Chapter 11
Analogical Modelling and Analytical Modelling: Different Approaches to the Same Context?

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11.1 Introduction

In this chapter, we present a set of models created with students in which we tackled problems from different projects in real-world contexts. Only three of those projects carried out between 2015 and 2021 are presented. These models occurred in complete modelling cycles (reality, modelling, reality) and they allowed both students and professors to significantly improve their understanding of the analyzed phenomena. Moreover, several of those models were fundamental for the decisions taken in the projects.

The types of models are different as well. Some are characterized for retaining elements or objects from the context, whereas others favour more abstract relations. They result from how students establish connections between the problem's context and the mathematical world.

Besides, we could observe links between these types of models that evidence the evolution in the students' mental processes. The models could also be didactically considered as strategies for progressively difficult modelling; allowing students to deepen their understanding of the phenomena and the appropriation of relatively complex disciplinary concepts.

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11.2 Problem

The literature on the role of modelling in mathematics teaching highlights two major uses: modelling as a vehicle for learning disciplinary concepts, and modelling as a teaching object (Blum & Borromeo Ferri, 2009; Blomhøj, 2019; Brown & Ikeda, 2019; Caron, 2019; Czocher, 2019).

As regards the learning vehicle role, modelling has been acknowledged to show potential for the discussion of mathematical concepts. Indeed, the models, as an abstraction space for a given problem, become the appropriate place for the learning and resignification of mathematical concepts.

However, modelling a problem with mathematical objects and methods constitutes a non-evident cognitive act. Complex mental processes appear in the interaction between the elements of the problem, in the person's perception of them, in the representations of the mathematical concepts, and the available modelling tools and creative abilities, among others.

The relationship between these matters does not appear to be a linear and determined sequence. Constructing a mathematical modelling of a problem does not constitute a bijective relationship between the problem and the mathematical tools. A problem can be modelled in different ways and vice versa. This relationship between the problem and the mathematical tools requires certain abilities and competencies that are nowadays considered necessary. Therefore, learning how to create models becomes a teaching-learning object.

By modelling, we understand a cycle that begins with a problem, continues in a space of mathematical objects, and then returns to the problem. What is the purpose of carrying out this cycle? Going through the mathematical objects allows us to analyze and construct arguments supported by a rich mathematical scaffolding. Besides, these arguments are many times impossible to obtain through direct observation of reality. To sum up, using mathematical modelling elevates the level of comprehension of the problem and allows us to approach it with solid arguments.

In this sense, the literature distinguishes two different types of modelling: descriptive modelling and prescriptive modelling (Blum & Borromeo Ferri, 2009; Brown & Ikeda, 2019). Descriptive models allow us to understand how a problem works, whereas prescriptive models help us determine how to act in the problem in question.

In this chapter, we will present a set of models created with students for projects in real-world contexts. In the examples we detail, the two types of models as well as their constant interactions can be observed. In fact, it is possible to observe how the models constitute vehicles for learning mathematical concepts and how they can become learning objects.

However, our focus is on proposing a categorization for modelling based on the type of problem's elements considered. Thus, we will discuss the concepts of analogical modelling and analytical modelling. In the former, physical elements of the problem as well as its relations, connections and articulations are taken into consideration. In the latter, abstract relations non-observable through our senses are considered.

In this chapter we present these and other differences with illustrative examples of the models created with students for projects in real-world contexts. We will refer to projects rather than problems due to several reasons. We claim that a project, due to its complexity, contains a set of different problems and it is developed within a period bigger than a problem. Moreover, these projects are claimed to be in a real-world context because said context exists and students are embedded in it. Besides, their solutions are implemented in it; that is, the context is not created, it is experienced.

In general, the models presented here are prescriptive since they facilitated the construction of arguments to intervene in the projects' problems. However, they are also descriptive because they helped to understand the analyzed phenomena. To present the models in terms of being analogical or analytical, we will first introduce a summary of the projects in which they were created.

11.3 Method

The models discussed here were created in three projects carried out in real-world contexts: the "Savonius Windmills" project, the "Mobile Panel" project, and the "Water Purification Systems" project.

11.3.1 Savonius Windmills

Course: mathematics (semestral).

Level: tertiary education. First semester of the first year.

Students: aged between 18 and 50.

Project's summary: the students designed, calculated, and constructed windmills to pump underground water for irrigation and animals. The windmills were given to rural communities relying on a subsistence economy in Argentinian Patagonia.

Year: the project has been carried out since 2015. One or two windmills are built each year.

Funding: National University of Río Negro (2015–2017). Argentine National Ministry of Education (2018–2022).

Illustrative media: Fig. 11.1 shows the type of windmill built. The following hyperlink redirects to a video summarizing the installation stages of two windmills at two different rural facilities: https://www.youtube.com/watch?v=889fvPzVK1g.

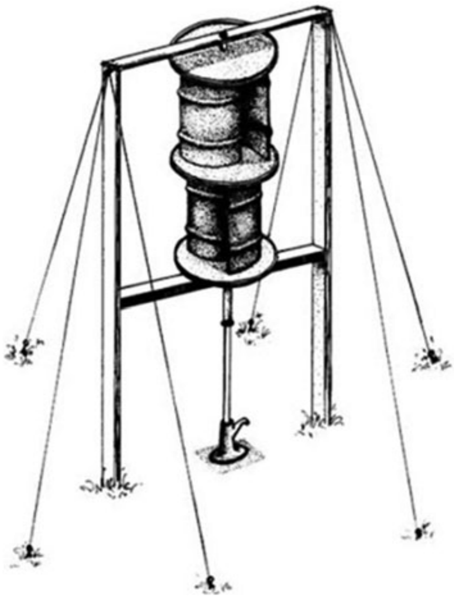


Fig. 11.1 Schematic representation of Savonius Windmill. (Source: http://cepechile.blogspot.com/)

11.3.2 Mobile Photovoltaic Panel

Course: extracurricular mathematics workshop (semestral).
Level: secondary education. Second semester of the first year.
Students: aged between 11 and 13.
Project's summary: the students designed, calculated, and built a mobile supporting structure for a photovoltaic panel. The purpose was to optimize the use of the available solar radiation. The panel and a water pump were given to a rural inhabitant in Argentinian Patagonia.
Year: 2019.
Funding: organizations and companies.
Illustrative media: Fig. 11.2 shows the supporting structure during the assembling stage. The students assembled the structure at the schoolyard.

