Chapter 20
Bending/Tension of Plate Reinforced by a System of Parallel Fiber

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Abstract We present a 3-D to 2-D dimension reduction procedure as applied to the periodicity cell problem (PCP) of the homogenization theory for plates reinforced with a unidirectional system of fibers. The original 3-D PCP is reduced to several 2-D problems. The reduction procedures are not trivial, in one case we encounter the incompatibility condition, which makes impossible to transform the 3-D elasticity problem to the 2-D elasticity problem (only the transformation to 2-D thermoelasticity problem is possible). Numerical analysis of 2-D periodicity cell problems demonstrates new phenomena: the boundary layers on the top and bottom surfaces of the plate and, as a result, the wrinkling of the top and bottom surfaces of the plate. Note that these phenomena never occur in uniform plates or plates made of uniform layers.

Keywords: Fiber-reinforced plates · Homogenization · Dimensional reduction

20.1 Introduction

We consider a plate reinforced by a periodic system of parallel fibers, see Fig.20.1. Assume the fibers are parallel to the Oy-axis and form a periodic structure in the Oxz-plane. The periodicity cell (PC) $P_3 = [0, L] \times [0, 1] \times [-h, h]$ of such structure and its 2-D cross-sections $P = [0, L] \times [-h, h]$ are displayed in Fig.20.1.

Since the plate under consideration is invariant with respect to translation along the Oy-axis, there is a reason to look for a 2-D model of the plate. The procedure

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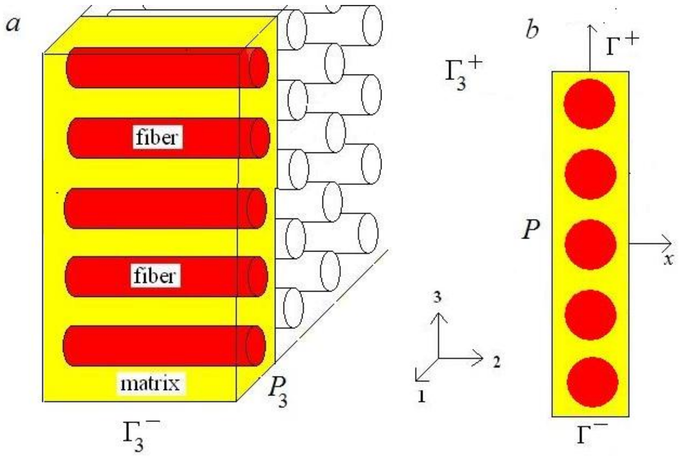

Fig. 20.1 a - fiber-reinforced plate, b - its 2-D periodicity cells

of dimension reduction is known for the solids with periodic systems of fibers or channels (Sendeckyj, 1974; Grigolyuk and Fil'Shtinskii, 1966; Grigolyuk and Fil'shtinskij, 1992; Lu, 1995; Mityushev and Rogosin, 2000; Gluzman et al, 2018; Drygaś et al, 2019) and for plates of complex geometry made of homogeneous materials (Annin et al, 2017; Kolpakov and Kolpakov, 2020; Grigolyuk et al, 1991).

The specific features of plates are:

- the free (top and bottom) surfaces;
- the bending/torsion modes of deformation.

These features distinguish the plates from the solids with the periodic structure considered in Sendeckyj (1974); Grigolyuk and Fil'Shtinskii (1966); Grigolyuk and Fil'shtinskij (1992); Lu (1995); Mityushev and Rogosin (2000); Gluzman et al (2018); Drygaś et al (2019). The inhomogeneity also brings some new effects.

There exists a great variety of approaches to the analysis of thin plates. In order to mention the recent papers, see Barchiesi and Khakalo (2019); Franciosi et al (2019); Yang et al (2020); Placidi and El Dhaba (2017); Altenbach et al (2010); Wang et al (2021) as well as references in them. As follows from the literature, the classical (Kirchoff–Love, Timoshenko, etc.) approaches work well for homogeneous plates or plates made of uniform layers. The classical theories do not work as applied to the inhomogeneous plates of general structure (for example, fiber-reinforced plates or plates with high-porous core). In some cases, even the basic notions of the classical theories are not well defined as applied to the inhomogeneous plates of general structure (for example, the inhomogeneous plate may have several "neutral" planes). The problem of construction of plate model was solved in the homogenization theory. The rigor homogenization theories as applied to elastic thin plates were developed first in Caillerie (1984); Kohn and Vogelius (1984). The papers (Caillerie, 1982, 1984) were devoted to the investigation of 3-D model of thin elastic periodic plate when the thickness of the plate and the size of the periods are small. In the paper (Kohn and Vogelius, 1984), the elastic thin body with rapidly varying thickness was considered.

Note that the homogenization theory is a mathematical theory and it does not answer the questions of mechanical nature on its own.

