An Excursion to Multiplications and Convolutions on Modulation Spaces

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Abstract We give a self-contained introduction to (quasi-)Banach modulation spaces of ultradistributions, and review results on boundedness for multiplications and convolutions for elements in such spaces. Furthermore, we use these results to study the Gabor product. As an example, we show how it appears in a phase-space formulation of the nonlinear cubic Schrödinger equation.

Keywords Time–frequency analysis · Modulation spaces · Convolutions · Multiplications

1 Introduction

Modulation spaces were introduced in Feichtinger's seminal technical report [17], and prove themselves as useful family of Banach spaces of tempered distributions in time-frequency analysis, [4, 10, 28]. The main purpose of this survey article is to enlighten some properties of modulation spaces in a rather self-contained manner. In contrast to the most common situation, our analysis includes both quasi-Banach and Banach modulation spaces within the framework of ultradifferentiable functions and ultradistributions of Gelfand–Shilov type. For that reason we collect necessary background material in a rather detailed preliminary section.

Motivated by recent applications of modulation spaces in the context of nonlinear harmonic analysis and its applications, cf. [4–6, 14, 22, 38, 39, 47, 54] we focus our attention to boundedness for multiplications and convolutions for elements in such spaces. The basic results in that direction go back to the original contribution [17],

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and were thereafter reconsidered by many authors in different contexts. Let us give a brief, and unavoidably incomplete account on the related results.

In Sect. 3 we formulate in Theorems 3.5 and 3.7 bilinear versions of more general multiplication and convolution results in [54, Section 3]. The contents of Theorems 3.5 and 3.7 in the unweighted case for modulation spaces $M^{p,q}$ can be summarized as follows.

Proposition 1.1 *Let* $p_j, q_j \in (0, \infty]$*,* $j = 0, 1, 2$*,*

$$\theta_1 = \max\left(1, \frac{1}{p_0}, \frac{1}{q_1}, \frac{1}{q_2}\right) \quad \textit{and} \quad \theta_2 = \max\left(1, \frac{1}{p_1}, \frac{1}{p_2}\right).$$

Then

$$M^{p_1,q_1} \cdot M^{p_2,q_2} \subseteq M^{p_0,q_0}, \qquad \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_0}, \qquad \frac{1}{q_1} + \frac{1}{q_2} = \theta_1 + \frac{1}{q_0},$$

$$M^{p_1,q_1} * M^{p_2,q_2} \subseteq M^{p_0,q_0}, \qquad \frac{1}{p_1} + \frac{1}{p_2} = \theta_2 + \frac{1}{p_0}, \quad \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_0}.$$

The general multiplication and convolution properties in Sect. 3 also overlap with results by Bastianoni, Cordero and Nicola in [2], by Bastianoni and Teofanov in [1], and by Guo et al. in [32].

The multiplication relation in Proposition 1.1 for $p_j, q_j \geq 1$ was obtained already in [17] by Feichtinger. It is also obvious that the convolution relation was well-known since then (though a first formal proof of this relation seems to be given first in [48]). In general, these convolution and multiplication properties follow the rules

$$\ell^{p_1} * \ell^{p_2} \subseteq \ell^{p_0}, \quad \ell^{q_1} \cdot \ell^{q_2} \subseteq \ell^{q_0} \quad \Rightarrow \quad M^{p_1,q_1} * M^{p_2,q_2} \subseteq M^{p_0,q_0}$$

and

$$\ell^{p_1} \cdot \ell^{p_2} \subseteq \ell^{p_0}, \quad \ell^{q_1} * \ell^{q_2} \subseteq \ell^{q_0} \quad \Rightarrow \quad M^{p_1,q_1} \cdot M^{p_2,q_2} \subseteq M^{p_0,q_0},$$

which goes back to [17] in the Banach space case and to [25] in the quasi-Banach case. See also [19] and [42] for extensions of these relations to more general Banach function spaces and quasi-Banach function spaces, respectively.

In Sect. 3 we basically review some results from [54]. To make this survey self-contained we give the proof of Theorem 3.7 in unweighted case. In contrast to [32], we do not deduce any sharpness for our results.

To show Proposition 1.1 in the quasi-Banach setting, apart from the usual use of Hölder's and Young's inequalities, additional arguments are needed. In our situation we discretize the situations in similar ways as in [2] by using Gabor analysis for modulation spaces, and then apply some further arguments, valid in non-convex analysis. This approach is slightly different compared to what is used in [32] which follows the discretization technique introduced in [55], and which has some traces of Gabor analysis.

We refer to [54] for a detailed discussion on the uniqueness of multiplications and convolutions in Proposition 1.1.

In Sect. 4 we apply the results from previous parts in the framework of the so called Gabor product. It is introduced in [14] in order to derive a phase space analogue to the usual convolution identity for the Fourier transform. The main motivation is to use such kind of products in a phase-space formulation of certain nonlinear equations. As noticed in [14], among other interesting characteristics of phase-space representations, the initial value problem in phase-space may be well-posed for more general initial distributions. This means that the phase-space formulation could contain solutions other than the standard ones. We refer to [11–13], where the phase-space extensions are explored in different contexts. Here we illustrate this approach by considering the nonlinear cubic Schrödinger equation, which appear for example in Bose-Einstein condensate theory [35]. We also refer to [4, Chapter 7] for an overview of results related to well-posedness of the nonlinear Schrödinger equations in the framework of modulation spaces, see also [3, 38, 39].

2 Preliminaries

In this section we give an exposition of background material related to the definition and basic properties of modulation spaces. Thus we recall some facts on the short-time Fourier transform and related projections, the (Fourier invariant) Gelfand-Shilov spaces, weight functions, and mixed-norm spaces of Lebesgue type. We also recall convolution and multiplication in weighted Lebesgue sequence spaces.

2.1 The Short-Time Fourier Transform

In what follows we let $\mathscr{F}$ be the Fourier transform which takes the form

$$(\mathscr{F} f)(\xi) = \widehat{f}(\xi) \equiv (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(x) e^{-i\langle x,\xi\rangle}\, dx$$

when $f \in L^1(\mathbb{R}^d)$. Here $\langle \cdot\, , \cdot \rangle$ denotes the usual scalar product on $\mathbb{R}^d$. The same notation is used for the usual dual form between test functions and corresponding (ultra-)distributions. We recall that map $\mathscr{F}$ extends uniquely to a homeomorphism on the space of tempered distributions $\mathscr{S}'(\mathbb{R}^d)$, to a unitary operator on $L^2(\mathbb{R}^d)$ and restricts to a homeomorphism on the Schwartz space of smooth rapidly decreasing functions $\mathscr{S}(\mathbb{R}^d)$, cf. (29). We also observe with our choice of the Fourier transform, the usual convolution identity for the Fourier transform takes the forms

$$\mathscr{F}(f \cdot g) = (2\pi)^{-\frac{d}{2}} \widehat{f} * \widehat{g} \quad \text{and} \quad \mathscr{F}(f * g) = (2\pi)^{\frac{d}{2}} \widehat{f} \cdot \widehat{g} \tag{1}$$

when $f, g \in \mathscr{S}(\mathbb{R}^d)$.

