Chapter 7
Variational Filter for Predictive Modeling of Structural Systems

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Abstract Bayesian inference offers a distinct advantage in predictive structural modeling as it quantifies the inherent epistemic uncertainties that arise due to observations of the system which are both finite in length and limited in representative behavioral information. Current research interest in the field of predictive structural modeling has emphasized analytical and sampling approaches to Bayesian inference, which have the respective advantages of either computational speed or inference accuracy. Recent work in optimization-based inference approaches have created new opportunities to balance these advantages and generate flexible, efficient, and scalable filters for joint parameter-state identification of complex nonlinear structural systems. These techniques, commonly referred to as variational inference, infer the hidden states of a system by attempting to match the true posterior and to a parameterized distribution. In this study we build on the theory of automatic differentiation variational inference to introduce a novel approach to variational filtering for the identification of complex structural systems. We evaluate our method using experimental observations from a nonlinear energy sink device subject to base excitation. Comparison between identification performed using our approach and the unscented Kalman filter reveals the utility of the variational filtering technique in terms of both flexibility in the stochastic model and robustness of the method to poor specification of prior uncertainty.

Keywords System identification · Nonlinear energy sink · Bayesian inference · Unscented Kalman filter · Variational inference

7.1 Introduction

Bayesian filtering approaches have become a powerful tool in predictive structural modeling as they provide a useful framework for interpreting damage and projecting system behavior under the various sources of uncertainty typical in practical structural systems. Current research typically emphasizes analytical or sampling approximations to the Bayesian framework, which, respectively, allow for fast approximations under restrictive modeling assumptions or flexible modeling at a steep computational cost. Recent advances in a third class of approximate Bayesian methods, known as *variational inference*, seek to balance the benefits of the analytical and sampling approximations and have been shown to provide a useful and comparatively efficient alternative to sampling techniques for problems involving complex computational models or large datasets [1].

Research into the adaptation of variational inference for time-series identification problems has most notably resulted in the work of Smidl and Quinn [2], who developed variational Bayesian filtering. Through this approach they coupled variational inference with the particle filter to infer the hidden states of generalized dynamical systems. Other works in this area have coupled variational inference with Kalman filter variants [3, 4]. Each of these approaches has shown some success on computational examples, though their validation on experimental systems with high levels of uncertainty remains to be seen.

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In this work, we examine the performance of a novel variational filter (VF) on the identification of an experimental nonlinear energy sink device. The proposed VF steps beyond previous filtering implementations by using state-of-the-art advances in variational inference, such as automatic differentiation variational inference (ADVI) [5] and KL-annealing [6], to enhance modularity and efficiency of the stochastic optimization process. The performance of the proposed filter and its robustness to experimenter uncertainty in the specification of the prior is benchmarked against that of the unscented Kalman filter (UKF).

7.2 Background

The VF algorithm proposed in this study operates in a two-step process aligned with the Bayesian filtering equations for the development of the marginal prior

$$p\left(\mathbf{x}_k, \boldsymbol{\theta} | \mathbf{y}_{1:k-1}, \mathbf{u}_{0:k-1}\right) = \int p\left(\mathbf{x}_k | \mathbf{x}_{k-1}, \boldsymbol{\theta}, \mathbf{u}_{k-1}\right) p\left(\mathbf{x}_{k-1}, \boldsymbol{\theta} | \mathbf{y}_{1:k-1}, \mathbf{u}_{0:k-2}\right) d\mathbf{x}_{k-1}, \tag{7.1}$$

and the marginal posterior

$$p\left(\mathbf{x}_k, \boldsymbol{\theta} | \mathbf{y}_{1:k}, \mathbf{u}_{0:k-1}\right) = \frac{p(\mathbf{y}_k | \mathbf{x}_k, \boldsymbol{\theta}) p\left(\mathbf{x}_k, \boldsymbol{\theta} | \mathbf{y}_{1:k-1}, \mathbf{u}_{0:k-1}\right)}{p\left(\mathbf{y}_k | \mathbf{y}_{1:k-1}, \mathbf{u}_{0:k-1}\right)}, \tag{7.2}$$

where $\mathbf{x}_k$ are the latent system states, $\boldsymbol{\theta}$ are the constant model parameters, $\mathbf{y}_k$ are the system observations, and $\mathbf{u}_k$ are the open-loop control inputs, measured at the discrete time intervals $t_k = k\Delta t$, where $k = 1, \ldots,$ K. Approximations of the marginal prior and posterior are generated at each filtering step over the batch of observation data $\mathcal{B}_i$. As shown in Fig. 7.1, the marginal prior is approximated by propagating Monte Carlo samples from the posterior of the states in $\mathcal{B}_{i-1}$ to describe the prior uncertainty over the states in $\mathcal{B}_i$. The approximate marginal posterior on $\mathcal{B}_i$ is then developed through ADVI, in which the family of distributions $g_{\mathcal{B}_i}\left(\mathbf{x}_{\mathcal{B}_i}, \boldsymbol{\theta}; \boldsymbol{\phi}\right)$ is optimized against the Kullback-Leibler divergence toward the true marginal posterior.