The homogenization theory justifies that solution of the elasticity theory problem in a thin inhomogeneous layer of complex geometry has the form $(A, B = 1, 2)$

$$\mathbf{u}^{\varepsilon} = u_A(x,z)\mathbf{e}_A + yw(x,z)_{,A}\mathbf{e}_A + \varepsilon\mathbf{N}^{AB0}(\mathbf{x}/\varepsilon)u_{A,B}(x,z) + \varepsilon\mathbf{N}^{AB1}(\mathbf{x}/\varepsilon)w_{,AB}(x,z), \tag{20.1}$$

where ε is the characteristic thickness of the plate; $u_A(x,z)$ are the global in-plane displacements, $w(x,z)$ is the global normal deflection; $e_{AB} = u_{A,B}$ are global in-plane strains and $\rho_{AB} = w_{,AB}$ are global curvatures/torsion $(A, B = 1, 2)$. These functions have the same meaning as in the classical theory. We use the notations $f_{,i}(\mathbf{x}) = \partial f(\mathbf{x})/\partial x_i$ and $f_{,i}(\mathbf{y}) = \partial f(\mathbf{y})/\partial y_i$ for the partial derivatives and assume summation with respect the repeating indices.

The term $\varepsilon\mathbf{N}^{AB0}(\mathbf{x}/\varepsilon)u_{A,B}(x,z) + \varepsilon\mathbf{N}^{AB1}(\mathbf{x}/\varepsilon)w_{,AB}(x,z)$ (known in the homogenization theory as "corrector" (Caillerie, 1984; Kohn and Vogelius, 1984) has the order of the thickness ε of the plate. Note that the plate may have rapidly varying thickness (top and/or bottom surfaces of the plate may be wavy). Therefore, the corrector has little effect on the global shape of the deformed plate. On the contrary, the derivatives of the functions $\varepsilon\mathbf{N}^{AB0}(\mathbf{x}/\varepsilon)$ and $\varepsilon\mathbf{N}^{AB1}(\mathbf{x}/\varepsilon)$ in $\mathbf{x}$ are not small and may strongly influence the local stress-strain state of the plate.

It is known from the homogenization theory (Caillerie, 1984; Kohn and Vogelius, 1984) that the functions $\mathbf{N}^{AB\mu}$ are solutions to the following so-called periodicity cell problems:

$$\begin{cases} (a_{ijkl}(x,z)N_{k,l}^{AB\mu} + (-1)^{\mu}a_{ijAB}(x,z)z^{\mu})_{,j} = 0 \text{ in } P_3, \\ (a_{ijkl}(x,z)N_{k,l}^{AB\mu} + (-1)^{\mu}a_{ijAB}(x,z)z^{\mu})n_j = 0 \text{ on } \Gamma_3, \\ \mathbf{N}^{AB\mu} \text{ is periodic in } x, y, \end{cases} \tag{20.2}$$

$\Gamma_3 = \Gamma_3^- \cup \Gamma_3^+$. Hereafter, $\mathbf{y} = (x,y,z) = \mathbf{x}/\varepsilon$. The variable-index correspondence: $x \leftrightarrow 1$, $y \leftrightarrow 2$, $z \leftrightarrow 3$. The Latin indices take values 1, 2, 3; the capital Greek indices takes values 1, 2; the indices μ, ν take values 0, 1. Γ_3 means the top and the bottom surfaces of the PC P_3.

The local stresses in the PC are (Caillerie, 1984; Kohn and Vogelius, 1984)

$$\sigma_{ij} = a_{ijkl}(x,z)N_{k,l}^{AB\mu} + (-1)^{\mu}a_{ijAB}(x,z)z^{\mu}$$

correspond to the in-plane strains ($\mu = 0$) of the unit magnitude or the bending/torsion ($\mu = 1$) of the unit magnitude.

In the plates subjected to the macroscopic stress-strain state (SSS) e_{AB}, ρ_{AB}, the local stresses are computed as

$$\sigma_{ij} = (a_{ijkl}(x,z)N^{AB0}_{k,l} + a_{ijAB}(x,z))e_{AB}+ \\ (a_{ijkl}(x,z)N^{AB1}_{k,l} - a_{ijAB}(x,z)z)\rho_{AB}. \tag{20.3}$$

This formula may be used for analysis of the local strength of plates, local stability of the constitutive elements of plate, etc. The effective rigidities of the plate are computed in accordance with the formulas (Caillerie, 1984; Kohn and Vogelius, 1984)

$$S^{\nu+\mu}_{\alpha\beta AB} = \frac{1}{|PrP_3|}\int_{P_3}(a_{\alpha\beta kl}(\mathbf{y})N^{AB\nu}_{k,l} + (-1)^\nu z^\nu a_{ijAB}(\mathbf{y})(-1)^\mu z^\mu d\mathbf{y} = \\ \frac{1}{L}\int_P (a_{\alpha\beta k\delta}(x,z)N^{AB\nu}_{k,\delta} + (-1)^\nu z^\nu a_{ijAB}(x,z))(-1)^\mu z^\mu dxdz. \tag{20.4}$$

2-D PC $P = [0,L]\times[-h,h]$ is projection PrP_3 of 3-D PC P_3 on Oxz-plane; $|PrP_3| = L\times 1$; L is the width of the 2-D periodicity cell P; Γ^+ and Γ^- are the top and the bottom of the PC P, correspondingly; $\Gamma = \Gamma^- \cup \Gamma^+$.