In several situations it is convenient to use a localized version of the Fourier transform, called the short-time Fourier transform, STFT for short. The short-time Fourier transform of $f \in \mathscr{S}'(\mathbb{R}^d)$ with respect to the fixed *window function* $\phi \in \mathscr{S}(\mathbb{R}^d)$ is defined by

$$(V_\phi f)(x, \xi) \equiv (2\pi)^{-\frac{d}{2}} (f, \phi(\,\cdot\, - x) e^{i\langle\,\cdot\,,\xi\rangle})_{L^2}. \tag{2}$$

Here $(\,\cdot\,,\,\cdot\,)_{L^2}$ denotes the unique continuous extension of the inner product on $L^2(\mathbb{R}^d)$ restricted to $\mathscr{S}(\mathbb{R}^d)$ into a continuous map from $\mathscr{S}'(\mathbb{R}^d) \times \mathscr{S}(\mathbb{R}^d)$ to $\mathbb{C}$.

We observe that using certain properties for tensor products of distributions,

$$(V_\phi f)(x, \xi) = \mathscr{F}(f \cdot \overline{\phi(\,\cdot\, - x)})(\xi). \tag{2$'$}$$

(cf. [33, 52]). If in addition $f \in L^p(\mathbb{R}^d)$ for some $p \in [1, \infty]$, then

$$(V_\phi f)(x, \xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(y) \overline{\phi(y - x)} e^{-i\langle y,\xi\rangle}\, dy. \tag{2$''$}$$

We observe that the domain of V_ϕ is $\mathscr{S}'(\mathbb{R}^d)$. The images are contained in $C^\infty(\mathbb{R}^{2d})$, the set of smooth functions defined on the phase space $\mathbb{R}^d \times \mathbb{R}^d \simeq \mathbb{R}^{2d}$.

The short-time Fourier transform appears in different contexts and under different names. In quantum mechanics it is rather common to call it the *coherent state transform* (see e.g. [37]). It is also closely related to the so-called Wigner distribution or radar ambiguity function (see e.g. [36]). In time-frequency analysis, it is also sometimes called the *Voice transform*.

The main idea with the design of short-time Fourier transform is to get the Fourier content, or the frequency resolution of localized functions and distributions. Roughly speaking, short-time Fourier transforms give a simultaneous information both on functions or distributions themselves as well as their Fourier transforms in the sense that the map

$$x \mapsto V_\phi f(x, \xi)$$

resembles on $f(x)$, while the map

$$\xi \mapsto V_\phi f(x, \xi)$$

resembles on $\widehat{f}(\xi)$.

As for the ordinary Fourier transform, there are several mapping properties which hold true for the short-time Fourier transform. As an elegant way to approach such properties in the framework of distributions, we may follow ideas given in [24] by Folland.

In fact, let T be the semi-conjugated tensor map

$$T(f,\phi) = f \otimes \overline{\phi}, \tag{3}$$

U be the linear pullback

$$(UF)(x,y) = U(y, y-x) \tag{4}$$

and $\mathscr{F}_2$ be the partial Fourier transform given by

$$(\mathscr{F}_2 F)(x,\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} F(x,y) e^{-i\langle y,\xi\rangle}\, dy. \tag{5}$$

Then

$$V_\phi f = (\mathscr{F}_2 \circ U \circ T)(f,\phi), \tag{6}$$

when $f, \phi \in \mathscr{S}(\mathbb{R}^d)$.

We observe that the mappings

$$T : \mathscr{S}(\mathbb{R}^d) \times \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^{2d}), \qquad U, \mathscr{F}_2 : \mathscr{S}(\mathbb{R}^{2d}) \to \mathscr{S}(\mathbb{R}^{2d}) \tag{7}$$

are continuous and uniquely extendable to continuous mappings

$$T : \mathscr{S}'(\mathbb{R}^d) \times \mathscr{S}'(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^{2d}), \quad U, \mathscr{F}_2 : \mathscr{S}'(\mathbb{R}^{2d}) \to \mathscr{S}'(\mathbb{R}^{2d}), \tag{8}$$

which in turn restricts to isometric mappings

$$T : L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d}), \qquad U, \mathscr{F}_2 : L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^{2d}). \tag{9}$$

Here that T is isometric means that

$$\|T(f,\phi)\|_{L^2(\mathbb{R}^{2d})} = \|f\|_{L^2(\mathbb{R}^d)} \|\phi\|_{L^2(\mathbb{R}^d)}.$$

It is now natural to define $V_\phi f$ as the right-hand side of (6) when $f, \phi \in \mathscr{S}'(\mathbb{R}^d)$, in which $V_\phi f$ is well-defined as an element in $\mathscr{S}'(\mathbb{R}^{2d})$.

Proposition 2.1 *The map*

$$(f,\phi)\mapsto V_\phi f : \mathscr{S}(\mathbb{R}^d)\times\mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^{2d}) \tag{10}$$

is continuous, which extends uniquely to a continuous map

$$(f,\phi)\mapsto V_\phi f : \mathscr{S}'(\mathbb{R}^d)\times\mathscr{S}'(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^{2d}), \tag{11}$$

which in turn restricts to an isometric map

$$(f,\phi)\mapsto V_\phi f : L^2(\mathbb{R}^d)\times L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d}). \tag{12}$$

If $\phi\in\mathscr{S}(\mathbb{R}^d)$ and $f\in\mathscr{S}'(\mathbb{R}^d)$, then (11) shows that $V_\phi f\in\mathscr{S}'(\mathbb{R}^{2d})$. On the other hand, it is easy to see that the right-hand side of (2) defines a smooth function. Consequently beside (11) and (10), we also have the continuous map

$$(f,\phi)\mapsto V_\phi f : \mathscr{S}'(\mathbb{R}^d)\times\mathscr{S}(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^{2d})\cap C^\infty(\mathbb{R}^{2d}). \tag{13}$$

