachers learn to design, program, and use interactive environments to investigate mathematics concepts, conjectures, and applications (Buteau & Muller, 2014; Buteau et al., 2015b). Our case study is part of a 5-year project aiming to understand how students learn to use programming for "authentic" mathematical investigations, identify if and how their use is sustained over time, and how instructors support that learning. Next, we provide a brief overview of the integration of programming within mathematics education.

2 Programming Integration and Teaching in the Mathematics Classroom: An Overview

The last decade has seen a resurgence of interest in education that integrates computer programming or more broadly "computational thinking". Indeed, we are witnessing this integration as part of compulsory elementary school programs in different parts of the world. For example, the integration is part of mathematics in Ontario, Canada (Ontario Ministry of Education, 2020) and in France (Direction générale de l'enseignement scolaire, 2020; Vandevelve & Fluckiger, 2020); part of computer science in England (Benton et al., 2017; Department for Education, 2013) and in New Zealand (New Zealand Ministry of Education, 2020); and a transversal approach throughout subjects in Finland (Bocconi et al., 2018). Although programming in secondary school programs has been integrated for some time (mostly as part of elective computer science courses), it now has started to change in some countries. For example, Poland has recently integrated programming in the compulsory secondary programs (Sysło & Kwiatkowska, 2015; Webb et al., 2017), while in France algorithmics has been integrated since 2009 into mathematics courses for students seeking a university STEM path (Lagrange & Rogalski, 2017; Ministère de l'Éducation nationale et de la Jeunesse, 2020).

Although the integration of programming in school mathematics curricula is relatively recent in some countries, programming for mathematics learning has a half-century legacy beginning with the design of the LOGO programming language (Papert, 1972) and constructionism (Papert & Harel, 1991). The fundamental premise of the constructionist paradigm is to create student-centered learning situations for students to consciously engage in constructing (e.g., through programming) shareable, tangible objects through meaningful projects. Studies of constructionism in mathematics education have shown how programming, used as an object-to-think-with (Papert, 1980), may support students' understanding of mathematical concepts (e.g., Noss & Hoyles, 1996; Wilensky, 1995). In Papert's (1980) vision, when integrating programming in this way, "the relationship of teacher to learner is very different: the teacher introduces the learner to the microworld in which discoveries will be made, rather than to the discovery itself" (p. 209). In other words, the role of the teacher in Papert's view is to create conditions that promote invention, creativity, and the pursuit of ideas (cf. Barabé & Proulx, 2017), particularly relevant in a context when programming is used by students as an object-to-think-with.

Overall, from the 1970s until the last decade, much research centered on programming's potential for engaging in mathematical thinking and learning (Benton et al., 2017), which evolved more recently into the broader context of "computational thinking" (e.g., diSessa, 2018; Gadanidis et al., 2018; Sinclair & Patterson, 2018). Until recently, research focused largely on learning, whereas pedagogical design mainly has been analyzed tangentially (e.g., Hoyles & Noss, 1992) due perhaps to programming's scant integration in schools. With the recent increased integration of programming in schools and (mathematics) curricula, we see a crucial need for more research about teaching programming as an object-to-think-with in mathematics. An example of such research is by Benton et al. (2018) who discuss professional development to support mathematical learning through programming in England's elementary schools.

At the university level, the integration of programming in mathematics has taken different forms, such as: within specific courses (e.g., modeling, numerical analysis), as a required skill (e.g., a computer science course requirement), as part of an interdisciplinary program (e.g., in data science, mathematical biology), or through a more integrated approach (e.g., a sequence of courses or throughout a program). In the U.K., for example, Sangwin and O'Toole (2017) found that 78% of undergraduate mathematics programs required training in computer programming, mostly in numerical analysis and statistics. As an example of a more integrated approach, the mathematics and physics programs at University of Oslo (Norway) are being revised to help students develop abilities to apply numerical methods to solve problems in courses throughout the program (Malthe-Sørenssen et al., 2015).

There are other examples of program-wide integration of programming whose foci are broader than numerical methods, such as at Carroll College, USA (Cline et al., 2020) and at Manchester Metropolitan University, England (Lynch, 2020), both of which use programming for modeling that may provide students an opportunity to experience programming as envisioned by Papert (1980). The integration of Python in the large-enrolment (>1000 students) calculus course for Life Sciences at McMaster University, Canada (Clements, 2020) is an example of such integration, as are the MICA sequence courses mentioned earlier (Buteau et al., 2015b), though the latter involve broader mathematical areas (number theory, dynamical systems, etc.). In fact, research has shown that programming can support students' learning of university mathematics in many areas such as algebra (Leron & Dubinsky, 1995), calculus (Clements, 2020), probability (Wilensky, 1995), combinatorics (Lockwood & De Chenne, 2019), and statistics (Mascaró et al., 2016). For instance, in their study, Lockwood and De Chenne (2019) found that "the Python programming, the representations of each of the outcomes, and the formulas that were reflected in the code together offered an effective approach by which to enhance students' combinatorial reasoning and activity" (p. 306).

Many of the approaches mentioned above concern meaningful, rich ways to integrate programming in university mathematics classrooms as described in the work of Weintrop et al. (2016), who detail various computational practices in which professional mathematicians and scientists engage in their work (see Fig. 1). Their detailed taxonomy gives insights into the engagement by research mathematicians as envisioned by the European Mathematical Society (2011): "Together with theory and experimentation, a third pillar of scientific inquiry of complex systems has emerged in the form of a combination of modeling, simulation, optimization and visualization" (p. 2). For example, "interpreting and preparing problems for mathematical modeling," "assessing different approaches," "developing modular solutions," and "using computational models to understand a concept."

Weintrop et al. (2016) argue that these practices should inform policy makers and instructors on how programming and computational thinking can be integrated in mathematics and science classrooms in order to be "more in line with current professional practices in these fields" (p. 143). In other words, these authors point to

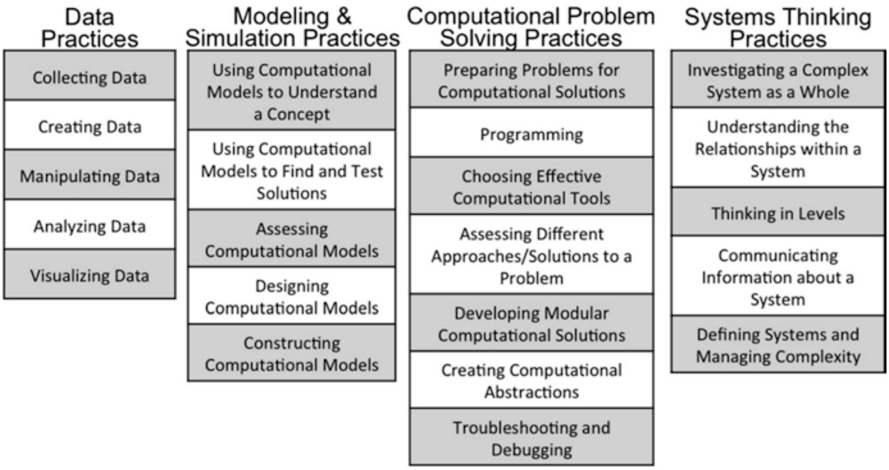


Fig. 1 Taxonomy of computational thinking in mathematics and science. (Weintrop et al., 2016, p. 135)

ways for instructors to integrate programming that facilitates diverse pure and applied "authentic" inquiry-based activities, such as in modeling and simulation projects. Broley (2014) found that Canadian research mathematicians' views of an ideal integration of programming in the mathematics classroom reinforces the affordances grounding the enactment of these computational practices. In their recent "Call for Research That Explores Relationships Between Computing and Mathematical Thinking," Lockwood and Mørken (2021) argue that "serious consideration of machine-based computing [including programming] is largely absent from much of our research in undergraduate mathematics education" (p. 2) and that the various integration approaches in the university mathematics classroom should provide opportunities for research on teaching. This chapter addresses this gap by discussing a study on the teaching of using programming as a tool for authentic pure or applied mathematical modeling projects. In the next section, we illustrate the use of programming for such mathematical investigation projects through a reconstitution of a student's engagement.

2.1 Programming for Authentic Mathematics Investigation Projects: An Example

As illustrated in this chapter's introduction, Cassie was asked by her MICA II instructor to investigate the bifurcation diagram of the logistic function $f(x) = kx(1 - x)$, where $x \in [0, 1]$, and $k \in [0, 4]$ is a constant. The year before in MICA I, she used VB.NET programming to investigate the dynamical system by fixing values for k and initial values of x, and observing, numerically and graphically, the convergence of the generated sequences (i.e., study of orbits). As she starts her work, she reads through this program, her current lecture notes about the logistic function, and her project guidelines; she is identifying what is common and what should be different. She now knows and understands; she will build on her previous year's code for the iterative process $x_{n+1} = f(x_n)$, but this year's input and output are to be different. Indeed, her input will include the range of values for k, and a minimum threshold for the number of terms in the sequence. The output will be a graph that shows for a k value (horizontal axis), the higher terms of the generated sequence for the initial x value (i.e., the bifurcation diagram on the selected range for k). See Fig. 2.

She now starts her work in VB.NET: she first creates her graphical user interface (GUI) using Visual Studio's drag 'n' drop menu (Fig. 3 left). She then uses her mathematical knowledge to program the mathematical process while regularly testing the accuracy of her code as she builds it (Fig. 3 right).

Cassie completes the last step of her programming after she verifies that her graphical output is correct by comparing it with the example given by the instructor. She then uses her program to investigate the dynamical system in two different ways: (i) she varies the range of k in order to identify in the diagram when the system first becomes chaotic; (ii) she also investigates fixed points of selected cases: of $f \circ f$ with $k = 3.3$ and of $f \circ f \circ f \circ f$ with $k = 3.5$, both analytically and visually. Finally, she submits her program and a written report of her exploration to her instructor. Throughout this activity, Cassie interacts at times with her peers and her instructor.

In a project like this, programming is not the sole goal but rather a way to engage in the mathematics made possible by the technology. Cassie confirms this by reflecting, when asked if engaging in programming or using the end program affected her learning:

> I feel like both probably affected my learning. Just because like then I understood what the graph was actually showing and then I could see it. So, it was like I was putting in the function and like telling it what to do and like how to form and then I saw it, so yeah I guess it- they both helped. (Cassie et al., 2019)

It appears that Cassie, as an undergraduate, was able to use programming as an object-to-think-with in a way that is similar to how some mathematicians use programming in their research (Broley et al., 2017). Other examples of such authentic investigation projects in MICA courses include a student designing and programming an interactive environment to investigate, visually and algebraically, the structure of hailstone sequences, or a student trio designing and programming an interactive environment to explore the changes in the water supply of the local Lake Erie (Canada) over time and explain why and how it changes (Buteau et al., 2016). In this chapter, we are interested in the ways an instructor creates conditions and teaches students to engage in such programming-based mathematical investigation projects. We frame our study with the theory of instrumental orchestration, which we introduce next.

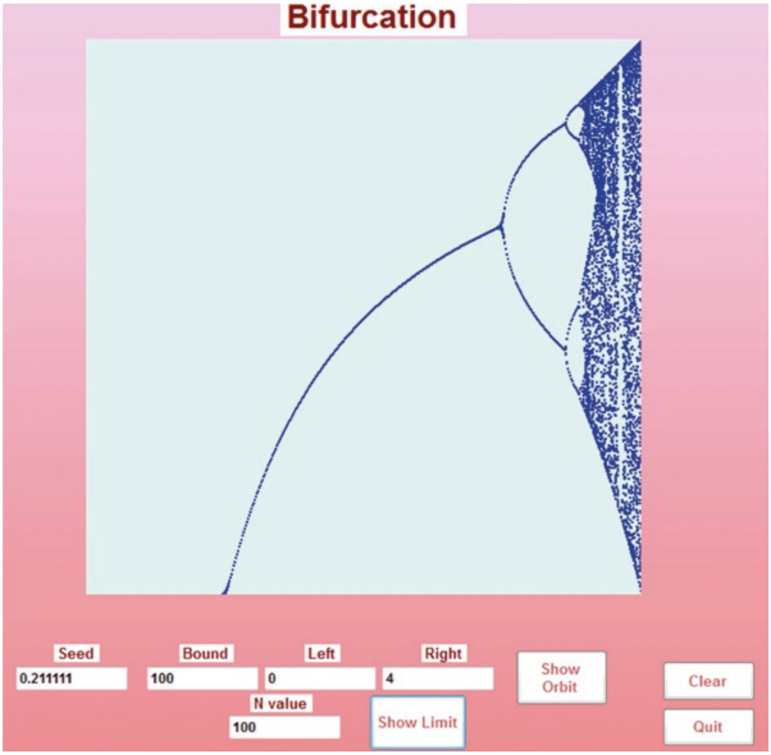


Fig. 2 Screenshot of the GUI of Cassie's program for her investigation of the bifurcation diagram

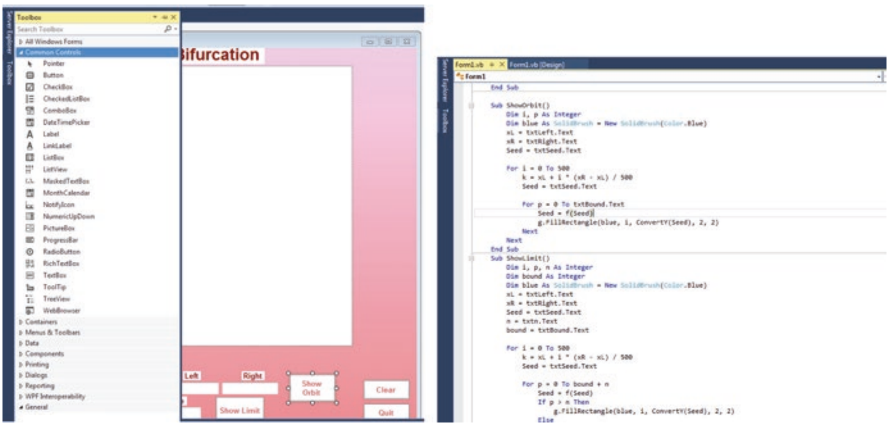


Fig. 3 Left: Screenshot of the drag 'n drop menu in Visual Studio to construction a GUI; Right: Screenshot of an excerpt of Cassie's VB.NET program for her investigation of the bifurcation diagram

3 Theoretical Framework

Our work is informed by the instrumental approach. We briefly describe the theory in Sect. 3.1, articulate student learning when considering programming for investigation projects as the artefact in Sect. 3.2, and finally in Sect. 3.3, briefly describe how this chapter extends the themes discussed in the 2014 chapter (Buteau & Muller, 2014).

3.1 Instrumental Genesis, Schemes, and Instrumental Orchestration

The instrumental approach was developed to conceptualise human activity with artefacts in the field of ergonomics (Rabardel, 1995) and it was further articulated to account for teaching and learning contexts that involve the use of artefacts in mathematics education (Artigue, 2002; Guin et al., 2005). The approach is grounded in the theoretical framework that brings together post Piagetian and Vygotskian perspectives of cognition—specifically, the post Piagetian concept of scheme as articulated in theory of conceptual fields (Vergnaud, 1998) and the Vygotskian concept of mediation by cultural artefacts as articulated in activity theories. In contrast with the dyadic subject–object interaction, the approach highlights the triad interactions among the subject, the instrument, and the object towards which instrumented action is directed.

From the instrumental approach perspective, there is a distinction between an artefact as a material or semiotic construct and an instrument as a psychological construct that emerges from the subject's activity with the artefact for a given goal through the process of instrumental genesis. The instrument is composed by a part of the artefact and a scheme of its use (Vergnaud, 1998). Vergnaud's (1998) reconceptualisation of the Piagetian concept of scheme provides a basis for a definition of instrumented action schemes. For Vergnaud, a *scheme*, defined as a stable organisation of the subject's activity for a given goal, comprises four components: (i) the *goal* of the activity; (ii) *rules-of-action* (RoA), generating the behaviour according to the features of the situation; (iii) *operational invariants*: concepts-in-action and theorems-in-action, which are propositions considered as true and governing the RoA; and (iv) possibilities of *inferences*.

Trouche (2004) considers that students' instrumental geneses in the mathematics classroom may need to be steered by a teacher, because the didactical introduction of artefacts impacts the students' activity and the teacher's work. To describe the ways that the teacher guides the students' instrumental geneses, and provides a learning environment, he proposes the concept of instrumental orchestration referring to the systematic and intentional organisation, arrangement, and didactical use of artefacts (including the digital ones) in the classroom. Trouche explains that, through the teacher's instrumental orchestration, not only individual aspects are mobilised in the class, but also social and collective aspects that influence students' instrumental geneses. The teacher steers the development of students' schemes by proposing tasks and resources, with certain goals for which the artefact can or must be involved, and by trying to foster the development of associated schemes.

Later, the concept of instrumental orchestration was enriched with the contributions of Drijvers et al. (2010), who considered Trouche's (2004) didactical configuration and exploitation mode components and introduced the didactical performance component to explain the modifications, adjustments, and changes made by the teacher in response to events and interactions in class. Drijvers et al. (2010) describe in detail the three components:

- Didactical configuration—"an arrangement of artefacts in the environment, or, in other words, a configuration of the teaching setting and the artefacts involved in it";
- Exploitation mode—"the way the teacher decides to exploit a didactical configuration for the benefit of his or her didactical intentions [it] includes decisions on the way a task is introduced and worked through, on the possible roles of the artefacts to be played, and on the schemes and techniques to be developed and established by the students"; and
- Didactical performance—which "involves the ad hoc decisions taken while teaching on how to actually perform in the chosen didactical configuration and exploitation mode" (p. 215).

As part of the exploitation mode, we see the teacher's decisions on the way tasks are worked through to possibly also include assessment and grading.

Instrumental orchestration has been used in mathematics education studies with various technologies such as the graphing calculator (e.g., Trouche, 2004), dynamic geometry software (e.g., Ndlovu et al., 2013), spreadsheets (e.g., Haspekian, 2014), and mathematics applets (Drijvers et al., 2010). Such studies have focused mostly on the elementary level (e.g., Gueudet et al., 2014; Santacruz & Sacristán, 2019) or the secondary level (e.g., Drijvers et al., 2014; Ruthven, 2014). It appears that only a few studies focus on the tertiary level (e.g., Ndlovu et al., 2013; Thomas et al., 2017). Other than in our own work (Buteau et al., 2020c; Sacristán et al., 2020), we haven't found any instrumental orchestration study focusing on programming technology. Furthermore, most studies examine aspects of the three orchestration components, whereas few concentrate on specific student schemes and their corresponding rules-of-action (e.g., Drijvers et al., 2010). To our knowledge, no study analyzes the instrumental orchestration of technology-rich investigation projects. In this chapter, we focus our orchestration analysis of teaching to use programming for authentic mathematics investigation projects by associating its elements to potentially intended students' schemes (i.e., what the instructor does with the intention to support students' instrumental geneses). In the next section, we describe a student's instrumental genesis in our context.

3.2 Instrumental Genesis of Using Programming for Mathematics Investigation Projects

The unit of analysis in the instrumental approach is the instrument-mediated activity. In our context, the students' instrument-mediated activity involves students using programming with a goal of mathematics investigations in project-based tasks. More precisely, the artefact is a programming language together with an integrated development environment (IDE), for example, Python in Jupyter Notebook IDE, or VB.NET in Visual Studio IDE (Buteau et al., 2019a). By using programming in these mathematics investigation projects, students transform programming into a mathematical instrument, associating some aspects of programming and schemes of use for specific sub-goals such as those described in the development process model illustrated in Fig. 4 (Buteau et al., 2019a; Buteau & Muller, 2010). For example, the *scheme of articulating a mathematics process in the programming language*, as a sub-scheme of the *scheme of designing and programming an object* (Step 3 in Fig. 4). (For a dynamic illustration of the model with a student activity, see Balt & Buteau, 2020b.)

The model represents the student engagement in the activity involving multiple steps that arise in a dynamic and non-linear way. The development process (DP) model, which aligns with mathematicians' process when using programming in their research work (cf. Broley et al., 2018), provides a basis for understanding (the structure of) a student's activity and schemes s/he mobilises and develops (Buteau et al., 2019a; Gueudet et al., 2020). Unlike studies that focus on one scheme, students in our context mobilise and develop a web of schemes whose ramifications include, among others, those shown in Fig. 4 (Buteau et al., 2020a, b). (For a dynamic visualisation of the web of schemes, see Balt & Buteau, 2020a.)

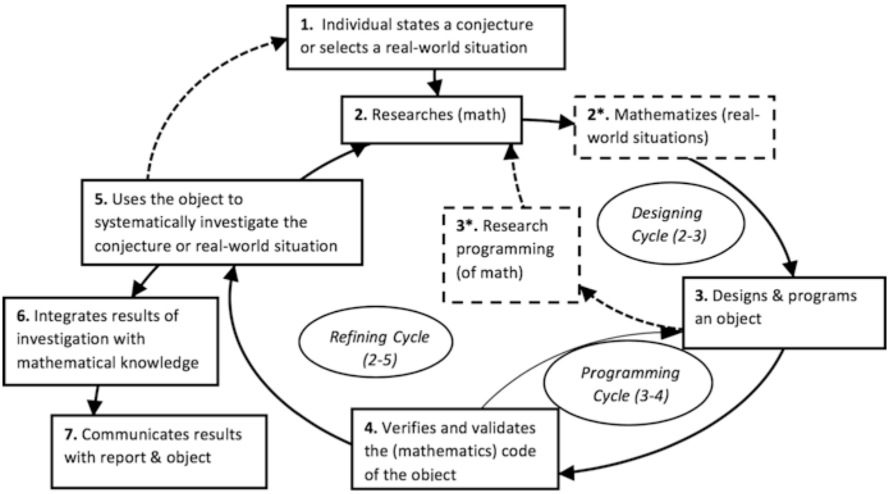


Fig. 4 Development process (DP) model of a student engaging in programming for an authentic pure or applied mathematical investigation. (Buteau et al., 2019a)

3.3 Research Question and Link with Our Previous Work

The question guiding the study presented in this chapter is: What do we learn about the teaching of using programming for authentic mathematical investigations by interpreting within the theoretical frame of instrumental orchestration, considering programming as an artefact? Building on our previous work on students' instrumental geneses when using programming (Buteau et al., 2019a; Gueudet et al., 2020) and on constructionist facets of related teaching (Buteau et al., 2019b; Sacristán et al., 2020), this chapter deepens and extends a preliminary study (Buteau et al., 2020c) focusing on teaching aspects that aim at steering students' instrumental geneses.

The chapter further develops the themes discussed in the first edition of this book (Buteau & Muller, 2014) in which we had discussed the dual role of university instructors, as teachers and policy makers, in relation to the classroom implementation of technology while guided by departmental policies. This role was illustrated through a discussion, guided by Assude's (2007) concept of instrumental integration, of the teaching of MICA I (which was extended later to student development stages of their web of schemes; see Buteau et al., 2020b). More precisely, this chapter extends the concepts of the instrumental approach used in (Buteau & Muller, 2014) by including instrumental orchestration, alongside building on our deeper and more recent understanding of the student activity and learning processes at stake (e.g., Buteau et al., 2019a; Gueudet et al., 2020). Furthermore, the analysis in this chapter mainly focuses on the teaching of MICA II whereas the 2014 chapter's main focus was on MICA I. Next, we present the methods used in our study.

4 Methods

We investigate the teaching of using programming for authentic mathematical investigations using the case of the instrumental orchestration of Bill, the MICA II instructor during Winter 2019. This is part of a 5-year non-design and non-interventionist-based research whereby some parts (participant recruitment and data collection) were designed in a way that would be least intrusive to (or constrained by) the naturalistic learning environment (i.e., the undergraduate MICA I–II–III/III* course sequence aforementioned).

4.1 Participant

Bill is a mathematician, a remarkable teacher with much teaching experience (over 30 years) who played a key role in the initial and ongoing development and teaching of MICA courses (Buteau et al., 2015b). This dual role of the instructor reflects that educators in a post-secondary context are often involved in the development and/or creation of courses, as well as in their teaching (Buteau & Muller, 2014). Bill is not researcher; rather, he volunteered to participate in this project. He was the only MICA II instructor when we first collected data for this second MICA course. By Winter 2019, Bill had taught this second course about 15 times.

4.2 Data Collection

Data in our research about Bill's teaching included all MICA II course materials (course outline, assignment guidelines) and semi-structured task-based interviews with Bill which were conducted shortly after the deadline for each of the four assigned and one original mathematics investigation projects (whose topic is chosen by the students). These 30–40-min interviews took place between January 2019 and April 2019. The interview questions prompted Bill to describe for each assignment: the learning objectives; the anticipated straightforward and more challenging parts (and why); and, for each part of the assignment, the expected knowledge to be mobilised by students. For this study, we use only one of Bill's project assignments, and we select the Assignment 1, a starting point for our long-term research. This assignment contains four short investigation problems involving the Monte Carlo integration method (we discuss Bill's problem choice in the next section; see Appendix or Ralph (n.d.) for a complete set of MICA II assignment guidelines):

- Problem 1. Buffon Needle problem;
- Problem 2. Area between two curves;
- Problem 3. Hypervolume of the unit hyper-sphere in R^4;
- One of the following: Problem 4. Buffon-Laplace problem; or Problem 5. Infinite limit of the probability that two randomly selected integers smaller than n are relatively prime.

Data for this study also included an additional post-course 90-min interview (in February 2020) that focused on Bill's expectations and guidance about two specific student schemes associated to the programming-based mathematics investigation project activity. We selected these two student schemes based on our detailed analysis of their components in our previous work (e.g., Buteau et al., 2019a). The interview was structured in two parts: (i) Bill's reflections arising from one open-ended question; and (ii) Bill's comments from a provided list of rules-of-action for each of the two student schemes (identified in Buteau et al., 2019a), viewed as a reflective process on Bill's part (akin to part of a reflective investigation, in terms of the methodology proposed by Gueudet & Trouche, 2012). We chose to focus on rules-of-action rather than operational invariants in order to align, in the future, Bill's reflection with the analysis of students' data about their explicit strategies (i.e., rules-of-action).

Finally, the data also included a departmental program description document published when the MICA courses were designed and adopted in 2000. This document contains a detailed course description, including learning objectives, suggested mathematics content, assessment practices, and how these courses fit with the undergraduate mathematics program principles (Buteau et al., 2015a).

Usual ethical protocols for this study were reviewed, received clearance, and were put into place accordingly. The instructor was formally invited to participate by an email invitation, and he confirmed his participation by signing an informed consent letter; as agreed, every interview was scheduled at a time that suited the instructor.

4.3 Data Analysis

Bill's Assignment 1 interview and project guidelines were analysed by identifying potential student schemes that might have, implicitly or explicitly, intentionally been promoted by Bill. This was done in two stages: first, the data was analysed by two coders, and second, two additional coders (all of whom are co-authors of this chapter) reviewed the coding to reach consensus. The first part of Bill's post-course interview focused on two student schemes, analysed similarly by identifying potential rules-of-action; the second part of the interview was analysed by first selecting excerpts representing his responses, summarised in a table, and categorised by associating his guidance with what he thinks of students' rule-of-actions enactment. This was also done in two stages similarly as mentioned above but involving one additional co-author. In most cases, we identified the schemes and rules-of-actions based on our previous work on students' instrumental geneses (e.g., Buteau et al., 2019a), and, in a few instances, based on our own understanding of and experience with the activity.

Finally, we looked through the 2000 document to provide us with the general guidance to instructors for the MICA courses (as policy document), including their main didactical intention. Moreover, throughout the analysis, we build on our multifaceted understanding of the teaching of MICA courses since their start (e.g., Buteau et al., 2015a, 2019b).

In order to deepen our analysis, we focus our examination on the development of schemes related to the programming artefact, rather than considering a system of artefacts involved in teaching (such as "mathematics," "internet," etc.). In our analysis, we do not elaborate on Bill's schemes; when we use the term "scheme," it always refers to students' schemes. Also, we did not directly observe Bill's teaching. Only a part of his teaching was accessible through the course material he produced. This is a limitation of our research, but we also collected student data from Bill's class (such as Cassie's), allowing to relate in the future their instrumental geneses with Bill's orchestration. Next, we present the results of our analysis of Bill's instrumental orchestration.

5 Instrumental Orchestration of Using Programming for Math Investigation Projects: A Case Study

Our analysis of Bill's instrumental orchestration is twofold: (i) by considering his guidance towards all student schemes involved in the activity (i.e., schemes represented by steps and cycles of the DP model, shown in Fig. 4); (ii) by examining Bill's teaching focused on his guidance towards two specific student schemes only, namely, the *scheme of articulating in the programming language a mathematics concept* and the *scheme of validating the programmed mathematics*.

5.1 Bill's Overall Instrumental Orchestration

In this section, we outline each orchestration component (didactical configuration, exploitation mode, and didactical performance) of Bill's teaching of MICA II by distinguishing his role as policy maker (of MICA courses) and as a teacher. Excerpts from Bill's interview data were selected to provide evidence of his orchestration associated to each step of the student DP model (Fig. 4).

5.1.1 Didactical Configuration

The didactical configuration of an instructor's orchestration concerns an organisation of the artefact in the course (Drijvers et al., 2010, p. 215). In the case of Bill's orchestration, we found that the choice of programming technology use in MICA courses and the teaching setting configuration was established in 2000 by the mathematics department (including Bill), and has since remained (Buteau & Muller, 2014; Buteau et al., 2015a).

The established teaching format involves 2 h of lecture (in a regular lecture room) and 2 h of computer lab (one computer per student) weekly, and with teaching assistants (with an instructor/teaching assistant–student ratio of about 1:10).

In regards to the artefact, there is an agreement among the instructors that MICA I–II courses mainly use VB.NET language with Visual Studio. There are now two MICA III courses; one for mathematics and science majors moving on to C++ programming language with GNU IDE, and the new MICA III* course for future mathematics teachers using VB.NET, Scratch, and Python with Jupyter Notebook. Specifically, in the case of Bill's teaching in 2019 of MICA II, he decided to also introduce Excel technology as part of his second assignment. Bill justifies: "I also think… that every math major has to be able to use Excel, because this is one of the standard tools in the outside world" (B.A2.143).

Finally, we also mention resources (textbook, lecture notes, internet, etc.) that are part of the didactical configuration yet not made explicit in the 2000 departmental document because these are assumed in any mathematics courses with a lab component.

5.1.2 Exploitation Mode

The exploitation mode concerns "the way the teacher decides to exploit a didactical configuration for the benefit of his or her didactical intentions" (Drijvers et al., 2010, p. 215). Once again, in the case of Bill's orchestration, we found that the main didactical intention grounding how programming technology would be integrated into a sequence of three MICA courses was established in 2000 by the mathematics department; students would learn to exploit programming for mathematical work (Buteau et al., 2015a). Indeed, the 2000 departmental document stipulates:

> [Students] will confront problems from pure and applied mathematics that require experimental and heuristic approaches. In dealing with such problems, students will be expected to develop their own strategies and make their own choices about the best combination of mathematics and computing required in finding solutions.

Furthermore, the core of each MICA course is the pure and applied programming-based mathematics investigation projects that account for 70−80% of a student's final MICA course grade. This is a key element of the exploitation mode (here again, the "instructor" is viewed as "policy maker faculty" rather than a "teacher"); through these projects, the department thus appears to expect that students engage in the process described in Fig. 4 (Buteau et al., 2015a). During lectures, the instructor introduces students to mathematics that is needed for the assigned individual mathematics investigation projects that are worked on during the labs. We interpret such projects as tasks designed to promote the development and reinforcement of various student schemes, such as the *scheme of articulating a math process in the programming language*. Because the instructor chooses the topic and direction of these mathematics investigations and communicates it through detailed guidelines, we interpret such projects to aim at students developing their web of schemes associated mainly to Steps 3–7 of the DP model (Fig. 4). Each MICA course also involves a final original project, in lieu of a final exam, in which students work individually or in pairs, and choose a topic of their own and the direction of the mathematics investigation. Such final projects can be viewed as an intention for students to develop further or mobilise their complete web of schemes including those associated to Steps 1 and 2.

For the individual MICA instructor (as a "teacher"), the way s/he decides to exploit the didactical configuration in order to meet the didactical intentions envisioned and decided by the department includes decisions about the mathematics content and related investigation projects. It also includes decisions about the ways the content is developed in lectures and synchronised with the investigation project work in the labs.

In terms of the choice of mathematics content in MICA II, the instructors over the years have selected various topics and areas of mathematics relevant to a computational approach for investigations, often according to their own evolving mathematics interests and research (Buteau et al., 2019b). In the case of Bill, he comments on the computational relevance of the Problem 5 investigation and his personal interest in this area:

> [Problem 5] generated a lot of great discussion… I love the idea… I like analytic number theory, I love the idea that, um, there are patterns… I jump up and down about that with the [students]... I can sell this assignment to my students. (B.A1.174)

We associate Bill's choice of "relevant topic" to Step 1 (Fig. 4), and interpret it as an implicit guidance to students (to develop a scheme of) identifying when a programming approach is an added value for the work, such as for math that cannot be done by hand.

Using various resources (including their own research), the MICA II instructor designs programming-based mathematics investigation projects and develops guidelines aligned with both the planned lecture content and planned guidance in lab as student work through the projects. In the case of Bill, he designs MICA II project assignments by "playing on the computer with some math" and decides on parts of an assignment (and guidelines) by thinking on the potential difficulties that MICA II students may confront, for example, when programming the mathematics (Steps 3−5 in the DP model). Bill mentions:

> I actually tried many many many things before we got the formula that you have here and I would try something and I'd say, "That's too hard, that comes too fast, this has to go, this has to be sequenced differently." (B.A1.43)

As Bill designs the investigations, he appears to have in mind a certain student background that he expects and wishes to build on. For example, we interpret Bill's expectations from students to be able to mobilise usage schemes of programming in VB.NET (Step 3) developed in MICA I. This is suggested by Bill, when he says about Problem 1 that it is "to keep them calm" as "there are no new programming tricks… it's all review." Furthermore, for this same Problem 1, Bill gives students a code to build from. Bill says that providing students with a "well written piece of code … helps them review… proper coding practice" (e.g., "how to change from math coordinates to graph coordinates… separately and clearly"). Also, Bill requires students to submit, for Problem 3, a print-out of their code rather than the program; he says: "I'm telling them I'm going to actually read the code on the page, it sends that signal" (B.A1.154). We interpret it as Bill's intention to steer students' mobilisation or development of their scheme of coding with rule-of-action "I write codes according to standards."

Bill suggests that he constantly has in mind the careful synchronisation of the mathematics content developed in lecture and the investigation project work initiated in lab sessions. Bill notes:

> Part of this course is to try to understand the idea that we can take a real-world situation, and we can distill from that the mathematics, and then take that mathematics and write a simulation based on that mathematics, so it's a kind of a two-part process, the real world, to the model, write the formal model that we do in the classroom, and then finally the computer simulation that we do in the lab. So, I'm thinking about that all the time, that, that sequence. (B.A1.96–98)

We associate these respectively to Steps 1, 2, and 3 in the DP model, and concerning the student *scheme of articulating in VB.NET a mathematics concept* which we elaborate more in Sect. 5.2.

The project guidelines developed by the MICA instructor outline the topic to be investigated, within the mathematics context developed (i.e., synchronised) in the lectures, together with some details of the investigation design (such as input and output) sometimes complemented with partial code (as for Problem 1 mentioned above). We associate these overall to Steps 1, 2, and 3 in the DP model. The guidelines sometimes also detail how to use the program for the mathematical investigation (e.g., by suggesting a range of parameter values) and/or emphasise the need to interpret output within their mathematics knowledge (e.g., by requiring to justify their conclusion from the investigation). We associate these to Steps 5 and 6 in the DP model, respectively. In Problem 3, Bill's guidelines read:

> The output should show the mean and standard deviation of the samples. Estimate the hypervolume accurate to one decimal place and use your observations to explain why you are confident that your first decimal place is correct.

This suggests to the students that they must apply their statistics knowledge in order to appropriately use their program and justify their answer (Step 5–6 cycle). Bill elaborates his view on this question:

> The final part of that [Problem 3] is that they need to think about what does it mean… what does confidence mean? How does the standard deviation have anything to do with that, how will you use the standard deviation and sample size to convince me that you have one decimal place of accuracy? (B.A1.117–119)

In fact, Bill mentions that he revised the guidelines due to his dissatisfaction from past students' poor interpretation of their program output. We interpret that Bill implicitly viewed that more student guidance was needed for this problem in relation to Step 6.

Bill deliberately includes a more challenging question (selection between Problem 4 or Problem 5) as part of Assignment 1 where he plans close to no extra guidance beyond the statement of the problem (i.e., Step 1): "But there has to be a question on every assignment that, is something to think about… This one is solo. Um, they don't get very much help from me" (A1.162–4). We interpret Bill's intention that students mobilise and develop, without his help, their whole web of schemes for this particular investigation task.

5.1.3 Didactical Performance

The didactical performance concerns the "ad hoc decisions taken while teaching on how to actually perform in the chosen didactic configuration and exploitation mode" (Drijvers et al., 2010, p. 215). In the case of Bill's orchestration, and of any MICA instructor's orchestration, teaching in lectures and labs involves ad hoc decisions aligned with how they have planned to support students' learning to use programming technology for pure and applied mathematical work mainly through their individual mathematical investigation projects. Based on interactions with students, individually or collectively, and on observations of interactions among students, the instructor takes decision as to how to respond. This response may take form as *individual help* addressing what we interpret as an identified student's difficulty *to develop/mobilise a certain scheme*, or as a class intervention aiming at what we interpret as steering the *collective development a certain scheme*.

Bill recalls many individual interventions during the labs. In one intervention, Bill expects from students what we interpret as mobilising their *scheme of debugging* (programming cycle in the DP model) when needed. He explicitly mentions it to students: "it will be unusual for either me or the TA [teaching assistant] to debug your code; that's not our job" (B.A1.207–8). He recalls an intervention with a student, aligned with his expectation, as he sits down beside the student: "Explain the principles and the ideas. … If you're desperate, we might look through your code" (B.A1.216). As Bill explains, "In first year they need much more support, but in second year they have to do it, you know, themselves" (B.A1.216). We interpret Bill's response as a reminder to the students of this scheme's effective rules-of-action: *step back from the code*, *think through the big picture of the code design*, and *think of the different parts of the code*.