One can conclude that the functions $\mathbf{N}^{AB\mu}$ are the key to the analysis of the macro and microscopic properties of the inhomogeneous plate.

The homogenization theory itself provides us with no information about the solution to the PCP (20.2). One can see that the PCP (20.2) is a special type of 3-D elasticity theory problem, which is the subject of the elasticity theory. It would be reasonable to regard the PCP as the point of torch transfer from the homogenization theory to the elasticity theory. In particular, it would be reasonable to apply the methods developed in the elasticity theory to the analysis of PCP.

20.2 Reduction of 3-D PCP (20.2) to 2-D problems

Although the dimension reduction procedures have a longstanding history, the first work (to the best knowledge of the authors) devoted to the dimension reduction in the bending problem for the 3-D elastic body of the periodic structure is in Grigolyuk et al (1991). Grigolyuk et al (1991) was devoted to the bending of an elastic layer with the periodic systems of tunnel cuts. Grigolyuk et al (1991) used the double periodic function technique, thus treated the body as a layer of "infinite" thickness. It means that Grigolyuk et al (1991) is not the case of the plate, which thickness is small in the original variables $\mathbf{x}$ or finite in the fast variables $\mathbf{y} = \mathbf{x}/\varepsilon$. The results in Grigolyuk et al (1991) can be used to predict the SSS inside the plate, but not near-surface phenomena. Do not confuse the dimension reductions in Grigolyuk et al (1991) and one discussed in this paper with the traditional dimensional reduction in the plane of the plate (Love, 2013). The dimensional reduction discussed in this paper is based on the transition to the problems on the cross-section of the plate.

Our starting point is the PCP (20.2) of the homogenization theory as applied to thin plates. The PC P is a cylinder parallel to the Oy-axis, see Fig.20.1, and the elastic constants a_{ijkl} do not depend on the variable y . In this case, the solution to the problem (20.2) does not depend on y and has the form $\mathbf{N}^{AB\mu} = \mathbf{N}^{AB\mu}(x,z)$. Substituting into (20.2), we arrive at the following 2-D PCP:

$$\begin{cases} (a_{i\alpha k\beta}(x,z)N^{AB\mu}_{k,\beta} + (-1)^\mu a_{i\alpha AB}(x,z)z^\mu)_{,\alpha} = 0 \text{ in } P, \\ (a_{i\alpha k\beta}(x,z)N^{AB\mu}_{k,\beta} + (-1)^\mu a_{i\alpha AB}(x,z)z^\mu)n_\alpha = 0 \text{ on } \Gamma, \\ \mathbf{N}^{AB\mu}(x,z) \text{ is periodic in } x. \end{cases} \tag{20.5}$$

Hereafter α, β=1,3; i,k=1,2,3; AB=11, 22, 12,21.

In the equations (20.5)

$$\begin{gathered} a_{i\alpha k\beta}(x,z)N^{AB\mu}_{k,\beta} + (-1)^\mu a_{i\alpha AB}(x,z)z^\mu = \\ a_{i\alpha\theta\beta}(x,z)N^{AB\mu}_{\theta,\beta} + a_{i\alpha 2\beta}(x,z)N^{AB\mu}_{2,\beta} + (-1)^\mu a_{i\alpha AB}(x,z)z^\mu. \end{gathered} \tag{20.6}$$

The boundary-value problem (20.5) decomposes into several 2-D problems. The form of the 2-D problems depends on the index i in (20.5). For $i = 2$, the original problems are reduced to scalar 2-D problems. For $i = \xi = 1,3$ the original problems are reduced to 2-D elasticity or thermoelasticity problems.

For this reason, we consider problem (20.2) for $i = 2$ and $i = \xi = 1,3$, separately. In this paper, we pay the main attention to the case $i = 2$, which leads to the analogs of the anti-plane elasticity problem. The case $i = \xi = 1,3$, leads to the analogs of the planar elasticity problem.

Problem (20.2) with index $i = 2$. We assume the fibers and matrix are made of isotropic materials. It is convenient to save the notations a_{ijkl} for the elastic constants in our analysis. In special cases below, we will use the technical constants, see formulas (20.31) below.

In the case, under consideration $a_{2\alpha\theta\beta} = 0$, $a_{2\alpha AB} = 0$ (Love, 2013) and expression in (20.6) takes the form ($\alpha = 1,3$)

$$\begin{gathered} a_{2\alpha\theta\beta}(x,z)N^{AB\mu}_{\theta,\beta} + a_{2\alpha 2\beta}(x,z)N^{AB\mu}_{2,\beta} + (-1)^\mu a_{i\alpha AB}(x,z)z^\mu = \\ a_{2\alpha 2\alpha}(x,z)N^{AB\mu}_{2,\alpha} + \begin{cases} (-1)^\mu a_{2121}(x,z)z^\mu \text{ if } AB = 12,21, \\ 0 \text{ else.} \end{cases} \end{gathered} \tag{20.7}$$

By virtue of (20.7), the solution to (20.2) $N^{AB\mu}_2(x,z) = 0$ if $AB \neq 12,21$. Only $N^{AB\mu}_2(x,z) \neq 0$. This is the case of in-plane shift (if $\mu = 0$) or torsion (if $\mu = 1$). The in-plane shift is also called anti-plane deformation (Love, 2013).