11.3.3 Water Purification Systems

Course: mathematics (annual).
Level: tertiary education. First year.
Students: aged between 18 and 40.
Project's summary: the students designed, calculated, built, and tested prototypes for low-cost water purification systems for families. These systems are based on technology developed by researchers at the University at Buffalo, in the United

Fig. 11.2 Photovoltaic supporting structure. (Source: Authors' own work)

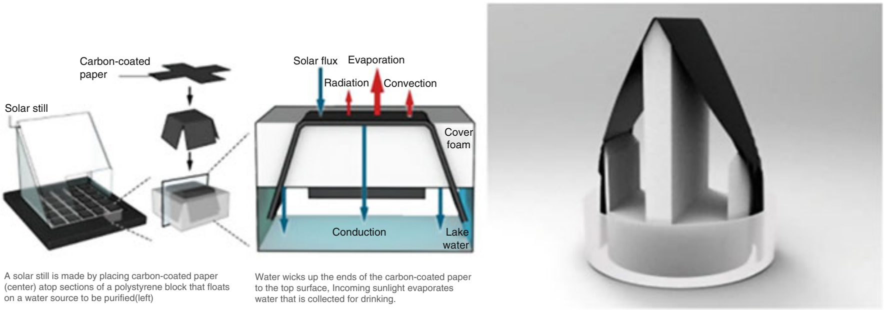


Fig. 11.3 Solar distillation. (Source: Buffalo University)

States (Gan & Zhang, 2017). The recipients are any inhabitants who are not connected to drinking water distribution systems.

Year: 2021.

Funding: organizations and university.

Illustrative media: Fig. 11.3 shows the first and second versions of the system's technological principle while Fig. 11.4 shows the culmination of the building stage of one of the five prototypes built by the students.

The set of models we will present were created in these three projects. All of them shared common ground: they allowed students to understand the phenomena in the projects and to construct rational arguments for their decisions. Although there are points in common between all the models exposed, they are differentiated by the type of elements they retain from the context.

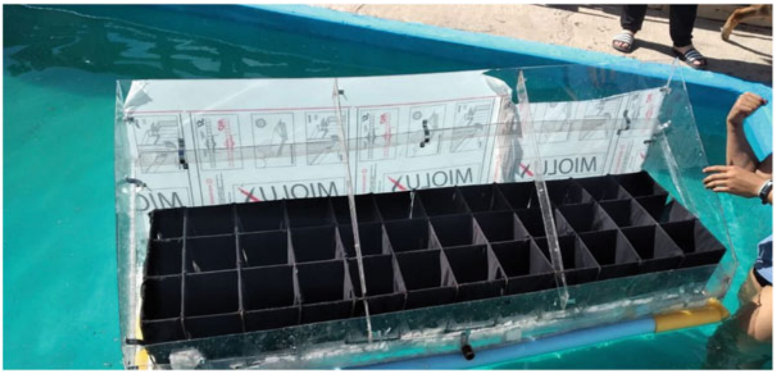


Fig. 11.4 Students' prototype of solar water distilling system. (Source: Authors' own work)

11.3.4 Modelling 1 of the Guy Wires in the Savonius Windmill: Analogical Modelling

The plateau region in Argentinian Patagonia is characterized by low precipitation levels (200 mm average per year) and strong winds. In this region, there live inhabitants relying on a subsistence economy where their main activity is sheep and goat farming. Usually, the necessary water for the animals, and for watering plants and crops is manually extracted from groundwater supplies (between 5 and 7 m deep).

The project's proposal was to develop low-cost, low-maintenance windmills that, by exploiting the wind resource, could facilitate the groundwater extraction tasks for the inhabitants. While the wind is a usable resource, strong gusts could damage the windmill. Therefore, the anchoring system had to be thoroughly analyzed. One of the topics particularly studied with the students referred to the location of the guy wires. In this regard, the professors usually ask one question:

Professors: "We will place the anchors of the guy wires 6 m away from the foot of the windmill. As you saw during your visit to the field, we may not be able to place the anchors where we planned. There were big shrubs and blocks of stone that may force us to relocate the anchors at a different place. The question is: if the anchor is placed nearer or farther from the original location, will the stress on the anchor and the guy wire change? Or will it remain the same?"

The students usually answer: "The force depends on the wind. If the wind does not vary, there is no reason for the stress to change". Figure 11.5 illustrates the windmill and one of the guy wires holding the mill to a concrete block buried on the ground 6 m away from its foot.

The purpose of the professors' questions was to analyze whether the stress on the anchor (concrete block and ground) changed according to the distance between the anchor and the windmill (6 m in the figure). Being one of the first considerations analyzed, professors usually propose a dynamic representation of stress in GeoGebra aiming to understand the relations between the vectors.

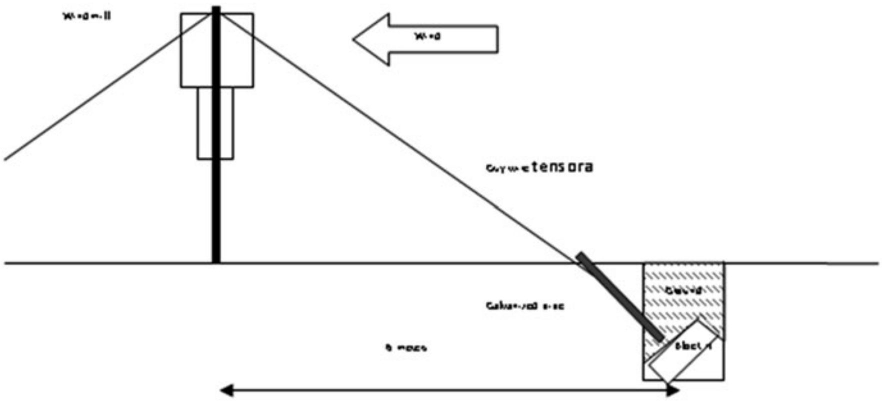


Fig. 11.5 The Savonius windmill and its support system. (Source: Authors' own work)

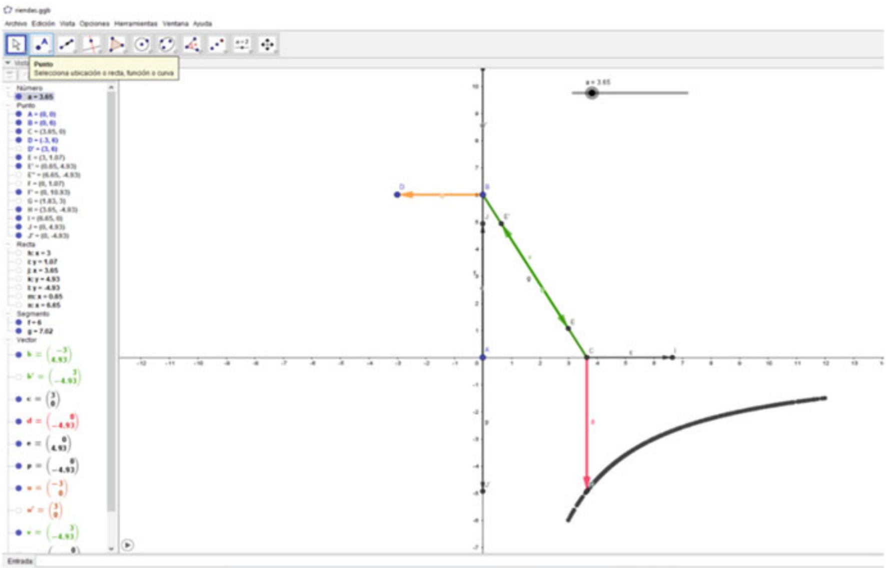


Fig. 11.6 Dynamic representation of the stress on the anchor in terms of distance. (Source: Authors' own work)

Briefly put, they started the modelling process and students progressively acquired the autonomy to complete it on their own. In this case, the modelling was descriptive because it allowed us to understand the relation between the stresses. Figure 11.6 shows the model produced. The slider controls the distance at which the guy wire is attached to the concrete block.