Bill recalls another individual intervention with a student who was communicating enthusiastically to him his finding about Problem 3:

> In the lab I had a very good student say, "Look, I've written the program, and uh, the area of the region is 4.25." And I said, "But, but, look, the region you're looking at sits inside an area of 4." And he said, "Yes but look, I've got an area of 4.25." And I kept saying, "Maybe you need to check that." (B.A1.102)

We interpret Bill's response as steering the student to the theorem-in-action, *An unexpected outcome from my code may indicate that there is something wrong with my code* of the *scheme of validating the programmed mathematics*.

Bill also recalls collective responses in lab, such as when addressing the students' difficulties in explaining their output from the program in Problem 3, which we interpret as steering the collective mobilisation and development, for this task, of schemes associated to Step 6:

> I'm very interactive in the lab… so when we get to working on this question I'll be talking about variability on the blackboard… you know, what does it mean in terms of the answers you're getting here. … It's an opportunity to work on the board with them… none of this sits by itself. (B.A1.140)

In the next section, we further our analysis by focusing on two specific student schemes.

5.2 *Bill's Orchestration of Two Selected Student Schemes*

We organise this section by presenting, for each scheme, first Bill's open-ended reflection and second his comments when he was shown a list of rules-of-action (RoA).

5.2.1 Student Scheme of Articulating a Mathematics Process in the Programming Language

Bill acknowledges there is a transition between abstract mathematics and operationalizing the mathematics as a code. He explains that "You have to understand things so much better if you're going to code it" (B.Post.33). Furthermore, Bill says "that's always a leap" (B.Post.24) and "so it's the translation really that's the hard part" (B.Post.47). He addresses this challenging transition for students by preparing and presenting in lecture the mathematics in a format that facilitates its coding:

> I would have done all the mathematics on the blackboard… very carefully and very clearly because I know that they're going to program it. So even when I'm teaching, I've oriented the lecture toward programming, in the sense that they can take what I've written on the board and that's the beginning in some sense of the code. (B.Post.24–25)

Furthermore, Bill expects students to have their lecture notes in front of them during the lab sessions and exploits them during individual interventions by pointing at them. We interpret Bill's action as part of his exploitation mode and as steering students to use the *I start by translating what I would do by hand* RoA.

In addition, Bill mentions developing his lecture notes throughout the course in such a way that the "leap" to be undertaken by students is purposely increased in later assignments. Bill justifies this didactical choice with his belief that there is no learning if there is no struggle: "It's a calculated thing how much frustration can my students stand before they give up" (B.Post.38).

As for during the lab sessions, Bill comments about the extent of his individual help to students to code mathematics processes, namely, devolving from a lot of direct help at the beginning to only giving hints later on. We interpret Bill's decision and action, which are involved in aspects of both his exploitation mode and didactical performance, as justified by his aforementioned belief. This appears to promote students to enact RoA relevant to the context, including *I start by translating what I would do by hand*. Furthermore, Bill reports sometimes sitting beside students, taking a sheet of paper, and saying "let's just think about how this all is going to work" (B.Post.43). This points to Bill's didactical performance where he appears to steer students to the *I organise on paper the programming of the concept* RoA.

Table 1 summarises selected excerpts from Bill's reactions when he was shown a list of RoA. We interpret Bill's responses about his guidance to each RoA as indicating what elements of his orchestration he viewed important in regards to the guidance of this scheme. Without having seen the list of RoA before, Bill promptly identified some of his actions corresponding to an obvious guidance of the RoA. We interpret that he implicitly knew about his guidance. For some RoA (d, g, h, i, and j), Bill mentions his guidance and that according to him, students often or even always enact the RoA. For others (a, e), Bill indicates he wished he had guided the students further, and mentions that according to him, only a few students enacted these RoA. As for RoA "f," Bill interestingly appears to realise that he might have unconsciously guided students to this RoA, whereas he believes this might have been too early for them as learners using programming for mathematics

Table 1 Selected excerpts from Bill concerning his guidance (middle column) of RoA (left column) pertaining to the scheme of articulating a mathematics process in the programming language, and his views/observations of students' enactment of these RoA (right column)

Student rules-of-action (RoA): When I program a mathematics concept I do the following:	Bill's responses to how he guided students to develop or prompt the corresponding RoA	Bill's views and observations of whether students enact or not the corresponding RoA
a. I work through one or more examples on paper	"I wish I had said more and more about pencil and sitting with a sheet of paper for five seconds and thinking about how this is all gonna work. I think that would be helpful" (306). "It's funny talking to you, it's, it's—I wish I had thought more about pushing them to the big picture. I have said things like that, but I haven't really really emphasised it, but I have emphasised structured programming" (304).	"Rare. … They want to dig into the code immediately." (135)
b. I work through one or more examples in my mind	[No comment]	"Most of them do that at some level." (138)
c. When I start typing in VB.NET I first declare all variables I think I will need	[No comment]	"Certainly they do a lot that." (139)
d. I translate into VB. NET what I think I do on paper	[Careful presentation of the math development in lectures—discussed above]	"Yes, they do a lot of that." (148)
e. I organise on paper the programming of the concept	"I wish I had said this in, a lot more strongly" (153).	"Oh, I wish!" (153).
f. I organise the programming of the concept as I go	"It's possible that they are following my possible 'bad example' [laughing)" (176).	"Definitely. They do it on the fly." (158)
g. I organise my code by using functions	"They do it partly because also I sit beside them and I say: 'don't you dare write all that code in one block.'… And this idea of breaking code down into small chunks is something… I really emphasise that" (188−192).	"They do a lot of that." (190)
h. I ask someone (e.g., a peer, a TA, or the instructor)	"A few students are shy to ask. ... And you have to watch for that. … If they're really confused and really upset, they won't talk to you. … So that requires… a very gentle kind intervention and support." (198–201)	"They're very good at it…In my tutorials, um I can't keep up; sometimes with the hands. It's that busy." (196, 205)
i. I review the mathematics concept (e.g., in my lecture notes, on internet, etc.)	"But you know because they're going to code it, there's extra emphasis [in lecture] on getting [the math] down not just correctly but so cleanly and clearly right on the board." (210)	"They almost always have their lecture notes open in front of them during the lab." (207)

(continued)

Table 1 (continued)

Student rules-of-action (RoA): When I program a mathematics concept I do the following:	Bill's responses to how he guided students to develop or prompt the corresponding RoA	Bill's views and observations of whether students enact or not the corresponding RoA
j. I search online or in textbook for how to use some programming concepts (e.g., loop, exit loop, functions, arrays, etc.)	"I encourage them to… I tell them where to go, how to find … where, I give them a direction… I sometimes teach it sometimes… depends on the student. … I just send them to look for the resources." (211–216)	"Yes, they go online. … All the time." (211–213)
k. [Other?] Open the blackbox. "Stop the program and look inside"	"I'm always telling my students to… pump out what's in there or put in a stop and just see what's happening with the code at that point." (220)	

investigations. Finally, Bill commented that one RoA (k) was missing in the list, which we interpret that to him, coding a mathematics process is done incrementally and that debugging is inherently part of this distributed process.

5.2.2 Student Scheme of Validating the Programmed Mathematics

This student scheme is related to the previous one as it concerns a complex process for students to control what they do by articulating a mathematics concept in VB.NET. Bill's immediate response about his guidance of students' control was that he demonstrates a working project in the lab and expects students to use the output of his program to validate their programmed mathematics by comparing with the feedback of their programs; he mentions: "I write [it]… at the beginning of the lab, I run it, and the data is sitting right there. Can you emulate this?" (B.Post.68, 80).

We interpret Bill's action as a collective steering of students to the *I use visual output to validate the programmed mathematics* and *I compare the output of my program with an example from a trustworthy source* RoA. Furthermore, still as part of his exploitation mode, Bill projects on the screen a "simple case" for students to replicate, "so that they can verify that they are doing it correctly in that simple case. … If they can duplicate that, then they feel confident that they can do more complicated things" (B.Post.85−86). We interpret Bill's action as to steer students to the *I trust that I translate in VB.NET what I do on paper* RoA. We interpret it also as guiding students to mobilise, adapt, and develop schemes involved in a simple case as they "work incrementally," such as the two schemes under discussion. To us, this RoA is the decision of not having to act at this time about checking the in-progress coding of the mathematics concept. It appears as the first entry in Table 2 that was shown to Bill after his open comments that we just discussed.

Table 2 Selected excerpts from Bill concerning his guidance (middle column) of RoA (left column) pertaining to the scheme of validating the programmed mathematics, and his views/observations of students' enactment of these RoA (right column)

Student rules-of-action (RoA): When I program a mathematics concept I do the following:	Bill's responses to how he guided students to develop or prompt the corresponding RoA	Bill's views and observations of whether students enact or not the corresponding RoA
A. I trust that I translate in VB.NET what I do on paper	"[Students] do like to know it corresponds with something that I've given to them as a start." (235)	"Umm, I don't think they much trust what they're coding" (234–235)
B. Once I have programmed it all, I run the program with a few different inputs and compare the output with my hand calculations	"They have to do it by hand and they have to do it by code, and the two things have to agree so I think that's a really important tool for us as teachers… you know theoretically… and then you watch it happen approximately." (238)	"Yes that's built in some… of the assignments" (236)
C. I check a few times as I program by compiling with a few input	"Okay so the different inputs and doing it by hand yes that's really important for a beginner and that's built into some of my assignments. I think that's really important that they do that." (248)	"Yeah they do that all the time." (255)
D. I compare my program with that of a peer	"[Students do this] constantly and that's encouraged" (257). "It's a public forum. Chat, talk, discuss" (261). "We are collaborating and I want to foster that atmosphere." (268)	"Constantly… [they're] functioning exactly the way I want them to function, they're sharing, they're working together, they're collaborating and this is our time. This is what people are doing in this age" (257, 267).
E. I compare the output of my program with that of a peer or with examples from the internet	"There are places where the answers are on the internet and I encourage them to check that their program gives that output." (270)	"Yes." (270)
F. I use other technology (Maple, Desmos, etc.) to generate an example and compare it with the output of my program	"Part of the problem is that Maple has no existence outside of academia" (273).	"No. I've never seen that.. I mean it'd be great actually if they did- if they used Maple … and Maple could do it." (271–272)
G. I ask someone (a peer, a TA, the instructor, etc.)	[Bill had previously mentioned unsolicited individual help and his encouragement of collaboration; see D above]	"Yes, constantly." (283)

(continued)

Table 2 (continued)

Student rules-of-action (RoA): When I program a mathematics concept I do the following:	Bill's responses to how he guided students to develop or prompt the corresponding RoA	Bill's views and observations of whether students enact or not the corresponding RoA
H.* I don't really know if it works **This is not a RoA proper but rather an indication of lack of mobilisation of the scheme*	"Well they'll ask me they say, 'I wrote it, I have no idea: is it working or not?' Right, and so then there's all kinds of discussions and… chat. 'Let's go through it together! Let's see how it's doing!'" (289–290)	"Yes, there's a good amount of that." (285)
I. [Other?]: "So one thing that's not on here is the idea of working incrementally. You put something in, you try it, and then you add to that, you try it, you add to that, you try it. So that incremental approach is very important to them."	[No comment on his guidance]	"They will often test… they develop programs incrementally, so they'll write, you know, just a couple of like a loop or something and they just look and see: does that make sense?" (294)

Again, we interpret Bill's responses as indicating what elements of his orchestration he viewed as important in relation to this scheme. For some RoA (B, C, D, E, G), Bill mentions about his guidance and that according to him, students often enact the RoA. For RoA "A," Bill reiterates his guidance mentioned during his open comment and adds that his view and observations is that students mostly do not yet enact this RoA. For RoA "F," he reflects that he does not guide students, has not observed any of them enacting it, and appears not to be concerned as he views Maple not a useful technology outside academia. As for what we could describe as a non-RoA "H," Bill answers by indicating individual interventions to assist students, step-by-step, in the process of controlling their work. Finally, just as for the previous scheme, Bill comments that the *working incrementally* RoA "I" is missing in the list. We interpret it as emphasising that for Bill, coding, testing/validating, and debugging are naturally done through a dynamical and cyclical process.

In the next section, we generalise the findings from Sect. 5.1 through a model highlighting the connections between teaching elements and intended students' instrumental geneses.

6 Model of Instructor's Instrumental Orchestration and Students' Instrumental Genesis Alignment

Our analysis in the previous section showed that Bill's didactical configuration, his main didactical intention, and the project element as part of his exploitation mode arose from his roles as designer and policy maker rather than as a course instructor.

It turns out that these three components are the same for all MICA courses adopted by the department in 2000. This finding is supported by an examination of the course material (course syllabi, project guidelines, lab guidelines, etc.) from all MICA I–II–III/III* courses. Furthermore, these three elements are followed and integrated by all MICA instructors. The finding is further supported by various previous background work about MICA, including about its teaching (e.g., Buteau & Muller, 2014, 2017; Buteau et al., 2015a, 2015b, 2019b).

Reviewing our analysis of Bill's orchestration from this broader perspective leads us to propose a model illustrating an instrumental orchestration of using programming for authentic mathematics investigation projects (see Fig. 5). More precisely, we are proposing a "model of (instructor's) instrumental orchestration and (students') instrumental genesis alignment" (OGA model) of using programming for authentic pure or applied mathematics investigation projects. The diagram on the left in Fig. 5 represents the main didactical intention and the three components of an instrumental orchestration. Selected key relations among those components are illustrated by the vertical arrows (which we elaborate further below). Written in black are elements of the orchestration deduced from the MICA instructor's role as policy maker, and written in orange, from their role as instructor. The horizontal yellow arrows link each of the instructor's orchestration elements to parts of the intended student activity represented by the student DP model (Fig. 4), in the diagram to the right. In other words, the yellow arrows link the instructor's teaching activity and decision-making to intended students' development of their web of schemes (instrumental geneses) associated to different steps of the student activity.

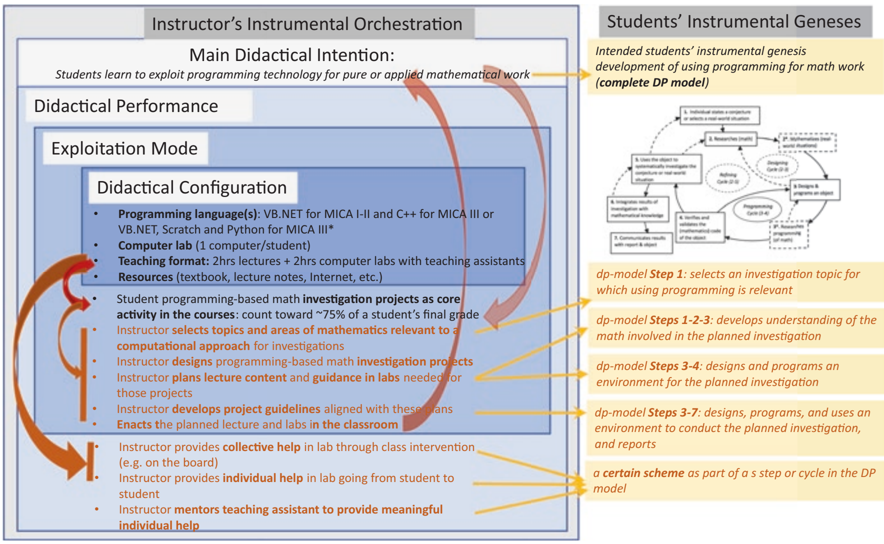


Fig. 5 The orchestration and genesis alignment (OGA) model of programming for authentic pure or applied mathematics investigation projects

In this model, certain elements of the orchestration components appear to be intrinsically connected (vertical arrows). Namely, the main didactical intention of *having students learn to exploit programming for authentic mathematical work* leads to establish individual investigation projects as the core activity of the course (exploitation mode). The latter, in turn, calls for a teaching format (didactical configuration) dedicating time to students in a computer lab that makes possible not only collective but also individual support from the instructor (exploitation mode and didactical performance), for example, Bill's collective and individual interventions in the lab sessions discussed in Sect. 5. These individual investigation projects as the core activity is the one element of the exploitation mode that seems to drive everything the instructor envisions, prepares, and does in his/her teaching (exploitation mode), all of which aligns with the main didactical intention established by the department (instructor as a policy maker).

7 Discussion

In this chapter we address the question: What do we learn about the teaching of using programming for authentic mathematical investigations by using the theoretical frame of instrumental orchestration, considering programming as an artefact? Drawing on the findings of our case study of MICA II teaching, we discuss here the different elements that together answer this question.

The identified didactical configuration, main didactical intention, and the project element as part of the exploitation mode turned out to be the same for all MICA courses adopted by the department and also followed by MICA instructors. This configuration and exploitation mode element underscore a "student-centered" and a mainly formative assessment approach, whereby the core of the courses is on individual student projects, which aligns with a constructionist approach (Papert, 1980). The instructor aligns with the collective exploitation mode, namely, through his/her choices of "content" through *project guidelines* and *planned guidance in lab and lectures* according to his/her intention of steering the collective students' instrumental geneses of their complex web of schemes associated with the programming-based mathematical investigation activity. This gives insights into how institutional decisions support individual instructors.

Instrumental orchestration has been used in some studies with various technologies and mainly focused on the school level (e.g., Drijvers et al., 2010; Trouche, 2004). Our study expands the instrumental orchestration literature at the tertiary level by illustrating the way that a university instructor orchestrates student learning of mathematics in programming-based investigations. The use of this frame, in tertiary education, allowed us to ascertain the complexity of the teacher work in the classroom and at the institutional level, identifying the key dual role of the teacher as policy maker and instructor.

The exploitation mode of MICA II teaching also highlights that, unlike most technology-rich mathematics courses, the choice of integrating programming comes *before* the choice of mathematics content. This has led to describe the mathematics content at the *individual* level, rather than the usual *collective* level (as a "policy maker level"). The didactical performance of MICA II teaching pointed to the significance of the lab setting as a key element of the didactical configuration to facilitate the MICA II instructor to steer both individual and collective students' development of schemes. Unlike the didactical configuration, these other two components of MICA instrumental orchestration seem to be evolving—for example, Bill's refining of the project guidelines to explicitly steer the students' mobilisation or development of the scheme to mathematically interpret the program output.

Our study contributes to the understanding of how an instructor at tertiary level attempts to support students' combination of programming and mathematics. Indeed, using the instrumental orchestration frame allowed us to describe Bill's action and decision-making with explicit or implicit intention of steering student schemes at the level of RoA, when the artefact is programming technology. Specifically, we learned which student RoA were emphasised by Bill and the different ways (as part of his exploitation mode and didactical performance) he guides students to enact those rules. This kind of analysis was made possible due to our detailed analyses of students' instrumental geneses at the level of schemes and their components (Buteau et al., 2019a; Gueudet et al., 2020). Our study contributes to the instrumental orchestration frame by expanding its use when the artefact is programming and by connecting the teacher's choices and the students' activity in terms of schemes the instructor intends the students to develop. Such an analysis of instrumental orchestration has rarely been done, particularly at the level of student scheme components (RoA).

Most studies about instrumental orchestration consider technology-mediated tasks in their analysis that involve a rather specific activity and are not as complex as tasks that involve investigation projects. As a consequence, orchestration analyses most often focus on one or two student schemes (e.g., use graphing calculator to find the infinite limit of a function; Trouche, 2004). Using the instrumental orchestration frame to examine Bill's teaching allowed us to identify different actions and decisions by Bill with the explicit or implicit intention to guide students through their whole investigation projects, and as such, associated to different steps (i.e., schemes) in the DP model (Fig. 4). In other words, the instrumental orchestration frame sheds light on how an instructor's decisions and actions may support the students' development of their web of schemes related to the project type of task.

Furthermore, by examining all MICA I–II–III/III* courses and building on the case study of Bill, we proposed a model—the OGA model (see Fig. 5)—of an instructor's teaching of authentic programming-based mathematics investigation projects, highlighting relationships between the different orchestration modes and between these modes and the students' activity as illustrated in the DP model. More precisely, the OGA model associates different elements of an instructor's orchestration to the intended students' development of specific schemes as part of their web of schemes. It contributes to the instrumental orchestration frame particularly by expanding to its use when the type of task is programming-based mathematics investigation projects. This model was made possible due to the sustained long-term implementation of the studied learning environment (20 years, different instructors, multiple courses, different programming languages).

Our study also highlights a methodological contribution to the instrumental orchestration approach. Showing a list of potential student RoA to the instructor fostered a reflective participation by Bill in the identification of his own orchestration (e.g., which RoA he observes his students using and which one(s) he emphasises in his teaching). We invite researchers to explore further this method.

8 Recommendations and Perspectives

Programming is increasingly being integrated in school education, with connections to mathematics (e.g., Benton et al., 2018; Webb et al., 2017). One can foresee opportunities and the need to bridge its integration in university mathematics education. Following the study presented in this chapter, we determined three recommendations for practice at the tertiary level.

First, our study highlighted that Bill implicitly knows about the student schemes, as he uses programming daily in his mathematical work as an expert. However, he mentions how having them made explicit to him was enlightening, for example, he mentions:

> It's funny talking to you. … I wish I had thought more about pushing [students] to the big picture. I have said things like that, but I haven't really really emphasised it. (Bill.Post.304)

We thus recommend that efforts be made to better communicate to tertiary instructors the results of research concerning the use of programming for mathematical investigation projects.

Second, the 20 years of sustained MICA implementation, through different instructors, suggests that this didactical configuration and exploitation mode element support well the teaching of programming-based mathematics investigations. In particular, we recommend that the instructor focuses on selecting "authentic" mathematics investigation projects with a level of programming appropriate to the students' abilities (either with or without guidance from the instructor), rather than trying to impose programming to a mathematics topic in mind.

Third, the MICA experience centered on four or five course projects stresses that such authentic mathematics investigation projects require significant time within a course. For example, Franklin et al. (2020) discuss the tension between covering the planned content and engaging students in constructionist experiences. We recommend the following for planned courses integrating programming-based mathematics investigation projects: (i) for curriculum policies (i.e., instructor as a policy maker) not to overload course curriculum; and (ii) for the instructor to plan for significant classroom time dedicated to authentic projects.

We also mention two perspectives from the research. First, we note that the project guidelines, as a collection, appear to steer students to develop or mobilise their whole complex web of schemes associated with the mathematics investigation activity. This includes their implicit guidance to the *scheme of identifying whether programming is an added value for the mathematical work* essential for conducting independent investigations (e.g., MICA final projects). Studying aspects of investigation project tasks, as part of the whole task collection, that affect which and how different schemes are guided in the project guidelines (and in lectures and labs), could lead to essential recommendations for practice in terms of key characteristics of individual project guidelines and for sequences of project guidelines within a course and among courses. Second, our research gathers both student (such as Cassie) and teacher data. This provides us with a unique opportunity to examine in a naturalistic learning environment the alignment between Bill's instrumental orchestration and students' actual instrumental geneses at the level of both schemes and scheme components, such as RoA.

We end our chapter with a quote from Bill that we associate with his view of the activity of using programming for authentic pure or applied mathematics investigation projects: "There's these two sides… they are doing things computationally, doing things theoretically, and how much does it need to agree?... It's a beautiful ballet of theoretical mathematics."

Acknowledgements This work is funded by the Social Sciences and Humanities Research Council of Canada (#435-2017-0367). It has received ethics clearance from the Research Ethics Board at Brock University (REB #17-088). We thank Bill Ralph who kindly accepted to participate in the research and whose generosity by sharing his insights with our team has been invaluable. We also thank all of the research assistants involved in our project for their valuable work toward our research, in particular Sarah Gannon for her assistance with the literature review, and Kelsea Balt, Nina Krajisnik, and Danielle Safieh who have directly been involved with the data used in this chapter.

Appendix: Bill's MICA II Assignment 1 Guidelines, Winter 2019

Note: All of your code should be carefully structured and very easy to read with all variables, functions and subroutines labeled in a helpful way. In addition, the interface should be user friendly and attractive.

1. Suppose that a needle of length 1/2 is dropped onto a plane of parallel lines that are 1 unit apart. By modifying the Buffon Needle program given in class, find the probability that the needle touches a line. Hand in your finished program which should look like the one given in class but with the appropriate modifications. Label this program as "Buffon Needle Problem". (25 marks)
2. Consider the region R in [0,2] x [0,2] for which

$$\sin\left(a*x*x+b*y*y*y \right) > \sin\left(c*x*x+d*y*y \right)$$

Hand in a program that makes this area appear on the screen for different values of a,b,c and d and estimates its area using n points chosen at random in R. The user should be able to input a,b,c,d and n. Label this program as "Area In A Square". (25 marks)

3. By choosing n points at random inside $[-1,1]^4$, write a program to estimate the hypervolume of the unit hypersphere in R^4 which is the set of points for which $x^2 + y^2 + z^2 + w^2 <= 1$. The user should be able to input the sample size n and the number of samples w. The output should show the mean and standard deviation of the w samples. Estimate the hypervolume accurate to one decimal place and use your observations to explain why you are confident that your first decimal place is correct. Also hand in a printout of your code which should be in the simplest possible form. Do not hand in the working program. (25 marks)

Do **either** question (4) or question (5). Your choice!

4. Suppose that a needle of length 1 is dropped onto a plane of parallel horizontal and vertical lines that are 1 unit apart. By modifying the code for the Buffon Needle program given in class, find the probability that the needle touches any of the lines. Hand in your written explanation of the mathematics behind your method as well as your working program. This program does not have to have a graphical component (unless you'd enjoy giving it one). Label this program "Buffon-Laplace Problem". (25 marks)
5. Suppose that two numbers a and b are chosen at random from {1, 2, ..., n}. Let Pn be the probability that they are relatively prime. As n goes to infinity, does the limit of Pn exist? Hand in the program you write to investigate this question and a discussion of what you observed. Can you guess the **exact** limit? (25 marks)

References

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning, 7*(3), 245–274. https://doi.org/10.1023/A%3A1022103903080

Assude, T. (2007). Teachers' practices and degree of ICT integration. In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of the fifth congress of the European Society for Research in Mathematics Education* (pp. 1339–1348). ERME. http://www.mathematik.uni-dortmund.de/~erme/CERME5b/WG9.pdf

Balt, K., & Buteau, C. (2020a, September 4). *A complex web of schemes development for authentic pure/applied mathematics investigations* [Video]. YouTube. https://youtu.be/teJAd3TDw9E

Balt, K., & Buteau, C. (2020b, September 4). *Process of using programming for pure/applied mathematics investigations* [Video]. YouTube. https://youtu.be/irTlCE-eXhc

Barabé, G., & Proulx, J. (2017). Révolutionner l'enseignement des mathématiques: Le projet visionnaire de Seymour Papert [Revolutionizing mathematics education: Seymour Papert's visionary project]. *For the Learning of Mathematics, 37*(2), 25–30. https://flm-journal.org/Articles/318D3351495F8F94E785130E59D4CE.pdf

Benton, L., Hoyles, C., Kalas, I., & Noss, R. (2017). Bridging primary programming and mathematics: Some findings of design research in England. *Digital Experiences in Mathematics Education, 3*, 115–138. https://doi.org/10.1007/s40751-017-0028-x

Benton, L., Saunders, P., Kalas, I., Hoyles, C., & Noss, R. (2018). Designing for learning mathematics through programming: A case study of pupils engaging with place value. *International Journal of Child−Computer Interaction*, *16*, 68−76. https://doi.org/10.1016/j.ijcci.2017.12.004

Bocconi, S., Chioccariello, A., & Earp, J. (2018). *The Nordic approach to introducing computational thinking and programming in compulsory education*. Nordic@BETT2018 Steering Group. https://doi.org/10.17471/54007

Broley, L. (2014, December 5−8). *Computer programming and the ideal undergraduate mathematics program: Some mathematicians' perspectives* [Paper presentation]. Canadian Mathematical Society (CMS) winter meeting, Hamilton, ON, Canada. https://www2.cms.math.ca/Reunions/hiver14/abs/ume

Broley, L., Buteau, C., & Muller, E. (2017, February 1−5). (Legitimate peripheral) computational thinking in mathematics. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education* (pp. 2515−2523). DCU Institute of Education & ERME. https://hal.archives-ouvertes.fr/CERME10/public/CERME10_Complete.pdf

Broley, L., Caron, F., & Saint-Aubin, Y. (2018). Levels of programming in mathematical research and university mathematics education. *International Journal of Research in Undergraduate Mathematics Education, 4*(1), 33–55. https://doi.org/10.1007/s40753-017-0066-1

Buteau, C., & Muller, E. (2010). Student development process of designing and implementing exploratory and learning objects. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of the sixth congress of the European mathematical society for research in mathematics education* (pp. 1111–1120). Institut National de Recherche Pédagogique and ERME. http://ife.ens-lyon.fr/publications/edition-electronique/cerme6/wg7-07-buteau-muller.pdf

Buteau, C., & Muller, E. (2014). Teaching roles in a technology intensive core undergraduate mathematics course. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era* (pp. 163–185). Springer. https://doi.org/10.1007/978-94-007-4638-1_8

Buteau, C., & Muller, E. (2017). Assessment in undergraduate programming-based mathematics courses. *Digital Experiences in Mathematics Education, 3*(2), 97–114. https://doi.org/10.1007/s40751-016-0026-4

Buteau, C., Muller, E., & Marshall, N. (2015a). When a university mathematics department adopted core mathematics courses of an unintentionally constructionist nature: Really? *Digital Experiences in Mathematics Education, 1*(2/3), 133–155. https://doi.org/10.1007/s40751-015-0009-x

Buteau, C., Muller, E., & Ralph, B. (2015b, June 19−21). *Integration of programming in the undergraduate mathematics program at Brock University* [Paper presentation]. Math + *Coding Symposium*, London, ON, Canada. https://researchideas.ca/coding/docs/ButeauMullerRalph-Coding+MathProceedings-FINAL.pdf

Buteau, C., Muller, E., Marshall, N., Sacristán, A., & Mgombelo, J. (2016). Undergraduate mathematics students appropriating programming as a tool for modelling, simulation, and visualization: A case study. *Digital Experiences in Mathematics Education, 2*(2), 142–166. https://doi.org/10.1007/s40751-016-0017-5

Buteau, C., Gueudet, G., Muller, E., Mgombelo, J., & Sacristán, A. I. (2019a). University students turning computer programming into an instrument for "authentic" mathematical work. *International Journal of Mathematical Education in Science and Technology, 57*(7), 1020–1041. https://doi.org/10.1080/0020739X.2019.1648892

Buteau, C., Sacristán, A. I., & Muller, E. (2019b). Roles and demands for constructionist teaching of computational thinking in university mathematics. *Constructivist Foundations*, *14*(3), 294−309. https://constructivist.info/14/3/294

Buteau, C., Gueudet, G., Dreise, K., Muller, E., Mgombelo, J., & Sacristán, A. (2020a). A student's complex structure of schemes development for authentic programming-based mathematical investigation projects. *Proceedings of INDRUM 2020: Third conference of the International Network for Didactic Research in University Mathematics*. Bizerte, Tunisia. https://hal.archives-ouvertes.fr/hal-03113837/document

Buteau, C., Muller, E., Mgombelo, J., Sacristán, A. I., & Dreise, K. (2020b). Instrumental genesis stages of programming for mathematical work. *Digital Experiences in Mathematics Education, 6*(3), 367–390. https://doi.org/10.1007/s40751-020-00060-w

Buteau, C., Muller, E., Rodriguez, M. S., Gueudet, G., Mgombelo, J., & Sacristán, A. I. (2020c). Instrumental orchestration of using programming for mathematics investigations. In A. Donevska-Todorova, E. Faggiano, J. Trgalova, Z. Lavicza, R. Weinhandl, A. Clark-Wilson, & H.-G. Weigand (Eds.), *Proceedings of the Tenth ERME Topic Conference (ETC 10) on Mathematics Education in the Digital Age* (pp. 443−450). ERME. https://hal.archives-ouvertes.fr/hal-02932218/document

Clark-Wilson, A., Robutti, O., & Thomas, M. (2020). Teaching with digital technology. *ZDM*, 52(7), 1223−1242. https://doi.org/10.1007/s11858-020-01196-0

Clements, E. (2020). *Investigating an approach to integrating computational thinking into an undergraduate calculus course* [Doctoral dissertation, Western University]. Scholarship@Western. https://ir.lib.uwo.ca/etd/7043

Cline, K., Fasteen, J., Francis, A., Sullivan E., & Wendt T. (2020). Integrating programming across the undergraduate mathematics curriculum. *Primus*, *30*(7), 735−749. https://doi.org/10.1080/10511970.2019.1616637

Department for Education. (2013). *National curriculum in England: Computing programmes of study*. https://www.gov.uk/government/publications/national-curriculum-in-england-computing-programmes-of-study

Direction générale de l'enseignement scolaire. (2020). *Programme du cycle 3—En vigueur à la rentrée 2020* [Cycle 3 program—Effective at the start of the 2020 academic year]. https://cache.media.eduscol.education.fr/file/A-Scolarite_obligatoire/37/5/Programme2020_cycle_3_comparatif_1313375.pdf

diSessa, A. A. (2018). Computational literacy and "the big picture" concerning computers in mathematics education. *Mathematical Thinking and Learning, 20*(1), 3–31. https://doi.org/10.1080/10986065.2018.1403544

Drijvers, P., Doorman, M., Boon, P., Reed, H., & Gravemeijer, K. (2010). The teacher and the tool: Instrumental orchestrations in the technology-rich mathematics classroom. *Educational Studies in Mathematics, 75*(2), 213–234. https://doi.org/10.1007/s10649-010-9254-5

Drijvers, P., Tacoma, S., Besamusca, A., van den Heuvel, C., Doorman, M., & Boon, P. (2014). Digital technology and mid-adopting teachers' professional development: A case study. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era* (pp. 189−212). Springer. https://doi.org/10.1007/978-94-007-4638-1_9

European Mathematical Society. (2011). *Position paper of the European Mathematical Society on the European Commission's contributions to European research*. EMS Secretariat. http://ec.europa.eu/research/horizon2020/pdf/contributions/post/european_organisations/european_mathematical_society.pdf

Franklin, D., Palmer, J., Coenraad, M., Eatinger, D., Zipp, A., Anaya, M., White, M., Pham, H., Gokdemir, O., & Weintrop, D. (2020). An analysis of use−modify−create pedagogical approach's success in balancing structure and student agency. In *ICER '20*: *Proceedings of the 2020 ACM Conference on International Computing Education Research* (pp. 14–24). https://doi.org/10.1145/3372782.3406256

Gadanidis, G., Clements, E., & Yiu, C. (2018). Group theory, computational thinking, and young mathematicians. *Mathematical Thinking and Learning, 20*(1), 32–53. https://doi.org/10.1080/10986065.2018.1403542

Gueudet G., & Trouche L. (2012). Teachers' work with resources: Documentational geneses and professional geneses. In G. Gueudet, B. Pepin, & L. Trouche (Eds.), *From text to "lived" resources. Mathematics teacher education* (vol. 7). Springer. https://doi.org/10.1007/978-94-007-1966-8_2

Gueudet, G., Bueno-Ravel, L., & Poisard, C. (2014). Teaching mathematics with technology at the kindergarten level: Resources and orchestrations. In A. Clark-Wilson, O. Robutti, &

N. Sinclair (Eds.), *The mathematics teacher in the digital era* (pp. 213−240). Springer. https://doi.org/10.1007/978-94-007-4638-1_10

Gueudet, G., Buteau, C., Muller, E., Mgombelo, J., & Sacristán, A. (2020, March). Programming as an artefact: What do we learn about university students' activity? *Proceedings of INDRUM 2020 Third Conference of the International Network for Didactic Research in University Mathematics*. https://hal.archives-ouvertes.fr/hal-03113851/

Guin, D., Ruthven, K., & Trouche, L. (Eds.). (2005). *The didactical challenge of symbolic calculators: Turning a computational device into a mathematical instrument*. Springer. https://doi.org/10.1007/b101602

Haspekian, M. (2014). Teachers' instrumental geneses when integrating spreadsheet software. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era* (pp. 241–275). Springer. https://doi.org/10.1007/978-94-007-4638-1_11

Hoyles, C., & Noss, R. (1992). A pedagogy for mathematical microworlds. *Educational Studies in Mathematics, 23*(1), 31–57. https://doi.org/10.1007/BF00302313

Lagrange, J. B., & Rogalski, J. (2017). Savoirs, concepts et situations dans les premiers apprentissages en programmation et en algorithmique [Knowledge, concepts and situations in early learning in programming and algorithmics]. *Annales de Didactiques et de Sciences Cognitives, 22*. https://hal.archives-ouvertes.fr/hal-01740442/document

Leron, U., & Dubinsky, E. (1995). An abstract algebra story. *American Mathematical Monthly, 102*(3), 227–242. https://doi.org/10.1080/00029890.1995.11990563

Lockwood, E., & De Chenne, A. (2019). Enriching students' combinatorial reasoning through the use of loops and conditional statements in Python. *International Journal of Research in Undergraduate Mathematics Education, 6*, 303–346. https://doi.org/10.1007/s40753-019-00108-2

Lockwood, E., & Mørken, K. (2021). A call for research that explores relationships between computing and mathematical thinking and activity in RUME. *International Journal of Research in Undergraduate Mathematics Education*, 1−13. https://doi.org/10.1007/s40753-020-00129-2

Lynch, S. (2020). Programming in the mathematics curriculum at Manchester Metropolitan University. *MSOR Connections*, *18*(2), 5−12. https://doi.org/10.21100/msor.v18i2.1105

Malthe-Sørenssen, A., Hjorth-Jensen, M., Langtangen, H. P., & Mørken, K. (2015). Integration of calculations in physics teaching. *Uniped*, *38*, Article 6. https://www.idunn.no/uniped/2015/04/integrasjon_av_beregninger_ifysikkundervisningen

Mascaró, M., Sacristán, A. I., & Rufino, M. M. (2016). For the love of statistics: Appreciating and learning to apply experimental analysis and statistics through computer programming activities. *Teaching Mathematics and Its Applications, 35*(2), 74–87. https://doi.org/10.1093/teamat/hrw006

Ministère de l'Éducation nationale et de la Jeunesse. (2020). *Programme de mathématiques de seconde générale et technologique*. https://cache.media.eduscol.education.fr/file/SP1-MEN-22-1-2019/95/7/spe631_annexe_1062957.pdf

Ndlovu, M., Wessels, D., & de Villiers, M. (2013). Competencies in using sketchpad in geometry teaching and learning: Experiences of preservice teachers. *African Journal of Research in Mathematics, Science and Technology Education, 17*(3), 231–243. https://doi.org/10.1080/10288457.2013.848536

New Zealand Ministry of Education. (2020). *Technology in the New Zealand curriculum (revised Technology learning area)*. https://nzcurriculum.tki.org.nz/The-New-Zealand-Curriculum/Technology

Noss, R., & Hoyles, C. (1996). *Windows on mathematical meanings: Learning cultures and computers*. Kluwer Academic. https://doi.org/10.1007/978-94-009-1696-8

Ontario Ministry of Education. (2020). *Elementary mathematics curriculum*. https://www.dcp.edu.gov.on.ca/en/curriculum/elementary-mathematics

Papert, S. (1972). Teaching children to be mathematicians versus teaching about mathematics. *International Journal of Mathematics Education, Sciences and Technology, 3*(3), 249–262. https://doi.org/10.1080/0020739700030306

Papert, S. (1980). *Mindstorms: Children, computers, and powerful ideas*. Basic Books. https://tinyurl.com/3kvnjauz

Papert, S., & Harel, I. (1991). Situating constructionism. In I. Harel & S. Papert (Eds.), *Constructionism* (pp. 1–11). Ablex. http://www.papert.org/articles/SituatingConstructionism.html

Rabardel, P. (1995). *Les hommes et les technologies: Approche cognitive des instruments contemporains* [People and technology: A cognitive approach to contemporary instruments]. Armand Colin. https://hal.archives-ouvertes.fr/hal-01017462/document

Ralph, B. (n.d.). *Mathematics integrated with computers and applications II: Five programming-based math project assignments*. Brock University. https://ctuniversitymath.files.wordpress.com/2020/08/mica-ii-assignment.pdf

Ruthven, K. (2014). Frameworks for analysing the expertise that underpins successful integration of digital technologies into everyday teaching practice. In I. A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era* (pp. 373–393). Springer. https://doi.org/10.1007/978-94-007-4638-1_16

Sacristán, A., Santacruz-Rodríguez, M., Buteau, C., Mgombelo, J., & Muller, E. (2020). The constructionist nature of an instructor's instrumental orchestration of programming for mathematics, at university level. In B. Tangney, J. K. Byrne, & C. Girvan (Eds.), *Proceedings of the 2020 Constructionism conference* (pp. 523–536) http://www.constructionismconf.org/wp-content/uploads/2020/05/C2020-Proceedings.pdf

Sangwin, C. J., & O'Toole, C. (2017). Computer programming in the UK undergraduate mathematics curriculum. *International Journal of Mathematical Education in Science and Technology, 48*(8), 1133–1152. https://doi.org/10.1080/0020739X.2017.1315186

Santacruz, M., & Sacristán, A. (2019). Una mirada al trabajo documental de un profesor de primaria al seleccionar recursos para enseñar geometría [A look at the documentary work of an elementary school teacher selecting resources to teach geometry]. *Educación Matemática, 31*(3), 7–38. https://doi.org/10.24844/EM.