For short-time Fourier transform, the Parseval identity is replaced by the so-called Moyal identity, also known as the *orthogonality relation* given by

$$(V_\phi f, V_\psi g)_{L^2(\mathbb{R}^{2d})} = (\psi,\phi)_{L^2(\mathbb{R}^d)}(f,g)_{L^2(\mathbb{R}^d)}, \tag{14}$$

when $f,g,\phi,\psi\in\mathscr{S}(\mathbb{R}^d)$. The identity (14) is obtained by rewriting the short-time Fourier transforms by $(2)'$ and then applying the Parseval identity in suitable ways. We observe that the right-hand side makes sense also when f, g, ϕ and ψ belong to other spaces than $\mathscr{S}(\mathbb{R}^d)$. For example we may let

$$\begin{aligned} &(f,g,\phi,\psi)\in\mathscr{S}'(\mathbb{R}^d)\times\mathscr{S}(\mathbb{R}^d)\times\mathscr{S}(\mathbb{R}^d)\times\mathscr{S}'(\mathbb{R}^d),\\ &(f,g,\phi,\psi)\in\mathscr{S}(\mathbb{R}^d)\times\mathscr{S}'(\mathbb{R}^d)\times\mathscr{S}'(\mathbb{R}^d)\times\mathscr{S}(\mathbb{R}^d),\\ &(f,g,\phi,\psi)\in\mathscr{S}'(\mathbb{R}^d)\times\mathscr{S}(\mathbb{R}^d)\times L^q(\mathbb{R}^d)\times L^{q'}(\mathbb{R}^d)\\ \text{or}\quad &(f,g,\phi,\psi)\in L^p(\mathbb{R}^d)\times L^{p'}(\mathbb{R}^d)\times L^q(\mathbb{R}^d)\times L^{q'}(\mathbb{R}^d), \end{aligned} \tag{15}$$

when $p,p',q,q'\in[1,\infty]$ satisfy

$$\frac{1}{p}+\frac{1}{p'}=\frac{1}{q}+\frac{1}{q'}=1.$$

By Moyal's identity (14) it follows that if $\phi \in \mathscr{S}(\mathbb{R}^d) \setminus \{0\}$, then the identity operator on $\mathscr{S}'(\mathbb{R}^d)$ is given by

$$\mathrm{Id} = \left(\|\phi\|_{L^2}^{-2}\right) \cdot V_\phi^* \circ V_\phi, \tag{16}$$

provided suitable mapping properties of the (L^2-)adjoint V_ϕ^* of V_ϕ can be established. Obviously, V_ϕ^* fullfils

$$(V_\phi^* F, g)_{L^2(\mathbb{R}^d)} = (F, V_\phi g)_{L^2(\mathbb{R}^{2d})} \tag{17}$$

when $F \in \mathscr{S}(\mathbb{R}^{2d})$ and $g \in \mathscr{S}(\mathbb{R}^d)$.

By expressing the scalar product and the short-time Fourier transform in terms of integrals in (17), it follows by straight-forward manipulations that the adjoint in (17) is given by

$$(V_\phi^* F)(x) = (2\pi)^{-\frac{d}{2}} \iint_{\mathbb{R}^{2d}} F(y,\eta)\phi(x-y)e^{i\langle x,\eta\rangle}\, dy d\eta, \tag{18}$$

when $F \in \mathscr{S}(\mathbb{R}^{2d})$. We may now use mapping properties like (11)–(12) to extend the definition of $V_\phi^* F$ when F and ϕ belong to various classes of function and distribution spaces. For example, by (11), (10) and (12), it follows that the map

$$(F, g) \mapsto (F, V_\phi g)_{L^2(\mathbb{R}^{2d})}$$

defines a sesqui-linear form on $\mathscr{S}(\mathbb{R}^{2d}) \times \mathscr{S}'(\mathbb{R}^d)$, $\mathscr{S}'(\mathbb{R}^{2d}) \times \mathscr{S}(\mathbb{R}^d)$ and on $L^2(\mathbb{R}^{2d}) \times L^2(\mathbb{R}^d)$. This implies that if $\phi \in \mathscr{S}(\mathbb{R}^d)$, then V_ϕ^* in (17) is continuous from $\mathscr{S}(\mathbb{R}^{2d})$ to $\mathscr{S}(\mathbb{R}^d)$ which is uniquely extendable to a continuous map $\mathscr{S}'(\mathbb{R}^{2d})$ to $\mathscr{S}'(\mathbb{R}^d)$, and to $L^2(\mathbb{R}^{2d})$ to $L^2(\mathbb{R}^d)$. That is, the mappings

$$\begin{aligned} V_\phi^* : \mathscr{S}(\mathbb{R}^{2d}) \to \mathscr{S}(\mathbb{R}^d), \qquad & V_\phi^* : \mathscr{S}'(\mathbb{R}^{2d}) \to \mathscr{S}'(\mathbb{R}^d) \\ \text{and} \qquad & V_\phi^* : L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^d) \end{aligned} \tag{19}$$

are continuous.

2.2 STFT Projections and a Suitable Twisted Convolution

If $\phi \in \mathscr{S}(\mathbb{R}^d)$ satisfies $\|\phi\|_{L^2} = 1$, then (16) shows that $V_\phi^* \circ V_\phi$ is the identity operator on $\mathscr{S}'(\mathbb{R}^d)$. If we swap the order of this composition we get certain types of projections. In fact, for any $\phi \in \mathscr{S}(\mathbb{R}^d) \setminus \{0\}$, let P_ϕ be the operator given by

$$P_\phi \equiv \|\phi\|_{L^2}^{-2} \cdot V_\phi \circ V_\phi^*. \tag{20}$$

We observe that P_ϕ is continuous on $\mathscr{S}(\mathbb{R}^{2d})$, $L^2(\mathbb{R}^{2d})$ and $\mathscr{S}'(\mathbb{R}^{2d})$ due to the mapping properties for V_ϕ and V_ϕ^* above.

It is clear that $P_\phi^* = P_\phi$, i.e. P_ϕ is self-adjoint. Furthermore, P_ϕ is an projection:

$$P_\phi^2 = \|\phi\|_{L^2}^{-2} \cdot V_\phi \circ \underbrace{\left(\|\phi\|_{L^2}^{-2} \cdot V_\phi^* \circ V_\phi\right)}_{\text{The identity operator}} \circ V_\phi^* = \|\phi\|_{L^2}^{-2} \cdot V_\phi \circ V_\phi^* = P_\phi.$$

Hence,

$$P_\phi^* = P_\phi \quad \text{and} \quad P_\phi^2 = P_\phi, \tag{21}$$

which shows that P_ϕ is an orthonormal projection.