Due to space limitations, we will not share the details of the creation of the modelling. The curve observed shows the evolution of the rear end of the d vector. This vector represents the stress caused on the concrete block and the ground above it when holding the windmill in its vertical position.

The modelling, iterated many times in class, helped students understand that the anchor's distance is a variable to consider when digging the pits for the concrete blocks. To assist their comprehension, some of the iterations included a visualization of the geometrical elements that structure those relations (symmetries could only be visualized through the objects).

This model was created based on the reproduction of fundamental physical elements of the context and the relations between them:

- Windmill's main pipe.
- Guy wire.
- Pit location.
- Associated vectors.

Therefore, it was a direct reproduction of the elements from the context: the pipe was represented by a segment, the wire was represented by another segment and the anchors appeared as intersection points. Indeed, there is a replication of the elements from reality in the modelling. The only elements that could not be observed through our senses were the vectors. The relations between them were mathematical constructions based on elementary geometry and rigid transformations.

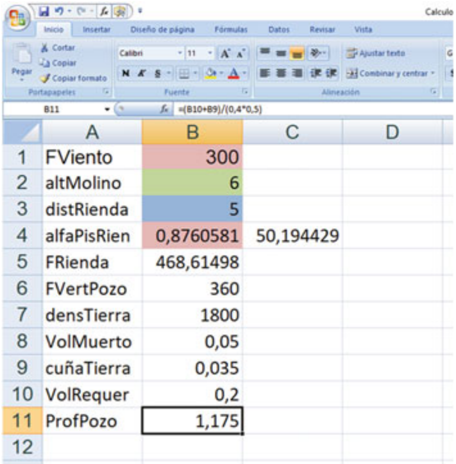
We argue this modelling was analogical since it was a replication of the objects from the context, which was done by geometrical objects. At a semantic level, the students could establish a direct, bijective relation between the elements from the context seen through their senses and their representations in the model. We will present another model referred to the problem of the guy wires; it will be a continuation of Modelling 1.

11.3.5 Modelling 2 of the Guy Wires in the Savonius Windmill: Analytical Modelling

The previous modelling done in GeoGebra helped the students understand a phenomenon relevant to the project: the stress on the guy wires was influenced not only by the wind speed but also by the distance at which the pits for the concrete blocks were dug. Of the various conclusions drawn, here we focused on one in particular: *the nearer the pits, the deeper we must dig them.*

That modelling was descriptive as well as analogical and was aimed at understanding the phenomenon. Once it was understood, the next step was its quantification. That is, it was necessary to determine with precision the pits' depth according to their distance to the foot of the windmill. It is worth mentioning that, while students were constructing the windmills, the inhabitants had to dig the pits for the concrete blocks. For this purpose, an explanatory plan with a schematic diagram of the distance between the pits and the foot of the windmill was given.

However, sometimes, they could not maintain the designated distances because of large shrubs' roots or blocks of stone. In those cases, they decided to relocate the pits. As a result, during the windmill's installation stage, students had to revise the



	A	B	C	D
1	FViento	300		
2	altMolino	6		
3	distRienda	5		
4	alfaPisRien	0,8760581	50,194429	
5	FRienda	468,61498		
6	FVertPozo	360		
7	densTierra	1800		
8	VolMuerto	0,05		
9	cuñaTierra	0,035		
10	VolRequer	0,2		
11	ProfPozo	1,175		
12				

Fig. 11.7 A spreadsheet on the quantification of the variables affecting the guy wires. (Source: Authors' own work)

pits' depth according to their final location. Moreover, the context determined that this revision task had to be expeditious. There was no time to make the necessary calculations at the location.

Likewise, it was not convenient to use the previous modelling in GeoGebra due to the location's conditions: the ambient light and the risks of damage for a laptop. Since that model was not appropriate for revising the pits' depth at the location, a new modelling was created on a spreadsheet that students would be able to access from their smartphones (Fig. 11.7).

Most of the cells for this modelling already had input values or formulas, the main data to input during the installation at the location was the pits' distance (B3 cell). Once it was introduced, the B11 cell showed, in meters, the optimum pits' depth. For the purpose of this chapter, we considered some characteristics of this modelling in relation to the type of elements from the context retained. This time, there were no physical (pipe and guy wires) or geometrical objects to represent in the modelling.

In this case, we retained, mainly, the abstract relations between the objects that could not be seen through our senses. As regards mathematics, algebra (Haspekian, 2005; Bruillard & Haspekian, 2009) and trigonometry are introduced instead of geometry. Likewise, the modelling abstraction level required the students to identify the trigonometric relations using paper and pencil; then, they had to materialize them in a spreadsheet. In this case, there is no longer a direct relationship between the objects from the context and the objects from the modelling. The abstraction level is greater.

Due to its role in the context, this modelling was prescriptive since it helped the students determine how to proceed (as regards pits' depth). Besides, it was analytical because of the type of elements considered. Additionally, it is worth highlighting the transitional role of the analogical modelling regarding the analytical modelling. The geometrical representation of the objects in the modelling in GeoGebra introduced mathematical objects that allowed the students to identify the trigonometric relations of the analytical modelling.

In other words, to identify those relations that would be used in the modelling on spreadsheets, the students no longer used the windmill's context as a reference; they used the modelling in GeoGebra. By observing this model on their laptops and the board, they recognized the triangles that consequently led to considering the trigonometric relations. Hence, the analogical modelling assisted in the comprehension of the stress variation concerning distance. What is more, it also helped transition to the analytical modelling that demanded greater abstractions.

11.3.6 Modelling 3 of the Lifting of the Savonius Windmill: Analogical Modelling

Among the analyses carried out with the students, there was one referred to hoisting the windmill from its horizontal assembling position to its vertical functional position. For this purpose, they watched footage from previous years showing how windmills were lifted. The hoisting stage at the location is critical due to the risk of the mill falling and the consequent damage if one of the parts involved in the hoisting process broke. Figure 11.8 shows a schematic diagram of the windmill's hoisting process. The structure is lifted to its vertical position using the synthetic cable of a vehicle's electric winch. Figure 11.8 shows a schematic diagram of a vehicle hoisting the Savonius windmill.

The winch's synthetic cable is attached to the superior part of the windmill (A). Upon watching the footage, one student proposed changing the attachment point from point A to point B to reduce stress. Other students proposed placing the vehicle as far as possible from the windmill, while others suggested the opposite; others also questioned whether the synthetic cable would resist the strain. In the end, the proposals were analyzed through a modelling.