The problem for $N^{21\mu}_2(x,z)$ takes the form

$$\begin{cases} (a_{2\alpha 2\alpha}(x,z)N^{21\mu}_{2,\alpha} + (-1)^\mu a_{2121}(x,z)z^\mu\delta_{\alpha 1})_{,\alpha} = 0 \text{ in } P, \\ (a_{2\alpha 2\alpha}(x,z)N^{21\mu}_{2,\alpha} + (-1)^\mu a_{2121}(x,z)z^\mu\delta_{\alpha 1})n_\alpha = 0 \text{ on } \Gamma, \\ N^{21\mu}_2(x,z) \text{ periodic in } x. \end{cases} \tag{20.8}$$

It is convenient to eliminate the "mass" and "surface" forces in (20.8). It may be done if there exists a function w , such that ($\nu = 0,1$)

$$a_{2\delta 2\delta}(x,z)w_{,\delta} = (-1)^{\mu} a_{2121}(x,z)z^{\mu}. \tag{20.9}$$

For $\delta = 2$ and $\delta = 2$, we obtain from (20.9) $a_{2\delta 2\delta}(x,z)w_{,1} = (-1)^{\mu} a_{2121} z^{\mu}$ and $a_{2\delta 2\delta} w_{,2} = (-1)^{\mu} a_{2121}(x,z)z^{\mu}$, correspondingly. From these equalities, we obtain the following system of differential equations

$$w_{,1} = (-1)^{\mu}, w_{,3} = 0. \tag{20.10}$$

In-plane shift ($\mu = 0$). For $\mu = 0$, the system (20.9) takes the form $w_{,1} = 1, w_{,3} = 0$. The solution to this system is $w(x,z) = x$. Introduce function

$$M(x,z) = N_2^{120}(x,z) + x,$$

and write (20.8) in the form of a boundary-value problem without "mass" and "surface" forces:

$$\begin{cases} (a_{2\alpha 2\alpha}(x,z)M_{,\alpha})_{,\alpha} = 0 \text{ in } P, \\ a_{2\alpha 2\alpha}(x,z)M_{,\alpha} n_{\alpha} = 0 \text{ on } \Gamma, \\ M(x,z) - x \text{ periodic in } x. \end{cases} \tag{20.11}$$

The problem (20.11) is the anti-plane elasticity theory problem.

After some algebra, we obtain the following formulas for the local stresses:

$$\sigma_{ij} = a_{ij2\alpha}(x,z)N_{2,\alpha}^{120} + a_{ij21}(x,z) = a_{ij2\alpha}(x,z)M_{,\alpha}, \tag{20.12}$$

and the homogenized shift rigidity

$$\begin{aligned} S_{2121}^0 &= \frac{1}{L}\int_P (a_{212\alpha}(x,z)N_{2,\alpha}^{210} + a_{2121}(x,z))dxdz = \\ &\frac{1}{L}\int_P (a_{212\alpha}(x,z)M_{,\alpha} dxdz, \\ S_{2121}^1 &= \frac{1}{L}\int_P (a_{212\alpha}(x,z)N_{2,\alpha}^{210} + a_{2121}(x,z))zdxdz = \\ &\frac{1}{L}\int_P (a_{212\alpha}(x,z)M_{,\alpha} zdxdz. \end{aligned} \tag{20.13}$$

The local stresses (20.12) and the homogenized shift rigidity (20.13) depend on the elastic constants of the composite plate.

These formulas may be used for the analysis of the local strength of the plates, local stability of the constitutive elements of the plate, etc. The effective rigidities of the plate are computed in accordance with the formulas (Caillerie, 1984; Kohn and Vogelius, 1984).

Torsion ($\mu = 1$). In this case, we meet a problem, which has no analog in the classical theory of elasticity or classical plate theory.

For $\mu = 1$, the system (20.10) takes form $w_{,1} = -z, w_{,3} = 0$. This system is not integrable. Really, the necessary integrability condition (Love, 2013) is not satisfied for this system because $w_{,13} = -z_{,3} = -1 \neq w_{,31} = 0$.