The ranks of P_ϕ are given by

$$\begin{aligned} P_\phi(\mathscr{S}(\mathbb{R}^{2d})) = V_\phi(\mathscr{S}(\mathbb{R}^d)), \qquad & P_\phi(L^2(\mathbb{R}^{2d})) = V_\phi(L^2(\mathbb{R}^d)), \\ \text{and} \qquad & P_\phi(\mathscr{S}'(\mathbb{R}^{2d})) = V_\phi(\mathscr{S}'(\mathbb{R}^d)). \end{aligned} \tag{22}$$

In fact, if $F \in \mathscr{S}'(\mathbb{R}^{2d})$, then

$$P_\phi F = V_\phi f,$$

where $f = \|\phi\|_{L^2}^{-2} V_\phi^* F \in \mathscr{S}'(\mathbb{R}^d)$. This shows that $P_\phi(\mathscr{S}'(\mathbb{R}^{2d})) \subseteq V_\phi(\mathscr{S}'(\mathbb{R}^d))$. On the other hand, if $f \in \mathscr{S}'(\mathbb{R}^d)$ and $F = V_\phi f$, then

$$P_\phi F = \left(V_\phi \circ \left(\|\phi\|_{L^2}^{-2} \cdot V_\phi^* \circ V_\phi\right)\right) f = V_\phi f,$$

which shows that any element in $V_\phi(\mathscr{S}'(\mathbb{R}^d))$ equals an element in $P_\phi(\mathscr{S}'(\mathbb{R}^{2d}))$, i.e. $P_\phi(\mathscr{S}'(\mathbb{R}^{2d})) = V_\phi(\mathscr{S}'(\mathbb{R}^d))$. This gives the last identity in (22). In the same way, the first two identities are obtained.

Remark 2.2 Let $F \in \mathscr{S}'(\mathbb{R}^{2d})$. Then it follows from the last identity in (22) that $F = V_\phi f$ for some $f \in \mathscr{S}'(\mathbb{R}^d)$, if and only if

$$F = P_\phi F. \tag{23}$$

Furthermore, if (23) holds, then $F = V_\phi f$ with

$$f = (\|\phi\|_{L^2}^{-2}) \cdot V_\phi^* F. \tag{24}$$

There is a twisted convolution which is linked to the projection in (20). In fact, if $F \in \mathscr{S}(\mathbb{R}^{2d})$ and $\phi \in \mathscr{S}(\mathbb{R}^d) \setminus \{0\}$, then it follows by expanding the integrals for V_ϕ and V_ϕ^* in (20), and performing some straight-forward manipulations that

$$P_\phi F = \|\phi\|_{L^2}^{-2} \cdot V_\phi \phi *_V F, \qquad F \in \mathscr{S}'(\mathbb{R}^{2d}), \tag{25}$$

where the *twisted convolution* $*_V$ is defined by

$$\begin{aligned}(F *_V G)(x,\xi) &= (2\pi)^{-\frac{d}{2}} \iint_{\mathbb{R}^{2d}} F(x-y,\xi-\eta)G(y,\eta)e^{-i\langle y,\xi-\eta\rangle}\, dyd\eta.\\ &= (2\pi)^{-\frac{d}{2}} \iint_{\mathbb{R}^{2d}} F(y,\eta)G(x-y,\xi-\eta)e^{-i\langle x-y,\eta\rangle}\, dyd\eta,\end{aligned} \tag{26}$$

when $F, G \in \mathscr{S}(\mathbb{R}^{2d})$. We observe that the definition of $*_V$ is uniquely extendable in different ways. For example, Young's inequality for ordinary convolution also holds for the twisted convolution. Moreover, the map $(F, G) \mapsto F *_V G$ extends uniquely to continuous mappings from $\mathscr{S}(\mathbb{R}^{2d}) \times \mathscr{S}'(\mathbb{R}^{2d})$ or $\mathscr{S}'(\mathbb{R}^{2d}) \times \mathscr{S}(\mathbb{R}^{2d})$ to $\mathscr{S}'(\mathbb{R}^{2d})$. By straight-forward computations it follows that

$$(F *_V G) *_V H = F *_V (G *_V H), \tag{27}$$

when $F, H \in \mathscr{S}(\mathbb{R}^{2d})$ and $G \in \mathscr{S}'(\mathbb{R}^{2d})$, or $F, H \in \mathscr{S}'(\mathbb{R}^{2d})$ and $G \in \mathscr{S}(\mathbb{R}^{2d})$.

Let $f \in \mathscr{S}'(\mathbb{R}^d)$ and $\phi_j \in \mathscr{S}(\mathbb{R}^d)$, $j = 1, 2, 3$. By straight-forward applications of Parseval's formula it follows that

$$\big((V_{\phi_2}\phi_3) *_V (V_{\phi_1} f)\big)(x,\xi) = (\phi_3, \phi_1)_{L^2} \cdot (V_{\phi_2} f)(x,\xi), \tag{28}$$

which is some sort of reproducing kernel of short-time Fourier transforms in the background of $*_V$.

2.3 Gelfand-Shilov Spaces

Before defining the Gelfand-Shilov spaces, we recall that the Schwartz space $\mathscr{S}(\mathbb{R}^d)$ consists of all (complex-valued) smooth functions $f \in C^\infty(\mathbb{R}^d)$ such that

$$\sup_{x\in\mathbb{R}^d} \big(|x^\beta \partial^\alpha f(x)|\big) \le C_{\alpha,\beta}, \tag{29}$$

for some constants $C_{\alpha,\beta} > 0$, which only depend on the multi-indices $\alpha, \beta \in \mathbb{N}^d$. The Schwartz space possess several convenient properties, and is heavily used in mathematics, science and technology. For example, the Schwartz space is invariant under Fourier transformation. By duality the same holds true for its (L^2-)dual $\mathscr{S}'(\mathbb{R}^d)$, the set of tempered distributions on $\mathbb{R}^d$.

On the other hand, we observe that there are no conditions on the growths of the constants $C_{\alpha,\beta}$ with respect to $\alpha, \beta \in \mathbb{N}^d$. This implies that in the context of the spaces $\mathscr{S}(\mathbb{R}^d)$ and $\mathscr{S}'(\mathbb{R}^d)$, it is almost impossible to investigate important properties like analyticity or related regularity properties which are stronger than pure smoothness. For investigating such stronger regularity properties, we need to modify $\mathscr{S}(\mathbb{R}^d)$ and the estimate (29) by imposing suitable growth conditions on the constants $C_{\alpha,\beta}$. This leads to the definition of Gelfand-Shilov spaces, [26, 40].