To facilitate the comprehension of the stress dynamics at the hoisting moment, the professors induced a modelling we consider as analogical. That is, a reproduction of the physical elements from the context in, in this case, GeoGebra. Figure 11.9 shows this modelling.

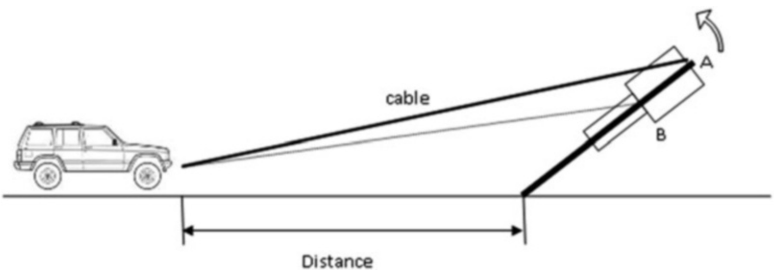


Fig. 11.8 Schematic diagram of a vehicle hoisting the Savonius windmill. (Source: Authors' own work)

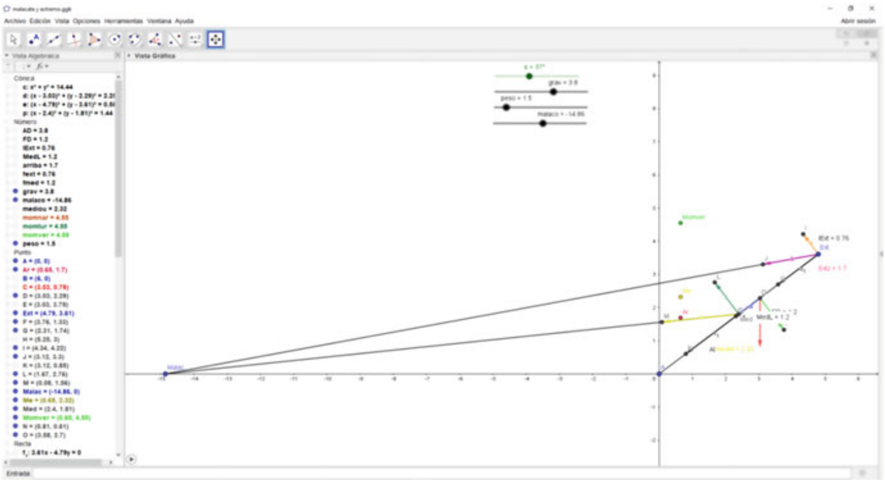


Fig. 11.9 Modelling of the stress on hoisting the Savonius windmill. (Source: Authors' own work)

This model assisted the comprehension of the stress produced during the hoisting stage and its changes as the windmill is closer to its vertical position. The lifting is controlled by the slider *alpha angle* and the location of the vehicle is controlled by the slider named *malaco*.

To quantify stress, the vectors' modules were made in scale. This allowed us to determine, for example, if the winch's cable would resist the strain during the hoisting. Unlike the modelling of the stress on the guy wires in GeoGebra, in this case, it was necessary to draw on elementary algebra since the physical concept that relates the vectors is torque ($M = Fxd$).

We consider this modelling as analogical because it consisted of the replication of objects from the context. This replication was achieved by using, mainly, geometrical elements. Figure 11.10 shows the graphic view with all the geometrical elements involved.

The modelling allowed us to understand the stress dynamics as well as to reject the idea of attaching the winch's cable to the windmill's point B, the stress would be greater if the cable was attached there and not in point A.

Likewise, the modelling confirmed that the winch's cable was appropriate for hoisting the windmill (according to its specifications, it resists 4 times more strain than the one caused at the hoisting stage). Besides, it allowed us to answer, among other matters, the question on the distance between the vehicle and the windmill, the farther the vehicle, the lesser the stress produced.

This modelling was descriptive because it assisted the comprehension of the stress dynamics. However, it was also prescriptive since it helped determine the decisions on how to act when hoisting the windmill. Moreover, because it was strongly based on the representation of objects from the context, it was analogical. Its creation was mainly based on geometrical elements, although elementary algebra was introduced for the physical concept discussed: torque.

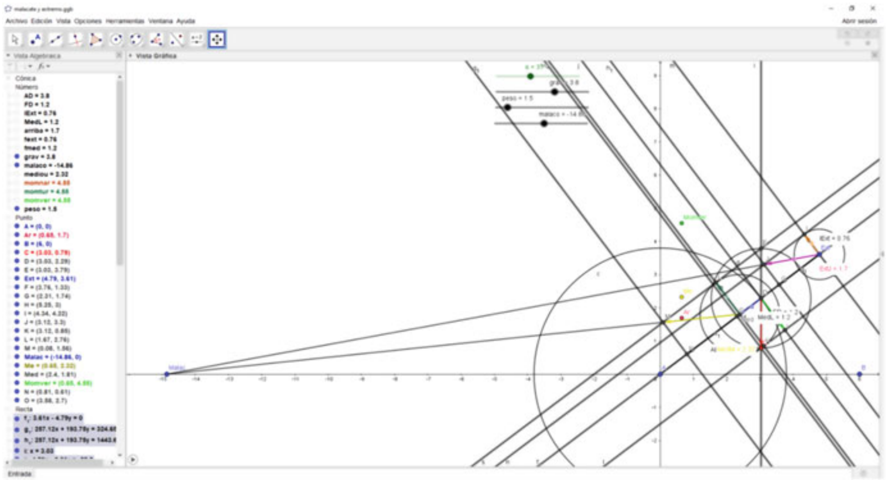


Fig. 11.10 Graphic view of all geometrical elements involved in modelling. (Source: Authors' own work)

11.3.7 Modelling 4: Analysis of the Savonius Windmill's Rotor—Analogical Modelling

One of the objectives of the Savonius windmills project is to let the inhabitants produce their solutions to the underground water extraction problem. For this purpose, we chose an easy, low-cost design. Additionally, given the distance between the rural facilities and the urban centres, we opted for a design that could be built using disused objects from the fields. That is why the windmill's rotor is built using recycled 200 L barrels.

As regards the rotor, various questions usually arise among the students: "Why are we using two rows of barrels instead of one?", "Will we produce more power if we use more rows?", and "Is it true that the rotor vibrates?". In these situations, students are usually asked to create a modelling for the rotor in GeoGebra that, even if it is simplified, allows them to understand the functionality and consequences of the design.

The modelling induced by the professors was analogical: physical elements of the rotor and their relations were taken into consideration, and then, a model was created introducing, mainly, geometrical elements and rigid transformations (rotations, translations, and symmetries). Figure 11.11 shows the simplified modelling of the rotor built using four half-barrels placed in two rows (two up and two down).