For $\mu = 1$, (20.8) takes the form

$$\begin{cases} (a_{2\alpha 2\alpha}(x,z)N^{211}_{2,\alpha} - a_{2121}(x,z)z\delta_{\alpha 1})_{,\alpha} = 0 \text{ in } P, \\ (a_{2\alpha 2\alpha}(x,z)N^{211}_{2,\alpha} - a_{2121}(x,z)z\delta_{\alpha 1})n_\alpha = 0 \text{ on } \Gamma, \\ N^{211}(x,z) \text{ is periodic in } x. \end{cases} \tag{20.14}$$

To write (20.14) in compact form, we introduce function as

$$\varphi_{,3} = a_{2121}(x,z)(N^{211}_{2,1} - z), \varphi_{,1} = -a_{2323}(x,z)N^{211}_{2,3}. \tag{20.15}$$

The function $\varphi(x,z)$ introduced by (20.15) is similar to the conjugate function (Sedov, 1971). The equality

$$\varphi_{,31} - \varphi_{,13} = (a_{2121}(x,z)(N^{211}_{2,1} - z))_{,1} + (a_{2323}(x,z)N^{211}_{2,3})_{,3} = 0, \tag{20.16}$$

follows from (20.14). This equality justifies the existence of the function $\varphi(x,z)$.

Express $N^{211}_2(x,z)$ from (20.15)

$$N^{211}_{2,1} = \frac{1}{a_{2121}(x,z)}\varphi_{,3} + z, N^{211}_{2,3} = -\frac{1}{a_{2323}(x,z)}\varphi_{,1}. \tag{20.17}$$

Differentiation of (20.17) yields

$$0 = N^{211}_{2,13} - N^{211}_{2,31} = (\frac{1}{a_{2121}(x,z)}\varphi_{,3} + z)_{,3} + (\frac{1}{a_{2323}(x,z)}\varphi_{,1})_{,1}. \tag{20.18}$$

Taking into account that for the isotropic materials $a_{2121} = a_{2323}$, we obtain

$$(\frac{1}{a_{2121}(x,z)}\varphi_{,3})_{,3} + (\frac{1}{a_{2121}(x,z)}\varphi_{,1})_{,1} = 1. \tag{20.19}$$

With the use of the function $\varphi(x,z)$, the boundary conditions on the top and bottom boundaries Γ^+ and Γ^- (20.8) can be written as

$$\begin{gathered}(a_{2121}(x,z)N^{21\nu}_{2,1} - a_{2121}(x,z)z)n_1 + a_{2323}(x,z)N^{21\nu}_{2,3}n_3 = \\ \varphi_{,3}n_1 - \varphi_{,1}n_3 = \frac{\partial\varphi}{\partial s} = 0 \text{ on } \Gamma,\end{gathered} \tag{20.20}$$

where $\partial\varphi/\partial s$ is the derivative along the boundary Γ^+ or Γ^- . Because of (20.20), the function $\varphi(x,z)$ takes constant values on the top and bottom boundaries Γ^+ and Γ^- :

$$\varphi(x,z) = const_\pm \text{ on } \Gamma^\pm. \tag{20.21}$$

Without loss of generality, we can assume that $\varphi(x,z) = 0$ at the bottom boundary Γ^-.

Integrating the first equality in (20.15) over z from $-h$ to h, we have

$$\varphi(x,L)=\varphi(x,-L)+\int_{-h}^{h}a_{2121}(x,z)(N_{2,1}^{211}-z)dz. \qquad (20.22)$$

Integrating the (20.23) over x from 0 to L, we have

$$\varphi(-h,L)L=\varphi(-h,-L)L+\int_{P}a_{2121}(x,z)(N_{2,1}^{211}-z)dxdz. \qquad (20.23)$$

Writing (20.23), we take into account that $\varphi(x,L)$ and $\varphi(x,-L)$ are constants.

Multiplying (20.14) by x and integrating by parts, we have

$$\int_{P}a_{2121}(x,z)(N_{2,1}^{211}-z)dxdz=\int_{P}\varphi_{,3}dxdz.$$

As the result

$$S_{2121}^{1}=\frac{1}{L}\int_{P}a_{2121}(x,z)(N_{2,1}^{211}-z)dxdz=\frac{1}{L}\int_{P}\varphi_{,3}dxdz. \qquad (20.24)$$

Comparing (20.23) and (20.24), we find that the RHP (20.23) is equal to

$$\varphi(h,-L)=\phi(-h,-L)+S_{2121}^{1}.$$

We have assumed that $\varphi(x,z)=0$ on the bottom boundary Γ^{+}, in particular, $\varphi(-h,-L)=0$ and $\varphi(x,z)=S_{2121}^{1}$ on the top boundary Γ^{+} . As a result, we arrive at the following boundary value problem:

$$\begin{cases}(\frac{1}{a_{2121}(x,z)}\varphi_{,1})_{,1}+(\frac{1}{a_{2121}(x,z)}\varphi_{,3})_{,3}=1 \text{ in } P,\\ \varphi=0 \text{ on } \Gamma^{-}, \varphi=S_{2121}^{1} \text{ on } \Gamma^{+},\\ \varphi(x,z) \text{ is periodic in } x.\end{cases} \qquad (20.25)$$

The problem (20.25) involves the asymmetric effective stiffness S_{2121}^{1} of the plate, which has been expressed in (20.24) through the solution φ to the BVP (20.25). The problem (20.25) with the condition (20.24) has a not usual form. This is lucky for us that the asymmetric effective stiffness S_{2121}^{1} also may be computed by using the second formula in (20.13). As a result, we have the problem (20.25) with S_{2121}^{1} known after the BVP (20.11) be solved.