Sinclair, N., & Patterson, M. (2018). The dynamic geometrisation of computer programming. *Mathematical Thinking and Learning, 20*(1), 54–74. https://doi.org/10.1080/10986065.2018.1403541

Sysło, M. M., & Kwiatkowska, A. B. (2015). Introducing a new computer science curriculum for all school levels in Poland. In A. Brodnik & J. Vahrenhold (Eds.), *Informatics in schools. Curricula, competences, and competitions* (pp. 141–154). Springer. https://doi.org/10.1007/978-3-319-25396-1_13

Thomas, M. O., Hong, Y. Y., & Oates, G. (2017). Innovative uses of digital technology in undergraduate mathematics. In E. Faggiano, F. Ferrara, & A. Montone (Eds.), *Innovation and technology enhancing mathematics education* (pp. 109–136). Springer. https://doi.org/10.1007/978-3-319-61488-5_6

Trouche, L. (2004). Managing the complexity of human/machine interactions in computerized learning environments: Guiding students' command process through instrumental orchestrations. *International Journal of Computers for Mathematical Learning, 9*(3), 281–307. https://doi.org/10.1007/s10758-004-3468-5

Vandevelde, I., & Fluckiger, C. (2020). L'informatique prescrite à l'école primaire. Analyse de programmes, ouvrages d'enseignement et discours institutionnels [Computers prescribed in elementary school. Analysis of programs, educational works and institutional discourse]. *Colloque Didapro-Didastic, 8*. https://hal.univ-lille.fr/hal-02462385/document

Vergnaud, G. (1998). Towards a cognitive theory of practice. In A. Sierpinska & J. Kilpatrick (Eds.), *Mathematics education as a research domain: A search for identity* (pp. 227–240). Springer. https://doi.org/10.1007/978-94-011-5470-3_15

Webb, M., Davis, N., Bell, T., Katz, Y. J., Reynolds, N., Chambers, D. P., & Syslo, M. M. (2017). Computer science in K-12 school curricula of the 21st century: Why, what and when? *Education and Information Technologies, 22*, 445–468. https://doi.org/10.1007/s10639-016-9493-x

Weintrop, D., Beheshti, E., Horn, M., Orton, K., Jona, K., Trouille, L., & Wilensky, U. (2016). Defining computational thinking for mathematics and science classrooms. *Journal for Science Education and Technology, 25*, 127–147. https://doi.org/10.1007/s10956-015-9581-5

Wilensky, U. (1995). Paradox, programming and learning probability. *Journal of Mathematical Behavior, 14*(2), 253–280. https://doi.org/10.1016/0732-3123(95)90010-1

Researching Professional Trajectories Regarding the Integration of Digital Technologies: The Case of Vera, a Novice Mathematics Teacher

Mónica E. Villarreal and Cristina B. Esteley

Abstract This study examines the professional trajectory of a novice mathematics teacher, Vera, concerning her integration of digital technologies (DTs). This case study aims to: (1) characterise Vera's initial views and experiences as a student regarding the use of technologies for mathematics education and (2) analyse in depth how Vera integrates technologies while acting as a mathematics teacher. Vera was invited to participate in the study due to her positive attitude towards the use of digital technologies. Different moments in her trajectory as a preservice teacher and as novice teacher are analysed to highlight the different kinds of relationships she established with DTs. To analyse those kinds of relationships, we used a taxonomy of four metaphors that represent different types of relationships with technology: as *master*, *servant*, *partner*, and *extension of self*. The integration of new technologies into Vera's teaching practices is then analysed using five dimensions: *working environment, resource system, activity structure, curriculum script,* and *time economy.* Our results show a prevalence of Vera's relationship with technology as a partner or extension of herself, moving on to more sophisticated ways of integrating new technologies when she started teaching regularly at school. The study provides useful insights to support the rethinking of how technology use is introduced and taught within teacher education programmes.

Keywords Professional trajectories · Novice mathematics teachers · Integration of digital technologies · Teaching practices

---

M. E. Villarreal (✉)
Consejo Nacional de Investigaciones Científicas y Técnicas, Argentina and Facultad de Matemática, Astronomía, Física y Computación, Universidad Nacional de Córdoba, Córdoba, Argentina
e-mail: monica.ester.villarreal@unc.edu.ar

C. B. Esteley
Facultad de Matemática, Astronomía, Física y Computación, Universidad Nacional de Córdoba, Córdoba, Argentina

A. Clark-Wilson et al. (eds.), *The Mathematics Teacher in the Digital Era*, Mathematics Education in the Digital Era 16,
https://doi.org/10.1007/978-3-031-05254-5_12

1 Introduction

Policies to incorporate DTs across the Latin American educational systems began in the 1990s following different models and with varied impact, depending on the country. Lugo and Delgado (2020) analyse the implementation of information and communication technology (ICT) policies in the regional education systems in the period since 2007, concluding that despite these policies, “it has not been possible to transform educational practices in such a way that they take advantage of DTs to improve teaching practices” (p. 22, our translation). This fact has a direct association with teacher education, particularly for initial teacher education where preparation in the pedagogical use of DTs is an ongoing demand. Thus, the authors recognise the need to “strengthen teacher education policies in the pedagogical use of digital technologies” (p. 22, our translation). This regional panorama related to teacher education includes the particular case of mathematics teacher education. Ruiz (2017) reports on the scarce training of mathematics teachers in the use of DTs in some Latin American countries. At the international level, Clark-Wilson, et al. assert that:

> …despite over 20 years of research and curriculum development concerning the use of technology in mathematics classrooms, there has been relatively little impact on students’ experiences of learning mathematics in the transformative way that was initially anticipated (2014, p. 1).

Six years after that publication, it seems that, at least in our country, Argentina, the problems associated with the integration of DTs in mathematics teacher education have still not yet been sufficiently addressed.

Alongside, the use of DTs has been recommended in the mathematics curriculum for secondary schools in Argentina (12–17-year-old students) since 2011, as well as in documents with didactic recommendations, since 2009, or the standards for mathematics teacher education since 2011. However, the use is still scarce in both contexts. Some reasons for this lack of use are related to institutional constraints, such as the lack of updated equipment or the existence of a conservative academic culture that does not promote the use of DTs at school. Other reasons are related to difficulties of access or insufficient digital literacy for both students and teachers. Furthermore, there are other reasons related to the limited use of DTs in schools such as poor trajectories in the use of DTs during the initial or in-service education of mathematics teachers and the low value attributed to DTs for mathematics education. At the same time, it is possible to find educational practices in mathematics classes in which DTs are significantly integrated into the didactical proposal of certain teachers (Mina, 2018; Sessa, 2018; Esteley, 2014). Therefore, we can find schools with absent or poorly integrated DTs, and schools with significantly integrated DTs.

Focusing on the last case, this chapter proposes an in-depth study of a novice female mathematics teacher's trajectory, called Vera, who has succeeded in integrating DTs in her mathematics classes.

Our study has two aims and a set of research questions that align with each aim.

Aim 1: To characterise Vera' initial views and experiences as a student regarding the use of technologies for mathematics education.

RQ1. What kind of experiences with DTs did Vera have while she was a student in high school or in the undergraduate teacher education programme at the university?

RQ2. What spontaneous views about the use of technologies for teaching and learning mathematics did Vera have?

RQ3. How did Vera use DTs while taking certain mathematics education courses?

Aim 2: To analyse in depth how Vera integrates technologies while acting as a mathematics teacher.

RQ4. In which ways did she integrate DTs in her first teaching practice, and later, as a novice teacher?

In the next section, we present a brief review of research related to our study.

2 Teacher Education and Digital Technologies

As highlighted in several studies, the integration of technology in mathematics classes responds to diverse and complex factors. These factors sometimes go beyond the mere disposition of the teacher, and some of them can become barriers to such integration. For instance, Desimone et al. (2013), and Ertmer and Ottenbreit-Leftwich (2010), identify different barriers such as: *resources*, *knowledge and skills*, *institution*, *assessment, subject culture* and teachers' *attitudes* and *beliefs.* Additionally, Drijvers (2015) identifies three factors which might inhibit or promote successful integration of DTs: the *teaching design*, the *role of the teacher*, and the *educational context*.

Ndlovu et al. (2020) report results from a study in which they investigated South African preservice teachers' (PSTs) beliefs about their intentions to integrate ICTs in their future mathematics classes. The study shows that attitudinal beliefs, and control over ICT use, shape PSTs' intentions to use technologies in classes. Based on their analysis, the authors recommend that technologies should be integrated into teacher education curricula, as early as possible, if we intend PSTs to integrate ICTs in their future teaching.

Gurevich et al. (2017) conducted a study involving a group of novice Israeli mathematics teachers who had attended a teacher education programme in which the use of technologies was encouraged. The authors conducted a longitudinal study to trace the participants' choices of technological tools and their attitudes toward the integration of technologies in mathematics teaching at three stages of their professional development: two stages as trainee teachers and a third as novice teachers. The paper reports a significant increase in the recognition of the technological benefits, and the incorporation of new technological tools for teaching when the participants became practicing teachers.

Drijvers et al. (2014) assert that the integration of DTs in secondary school mathematics classes has not yet been successful. Moreover, the authors consider that teachers are "crucial players" for such integration. Their study reports a gradual introduction of DTs in the classes of two experienced teachers in the Netherlands. They conclude that the process of integrating technology in mathematics classes is not simple and requires that teachers have an early immersion in the use of technology for teaching. In this sense, Ertmer and Ottenbreit-Leftwich (2010), and Hammond et al. (2009) point out that preservice teacher education that includes the use of technologies influences the type of teachers' future instruction and their capability to manage technological challenges.

Stein et al. (2019) study attitudes of novice mathematics teachers towards the use of technological tools in their teaching. They researched 14 novice teachers from Israel who studied in a technologically rich environment. In their results, they conclude that the novice teachers adopted digital tools for teaching and learning in a deliberate and rational way. Although the novice teachers recognised the benefits of using ICTs in their classes, they also pointed out some difficulties linked to institutional aspects.

Goos (2005) reports on PSTs' and novice teachers' pedagogical practices and beliefs about the integration of technologies into the teaching of mathematics in Australian secondary schools. The study focuses on cases of novice teachers who graduated from a technology-enriched teacher education programme. The author conducted her study from a sociocultural perspective in which teacher's beliefs can change in relation to the social environment, and the teachers' related goals and actions. The cases examined show that the development of the pedagogical identities of novice teachers related to the use of technology is shaped by constant negotiations between their teaching environments, actions, and beliefs. The study also evidences that novice teachers had to overcome some constraints of their working environments to integrate the technologies in their classes.

Considering the development of mathematics teachers' professional identities and agency, Losano et al. (2018) conducted an interpretative case study centred on one novice mathematics teacher who worked in a secondary school in the city of Córdoba (Argentina). As a student teacher, she had experienced the possibility of working in technologically rich environments. In her first years as a teacher, despite certain constraints in her working environment, she found ways to integrate technologies into her mathematics classes. She mainly developed her professional identity and agency in relation to technologies by incorporating the feasible teaching practices of her school, the positions she could occupy as a newcomer in the institution, and also the cultural practices and discourses embodied during her preservice education.

This literature review reports a small sample of studies that, in placing the focus on teachers and their educational trajectories, identify factors that hinder or favour the process of technological integration in mathematics classes. Teachers are recognised as central actors in this integration. Hence, we consider that our study complements existing research findings by providing evidence of how a novice teacher's initial educational process impacts the ways in which she integrates DTs in her later classes.

3 Theoretical Framework

In this section, we describe some framing ideas regarding teacher education related to DTs, and outline the dimensions that are considered to support the analysis of Vera's integration of technologies.

Teacher education and the relationships that teachers establish with DTs can be recognised as influential factors for the integration of DTs in mathematics classes. These relationships are forged by their experiences within different environments. More specifically, (preservice or in-service) teachers' trajectories and previous personal experiences constitute the matrix from which teachers interpret, and make sense of themes, debates and tasks that are pertinent in their respective learning environments (Menghini, 2015; Edelstein, 2011; Vezub, 2009).

For Vezub (2013), teachers' professional trajectories[1] result from the interaction of multiple objective elements (e.g., the context of teachers' performance, initial and continuous education) and subjective ones (e.g., their motivations and expectations). The author proposes the idea of trajectory to emphasise teacher education as a complex and long-term process that articulates initial and in-service education. Vezub considers that professional trajectories are neither linear nor uniform, and that they can be seen as the result of actions and practices developed by individuals in specific situations over time. The interactions between the existing structures of opportunities, and the appropriations that individuals achieve, according to their own objective and subjective possibilities, are synthesised through these trajectories.

Vezub's ideas help us to understand that the integration (or not) of DTs in teachers' professional trajectories is conditioned by the existing opportunities and the possibility they have to take advantage of such opportunities. According to Borba and Villarreal (2005), technologies may reorganise teaching and learning practices, curriculum content, and ways of thinking and knowing. Particularly, in mathematics

[1] Vezub (2013) proposes the term *trajectory* considering Bourdie's use and definition of the term when referring to the trajectories of subjects in a given social field.

classes, DTs are understood as actors that demand new pedagogical approaches. However, such pedagogical approaches that seek to integrate DTs[2] in the mathematics classes may not be implemented, due to the multiple factors listed above, if the actors involved in the classroom do not form significant relationships with the technologies.

Goos et al. (2000) developed four metaphors to form a "taxonomy of sophistication with which teachers and students work with technology" (p. 307). This taxonomy describes four different roles for technology in relation to teaching and learning interactions:

- *Technology as master*: the user is subordinate to the technology and is only able to make use of some features due to the limited individual knowledge and the force of circumstance.
- *Technology as servant*: the user knows the technology but uses it in a limited way to support their usual way of performing tasks.
- *Technology as partner*: the user makes creative use of technology to increase the power over their learning.
- *Technology as extension of self*: the user incorporates technological expertise as an integral part of their repertoire as teacher or student. In this case, powerful use of technologies "forms an extension of the user's mathematical prowess" (p. 312), and also of the pedagogical skills of teacher users. This is a type of relationship with technology that involves the highest level of functioning.

The metaphors of Goos et al. (2000) constitute an analytical tool that allows us to analyse the type of relationship with technology that Vera built throughout her educational trajectory. Alongside, the interplay between the *five structuring features of classroom practice* for understanding the integration of new technologies into daily mathematics classroom practice proposed by Ruthven (2009) offers a rich analytical lens to scrutinise Vera's teaching practices both as a preservice teacher and as a novice teacher. Working from a perspective that focuses on the daily work of teachers, Ruthven identifies five key structural characteristics of classroom practice that relate to technology use: *working environment, resource system, activity structure, curriculum script,* and *time economy.*

The *working environment* concerns the physical arrangements and class organisation required for the introduction of technologies for teaching. Technologies have not only the potential to expand the range of tools and materials available to support school mathematics, but they also imply the need to build a consistent *resource system*. The use of DTs may require a set of adaptations of certain established repertoires in the construction of classroom activities that frame the actions and interactions between teachers and students or among students, which would imply the creation of prototypical *activity structures* or cycles for some types of lessons.

---

[2] Briefly, for the purposes of this article, we are specifically interested in DTs, which include the internet, any type of software (GeoGebra, spreadsheets) and programming languages (such as Python).

The incorporation and integration of instruments and resources require that teachers make new choices, organisation, and sequencing of the content to be taught, alongside the activities and resources for teaching. This implies that teachers will need to reorganise their *curricular scripts,* into which the activities and tasks are contemplated. These scripts are influenced by the resources that are incorporated, the possible students' difficulties and the learning environment.

Finally, the introduction of DTs can influence the *economy of time* in the class, changing the "rhythms" of work and the creation of "didactic time". This didactic time is measured in terms of the advancement of knowledge within the classroom.

4 The Contextual Frame of the Study

Our research was conducted with the participation of Vera, a novice mathematics teacher, who graduated from the mathematics teacher education programme at the University of Córdoba (UNC) at the end of 2017. For this study, it is necessary to consider two main contexts that frame our results and analysis: the UNC teacher education programme, and the school where Vera has been working as a mathematics teacher since 2018.

4.1 Vera's Educational Context as a Preservice Teacher

The teacher education programme at UNC that prepares mathematics teachers for secondary schools lasts 4 years. Sixty six percent of the curriculum courses are devoted to mathematics and are mainly taught by mathematicians. The remaining courses deal with educational issues and are taught by pedagogues or mathematics educators. Within this second group of courses, there are two annual courses which are central for the PSTs' education: Mathematics Education (ME) and Teaching Methodology and Practice (TMP). They are included in the third and fourth year of the programme, respectively. Both are 30 weeks courses of two 4-h classes per week. Every year since 2011, at least one of the authors of this chapter has taught on these courses.

Within the mathematics courses of the teacher education programme, PSTs experience few mathematical activities in which DTs are significantly integrated. Consequently, work with technologies during the ME and TMP courses is essential if future teachers are to be expected to integrate technologies into their school mathematics classes.

In the ME course, several themes are studied, among them, the use of technologies. For this course, we adopt the epistemological perspective presented by Borba and Villarreal (2005), which assumes that knowledge is produced by collectives of humans-with-media[3] and that cognition is a social enterprise that includes the media with which knowledge is produced. During the course, both the teachers and the PSTs discussed these ideas, analysed examples, and recognised the role of the media in the processes of knowledge production as well as in the mathematical teaching and learning processes. Moreover, they discussed possibilities, scope, and conditions for using technologies in educational contexts, and they analysed synergistic pedagogical approaches for the use of DTs. Mathematical tasks are solved using different technologies, such as calculators, GeoGebra, or PhET Interactive Simulations.[4] The advantages and disadvantages of the use of technology in different contexts arise from the texts studied, personal experiences, discussions, and the tasks solved during the course.

Although the topic “DTs in mathematics education” is specifically addressed over a period of one and a half months, DTs are also present in the treatment of different curriculum topics throughout the ME course. For example, when mathematical modelling is studied as another important theme, the synergy between technologies and the modelling process arises naturally when the PSTs develop free open modelling projects (Villarreal et al., 2018).

The path proposed in the ME course provides some foundational tools and strategies for the TMP course, which many PSTs attend the following year. The TMP course’s central aim is for student teachers to develop their first teaching practice in secondary school classes (which lasts for 1 month). During the first four-month period of the TMP course, issues corresponding to the macro-educational and the micro-didactical levels of the curriculum are addressed. The analysis of certain learning environments (previously introduced during the ME course) are deepened and the main variables that influence lesson planning are discussed.

The PSTs first teaching practice at secondary schools is carried out in groups of two or three within the same school and grade, under the supervision of one of the TMP course teachers and the secondary school teacher of the grade assigned for the teaching practice. Before teaching starts, each group conducts observations in the assigned classes, develops lesson plans, prepares materials and elaborates scripts for each class, anticipating possible students’ actions or difficulties, interventions and concerns. The overarching work of teaching is continuously under revision according to the emerging conditions and requirements of the school. If schools are richly equipped with DTs, PSTs must integrate them into their teaching. When conducting the teaching practices, one PST oversees the class, another PST observes it, acting as an assistant, if necessary. When the teaching practices conclude, each group of PSTs writes a report and prepares an oral presentation to share their work with teachers and classmates on the PST programme.

[3] For these authors, media means any kind of tool, device, equipment, instrument, artefact, or material resulting from technological developments, but also includes orality and writing.

[4] PhET Interactive Simulations (https://phet.colorado.edu/) is a non-profit open educational resource project at the University of Colorado Boulder (USA), which provides a suite of research-based interactive computer simulations for teaching and learning physics, chemistry, biology, earth science, and mathematics.

4.2 The Context of Vera's Current School

At the beginning of the 2018 school year, Vera worked temporarily in two schools in the city of Córdoba for a short period of time. They were part-time jobs. Then, she quit those jobs to concentrate her teaching at the secondary school where she is currently working, because this institution offered better working conditions. At this privately managed public school,[5] Vera teaches mathematics to 2nd- and 3rd-year students (13–15 years old). During the last 10 years, this institution has been encouraging the use of DTs for the teaching and learning of all subjects and for all courses. Each classroom has wifi internet access and is equipped with a digital board connected to a data projector and a chalkboard. Each student has a tablet or a personal mobile phone with specific software for mathematics, such as GeoGebra, and classic office suite.

Knowing this context supports us to understand and make sense of Vera's answers, ideas, and different decisions within the context of her trajectory as a teacher with respect to DTs.

5 Methodological Procedures

We conducted qualitative research within the interpretative paradigm (Denzin & Lincoln, 2018) as an in-depth study of Vera's case. She was selected to participate in the study for several reasons. We had both taught Vera during the teacher education programme at UNC and so had observed her trajectory as a student and were aware of her positive disposition towards the use of DTs for learning or teaching mathematics. Two years after finishing her studies at the university, we learned that she had started working in a school that has both the all-important technological infrastructure and a favourable position towards the pedagogical use of DTs. Consequently, Vera had now begun to integrate DTs into her classes extensively and we were keen to research the trajectory of her development in this respect.

The study comprises two parts which align to the aims for the study, which focus on Vera's educational and professional trajectories. The former (addressing RQs 1, 2 and 3), refers to Vera's experiences with DTs as a high school student and her first 3 years as a PST. The latter (addressing RQ 4), is based on the analysis of data collected during Vera's first teaching practice as a PST at UNC and an interview conducted in her role as an in-service teacher.

[5] In Argentina, privately managed public schools are those in which the school building and its entire infrastructure belong to a private entity. However, the school employees' salaries are paid by the State. In these schools, students pay a monthly fee that is not as high as the fees of a fully private school.

The taxonomy proposed by Goos et al. (2000), and Ruthven's (2009) conceptual framework are used as analytical tools.

5.1 Vera as a Student

We draw on data from the ME course that Vera attended in 2016, which comprised: individual written answers to tasks related to DTs, the written report of the modelling project carried out by Vera's group, the files produced with software (GeoGebra, spreadsheets, Python, etc.) when solving the problems of the modelling project or other mathematical problems, the slides for the oral presentation of their modelling project, videotape recordings of such oral presentation, and our field notes.

To address RQs 1 and 2, we analyse Vera's written response to a task posed by the teacher to the whole class, when the "DTs in mathematics education" topic was studied in the ME course. In this task, the PSTs were asked to explain (a) their prior experiences with the use of technologies in mathematics classes, and (b) their views about the relationships between the use of technologies and the teaching and learning of mathematics. From this analysis, we aimed to characterise Vera's initial views and experiences regarding the use of technologies for mathematics education.

To address RQ 3, we focus on the analysis of Vera's productions resulting from an intensive use of DTs during a modelling project.

5.2 Vera Acting as a Mathematics Teacher

The data associated with the second aim of the study originate from the written report[6] (Lovaiza & Marchesini, 2017) on Vera's first teaching practice and a semi-structured interview protocol. The interview, which was conducted virtually in 2020 by the second author, probes Vera's reflections and views on her integration of DTs into the ME course, her first teaching practice and her current practice as a novice teacher. The initial questions were:

1. Do you remember in which of the UNC courses you used DTs and for what purpose?
2. The year after you graduated from UNC, you taught at two schools. In those schools, did you use DTs?

[6] This report is open access and published under Creative Commons license.

3. Regarding the school where you are currently working:
   (a) What do you think of the infrastructure offered by the school for the use of DTs?
   (b) Do you incorporate the DTs into your daily class routines?
   (c) How do you organise and complement the resources you use?
   (d) How do you select and organise the activities?
   (e) How do you think that the use of DTs influences the interactions in the classroom or the didactic time?

Before concluding this section, we include a statement of compliance with ethical standards. Four PSTs including Vera gave us their consent, via email, to use in this study the information from their modelling project written report. Moreover, they were also informed by email about this chapter and authorised us to publish their report's images and content. Vera was informed about the aims and scope of the study and the use of her responses during the interview, both at the time of the invitation to the interview and prior to its start. Both Vera and the other three PSTs were given the opportunity to request clarification on what would be reported. Anonymity was also guaranteed.

6 Results

In this section, we present the results with respect to how Vera was developing her relationship with the DTs. We consider instances of her approach to technologies as a high school student and as an undergraduate student. Then, we report ideas and experiences of the integration that Vera developed as a novice teacher. Figure 1 presents a timeline showing Vera's trajectory.

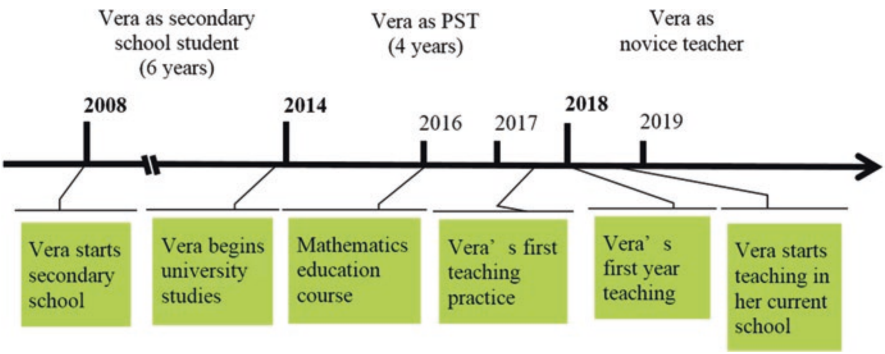


Fig. 1 A timeline of Vera's trajectory

6.1 Vera's Relationship with DTs in Different Educational Contexts

Vera had no experience of using DTs at her high school. She admitted, "…I have not used much technology in mathematics classes. In other subjects, yes, we had to prepare PowerPoint […] as the main task". Besides, she stated that in her mathematics classes, scientific calculators were allowed only to solve computations involving trigonometrical, logarithm or exponential functions. However, when students had to analyse and plot functions, the calculator was not allowed as the task had to be handwritten. According to Goos et al.'s (2000) taxonomy, the role of technology promoted in Vera's high school can be associated with the metaphor of *technology as servant* since it is not used in a creative way, but only for making routine computations.

Vera was asked about the use of calculators when she was a preservice teacher, on which she reflected:

> …the fact that they [the students] are allowed to use the calculator in certain topics to be taught or studied does not make the student think or reason less. I simply see it as a tool that was made to be used as the ruler and the compass.

These words show Vera's special disposition towards the use of DTs, revealing her open-minded position in relation to the use of the calculator which differed from her college classmates' predominant position.

When Vera took the mathematics courses during the teacher education programme, the use of DTs was limited. For instance, not all mathematics faculty embraced the use of DTs, the mathematical tradition was still handwritten mathematics, and the examinations did not allow DTs (including calculators) to be used. More specifically, GeoGebra software was used to make geometrical constructions in a course on Euclidean geometry, but not to experiment or make conjectures. Alongside the mathematics courses, basic notions of programming using Python were introduced in a course on computer programming and applications of numerical methods taught by physicists. At the same time, during the ME course, Vera used DTs profusely. For example, during the aforementioned collaborative mathematical modelling project, and for communicating the progress of such work throughout her different classes.

For the modelling project, Vera and a group of other three students decided to study the following theme: "Looking for letters in magazines for school homework". A common homework for children in the first grades of primary schools in Argentina is to find and cut out, from different magazines, letters or words that contain a combination of certain letters. This theme was chosen since one of the members of the group remembered that she always found it difficult to complete this activity because she had very few magazines at home. The group established some assumptions to solve some typical school tasks related to this theme and posed some questions such as:

- what is the minimum number of magazines needed to find all the letters of our alphabet?

- or, given a specific task such as to cut out 3–5 words that are written with MB or with, BR or with CH, what is the probability of finding that number of words on one page of a magazine?,
- or what is the probability of finding that number of words on more than one page of a magazine?, and what about using more than one magazine?

The group established a set of assumptions, which included: a standard magazine has about 45 pages, a child can look for words or letters in 4 magazines at most, and a child can cut letters of at least 1 cm high. The group also defined the following probabilistic condition: an event is, in their words, "sufficiently probable" when its probability is greater than or equal to 0.7. The selected tasks and the assumptions were established following consultation with elementary school teachers.

To respond to their questions regarding the probability of accomplishing a given specific school task, Vera and her group turned to Python programming. For this, having used a word processor to reproduce the sentences (with letters of at least 1 cm high) from a sample of 30 magazines of 45 pages each, they programed a code to count words according to the selected task. Once the database was built, they used a spreadsheet to calculate probabilities, make charts and graphs. In some combined tasks, they also wrote programs within the spreadsheet itself. Figures 2 and 3 illustrate the work developed within Python and the spreadsheet, respectively.

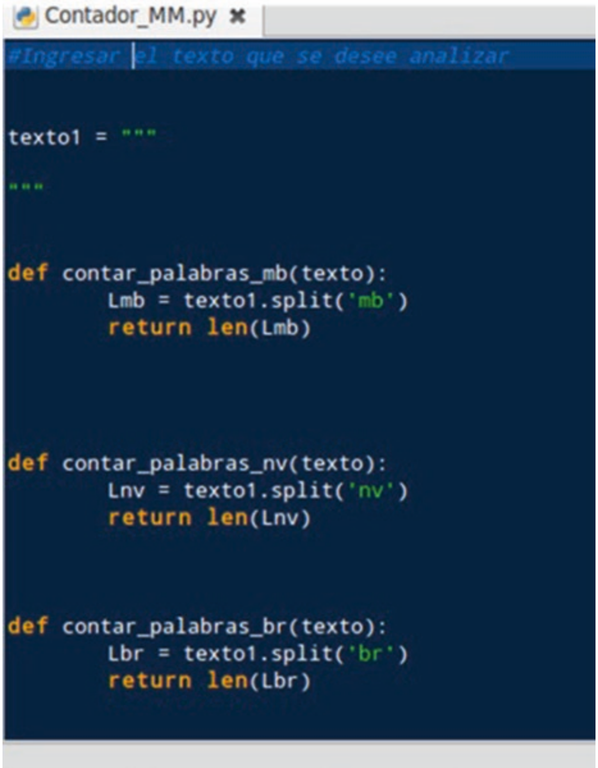


Fig. 2 Python Code. (Source: students' written report)

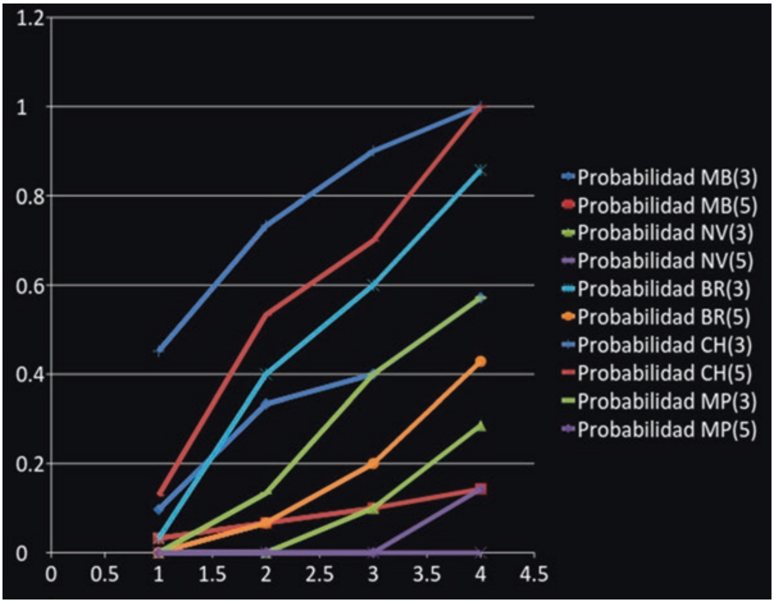


Fig. 3 Graph of probabilities of achieving the tasks (finding 3 or 5 words) versus number of magazines used. (Source: students' written report)

Among their conclusions, the group indicated that, to be able to find all 27 letters of the alphabet with "sufficient probability", it was necessary to have at least 3 magazines to cut. It would not have been "sufficiently probable" to perform any of the proposed tasks with only one magazine. Moreover, they were not aware of the difficulties that these types of activities could bring to the children and their parents, especially at the current time when fewer and fewer magazines are available at home.

In the final report of the modelling project, Vera and her colleagues recognised the importance and support of the various technologies (Python, office suite, and Google docs) used for various purposes at different phases of the modelling project. They wrote: "They [the DTs] allowed us to streamline the organisation and analysis of data and calculations as well as to visualise the results obtained in a clearer and more concise graphic way". In this case, DTs were *partners* that contributed to the efficiency of the group's work, facilitating and enhancing the mathematical production. In addition, the students stressed that they had learned how to use the technologies they needed for their modelling project, more specifically programming in Python and spreadsheet, in a collaborative way. The project was carried out by a thinking collective of *humans-with-DTs*, showing that the technologies were integrated not only as an *extension of* each isolated individual, but as an *extension of themselves*.

In the following subsections, we will focus on Vera's use of DTs to teach mathematics in the school setting as a PST and as in-service teacher.

6.2 *Vera's First Teaching Practice: DTs as Media to Teach and Facilitate Mathematical Production*

During 2017, Vera and her pedagogical partner carried out their first teaching practice in a public high school. They worked with students of 12–13 years of age. The mathematics theme for their practice was angles and triangles, which included the study of pairs of special angles formed when parallel lines are intersected by a transversal line, the classification of triangles, and congruence in triangles: definition and properties.

For the lesson plan, as detailed in the written report of their practice (Lovaiza & Marchesini, 2017), they considered a general objective: "…to generate a teaching proposal that covers the required knowledge…, taking into account the importance assigned by the institution to the development of critical and argumentative thinking of students" (p. 31). Therefore, they claimed: "[we] decided that our plan should be permeated by the use of technologies" (p. 32). The resources and didactical materials used in the classroom were: textbooks, PSTs' notes, blackboard and chalk, a data projector, PowerPoint software, computers, students' mobile phones, GeoGebra software, rulers, pairs of compasses, protractors, and photocopied texts with activities and definitions.

Guided by their general objective and decision to use DTs, they organised the lesson script by considering three didactic units: definition, argumentation, and the congruence of triangles.

For the first didactic unit, the notion of mathematical definition was considered as a teaching object. For this purpose, they worked on what a definition is, the parts of which it is composed, and examples and counterexamples generated for a particular concept. They also designed a teaching plan in which the knowledge about definitions was then applied to define, and then classify angles.

The second didactic unit focused on the justification of mathematical statements. They designed definition analysis activities to support arguments about the validity of any given statement. In this case, they appealed to concepts and relationships between angles defined by two parallel lines and a transversal one. They also proposed activities for students to elaborate written and oral arguments, validating their own and other classmates' statements, by using geometric figures, hand drawn on the chalkboard, or cut from paper, or constructed in GeoGebra and GeoGebra hand-animated dynamic figures as didactical resources.

In the third didactic unit, they applied the concept of congruence already developed within the first unit, which addressed mathematical definitions. They designed congruent triangle construction activities "using GeoGebra as a teaching resource to elaborate conjectures about the properties of and criteria for the congruence of triangles" (p. 33).