In this modelling, the rotor's spin is controlled by an angle-type slider named alpha. The modelling represents the four half-barrels as well as each half-barrel's contribution to the total area, although it is a simplified representation. The contributions of each half-barrel are represented by points; their traces allow us to understand the dynamics of the rotor's swept area. Figure 11.12 shows, in a

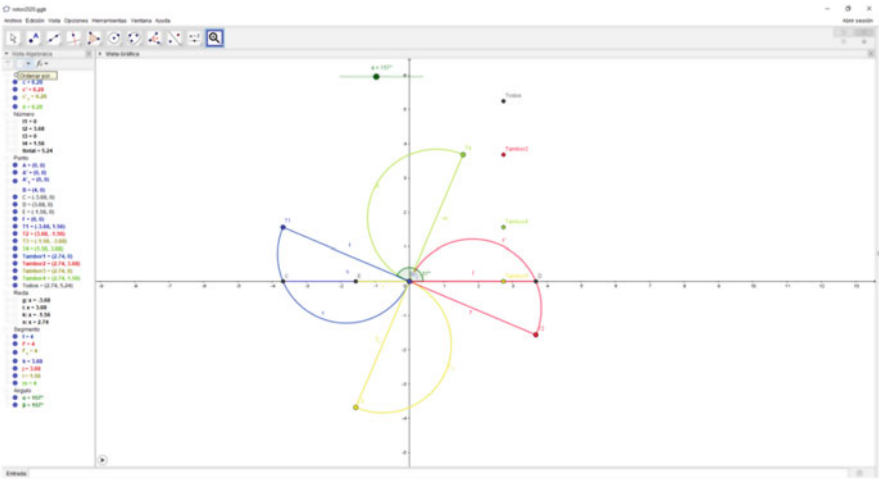


Fig. 11.11 Analysis of the contribution of each half barrel in the swept area (first steps). (Source: Authors' own work)

simplified manner, the swept area of each half-barrel and the total area of the four half-barrels when they are all spinning.

This modelling was descriptive for assisting us in understanding the rotor's functionality: in this regard, a fundamental aspect is the one referred to the variable rotor's area (superior trace). This variation in the rotor's swept area led to warning the students about the windmill's vibration. The reason for such vibration was the variability of the swept area, not the supposed irregular centres of gravity of the axis of rotation.

Thanks to the modelling, we could answer the question on the (in)convenience of using only two half barrels instead of four. Not only the total area would be minor but there would also be greater vibration as a result of the increased variability of the area. Therefore, we could determine that placing the four half-barrels in the same row would be inconvenient since they would obstruct each other.

This model was also prescriptive. Among other conclusions, the ones related to the unavoidable vibration of the rotor led the students to prioritize bolting rather than welding. From the current perspective, the model was analogical: there is a replication of objects from the real-world using geometrical objects; besides, the rigid transformations are predominant.

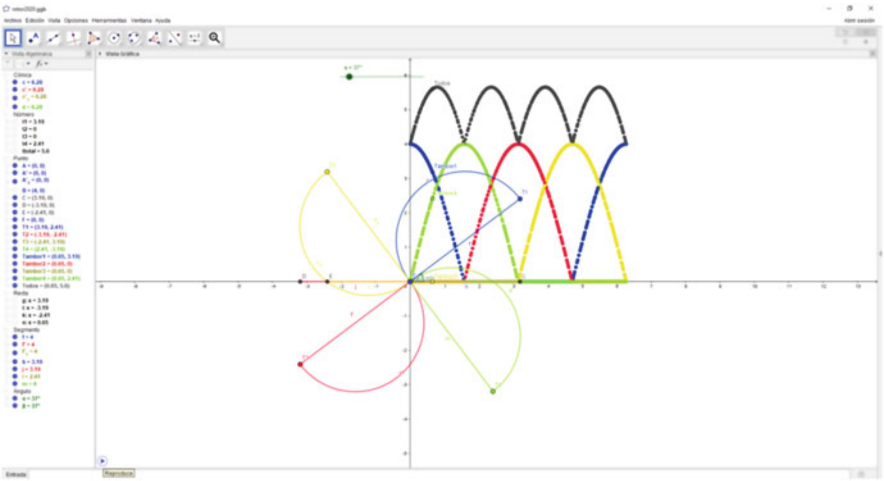


Fig. 11.12 Analysis of the contribution of each half barrel in the swept area. (Source: Authors' own work)

11.3.8 Modelling 5: Analysis of the Possible Changes to the Savonius Windmill's Rotor—Analytical Modelling

In the previous modelling, the students discovered that the windmill would vibrate due to certain matters inherent in the design: the variable swept area. A group of students suggested analyzing a possible modification to reduce or, in the best-case scenario, eliminate the vibration. The conversation ensued as follows:

Group 1: "Since there is increased variability using two barrels, and less variability using four barrels. . .is it possible that there will be even less variability using six barrels?"
Group 2: "But they shouldn't be at the same level, otherwise they would collide."
Group 1: "It could be a windmill with three rows of barrels".
Professors: "Are you sure that that is how you reduce vibration? That is, by increasing the number of barrels?"

This debate continued and encouraged performing a new analysis about increasing the number of half barrels symmetrically placed and their effect on the variability of the total swept area. Unlike the previous modelling created for four half-barrels, in this case, we tried to create a model for six, eight or more half-barrels.

The constructive method of analogy with the real world was now perceived a tedious due to the costs in terms of time and the elements involved. Meanwhile, in the construction of the modelling of the four half barrels, more precisely on the section of the fourth half-barrel, some students identified the trigonometric relation

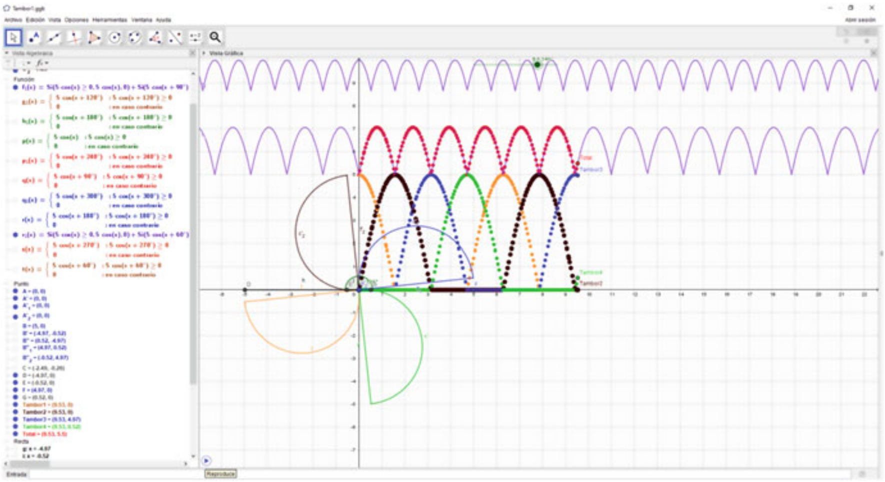


Fig. 11.13 Analysis of the contribution of each half barrel in the swept area (trigonometric functions). (Source: Authors' own work)

that described the area contribution of each half-barrel. Two elements constituted clear evidence: the shape of the curve that described the tracing and the triangular geometry of the contribution of each half-barrel.