The local stresses, corresponding to the case under consideration, are expressed in the form

$$\sigma_{ij}=a_{ij2\alpha}(x,z)+a_{ij21}(x,z)z=\frac{a_{ij21}(x,z)}{a_{2121}(x,z)}(\varphi_{,3}-\varphi_{,1}),$$

and the homogenized torsion rigidity is expressed in the form

$$S_{2121}^{2}=-\int_{P}(\varphi_{,3}-\varphi_{,1})zdxdz. \qquad (20.26)$$

Problem (20.2) with index $i = \xi = 1,3$**. Deformation perpendicular to the fibers**. In this case, $a_{\xi\alpha 2\beta}(\mathbf{y}) = 0$ and expression in (20.6) takes the form $(\alpha, \beta, \theta, \xi = 1, 1$).

In the equations (20.5)

$$\begin{gathered} a_{i\alpha k\beta}(\mathbf{y})N_{k,\beta}^{AB\mu} + (-1)^{\mu}a_{i\alpha AB}(x,z)z^{\mu} = \\ a_{\xi\alpha\theta\beta}(x,z)N_{\theta,\beta}^{AB\mu} + a_{\xi\alpha 2\beta}(x,z)N_{2,\beta}^{AB\mu} + (-1)^{\mu}a_{\varsigma\alpha AB}(x,z)z^{\mu} = \\ a_{\xi\alpha\theta\beta}(x,z)N_{\theta,\beta}^{AB\mu} + (-1)^{\mu}a_{\xi\alpha AB}(x,z)z^{\mu}. \end{gathered} \tag{20.27}$$

Here $AB = 11, 22, 12, 21$. Then the PCP (20.5) takes the form

$$\begin{cases} (a_{\xi\alpha\theta\beta}(x,z)N_{\theta,\beta}^{AB\mu} + (-1)^{\mu}a_{\xi\alpha AB}(x,z)z^{\mu})_{,\alpha} = 0 \text{ in } P, \\ a_{\xi\alpha\theta\beta}(x,z)N_{\theta,\beta}^{AB\mu} + (-1)^{\mu}a_{\xi\alpha AB}(x,z)z^{\mu})n_{\alpha} = 0 \text{ on } \Gamma, \\ (N_1^{AB\mu}, N_3^{AB\mu})(x,z) \text{ periodic in } x. \end{cases} \tag{20.28}$$

Note that $a_{\xi\alpha 12} = 0$ and $a_{\xi\alpha 21} = 0$ for $i = \xi = 1,3$, then $(N_1^{21\mu}, N_3^{21\mu})$=0 and $(N_1^{12\mu}, N_3^{12\mu})$=0. The problem (20.28) is non-trivial for $AB = 11, 22$.

In some cases, (20.28) for $AB = 11, 22$ may be transformed into problems without free terms. We shall check if it is possible to represent the free term $(-1)^{\mu}a_{\xi\alpha AB}(x,z)z^{\mu}$ in (20.28) in the form $(-1)^{\mu}a_{\xi\alpha\theta\beta}(x,z)e_{\theta\beta}^{AB\mu}$ with the strains $e_{\theta\beta}^{AB\mu} = v_{\theta,\beta}^{AB\mu}$ corresponding proper displacements $\mathbf{v}_{\theta,\beta}^{AB\mu}$ (μ =0, 1):

$$a_{\xi\alpha\theta\beta}(x,z)v_{\theta,\beta}^{AB\mu} = a_{\xi\alpha AB}(x,z)z^{\mu}. \tag{20.29}$$

Index $AB = 22$**. Tension and bending along the fibers**. Equation (20.29) takes the form $a_{\xi\alpha\theta\beta}(x,z)e_{\theta,\beta} = a_{\xi\alpha 22}(x,z)z^{\mu}$. In the coordinate-wise form, it is

$$\begin{gathered} a_{1111}e_{11} + a_{1133}e_{33} = -a_{1122}z^{\mu}, \\ a_{3311}e_{11} + a_{3333}e_{33} = -a_{1122}z^{\mu}, \\ a_{1313}e_{13} = 0, a_{1111}e_{31} = 0. \end{gathered} \tag{20.30}$$

Write the elastic constants in terms of Young's E modulus and Poisson's ratio ν (Love, 2013)

$$\begin{gathered} a_{1111} = a_{1111} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}, \\ a_{1133} = a_{3311} = a_{1122} = a_{3322}\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}. \end{gathered} \tag{20.31}$$

In this case, the first two equations in (20.30) take the form

$$\begin{gathered} (1-\nu)e_{11} + \nu e_{33} = -\nu(x,z)z^{\mu}, \\ \nu e_{11} + (1-\nu)e_{33} = -\nu(x,z)z^{\mu}. \end{gathered} \tag{20.32}$$

The solution to (20.32) is

$$e_{11} = e_{33} = -\nu(x,z)z^{\mu}. \tag{20.33}$$

Taking into account that $e_{13} = e_{31} = 0$, we arrive at the following system:

$$\frac{\partial v_1}{\partial x} = -\nu(x,z)^{\mu}, \frac{\partial v_3}{\partial z} = -\nu(x,z)^{\mu}, \frac{\partial v_1}{\partial z} + \frac{\partial v_3}{\partial x} = 0. \tag{20.34}$$

Generally, the compatibility condition (Love, 2013) is not satisfied for the system (20.34) for arbitrary $\nu(x,z)$.