Some of the GeoGebra files designed by Vera and her partner are shown in Figs. 4 and 5. Figure 4 shows an image of the animation in GeoGebra used to study the congruence of the corresponding angles between two parallel lines cut with an intersecting transversal line. Figure 5 shows part of the process demonstrating the

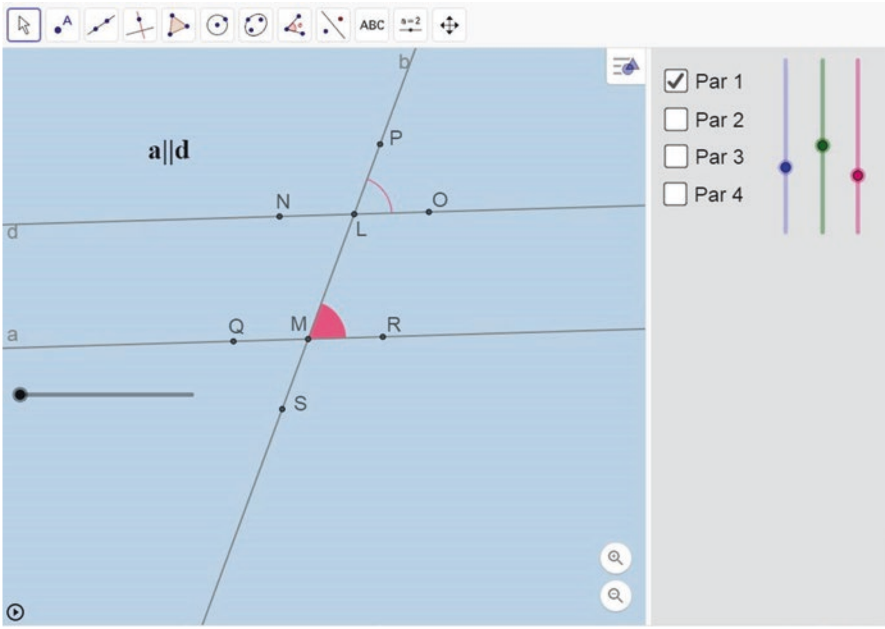


Fig. 4 Image of an animation in GeoGebra. (Source: Lovaiza & Marchesini, 2017, p. 48. Accessible at https://www.geogebra.org/m/esPBcwR3)

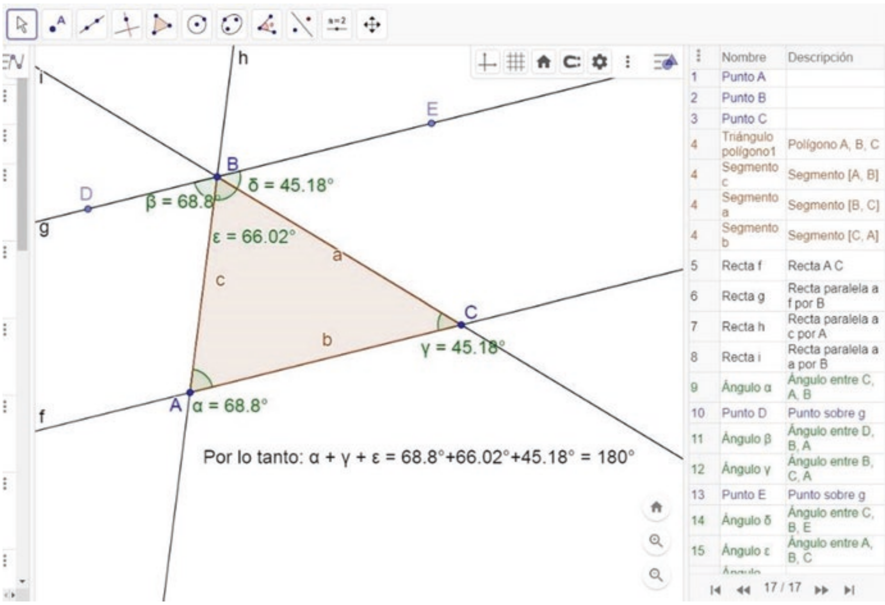


Fig. 5 Sum of the interior angles of a triangle and the construction protocol, on the right. (Source: Lovaiza & Marchesini, 2017, p. 55. Accessible at https://www.geogebra.org/m/naaEb7xY)

property of the sum of the interior angles of a triangle referring to the argument of the congruence of angles determined by two parallel lines cut with an intersecting transversal.

During her first teaching practice, Vera and her partner made an intensive use of DTs. This use was not only for the purpose of exposing or discussing ideas or arguing with their students, but they also proposed tasks mediated by DTs for them. Such tasks invited the use of DTs in small groups or collectively in order to explore situations, propose conjectures and move forward with arguments to support the conjectures. Among the multiple reflections made by Vera and her pedagogical partner in their final report, they highlighted:

> The use of technologies was fundamental for our practices as they permeated and conditioned all the didactic units. Software such as PowerPoint and GeoGebra allowed students to construct figures, analyse them, generate conjectures and make assumptions, question them, and validate them. They facilitated the creation of experimentation and debate scenarios that favoured group interactions and the cooperative generation of knowledge as collective knowledge… (Lovaiza & Marchesini, 2017, p. 108).

As highlighted above, during Vera's first teaching practice, technologies were essential, not only for her, but also for her students. As reported by the pair, the initial didactic decisions that related to the selection, organisation and sequencing of the activities and corresponding resources played a fundamental role in influencing the quality and type of knowledge produced by the students. For Vera, technology had become an *extension of herself* as a teacher, thus encouraging students to *think collectively* among themselves and with the DTs.

It is worth noting that, whilst Vera was answering the first interview question, she explained that both her group's mathematical modelling project and her first teaching practice were instances that contributed and motivated her to use various technological resources as a teacher.

6.3 The Integration of DTs into Current Vera's Daily Work

Vera began teaching in two high schools in Córdoba for a short period. When asked if she had used DTs in those schools, her answer was short and clear: "[…] no, because the resources [DTs] were not available. That's why". And she quickly pointed out that in contrast, at the school where she is currently working, she frequently uses DTs, asserting that "[…] in this school, I use Excel, GeoGebra… a lot, for the students to explore and conjecture, mostly […] I also use the digital screen, the Activ[7] […]".

In Vera's current school, the work with DTs did not require students and teachers to move to a special place away from their usual mathematics classroom. The classroom had wifi internet access, the students had tablets or mobile phones for personal

[7] She refers to ActivInspire, a lesson delivery software for interactive displays. The digital Screen and the Activ are installed in the classroom.

use and the teacher had a desktop computer connected to a digital screen. Since the available infrastructure facilitated the introduction of the DTs and it did not involve changes in the working environment or in the physical layout, it was not necessary to modify the rhythm of the work routine in the classroom.

For Vera, this new *work environment* was conducive for using DTs, enabling her to retrieve familiar knowledge and skills. But this new environment also meant that she had to broaden her knowledge. As Vera explained, while she was teaching, she was learning to use new resources such as the digital screen. She began to learn by observing her colleague's work, and then attended a course offered at the school in which she learned to use the digital screen and the school's Moodle platform.

Although this *work environment* allowed her to access all the DTs needed for the daily work in the classroom, she did encounter some technical problems such as "the electricity gets disconnected, the internet doesn't work, or the screen doesn't turn on…" For Vera, what was important for *the daily routine of the mathematics classes* was not only the access to DTs, but the activities proposed that included the use of DTs. She explained:

> […] I find it hard to imagine my classes without technology as a medium. Obviously, not always the same type of use, nor with the same intensity or the same role. That is, there are classes where it is auxiliary. For example, the use of a calculator or designing a small presentation to share. And I make other [uses] in which the class is based on observing a GeoGebra animation and making conjectures, or making a construction which we could not do without the tool [the DTs] […] that depends on the goals established for the class.

In the *scripts* for her daily classes, Vera incorporated activities, purposes, DTs and other resources. However, the *curriculum* is shaped through the students' interactions with DTs. For instance, Vera indicated that she usually tried to follow a structure for each class. She begins by presenting the objectives and the organisation of the class. In the classes, the teacher and students interact using presentation slides (previously prepared by Vera) to introduce a new topic or to assign tasks for her students that are displayed using the digital screen. For group activities during the class, each group prepares presentation slides and some groups are chosen to presentation their work to their classmates using both the digital display board and the chalkboard. When taking notes or completing activities, students can use either pen and paper or a digital pen and tablets, or a combination of both.

Although Vera and her students have a wide range of available materials or tools, she preferred to develop her lesson material using the digital screen since the resulting work can always be saved as a pdf file. Following the lesson, she uploads the files to the Moodle virtual classroom so that everyone could have access to these materials. In some instances, she also promotes the use of compasses, rulers, or other geometric tools, or pencil and paper to encourage the development of fine motor skills. This integration of non-digital media such as these is also motivated by the institutional factor that tests must be handwritten to conform to the school rules.

By recognising the school's specific demands, Vera was able to overcome the challenge of building a *coherent resource system* composed of digitals and non-digital tools.

When dealing with DTs in the classroom, Vera highlighted: "I think that the DTs encourage interactions in the classroom, well now [she referred to the classes during the COVID-19 pandemic], in this virtual model, the Moodle platform favours discussions, […] the production of conjectures and explorations".

When referring to the production of conjectures, she explained, "that was the theme of my first teaching practice, and I continue to implement it", referring to her current pedagogic approach. When asked about the relationships between DTs and time, Vera noted that the time spent working with DTs is based on its contributions to students' learning and not to the *economy of time*. She illustrated this idea with an example: when teachers have to explain the property of the sum of the interior angles of a triangle, they may resort to a time-saving strategy of informing that the sum is 180°. But, if they want the students to discover such property, they can propose exploratory activities using DTs. In this last case, the nature of the knowledge and learning will be more meaningful, and the time invested will become a *didactic time*.

Didactic time for Vera seems to be measured in terms of the richness of the interactions between the participants in the classroom as well as their progress in learning mathematical concepts and processes. The economy of time and rhythm goes beyond the time measured by the clock, but it seems to be measured in terms of the advances in the students' learning.

Vera's interview responses evidence a *work environment* that is conducive to the integration of DTs, which seems to start when she produces the *curriculum script* for her classes. The *script assembles goals, a structure of activities*, a *coherent system of resources*, and a *didactic time* that privileges the students and their learning as well as the production of rich mathematical and technological knowledge. In that sense, most of Vera's answers put into play an interesting network of ideas in which the dimensions proposed by Ruthven (2009) are evident.

Finally, Vera summarises her vision of the deep sense she perceives about the integration of the DTs in her words, "[…] technology is omnipresent at this time. Kids always have their mobile phones in their hands, so they always have their maths folder[8] in hand".

7 Discussion and Conclusions

Our study contributes to the body of research on the professional trajectories of mathematics teachers by considering one teacher's passage from a PST to becoming a teacher into the first years of teaching and focuses on issues related to the integration of DTs in this context.

[8] It has been translated as *math folder* to refer to what in Argentina we know as *carpeta de matemática*, which is a school material in which a student can store their mathematics work, written assignments, class notes, etc. In the case of a mobile phone, the *carpeta de matemática* would be a folder containing files with all the work done in mathematics class.

As reported in the literature, current integration of DTs in mathematics classes, in which teachers are recognised as crucial players, is not satisfying the initial expectations with respect to improving mathematical learning (Drijvers et al., 2014). Many authors (Desimone et al., 2013; Ertmer & Ottenbreit-Leftwich, 2010; Drijvers, 2015) have listed factors that can act as barriers or promoters of such integration. In the analysis of Vera's case, it is possible to observe some of these promotional factors in action. For example, the presence of an *educational context* that favours the integration of DTs in the school environment resulting from an infrastructure that guarantees access conditions or Vera's positive *attitude* towards the use of DTs. This attitude was evidenced in: her position towards the use of calculators; the profuse and creative use of DTs during the development of a group modelling project; the type of tasks she proposed in her first teaching practice; and her statements about the evolving role of DTs in her current professional life. Our study provides evidence that supports the points made by Ertmer and Ottenbreit-Leftwich (2010) and Hammond et al. (2009) regarding how experiences with DTs during teachers' initial education influence and impact upon their emerging teaching style.

In Vera's case, the use of DTs in the mathematics teacher programme at the UNC was particularly intensive within the ME course and during her first teaching practice. In the ME course, PSTs carried out a mathematical modelling project in groups. The project conducted by Vera and her colleagues using multiple technologies, discussed in this chapter, was part of the data used in a previous study we reported on Villarreal et al. (2018). In that article, we showed evidence of the synergy between the use of technologies and the development of modelling tasks. The analysis we present in this chapter constitutes new evidence of such a synergic relationship and brings an example of a type of task that allows a natural integration of DTs in mathematics classes. Borba and Villarreal (2005) point out that the association of exploratory activities, technology and modelling exhibits a natural synergy. Another example of tasks that call for the integration of DTs in a significant way is the creation of learning scenarios where technologies allow exploration, production of conjectures and arguments to justify their validity, as Vera and her pedagogical pair did during their first teaching practice.

Throughout Vera's trajectory, we could observe the different types of relationships she established with technologies. Following the taxonomy of Goos et al. (2000), we found instances in which technology was assumed as a *servant*, but it also assumed other significant roles. For example, when a collaborative modelling project was being developed, a collective of humans-with-DTs was constituted, and technology was assumed as a *partner*, and even became an *extension of selves*. As observed by Goos (2005), the ways of working with DTs can become more varied and sophisticated over time, moving from using technology *as a servant* to *an extension of self*.

Despite the limitations of the type of research carried out (an individual case study) in terms of the possibilities of wider generalisations, details of a novice teacher's professional trajectory in relation to DTs provide clues for rethinking the actions in our teacher education programme in which DTs are completely absent from the teaching of several mathematics courses. In other courses, DTs have a mere auxiliary role, acting as *servants*, which means that they are neither integrated meaningfully into the teaching or learning processes nor does their presence result in changes in the task, DTs merely accompany the usual performance of tasks. It is therefore necessary to change the approach from using technology *as a servant* to using technology as a *partner* and as *an extension of self* (Goos et al., 2000). However, as pointed out by Goos, this is not an easy task. For this to happen, during initial teacher education, it is necessary to create learning scenarios involving tasks that demand technology use that go beyond the teacher-centred lecturing pedagogy that is typical of the Argentinian university context. As previously highlighted, modelling or explorations-with-technologies scenarios are a possible option to generate another type of relationship with the DTs.

Vera's case was chosen due to its uniqueness. The relationship that Vera has established with DTs can be characterised as an *extension of self*. In her classes, this underpins her significant integration of a variety of DTs, both in terms of the types of tasks posed and in the ways that students participate. Vera's earlier experiences as a student alongside the current *work environment* in which she is immersed, facilitate the organisation and management of her students' access to DTs. The *system of resources* developed by Vera, enables coordinated work between digital and non-digital technologies. Such coordination is related to the choice of a *system of activities* declared in her *curricular script*. The time spent in the classroom is for Vera a true *didactic time* measured in terms of students' learning and knowledge.

Most recently, Vera's knowledge, the available school infrastructure, and her own disposition towards the integration of DTs, facilitated a quick adaptation of her practice to the new requirements of a wholly virtual teaching environment in the context of the COVID-19 pandemic, which hit the world in 2020. During her interview, Vera specified that, although she reduced the number of activities when she worked with students, she continued to privilege joint interactions and the production of arguments. This required some changes in the interaction rules. The school Moodle platform continued to be the space where the memory of the schoolwork was preserved.

Vera's current conditions are favourable for the adoption of DTs. By contrast, Vera had stated that in the first schools where she worked, she did not employ DTs due to a lack of technological resources in the classroom. This shows how the same teacher can act in different ways according to the context in which the teacher is immersed. For this reason, guaranteeing equity of access to DTs in schools is indispensable for the achievement of a quality education that integrates DTs in the teaching and learning of school subjects for all learners.

In the current socioeconomic conditions of our country, with more than 40% of the population below the poverty line,[9] public policies for digital inclusion are absolutely indispensable to guarantee the right of access to technologies for all students in public schools, which are the ones that concentrate the poorest student

[9] Data published by the National Institute of Statistics and Census of the Argentine Republic (INDEC). Available in: https://www.indec.gob.ar/indec/web/Nivel4-Tema-4-46-152. Accessed on 19 Dec 2020.

population. The situation of inequality was aggravated by the COVID-19 pandemic. The sudden change to remote education exposed the inequity in access to DTs that was necessary for educational continuity. This shift towards distance education was detrimental to students from the most impoverished sectors of society.

Under these conditions, the integration of DTs in the classroom cannot be the exclusive responsibility of the teacher. However, when the conditions of access are guaranteed at school, it is absolutely necessary that teachers are prepared to integrate DTs in their classes, overcoming personal prejudices, and understanding access to DTs as a citizen's right. For this to happen, and in accordance with the recommendations of Ndlovu et al. (2020) or Lugo and Delgado (2020), it is essential to integrate the use of DTs into initial teacher education at an early stage.

We suggest that, through these research results, it is feasible to highlight how an initial professional development course can have an impact on how PSTs are able to both appropriate and use DTs in their practice. Within the boundaries of the reported case, it is hoped that the advances of this study can provide input for researchers, teacher educators, and curriculum developers.

Our study leaves open questions. Having investigated this particular case leads us to highlight the need of widen the horizon of the study towards: an investigation of other possible trajectories and the level of integration in the DTs in other Argentinean or Latin American educational contexts involving other teachers. For example, if the pre-condition of access to DTs in schools is not satisfied, but students have access to mobile phones, what kinds of tasks can a mathematics teacher propose to integrate such technology in the classroom? How is it possible to integrate mobile phone technologies into mathematics teacher education programmes so that teachers can later integrate them into their teaching? How can we educate mathematics teachers to be intelligent and knowledgeable users of technology such that they can integrate the applications available on a mobile phone in their teaching? These questions open new horizons for research that we look to explore in the future.

Acknowledgments This research was carried out with the financial support of the *Secretaría de Ciencia y Técnica* (UNC), the *Agencia Nacional de Promoción de la Investigación, el Desarrollo Tecnológico y la Innovación*, and the *Consejo Nacional de Investigaciones Científicas y Técnicas*. We would like to thank the meaningful and insightful suggestions of the reviewers in previous versions of this paper. Finally, we would like to thank our students who made this work possible.

References

Borba, M., & Villarreal, M. (2005). *Humans-with-media and the reorganization of mathematical thinking: Information and communication technologies, modeling, experimentation and visualization*. Springer.

Clark-Wilson, A., Robutti, O., & Sinclair, N. (Eds.). (2014). *The mathematics teacher in the digital era. An international perspective on technology focused professional development*. Springer.

Denzin, N. K., & Lincoln, Y. S. (Eds.). (2018). *The SAGE handbook of qualitative research* (5th ed.). SAGE.

Desimone, L. M., Bartlett, P., Gitomer, M., Mohsin, Y., Pottinger, D., & Wallace, J. D. (2013). What they wish they had learned. *Phi Delta Kappan, 94*(7), 62–65. https://doi.org/10.1177/003172171309400719

Drijvers, P. (2015). Digital technology in mathematics education: Why it works (or doesn't). In S. Cho (Ed.), *Selected regular lectures from the 12th international congress on mathematical education* (pp. 135–151). Springer. https://doi.org/10.1007/978-3-319-17187-6_8

Drijvers, P., Tacoma, S., Besamusca, A., van den Heuvel, C., Doorman, M., & Boon, P. (2014). Digital technology and mid-adopting teachers' professional development: A case study. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era* (pp. 189–212). Springer. https://doi.org/10.1007/978-94-007-4638-1_9

Edelstein, G. (2011). *Formar y formarse en la enseñanza*. Paidós.

Ertmer, P. A., & Ottenbreit-Leftwich, A. T. (2010). Teacher technology change: How knowledge, confidence, beliefs, and culture intersect. *Journal of Research on Technology in Education, 42*(3), 255–284.

Esteley, C. (2014). *Desarrollo Profesional en Escenarios de Modelización Matemática: Voces y Sentidos* [Doctoral dissertation, Facultad de Filosofía y Humanidades – Universidad Nacional de Córdoba]. Editorial Filosofía y Humanidades. E-books. https://ffyh.unc.edu.ar/publicaciones/wp-content/uploads/sites/35/2022/05/EBOOK_ESTELEY.pdf?

Goos, M. (2005). A sociocultural analysis of the development of pre-service and beginning teachers' pedagogical identities as users of technology. *Journal of Mathematics Teacher Education, 8*, 35–59. https://doi.org/10.1007/s10857-005-0457-0

Goos, M., Galbraith, P., Renshaw, P., & Geiger, V. (2000). Reshaping teacher and student roles in technology-enriched classrooms. *Mathematics Education Research Journal, 12*, 303–320. https://doi.org/10.1007/BF03217091

Gurevich, I., Stein, H., & Gorev, D. (2017). Tracking professional development of novice teachers when integrating technology in teaching mathematics. *Computers in the Schools, 34*(4), 267–283. https://doi.org/10.1080/07380569.2017.1387470

Hammond, M., Fragkouli, E., Suandi, I., Crosson, S., Ingram, J., Johnston-Wilder, P., Johnston-Wilder, S., Kingston, Y., Pope, M., & Wray, D. (2009). What happens as student teachers who made very good use of ICT during pre-service training enter their first year of teaching? *Teacher Development, 13*(2), 93–106. https://doi.org/10.1080/13664530903043939

Losano, L., Fiorentini, D., & Villarreal, M. (2018). The development of a mathematics teacher's professional identity during her first year teaching. *Journal of Mathematics Teacher Education, 21*, 287–315. https://doi.org/10.1007/s10857-017-9364-4

Lovaiza, P., & Marchesini, V. (2017). *Una propuesta para aprender a argumentar en geometría con alumnos de segundo año del nivel secundario* [Final teaching practice report, Facultad de Matemática, Astronomía, Física y Computación – Universidad Nacional de Córdoba]. Repositorio Diginal UNC. http://hdl.handle.net/11086/5784

Lugo, M. T., & Delgado, H. (2020). *Hacia una nueva agenda educativa digital en América Latina* (Documento de Trabajo n° 188). Centro de Implementación de Políticas Públicas para la Equidad y el Crecimiento. https://www.cippec.org/wp-content/uploads/2020/03/188-DT-EDU-Hacia-una-nueva-agenda-digital-educativa-en-Am%C3%A9rica-Latina-L....pdf. Accessed on 1 April 2021.

Menghini, R. A. (2015). Los profesores principiantes frente a su formación inicial: entre la información y la construcción de herramientas intelectuales para enseñar. *Polifonías Revista de Educación*. Año IV, *6*(2015), 103–126.

Mina, M. (2018). *Simulaciones-con-Scratch como proceso de modelización matemática: Un estudio de caso acerca de la construcción de conocimiento matemático con alumnos de nivel secundario* [Master thesis, Facultad de Ciencias Sociales – Universidad Nacional de Córdoba]. Repositorio Diginal UNC. http://hdl.handle.net/11086/12854

Ndlovu, M., Ramdhany, V., Spangenberg, E., & Govender, R. (2020). Preservice teachers' beliefs and intentions about integrating mathematics teaching and learning ICTs in their classrooms. *ZDM Mathematics Education, 52*, 1365–1380. https://doi.org/10.1007/s11858-020-01186-2

Ruiz, A. (Ed.). (2017). *Mathematics teacher preparation in Central America and the Caribbean. The cases of Colombia, Costa Rica, the Dominican Republic and Venezuela*. Springer.

Ruthven, K. (2009). Towards a naturalistic conceptualisation of technology integration in classroom practice: The example of school mathematics. *Éducation et Didactique, 3*(1), 131–159. https://doi.org/10.4000/educationdidactique.434

Sessa, C. (2018). About collaborative work: Exploring the functional world in a computer-enriched environment. In G. Kaiser, H. Forgasz, M. Graven, A. Kuzniak, E. Simmt, & B. Xu (Eds.), *Invited lectures from the 13th international congress on mathematical education. ICME-13 monographs* (pp. 581–599). Springer. https://doi.org/10.1007/978-3-319-72170-5_32

Stein, H., Gurevich, I., & Gorev, D. (2019). Integration of technology by novice mathematics teachers–what facilitates such integration and what makes it difficult? *Education and Information Technologies, 25*, 141–161. https://doi.org/10.1007/s10639-019-09950-y

Vezub, L. F. (2009). Notas para pensar una genealogía de la formación permanente del profesorado en la Argentina. *Revista Mexicana de Investigación Educativa, 14*(42), 911–937. http://www.scielo.org.mx/pdf/rmie/v14n42/v14n42a14.pdf

Vezub, L. F. (2013). Hacia una pedagogía del desarrollo profesional docente: modelos de formación continua y necesidades formativas de los profesores. *Páginas de Educación, 6*(1), 97–124. http://www.scielo.edu.uy/pdf/pe/v6n1/v6n1a06.pdf

Villarreal, M., Esteley, C., & Smith, S. (2018). Pre-service teachers working in mathematical modelling scenarios with digital technologies. *ZDM Mathematics Education, 50*, 327–341. https://doi.org/10.1007/s11858-018-0925-5

The Abrupt Transition to Online Mathematics Teaching Due to the COVID-19 Pandemic: Listening to Latin American Teachers' Voices

Mario Sánchez Aguilar, Danelly Susana Esparza Puga, and Javier Lezama

Abstract This study explores the way in which a group of Latin American mathematics teachers cope with the abrupt integration of digital technology into their mathematics teaching caused by the COVID-19 pandemic, which began in 2020. The study gives *voice* to mathematics teachers who have experienced first-hand the digital transition caused by the pandemic. Through an open survey of 179 mathematics teachers from different Latin American countries, teachers are asked how they adapted their mathematics lessons to the new context, how they felt in this transition, and if they received associated material support or guidance. The questionnaire recognises teachers' knowledge and experience by asking them about suggestions or recommendations to other colleagues who are experiencing the same digital transition. This study contributes to broadening our knowledge about the way mathematics teachers deal with the integration of technology in their teaching practices, particularly in situations where such integration is imposed.

Keywords Online mathematics teaching · Digital migration · Latin American teachers' voices · COVID-19 pandemic

M. S. Aguilar (✉)
Instituto Politécnico Nacional, CICATA Legaria, Programa de Matemática Educativa, Mexico City, Mexico
e-mail: mosanchez@ipn.mx

D. S. Esparza Puga
Universidad Autónoma de Ciudad Juárez, Instituto de Ingeniería y Tecnología, Ciudad Juárez, Mexico

J. Lezama
Universidad Autónoma de Guerrero, Guerrero, Mexico

A. Clark-Wilson et al. (eds.), *The Mathematics Teacher in the Digital Era*, Mathematics Education in the Digital Era 16,
https://doi.org/10.1007/978-3-031-05254-5_13

1 Introduction

The infectious COVID-19 disease is caused by the virus SARS-CoV-2 and can lead to serious respiratory complications, and even death. The first COVID-19 outbreak was first identified in Wuhan, China, in December 2019. Since then the virus gradually spread around the world, including Latin America, the region where the teachers who participated in this study live. The global COVID-19 pandemic brought about profound changes in the social, political and economic dynamics of nations around the world. In the particular case of school mathematics instruction, many teachers experienced an abrupt transition from a face-to-face teaching setting to a completely different scenario of remote instruction based on the use of digital tools. However, such a transition is not without complications. It is possible to hear diverse stories from teachers expressing the difficulties resulting from this transition. Several of those difficulties are related to the implementation of technological elements in their mathematics teaching, which in many cases is carried out under adverse conditions.

The digital transition also evidenced a heterogeneous digital culture reflected in a diversity of teaching practices and competencies. Some mathematics teachers seemed to be better prepared for the implementation of digital tools in their teaching practice. This is mainly due to the access and previous experiences with the use of digital tools in mathematics that these teachers had (Csachová & Jurečková, 2020).

This digital transition led the mathematics teachers to make improvements—not without difficulties—at the personal, collective and institutional level that would allow them to build teaching solutions to continue the educational act. Training themselves in the use of digital tools, improving the production of digital materials, designing ways to promote and evaluate mathematical learning in virtual settings, as well as trying to rescue those students lost during the digital transition (Chirinda et al., 2021).

Research on mathematics teacher education and development has studied the limitations, constraints and obstacles related to the implementation and adoption of digital technologies into mathematics teaching (e.g., Abboud-Blanchard, 2014; Thomas & Palmer, 2014); however, such studies were not developed under the extraordinary conditions experienced as a result of the COVID-19 pandemic. In this particular scenario, the implementation and adoption of digital tools was suddenly imposed on many teachers in regions of the world where the economic and social conditions were not always optimal. The study reported in this chapter expands the discussion on the constraints of, and obstacles to, implementing digital technologies in mathematics teaching, by exploring the way teachers cope with the abrupt integration of digital technology (mathematical software, video-conferencing software, learning management systems, YouTube, among others) into their daily teaching practices due to the COVID-19 pandemic.

The aim of this study is to explore the way in which Latin American mathematics teachers deal with the abrupt implementation of digital technology in their teaching practices. The study gives *voice* to mathematics teachers in order to know more about not only how they deal with the digital implementation, but also their feelings about this digital transition caused by the COVID 19 pandemic and the support they have received to navigate it. The study contributes to broadening our knowledge about the way mathematics teachers deal with the integration of technology in their teaching practices, particularly in situations where such integration is imposed upon them.

2 On the Notion of Teachers' Voice

The notion of *teachers' voice* arises as a critical response to a type of educational research focused on producing knowledge about teachers and their work, but which paradoxically tended to ignore the teachers' inquiries and lived experiences of their own teaching practice as a possible source of knowledge (Atkinson & Rosiek, 2008). Thus, the interest in giving teachers a *voice* arises from the need to produce research knowledge that considers the experiences lived by teachers, the teaching contexts that give rise to their voices, and the different things they may have to say about teaching and learning (Hargreaves, 1996).

The notion of "teachers' voice" has been used and conceptualised in different ways within the field of teacher education research. For instance, in the case of mathematics teacher education, the development of the teachers' own voice has been associated with the evolution of a professional identity as a mathematics teacher (Brown & McNamara, 2011). It has also been used as a tool to articulate critical and dissenting thoughts in connection to the teaching of mathematics (De Freitas, 2004). In educational research where the notion of teachers' voice has been most widely used, there are also varied and nuanced definitions. For instance, Frost (2008) defines teacher voice as "the views, experience, and perspective of teachers on educational policy and practice" (p. 347), while Hargreaves (1996) defines it as "the place teachers occupy and the role they play in school restructuring and reform" (p. 12). Another definition is provided by Gyurko (2012): "the expression by teachers of knowledge or opinions pertaining to their work, shared in school or other public settings, in the discussion of contested issues that have a broad impact on the process and outcomes of education." (p. 4).

In this study the notion of teacher's voice is understood as the values, beliefs, emotions, practical experiences, and perspectives of teachers about their work alongside the degree to which those elements are considered, included, listened to, and acted upon when important decisions and changes are being made in the educational context where teachers carry out their work. Some authors have argued about the importance of referring to the teachers' *voices* (in plural) instead of the "teacher's voice" as a representative and unifying entity (e.g., Atkinson & Rosiek, 2008; Hargreaves, 1996). Referring to *teachers' voices* emphasises the individuality and even dissonance of such voices, which have been shaped "by immense variations in the context in which they teach" (Atkinson & Rosiek, 2008, p. 177).

Before introducing our methods to elicit the voices of the teachers in this qualitative study, we present a brief review of previous research focused on analysing the obstacles and constraints that mathematics teachers experience when integrating digital technology into their teaching. To promote the articulation and continuity among research studies, we frame our study by focusing on relevant research published in the first edition of this book (Clark-Wilson et al., 2014).

3 Previous Research on Obstacles and Constraints to Digital Technology Integration

The research focused on analysing the obstacles and constraints that mathematics teachers experience when integrating digital technology into their own teaching is well represented in the first edition of this book (Clark-Wilson et al., 2014). In particular, two chapters address this problem from different perspectives (Abboud-Blanchard, 2014; Thomas & Palmer, 2014).

The study by Thomas and Palmer (2014) reviews research that identifies obstacles to, and constraints on, secondary teachers' implementation of digital technology into their mathematics teaching. Based on the results of a longitudinal study in New Zealand, the researchers identify the following obstacles and constraints to technology use (particularly computers and calculators):

- Time constraints (e.g., the time needed to become familiar with the technology).
- Access to technology (e.g., unavailability of calculators, computers and software).
- Lack of training in the use of technological tools.
- Lack of confidence in the use of technological tools.
- Government and school policies.

Thomas and Palmer (2014) point out that the obstacles and constraints identified can be divided into extrinsic factors (such as the lack of access to technological resources) and intrinsic factors (such as the lack of confidence in the use of technological tools in teaching mathematics). The authors also indicate that their results coincide with the factors influencing teacher adoption and implementation of technology in mathematics teaching identified by other researchers (e.g., Forgasz, 2006; Goos, 2005).

The approach followed by Abboud-Blanchard (2014) to the study of the integration of digital technology (such as dynamic geometry software and online electronic exercise portals) into mathematics teaching is different from that taken by Thomas and Palmer (2014). Based on a synthesis of three studies developed in France, the researcher identifies common characteristics in terms of common responses to shared constraints related to the integration of technology by ordinary mathematics teachers. By *ordinary teachers* she refers to "teachers who are not technology-experts and who are not involved in experimental projects" (Abboud-Blanchard, 2014, p. 298).

Abboud-Blanchard (2014) acknowledges that the practices analysed in the three considered studies are shaped by the socio-educational and institutional conditions in which teachers develop their job, as well as by their personal trajectories. Nevertheless, she claims that it is possible to find regularities in the teachers' responses reported in these studies:

> These regularities seem to be directly related to the common constraints and difficulties that teachers face when using technology and the way that they handle them […] these are choices (though certainly related to the personal component) that reflect how teachers invest the few options left, given the institutional and social constraints. (p. 304)

Abboud-Blanchard (2014) analysed the teachers' common responses to the shared constraints into three axes, which are the:

- *Cognitive axis*. How to simultaneously teach mathematics and use technology in class (related to the mathematical content taught with technology)
- *Pragmatic axis*. How to teach mathematics in new teaching environments (related to what the teacher does and says when implementing a classroom situation using technology)
- *Temporal axis*. How to manage the time for teaching and learning when using technology (related to different aspects of time management)

There are some similarities in the research findings reported by Abboud-Blanchard (2014) and Thomas and Palmer (2014). For example, both researchers coincide in pointing to teachers' lack of confidence in using digital technology as a constraint of, or an obstacle to, its implementation in the teaching of mathematics alongside a (lack of) time.

In our study we also focus on the way teachers implement digital technology in their mathematics teaching and the obstacles they find to doing so; however, our study maintains important differences with its predecessors. Firstly, we not only consider the implementation of digital technologies typically associated with the teaching of mathematics (such as calculators or dynamic geometry software), but we also consider the implementation of more general digital technologies such as videoconferencing software, learning management platforms, and YouTube, among others.

Another important difference with previous studies is the context and conditions in which the process of implementation of digital technologies takes place. The COVID-19 pandemic forced mathematics teachers to abruptly implement the use of digital tools in their teaching, regardless of their institutional and social constraints and conditions. This study explores how mathematics teachers navigate the early stages of this difficult and demanding implementation process.

Finally, another significant difference between this study and those reported by Abboud-Blanchard (2014) and Thomas and Palmer (2014) is the fact that this study does not directly observe the teacher's practice (the "pragmatic axis" according to Abboud-Blanchard, 2014). As explained in more detail in the next section on the research method, this study approximates mathematics teachers' practices through their self-reports captured through an open questionnaire, which were analysed using a grounded (inductive) approach. Consequently, our research enriches the inquiry of the limitations, constraints, and obstacles related to the implementation and adoption of digital technologies into mathematics teaching. In particular, this study aims to broaden our understanding of the way in which mathematics teachers act and feel when faced with a phenomenon of abrupt implementation of digital technology in their own teaching.

4 Method

This chapter and the study that it reports were developed under lockdown during the COVID-19 global pandemic. This section describes the method that was followed to develop the study under these conditions, beginning with a brief description of the pandemic context in which it was developed.

4.1 Context of the Study

The global COVID-19 pandemic triggered social and economic crisis around the world. Among the effects of the pandemic is the global digital migration of thousands of teachers and students from face-to-face mathematics instruction to online mathematics instruction. This research took place during the first months of this massive digital migration.

Although with different issuing dates, the institutional orders to begin distance instruction in Latin American countries took place during the first semester of 2020. The abrupt nature of the digital transition made evident the heterogeneity between, and within, Latin American countries in terms of quality of internet connections, access to digital tools, and digital competencies. Consequently, students and teachers in geographically distant or isolated regions experienced greater difficulties in continuing with their mathematics instruction. Even in large urban areas in Latin America where access to the internet and digital tools is more widespread, access to online instruction was not guaranteed for all students. The socio-economically disadvantaged students faced greater obstacles to staying connected, due to a limited access to digital resources exacerbated by the economic crisis that accompanied the pandemic. In the more severe cases, students and teachers from some particularly underprivileged areas of Latin America were simply detached from the educational system—interrupting their mathematics instruction completely.

4.2 Study Participants

We were interested in hearing the voices of Latin American teachers about their experience teaching mathematics in this new context of instruction. An opportunity sample was constituted by sending email invitations to approximately 800 Latin

American mathematics teachers, asking them to voluntarily answer a questionnaire (see next section) related to the digital transition of their teaching practice due to the COVID-19 pandemic. Despite the pressing moment in which the questionnaire was sent, 179 teachers agreed to answer it.

The teachers who answered the questionnaire were contacted through two main means. On the one hand, teachers associated with the social network *DocenMat* (a regional social network of mathematics teachers interested in mathematics education; see https://docenciaenmatematicas.ning.com) were contacted and invited to answer the questionnaire; on the other hand, teachers who had graduated from an online postgraduate program in mathematics education were invited to participate and to use their personal networks to extend the invitation to other teachers. Although the online postgraduate program is located in Mexico City, it receives in-service mathematics teachers from different Latin American countries (see Gómez-Blancarte et al., 2019).

The 179 teachers who responded were men and women from Argentina, Chile, Colombia, Mexico and Uruguay. One hundred forty-nine of these teachers reported working in public schools and 28 in private schools (two teachers declined to provide this information). The participating teachers work at the university level (45), upper secondary level (66), primary level (44), and some of them (24) declared that they work at more than one educational level, without stating the levels. All the schools where these teachers work are located in urban and semi-urban areas, except for six teachers who through their responses stated that they work in rural areas.

4.3 The Questionnaire

Due to its potential for gathering information from large audiences, and due to the mobility and social distancing restrictions imposed by the COVID-19 pandemic, we designed an online questionnaire. It was constructed with an empathetic spirit that invited teachers to express their views. The questionnaire (see Appendix 1) was designed to elicit:

- general contextual questions not directly related to mathematics instruction (questions 1, 2 and 3).
- how the teachers were adapting their mathematics lessons to the new context (questions 7, 8 and 9)
- how the teachers felt about this transition and if they received related material support or guidance (questions 5, 6 and 10).
- the knowledge and experience of the teachers by asking about the suggestions or recommendations that they would make to other colleagues who are in the same digital transition (question 11).

Of special relevance to the focus of this book are the questions related to teachers' adaptations of the mathematical content and its approach to the virtual format, the time this took and the format of the resulting lessons.

4.4 Implementation of the Questionnaire

The final questionnaire was distributed to the participating teachers using the survey administration software Google Forms. This software automatically anonymised the responses and organised them in a spreadsheet, which facilitated the capture and subsequent analysis of the empirical data. The questionnaire was distributed between May 22 and June 3, 2020 a few weeks after the teachers had begun the digital migration, to gather data on their emotions and experiences at this early stage of the process of transformation.