We only discuss Fourier invariant Gelfand-Shilov spaces and their properties. Let $0 < s \in \mathbb{R}$ be fixed. We have two different types of Gelfand-Shilov spaces. The Gelfand-Shilov space $\mathcal{S}_s(\mathbb{R}^d)$ of Roumieu type with parameter $s > 0$ consists of all $f \in C^\infty(\mathbb{R}^d)$ such that

$$\sup_{x\in\mathbb{R}^d} \left(|x^\beta \partial^\alpha f(x)|\right) \le C h^{|\alpha+\beta|}(\alpha!\beta!)^s, \tag{30}$$

for some constants $C, h > 0$. In the same way, the Gelfand-Shilov space $\Sigma_s(\mathbb{R}^d)$ of Beurling type with parameter $s > 0$ consists of all $f \in C^\infty(\mathbb{R}^d)$ such that for every $h > 0$, there is a constant $C = C_h > 0$ such that (30) holds. Hence, in comparison with the definition of Schwartz functions, we have limited ourself to constants $C_{\alpha,\beta}$ in (29) which are not allowed to grow faster than those of the form

$$C h^{|\alpha+\beta|}(\alpha!\beta!)^s$$

when dealing with Gelfand-Shilov spaces.

It can be proved that $\mathcal{S}_s(\mathbb{R}^d)$ and $\Sigma_t(\mathbb{R}^d)$ are dense in $\mathscr{S}(\mathbb{R}^d)$ when $s \ge \frac{1}{2}$ and $t > \frac{1}{2}$. We call such s and t admissible. On the other hand, for the other choices of s and t we have

$$\mathcal{S}_s(\mathbb{R}^d) = \Sigma_t(\mathbb{R}^d) = \{0\}, \quad \text{when} \quad s < \frac{1}{2},\ t \le \frac{1}{2}.$$

One has that $\mathcal{S}_1(\mathbb{R}^d)$ consists of real analytic functions, and that $\Sigma_1(\mathbb{R}^d)$ consists of smooth functions on $\mathbb{R}^d$ which are extendable to entire functions on $\mathbb{C}^d$. The topologies of $\mathcal{S}_s(\mathbb{R}^d)$ and $\Sigma_s(\mathbb{R}^d)$ are defined by the semi-norms

$$\|f\|_{\mathcal{S}_{s,h}} \equiv \sup \frac{|x^\beta \partial^\alpha f(x)|}{h^{|\alpha+\beta|}(\alpha!\beta!)^s}. \tag{31}$$

Here the supremum should be taken over all $\alpha, \beta \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$. We equip $\mathcal{S}_s(\mathbb{R}^d)$ and $\Sigma_s(\mathbb{R}^d)$ by the canonical inductive limit topology and projective limit topology, respectively, with respect to $h > 0$, which are induced by the semi-norms in (31).

Let $\mathcal{S}_{s,h}(\mathbb{R}^d)$ be the Banach space which consists of all $f \in C^\infty(\mathbb{R}^d)$ such that $\|f\|_{\mathcal{S}_{s,h}}$ in (31) is finite, and let $\mathcal{S}'_{s,h}(\mathbb{R}^d)$ be the (L^2-)dual of $\mathcal{S}_{s,h}(\mathbb{R}^d)$. If $s \ge \frac{1}{2}$, then the *Gelfand-Shilov distribution space* $\mathcal{S}'_s(\mathbb{R}^d)$ of *Roumieu type* is the projective limit of $\mathcal{S}'_{s,h}(\mathbb{R}^d)$ with respect to $h > 0$. If instead $s > \frac{1}{2}$, then the *Gelfand-Shilov distribution space* $\Sigma'_s(\mathbb{R}^d)$ of *Beurling type* is the inductive limit of $\mathcal{S}'_{s,h}(\mathbb{R}^d)$ with respect to $h > 0$. Consequently, for admissible s we have

$$\mathcal{S}'_s(\mathbb{R}^d) = \bigcap_{h>0} \mathcal{S}'_{s,h}(\mathbb{R}^d) \quad \text{and} \quad \Sigma'_s(\mathbb{R}^d) = \bigcup_{h>0} \mathcal{S}'_{s,h}(\mathbb{R}^d).$$

It can be proved that $\mathcal{S}'_s(\mathbb{R}^d)$ and $\Sigma'_s(\mathbb{R}^d)$ are the (strong) duals to $\mathcal{S}_s(\mathbb{R}^d)$ and $\Sigma_s(\mathbb{R}^d)$, respectively.

We have the following embeddings and density properties for Gelfand-Shilov and Schwartz spaces

$$\begin{aligned} &\mathcal{S}_s(\mathbb{R}^d) \hookrightarrow \Sigma_t(\mathbb{R}^d) \hookrightarrow \mathcal{S}_t(\mathbb{R}^d) \hookrightarrow \mathscr{S}(\mathbb{R}^d), \\ &\mathscr{S}'(\mathbb{R}^d) \hookrightarrow \mathcal{S}'_t(\mathbb{R}^d) \hookrightarrow \Sigma'_t(\mathbb{R}^d) \hookrightarrow \mathcal{S}'_s(\mathbb{R}^d), \qquad t > s \ge \frac{1}{2}, \end{aligned} \tag{32}$$

with dense embeddings. Here $A \hookrightarrow B$ means that the topological spaces A and B satisfy $A \subseteq B$ with continuous embeddings.

The Fourier transform possess convenient mapping properties on Gelfand-Shilov spaces and their distribution spaces. In fact, the Fourier transform extends uniquely to homeomorphisms on $\mathcal{S}'_s(\mathbb{R}^d)$ and on $\Sigma'_s(\mathbb{R}^d)$ for admissible s. Furthermore, $\mathscr{F}$ restricts to homeomorphisms on $\mathcal{S}_s(\mathbb{R}^d)$ and on $\Sigma_s(\mathbb{R}^d)$.