This incompatibility indicates that a simple transfer from 3-D to the 2-D problem is impossible in the general case.

The case $\nu(x,z) = const$. If $\nu(x,z) = \nu = const$, the system (20.34) is compatible. In this case, the solution to (20.34) may be obtained in the explicit form. For $\mu = 0$, $v_1 = -\nu x$ and $v_3 = -\nu z$.

Introduce $M_1^{22\mu} = N_1^{22\mu} + v_1$ and $M_2^{22\mu} = N_2^{22\mu} + v_2$. By using these functions, we can transform the problem (20.28) to the following:

$$\begin{cases} (a_{\xi\alpha\theta\beta}(x,z)M_{\theta,\beta}^{22\mu})_{,\alpha} = 0 \text{ in } P, \\ a_{\xi\alpha\theta\beta}(x,z)M_{\theta,\beta}^{22\mu}n_{\alpha} = 0 \text{ in } \Gamma, \\ [M_1^{22\mu}]_x = -\nu z^{\mu}[x]_x, [M_3^{22\mu}]_x = 0, \end{cases} \tag{20.35}$$

where $[]_x$ means the jump of the function value at the opposite sides of the PC in the direction Ox.

The case $\nu(x,z) \neq const$. In this general case, we can transform the problem (20.28) into a thermoelasticity problem.

In (20.28)

$$a_{\xi\alpha\theta\beta}(x,z)N_{\theta,\beta}^{AB\mu} + (-1)^{\mu}a_{\xi\alpha AB}(\mathbf{y})z^{\mu} =$$
$$a_{\xi\alpha\theta\beta}(x,z)N_{\theta,\beta}^{AB\mu} + a_{\xi\alpha\theta\beta}e_{\theta\beta} = a_{\xi\alpha\theta\beta}(x,z)(N_{\theta,\beta}^{AB\mu} + e_{\theta\beta}),$$

where $e_{\theta\beta}$ are given by (20.33). Then (20.28) may be written in the form

$$\begin{cases} (a_{\xi\alpha\theta\beta}(x,z)(N_{\theta,\beta}^{AB\mu} + e_{\theta\beta}))_{,\alpha} = 0 \text{ in } P, \\ a_{\xi\alpha\theta\beta}(x,z)(N_{\theta,\beta}^{AB\mu} + e_{\theta\beta})n_{\alpha} = 0 \text{ on } \Gamma, \\ (N_1^{AB\mu}, N_3^{AB\mu})(x,z) \text{ periodic in } x. \end{cases} \tag{20.36}$$

Problem (20.36) is the thermoelasticity problem with the coefficients of thermal expansion $e_{\theta\beta}$. Since $e_{11} = e_{22} = -\nu(x,z)z^{\mu}$ and $e_{13} = e_{31} = 0$, this tensor is isotropic. For $\nu = 1$, coefficients $e_{11} = e_{22} = -\nu(x,z)z$, where $\nu(x,z)$ takes constant values in the fibers and the matrix. Some ANSYS APDL programming is required to input such kind coefficients. The local stresses are

$$\sigma_{\xi\alpha} = a_{\xi\alpha\theta\beta}(x,z)(N_{\theta,\beta}^{AB\mu} + e_{\theta\beta}))n_{\alpha}. \tag{20.37}$$

The effective rigidities are

$$S^{\mu+\nu}_{\xi\alpha\theta\beta} = \frac{1}{L}\int_P (a_{\xi\alpha\theta\beta}(x,z)N^{AB\mu}_{\theta,\beta} + a_{\xi\alpha AB}(x,z))z^\nu dxdy =$$
$$\frac{1}{L}\int_P (a_{\xi\alpha\theta\beta}(x,z)(N^{AB\mu}_{\theta,\beta} + e_{\theta\beta}(x,z))z^\nu dxdy.$$

Index $AB = 11$**. Tension and bending perpendicular to the fibers**. In this case, we arrive at the problem

$$\frac{\partial v_1}{\partial x} = -z^\mu, \frac{\partial v_3}{\partial z} = 0, \frac{\partial v_1}{\partial z} + \frac{\partial v_3}{\partial x} = 0. \qquad (20.38)$$