To model the six-half-barrels rotor's swept area, the students proposed a more economical option as opposed to the analogical model: using trigonometric functions. A brief representation of the type of modelling created by the students with the professors' assistance can be seen below (Fig. 11.13).

Figure 11.13 shows the graphic and algebra view of the original modelling (analogical) of four half-barrels, the curve of the total area of the four half-barrels obtained by the analytical model and the curve of the total area of a hypothetical windmill with six half-barrels. The trigonometrical models of each half barrel are hidden in the graphic view, but they can be seen in the algebra view.

Similar to the guy wires case, we observed a switch in the type of modelling: from analogical to analytical. In the guy wires case, due to the characteristics of the context, it was convenient to carry out the modelling of the pits depth calculation on a different support (Spreadsheets). In this case, the support (GeoGebra) was appropriate, although that was not the case for the type of modelling used (analogical).

It is worth mentioning that the students found in the analogical modelling the indications to create an analytical modelling: the shape of the curve of the area of each half-barrel, and the triangular geometry of the area of each half-barrel. Figure 11.14 shows this triangular geometry.

Considering the wind comes from below, the simplified wind area of the half-barrel in the figure is determined by the projection of the barrel's diameter on the X-axis (multiplied by the height of the barrel). This geometrical representation and the shape of the curve that describes the projection based on the angle constituted

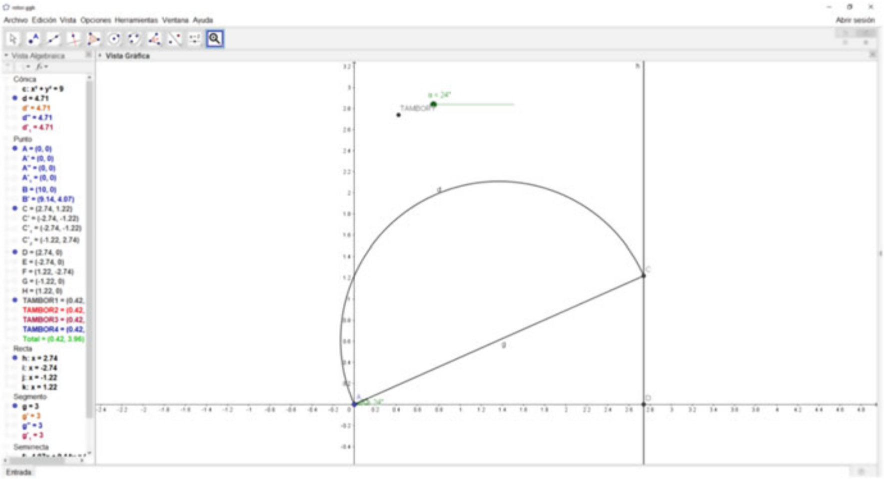


Fig. 11.14 Triangular geometry of the area of each half-barrel. (Source: Authors' own work)

enough evidence for the students to propose the trigonometric functions as more economic models to address the problem.

Therefore, due to economy reasons and, again, parting from the indications of the analogical modelling, the students created an analytical modelling of the rotor's swept area. As a result, they could confirm that as the number of half barrels increases, the variability in the swept area decreases, which results in a reduction of the vibration. Parting from this analysis, the students could provide a basis for a helical design (Fig. 11.15) that would have no vibration in the rotor.

11.3.9 Modelling 6: Building a Supporting Structure for a Solar Panel—Analogical Modelling

Within the framework of an extracurricular workshop conducted at a secondary school, we presented the students (aged between 11 and 13) a project to build a mobile supporting structure for a photovoltaic panel. The purpose was to improve the usage of solar radiation. The panel would supply energy to an electric 12 V water pump used to extract water from an unconfined aquifer at a rural facility in Argentinian Patagonia.

After weeks of exploring the causal relationships that optimize the performance of the solar panel (see Chap. 3), the students discovered that the optimal amperage was obtained when the photovoltaic panel was in a perpendicular position to the solar rays.

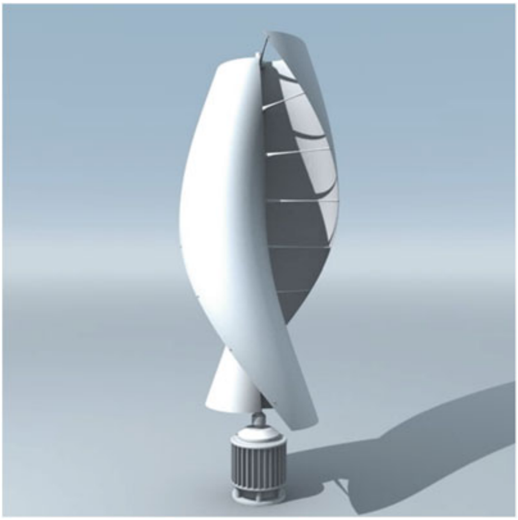


Fig. 11.15 Helical windmill. (Source: https://www.turbosquid.com/es/3d-models/3ds-max-helical-savonius-wind-turbine/804711)

Students and teachers then determined to provide four different positions for the photovoltaic panel, one for each season of the year. However, given the proximity between the angles for spring and autumn, only one position was considered for those two seasons. As a result, the supporting structure presented three positions: summer, autumn-spring, and winter.

One of the potentialities of the analogical models in GeoGebra is that the scale work leads to quantifiable results based on basic geometrical elements. Considering the students' age and their difficulties in mathematics, we opted for that strategy to determine the placement of the holes for each of the panel's positions. Figure 11.16 shows the construction made in GeoGebra that allowed them to determine the distance between the holes for the photovoltaic panel.

This modelling helped the students determine placement by simply moving the slider (sun's angle) until reaching the angle for each season and then observing the distance of the CG segment. This representation was descriptive since it allowed the students to understand how the mobile support worked. It was also prescriptive because it helped determine the actions to take.

From the perspective of the type of elements considered, it is an analogical modelling. In its construction, physical objects from the context were replicated as well as the movement and contact relations between them. Moreover, this modelling led to identifying—through elementary geometry elements—the distances at which the holes had to be punched to guarantee the panel's perpendicular position to the solar rays.