One of the most important characterizations of Gelfand-Shilov spaces is performed in terms of estimates of the functions and their Fourier transforms. More precisely, in [8, 15] it is proved that if $f \in \mathscr{S}'(\mathbb{R}^d)$ and $s > 0$, then $f \in \mathcal{S}_s(\mathbb{R}^d)$ ($f \in \Sigma_s(\mathbb{R}^d)$), if and only if

$$|f(x)| \lesssim e^{-r|x|^{\frac{1}{s}}} \quad \text{and} \quad |\widehat{f}(\xi)| \lesssim e^{-r|\xi|^{\frac{1}{s}}}, \tag{33}$$

for some $r > 0$ (for every $r > 0$). Here $g_1 \lesssim g_2$ means that $g_1(\theta) \le c \cdot g_2(\theta)$ holds uniformly for all θ in the intersection of the domains of g_1 and g_2 and for some constant $c > 0$, and we write $g_1 \asymp g_2$ when $g_1 \lesssim g_2 \lesssim g_1$.

The analysis in [8, 15] can also be applied on the Schwartz space, from which it follows that an element $f \in \mathscr{S}'(\mathbb{R}^d)$ belongs to $\mathscr{S}(\mathbb{R}^d)$, if and only if

$$|f(x)| \lesssim \langle x \rangle^{-N} \quad \text{and} \quad |\widehat{f}(\xi)| \lesssim \langle \xi \rangle^{-N}, \tag{34}$$

for every $N \ge 0$. Here and in what follows we let

$$\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}.$$

Remark 2.3 Several properties in Sects. 2.1–2.3 in the background of $\mathscr{S}(\mathbb{R}^d)$ and $\mathscr{S}'(\mathbb{R}^d)$ also hold for the Gelfand-Shilov spaces and their distribution spaces. Let $s \geq \frac{1}{2}$. By similar arguments which lead to Proposition 2.1 and (13), it follows that

$$(f,\phi) \mapsto V_\phi f : \mathcal{S}_s(\mathbb{R}^d) \times \mathcal{S}_s(\mathbb{R}^d) \to \mathcal{S}_s(\mathbb{R}^{2d}) \tag{35}$$

is continuous, which extends uniquely to continuous mappings

$$(f,\phi) \mapsto V_\phi f : \mathcal{S}_s'(\mathbb{R}^d) \times \mathcal{S}_s(\mathbb{R}^d) \to \mathcal{S}_s'(\mathbb{R}^{2d}) \cap C^\infty(\mathbb{R}^{2d}) \tag{36}$$

and

$$(f,\phi) \mapsto V_\phi f : \mathcal{S}_s'(\mathbb{R}^d) \times \mathcal{S}_s'(\mathbb{R}^d) \to \mathcal{S}_s'(\mathbb{R}^{2d}). \tag{37}$$

It follows that (14) makes sense after each $\mathscr{S}$ in (15) are replaced by $\mathcal{S}_s$. Let $\phi \in \mathcal{S}_s(\mathbb{R}^d) \setminus \{0\}$ be fixed. Then by similar arguments which lead to (19) give that the mappings

$$V_\phi^* : \mathcal{S}_s(\mathbb{R}^{2d}) \to \mathcal{S}_s(\mathbb{R}^d), \qquad V_\phi^* : \mathcal{S}_s'(\mathbb{R}^{2d}) \to \mathcal{S}_s'(\mathbb{R}^d) \tag{19$'$}$$

are continuous. For P_ϕ in (20) we have that (21) still holds true and that (22) can be completed with

$$P_\phi(\mathcal{S}_s(\mathbb{R}^{2d})) = V_\phi(\mathcal{S}_s(\mathbb{R}^d)) \quad \text{and} \quad P_\phi(\mathcal{S}_s'(\mathbb{R}^{2d})) = V_\phi(\mathcal{S}_s'(\mathbb{R}^d)). \tag{38}$$

We also have that the twisted convolution in (26) is continuous from $\mathcal{S}_s(\mathbb{R}^{2d}) \times \mathcal{S}_s(\mathbb{R}^{2d})$ to $\mathcal{S}_s(\mathbb{R}^{2d})$ and uniquely extendable to a continuous map $\mathcal{S}_s(\mathbb{R}^{2d}) \times \mathcal{S}_s'(\mathbb{R}^{2d})$ or $\mathcal{S}_s'(\mathbb{R}^{2d}) \times \mathcal{S}_s(\mathbb{R}^{2d})$ to $\mathcal{S}_s'(\mathbb{R}^{2d})$, and that the formulae (25)–(28) still hold true after each $\mathscr{S}$ is replaced by $\mathcal{S}_s$ in the attached assumptions.

If instead $s > \frac{1}{2}$, then similar facts hold true with Σ_s in place of $\mathcal{S}_s$ above, at each occurrence.

Remark 2.4 In similar ways as characterizing Gelfand-Shilov spaces in terms of Fourier estimates (see (33)), we may also use the short-time Fourier transform to perform similar characterizations. Moreover, the short-time Fourier transform can in addition be used to characterize spaces of Gelfand-Shilov distributions.

In fact, let $\phi \in \mathcal{S}_s(\mathbb{R}^d) \setminus \{0\}$ ($\phi \in \Sigma_s(\mathbb{R}^d) \setminus \{0\}$) be fixed and let f be a Gelfand-Shilov distribution on $\mathbb{R}^d$. Then the following is true:

1. $f \in \mathcal{S}_s(\mathbb{R}^d)$ ($f \in \Sigma_s(\mathbb{R}^d)$), if and only if

$$|V_\phi f(x,\xi)| \lesssim e^{-r(|x|^{\frac{1}{s}} + |\xi|^{\frac{1}{s}})} \tag{39}$$

   for some $r > 0$ (for every $r > 0$);

2. $f \in \mathcal{S}_s'(\mathbb{R}^d)$ ($f \in \Sigma_s'(\mathbb{R}^d)$), if and only if

$$|V_\phi f(x,\xi)| \lesssim e^{r(|x|^{\frac{1}{s}}+|\xi|^{\frac{1}{s}})} \tag{40}$$

for every $r > 0$ (for some $r > 0$).

We refer to [31, Theorem 2.7] for the characterization 1. concerning Gelfand-Shilov functions and to [51, Proposition 2.2]) for the characterization 2. concerning Gelfand-Shilov distributions.

2.4 Weight Functions

A *weight* or *weight function* on $\mathbb{R}^d$ is a positive function $\omega \in L^\infty_{loc}(\mathbb{R}^d)$ such that $1/\omega \in L^\infty_{loc}(\mathbb{R}^d)$. The weight ω is called *moderate*, if there is a positive weight v on $\mathbb{R}^d$ and a constant $C \geq 1$ such that

$$\omega(x+y) \leq C\omega(x)v(y), \qquad x, y \in \mathbb{R}^d. \tag{41}$$

If ω and v are weights on $\mathbb{R}^d$ such that (41) holds, then ω is also called v-*moderate*. We note that (41) implies that ω fulfills the estimates

$$C^{-1}v(-x)^{-1} \leq \omega(x) \leq Cv(x), \quad x \in \mathbb{R}^d. \tag{42}$$

We let $\mathscr{P}_E(\mathbb{R}^d)$ be the set of all moderate weights on $\mathbb{R}^d$.