The problem (20.38) may be solved in the explicit form. Its solution is

$$v_1 = -x, v_1 = 0 \text{ for } \mu = 0,$$

$$v_1 = -xz, v_1 = x^2/2 \text{ for } \mu = 1.$$

Introduce $M^{11\mu}_1 = N^{11\mu}_1 + v_1$ and $M^{11\mu}_2 = N^{11\mu}_2 + v_2$. By using the functions, we can transform the problem (20.28) to the following:

$$\begin{cases} (a_{\xi\alpha\theta\beta}(x,z)M^{11\mu}_{\theta,\beta})_{,\alpha} = 0 \text{ in } P, \\ a_{\xi\alpha\theta\beta}(x,z)M^{11\mu}_{\theta,\beta}n_\alpha = 0 \text{ on } \Gamma, \\ [M^{11\mu}_1]_x = -\nu z^\mu [x]_x, [M^{11\mu}_3]_x = 0. \end{cases} \qquad (20.39)$$

Index $AB = 12, 21$**. Shift/torsion**. For $AB = 12, 21$, equation (20.29) takes the form $a_{\xi\alpha\theta\beta}e_{\theta,\beta} = a_{\xi\alpha 12}z^\mu = 0$, $\xi, \alpha = 1, 3$. Its solution is $e_{\theta\beta} = 0$. Then $v_1 = v_3 = 0$ and solution to (20.28) is trivial.

20.3 Numerical Computations

In this section, we present several numerical solutions interesting from mechanic's point of view. In our computations the fibers Young's modulus E=170GPa and Poisson's ratio ν=0.3; and the matrix E=2GPa and Poisson's ratio ν=0.36. These values correspond to carbon/epoxy composite (Agarwal et al, 2017).

The computer program was developed by using the APDL programming language (Thompson and Thompson, 2017). The finite elements PLANE183 are used for the fibers and the matrix, the characteristic size of the finite elements is 0.03, the total number of finite elements is about 10000.

Figure 20.2 displays the solution to the PCP corresponding to the bending in the direction perpendicular to the fibers. We have observed edge effects near the top and the bottom surfaces of the plate. The edge effect zone thickness is less than the thickness of one structural layer (fiber + surrounding matrix). To the best knowledge of the authors, such kind edge effect was not reported before.

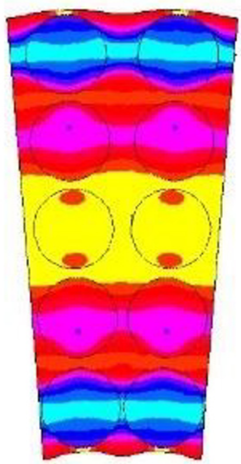

Fig. 20.2 PC formed of two adjacent PCs of 5-layer fiber-reinforced plate

If the plate is thick, these top/bottom edge layers do not influence the effective rigidity of the plate. But they influence the local SSS, thus, the strength of the plate.

An analysis of Fig.20.2 leads to the conclusion that the found edge effect does not lead to a stress concentration in the edge effect zone. In Fig.20.2, we observe the von Mises stress decrease in the edge effect zone. The stress concentration between the fibers is the result of the dense packing on the fibers (Flaherty and Keller, 1973; Kolpakov and Kolpakov, 2009; Kang and Yu, 2020; Kolpakov, 2007; Rakin, 2014).

One result of the edge effect is the wrinkling of the top/bottom boundaries of the plate. The wrinkling is especially good seen for the PCP formed of two adjacent PCs, see Fig.20.2. The wrinkling may influence the plate-to-surrounding media interaction. The various kinds of wrinkling were discussed in the literature on the composite materials (Boisse et al, 2018, 2021; Giorgio et al, 2018). The authors find no analogs between the wrinkling effects describer early and the wrinkling described in this paper.

The top/bottom edge effect (including wrinkling) described above never occurs in uniform plates or plates made of uniform layers. The solutions to the PCPs for the homogeneous plates and plates made of uniform layers are well known and may be easily computed.

20.4 Conclusion

We developed a procedure of transition from the original 3-D PCP (20.2) in a thin domain with a system of parallel cylindrical inhomogeneities to 2-D boundary-value problems. We arrive at 2-D boundary-value problems (20.11) and (20.25) corresponding to the shift and torsion of the plate. These problems have the forms of the anti-plane elasticity problems with and without mass forces (Laplace-type and Poisson-type problems). 2-D boundary-value problem (20.25) is a new problem. 2-D boundary-value problems (20.35) and (20.36) correspond to the tension/bending. They have the forms of planar elasticity and thermo-elasticity problems.

Our numerical analysis of the obtained 2-D problems demonstrates the existence of boundary layers near the top and bottom surfaces of the plate. The boundary layer thickness is less than the thickness of one structural layer (the diameter of the fiber + the thickness of the surrounding matrix).

One of the manifestations of the found boundary layer is the wrinkling of the top and the bottom of the plate. To the best knowledge of the authors, such kind boundary layers and the wrinkling effect did not refer earlier. Note that the boundary layers and the wrinkling effect described in this paper cannot occur in uniform plates or plates made of homogeneous layers.

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