4.5 Analysis of the Teachers' Responses

The analysis process for the teachers' responses was different for each type of question. In the case of questions that were answered with a "yes/no" or with a small set of possible answers, a frequency count of the answers expressed by the teachers was made (i.e., questions 1, 6 ad 9).

The open questions, designed to give voice to the mathematics teachers, were subjected to open coding (Saldaña, 2013). As mentioned before, this type of questions not only ask teachers about the adaptation of their mathematics lessons to the new context, but also ask about the way they feel and the support they have received during the transition (i.e., questions 7, 8 and 10). This open coding enabled regularities in the teachers' answers to be found, which could then be grouped into categories. In a first level of coding, similar keywords or phrases were identified within the teachers' responses and a code was assigned to each, which would enable later categorical grouping. For example, in several responses, teachers reported difficulties they were experiencing related to the digital transition. The utterances in which the teachers expressed such difficulties were coded according to their nature (see Table 1). These codes were subsequently grouped into the category of *implementation obstacles* that teachers face during the digital transition caused by the COVID-19 pandemic.

5 Results

In this section we present the results of the analysis of the teachers' responses to the open questions. The results are organised into six categories: *implementation obstacles*, *time needed for adapting the lessons*, *teachers' lessons descriptions*, *implementation of digital tools*, *teachers' emotions*, and *teachers' suggestions and recommendations*. These categories represent different aspects of what teachers did and felt during the abrupt digital transition that their mathematics teaching work had undergone. The categories are illustrated with extracts from the teachers' responses to the questionnaire.

Table 1 Codes and responses that constitute the category of implementation obstacles faced by teachers

Code and number of respondents	Code description	Sample response
Lack of training (92)	Teachers stated that they did not receive training from the authorities of their educational institutions	R127: **I did not receive advice**. Only between colleagues and friends from the school. Remains the same until now. Then I resorted to what was available on YouTube
Lack of computer equipment (8)	Teachers stated that either their students or themselves did not have the necessary computer equipment to teach or study online	R147: I have had great difficulty adapting myself to the virtual mode, and in acquiring the appropriate equipment to perform it (**I have old and obsolete computer equipment**)
Limited access to the internet by students (69)	Teachers declare that, for various reasons, students have limited or no access to the internet	R117: As the students come from a rural community, **they do not have unlimited access to the internet** and they do not have computers. We generally work by WhatsApp
Deficiencies in the teachers' quality of internet services (97)	Although most teachers had internet access, some commented that the quality of their service was poor	R131: The internet service is average. **I have had some problems due to the total loss of the internet signal**, one of them precisely on the day of the lesson

5.1 Implementation Obstacles

As mentioned before, in the teachers' responses it was possible to identify obstacles that they faced when trying to migrate from face-to-face mathematics instruction to online mathematics instruction (see the sample responses and number of respondents in Table 1). In particular, two types of obstacles were identified. The first of these is the *lack of training*, which refers to the lack of support that teachers received from the authorities to train themselves in the use of digital tools.

R35: We were not given assistance and I had to do some research around the handling of online whiteboards.

R108: I did not receive support, but with my colleagues we supported each other.

The second type of obstacle refers to the *lack of internet access*. Teachers refer to how the lack of internet access—for both their students and themselves—hampered the development of their courses.

R67: It is very difficult to do an online class as such, because in the environment where I work, students do not have enough financial resources to be in virtual lessons.

R167: I have internet access, but the access to the service is intermittent. There are times of the day where the connection does not allow you to work smoothly.

R74: The internet access in my community is null, so I limit myself to working on WhatsApp.

There were eight teachers who identified the lack of access to adequate computer equipment as an obstacle to implementing their mathematics lessons online (for instance, see sample response R147 in Table 1).

5.2 Time Needed to Adapt Mathematics Lessons

Teachers wrote about the time needed to adapt their mathematics lessons to the online modality. A recurring complaint from teachers was that they had to invest much more time preparing their teaching materials for the online setting than the time needed for planning face-to-face lessons (91 of the teachers reported this). However, there were a few teachers (24) who stated that they did not need much time to plan their lessons, due to their previous experiences of using digital tools.

R72: It takes me a long time. Because I look for activities and also explanatory videos on the topic. If I calculate the time, it would be triple the time I did before. Practically I am working for most of the day.

R149: I changed the topics to start with the simplest ones. It takes a lot of time, since it not only involves preparing each activity in a digital format and explaining it in a way that the student can understand it, but also involves learning to use the [digital] tools and provide feedback to each student.

R90: I do not need much [time] because I constantly use technology in my lessons.

5.3 Teachers' Lessons Descriptions

The analysis of the teachers' descriptions of their own mathematics lessons allowed us to identify a number of similarities within the descriptions. Some teachers (42) declared that, for different reasons, their lessons had to be asynchronous. Teachers (76) also indicated that they had the possibility to communicate synchronously with their students. However, a dominant trait in the teachers' descriptions was the tendency to mimic or reproduce the form of their face-to-face mathematics lessons but in an online setting (53 teachers declared this).

R97: They are lessons recorded and posted online, and I have consultation hours every day.

R64: I only send emails with activities and delivery deadlines.

R18: Two platforms are available for the lessons: Zoom to teach live every day of the week, both morning and evening. The Schoology platform is used to post recorded videos and activities to be solved online (questionnaires, crosswords, word searches). WhatsApp and email are used to send instructions and have contact with parents.

R168: They are lessons very similar to the face-to-face ones, with a traditional explanation supported by presentations, videos and the digital whiteboard. The students ask questions whenever they want, and they use the platform to reinforce knowledge and hand in their homework.

R25: I try to make them as similar as possible to the face-to-face lessons.

R180: I have two groups of students, undergraduate (Geometry) and graduate (Research Methodology and thesis supervision). In both cases we work through meetings via Zoom. At undergraduate level I present the subject. We use the sharing tool to look at the book in PDF, or the whiteboard, or GeoGebra. At first, I used an auxiliary camera to focus on a notebook in which (using it as a blackboard) I made operations or sketches, as required. But then I discovered a tool that has been very useful to me and allows me to write or draw over the file.

5.4 Implementation of Digital Tools

The digital tools that the mathematics teachers report having implemented can be divided into *mathematical digital tools*, that is, digital tools associated with the teaching of mathematics (n = 68) and *non-mathematical digital tools*, which are not necessarily associated with the teaching of mathematics but that allow for communication between teachers and students (n = 111). Example responses include:

R90: We use applications and applets such as GeoGebra, the games on the Spanish page of the Canary Islands, interaction on the Zoom whiteboard to build answers, a digital book, and YouTube to explain topics but with videos made by myself.

R114: The subject of numerical analysis has three tools: Visual Studio (C++), Mathematica and Simulink. The difference lies in the use of Blogger that is no longer used and YouTube videos. An example accompanied with exercises, and a problem to be solved as a team. Their homework is sent via WhatsApp Web. They [the students] do not like Blogger, they told me it is more complicated.

R9: I send the activity through Google Classroom and then I invite the students to share their doubts through Google Meet. But it is very difficult due to the limited time that the application gives you and not being able to use a blackboard.

R11: Brief explanations, video recordings on the YouTube platform, answering questions through Google Classroom and social networks.

5.5 Teachers' Emotions

When mathematics teachers were asked how they felt in relation to the transition from face-to-face to online mode of instruction, most of them (148) concurred in their expression of negative feelings such as stress, frustration, uncertainty, and worry.

R143: It was very abrupt, it is very stressful; there are no equal conditions for [internet] connection, neither between students, nor between teachers.

R32: Frustration, since I cannot reach even half of my students.

R42: Uncertainty, because we don't know how long we will be working like this, worried because not all of our students have access to digital tools.

R3: I am very concerned about those who do not have internet service, there are many and this greatly limits my interaction with them. The authority pretends that this does not matter.

R169: Only 10% of the students who enter into the virtual class participate. I sit talking alone. Human warmth is lacking. The order, discipline, organisation, participation, attention, punctuality and other factors that help the schooling of students, has been lost.

5.6 Teachers' Suggestions and Recommendations

Through the questionnaire, the mathematics teachers gave suggestions and recommendations for their colleagues who were going through the same transition to online teaching. These recommendations focus on time management (n = 65), the exploration of digital tools (n = 77), self-initiated training (n = 43), and emotional issues (n = 68).

R108: The most important thing is to put limits on the consulting hours for students, because once they have access to your mobile number or Facebook, they will send you messages at any time, any day and that is not very healthy. We must educate ourselves in organising our time and activities.

R155: Set defined hours to work. The virtuality has no end.

R6: Explore options, there are many resources available that we are not aware of and can contribute with something different to our lessons.

R2: We must take advantage of the possibilities that technology offers us. This is going to be a great opportunity to further explore tools that can improve the lessons even when we return to the [face-to-face] classroom environment.

R31: As teachers, we need to train ourselves in the use of digital tools.

R18: Learn, take courses, understand or try to understand how students think to determine what resources can facilitate their learning processes.

R11: Be patient, do not get discouraged if the percentage of active students decreases or does not increase.

R47: Be patient with ourselves, seek support and accept all possible help, be ready to look for information and be patient with parents as well.

R69: Do not be afraid and dare to address this new technological challenge.

6 Discussion

The aim of this study is to explore the way in which Latin American mathematics teachers deal with the abrupt implementation of digital technology in their teaching practices. Mathematics teachers were given a voice to express their feelings about this abrupt implementation and how they are responding to it in their practice.

The notion of teacher's voice is understood as the values, beliefs, emotions, practical experiences, and perspectives of teachers about their work. It also considers the degree to which those elements are considered, included, listened to, and acted upon when important decisions and changes are being made in the educational context where teachers develop their work. Thus, an open questionnaire was designed and implemented through which teachers could freely describe the material and technical difficulties experienced during the digital migration and suggestions they had to help other colleagues to navigate this digital transition. The open questionnaire also provided space for the mathematics teachers to express their emotional response to this abrupt implementation of technology in their teaching.

The analysis of the teachers' responses shows that they encounter obstacles to the implementation of digital technologies mainly related to poor or non access to the internet, and the lack of associated support and training from the educational authorities. The results of the study also indicate that the abrupt implementation of digital technology is time-consuming and can generate negative emotions for mathematics teachers. Finally, the teachers' responses suggest that during the first weeks of the abrupt implementation of digital technology, several of them focused on the problem of how to communicate and share information (files, videos) with their students, and not so much on the problem of how to represent and manipulate the mathematical content in the new instructional setting. This might be due to the abrupt conditions in which technology implementation took place, where the problem of establishing contact and fluid communication with students is a priority for the development of remote instruction and a prerequisite for online mathematics instruction.

Some of the results of this study coincide with the observations of Abboud-Blanchard (2014) and Thomas and Palmer (2014), who studied the phenomenon of the integration of digital tools into mathematics teaching. For example, we agree with Abboud-Blanchard's observation regarding the existence of commonalities in the way that teachers integrate technology into their classroom practices—regardless of whether the teachers come from different contexts. We have also identified regularities in the implementation practices and in the obstacles reported by the different teachers who participated in the study. One of the regularities is the fact that the integration of digital technology can be time-consuming, particularly for those mathematics teachers without prior experience with the use of digital tools in their teaching. Time constraints and time-related limitations connected to the implementation of digital technology into mathematics teaching have been also identified in the studies of Abboud-Blanchard and Thomas and Palmer. The significant increase in the time required to prepare and teach an online mathematics lesson during the pandemic has also been reported by Italian university mathematics teachers (see Cassibba et al., 2021).

As in the work of Thomas and Palmer (2014), in this study the lack of training and the lack of access to digital resources were identified as obstacles to the implementation of digital technology in the teaching of mathematics. However, we argue that the impact that the lack of access to digital resources—particularly the internet—has on the development of mathematics instruction in the pandemic scenario is much greater than the impact that such lack of access to digital resources could have in the mathematics instruction of the pre-pandemic era. Here the data shows how the lack of access to a stable internet connection can have serious consequences for the relationship between the teachers and their students, such as the impoverishment of their interaction and feedback, and, in some cases, the inability to continue attending the mathematics lessons. The lack of access to a stable internet connection or to basic digital tools such as a computer or a tablet is probably not a common problem in well-developed countries. However, there are wide sectors in Latin America where these shortages are part of everyday life.

What we witnessed is a massive digital transition focused on overcoming the disruption in the teaching process caused by the pandemic. However, this digital transition exacerbates the already profound inequality between students from different regions of the world. The transition allows those students in better geographic and socioeconomic conditions to somehow continue with their mathematical education, but leaves behind thousands of students who cannot be part of this digital transition. This poses a huge problem of inequality in access to mathematical instruction that will profoundly influence the mathematical literacy of these future adults and the societies to which they belong.

The results of this study contribute to expand the perspectives developed by Abboud-Blanchard (2014), Thomas and Palmer (2014) and other scholars about the affective elements related to the integration of digital tools into mathematics teaching. Research on the adoption and integration of digital tools into mathematics teaching has usually focused on teachers' *beliefs* on the use of digital tools. Different mathematics teachers' belief systems have been identified, some more compatible than others with the integration of digital tools. An example of this is the work of Erens and Eichler (2015) who identified two general teachers' beliefs systems, which they called "the old school" and "technology supporter", and relate such beliefs systems to teachers' ways of integrating graphing and computer algebra systems in their calculus teaching. On the other hand, Abboud-Blanchard (2014) and Thomas and Palmer (2014) point to teachers' lack of confidence in using digital technology as an obstacle to its implementation in the teaching of mathematics. Such lack of confidence could be interpreted as a self-efficacy belief (Bandura, 1993), i.e., teachers' beliefs in their personal efficacy to put digital technologies to good use in the mathematics classroom. However, the exploratory study reported in this chapter brings to the fore the *emotions* experienced by mathematics teachers during the abrupt process of implementing digital tools into their teaching. Emotions are a more intense and less stable affective element than beliefs, however, beliefs and attitudes are thought to arise from emotions (McLeod, 1992; Schukajlow et al., 2017). Most of the emotions expressed by the teachers who participated in this study were characterised by a negative valence (stress, frustration, worry) and the object of these emotions was the abrupt process of implementation of digital tools into their teaching. Since emotions are the basis on which beliefs and attitudes are consolidated, we think it is necessary to pay more attention to these affective elements that are triggered by the process of implementing digital tools. It is important that mathematics teachers are heard with respect to what they feel and do during processes of digital transformation.

As noted earlier, this study did not directly observe the implementation of digital tools into the teachers' practices during the COVID-19 pandemic. This is an aspect that needs to be addressed by future studies to corroborate and complement the findings reported in this exploratory work. We believe that the study of mathematics teachers' practices in the post-pandemic digital era is one of the topics that will require the attention of researchers in the years to come.

Acknowledgments We express our warmest thanks to the mathematics teachers who answered the questionnaire that served as the basis for this study.

Appendix: Questionnaire Given to the Mathematics Teachers Who Participated in the Study

1. The educational system(s) where you work, is it public or private?
2. Indicate the educational levels in which you work
3. Do you have internet access at home?
4. Before the health emergency and the suspension of face-to-face instruction, was it usual for you to use digital tools in your courses? If your answer is yes, indicate which ones you used.
5. Were you instructed to change to the online teaching format? If your answer is yes, from what authority did you receive the instruction?
6. Upon receiving the instruction, was any digital tool provided to you to develop your work? Did you receive support on this respect?
7. How did you adapt the mathematical content and its approach to the virtual format? How long did this take?
8. Describe your mathematics lessons in the virtual environment.

9. Specify the digital tools that you currently use with your students to develop your courses.
10. How do you feel about the transition from face-to-face to online mode of instruction?
11. What suggestions or recommendations would you make to other colleagues who are undergoing the same digital transition?

References

Abboud-Blanchard, M. (2014). Teachers and technologies: Shared constraints, common responses. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era: An international perspective on technology focused professional development* (pp. 297–317). Springer. https://doi.org/10.1007/978-94-007-4638-1_13

Atkinson, B. M., & Rosiek, J. (2008). Researching and representing teacher voice(s). A reader response approach. In A. Y. Jackson & L. A. Mazzei (Eds.), *Voice in qualitative inquiry: Challenging conventional, interpretive, and critical conceptions in qualitative research* (pp. 175–196). Routledge.

Bandura, A. (1993). Perceived self-efficacy in cognitive development and functioning. *Educational Psychologist, 28*(2), 117–148. https://doi.org/10.1207/s15326985ep2802_3

Brown, T., & McNamara, O. (2011). *Becoming a mathematics teacher: Identity and identifications*. Springer. https://doi.org/10.1007/978-94-007-0554-8

Cassibba, R., Ferrarello, D., Mammana, M. F., Musso, P., Pennisi, M., & Taranto, E. (2021). Teaching mathematics at distance: A challenge for universities. *Education Sciences, 11*(1), Article number 1. https://doi.org/10.3390/educsci11010001

Chirinda, B., Ndlovu, M., & Spangenberg, E. (2021). Teaching mathematics during the COVID-19 lockdown in a context of historical disadvantage. *Education Sciences, 11*(4), Article number 177. https://doi.org/10.3390/educsci11040177

Clark-Wilson, A., Robutti, O., & Sinclair, N. (Eds.). (2014). *The mathematics teacher in the digital era: An international perspective on technology focused professional development*. Springer. https://doi.org/10.1007/978-94-007-4638-1

Csachová, L., & Jurečková, M. (2020). Mathematics teaching in Slovakia during COVID-19 quarantine season in spring of 2020. *Open Education Studies, 2*(1), 285–294. https://doi.org/10.1515/edu-2020-0131

De Freitas, E. (2004). Plotting intersections along the political axis: The interior voice of dissenting mathematics teachers. *Educational Studies in Mathematics, 55*(1–3), 259–274. https://doi.org/10.1023/B:EDUC.0000017694.18374.d6

Erens, R., & Eichler, A. (2015). The use of technology in calculus classrooms – Beliefs of high school teachers. In C. Bernack-Schüler, R. Erens, T. Leuders, & A. Eichler (Eds.), *Views and beliefs in mathematics education: Results of the 19th MAVI conference* (pp. 133–144). Springer. https://doi.org/10.1007/978-3-658-09614-4_11

Forgasz, H. (2006). Factors that encourage and inhibit computer use for secondary mathematics teaching. *Journal of Computers in Mathematics and Science Teaching, 25*(1), 77–93.

Frost, D. (2008). 'Teacher leadership': Values and voice. *School Leadership & Management, 28*(4), 337–352. https://doi.org/10.1080/13632430802292258

Gómez-Blancarte, A., Romo-Vázquez, A., Miranda, I., Aguilar, M. S., Castañeda, A., & Lezama, J. (2019). An online learning community for the professional development of mathematics teachers in Mexico. *Interciencia, 44*(4), 247–252.

Goos, M. (2005). A sociocultural analysis of the development of pre-service and beginning teachers' pedagogical identities as users of technology. *Journal of Mathematics Teacher Education, 8*(1), 35–59. https://doi.org/10.1007/s10857-005-0457-0

Gyurko, J. S. (2012). *Teacher voice* [Doctoral dissertation, Columbia University]. Columbia University Libraries. https://doi.org/10.7916/D8542VJ7

Hargreaves, A. (1996). Revisiting voice. *Educational Researcher, 25*(1), 12–19. https://doi.org/10.3102/0013189X025001012

McLeod, D. B. (1992). Research on affect in mathematics education: A reconceptualization. In D. A. Grouws (Ed.), *Handbook of research on mathematics, teaching and learning* (pp. 575–596). Macmillan.

Saldaña, J. (2013). *The coding manual for qualitative researchers* (2nd ed.). Sage.

Schukajlow, S., Rakoczy, K., & Pekrun, R. (2017). Emotions and motivation in mathematics education: Theoretical considerations and empirical contributions. *ZDM Mathematics Education, 49*(3), 307–322. https://doi.org/10.1007/s11858-017-0864-6

Thomas, M. O. J., & Palmer, J. M. (2014). Teaching with digital technology: Obstacles and opportunities. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era: An international perspective on technology focused professional development* (pp. 71–89). Springer. https://doi.org/10.1007/978-94-007-4638-1_4

Meta-Didactical Transposition.2: The Evolution of a Framework to Analyse Teachers' Collaborative Work with Researchers in Technological Settings

Annalisa Cusi, Ornella Robutti, Monica Panero, Eugenia Taranto, and Gilles Aldon

Abstract Meta-Didactical Transposition is a framework created to interpret and analyse the interactions between teachers and researchers in the general context of teachers' professional development. The use of this framework in specific contexts (not only professional development face-to-face courses, but also within MOOCs or collaborative research projects) triggered the need to develop the main ideas of the framework, intertwining them with ideas from other theoretical frameworks that analyse interactions between actors in education. In this chapter we present the evolution of the Meta-Didactical Transposition framework, focusing, in particular, on the integration of new theoretical elements with the aim to deepen the analysis and interpretation of the so-called phenomenon of "internalisation", which allows teachers and researchers to introduce external components to their own praxeologies. Specifically, we show, by means of three examples in which digital technolo-

A. Cusi (✉)
Mathematics Department, Sapienza University of Rome, Rome, Italy
e-mail: annalisa.cusi@uniroma1.it

O. Robutti
Dipartimento di Matematica, Università di Torino, Torino, Italy
e-mail: ornella.robutti@unito.it

M. Panero
Department of Education and Learning, University of Applied Sciences and Arts of Southern Switzerland (SUPSI), Locarno, Switzerland
e-mail: monica.panero@supsi.ch

E. Taranto
Department of Mathematics and Computer Science, University of Catania, Catania, Italy
e-mail: eugenia.taranto@unict.it

G. Aldon
IFÉ-ENS de Lyon, Université de Lyon S2HEP EA 4148, Lyon, France
e-mail: gilles.aldon@ens-lyon.fr

A. Clark-Wilson et al. (eds.), *The Mathematics Teacher in the Digital Era*, Mathematics Education in the Digital Era 16,
https://doi.org/10.1007/978-3-031-05254-5_14

gies play complementary roles, how this theoretical integration has supported the investigation of the internalisation process from different perspectives, that is, “the where”, “the why” and “the how”.

Keywords Mathematics teachers’ professional development · Meta-didactical transposition · Collaborative work · Praxeology · Agents · Boundary object · MOOCs

1 Introduction: The Meta-Didactical Transposition Framework

All the authors of this chapter have been involved in research contexts characterised by interactions between teachers and researchers in mathematics education involving uses of technology. All of our investigations have been shaped within the Meta-Didactical Transposition framework (indicated here as MDT.1), described in the previous edition of this book (Arzarello et al., 2014), grouping ourselves in smaller authoring groups with the indirect result of differently extending and deepening both the use and the characterisation of the framework.

In this chapter, we first summarise the main elements that characterise MDT.1, before discussing the advances we have made in the elaboration of this framework, through three different examples, how the integration of specific new theoretical elements within MDT.1 have enabled us to deepen our investigations. As the result of this integration, we present a new framework MDT.2.

MDT.1 was created to analyse the evolution of mathematics teachers’ and researchers’ practices within institutional contexts, when they are jointly engaged in professional development programmes that had been designed by researchers (Aldon et al., 2013; Arzarello et al., 2014). It was based on Chevallard’s Anthropological Theory of Didactics (Chevallard, 1985), which conceives mathematics teaching as a human activity, carried out within institutions, and shapeable through different praxeologies. A praxeology is constituted by 4 different *components*: the *task* and one or more *techniques* for solving the task (together they consist in the “praxis” or “know how”); the justification of the technique, which is called *technology*, as a discourse (λόγος: logos) on the technique (τέχνη: tekhnè), and a *theory* (technology and theory form the “logos” or “know why”). Different techniques could be used to address the same type of tasks and the justifications for these *praxis* could refer to different *logos*, depending on one’s institutional position (think, for example, of the task of teaching fractions at primary or secondary school level). In contexts in which researchers and teachers are involved, such as teacher education, research projects or collaborative design, MDT.1 takes into account the relationships and reciprocal influences of the different communities with respect to their professional practices. It models the evolution of their praxeologies, since they could build new *praxis* or *logos*, internalising (in the sense of making them part of their activity) those components that were initially external to their respective praxeologies.

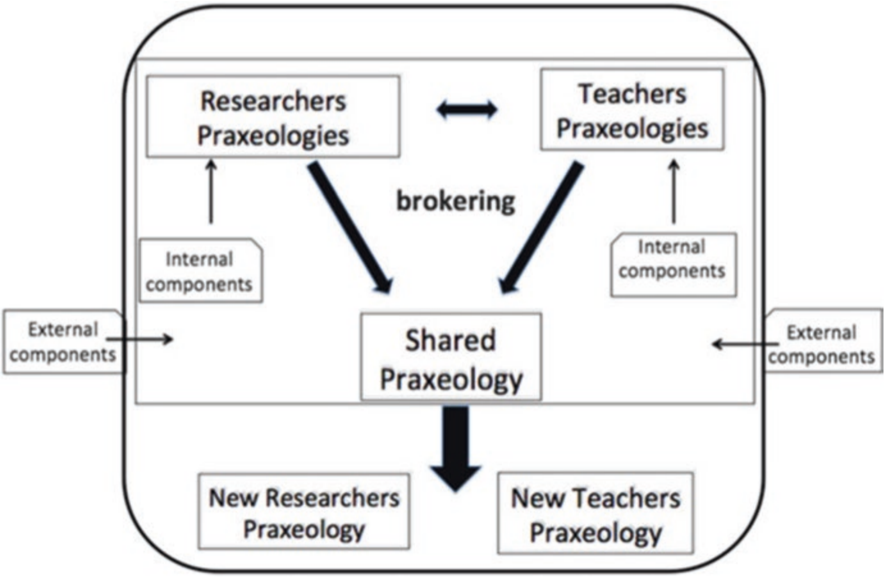


Fig. 1 An illustration of MDT.1 (Arzarello et al., 2014, p. 355)

The frame of MDT.1 (Fig. 1) has been structured on five different cornerstones (Arzarello et al., 2014; Robutti, 2020):

1. *Institutional aspects*, since the actors involved come from different institutions whose expectations determine specific operating rules, codes, objectives and goals. For example, in a national teacher education programme such as M@t.abel (see Arzarello et al., 2014), the actors involved are teachers, who belong to specific institutions (within the Italian school system), researchers from universities, which are another type of institution, and educators—experienced teachers who have been trained by researchers.
2. *Didactical praxeologies* referring to tasks related to the knowledge to be taught and the technique being recognised and justified within a specific institution, and *meta-didactical praxeologies* focused on teachers' and researchers' reflections on contents to be taught and corresponding didactical praxeologies. Praxeologies are dynamic in their nature: they continuously evolve, and their evolution is the result of the reflections developed by actors from different institutions when they interact.
3. The dynamics of praxeological components from external to internal, which constitute the *internalisation* phenomenon, at the base of the evolution of both didactical and meta-didactical praxeologies. Internalisation is linked to the knowledge at stake, whether practical or theoretical, and could not exist if the actors do not take advantage of their interactions to modify their knowledge systems.

4. The role of the *broker*, the “mediator” who links the different communities and facilitates dialogue between them. Often used in the medical field (e.g., Boissel et al., 2010; Gagnon et al., 2016; Lomas, 2007), but not only (Rasmussen et al., 2009), the concept of “brokering” is defined by the actions that an actor carries out at a certain point in time in a given institutional context to make the link and ensure good communication between the actors involved. For example, within M@t.abel programme, educators played the role of brokers between the community of teachers and the community of researchers by supporting the interpretation of concepts introduced by researchers referring to specific ideas set within the institutional context to which the teachers belong.
5. The *double dialectics*, the first—didactical—referred to teaching into the class and the second—meta-didactical—referred to the experience of professional development, in relation to the didactical experience.

In Fig. 1, praxeologies of researchers and of teachers are represented at the top (as initial praxeologies of the communities) and at the bottom (as new praxeologies), after the process of meta-didactical transposition. This process may give rise to a convergent and partially overlapping status of praxeologies of the two communities, called “shared”. MDT.1 is intended to be a framework for describing and analysing the evolution of praxeologies in communities working together. This evolution consists in modifying the techniques and justifying discourses—as components of praxeologies—that each community has at its disposal, as professional experience and knowledge. MDT.1 has been introduced to analyse the processes involved in professional development environments, but it has been quickly used to analyse collaborative research contexts (e.g., Aldon et al., 2017; Robotti et al., 2019; Robutti et al., 2019; Sanchez & Monod-Ansaldi, 2015), focusing also on members of other communities, such as computer scientists and designers. These new studies have triggered our reflections on the potential uses of MDT.1 to deepen the investigation of these new environments, fostering the integration of new theoretical elements within the framework. In the sections that follow, we present this integration through three examples, focused on the identification of different new theoretical lenses aimed at analysing the effects of the interaction between different communities of teachers and researchers within different settings.

2 Three Examples to Chart the Evolution from MDT.1 to MDT.2

The focus of the three narratives is on the process of internalisation. These examples aim to show how the analysis of teachers’ and researchers’ learning can be deepened with respect to internalisation processes, as they interact within professional development environments or collaborative research settings. We then show how new theoretical elements have been identified to be included within MDT.1 to better describe, analyse and interpret this process of internalisation.

Specifically, each example will deepen understanding of a specific focus of the analysis of the internalisation process:

- The first example focuses on "the *where*" of internalisation: Connectivism is the theoretical element integrated within MDT.1 to conceive internalisation as a process that happens within a context that involves not only the institutions in which the groups of actors live, but also the complex networks within which they interact.
- The second example focuses on "the *why*" of internalisation: the integrated theoretical element is the notion of agents, used to develop an analysis aimed at explaining different effects of internalisation. In particular, we show how developing a micro-level analysis could enable to identify the driving forces that trigger, or sometimes inhibit, an evolution of specific praxeological components that characterise an internalisation process.
- The third example focuses on "the *how*" of internalisation: the notion of a boundary object and a framework to support interpretations of the processes that are developed at the boundary are integrated within MDT.1 to support the analysis of the discourses that the actors develop when working on a common object and deepen the interpretation of the evolution of specific praxeological components.

The following table introduces the three examples, focusing on the contexts within which they are set, and on the main characteristics of these contexts (Table 1).

2.1 Example 1: A Focus on the Where of Internalisation

2.1.1 Aim of the Research Regarding MDT and Focus of the Analysis

In this example, MDT.1 is used to frame teachers' professional development context that takes place entirely online: the *Math MOOC UniTo* project (2015–2020), at the Mathematics Department of the University of Turin, where a MOOC on a specific mathematical topic is designed and implemented every year. Two communities interacted within each MOOC: a community of researchers (3 researchers and 10 researcher-teachers, who we refer to as educators) and a community of teachers (424 teachers from across Italy, working at different school levels and with different levels of seniority in their institutions).

Within the MOOC, the teachers can interact in the virtual environment in synchronous or asynchronous mode, with no orchestration by the educators (as an educational choice). This choice implies that the MOOC is enriched by teachers' observations, ideas, reflections, and shared resources, expanding the number of stimuli available to teachers, fostering the internalisation of components of teachers' praxeologies, which represents the focus of our analysis. Our aim is, in fact, to investigate "*the where*" of internalisation, by researching the influence of the MOOC context on the internalisation of specific components within teachers' praxeologies.

Table 1 The context in which the three examples are set

	Example 1	Example 2	Example 3
Context	A professional development course focused on the didactics of geometry, as a MOOC for in-service mathematics teachers of all school levels.	A professional development course, realised in Italy and in Australia, to integrate GeoGebra in classroom practices for in-service secondary mathematics teachers. The example focuses on Italian data.	A collaborative research project (the European project FaSMEd) involving teams of researchers and teachers in 7 countries. The example focuses on data collected by the Italian and French teams.
Aim of the course/ project	Researchers and researcher-teachers, as educators, offer an online professional development experience for a large number of teachers, on a national basis (see, for instance, Taranto, 2020; Taranto et al., 2020)	Researchers, as educators, have designed the course with the aim of developing teachers' competencies for the integration of GeoGebra in mathematics problem-solving activities at secondary level and supporting teachers to implement student-oriented teaching practices (for further details, see Prodromou et al., 2018).	Teachers and researchers work together to study the potential for the design and implementation of digital tools and resources that aim to foster the activation of formative assessment strategies within the classroom (see, for instance, Aldon & Panero, 2020; Cusi et al., 2017).
Course/ project duration	The MOOC lasted one semester of a school year.	The teacher education programme lasted 4 months.	The overall project lasted 3 years. Collaborative work between teachers and researchers lasted officially 2 years, although their collaboration continued beyond the end of the project.
Data collection	The data was collected from the Moodle platform that is hosting the MOOC and comprising teachers' contributions to communication message boards and their uploaded resources.	Data has been collected from initial and final interviews and from teachers' logbooks of the whole experience.	The data, collected throughout the whole project, consisted of students' written work; video-recordings of classroom activities and of the meetings between teachers and researchers; teachers' post-lesson interviews; and students' final interviews and questionnaires.
The type/ role of digital technology within the course/ project	Three forms of technology are present: The platform, its communication message boards and the specific technologies/resources for didactical activities (e.g., GeoGebra, videos).	The course was planned specifically around GeoGebra and its sustainable integration into teachers' practices.	Digital technology is the core of the project, since its focus is on the identification of specific digital tools and on a careful design of their use to support formative assessment processes.

2.1.2 Actors Involved and Initial State of the Praxeologies

With respect to the MOOC the educators focus on the task of transposing, totally online, teaching approaches for Geometry (in relation to the national curriculum), by means of a set of resources (activities for teachers and for students, teaching methods and suggestions for integration of technologies), and the different techniques adopted to implement them. Educators' practical choices in designing the MOOC are informed by theories such as Didactical Transposition and community of practice.

Teachers participating in the MOOC have their own didactical praxeologies, based on their professional experience. They address the tasks proposed within the MOOC (reading texts, looking at videos and commenting on them, experiencing activities in their classes) making use of different techniques, accompanied by theoretical justifications, which may be explicit or implicit in teachers' interventions.

2.1.3 What is the Theoretical Gap in the MDT.1 Framework?

The dynamics of the interactions between participants and their related learning in a MOOC are different from those of other educational environments explored previously with MDT.1. So this example helps to characterise new aspects of "the where" of internalisation. Since a MOOC constitutes a new environment, the study of the particular internalisation processes requires new theoretical ideas that support interpretations of the complexity of the interactions.

2.1.4 What Do We Add and Why?

MDT.1 has been hybridised (Arzarello, 2016) with other theoretical frames (Taranto, 2018) and here we present the integration of the construct of Connectivism (Siemens, 2005). A MOOC is a particular learning environment in which Connectivism can be applied to account for its complexity. Each teacher is connected with colleagues and has the opportunity to share ideas/questions/resources with them. Connectivism enables the interpretation of the learning processes, activated in a MOOC by participants' interaction, as the evolution of a network of knowledge (Siemens, 2005). This evolution is characterised by different phases: self-organisation of participants in selecting resources and ways of using them (Siemens, 2005), integration of these resources/uses in participants' own cognitive structures and sharing of uses/resources inside the interactive tools at their disposal (Taranto, 2020; Taranto et al., 2017; Taranto & Arzarello, 2020).

According to Connectivism, knowledge is defined as a particular type of network, whose nodes are "any entity that can be connected with another node" (Siemens, 2005, p. 4), including information, data, images, ideas, and feelings. The network is dynamic and may change over time, so learning is conceived as a continuous process of network exploration, involving construction, development, and self-organisation of knowledge. Hence, according to Connectivism, learning implies: (a) adding a new node to one's own network of knowledge; (b) connecting (in the sense of relating) old nodes of one's own network of knowledge in a new way.

Connectivism supports both the observation of teachers' activity in MOOCs and the peculiarities of the learning that is fostered through this activity. Learning within a MOOC could be, in fact, interpreted in terms of the evolution of teachers' own network of knowledge. This evolution is triggered by the creation of new nodes when a teacher perceives the MOOC resources as new within their network. Moreover, teachers' interaction with other teachers in the virtual environment enables them to create new and different connections. This evolution of teachers' networks of knowledge have been interpreted within the MDT framework in this way (Taranto et al., 2020):

(a) when a teacher adds a new node to his/her own network of knowledge, it means that one or more components become internal to their meta-didactical praxeologies;
(b) when a teacher connects old nodes of his/her own network of knowledge in a new way, it means that s/he looks at his/her didactical praxeologies in a fresh way and possibly modifies them, so changing also his/her meta-didactical praxeologies.

2.1.5 Data

Module 1 of the first MOOC delivered (*MOOC Geometria* on geometry contents) is focused on the topic of distance between a point and a line and is aimed at overcoming students' typical misconceptions related to this topic. The communication message board embedded into Module 1 was the forum where the educators have inserted an assignment to stimulate discussion among the teachers: *"Share your ideas and/or teaching experiences related to the topics [in Module 1]"*. The following interventions are taken from the discussion triggered by the educators' assignment within the forum:

T1 - 27/10/15; 6:50 p.m. - *The idea is to play with the heights of the triangles and I half minded proposing it to my students :) This is a draft of text. 3 male friends Antonio, Bruno and Carlo are at the top of the triangle in the figure* (Figure 2). *3 female friends Antonella, Barbara and Carlotta are also at the top of the triangle in the picture. Friends via WhatsApp agree to find themselves in the orthocentre of the triangle while the friends will meet in the centroid of the triangle. Draw the meeting points of the two groups.*

PS: I used the map of Latina, my city.

Didactic note: I deliberately chose an obtuse triangle and the position of the triangle is not the stereotyped one.