We say that v is *submultiplicative* if

$$v(x+y) \leq v(x)v(y) \quad \text{and} \quad v(-x) = v(x), \qquad x, y \in \mathbb{R}^d. \tag{43}$$

We observe that if $v \in \mathscr{P}_E(\mathbb{R}^d)$ is even and satisfies

$$v(x+y) \leq Cv(x)v(y), \qquad x, y \in \mathbb{R}^d, \tag{44}$$

for some constant $C > 0$, then for $v_0 = C^{1/2}v$, one has that $v_0 \in \mathscr{P}_E(\mathbb{R}^d)$ is submultiplicative and $v \asymp v_0$ (see e.g. [17, 19, 28]).

We also recall from [29] that if v is positive and locally bounded and satisfies (44), then $v(x) \leq C_0 e^{r_0|x|}$ for some positive constants C_0 and r_0. In fact, if $x \in \mathbb{R}^d$,

$$r = \sup_{|x|\leq 1} \log v(x), \quad c = \log C$$

and n is an integer such that $n-1 \le |x| \le n$, then (44) gives

$$v(x) = v(n \cdot (x/n)) \le C^n v(x/n)^n \le C^n e^{rn} = e^{(r+c)n} \le e^{(r+c)(|x|+1)},$$

which gives the statement.

Therefore, if v is a submultiplicative weight, then

$$v(x) \lesssim e^{r|x|}, \qquad x \in \mathbb{R}^d, \tag{45}$$

for some $r \ge 0$. Hence, if $\omega \in \mathscr{P}_E(\mathbb{R}^d)$, then (41) and (45) imply

$$\omega(x+y) \lesssim \omega(x) e^{r|y|}, \quad x, y \in \mathbb{R}^d \tag{46}$$

for some $r > 0$. In particular, (42) shows that for any $\omega_0 \in \mathscr{P}_E(\mathbb{R}^d)$, there is a constant $r > 0$ such that

$$e^{-r|x|} \lesssim \omega_0(x) \lesssim e^{r|x|}, \quad x \in \mathbb{R}^d.$$

If (41) holds, then there is a smallest positive even function v_0 such that (41) holds with $C = 1$. We remark that this v_0 is given by

$$v_0(x) = \sup_{y \in \mathbb{R}^d} \left(\frac{\omega(x+y)}{\omega(y)}, \frac{\omega(-x+y)}{\omega(y)} \right),$$

and is submultiplicative (see e.g. [19, 27, 49]). Consequently, if ω is a moderate weight, then it is also moderated by a submultiplicative weight. In the sequel, v and v_j for $j \ge 0$, always stand for submultiplicative weights if nothing else is stated.

We also remark that in the literature it is common to define submultiplicative weights as (43) should hold, without the condition $v(-x) = v(x)$, i.e. that v does not have to be even (cf. e.g. [17, 19, 25, 28]). However, in the sequel it is convenient for us to include this property in the definition.

There are several subclasses of $\mathscr{P}_E(\mathbb{R}^d)$ which are interesting for different reasons. Though our results later on are formulated in background of weights in $\mathscr{P}_E(\mathbb{R}^d)$, we here mention some subclasses which especially appear in time-frequency analysis. First we observe the class $\mathscr{P}_E^0(\mathbb{R}^d)$, which consists of all $\omega \in \mathscr{P}_E(\mathbb{R}^d)$ such that (46) holds for every $r > 0$.

The class $\mathscr{P}_E^0(\mathbb{R}^d)$ is important when dealing with spectral invariance for matrix or convolution operators on $\ell^2(\mathbb{Z}^d)$ (see e.g. [30]). If $v \in \mathscr{P}_E(\mathbb{R}^d)$ is submultiplicative, then $v \in \mathscr{P}_E^0(\mathbb{R}^d)$, if and only if

$$\lim_{n \to \infty} v(nx)^{\frac{1}{n}} = 1 \tag{47}$$

(see e.g. [23]). The condition (47) is equivalent to

$$\lim_{n\to\infty} \frac{\log(v(nx))}{n} = 0, \tag{47$'$}$$

and is usually called the *GRS condition*, or *Gelfand-Raikov-Shilov condition*.

A more restrictive condition on v compared to (47)$'$ is given by the Beurling-Domar condition

$$\sum_{n=1}^{\infty} \frac{\log(v(nx))}{n^2} < \infty. \tag{48}$$

This condition is strongly linked to non quasi-analytic classes which contain non-trivial compactly supported elements (see e.g. [29]). Any subexponential submultiplicative weight satisfies the Beurling-Domar condition. That is, suppose that $\theta \in (0,1)$ and that $v(x) = e^{r|x|^{\theta}}$, $x \in \mathbb{R}^d$, then (48) is fulfilled. We let $\mathscr{P}_{\rm BD}(\mathbb{R}^d)$ be the set of all weights which are moderated by submultiplicative weights which satisfy the Beurling-Domar condition.

Finally we let $\mathscr{P}(\mathbb{R}^d)$ be the set of all weights on $\mathbb{R}^d$ which are moderated by polynomially bounded functions. That is, $\omega \in \mathscr{P}(\mathbb{R}^d)$, if and only if there are positive constants r and C such that

$$\omega(x+y) \le C\omega(x)(1+|y|)^r, \qquad x,y \in \mathbb{R}^d.$$

Here we observe that $v(x) = (1+|x|)^r$ is submultiplicative.

Among these weight classes we have

$$\mathscr{P}(\mathbb{R}^d) \subsetneq \mathscr{P}_{\rm BD}(\mathbb{R}^d) \subsetneq \mathscr{P}_E^0(\mathbb{R}^d) \subsetneq \mathscr{P}_E(\mathbb{R}^d). \tag{49}$$

In fact, it is clear that the ordering in (49) holds. On the other hand, if $r > 0$ and $\theta \in (0,1)$, then due to

$$\begin{aligned} e^{r|x|^{\theta}} &\in \mathscr{P}_{\rm BD}(\mathbb{R}^d) \setminus \mathscr{P}(\mathbb{R}^d), \\ e^{r|x|/\log(e+|x|)} &\in \mathscr{P}_E^0(\mathbb{R}^d) \setminus \mathscr{P}_{\rm BD}(\mathbb{R}^d), \\ \text{and} \quad e^{r|x|} &\in \mathscr{P}_E(\mathbb{R}^d) \setminus \mathscr{P}_E^0(\mathbb{R}^d), \end{aligned} \tag{50}$$

it also follows that the inclusions in (49) are strict.