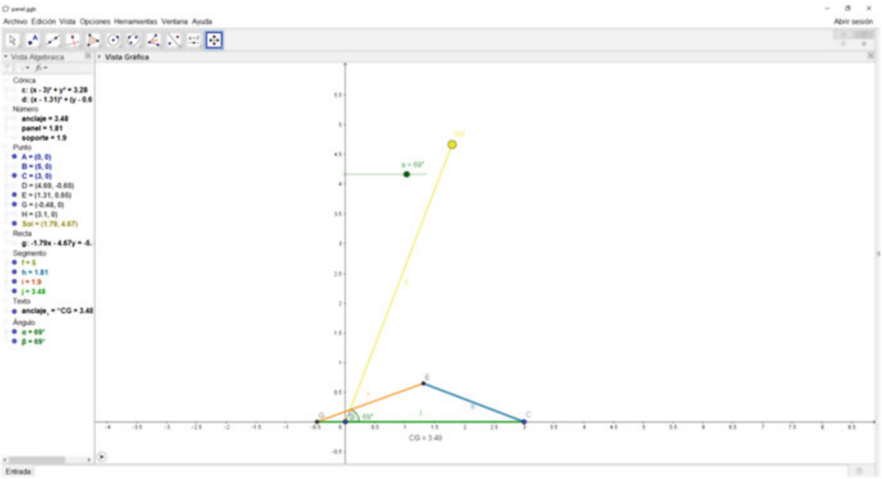


Fig. 11.16 Supporting structure for photovoltaic panel. (Source: Authors' own work)

11.3.10 Modelling 7: Optimization of the Lateral Sides of the Water Purification System—Analogical Modelling

In the framework of a project carried out in the Geology and Palaeontology programs (Bachelor's degree), first-year students (in 2021) designed, calculated and, built and tested prototypes for low-cost, solar-powered water purification systems. The systems' technological principle was based on research carried out at the University at Buffalo (the USA). The professors involved in the project maintained contact with the researchers under memorandums of understanding between the two universities. This agreement authorized the non-profit use of this technological development (Gan & Zhang, 2017).

The project with students consisted of implementing that technology into prototypes to be tested. Therefore, while the technological principle was already determined, designing the prototypes involved an important number of decisions to make. Especially considering the shortage of the needed energy source (solar radiation) in Argentinian Patagonia.

The system consisted of using solar radiation to evaporate water retained in an absorbent fabric. Then, that steam was condensed in a transparent cover and transported in gutters into a container. The water collected (distilled) was then purified using the appropriate salts. Figure 11.17 shows a picture of the first version of this technology. It was published by the researchers at the University at Buffalo.

To optimize the system's functionality, one of the variables analyzed was the surface of the lateral sides of the cover. The modelling we present here was created with students from the mathematics course. Its purpose was to determine the possible existence of a minimum area of lateral sides for a prototype with a base area of 1 m^2.

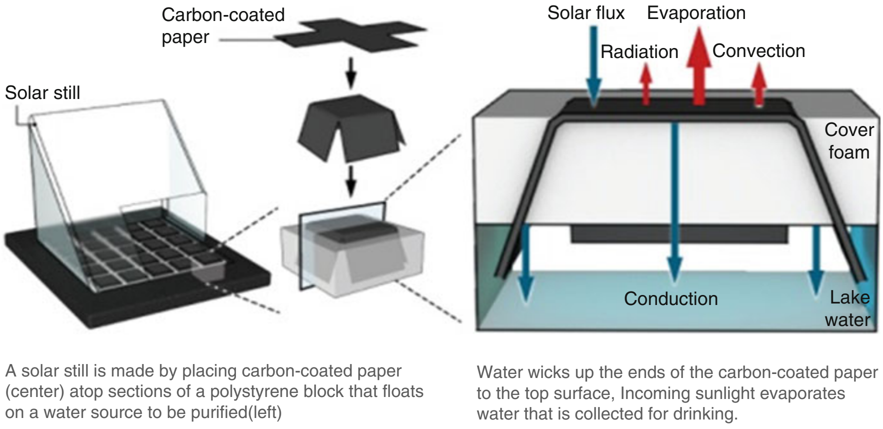


Fig. 11.17 Solar distillation. (Source: Buffalo University)

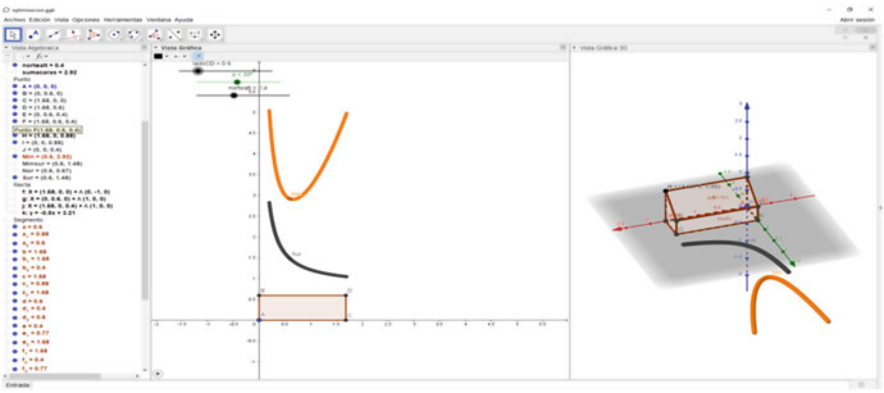


Fig. 11.18 Modelling the surface of the lateral sides of the cover. (Source: Authors' own work)

Figure 11.18 shows the students' construction in GeoGebra; they were assisted by their professors.

While performing the analysis with the students, a set of variables remained undefined. For instance, the angle of the cover's upper part and the height of the north side. Therefore, the professors induced a modelling with sliders on three variables: the measure of a side of the base (lado CD), upper part's angle (alfa) and north side's height (nortealt). This modelling was also constructed based on physical objects and their geometrical relations.

The 2D graphic view shows the rectangular base that allowed us to obtain the minimal surface of lateral sides for the 39° upper part's angle and the 40 cm height of the north side. The minimal surface of the lateral sides was obtained by placing the lado CD slider at 60 cm, approximately. The inferior curve shows the evolution of the south side's surface. This side was also relevant for not receiving direct radiation, being the coldest one and probably the one that condenses more water.

The 3D graphic view shows the cover. It was created replicating the physical objects and their geometrical relations. This modelling was descriptive since it allowed the students (and professors) to understand the problem in question. At the same time, it was prescriptive because, by finding the dimensions that determine the minimal area of lateral sides, it facilitated determining the criteria for the actions to take. Moreover, from our perspective, it was analogical due to the objects retained from the context and their geometrical relations.

11.3.11 Modelling 8: Optimization of the Lateral Sides of the Water Purification System—Analytical Modelling

For this project, the students (around 40) were divided into teams. In total, there were five teams and each of them made their own decisions, which in some cases were different. Therefore, there were five prototypes with some variations. For instance, as regards the strategy of keeping the upper part of the cover in a perpendicular position to the solar rays, some preferred to place regulable floaters based on the summer position, while others were based on the winter position. This led to certain differences in some of the variables, in particular, the angle of the upper part of the cover.

In the analysis of the optimal measures, it was necessary to work on the three variables (sliders). Each team assigned their values to the upper part's angle (alfa) and the height of the north side (nortealt) to determine the measure of the side named caraslat. For a more evident analysis of the change of the minimal surface of the lateral faces (caraslat) according to the other variables, the teachers proposed making an algebraic representation of the addition of the lateral sides' surfaces.

Thus, the independent variable would be lado CD, while the dependent variable would be the addition of the lateral sides' surfaces, and the parameters would be the upper part's angle and the height of the north side. Figure 11.19 shows the curve obtained; first, supported on paper and then in GeoGebra (p(x)).