T2 - 27/10/15; 11:16 p.m. - *This activity is beautiful: I will propose it next week (obviously using a map of a city closer to my students, like Turin) to see how they*

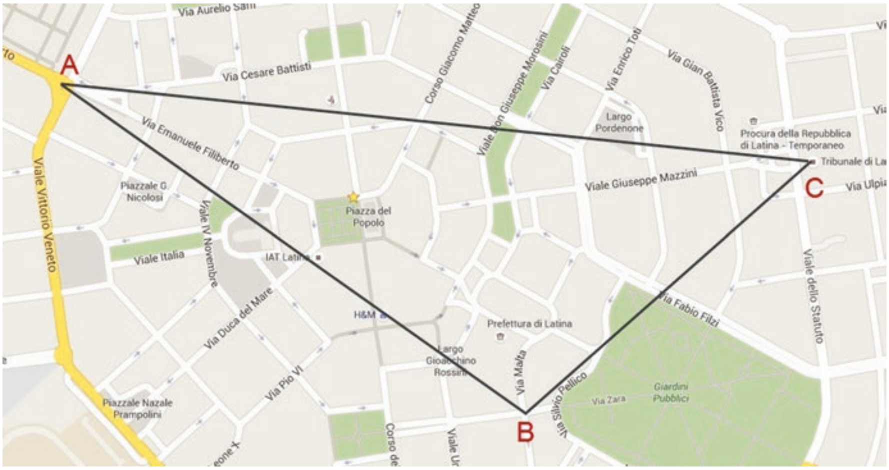


Fig. 2 Latina's map created by T1

have internalised the concepts of orthocentre and centroid, since they have just discovered heights and medians […]

T3 - 28/10/15; 10:40 a.m. - *I really like the proposal and I propose a variant of the text: in a treasure hunt the competitor Alberto of the team is in A, Bruno in B and Caterina in C. The next clue will be given only when all three competitors will meet in the orthocentre of the triangle and communicate the position to the director ... etc ... it could also be said that there is a tolerance of a certain amount of meters for the possible presence of buildings on the geometrically found point. Other points of discussion could arise on the comparison between the mutual positions of centre of gravity and orthocentre. What do you think?*

T1 - 29/10/15; 4:45 p.m. - *I really like the use of tolerance! [...] Thanks for the idea :)*

2.1.6 Analysis

The intervention with which T1 opens the discussion shows that the stimuli received by the materials allowed him to produce his own resource and share it with others. T1 was inspired by a concrete situation: he took from Google Maps a map of his city, Latina, and drew a triangle (Fig. 2) to design a task for his students.

T2 congratulated T1 and, in a process of self-organisation, makes visible the fact that she has added a new node to her network of knowledge ("I will propose it"), but at a specific time ("next week") to link it to mathematical concepts that she has already addressed with her class. T3 positively evaluated T1's idea and, in making it her own, proposed a variant to the original task, to stimulate students' reflection and argumentation. T1's reply to T3 testifies not only his appreciation for the suggestion, but also that he had updated his network of knowledge.

Although he had not been asked to produce and share materials, T1 used the ideas he gained from the MOOC to develop a new and original product. In this way, the other teachers were provided with additional sources of learning and led to embedding T1's product as a new node in their own networks and highlighting connections with other nodes.

2.1.7 Final State of the Praxeologies

The dynamics between internal and external components of the praxeologies, namely, the phenomenon of internalisation, is addressed by calling into question the network of knowledge of Connectivism, specifying what it means when a new node is added to one's network or new connections are generated. Since the knowledge at stake in a MOOC is not only the knowledge transposed by the instructors, but also the result of a process that takes into account the countless interactions between teachers within the communication message boards, Connectivism's terminology allows us to better interpret these virtual and massive phenomena.

As far as the educators are concerned, the evolution of their meta-didactical praxeologies occurs at the end of the MOOC, that is when they go to reflect on the results of this training from the point of view of the homework carried out by the teachers, the requests made by the teachers in questionnaires or interviews, and by carefully reading of the interactions that took place on the communication message boards. They have the possibility to reflect on their own praxeologies and possibly arrive at an evolution of them (e.g., contents to be proposed to teachers of different school levels; presentation of pedagogical suggestions to be maintained or made more explicit in the proposals for teachers; variation in the time allowed to teachers to complete the homework assigned during the training).

2.1.8 Conclusion of This Example

MOOCs constitute professional development environments that are new with respect to those in which MDT.1 has originally been developed. This example has shown that integrating Connectivism within MDT.1 enabled us to better characterise "*the where*" of internalisation. Interpreting learning as the evolution of the individual's own network of knowledge enables, in fact, to conceive internalisation as a phenomenon that happens within a wide context, which includes not only the institutions to which the individual belongs, but also the complex networks of his/her interactions. Moreover, Connectivism frames the analysis of both the activities that teachers perform within a MOOC (the exploration of resources, their self-organisation in cognitive structures, and the sharing of uses/resources through interactive tools), and the peculiarities of the dynamics of the learning that is promoted through each activity.

2.2 *Example 2: A Focus on the Why of Internalisation*

2.2.1 Aim of the Research Regarding MDT and Focus of the Analysis

In this example, we use the MDT.1 framework in the same context in which it was originally created, a teacher education programme. Our aim is to explore the effectiveness of this programme by focusing on the evolution of teachers' praxeologies during the course over time. The analysis focuses, in particular, on teachers' praxeologies concerning the use of GeoGebra and on the internalisation process of specific praxeological components (i.e., techniques related to a specific use of GeoGebra and corresponding justifications within their meta-didactical praxeologies). In particular, we aim to provide evidence for the possible causes that might help to determine a successful or unsuccessful internalisation process, that is, "*the why*" of internalisation.

2.2.2 Actors Involved and Initial State of the Praxeologies

Our reflections draw on data from a common project developed in Italy and Australia. In this chapter, we focus only on the Italian case, consisting in a secondary school teachers' professional development course with GeoGebra as part of a national programme, promoted by the Ministry of Education and implemented by the Department of Mathematics of the University of Torino. The professional development comprised three face-to-face meetings (of 3 hours each) between teachers and educators; collaborative work through the Moodle platform; and teachers' implementations of the activities in their classrooms.

The researcher-educators manage the task, which is to "stimulate teachers' use of GeoGebra in a mathematics laboratory modality", where "mathematics laboratory" (Robutti, 2006) represents a particular student-centred teaching method. The techniques adopted by researcher-educators include proposing significant activities with an active use of GeoGebra for exploratory purposes, discussing the proposed activities with the teachers, leaving the teachers to work independently on tasks, as if they were the students (the "mathematics laboratory" teaching method to cope with problem solving activities).

The participating teachers were not all new to the use of GeoGebra. Some of them used it within teacher-centred methodologies, by exploiting GeoGebra mainly to show properties to students. However, these teachers were less familiar with the use of GeoGebra in a "mathematics laboratory" modality. This implies that their initial use of GeoGebra was based on specific logos that consider this digital tool as useful for demonstration purposes.

2.2.3 Data: Two Contrasting Cases

We present two paradigmatic cases (further described in Prodromou et al., 2018): one of internalisation (Riccardo) and one of lack of internalisation (Lara).

Riccardo's case. At the beginning of the programme, Riccardo explicitly declared, in the pre-training interview, that he used GeoGebra in his classrooms only on some sporadic occasions and "in an even less than basic way":

> "*With grade 13 students, I went [to the digital laboratory] in order to show them how to draw graphs of functions with GeoGebra, whereas, with grade 10 students I made some statistics. I haven't gone yet [to the digital laboratory] with grade 9 students, but I intend to show them some geometrical topics with GeoGebra. The training course can help me in this sense*".

At the end of the course, Riccardo explicitly declared that he had discovered a different use of GeoGebra, aimed at enhancing students' imagination:

> "*there is a childish attitude in me, that is … the child who has not discovered geometry has to manipulate [GeoGebra]…You can use this with students […] In my opinion, working with GeoGebra, you can build upon their imagination. And some of them have got a lot of it [imagination], it is sufficient to train it someway*".

The analysis of other interviews and meetings, and of Riccardo's logbook, enabled us to show that Riccardo had started to adopt a more student-centred methodology during his lessons with GeoGebra as a result of his participation in the PD programme.

Lara's case. Lara has experience in using GeoGebra on the IWB to reproduce the sketches she would have otherwise drawn on the blackboard to stress invariant properties of geometrical constructions. However, after having implemented the activities in her classroom, she answered negatively to the question "Would you repeat this experience?":

> *I don't believe I would repeat it: number one for time reasons, number two because I have neither the availability of [a] digital laboratory nor of a technician who could help me. (…) GeoGebra, as after all I'm often using it in the classroom with the IWB, but, indeed, everything is constructed by me. Here the only difference for them is that I don't show a ready-made construction, we build it together, starting from zero.*

2.2.4 What is the Theoretical Gap in the MDT.1 Framework?

In its current form MDT.1 does not help us to explain why the internalisation of some components of teachers' praxeologies occurs in some cases (as in Riccardo's case) and not in others (as in Lara's case). Our hypothesis is that, if praxeologies are the observable variables, there may exist other underlying variables, which can influence the evolutions of the praxeologies. The investigation of these underlying variables, through lenses of more specific magnitudes, could support an analysis aimed at identifying the possible causes that might help to determine the internalisation process of praxeologies' components.

2.2.5 What Do We Add and Why?

We introduce a new level of analysis of the professional development process: the micro-level, in addition to the already existing macro-level of praxeologies.

This approach recalls the description of physical phenomena related to gas distinguishing between macro variables (e.g., temperature, pressure or volume of the gas) and micro variables (e.g., mass or velocity of a particle). To this purpose, we identified agents (at micro-level) that may determine teachers' praxeologies' development and evolution (at macro-level). With the term "agents" we refer to human or non-human entities involved in the mathematical activity (De Freitas & Sinclair, 2014). The result of the interaction of agents at the micro-level can be observed at the macro-level, as it happens for micro and macro variables of the gas.

Similar phenomena happen within professional development environments, where interactions between numerous smaller and simpler agents at the micro-level can influence the internalisation process, at the macro-level (Goldstein, 1999). When a teacher is planning and teaching, the interaction between different agents, at the micro-level, contributes to shaping the teacher's praxeologies or some of their components (technique or justifying discourses), at the macro-level. Among others, we identified methodological agents, institutional agents, material and technological agents, motivational agents (Prodromou et al., 2018). To exemplify, teacher-centred (explaining, or lecturing, demonstration, and direct instruction) and student-centred (class participation, inquiry-based learning, cooperative learning, discussions, mathematics laboratory, etc.) teaching methods are methodological agents, while, for instance, GeoGebra is considered within material and technological agents.

2.2.6 Analysis

The technique of using GeoGebra underpinned by a specific justifying discourse (such as the activation of the mathematics laboratory methodology) was external for the teachers' community at the beginning of the professional development course. To foster the transformation of these praxeological components from external to internal for the teachers' community (or at least for some of the individual teachers), the researchers implemented activities involving different kinds of agents: material (e.g., paper, pencil); technological (e.g., GeoGebra, IWB); methodological (e.g., mathematics laboratory); institutional (national curriculums, other teachers); and motivational (e.g., personal beliefs).

For each participating teacher, these agents could be already active or needed to be activated by the professional development programme,which fosters their interaction with other agents, observable at a micro-level of analysis. When different independent agents are active in teachers' activities and reflections, the researchers observe some changes in teachers' praxeologies at the macro level. For Riccardo and Lara, GeoGebra was already active as a technological agent in their didactical praxeologies. However, it was interacting at a micro-level with more teacher-centred teaching practices as methodological agents. In particular, the methodological agent of the mathematics laboratory was not active, which is fundamental to teachers' internalisation of the desired praxeological components, namely, the use of GeoGebra for mathematics laboratory activities.

2.2.7 Final State of Praxeologies

As a result of the professional development course and the stimuli introduced by educators, some teachers (i.e., Riccardo) experienced the interaction of GeoGebra as a technological agent with the newly explored mathematics laboratory methodology, which fostered the internalisation of specific praxeological components at both the praxis and the logos level. In particular, Riccardo had begin to use a more student-centred methodology during his lessons with GeoGebra (technique), justifying this choice declaring that the use of GeoGebra can enhance students' imagination (logos). This internalisation concerns the meta-didactical praxeologies since our data concern the teachers' reflections about their didactical praxeologies (that is not observable in the classroom context).

Lara's case is completely different from Riccardo's case, since, although she tested the use of GeoGebra in a mathematics laboratory modality with her students, she decided not to repeat the experience after the PD course. Her intervention during the meeting highlights that her technique has probably remained the same (use of GeoGebra for demonstrative purposes) as well as the logos behind. Such a difference can be interpreted as the intervention of motivational agents coming from the teacher's beliefs and personal experience, as well as of institutional agents (e.g., time, school context and curriculum constraints), which have interacted with GeoGebra agent and mathematics laboratory agent, inhibiting, in this case, the shift of the component "use of GeoGebra in a laboratory way" from external to internal.

From a different perspective, the researcher-educators highlighted the importance of explicitly discussing, within the different institutions, the role of time as a fundamental variable which could foster (or not) the integration of GeoGebra in a mathematics laboratory modality.

2.2.8 Conclusion of This Example

The internalisation of praxeological components (initially new or little explored by teachers), at both the praxis and the logos level, is an emergent phenomenon that may occur or not at the macro-level of professional development, even if stimulated in a specific educational programme. To investigate why it occurs successfully for some individuals and does not occur for others, we have deepened the analysis considering what happens at the micro-level of professional development, where several agents of different kinds interact. Some of them are activated by rooted and spontaneous teachers' meta-didactical praxeologies, some others are added or differently stimulated by the course that teachers are attending. The focus on agents represents a key idea to deepen the investigation of "*the why*" of internalisation, since the interaction of agents (at the micro-level) could foster or inhibit teachers' internalisation of specific praxeological components that a professional development course intends to promote (at the macro-level). This result has an impact on researcher-educators' meta-didactical praxeologies, suggesting which agents have to be activated and taken into account to trigger and support the evolution of teachers' praxeologies.

2.3 Example 3: A Focus on the How of Internalisation

2.3.1 Aim of the Research Regarding MDT and Focus of the Analysis

In this example, the frame of MDT.1 is used within a context of a collaborative research project involving researchers and teachers within the European project FaSMEd (where collaborative research is intended in the sense proposed by Robutti et al., 2016). This represents a new application of the framework, since here professional development is conceived as the consequence of the collaborative work.

The research aim is to interpret, through a MDT.1 lens, data from teachers and researchers' interactions within the project to analyse how their participation in the project has affected the teachers' and researchers' learning. In alignment with the FaSMEd project's aim and methodology, the focus of the analysis is on the participants' learning about formative assessment (FA) and on their reflections on the use of technologies to support FA processes.

The data is analysed with the aim to highlight "the how" of the internalisation process, specifically how it could develop as a result of the participants' collaborative work.

2.3.2 Actors Involved and Initial State of the Praxeologies

The data on which we base our reflections was collected by two of the partners of the FaSMEd project, in Italy (University of Turin) and France (Ecole Normale Supérieure de Lyon). The research particpants are: (1) a team of Italian teachers who work in a primary school (grades 4 and 5) in Turin and the researchers who collaborated with them; (2) a team of French teachers who work in a primary school in the suburb of Lyon (grade 3 to 5) and the researchers who collaborated with them. Both teachers and researchers are focusing on the same meta-task: designing the use of digital tools within classroom activities in a way that fosters FA processes.

Initially, researchers share their ideas about the techniques to be used (specific digital tools that could be used to address the task and possible ways of using them) and the justifying discourses corresponding to the reasons why these techniques could be effective (elements of a framework on FA, Black & Wiliam, 2009).

In particular, during the first cycle of teaching experiments, Italian researchers planned to equip all of the classes with computers and tablets and to use a connected classroom technology to create a network between the teacher's computer and the students' tablets. Moreover, researchers suggested the organisation of the lessons according to a common structure focused on teachers' construction of a digital document containing groups of students' answers to be shown through the IWB (for more details, see Cusi et al., 2017).

French researchers equipped the classes with Interactive Whiteboards, electronic voting keypads and Primary Plus TI calculator.

The Italian teachers had a previous long experience within a regional project focused on FA, which represented, for this reason, an internal component of their didactical praxeologies. As a result, they were used to collecting students' answers (written with paper and pencil), analysing them at home and writing these answers on the blackboard during the following lesson to open a discussion with students. However, they were not familiar with any digital technology, so the techniques proposed by researchers were external to their praxeologies.

For French teachers initially, the elements belonging to the technique's component of the researchers' praxeologies were external as well as the justifying discourses.

In both cases, researchers shared their ideas and were confronted with teachers' didactical praxeologies based on their habitual assessment behaviours and paper and pencil techniques.

2.3.3 The Data

Both in France and in Italy, the methodology of collaborative work during the FaSMed project consisted of periodic meetings, between teachers and researchers, aimed at reflecting on the results of the teaching experiments that were being carried out in the schools.

In this example we focus on two meetings: (1) the first took place in Italy, at the end of the first year of the project following a first round of teaching experiments; (2) the second took place in France, at the end of the project, and following two cycles of teaching experiments.

For each meeting, we focus on one scene and include the interventions proposed by some teachers during the discussions with researchers.

Scene 1: A Meeting in Italy at the End of the First Year of the Project
The following transcript describes some teachers' interventions when discussing, with researchers, the use of connected classroom technologies and the IWB to plan and conduct the discussions on students' answers.

T1: "*We are used to carrying out classroom discussions and we work a lot on argumentation, but, usually, we have to write down all the students' answers and to read them with students. This [the use of the digital technology] is immediate:*

you have all the students' answers over there [on the computer]! The difference is undeniable!"

T2: "*Students appreciated a lot to have the possibility to see their answers written [on the IWB] and to discuss them. …They would have wanted to see all the answers. It is different to listen to something or to read it. This is an important support provided by technology*."

T3: "*I remember one of the first lessons, when a student, who previously gave a wrong answer, referred to what his mates said during the discussion and, in this way, was able to correctly answer. Discussing in-the-moment was really useful for him. If we had had discussed the following week, maybe we would have lost that moment*."

Scene 2: A Meeting in France at the End of the Whole Project
At the beginning of the project, the analysis of the discussions between French researchers and teachers revealed a misunderstanding between the two communities, due to their different goals. The teachers wanted to use the digital materials and tools to avoid students' mistakes and prevent any pupil from falling behind, while the researchers focused on the ways in which these materials could be used to foster multiple FA strategies.

The following reflections were developed by teachers when they were asked to comment on their new ways of interpreting the use of digital tools during their lessons:

T4: *As a teacher, I found myself well suited to this because I found that it allowed me to be very precise [...] it allowed me to take time for students with difficulties and also to put those who were doing well in situations where they could deepen their competences.*

T5: *The goal is that we changed groups, that we became more green than red, and even at the level, uh, at the level of the class, uh, not just at the individual level.*

T6: *[...] they exchange with each other on this competence, there are exchanges because it creates, as you say, emulation. [...] It's rather positive, in fact, in terms of assessment!*

2.3.4 What is the Theoretical Gap in the MDT.1 Framework?

The MDT.1 framework supports the analysis of the short excerpts presented in the previous section by highlighting the changes within the praxeologies of the teachers who participated during the meetings. However, although an evolution of praxeologies could be described to make the changes in praxeological components explicit, we could see an opportunity to deepen the interpretation and analysis of how the evolution of praxeologies has been triggered. In other words, what kind of analysis could we perform to show "*the how*" of internalisation?

2.3.5 What Do We Add and Why?

In our goal to reveal the how of internalisation, we identified new theoretical elements to be integrated within MDT.1 to support the investigation of the ways in which the teachers and researchers' joint actions on the object of the collaborative work (in this case, FA through digital technologies) to affect the internalisation processes of specific components of their meta-didactical praxeologies.

We interpret the object of the collaborative work in terms of a boundary object (hereafter, BO), referring to Star and Griesemer's (1989) and Star (2010) characterisation of BOs as "objects which are more plastic enough to adapt to local needs and constraints of the several parties employing them, yet robust enough to maintain a common identity across sites" (p. 393). Moreover, we adopt Star's (2010) perspective on BO, which stresses that the BOs are the objects of common actions developed by different communities within a shared place. This new perspective enables us to conceive the boundaries of the BO as the collaborative contexts within which the people involved in the work on BO have opportunities to exchange and pursue common aims. The transition from an object of work to a BO thus can be interpreted as a process that requires acceptance, interest and the development of common joint actions by the various actors involved in the collaborative work.

To highlight the internalisation processes boosted by the joint actions on BOs, we refer also to Carlile's (2004) characterisation of the three actions that could be carried out on BOs, at different levels of interaction:

- *Transfer*, an action carried out at a syntactic level and aimed at sharing a common vocabulary on specific BO's components;
- *Translation*, an action carried out at a semantic level and aimed at negotiating common meanings related to specific BO's components;
- *Transformation*, an action carried out at a pragmatic level and aimed at enabling the actors involved to integrate specific BO's components within their own practices.

2.3.6 Analysis

The two scenes are complementary in terms of elements that characterise the internalisation process, since they highlight different dynamics connected to the meta-didactical praxeologies' components that are the object of this process.

In fact, in *scene 1*, the teachers' reflections focus on new techniques within their didactical praxeologies, while in *scene 2* the new elements that are internalised mainly enrich the logos part of teachers' meta-didactical praxeologies.

The excerpt in *scene 1* highlights the effects of an action that teachers performed on the BO at the *semantic level*, since they are developing their reflections with the aim of giving meaning to the new techniques they are experimenting (initially external to their praxeologies), by referring to their previous experiences to justify these new techniques. It is evident, for example, when T1 highlights the effectiveness of accessing and storing all students' answers through the connected classroom technology, in contrast with the less effective strategy of writing down students' answers on the blackboard. Also T2's reflection stresses on the comparison between the previous approach of reading students' answers during classroom discussion and the power of displaying students' answers on the IWB. The new praxis is defined in light of the previous one through a *translation* action aimed at carrying out a process of sense making that highlights the positive effects of the cooperation between previous practices and new ones, as in T3's reflection, which stresses the potential related to being able to develop an in-the-moment discussion thanks to the support given by digital tools. Other hints that testify that a translation action has been carried out are related to teachers' need to take multiple perspectives when they reflect on the use of digital tools to support FA processes. It is evident, for example, in T2 and T3's interventions, explicitly aimed at considering the point of view of their students. In our opinion, this hint of an ongoing expansion of perspectives could be interpreted as the effect of actions of *transformation* that were developed at the *pragmatic level*, since the collaborative work at the boundary has fostered the internalisation of new elements within the technique component of teachers' didactical praxeologies.

The excerpt in *scene 2* highlights how the actions on components of the BO have been illuminated not only at the pragmatic level, since they stress on the more effective organisation of the class, but also at the theoretical level, since they interpret the strategies developed during the lessons according to the new elements provided by the FA framework. In T4's reflection, for example, we could highlight implicit references to Black and Wiliam's (2009) FA strategies, such as, for example, clarifying learning intentions and criteria for success ("*it allowed me to be very precise*") and engineering effective classroom discussions and other learning tasks that elicit evidence of student understanding ("*in situations where they could deepen their competences*"). Moreover, in T5's reflections the shift of focus from an individual perspective on assessment to a collective perspective is made evident, referring, as in T6's reflections, to the FA strategy of activating students as resources for each other. Although the reference to the FA strategies is not explicit, we could observe that new didactical praxeologies are justified referring to the theoretical lenses shared by researchers, highlighting an ongoing process of teachers' internalisation of these new components.

Therefore, the phenomenon of internalisation shown in scene 2 has started through actions on the BO at a *syntactic* level, due to the teachers and researchers' need of better understanding each other, then the actions shifted to a *pragmatic* level, to modify the techniques to be implemented in the classroom, and to a *semantic* level, to develop shared meanings and a consequent better understanding of the BO at stake.

2.3.7 Final State of the Praxeologies of Teachers and Researchers

The performed analysis has shown that the experience in the use of specific digital technologies and the reflections on the comparison between the previous methodologies adopted to foster FA and the actual ones have enabled teachers to internalise the new technique component and to enrich the justifying discourses related to this technique. Due to space limitations, we were not able to document also the evolution of researchers' praxeologies. However, we can state that the dialogue with teachers enriched researchers' praxeologies since the FA practices adopted by teachers before participating in FaSMEd have made the methodology of using specific digital tools evolve. So the technique component of researchers' praxeologies have evolved, together with the justifying discourses behind it. Both examples show the evolution of actors' praxeologies into a "shared praxeology", in the sense of a mutual understanding of each other's praxeologies and the internalisation of components allowing praxeologies to evolve.

2.3.8 Conclusion of This Example

The work on BO represents the driving force that triggered the evolution of both teachers' and researchers' praxeologies. To describe how this evolution happens, we referred to Carlile's levels of communication on BO to analyse the reflections developed by teachers (and also by researchers, even if not documented here), interpreted as the justifying discourses through which they explain what they have learnt from the experience. This analysis shows: (a) the role played by the syntactic level of communication in fostering the appropriation of the task by both communities; (b) the link between the semantic level of communication and the internalisation of the logos' components of praxeologies; (c) the connection between the discourses developed at the pragmatic level of communication and the internalisation of the technical components of praxeologies.

Therefore this example has highlighted that the integration of the idea of BO and of the framework useful to interpret the work developed on BO within MDT.1 has enriched our analysis, since it has enabled to deepen the investigation of "*the how*" of internalisation by highlighting evidence of this process by means of focusing on the actors' discourses.

3 The Evolution of the MDT Framework from MDT.1 to MDT.2

In this chapter, we discuss the evolution of the MDT framework from MDT.1 to MDT.2 through our three examples, developed around the concept of internalisation (of praxeological components), to show how the integration of new theoretical elements into the MDT.1 framework supported the investigation of this process from different perspectives, that is, "the where" (first example), "the why" (second example) and "the how" (third example) of internalisation.

The final results of the various integrations are shown diagrammatically in Fig. 3, which is built around the process of internalisation.

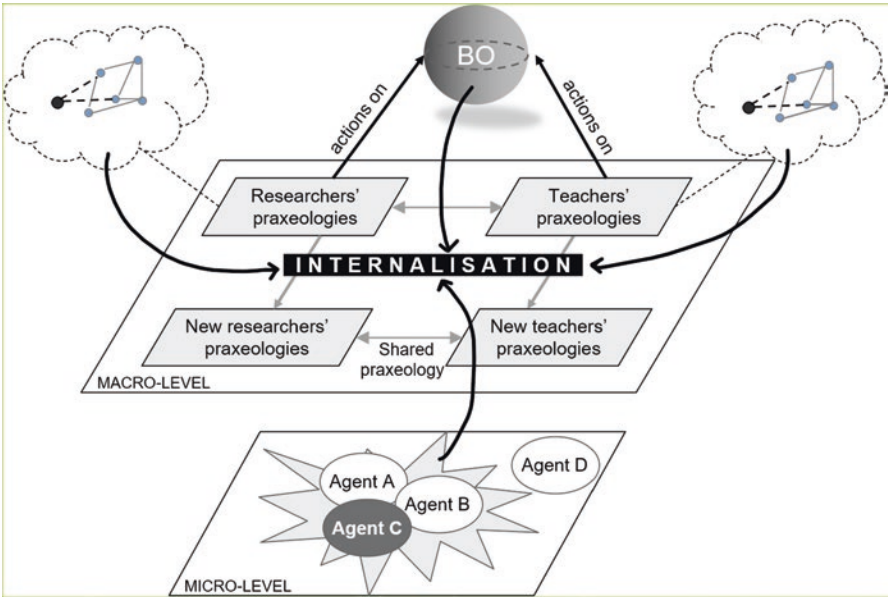


Fig. 3 Diagram representing the MDT.2 framework

Internalisation is conceived in this chapter as an ongoing process that determines a continuous evolution of the praxeologies of the actors collaboratively interacting in the research or PD settings described in the three examples. Therefore, without the internalisation of praxeological components, the professional development, conceived as the result of interaction, whether in a specific educational programme or in participation in a research group, cannot take place.

Through teachers' participation within PD programmes or collaborative research projects, their initial praxeologies, grounded on their professional experiences and knowledge, could evolve into new ones, integrating, thanks to the internalisation process, "new" components within them.

The *first example* focuses on "*the where*" of internalisation and introduces Connectivism as a theoretical lens that complements MDT.1, since it enables to interpret the internalisation process in terms of extension of the actors' network of knowledge which characterised the dynamics observed within MOOCs or other PD settings. The result of the internalisation process is, in fact, the addition of a new node or a new connection between two nodes in the teachers' network of knowledge. The clouds in Fig. 3 represent the teachers' or researchers' networks of knowledge before their participation in the research project or PD programme. In particular, the grey dots represent the nodes that have already settled in the network and the lines connecting them are the current connections linking the nodes to each other. The black dot linked to dashed segments indicates the possibility of inserting a new node in the network of knowledge and/or generating new connections. The arrows that link these clouds to the word internalisation highlight the interrelation between the evolution of the network of knowledge and the internalisation phenomenon, since this evolution enables the shifts of specific praxeological components from external to internal. The integration of the connectivitistic approach within MDT.1 allows, therefore, researchers to interpret the internalisation phenomenon within a wider context in which learning is also analysed in terms of expansion and evolution of networks of knowledge.

The *second example* shows that the internalisation process could be investigated from the point of view of the factors and the conditions that activate it, that is, "*the why*" of internalisation. The main theoretical element, which is integrated within the MDT.1 framework, is the notion of agent, which provides opportunities for understanding how the complex process of teachers' professional development is generated. This example highlights, in fact, the role played (at a micro-level) by specific agents (C in Fig. 3), in interaction with each other (A and B in Fig. 3), in determining (or not) the internalisation of new components (at the macro-level). The analysis developed in this example shows the value of studying the internalisation process at both micro and macro-level, to highlight the driving forces (the agents) that determine or inhibit the internalisation of different components within teachers' meta-didactical praxeologies.

The *third example* enables us to deepen "*the how*" of the internalisation process, by showing that this process could be made visible through the analysis of the teachers and researchers' joint actions on a common object of work which becomes a boundary object (BO). The work on BO is conceptualised as an opportunity to initiate a joint work between different communities (represented through the arrows in Fig. 3 that connect teachers' and researchers' praxeologies to the BO). The joint work on the BO fosters the two communities' reflections and makes them refer to properties, concepts, pieces of knowledge or experiences to develop a better understanding of the BO itself. Internalisation is conceived as the result of this process, since it makes the objects of reflection become internal components of the two communities' didactical or meta-didactical praxeologies.

In particular, each level of action on the BOs is related to the internalisation of specific components, as we highlighted at the end of sect. 3.3. Therefore, the two communities' common aim of understanding the BO contributes to modify their praxeologies, fostering the internalisation process (as highlighted by the arrow which connects the BO to the word internalisation, in Fig. 3) and highlighting a "shared praxeology", characterised by those components that are understandable and familiar by both the teachers' and researchers' new praxeologies (Fig. 3).

The MDT.2 framework has been used to investigate the dynamics that characterise not only the collaborative PD settings presented in the three examples, but also other PD contexts with different cultural backgrounds (see Capone et al., 2020; Otaki et al., 2020; Shinno & Yanagimoto, 2020). The flexibility of the framework in supporting the investigations developed within these different contexts highlights its robustness.

We showed through our examples that, although not created specifically to analyse processes that involve the broad use of technology in mathematics teaching and in teacher education, the MDT.2 framework proved to be flexible enough to enable us to deepen this analysis. In the three analysed examples, digital technology plays various roles: (a) the context in which the interaction between communities happens; (b) a tool used to perform specific tasks; (c) a common object of work. As regards points (b) and (c), in particular, the three examples clearly show that MDT.2 can be efficiently applied to the study of didactical and meta-didactical praxeologies related to the introduction, use and reflection on the role of digital tools in mathematics teaching, learning and teacher education contexts.

Even if the new framework MDT.2 presents an evolution with respect to MDT.1, research is still in progress to enhance the complete framework, both from a theoretical and from a methodological point of view. Firstly, in order to further study the interactions between the didactical and the meta-didactical levels, we aim at deepening the investigation of the double dialectic to analyse the long-term effects of the internalisation process, focusing on the evolution of the practices activated by the different actors involved in a collaborative work. Second, we want to deepen the study of motivational agents at the micro level by analysing, through the double dialectics, the role played by beliefs, emotions, attitudes and values in influencing the elaboration of the justifying discourses within the teachers' and researchers' meta-didactical praxeologies.

References

Aldon, G., & Panero, M. (2020). Can digital technology change the way mathematics skills are assessed? *ZDM – The International Journal on Mathematics Education, 52*(7), 1333–1348.

Aldon, G., Arzarello, F., Cusi, A., Garuti, R., Martignone, F., Robutti, O., & Soury-Lavergne, S. (2013). The meta-didactical transposition: A model for analysing teachers education programs. In A. M. Lindmeier & A. Heinze (Eds.), *Proceedings of the 37th conference of the international group for the psychology of mathematics education. — Mathematics learning across the life span, July 28-Aug 02, Vol. 1* (pp. 97–124). Kiel, Germany.

Aldon, G., Cusi, A., Morselli, F., Panero, M., & Sabena, C. (2017). Formative assessment and technology: Reflections developed through the collaboration between teachers and researchers. In G. Aldon, F. Hitt, L. Bazzini, & U. Gellert (Eds.), *Mathematics and technology, a CIEAEM sourcebook* (pp. 551–578). Springer.

Arzarello, F. (2016). Le phénomène de l'hybridation dans les théories en didactique des mathématiques et ses conséquences méthodologiques, *Conférence au Xème séminaire des jeunes chercheurs de l'ARDM*, Mai 7–8, 2016, Lyon.

Arzarello, F., Robutti, O., Sabena, C., Cusi, A., Garuti, R., Malara, N., & Martignone, F. (2014). Meta-didactical transposition: A theoretical model for teacher education programmes. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era* (pp. 347–372) Springer.

Black, P., & Wiliam, D. (2009). Developing the theory of formative assessment. *Educational Assessment, Evaluation and Accountability, 21*(1), 5–31.

Boissel, J.-P., Riondet, O., Cucherat, M., Stagnara, J., Wazné, H., & Nony, P. (2010). Le courtage des connaissances en thérapeutique. Une étude pilote de faisabilité. *Pratiques et Organisation des Soins, 41*(1), 55–64.

Capone, R., Manolino, C., & Minisola, R. (2020). Networking of theories for a multi-faceted understanding on lesson study in the Italian context. In H. Borko & D. Potari (Eds.), *Proceedings of the 25th ICMI study (pp. 102–109)*. National and Kapodistrian University of Athens. http://icmistudy25.ie.ulisboa.pt/wp-content/uploads/2020/11/201114-ICMI25Proceedings6.13.2020.pdf.

Carlile, P. (2004). Transferring, translating, and transforming: An integrative framework for managing knowledge across boundaries. *Organization Science, 15*(5), 555–568.

Chevallard, Y. (1985). *La transposition didactique du savoir savant au savoir enseigné*. La Pensée Sauvage.

Cusi, A., Morselli, F., & Sabena, C. (2017). Promoting formative assessment in a connected classroom environment: Design and implementation of digital resources. *ZDM Mathematics Education, 49*(5), 755–767.

De Freitas, E., & Sinclair, N. (2014). *Mathematics and the body: Material entanglements in the classroom*. Cambridge University Press.

Gagnon, J., Lapierre, J., Gagnon, M.-P., Lechasseur, K., Dupéré, S., Gauthier, M., Lechasseur, K., Dupéré, S., Gauthier, M., Farman, P., & Lazure, G. (2016). Processus de transfert et d'appropriation des savoirs d'étudiantes en sciences infirmières et de milieux de soins Africains: une étude de cas multiples. *Recherche en Soins Infirmiers, 124*(1), 53–74.

Goldstein, J. (1999). Emergence as a construct: History and issues. *Emergence, 1*(1), 49–72.

Lomas, J. (2007). The in-between world of knowledge brokering. *BMJ: British Medical Journal (International Edition), 334*(7585), 129–132. http://ezproxy.usherbrooke.ca/login?url=https://search.ebscohost.com/login.aspx?direct=true&db=a9h&AN=23834807&site=ehost-live

Otaki, K., Asami-Johansson, Y., & Hakamata, R. (2020). Theoretical preparations for studying lesson study: Within the framework of the anthropological theory of the didactic. In H. Borko & D. Potari (Eds.), *Proceedings of the 25th ICMI study* (pp. 150–157). National and Kapodistrian University of Athens. http://icmistudy25.ie.ulisboa.pt/wp-content/uploads/2020/11/201114-ICMI25Proceedings6.13.2020.pdf

Prodromou, T., Robutti, O., & Panero, M. (2018). Making sense out of the emerging complexity inherent in professional development. *Mathematics Education Research Journal, 30*(4), 445–473.

Rasmussen, C., Zandieh, M., & Wawro, M. (2009). How do you know which way the arrows go. *Mathematical Representation at the Interface of Body and Culture*, 171–218.

Robotti, E., Grange, T., & Peloso, S. (2019). Recherche action et développement professionnel des enseignants de maths en maternelle et primaire. Le cas d'EduMathVallée (Italie). *Actes COPIRELEM, 46° colloque international sur la formation en mathématiques des professeurs des écoles*, 547–559, HEP Vaud, Lausanne, Suisse.

Robutti, O. (2006). Motion, technology, gestures in interpreting graphs. *International Journal for Technology in Mathematics Education, 13*(3), 117–125.

Robutti, O. (2020). Meta-didactical Transposition. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 611–619). Springer.

Robutti, O., Cusi, A., Clark-Wilson, A., Jaworski, B., Chapman, O., Esteley, C., Goos, M., Isoda, M., & Joubert, M. (2016). ICME international survey on teachers working and learning through collaboration. *ZDM Mathematics Education, 48*, 651–690.

Robutti, O., Aldon, G., Cusi, A., Olsher, S., Panero, M., Cooper, J., Carante, P., & Prodromou, T. (2019). Boundary objects in mathematics education and their role across communities of teachers and researchers in interaction. In G. M. Liloyd & O. Chapman (Eds.), *International handbook of mathematics teacher* (Participants in mathematics teacher education) (Vol. 3, 2nd ed., pp. 211–240). Brill-Sense.

Sanchez, E., & Monod-Ansaldi, R. (2015). Recherche collaborative orientée par la conception. Un paradigme méthodologique pour prendre en compte la complexité des situations d'enseignement-apprentissage. *Education & Didactique, 9*(2), 73–94.

Shinno, Y., & Yanagimoto, T. (2020). An opportunity for preservice teachers to learn from in service teachers' lesson study: Using meta-didactic transposition. In H. Borko & D. Potari (Eds.), *Proceedings of the 25th ICMI study* (pp. 174–181). National and Kapodistrian University of Athens. http://icmistudy25.ie.ulisboa.pt/wp-content/uploads/2020/11/201114-ICMI25Proceedings6.13.2020.pdf

Siemens, G. (2005). Connectivism: A learning theory for the digital age. *International Journal of Instructional Technology and Distance Learning, 2*(1), 3–10.

Star, S. L. (2010). This is not a boundary object: Reflections on the origin of a concept. *Science, Technology, & Human Values, 35*(5), 601–617.

Star, S. L., & Griesemer, J. R. (1989). Institutional ecology, 'Translations' and boundary objects: Amateurs and professionals, Berkeley's Museum of Vertebrate Zoology, 1907-39. *Social Studies of Science, 19*(3), 387–420.