The performance of the proposed variational filter is compared against that of the UKF [7] in the identification of the states and parameters of an experimental nonlinear energy sink device, which can be modeled by the dynamic equation of motion

$$m\ddot{x} + c_v\dot{x} + c_f \tanh\left(200\dot{x}\right) + kx + zx^3 = -m\ddot{x}_g. \tag{7.3}$$

For the identification problem, the system mass m is assumed known at 0.6637 kg, whereas the viscous damping coefficient (c_v Ns/m), friction damping coefficient (c_f N), stiffness coefficient (k N/m), and nonlinear stiffness coefficient (z N/m^3) are unknown parameters which will be inferred in conjunction with the states. Observations of the system response to a sine wave input with linearly varying amplitude are made in terms of its relative displacement (x) and absolute acceleration ($\ddot{x} + \ddot{x}_g$).

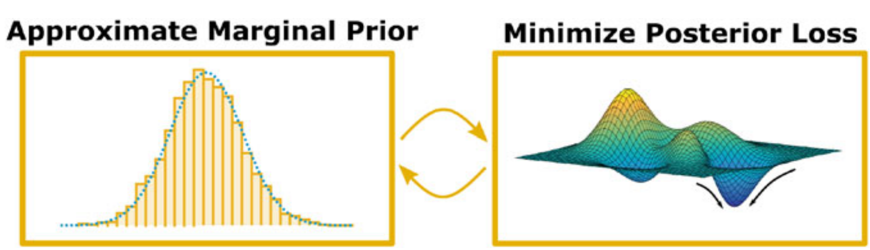

Fig. 7.1 Representation of the operation of the proposed variational filter

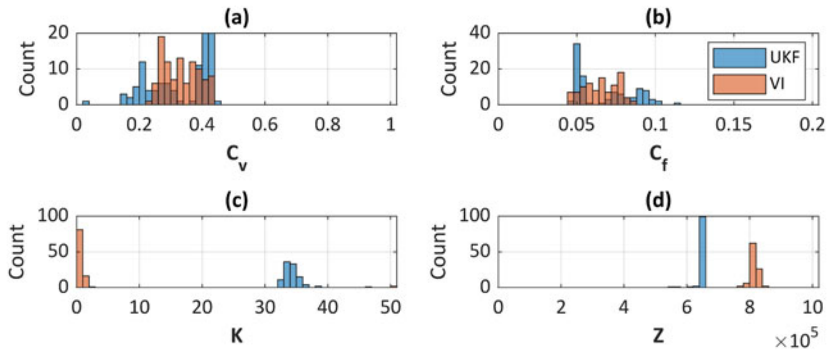

Fig. 7.2 Modes of the (**a**) viscous damping, (**b**) friction damping, (**c**) linear stiffness, and (**d**) nonlinear stiffness for the 100 identified models

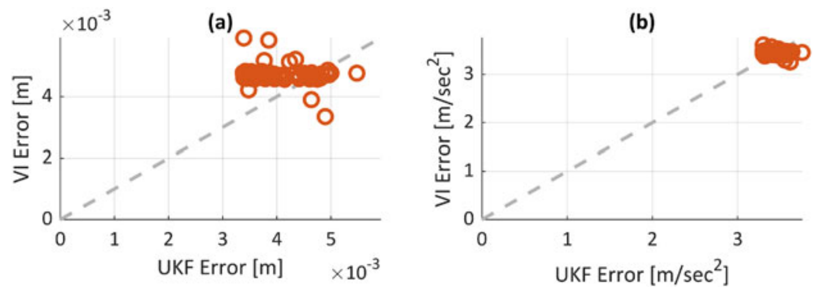

Fig. 7.3 Error in the simulated (**a**) displacement and (**b**) acceleration responses developed from the 100 candidate models

7.3 Analysis

Our objective is to investigate the extent to which variability in the prior distribution produces variability in the candidate model obtained from the Bayesian identification algorithm. To this end, we compare the relative performance of the VF and UKF algorithms on a series of inference trials run using 100 possible prior distributions on the parameters. These distributions were selected to represent the variability with which different experimentalists might approach the specification of prior uncertainty. The results of these inference trials are given with respect to the modes of the posterior distributions on the parameters (Fig. 7.2) and the root-mean-square (RMS) error of the simulated displacement and acceleration responses of the generated candidate models over the observed system responses (Fig. 7.3).

The results in Fig. 7.2 show a similar concentration in the parameters developed through the candidate models between the VF an UKF approaches. Though the algorithms generate different interpretations of the stiffness parameters, these differences do not correlate to large differences in the resulting RMS error values, as one might expect. Rather, these results demonstrate the gap between the computational model form selected for the experimental system and the actual response behavior of that system. Both approaches employ slightly different modeling assumptions and tune the identification process for slightly different objectives, resulting in two high-quality, but different, calibrated models.

7.4 Conclusion

The results of this study reveal that the proposed VF algorithm experiences similar levels of success in identification as the more commonly used UKF, though the two converge to different high-quality models. The key functional difference between these algorithms in the identification of models of similar complexity to the nonlinear energy sink system described herein then becomes their respective computational efficiencies, where the UKF currently runs orders of magnitude faster the VF. This computational constraint in the VF has the potential for improvement in future work.

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