The tangent line to the curve for a given x value became a visual indicator of the location of the minimal point. The function p(x) that models the total surface of the lateral sides was done on paper based on the analogical modelling from the representation in GeoGebra's 3D algebra view.

Again, this modelling was descriptive as well as prescriptive due to the same reasons detailed for the analogical modelling. In this case, resorting to algebra for the relation between the measure of lado CD and the total surface of the lateral sides constituted a conceptual leap different from the previous modelling. This model was then analytical. It was no longer based on the physical relations between the objects. Instead, it was based on the more abstract relations that exceeded sensorial perception. These relations had to be analyzed in the algebraic and functional plane, among others.

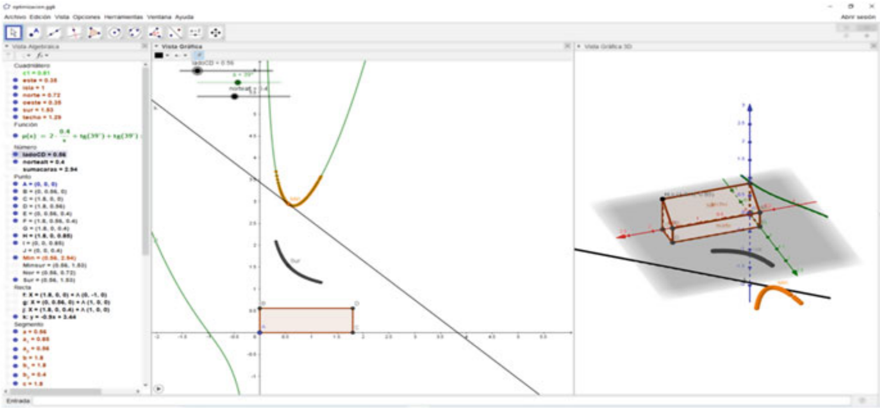


Fig. 11.19 Optimization of the lateral sides of the water purification system. (Source: Authors' own work)

11.4 Results and Perspectives

In terms of preliminary results, we claim that, as the literature states, modelling is an appropriate place for learning disciplinary concepts; in particular, mathematical. The eight models presented here have allowed us to discuss new concepts while re-significating concepts already known by the students. The list of mathematical concepts introduced in each modelling will not be listed here as it is not relevant for this chapter.

We do wish to highlight the diversity of the branches of mathematics that are interrelated in the same modelling and the possibilities for the resignification of elementary concepts that become fundamental to characterize objects as well as parts' movements. To illustrate, we could consider the circle; a geometrical object which at the same time is used as an instrument to measure distance and even to determine rotations. The modelling experiences with the students have enabled them to interpret and reinterpret concepts not only by the semiotic registers in which they appear (Duval, 1993, 2006; Hitt, 2004) but also by the phenomena that they can explain and the objects that can represent the real-world.

Another interesting dynamic observed in the models with students refers to the tool-object dialectic (Douady, 1986, 2002; Czocher, 2019). Because of the models, it was possible to deepen the learning of concepts in that dialectic. The concepts are tools for understanding and describing phenomena, but they are also considered learning objects. As such, one can be familiarized with their characteristics, rules and types of use leading to instances of institutionalization of the characteristics of the mathematical objects.

As regards modelling as a teaching object connected to abilities and competencies, we could verify a considerable number of difficulties among the students at the beginning of the modelling. Partly, this could be explained by their lack of experience. The mathematics they experienced in their prior education could be described as a discipline where the concepts are exercised but not applied. Therefore, the abilities referring to the associations between mathematical objects and real-world relations were scarce at the beginning of the models. As a consequence, the professors had to intervene and induce relationships between the objects of these two worlds (real and model).

Another difficulty observed is related to the supports where the models were carried out. For all the students, GeoGebra was a new tool at the beginning of the activities, although the spreadsheet was familiar to them. We understand that the supports on which the models are produced are not transparent. This lack of transparency is understood in two directions: the knowledge of both the existence and use of the supporting tools, and the possibilities they enable. In this regard, we argue that the support is not transparent when modelling and, thus, some representations are feasible on a certain support while others are expensive in terms of procedure, or plainly impossible to carry out. Hence, a support has a space of possibilities linked and this space conditions what can be modelled with it. Thus, the students also presented difficulties due to their initial lack of knowledge about the GeoGebra support.

There were two more phenomena linked to difficulties observed at the initial stage. On the one hand, there was confusion caused by the change in the didactic contract. On the other, there was a lack of experience working in teams in complex contexts. As regards the didactic contract, the students entered their project classes with habits quite different from the ones in project-based learning. Their prior experiences were marked by doing individual activities using pencils and paper, their educators' expository lessons and final exams at the end of the course. The dynamic of working in projects in a real-world context and, particularly, the models where they had to produce arguments caused surprises at the initial stages of the projects. The same applies to teamwork and collective commitments.

In addition to that, the proposals presented here were carried out throughout the first year of their education. It is important to highlight that dropouts characterize this period. Thus, the students who remained in the courses also had to learn how to solve complex issues in the projects related to their classmates' absence. In this sense, we highlight that the models presented were carried out in educative experiences (projects) that lasted 4 or 8 months, depending on the case. This relatively long-time frame allowed the students to improve their previously mentioned difficulties.

We also consider relevant the interdisciplinarity, unavoidable at times, to discuss modelling. In particular, although not limited to, the analogical models. However, since analogical models replicate characteristics and phenomena of the context, they appear to have an evident tendency to resort to interdisciplinarity in comparison to analytical models. This could be observed in the development of all the analogical models presented here.

Moreover, the analogical models seemed more appropriate than the analytical for the general and basic understanding of the analyzed phenomena. The analogical modelling possibility to establish direct relations between the objects from the context and the objects of the model appear to have facilitated the students' appropriation of the problems as well as their understanding.

This potentiality presented in the analogical models did not pose any difficulties in the possibilities of constructing arguments or making decisions based on them. On the contrary, precisely for establishing relations between the model and reality, they helped the students construct the arguments for their decisions. In this direction, we have observed certain difficulties in the analytical models. Their level of abstraction posed difficulties in the students' understanding and production of potential analyses relevant to the problems being discussed. This was verified in the answers to open questionnaires we gave to them.

In those questionnaires, they were asked about matters observed in an analytical modelling. In general, their answers did not contemplate elements of the model. Instead, there was a tendency of returning to the real-world context to explain it, which was inappropriate considering the impossibility to observe the phenomenon in the real-world context. Thus, the analogical models appear as an interesting intermediate instance in relation to the analytical models. In fact, the analytical models presented in this chapter were produced based on analogical models.

There appears a sort of sequence of levels of abstraction where the analogical modelling constitutes the first step of abstraction on which it is convenient to support more abstract models that enable other types of analyses. As a result, at least two potentialities can be observed. On the one hand, the possibility of creating different interpretations and arguments of each type. On the other, the staggering or sequencing where analogical modelling is relevant for two reasons: its results and its support for a more complex analytical model.

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