Taranto, E. (2018). *MOOC's zone theory: Creating a MOOC environment for professional learning in mathematics teaching education*. Doctoral Thesis. Turin University.

Taranto, E. (2020). MOOCs for mathematics teacher education: New environments for professional development. In J. P. Howard II & J. F. Beyers (Eds.), *Teaching and learning mathematics online* (pp. 359–384). CRC Press.

Taranto, E., & Arzarello, F. (2020). Math MOOC UniTo: An Italian project on MOOCs for mathematics teacher education, and the development of a new theoretical framework. *ZDM – The International Journal on Mathematics Education, 52*(5), 843–858.

Taranto, E., Arzarello, F., Robutti, O., Alberti, V., Labasin, S., & Gaido, S. (2017). Analyzing MOOCs in terms of their potential for teacher collaboration: The Italian experience. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the tenth congress of the European Society for Research in mathematics education* (pp. 2478–2485). Dublin.

Taranto, E., Robutti, O., & Arzarello, F. (2020). MOOCs-UniTo: Theoretical framework and research lines on teachers and researchers. *Quaderni di Ricerca in Didattica (mathematics)*, *Numero speciale n. 8*, 31–42. Palermo, Italy: G.R.I.M., Dipartimento di Matematica e Applicazioni. http://math.unipa.it/~grim/quaderno_2020_numspec_8.htm

Revisiting Theories That Frame Research on Teaching Mathematics with Digital Technology

Nathalie Sinclair, Mariam Haspekian, Ornella Robutti, and Alison Clark-Wilson

Abstract In this chapter, we offer an overview of some of the major trends in theory development and use in relation to teaching mathematics with digital technology. We showcase some of the developments that have occurred since the first edition of this book (2014). We also provide a deep review of the multiple ways in which the instrumental approach has evolved over time, as a way to exemplify how theory development responds to new questions and new theoretical insights. Throughout the chapter, we make explicit the philosophical assumptions on which these theories depend—particularly the binaries they reify—and use these to open up consideration of different assumptions and how they might matter to our field of research.

Keywords Theories in mathematics education · Evolution of theories · Teaching with technology · Instrumental approach · Philosophical considerations · Axiology · Ontology · Epistemology

N. Sinclair (✉)
Faculty of Education, Simon Fraser University, Burnaby, BC, Canada
e-mail: nathalie_sinclair@sfu.ca

M. Haspekian
Université Paris Cité, EDA, F-75006, Paris, France

O. Robutti
Dipartimento di Matematica, Università di Torino, Torino, Italy

A. Clark-Wilson
UCL Institute of Education, University College London, London, UK
e-mail: a.clark-wilson@ucl.ac.uk

A. Clark-Wilson et al. (eds.), *The Mathematics Teacher in the Digital Era*, Mathematics Education in the Digital Era 16,
https://doi.org/10.1007/978-3-031-05254-5_15

1 Introduction

In the 2014 edition of this book, theorising around the practice of teaching mathematics with digital technology was relatively new. Ruthven (2014) provided an overview of three of the main theories being used in the field at the time and, significantly, highlighted the different insights they offered. These three theories were interesting in that they represented three different provenances: one that was rooted in more general theories about teaching (TPACK); one that drew from psycho-ergonomic studies of the human activity (Instrumental Approach), and one that was 'homegrown' in the specific context of teaching mathematics with digital technology (Structuring Features of Classroom Practice[1]).

Now, eight years later, we can see that some of these theories have evolved and that new theories have emerged. In a review of the articles reporting research on teaching mathematics with technology published in seven prominent mathematics education journals (see Appendix) between 2014 and 2020,[2] we have found that the Instrumental Approach dominates the recent research landscape. Several of these articles also extended this approach with other theories. The second most dominant frame was TPACK. Setting the Instrumental Approach and TPACK aside, researchers adopted a wide range of theories that were not technology specific. To put this in context, we noted that over a half of these articles *made no reference to technology-specific theories* and that 11/67 articles offered no explicit theoretical grounding. Despite the slow uptake of theories to support understanding of teaching mathematics with technology, there has been significant evolution of existing theories—especially with respect to the Instrumental Approach—as well as the growing networking of theories, as we will show in the Sects. 1 and 2 of this chapter.

Theories in mathematics education research pertain both to mathematics itself, as well as to mathematics education, and they concern their epistemological assumptions (about what can be known and how), ontological assumptions (about the nature of things, such as concepts) and axiological assumptions (about ethical and aesthetic values). Any theory makes certain choices about these assumptions, even if they are only inferred implicitly. As theories diversify and develop over time, we think it is important to gain explicit appreciation of these assumptions, not only to be able to compare and possibly network them, but also in order to gain awareness of how the theories we use shape our interpretation of the world and, with it, the

[1] Although the component ideas were drawn from different sources in the general literature on teaching, as Ruthven (2014) writes: "The Structuring Features of Classroom Practice framework (Ruthven 2009) was devised by bringing a range of concepts from earlier studies of classroom organisation and interaction and of teacher craft knowledge and thinking to bear on this specific issue of technology integration" (p. 386).

[2] For each journal, we searched for articles that had the word "teacher", "teaching" or words related to teachers' classroom practices (e.g., questioning, assessment, etc.) and words related to technology (e.g., ICT, software, DGE, etc.). We also read the abstract and research questions to determine whether the article related to aspects of teaching mathematics with technology. This produced a final sample of 67 articles. We thank Canan Gunes for her help with this research.

goings-on in a mathematics classroom. This awareness is also necessary to better appreciate the research results that these theories allow us to obtain. Additionally, while many theories that have been traditionally used in mathematics education research articulated particular epistemological assumptions (we know by developing/constructing mental schemas or we know by participating in a community of practice or we know through our social, economic, or historical–cultural positionings), it is only recently that theories making explicit ontological and axiological assumptions have emerged. In Sect. 3, we will show how such assumptions might be relevant to theorising in relation to teaching mathematics with technology.

Overall, this chapter will point to some emerging trends in theory development. These include, for example, a growing interest in calling into question the many binaries that have long shaped our understanding of mathematics teaching and learning. For example, the mind–body binary (of Cartesian origin) that separates the thinking mind from the active body is challenged by theories that seek to understand embodied ways of knowing, that is, how bodily actions (e.g., gestures) shape thinking. Other commonly found binaries include: nature–culture, cognition–affect, process–product, procedural–conceptual, intuition–logic, technocentric–anthropocentric points of view or, more generally, human–non-human.[3] In the context of teaching with digital technology, the relaxing of the mind–body binary has led to theoretical elaborations that attend to the ways that teachers notice and use gestures and other bodily actions in the classroom, as can be seen in the Theory of Semiotic Mediation (Bartolini-Bussi & Mariotti, 2008) or in the notion of the Semiotic Bundle (Arzarello, 2006). Another binary is that between thinking and feeling, which makes hard distinctions between cognitive and affective processes. This binary can be seen in action when the focus on research is entirely on one or the other, as in the *Second Handbook on the Psychology of Mathematics Education* (2016), which has a large section on cognitive aspects of teaching (and learning) and a small section on social and affective aspects.

In the specific context of teaching mathematics with digital technology, theories seeking to challenge these binaries might take seriously the affective impact of technology use in the classroom. For example, how do existing theories attend to a teacher's uncertainty around how to deal with the changing status of school mathematics when digital technology can be used to do so many things that used to be done by hand, such as long division, solving equations and graphing functions? We think that re-visiting some mathematics education theories to better understand which binaries they mobilise (or, in some cases, the binaries that uses or interpretations of these theories introduce), even implicitly, might lead to productive growth in our knowledge of the field from a more general point of view.

Finally, in this chapter we have attempted to provide detailed and trustworthy description of various theories and how they are used. However, given our own

[3] This has been a long-standing binary that has served to distinguish those who can think from whose who cannot (animals, plants, stones, etc.). In mathematics, this binary has been at stake in discussions of computer-based proofs—can machines think and know and learn, and therefore produce acceptable proofs, or must this be done by humans in order to be valid? A whole part of the research activity in computer science, via the field of semantics, is occupied with precisely this question and the search for rigorous justifications in computer science.

philosophical commitments, we cannot assert that it is possible to write from a completely unbiased, neutral point of view, since our own values and interests, as well as our own cultural backgrounds, shape what we see as being of importance or interest. This does not mean that we will put forward the claim that *any* theory is worth using or that *all* theories are somehow valid; indeed, we are committed to sustaining a critical attitude towards every theory, and particularly those that aim, implicitly or not, to depreciate or marginalise teachers, students or researchers.

2 Evolutions of Theories Related to Teaching with Technology

While we find ourselves today with multiple theories supporting research on *teaching* practices in *instrumented environments*, a retrospective panorama of the evolution of these theories in mathematics education shows that these theories have roots in the 1960s with research and general theories on *students' learning*, as shown in Fig. 1. From the 1960s to the 1990s, technology-related research in mathematics education progressively drew from these general developments, adapting them to the use (and, at the time, rather the *usefulness*) of a specific digital technology. Thus, this corpus mainly investigated *learning processes*. Although this chapter is focused on teachers, we insist on this history because it helps illuminate the persistence of well-entrenched theories of learning in the current landscape of theories on teaching. In this section, we chart the broad strokes of theory development as it pertains to the use of technology in mathematics education, therefore sweeping through the bottom row of Fig. 1. In Sect. 2, we consider the evolution of a particular theory, the instrumental approach and therefore show some of the links across the three rows, as scholarship from outside of mathematics education as well as outside of technology-specific concerns, have shaped theory development. Section 3 steps aside from current mainstream theories—and therefore past the right-most column—to show how future theory development might evolve, that is, where new considerations and assumptions might lead.

As is evident in Fig. 1, since the mid-2000s, an increasing number of teacher-specific theories have emerged. One large group of these are linked to the instrumental approach in didactics, based on the cognitive-ergonomic instrumental approach (Vérillon & Rabardel, 1995), which has roots in the work of Piaget and Vergnaud (for the concept of scheme), as well as Vygotsky (for the concepts of mediation, and the social aspects of the instrumentation). Over time, it is also possible to see how different influences from outside of mathematics education research, such as the notion of boundary objects (Star, 1989) and the role of gestures in teaching and learning (Goldin-Meadow, 2005), have been conjugated into teacher-specific theories. In addition, changes in the software and hardware technological infrastructure, such as the emergence of touchscreen tools, which have shaped broader theories in mathematics education, may eventually come to affect teacher-specific theories. In this section, we will trace some, but not all, of the evolutions represented in Fig. 1.

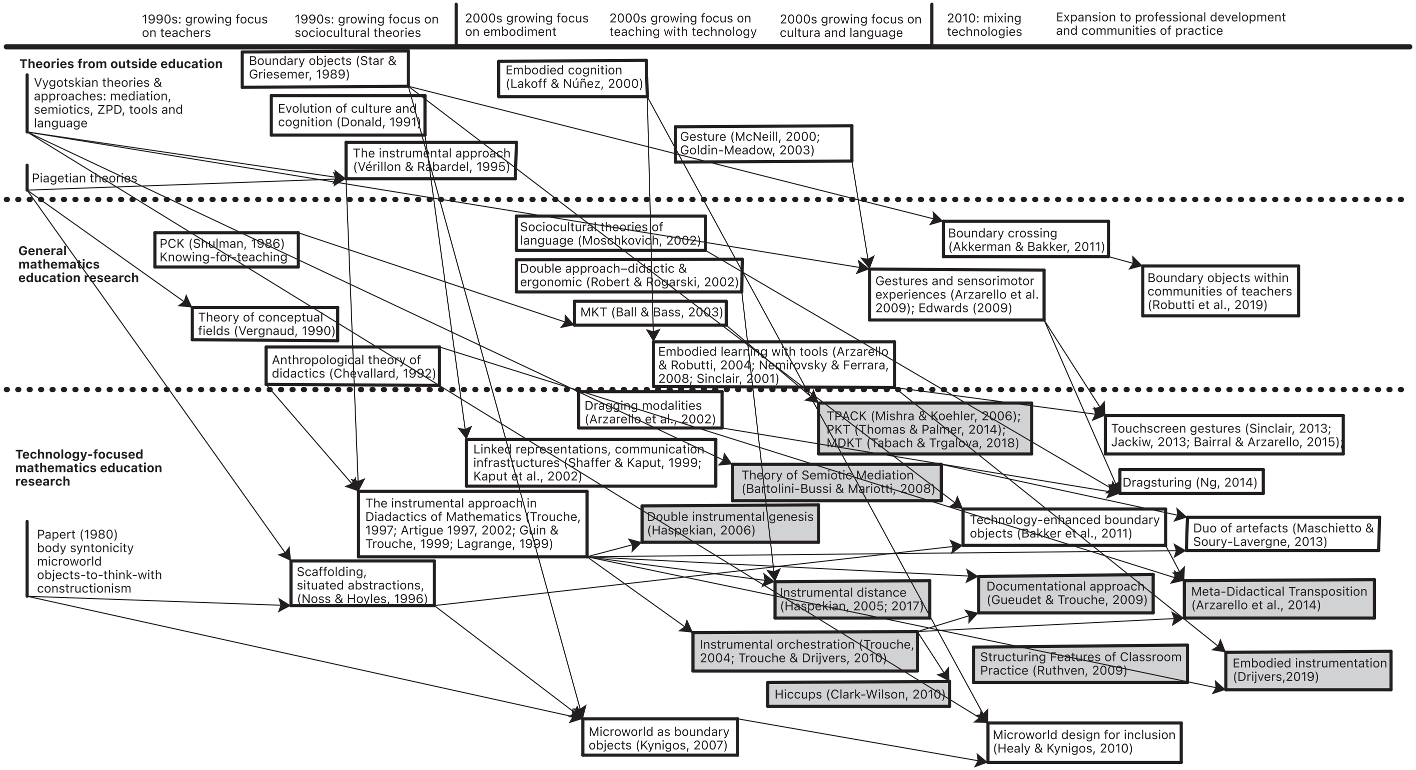


Fig. 1 Some episodes in the emergence of constructs and theories over time. The graph is not meant to be exhaustive, but instead to show large trends in the evolution of and connections between theories. Teacher-specific theories are shaded. Loose decadal shifts are shown in columns along the top

The early work of Papert (1980) offered many new concepts for understanding the role of technology in learning, such as body syntonicity (which would later become central to theories of embodiment), microworlds, bricolage, and objects-to-think-with. With Logo, and then the proliferation of new digital technologies such as Computer Algebra Systems (CAS), Spreadsheets and Dynamic Geometry Environments (DGEs), the technology-related literature was initially mainly limited to descriptions of digital technology's potential to change mathematics learning (see Lagrange et al., 2003).

From the 1990s, new theorizing attempted to account for *how* that learning might take place, using concepts such as *scaffolding* and *situated abstraction* (see Noss & Hoyles, 1996). These concepts were adapted from more general theories (such as Piaget, but also theories of situated cognition). Similarly, the concept of representation, which began as a general-level theoretical idea from scholars such as Bruner, was transposed into the technology-specific literature through the work of Kaput (1989) and colleagues, who were concerned with the particular linked and distributed forms of representation that digital technology offered. In return, this contributed to develop more general theories about representation. In the words of Drijvers et al. (2010) "The demand for clarification coming from the new technologies and their representational potential contributed to an effort to outline a unifying theoretical frame for representation" (p. 97).

Yet, these elaborations were still mostly addressing *learning phenomena*. The word "teaching" was missing, which says something not only about the conception of learning at that time, but also about assumptions made about teachers. Of course, concerns about teacher education have always existed—indeed, Celia Hoyles highlighted this in her 1992 Plenary talk at PME—but were rather oriented towards defining the mathematical knowledge required to teach. With the emergent question of content transposition, research on *teachers* emerged gradually. So did research on teaching practices *with* digital technology. It was also in 1992 that Chevallard emphasised the importance of the teacher in the integration and viability of digital technology, warning that innovation will fail unless "a functionally integrative didactic stewardship" is explicitly managed (Chevallard, 1992, p. 195, *our translation*). Ten years later, in her Handbook chapter, Mariotti (2002) also points out, even if in a very small section, that the role of the teacher is a central consideration in students' mathematics learning with technology. Some years later, she studied the role of the teacher in the processes mediated by DGE (Mariotti, 2009).

From the 2000s onwards, teacher concern expanded. For example, some early studies focussed on teachers' perception of graphing calculators (Simonsen & Dick, 1997; Tharp, Fitzsimmons & Ayers, 1997). By this time, the concept of instrumental orchestration had evolved and developed to the point of sometimes being called a frame *per se*, extending the instrumental approach to a focus on the teacher. In CERME conferences, for example, there was a shift from one TWG on technologies in CERME8 (2013) (called Technologies and resources in mathematics education), to two TWGs in CERME9 (2015) (called, respectively, TWG15 on Teaching mathematics with resources and technology and TWG16 on Students learning mathematics with resources and technology). By 2014, in the first edition of this book

(Clark-Wilson et al., 2014), various theories relating to teaching with digital technology appeared, which showed an increased awareness of the complexity that new technologies introduce into teaching practices.

Research on technology integration in mathematics *teaching* has now contributed to general research on teaching. An example is the evolution of a new research field, Meta-Didactical Transposition (MDT) (Arzarello et al., 2014; Robutti, 2020), this time focused on teacher professional development and teacher education. MDT can model the evolution of praxeologies of teachers (rather than focus on teacher knowledge at a certain time, or to teachers' practices in the class) and researchers working together in an institutional educational setting, towards the sharing of common practices and corresponding theoretical justifications (the so-called shared praxeology). This framework evolves from the anthropological theory of didactics (Chevallard, 1989), concentrating the attention on the teachers not in classrooms (as Chevallard did), but in their professional education.

While it is possible to distinguish theories in terms of their historical genesis, it is also interesting and important to understand their differences in terms of what they stress and ignore in the phenomena they seek to understand or interpret. For example, as Ruthven (2014) points out, whereas TPACK focusses on the explicit knowledge of the teacher, the Structuring Features of Classroom Practice focusses on teacher expertise and practice, much of which may be quite implicit. While the Structuring Features of Classroom Practice "provides a more differentiated characterisation" of the incorporation of a new tool into the resources system and adapting activity structures (p. 391), the Instrumental Orchestration theory identifies specific patterns of teacher activity in the classroom, around the instrumentation of student mathematical knowledge. The Documentational Approach considers the teacher activity as a process that traces the whole span of professional work (not just in the classroom), in relation to the use of different tools and resources (Trouche et al., 2019).

Theories can also be compared in terms of the binaries they create and/or maintain, which are often implicitly assumed rather than explicitly stated. In relation to the three theories discussed by Ruthven (2014), for example, the cognition–affect binary is maintained in the sense that no attention is paid to affect, which suggests that teaching and learning is primarily a cognitive enterprise. Obviously, teachers respond in a variety of ways to the emotional and aesthetic cues from both of their students and of the discipline and among the variety of tasks potentially favorising mathematics learning, they choose certain tasks because they know their students will experience pleasure or surprise or satisfaction; they make certain instructional decision to avoid student anxiety or frustration, but these phenomena are not considered to be significant in current theories of teaching with technology. Another example, which we will return to in Sect. 3, is the mind–matter binary. All the theories discussed so far maintain this binary by centring the human subject as the intentional, knowledge-producing actor, whether as a teacher or learner, while the technology is given a more passive, mediating role. This binary has been challenged by some scholars, who argue that the material world (which can include physical objects as well as digital ones, but also various aspects of the environment) cannot be separated from the knowing subject and that knowledge emerges from human–tool relations (see Latour, 1987) or in a 'dance of agency' between the discipline, the material world and the mathematician (see Pickering, 1995). We mention this point about binaries because they are important to keep in mind when considering theory networking and because the attempt to dissolve a particular binary can be theoretically very productive, as we aim to show in the next section.

3 An Example of a Theory and its Evolution: The 'Instrumental Approach'

One of the arguments for the educational integration of digital technologies in mathematics is that they can relieve students from the more computational aspects of mathematical activity and allow them to focus on the conceptual ones. These oppositions are part of the broader binary between body and mind, which is often hierarchised by assigning cognitive activities a value that is traditionally more 'noble' than that assigned to bodily activities (a point of view inherited from Descartes, and repeated throughout history by philosophers, psychologists and educators). The Instrumental Approach questions the binary between the 'technical' use of instruments and the 'cognitive' conceptualisations that this use brings into play.

3.1 An Overview of the Instrumental Approach

For Rabardel (1995), the study of the relation between cognition and action constitutes one of the important tasks of contemporary psychology. The instrumental approach, born in the context of cognitive ergonomics by Vérillon and Rabardel (1995), follows the work of Vygotsky's theorisation, focussing on the learning processes involving tools. Going beyond classical binary models (subject/object), they introduce a third pole between the subject and the object, which is that of the instrument: a mixed entity that pertains to both the subject and the artefact: "The instrument does not exist in itself, it becomes an instrument when the subject has been able to appropriate it for himself and has integrated it" (Vérillon & Rabardel 1995, p. 84). The cognitive component refers to what a subject learned from/for using the artefact in this context. Consequently, an instrument is always something that belongs to somebody, for performing a given type of task, at a given step of its development (Trouche, 2020). The development of an instrument from an artefact, along with the actions of the subject, is called instrumental genesis.

The instrumental approach was introduced in mathematics didactics after a period of "disappointed" hopes regarding the integration of digital technologies in the mathematics classroom, while they were, paradoxically, very present both in the resources and in the curricula. In France, the first studies aiming to understand these paradoxes involved formal calculation software (a Computer Algebra System, CAS) and symbolic calculators (Guin & Trouche, 1998). CAS research has shown that the real teaching situation was much more complex than that which the favourable arguments above were suggesting. For example, the introduction of such tools into the classroom considerably increased the number of resolution techniques available to the students, an increase that was not necessarily anticipated, nor intended, by the teacher. The symbolic calculator was also introducing new types of tasks and, more disturbing, new instrumental objects that involved mathematical concepts different from those intended by the curriculum. Moreover, the articulation of the work and the tasks within the instrumented environment, especially with the traditional paper-based techniques (to which the didactical objectives of curricula and institutional assessments continued to refer) was not self-evident either.

A double complexity, underestimated or even denied until then, was thus coming to light, which was both institutional and instrumental:

- Institutional because the knowledge taught depends on an institutional context: it has undergone a didactic transposition (Chevallard, 1985) regarding the knowledge of 'mathematicians'—this transposition being specific to the institution in which it is taught.
- Instrumental because the artefact does not intervene in a neutral way in the teaching-learning situation. It carries its own techniques, instrumental knowledge, signs, meanings, and a technological transposition of mathematical knowledge also takes place (see Balacheff's notion of computer transposition, 1994). The instrument transforms the knowledge into knowledge that is distant from that institutionally intended—this instrumental distance (Haspekian, 2005) being more or less significant, depending on the artefact.

To reckon with this double complexity, researchers in this field (Artigue, 1996; Trouche, 1997; Lagrange 1999; Defouad, 2000) have combined the two theoretical perspectives: the *anthropological theory of the didactic* (*ATD* – Chevallard, 1999), with this *instrumental approach* (Rabardel, 1995; Vérillon & Rabardel, 1995). ATD allows the institutional dimension with its conditions and constraints to be taken into account, while the *instrumental approach* takes into account the instrumental dimension and the non-neutral role of the 'gesture on the thought', i.e., of the use of the artefacts on the conceptualisation of the subjects (the human users).

At the basis of the ATD there is the notion of didactic transposition in mathematics education (Chevallard, 1985; Chevallard & Bosch, 2020), as the transformations that knowledge undergoes from the moment it is produced and theorised, to the moment in which it is put into use in a teaching situation, so it is selected and designed to be taught in a specific educational institution. The ATD theory is based on the assertion that mathematical activity has to be interpreted as an ordinary human activity, and gives a general model to describe human activities (the praxeologies), a model that links and gives the same importance to their theoretical (knowledge) and practical (know-how) dimension (Bosch & Gascon, 2006).

The combination of these two theoretical perspectives, usually called *the instrumental approach in didactics of mathematics*, is interesting in that it both revives and overcomes binaries by questioning the role that instruments play in the production of scientific knowledge. In particular, through its use of ATD, which does not include 'mind' or 'body' or related concerns, the instrumental approach highlights the idea of *instrumented techniques*. At the same time, it takes from the cognitive ergonomic perspective the concept of utilisation schemes (in the sense of Vergnaud, 1990). This introduces an ontological distinction between the subject and the tool, which requires the introduction of psychological processes (in this case, schemes) to explain how the subject knows and/or uses the tool. These schemes are inferential, of course – they are not directly visible, but *assumed to exist* (in the brain). The instrumental approach thus links mind and body through the notion of schemes of use, which give origin to the instrument, thanks to the use of an artefact made by a subject.

The question of techniques played an important role in the development of the instrumental approach, which challenges assumptions about technology being merely a scaffolding or mediating tool, and instead as co-implicated in mathematics itself. In other words, pencil-and-paper mathematics is not the fixed, universal given that technology merely attempts to represent, a point made repeatedly in the past by researchers such as Shaffer and Kaput (1999). This partly explains why it demands such radical changes on the part of the teacher.

3.2 Towards a Focus on the Teacher: Instrumental Orchestration, Teachers' Instrumental Geneses and the Documentational Approach

The development of the instrumental approach continued with changes in focus, by moving from a focus on the learning with an artefact (the student and the artefact, instrumental genesis, schemes of use, instrumental distance) towards a focus on the teaching with one or more artefacts. This led to several, and non-independent, extensions:

- extending the focus on the use of instruments by singular subjects to the focus on the teacher, managing a class of students, developing their own instrumental genesis, which led to the concept of *instrumental orchestration* (Trouche, 2004), which concerns these collective and collaborative dimensions of the uses;
- extending the instrumental genesis to the teacher (analysing also the instrumental genesis of the teacher);
- extending the notion of artefact to the teachers' resources, which led to the *documentational approach* (Gueudet & Trouche, 2009);
- enlarging the perspective of observation to a holistic approach—the *resource approach*—which investigates the teacher's documentational approachivity in all their multifaceted aspects.

In terms of the shift from the instrumentation of the student to that of the teacher, the multiplicity of student geneses and the need to unify social schemes of use of the artefact in the classroom raised the question of the management and guidance of these instrumental geneses by the teacher. Focussing then on the teacher's consideration of these phenomena and, more generally, on the management of the students' instrumental geneses, Trouche (2004) introduces the concept of instrumental orchestration to designate this necessary work of the teacher to devise an organisation of the teaching that manages the students' instrumental genesis. The notion of *instrumental orchestration* includes, according to Trouche, the *didactic configurations* and the *exploitation modes*. These ideas will be further developed (Trouche & Drijvers, 2010), for example, by Drijvers et al. (2010), who adds the idea of *didactical performance*. Moreover, Drijvers et al. (2013) proposes a classification of the orchestrations initially identified in research (whole-class/individual, more teacher-centred/more student-centred). This has been useful to subsequent authors in offering new types of orchestration, variations of existing ones and, later so-called *chains of orchestration* (Besnier & Gueudet, 2016).

The focus on teachers gains a reinforced position when Rabardel's concept of instrumental genesis is applied to the teacher and not just to the student. This extension of the theory leads to the identification of a *double instrumental genesis* (professional and personal) (Haspekian, 2011) on the teacher's side, intertwining in that case a personal instrumental genesis, leading to a personal instrument, with a professional one, leading to the didactical one. In this work, the growing attention to the teaching practices leads to the necessity of another theory: *the double approach—ergonomics and didactic—of teacher practices* (Robert & Rogalski, 2002). For example, this approach helps to better define and explain why and how the instrumental distance presents a problem for mathematics teachers as they work to integrate technologies in their teaching practices (Haspekian, 2017). The association with the instrumental approach shares the fact that this theory also draws on cognitive ergonomics roots, combined with a didactic perspective. Continuing in the focus on the teachers' activity in a technology-instrumented classroom, Gueudet and Trouche (2009) applied Rabardel's artefact/instrument distinction, this time to the teacher's resources. Thus, they introduced the distinction "resource/document" for the teacher and elaborated the *documentational approach* to didactics (also called DAD).

The fourth theoretical extension draws on the DAD and aims to describe teachers' work from a holistic approach: the *resource approach* considers all the things that feed teacher work. The focus is on activity, on what actors in the educational environment use and/or design for their work. A resource is what can be drawn on by a person or organisation in order to function effectively (Trouche, 2019). This approach is more directly linked to activity (something to be engaged in somebody's activity) rather than to characteristics or features of the actor (their knowledge or beliefs, for example). It has been used in a variety of contexts to study not just digital resources, but also curricular and cognitive resources, referring to teachers' professional work as a process, in which managing complexity among resources plays a fundamental role.

To the four theoretical extensions described above, we can add one more, introduced by Drijvers et al. (2019) in their book on orchestration: "five future perspectives of this notion, to further extend its value for mathematics educations, and for teacher training in particular: (1) a shift toward student-centred orchestrations, (2) extending the repertoire of orchestrations, (3) chaining orchestrations, (4) didactical performance, and (5) teachers' and students' gestures" (p. 400). The 5th point aims to incorporate into the instrumental approach, existing theories of embodied cognition (with their attention to the significant of gestures in mathematics thinking and learning).

Attention to embodiment in the context of learning with digital technology dates back to the work of Papert (1980) with his notion of body syntonicity. Since then, various theories of embodied cognition have been used, with some drawing on Lakoff and Núñez's (2000) metaphor approach (Sinclair, 2001; Edwards, 2009) and others drawing on semiotic approaches (Arzarello & Robutti, 2008; Arzarello et al., 2009; Arzarello et al., 2015) and still others drawing on phenomenological and new materialist approaches (Sinclair et al., 2013; Nemirovsky & Ferrara, 2008). Drijvers et al. (2019) adopt a more semiotic approach, where gestures are studied through their "symbolic" dimension (through mathematics techniques), and rarely from their "physical" dimension. This may be due to the allegiance of the instrumental approach in the didactics of mathematics to ATD, since the study of mathematical techniques is strongly inscribed in Chevallard's concept of praxeology.

4 Reflections on Theory Development in Mathematics Education

With the advent of digital technology in school mathematics (in the 80s, mainly through the use of turtle geometry, e.g., Papert, 1980), and the rapid evolution of different kinds of software, the use of digital tools spread out all over the world and researchers needed to find ways to theoretically frame their uses in school. This first began by attending to *students'* use of digital tools, drawing on theories used within mathematics education (Piaget, for example), and attempting to justify what mathematics were being learned. Theories then began drawing on ideas and points of view found outside of mathematics education, and were used to explain the interactions between students and tool (initially individually). This was the case, for example, with the instrumental approach's use of cognitive ergonomics.

We are now completely immersed in a third wave of digital technology use, with distributed, highly integrated, connected, and interconnectable infrastructures (that includes both the first and the second waves). Indeed, communication and representation infrastructures, which were previously separated (Hegedus & Moreno-Armella, 2009) are now more blended in a complex net with different affordances. How, as researchers, do we make sense of the use of tools by students and teachers in these interconnectable infrastructures? We are now seeing the emergence of theories responding to technology infrastructure changes. For example, new kinds of infrastructures are being explored by networking a theory relating to teachers—that of Meta-Didactical Transposition (Arzarello et al., 2014)—with connectivism (Siemens, 2005) within the context of Massive Open Online Courses (MOOCs), in which online connectivity provides the medium for teachers' collective work (Taranto et al., 2020). We are also seeing the emergence of theories responding to changing philosophical perspectives. For example, the use of the construct of boundary object (Star & Griesemer, 1989), as a socio-material and dynamic interface that attempts to dissolve the theory–practice binary, has been used in teacher professional development (Robutti et al., 2019) and in studies of teacher practice (Sinclair et al., 2020), to interpret and justify phenomena of interaction among communities, focusing attention on the processes that characterise them.

More generally, we see several specific opportunities for theoretical development, some of which have already begun:

- Adapting existing theories and theoretical concepts (such as instrumental orchestration) to account either for new phenomena (new kinds of digital experiences, such as hapticity, e.g., Sinclair & de Freitas (2014)) or for new aspects of existing phenomena (e.g., embodiment) (e.g., Drijvers et al., 2019).
- Networking existing and/or new theories to draw on established constructs and findings in a more inclusive and reliable way (see Prediger et al., 2008).
- Exploring and addressing the different scales of phenomena at play in the mathematics classroom, from the molecular circulation of affect, to the mid-level temporality of discussion and tool-use, to the larger-scale temporality of institutions and disciplines, including teacher professional development (Lemke, 2000). These scales can be also observed from an institutional point of view, looking at the context according to levels of codetermination (Chevallard, 2002; Barquero, Bosch & Gascón, 2013).
- Challenging existing philosophical assumptions (about the nature of learning; about the status of mathematical objects; about the role of values in mathematics and mathematics teaching and learning) to see how they give rise to new theoretical ideas that can productively address current opportunities and challenges.

In the next section, we show how re-thinking the philosophical assumptions of theories can give rise to new insights into the possibilities of networking, the different scales of phenomena that might matter in the teaching and learning of mathematics, and the challenging of existing binaries.

4.1 Some Philosophical Reflections on Theories Relation to Technology

Two main theoretical tendencies have dominated the past few decades of research in mathematics education. On the one hand, there are student-centred theories that adopt an epistemological stance in which the cognising subject constructs her own knowledge *in an autonomous way*, such as constructivism and radical constructivism (Radford, 2008). In these theories, knowledge is not passively received, which contravenes prior transmission theories of learning such as behaviourism, but actively constructed. There are also constructivist theories that attend more specifically to the social processes involved in this construction, such as interactionism. Epistemologically, these theories assume that it is not possible to have any certain knowledge of reality, opting instead to conceptualise knowledge as adaptive, serving to organise one's experiential reality—these theories do not assume that knowledge correspond to an ontological reality. This epistemological position finds its lineages in Kantian philosophy. However, whereas Kant saw mathematics as the paradigm of certain knowledge (therefore according it an ontological reality), Piaget, whose influence on constructivism in mathematics education was significant, assumed all knowledge to be hypothetical—it is about viability rather than certainty.

Even while recognising the importance of social processes, the theories described above depend on a binary that distinguishes the cognising subject from the culture in which the subjects act and know (Lerman, 1996). In socially oriented constructivist theories, the teachers play a crucial role, because they must enable children to construct their own knowledge, while at the same time guiding the negotiation of this knowledge in the classroom. Unlike constructivism, the Theory of Didactical Situations (TDS), which inherits the constructivist assumption of students constructing their own knowledge, and attends to the social, with its concepts of *situation* and *milieu*, assumes that there is a target, cultural knowledge at stake, which means that the teacher is focussed on straddling the border between student knowledge and cultural knowledge. In both cases, teachers integrating technologies that are novel (either historically, or in terms of a teacher's prior experiences) will face significant challenges, both in legitimising student knowledge and in linking student knowledge to cultural knowledge.

On the other hand, there are socio-cultural approaches, where autonomy is not a prerequisite for knowledge construction. These approaches view knowledge as coming from the historically generated knowledge that is mediated by language and tools, as can be seen in the Theory of Semiotic Mediation (TSM). Such approaches adopt a significantly different ontological position, which is that knowledge is historically generated, which contrasts with the realist assumption that mathematical exists independently of time and culture. Therefore, as Radford (2008) argues, "sociocultural research starts from the premise of a cultural (material and ideal) reality that precedes the cognitive activity of the individual, and as such transcends the individual from the outset" (p. 10).[4] This quote makes explicit the epistemological position of socio-cultural theories, which is that students make sense of certain modes of thinking, learn to participate or become fluent in them—knowledge comes from without, rather than from within. For the teacher, the goal is less to support

[4] Social constructivism recognises the importance of social interactions in the classroom in the processes of teaching and learning, but this does not change its epistemological commitment to knowledge arising from the individual.

students in autonomous cogitations, than to provide mediation—through tools, concepts, language—that enables a learner to achieve something that could not be achieved autonomously. In this sense, autonomy is the outcome of the learning process rather than its prerequisite. For teachers integrating novel technology, it can be challenging for them to know just what mathematically valid problem these technologies can enable students to accomplish, especially when that problem is not explicitly connected to curricular expectations. For example, to use a dynamic geometry environment example, dragging a point can quite evidently enable students to generate a family of dynamic triangles, it is less evident for teachers to identify both the cultural knowledge this action might mediate and the curriculum goal to which it relates.

The instrumental approach has been influenced by the constructivist orientation—as Ruthven writes, citing Rabardel (2002), the instrumental approach was "developed in cognitive ergonomics to study the typically non-proposition and action-oriented knowledge involved in making use of tools" (p. 380). The two processes of instrumental genesis, although oriented in two different directions in the subject-artefact (learner-tool) relationship, are both generated by the subject, within the affordances and constraints of the artefact. As such, most researchers in mathematics education are primarily interested in studying—in terms of processes—the evolution of students and teachers, both of whom begin as naïve users, but eventually become proficient ones. This approach disrupts both the conceptual–technical and thinking–acting binaries that commonly ground mathematics education research.

Instrumental Orchestration focusses on the teacher's handling of the myriad of processes of instrumental genesis in the classroom, and it adopts a socio-constructivist interest in studying the "collective path by means of which emergent knowledge is socialised into a shared form aligned with wider conventions and practices" (Ruthven, 2014, p. 381), thereby the term "orchestration". As Trouche (2005) points out, of the various didactic configurations identified in Trouche (2004), only one focusses on the adaptation of the tool itself, whereas the others only obliquely implicate the tool. Indeed, as Ruthven (2014) argues, much of the research using Instrumental Orchestration has focussed almost exclusively on the organisation of classroom activity, thereby saying little about tool adaptation. Additionally, by focusing on the exploitation modes of the teacher, which are "the way the teacher decides to exploit a didactical configuration for the benefit of his or her didactical intentions" (p. 215), Drijvers et al. (2010) study only the presumed intentional acts of teachers.

While the Instrumental Approach in didactics began as a learner-specific theory with the concept of instrumental genesis, and later gave birth to teacher-specific concepts, the theory of semiotic mediation (TSM), which draws on a socio-cultural approach, has always been a teacher-and-student theory (Bartolini Bussi & Mariotti, 2008). In TSM, teaching and learning are studied simultaneously, with the teacher having two specific aims, which are to design activities (which includes the choice of task, artefact and mathematical knowledge) and to make the activity function (which involves exploiting, monitoring and manages the students' behaviours, including utterances, gestures and artefact actions). Given the socio-cultural epistemological assumptions of TSM, knowledge is taken to be grounded in the historical practices, which includes the artefacts, language and mathematical objects in which the teacher acts as a cultural mediator. An important a priori stance is taken in TSM, through the work to identify the semiotic potential of an artefact, which is to recognise the links between the artefact, the task and mathematical knowledge. The teacher must then try to exploit this semiotic potential, and this can occur when the teacher manages to transform artefact signs (actions accomplished with the artefact, such as dragging a point along a specific path) into mathematical signs (such as a locus). The TSM includes certain specifications, called the didactic cycle, about how a lesson might be planned, as well as how it should be performed, including how a collective discussion should unfold (Bartolini Bussi & Mariotti, 2008). When it is used to study student learning (e.g., Bartolini Bussi & Baccaglini-Frank, 2015), it does so by attending also to the teacher's role, moreover specifying – in a sense – certain aspects of it.

While differing in terms of their epistemological assumptions, both constructivist and socio-cultural theories treat tools as being essentially passive with respect to human intention and action. However, both the Instrumental Approach and TSM challenge the passivity of tools and grant them a significant role in shaping students' conceptualisations, thus showing how changes in technology can also lead to changes in philosophical assumptions about how we come to know. For example, Rabardel (1995) writes about the "structuring effects of artefacts on activity" (p. 5). He identifies different types of structuring constraints of artefacts, which thus include "pre-organized forms that subjects are confronted with in their instrumented activities" (p. 6). Arguably, however, the ontological question about *what knowledge is* has not been explicitly addressed in either theory. In most constructivist approaches there is a sense that mathematical knowledge (of a given concept, such as number), exists, and that the cognising subjects attempts to construct it, perhaps with the help of a teacher or a tool, but the knowledge itself is an independent entity. Knowledge can therefore be acquired, transferred, developed, connected, etc. For socio-cultural theories, including TSM, knowledge is historically generated and cannot be dis-embedded from cultural and context. Knowledge is framed in terms of participation within a particular community, focussing on doing/talking/making more than having. Knowledge is not just situated within culture, it is subordinate to that culture—defined and determined by it.

There are many variations of both constructivist and sociocultural theories in mathematics education, which we will not explore here. Instead, for the sake of thinking generatively about theory, we are interested in the possibilities opened up by considering both ontological assumptions and axiological ones. This is not just idle theory play, but is instead motivated by the new perspectives that such considerations would give rise to, particularly in relation to issues of equity, but also of validity (accounting for the complexity of teaching and learning mathematics). For this chapter, we use the work of the French philosopher Gilbert Simondon to guide our exploration, not only due to his extensive writings on technology (some of which shaped Rabardel and Vérillon's thinking), but also for his commitment to challenging some of the binaries we have evoked earlier in this chapter.

4.2 An Exploration of Technology Related Theories from the Work of Simondon

Simondon can be described as a process philosopher, that is, he takes *becoming to be* more fundamental than *being*—seeing *change or difference* as the central feature of reality, rather than *identity*. He therefore seeks to replace substantialist assumptions (that students, teachers, tools, etc. can be adequately defined through a set of properties or characteristics) with operational ones (that what exists, what can be described, are processes and relations). By focusing on *process*, he does not need to make *a priori* distinctions between cognition and affect, for example, nor between the individual and the collective. Instead of asking questions about being—what does the teacher know? what are the tool's affordances?—which assume an intact subject to which things happen (like instrumentation, or internalisation or participation), Simondon (1958) would prompt us to ask ontogenetic questions such as: how does a knowing teacher emerge from interactions with textbooks or tools? what gives rise to the idea of dynamic geometry and how does it remain stable or change over time? In other words, rather than assuming the existence of the individual that one is seeking to account for, Simondon focusses on *individuation*, which is the process of constituting that individual, dynamic in itself. Individuation describes processes through which a tool becomes a tool or a teacher becomes a teacher, processes that occur through multiple domains, including the physical, the biological, the psychosocial and the technological. The concept of *being* concerns the process of mutation, a potential, rather than fixed categories. Simondon will assert that this process is fundamentally affective. He blurs the individual–collective by studying the process of *transindividuation, which* concerns individuation in relation to a collective.

This is interesting because it might help address important problems in our studies of teaching with technology. For example, it might help account for how teaching and learning mathematics arises from *both* individual *and* cultural forces. It might draw attention to how the individuation of tools impacts teachers' actions. It might shine light on the affective dimension of teachers' individuation, with respect to digital tools, which might help us understand their limited presence in the mathematics classroom. It might help us describe how digital tools participate in processes of collective individuation. Finally, it can help us appreciate the new relations made possible between living beings and digital tools, which exceed—and cannot be read in terms of—existing paper-and-pencil, alphanumeric relations.

A significant assumption of Simondon's is that individuation is not reserved for humans, and also occurs, for example, in technical objects (as well as crystals, which he uses as a paradigmatic example). Tools thus have a new "ontological dignity" in that they individuate—they have the potential to change or to become in their given milieus, much as humans do. The theory of instrumental genesis respects this perspective when it focuses on the genesis of instrument (through instrumentation and instrumentalisation). For Rabardel, one of the interests of the instrumental genesis is precisely to theoretically "found the articulation and the continuity between the institutional processes of conception of the artefacts and the continuation of the conception within the activities of use" (ibid, p. 5). However, it adopts a psychological approach: this *becoming* of the technological objects is a psychological construct, on the subject's side. It exists only through this human relation. Methodologically, it is crucial for Simondon that this process of individuation is not seen from a strictly psychological perspective, nor from a sociological one, each of which assumes that the human and the group, respectively, are pre-defined.

Simondon's insistence on affectivity—that is, the way we affect and are affected in the world, prior to any conscious or cognitive interpretation or inference—offers an interesting direction for research on teaching with technology. Affectivity, for Simondon, describes the liaison between the individual and the world. It establishes a relation with something that the individual brings with them, but that is felt as being exterior to themselves. This might be the relation that arises from a teacher's encounter with a pair of compasses (such as a haptic response of fixing, turning) or with dynamic geometry environment (such as a anticipation of movement). Affective life shows us we are more than individuals, we are parts of multiple, heterogeneous networks (other humans, groups, tools, mountains, rain, spiders) and this is why Simondon will refer to the "more-than-individual" life. Individuation can be described as the resolving of this affective world (the relation) and the perceptual one (the disjunction). As researchers, we might inquire into how is this tension resolved when new digital tools are used in the classroom, or while teachers participate in professional development programmes, or in the process of task design.

From Simondon's perspective, technical objects are not "in need of meaning, form, purpose and value which must be brought to it from the outside, through human intervention" (Grosz, 2012, pp. 52–53). In Trouche's (2000) parable of the "casserole à bec" (a saucepan with a spout used to pour milk) the technical object is described as not being adapted to its user, who is left-handed, which means that the user must make one of several (seven!) choices on how to proceed. The focus is on the point of view of the subject, on how the subject changes to accommodate the tool. And while the tool is granted some "affordances", its own individuation is rarely studied. What Simondon would have been interested in, is not only the genesis of the spouted saucepan, including the microtechnical level of detachable pieces (the handle, the spout) and the macrotechnical levels of webs of distribution and exchange (as it co-mingles with other objects), but how the technical object *individuates*. Doing so would confer a certain status on the technical object that is not merely instrumental (what it enables us to do), but that is also aesthetic (how it makes us feel). Simondon's perspectives align well with recent posthuman approaches to education that attempt to de-centre the human subject (see Snaza et al., 2014), and re-direct attention to how the material world, our geography, or classroom seating arrangements, our tools, the weather, etc., matters in teaching and learning. This may represent an actual Copernican revolution in mathematics education theories.

More generally, for Simondon, technical objects have modes of existences just as humans and cultures do; they are *becomings* rather than beings, and it thus makes sense to study them as actualisations of a process rather than as fixed, isolated entities—there is a dynamic reciprocity within the world of technical objects. Here, the documentational genesis of Gueudet and Trouche (2009) comes to mind since it focusses on the individuation of a teacher's resource, but in Simondon's hands, the genesis applies not only to the teacher (or to the student) but to the resources as well. In Simondon's ontology, the socio-cultural assumption about knowledge being fundamentally historical and cultural no longer applies, and therefore the physical, material world is not read in terms of this socio-cultural reality. This is the sense in which technology objects are proffered an ontological dignity. Simondon thus offers new considerations for how theories are used in mathematics education research, especially as they relate to digital technology, which will have geneses that are quite unique. In education, where we are concerned with the experiences of students and teachers, we may well gravitate to theories that concern psychological or sociological processes. However, we might draw on Simondon to see how our theories could position digital technologies in less instrumental or passive ways, so they can disrupt the subject–object and individual–society binaries. That they could use challenging is evident in the fact that our current theories have either been psychological or sociological, whereas lived experience is clearly both.

Simondon-inspired research might study the genesis of specific technologies, be they physical ones such as Cuisenaire rods or virtual ones such as dynamic geometry environments, which, like the automobile motor, have evolved in relation to their own modes of existence, and experienced multiple *individuations*, that is, multiple processes of *becoming*. Indeed, the simple action of triangle dragging has evolved into measuring angles and sides, plotting measurements related to the triangle onto a Cartesian coordinate system, collecting measures in a spreadsheet, etc. These processes enabled the dynamic geometry environments *become* dynamic mathematics environments, through complex assemblages with mathematics, teachers, school mathematics, and other technologies. Now there exist many large multipurpose dynamic mathematics environments, parts of which can be found—that have been re-used—in other technical objects, such as on-line textbooks or tablet-based microworlds. And these parts have different geneses as they become re-inscribed in new media, for example, the multi-touch tablet now makes dragging all three vertices of the triangle possible, which fundamentally changes the experience of dragging, as well as the mathematics (from single, sequential variation to simultaneous multi-dimensional variation).

In terms of theories concerned with teachers, Simondon invites a view of technology as being less epiphenomenal to issues of didactic configurations, exploitation schemes, classroom discussions, and so on. We might consider, for example, the non-intentional ways in which classroom activity unfolds, perhaps seeking to study how teachers respond to these agential forces of technical objects (see Carlsen et al., 2016; Chorney, 2017). Clark-Wilson's (2014) study of "hiccups" seems to be within this vein, in that the particular moments of instability in which she documents (the "hiccup"), are precisely those in which the technology is asserting itself, affecting the teacher in ways that demand new responses. Such moments of instability may be transformed into pedagogical opportunities. As with the other untheorised aspects of classrooms, periods of silence, side conversations about non-mathematical topics, rises in heartbeat, flushing of the cheeks—they might matter quite significantly to our understanding of the teaching and learning process.

5 Looking Forward

All three of the theories described in Ruthven (2014) fall within the *Understand* paradigm of educational theories discussed by Stinson and Walshaw (2017). Loosely following the *Predict* paradigm that is associated with behaviourism and other positivist theories, the *Understand* paradigm begins in the 1980s and continues to dominate mathematics education research. The two other paradigms, emerging from the socio-political turn, are characterised as *Emancipate* and *Deconstruct.* These make very different epistemological and ontological assumptions from constructivist and socio-cultural theories. Theories in the *Emancipate* paradigm assume, for example, that knowledge, power and identity are constituted in and through socio-political discourse. Theories in the *Deconstruct* paradigm challenge that transparency of language, the objectiveness of knowledge and "the idea that knowing is an outcome of human consciousness and interpretation and that individuals are autonomous and stable with agency to choose what kind of individual they might become" (Stinton & Walshaw, 2017, p. 139). Further, they may seek to decentre the human, and thus attend to the distributions of agency and knowledge across human *and* non-human entities – therefore troubling the primacy of the intact human body and passiveness of the material world.

Interestingly, there are extremely limited research studies in the mathematics education research related to the use of digital technology that adopts these second two paradigms of research. This may be a result of the historical divide between technology enthusiasts who align with scientific ideas of progress and detractors who align with humanist concerns (that technology will rob humans of agency, inequitably redistribute wealth and freedom, etc.) (Simondon, 1958/2017). This can be seen in Fig. 1, where there are no connections to theories emanating from the socio-political turn of the 2010s (Gútierrez, 2013). Very few articles[5] use a critical theory lens to examine, for example, how the use of digital technology may feed into the inequalities that exist between populations (rich/poor, male/female, black/white, abled/disabled). There is broad-level research that critiques the neoliberal, social engineering project of the STEM movement. Some authors argue that

[5] As a counterpoint though, two leading scholars of this socio-political turn have studied the role of interactive whiteboards, arguing that this technology exacerbates existing inequities—see Zevenbergen and Lerman (2008).

STEM-based activities must be more attentive to ethical and ecological forms of knowing (Glanfield et al., 2020; Wisemant et al., 2020). The research of feminist black studies, who draw on Simondon (among other theorists), to disrupt colonial assumptions about who counts as human and what counts as reason (see Keeling, 2019), might be generative for rethinking what it means to know and do and teach mathematics.

The current lack of research in these two paradigms points to an underlying feature of the framing of digital technology in education (and more generally, in society) as a largely utilitarian concern that relates very little to broader questions around politics, social, ethics and aesthetics. For Simondon, this trivialising of technology was a symptom of a significant paradoxical angst, namely, that the world divides into two camps: one on side, there are the technology cheerleaders and the technology detractors. (This is of course a simplistic binary, one that has surely been complicated in the past 60 years, but we use it merely to illustrate a tension.) The former are keen to show how technology is going to improve human life (including enthusiasts such as Francis Bacon and Karl Marx) and the latter are those who fear that technology will replace or control humans, Samuel Butler's *Erewhon* (1872) being a paradigmatic example. As the divide between the "two cultures" grew, the humanists and social scientists tended to belong to the latter group, while the scientists and engineers tended to the former, and were interested in the more technical aspects of technology.

In education, this angst plays out in a more nuanced way, particularly in relation to the detractors, since the very same digital technologies that have improved mathematical activity (such as the calculator that can compute more quickly and accurately, or the dynamic geometry environment that can continuously manipulate shapes) actually replace important aspects of some teachers' practices. While mathematics education researchers tend to celebrate the added opportunity offered by digital technology, they seldom acknowledge the associated losses, especially in terms of how they might affect teachers' professional or mathematical identities or, to put it in Simondon's language, how they problematise the liaison between affective and perceptual worlds.

Appendix: List of the Journals Reviewed

1. International Journal of Science, Mathematics & Technology Learning (From Volume 21 Issue 2 to Volume 27 Issue 1)
2. International Journal of Mathematical Education in Science and Technology (From Volume 46 Issue 1 to Volume 51 Issue 8)
3. Educational Studies in Mathematics (From Volume 88 Issue 1 to Volume 105 Issue 2)
4. Digital Experiences in Mathematics Education (From Volume 1 Issue 1 to Volume 6 Issue 3)

5. International Journal for Technology in Mathematics Education (From Volume 22 Issue 1 to Volume 27 Issue 3)
6. International Journal of Science and Mathematics Education (From Volume 13 Issue 1 to Volume 18 Issue 8)
7. ZDM Mathematics Education (From Volume 47 Issue 1 to Volume 52 Issue 7)

References

Abdu, R., Schwarz, B., & Mavrikis, M. (2015). Whole-class scaffolding for learning to solve mathematics problems together in a computer – Supported environment. *ZDM Mathematics Education, 47*(7), 1163–1178.

Akkerman, S., & Bakker, A. (2011). Boundary crossing and boundary objects. *Review of Educational Research, 81*(2), 132–169.

Aldon, G., & Panero, M. (2020). Can digital technology change the way mathematics skills are assessed? *ZDM Mathematics Education, 52*(7), 1333–1348.

Artigue, M. (1996). Using computer algebraic systems to teach mathematics: A didactic perspective. In E. Barbin & A. Douady (Eds.), *Teaching mathematics: The relationship between knowledge, curriculum and practice* (pp. 223–239). TOPIQUES éditions.

Artigue, M. (1997). Le logiciel DERIVE comme révélateur de phénomènes didactiques liés à l'utilisation d'environnements informatiques pour l'apprentissage. *Educational Studies in Mathematics, 33*, 133–169.

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *The International Journal of Computers for Mathematics Learning, 7*(3), 245–274. (hal-02367871).

Artigue, M. (2020). ICMI AMOR MOOC: *Michèle Artigue Unit*. https://www.mathunion.org/icmi/awards/amor/michele-artigue-unit. Accessed 6 Oct 2021.

Artigue, M., Cazes, C., Haspekian, M., Khanfour, R., & Lagrange, JB. (2013). Gestes, cognition incarnée et artefacts : une analyse bibliographique pour une nouvelle dimension dans les travaux didactiques au LDAR. *Cahiers du laboratoire de didactique André Revuz, n°8*, IREM Paris 7. ISSN 2105–5203. http://docs.irem.univ-paris-diderot.fr/up/publications/IPS13006.pdf

Arzarello, F. (2006). Semiosis as a multimodal process. *Revista Latinoamericana de Investigación en Matemática Educativa RELIME, 9*(Extraordinario 1), 267–299.

Arzarello, F., & Robutti, O. (2004). Approaching functions through motion experiments. *Educational Studies in Mathematics, 57*(3), 305–308.

Arzarello, F., & Robutti, O. (2008). Framing the embodied mind approach within a multimodal paradigm. In L. English (Ed.), *Handbook of International Research In Mathematics Education* (pp. 720–749).

Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A cognitive analysis of dragging practises in Cabri environments. *ZDM: Zentralblatt für Ditaktik derMathematik, 34*(3), 6–72.

Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics, 70*(2), 97–109.

Arzarello, F., Robutti, O., Sabena, C., Cusi, A., Garuti, R., Malara, N. A., & Martignone, F. (2014). Meta-didactical transposition: A theoretical model for teacher education programs. In A. Clark-Wilson et al. (Eds.), *The mathematics teacher in the digital era, mathematics education in the digital era 2* (pp. 347–372). Springer.

Arzarello, F., Robutti, O., & Thomas, M. (2015). Growth point and gestures: Looking inside mathematical meanings. *Educational Studies in Mathematics, 90*(1), 19–37.

Bairral, M., & Arzarello, F. (2015). The use of hands and manipulation touchscreen in high school geometry classes. In *CERME 9-Ninth Congress of the European Society for Research in Mathematics Education* (pp. 2460–2466).

Bakker, A., Kent, P., Hoyles, C., & Noss, R. (2011). Designing for communication at work: A case of technology-enhanced boundary objects. *International Journal of Educational Research, 50*(1), 26–32.

Balacheff, N. (1994). Didactique et intelligence artificielle. *Recherches en Didactique des Mathématiques, 14*(1–2), 9–42.

Ball, D. L., & Bass, H. (2003). Toward a practice-based theory of mathematical knowledge for teaching. In B. Davis & E. Simmt (Eds.), *Proceedings of the 2002 annual meeting of the Canadian Mathematics Education Study Group Edmonton* (pp. 3–14). CMESG/GDEDM.

Barquero, B., Bosch, M., & Gascón, J. (2013). The ecological dimension in the teaching of mathematical modelling at university. *Recherches en didactique des mathématiques, 33*(3), 307–338.

Bartolini Bussi, M. G., & Baccaglini-Frank, A. (2015). Geometry in early years: Sowing seeds for a mathematical definition of squares and rectangles. *ZDM, 47*(3), 391–405.

Bartolini Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. English et al. (Eds.), *Handbook of international research in mathematics education* (2nd ed., pp. 746–783). Routledge.

Besnier, S., & Gueudet, G. (2016). Usages de ressources numériques pour l'enseignement des mathématiques en maternelle: orchestrations et documents. *Perspectivas da Educação Matemática, 9*(21), 28 dez. 2016.

Bosch, M., & Gascon, J. (2006). Twenty-five years of the didactic transposition. *ICMI Bulletin, 58*, 51–65.

Bozkurt, G., & Ruthven, K. (2017). Classroom-based professional expertise: A mathematics teacher's practice with technology. *Educational Studies in Mathematics, 94*(3), 309–328. https://doi.org/10.1007/s10649-016-9732-5

Brousseau, G. (2002). Les doubles jeux de l'enseignement des mathématiques. *Questions éducatives, l'école et ses marges: Didactique des mathématiques*, Revue du Centre de Recherches en Education de l'Université Saint Etienne (22–23), pp. 83–155. hal-00516813.

Butler, S. (1872). *Erewhon*. Trübner & C.

Caniglia, B. J., & Meadows, M. (2018). Pre-service mathematics teachers' use of web resources. *International Journal for Technology in Mathematics Education, 25*(3), 17–34.

Carlsen, M., Erfjord, I., Hundeland, P. S., & Monaghan, J. (2016). Kindergarten teachers' orchestration of mathematical activities afforded by technology: Agency and mediation. *Educational Studies in Mathematics, 93*(1).

Chevallard, Y. (1985). *La transposition didactique – du savoir savant au savoir enseigné*. La Pensée Sauvage.

Chevallard, Y. (1989). Le passage de l'arithmétique à l'algèbre dans l'enseignement des mathématiques au collège. Deuxième partie. Perspectives curriculaires : la notion de modélisation. *Petit x, 19*, 43–72.

Chevallard, Y. (1992). Concepts fondamentaux de la didactique: perspectives apportées par une approche anthropologique. *Recherches en Didactique des Mathématiques, 12*(1), 73–112.

Chevallard, Y. (1999). L'analyse des pratiques enseignantes en théorie anthropologique du didactique. *Recherche en Didactique des Mathématiques, 19*(2), 221–266.

Chevallard, Y. (2002). Organiser l'étude: 3. Ecologie & régulation. *XIe école d'été de didactique des mathématiques* (Corps, 2130 de agosto de 2001) (4156). Grenoble: La Pensée Sauvage.

Chevallard, Y., & Bosch, M. (2020). Didactic transposition in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education*. Springer.

Chevallard, Y., & Sensevy, G. (2020). Anthropological approaches in mathematics education, French perspectives. In S. Lerman (Ed.), *Encyclopedia of mathematics education*. Springer.

Chorney, S. (2017). Circles, materiality and movement. *For the Learning of Mathematics, 37*(3), 45–49.

Clark-Wilson, A. (2014). A methodological approach to researching the development of teachers' knowledge in a multi-representational technological setting. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital age* (pp. 276–295). Springer.

Cobb, P. (1988). The tension between theories of learning and instruction in mathematics education. *Educational Psychologist, 23*(2), 87–103.

Defouad, B. (2000). *Etude de genèses instrumentales liées à l'utilisation d'une calculatrice symbolique en classe de première S* (p. 7). Thèse de doctorat. Université Paris.

Donald, M. (1991). *Origins of the modern mind: Three stages in the evolution of culture and cognition*. Harvard University Press.

Drijvers, P. (2004). Learning algebra in a computer algebra environment. *International Journal for Technology in Mathematics Education, 11*(3), 77–90.

Drijvers, P. (2012). Teachers transforming resources into orchestration. In G. Gueudet, B. Pepin, & L. Trouche (Eds.), *From text to 'lived resources': Curriculum material and mathematics teacher development* (pp. 265–281). Springer.

Drijvers, P. (2019). Embodied instrumentation: Combining different views on using digital technology in mathematics education. In *Eleventh Congress of the European Society for Research in Mathematics Education*. Utrecht University.

Drijvers, P., Doorman, M., Boon, P., Reed, H., & Gravemeijer, K. (2010). The teacher and the tool: Instrumental orchestrations in the technology-rich mathematics classroom. *Educational Studies in Mathematics, 75*(2), 213–234.

Drijvers, P., Tacoma, S., Besamusca, A., Doorman, M., & Boon, P. (2013). Digital resources inviting changes in mid-adopting teachers' practices and orchestrations. *ZDM Mathematics Education, 45*(7), 987–1001.

Drijvers, P., Gitirana, V., Monaghan, J., Okumus, S., Besnier, S., Pfeiffer, C., … Rodrigues, A. (2019). Transitions toward digital resources: Change, invariance, and orchestration. In L. Trouche et al. (Eds.), *The 'Resource' approach to mathematics education* (pp. 389–444). Springer Nature Switzerland AG.

Edwards, L. D. (2009). Gestures and conceptual integration in mathematical talk. *Educational Studies in Mathematics, 70*(2), 127–141.

Glanfield, F., Thom, J. S., & Ghostkeeper, E. (2020). Living landscapes, Archi-text-ures, and land-guaging Algo-rhythms. *Canadian Journal of Science, Mathematics, and Technology Education, 20*, 246–263.

Goldin-Meadow, S. (2005). *Hearing gesture: How our hands help us think*. Harvard University Press.

Grosz, E. (2012). The nature of sexual difference: Irigaray and Darwin. *Angelaki, 17*(2), 69–93.

Gueudet, G., & Trouche, L. (2009). Towards new documentation systems for mathematics teachers. *Educational Studies in Mathematics, 71*, 199–218.

Guin, D., & Trouche, L. (1998). The complex process of converting tools into mathematical instruments: The case of calculators. *International Journal of Computers for Mathematical Learning, 3*(3), 195–227.

Guin, K., Ruthven, R., & Trouche, L. (Eds.). (2005). *The didactical challenge of symbolic calculators: Turning a computational device into a mathematical instrument*. Springer.

Gutiérrez, R. (2013). The sociopolitical turn in mathematics education. *Journal for Research in Mathematics Education, 44*(1), 37–68.

Haspekian, M. (2005). An "instrumental approach" to study the integration of a computer tool into mathematics teaching: The case of spreadsheets. *The International Journal of Computers for Mathematics Learning, 10*(2), 109–141.

Haspekian, M. (2011). The co-construction of a mathematical and a didactical instrument. In M. Pytak, E. Swoboda, & T. Rowland (Eds.), *Proceedings of CERME 7* (pp. 2298–2307).

Haspekian, M. (2017). Computer science in mathematics new curricula at primary school: New tools, new teaching practices? In G. A. & J. Trgalova (Eds.), *Proceedings of ICTMT 13* (pp. 23–31).

Healy, L., & Kynigos, C. (2010). Charting the microworld territory over time: Design and construction in mathematics education. *ZDM Mathematics Education, 42*, 63–76.

Hegedus, S. J., & Moreno-Armella, L. (2009). Intersecting representation and communication infrastructures. *ZDM, 41*(4), 399–412.

Højsted, I. H. (2020). Teachers reporting on dynamic geometry utilization related to reasoning competency in Danish lower secondary school. *Digital Experiences in Mathematics Education, 6*(1), 91–105.

Hoyles, C. (1992). Illuminations and reflections: Teachers, methodologies and mathematics. In W. Geelin & K. Graham (Eds.), *Proceedings of the 16th Conference of the Internal Group for the Psychology of Mathematics Education* (Vol. 3, pp. 263–286). University of New Hampshire.

Jackiw, N. (2013). Touch and multitouch in dynamic geometry: Sketchpad explorer and 'digital' mathematics. In E. Faggiano & A. Montone (Eds.), *Proceedings of the 11th International Conference on Technology in Mathematics Teaching* (pp. 149–155). Università degli Studi di Bari Aldo Moro.

Kaput, J. (1989). Linking representations in the symbol systems of Algebra. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (pp. 167–194). NCTM.

Keeling, K. (2019). *Queer Times, Black Futures*. New York University Press.

Kynigos, C. (2007). Half-based Logo microworlds as boundary objects in integrated design. *Informatics in Education, 6*(2), 335–358.

Lagrange, J. B. (1999). Techniques and concepts in pre-calculus using CAS: A two year classroom experiment with the TI92. *The International Journal for Computer Algebra in Mathematics Education, 6*(2), 143–165.

Lagrange, J.-B., Artigue, M., Laborde, C., & Trouche, L. (2003). Technology and mathematics educaiton: A multidimensional study of the evolution of research and innovation. In A. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick, & F. Leung (Eds.), *Second international handbook of mathematics education* (pp. 237–269). Springer.

Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. Basic Books.

Latour, B. (1987). *Science in action*. Harvard University Press.

Lemke, J. (2000). Across the scales of time: Artifacts, activities, and meanings in ecosocial systems. *Mind, Culture, and Activity, 7*(4), 273–290.

Lerman, S. (1996). Intersubjectivity in mathematics learning: A challenge to the radical constructivist paradigm? *Journal for Research in Mathematics Education, 27*(2), 133–150.

Mariotti, M. A. (2002). *The influence of technological advances on students' mathematics learning* (pp. 695–723). Handbook of International Research in Mathematics Education.

Mariotti, M. A. (2009). Artifacts and signs after a Vygotskian perspective: The role of the teacher. *ZDM, 41*(4), 427–440.

Maschietto, M., & Soury-Lavergne, S. (2013). Designing a duo of material and digital artifacts: The pascaline and Cabri Elem e-books in primary school mathematics. *ZDM, 45*(7), 959–971.

Mishra, P., & Koehler, M. J. (2006). Technological pedagogical content knowledge: A framework for integrating technology in teacher knowledge. *Teachers College Record, 108*(6), 1017–1054.

Moschkovich, J. (2002). A situated and sociocultural perspective on bilingual mathematics learners. *Mathematical Thinking and Learning, 4*(2–3), 189–212.

Nemirovsky, R., & Ferrara, F. (2008). Mathematical imagination and embodied cognition. *Educational Studies of Mathematics, 70*(2), 159–174.

Noss, R., & Hoyles, C. (1996). *Windows on mathematical meanings: Learning cultures and computers*. Kluwer.

Panero, M., & Aldon, G. (2016). How teachers evolve their formative assessment practices when digital tools are involved in the classroom. *Digital Experiences in Mathematics Education, 2*(1), 70–86.

Papert, S. (1980). *Mindstorms: Children, computers and powerful ideas*. Harvester Press.

Pickering, A. (1995). *The mangle of practice: Time, agency and science*. University of Chicago Press.

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework. *ZDM – The International Journal on Mathematics Education, 40*(2), 165–178.

Rabardel, P. (1995). *Les hommes et les outils contemporains*. Armand Colin. English version accessible at https://halshs.archives-ouvertes.fr/file/index/docid/1020705/filename/people_and_technology.pdf
Rabardel, P. (2002). People and technology – A cognitive appraoch to contemporary instruments. https://hal.archivesouvertes.fr/file/index/docid/1020705/filename/people_and_technology.pdf. Accessed 17 July 2022.
Radford, L. (2008). Theories in mathematics education: A brief inquiry into their conceptual differences. In *Working paper. Prepared for the ICMI survey team 7*. The notion and role of theory in mathematics education research. Available: http://www.luisradford.ca/pub/31_radfordicmist7_EN.pdf
Robert, A., & Rogalski, J. (2002). Le système complexe et coherent des pratiques des enseignants de mathématiques: une double approche. *Revue Canadienne de l'enseignement des sciences, des mathématiques et des technologies, 2*(4), 505–528.
Robutti, O. (2020). Meta-didactical transposition. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 611–619). Springer.
Robutti, O., Aldon, G, Cusi, A., Olsher, S., Panero, M., Cooper, J., Carante, P. & Prodromou, T. (2019). Boundary objects in mathematics education and their role across communities of teachers and researchers in interaction. In G. M. Lloyd (Ed.). *Participants in mathematics teacher education* (Vol. 3, International Handbook of Mathematics Teacher Education). Sense Publishers.
Ruthven, K. (2009). Towards a naturalistic conceptualisation of technology integration in classroom practice: The example of school mathematics. *Education and Didactique, 3*(1), 131–152.
Ruthven, K. (2014). Frameworks for analysing the expertise that underpins successful integration of digital technologies into everyday teaching practice. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era: An international perspective on technology focused professional development* (pp. 373–393). Springer.
Schwarz, B. B., De-Groot, R., Mavrikis, M., & Dragon, T. (2015). Learning to learn together with CSCL tools. *International Journal of Computer-Supported Collaborative Learning, 10*(3).
Shaffer, D. W., & Kaput, J. (1999). Mathematics and virtual culture: A cognitive evolutionary perspective on technology and mathematics education. *Educational Studies in Mathematics, 37*(2), 97–119.
Shulman, L. (1986). Those who understand: Knowledge and growth in teaching. *Educational Researcher, 15*(2), 4–14.
Siemens, G. (2005). Connectivism: A learning theory for the digital age. *International Journal of Instructional Technology and Distance Learning, 2*(1), 3–10.
Simonsen, L. M., & Dick, T. P. (1997). Teachers' perceptions of the impact of graphing calculators in the mathematics classroom. *Journal of Computers in Mathematics and Science Teaching, 16*(2), 239–368.
Sinclair, N. (2001). The aesthetic is relevant. *For the Learning of Mathematics, 21*(1), 25–33.
Sinclair, N. (2013). Touch counts: An embodied, digital approach to learning number. In E. Faggiano & A. Montone (Eds.), *Proceedings of ICTMT12* (pp. 262–267). University of Bari.
Sinclair, N., & de Freitas, E. (2014). The haptic nature of gesture: Rethinking gesture with new multitouch digital technologies. *Gesture, 14*(3), 351–374.
Sinclair, N., & Heyd-Metzuyanim, E. (2014). Learning number with *Touch Counts*: The role of emotions and the body in mathematical communication. *Technology, Knowledge and Learning, 19*(1–2), 81–99.
Sinclair, N., & Jackiw, N. (2005). Understanding and projecting ICT trends. In S. Johnston-Wilder & D. Pimm (Eds.), *Teaching secondary mathematics effectively with technology* (pp. 235–252). Open University Press.
Sinclair, N., de Freitas, E., & Ferrara, F. (2013). Virtual encounters: The murky and furtive world of mathematical inventiveness. *ZDM – The International Journal on Mathematics Education, 45*(2), 239–252.

Sinclair, N., Chorney, S., Gunes, C., & Bakos, S. (2020). Disruptions in meanings: Teachers' experiences of multiplication in *Touch Times*. ZDM*: Mathematics Education*. [On-line first].

Smit, J., Van Eerde, H. A. A., & Bakker, A. (2013). A conceptualization of whole-class scaffolding. *British Educational Research Journal, 39*(5), 817–834.

Smythe, S., Hill, C., MacDonald, M., Dagenais, D., Sinclair, N., & Toohey, K. (2017). *Disrupting boundaries in education and research*. Cambridge University Press.

Snaza, N., Applebaum, P., Bayne, S., Carlson, D., Morris, M., Rotas, N., Standlin, J., Wallin, J., & Weaver, J. (2014). Toward a posthuman education. *Journal of Curriculum Theorizing, 30*, 39–55.

Star, S., & Griesemer, J. (1989). Institutional ecology, translations' and boundary objects: Amateurs and professionals in Berkeley's Museum of Vertebrate Zoology, 1907–39. *Social Studies of Science, 19*(3), 387–420.

Steinbring, H., Bartolini Bussi, M. G., & Sierpinska, A. (1998). *Language and communication in the mathematics classroom*. National Council of Teachers of Mathematics.

Stinson, D., & Walshaw, M. (2017). Exploring different theoretical frontiers for different (and uncertain) possibilities in mathematics education research. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 128–155). National Council of Teachers of Mathematics.

Tahta, D. (1981). Some thoughts arising from the new Nicolet films. *Mathematics Teaching, 94*, 25–29.

Taranto, E., Robutti, O., & Arzarello, F. (2020). Learning within MOOCs for mathematics teacher education. *ZDM – The International Journal on Mathematics Education.* https://doi.org/10.1007/s11858-020-01178-2

Tharp, M. L., Fitzsimmons, J. A., & Ayers, R. L. (1997). Negotiating a technological shift: Teacher perception of the implementation of graphing calculators. *Journal of Computers in Mathematics and Science Teaching, 16*(4), 551–575.

Thomas, M. O. J., & Palmer, J. M. (2014). Teaching with digital technology: Obstacles and opportunities. In A. Clark-Wilson, O. Robutti, & N. Sinclair (Eds.), *The mathematics teacher in the digital era: An international perspective on technology focused professional development* (pp. 71–89). Springer.

Tragalová, J., & Tabach, M. (2018). In search for standards: Teaching mathematics in technological environments. In L. Ball, P. Drijvers, S. Ladel, H.-S. Siller, M. Tabach, & C. Vale (Eds.), *Uses of technology in primary and secondary mathematics education: Tools, topics and trends* (pp. 387–397). Springer.

Trouche, L. (1997). *A propos de l'apprentissage de fonctions dans un environnement de calculatrices, étude des rapports entre processus de conceptualisation et processus d'instrumentation*. Thèse de doctorat. Université de Montpellier.

Trouche, L. (2000). La parabole du gaucher et de la casserole à bec verseur: Étude des processus d'apprentissage dans un environment de calculatrices symboliques. *Education Studies in Mathematics, 41*, 239–264.

Trouche, L. (2004). Managing complexity of human/machine interactions in computerized learning environments. *International Journal of Computers for Mathematics Learning, 9*(3), 281–307.

Trouche, L. (2005). Instrumental genesis, individual and social aspects. In D. Guin, K. Ruthven, & L. Trouche (Eds.), *The didactical challenge of symbolic calculators: Turning a computational device into a mathematical instrument* (pp. 197–230). Springer.

Trouche, L. (2019). Evidencing missing resources of the documentational approach to didactics. Toward ten programs of research/development for enriching this approach. In L. Trouche, G. Gueudet, & B. Pepin (Eds.), *The 'resource' approach to mathematics education* (pp. 447–489). Springer.

Trouche, L. (2020). Instrumentation in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 404–412). Springer.

Trouche, L., & Drijvers, P. (2010). Handled technology: Flashback into the future. ZDM. *The International Journal on Mathematics Education, 42*(7), 667–681.

Trouche, L., Gueudet, G., & Pepin, B. (2019). *The "resource" approach in mathematics education*. Springer.

Vergnaud, G. (1990). La théorie des champs conceptuels. *Recherches en Didactique des Mathématiques, 10*(2,3), 133–170.

Vergnaud, G. (2009). The theory of conceptual fields. *Human Development, 52*, 83–94.

Vérillon, P., & Rabardel, P. (1995). Cognition and artifacts: A contribution to the study of thought in relation to instrumented activity. *European Journal of Psychology of Education, 10*(1), 77–101.

Whitacre, I., Hensberry, K., Schellinger, J., & Findley, K. (2019). Variations on play with interactive computer simulations: Balancing competing priorities. *International Journal of Mathematical Education in Science and Technology, 50*(5), 665–681.

Wiseman, D., Lunney Borden, L., Beatty, R., et al. (2020). Whole-some artifacts: (STEM) teaching and learning emerging from and contributing to community. *Canadian Journal of Science, Mathematics, and Technology Education, 20*, 264–280.

Zevenbergen, R., & Lerman, S. (2008). Learning environments using interactive whiteboards: New learning spaces or reproduction of old technologies? *Mathematics Education Research Journal, 20*(1), 108–126.

Index

A. Clark-Wilson et al. (eds.), *The Mathematics Teacher in the Digital Era*, Mathematics Education in the Digital Era 16,
https://doi.org/10.1007/978-3-031-05254-5