We refer to [16, 28, 29, 49] for more facts about weights in time-frequency analysis.

2.5 Mixed Norm Spaces of Lebesgue Type

For every $p, q \in (0, \infty]$ and weight ω on $\mathbb{R}^{2d}$, we set

$$\|F\|_{L^{p,q}_{(\omega)}(\mathbb{R}^{2d})} \equiv \|G_{F,\omega,p}\|_{L^q(\mathbb{R}^d)}, \quad \text{where} \quad G_{F,\omega,p}(\xi) = \|F(\,\cdot\,,\xi)\omega(\,\cdot\,,\xi)\|_{L^p(\mathbb{R}^d)}$$

and

$$\|F\|_{L^{p,q}_{*,(\omega)}(\mathbb{R}^{2d})} \equiv \|H_{F,\omega,q}\|_{L^p(\mathbb{R}^d)}, \quad \text{where} \quad H_{F,\omega,q}(x) = \|F(x,\,\cdot\,)\omega(x,\,\cdot\,)\|_{L^q(\mathbb{R}^d)},$$

when F is (complex-valued) measurable function on $\mathbb{R}^{2d}$. Then $L^{p,q}_{(\omega)}(\mathbb{R}^{2d})$ ($L^{p,q}_{*,(\omega)}(\mathbb{R}^{2d})$) consists of all measurable functions F such that $\|F\|_{L^{p,q}_{(\omega)}} < \infty$ ($\|F\|_{L^{p,q}_{*,(\omega)}} < \infty$).

In similar ways, let Ω_1, Ω_2 be discrete sets, ω be a positive function on $\Omega_1 \times \Omega_2$ and $\ell'_0(\Omega_1 \times \Omega_2)$ be the set of all formal (complex-valued) sequences $c = \{c(j,k)\}_{j\in\Omega_1, k\in\Omega_2}$. Then the discrete Lebesgue spaces, i.e. the Lebesgue sequence spaces

$$\ell^{p,q}_{(\omega)}(\Omega_1 \times \Omega_2) \quad \text{and} \quad \ell^{p,q}_{*,(\omega)}(\Omega_1 \times \Omega_2)$$

of mixed (quasi-)norm types consist of all $c \in \ell'_0(\Omega_1 \times \Omega_2)$ such that $\|c\|_{\ell^{p,q}_{(\omega)}(\Omega_1\times\Omega_2)} < \infty$ respectively $\|c\|_{\ell^{p,q}_{*,(\omega)}(\Omega_1\times\Omega_2)} < \infty$. Here

$$\|c\|_{\ell^{p,q}_{(\omega)}(\Omega_1\times\Omega_2)} \equiv \|G_{c,\omega,p}\|_{\ell^q(\Omega_2)}, \quad \text{where} \quad G_{c,\omega,p}(k) = \|F(\,\cdot\,,k)\omega(\,\cdot\,,k)\|_{\ell^p(\Omega_1)}$$

and

$$\|c\|_{\ell^{p,q}_{*,(\omega)}(\Omega_1\times\Omega_2)} \equiv \|H_{c,\omega,q}\|_{\ell^p(\Omega_1)}, \quad \text{where} \quad H_{c,\omega,q}(j) = \|c(j,\,\cdot\,)\omega(j,\,\cdot\,)\|_{\ell^q(\Omega_2)},$$

when $c \in \ell'_0(\Omega_1 \times \Omega_2)$.

2.6 Convolutions and Multiplications for Discrete Lebesgue Spaces

Next we discuss extended Hölder and Young relations for multiplications and convolutions on discrete Lebesgue spaces. The Hölder and Young conditions on Lebesgue exponent are then

$$\frac{1}{q_0} \le \frac{1}{q_1} + \frac{1}{q_2}, \tag{51}$$

respectively

$$\frac{1}{p_0} \le \frac{1}{p_1} + \frac{1}{p_2} - \max\left(1, \frac{1}{p_1}, \frac{1}{p_2}\right). \tag{52}$$

Notice that, when $p_1, p_2 \in (0,1)$, then (52) becomes $p_0 \ge \max\{p_1, p_2\}$, while for $p_1, p_2 \ge 1$ it reduces to the common Young condition

$$1 + \frac{1}{p_0} \le \frac{1}{p_1} + \frac{1}{p_2}.$$

The conditions on the weight functions are

$$\omega_0(j) \le \omega_1(j)\omega_2(j), \qquad j \in \Lambda, \tag{53}$$

respectively

$$\omega_0(j_1 + j_2) \le \omega_1(j_1)\omega_2(j_2), \qquad j_1, j_2 \in \Lambda, \tag{54}$$

where Λ is a lattice of the form

$$\Lambda = \{ n_1 e_1 + \cdots + n_d e_d \, ; \, (n_1, \ldots, n_d) \in \mathbb{Z}^d \},$$

where $e_1, \ldots e_d$ is a basis for $\mathbb{R}^d$.

Proposition 2.5 *Let $p_j, q_j \in (0, \infty]$, $j = 0, 1, 2$, be such that* (51) *and* (52) *hold, let $\Lambda \subseteq \mathbb{R}^d$ be a lattice and let ω_j be weights on Λ, $j = 0, 1, 2$. Then the following is true:*

1. *if* (53) *holds, then the map $(a_1, a_2) \mapsto a_1 \cdot a_2$ from $\ell_0(\Lambda) \times \ell_0(\Lambda)$ to $\ell_0(\Lambda)$ extends uniquely to a continuous map from $\ell^{q_1}_{(\omega_1)}(\Lambda) \times \ell^{q_2}_{(\omega_2)}(\Lambda)$ to $\ell^{q_0}_{(\omega_0)}(\Lambda)$, and*

$$\|a_1 \cdot a_2\|_{\ell^{q_0}_{(\omega_0)}} \le \|a_1\|_{\ell^{q_1}_{(\omega_1)}} \|a_2\|_{\ell^{q_2}_{(\omega_2)}}, \qquad a_j \in \ell^{q_j}_{(\omega_j)}(\Lambda),\ j = 1, 2; \tag{55}$$

2. *if* (54) *holds, then the map $(a_1, a_2) \mapsto a_1 * a_2$ from $\ell_0(\Lambda) \times \ell_0(\Lambda)$ to $\ell_0(\Lambda)$ extends uniquely to a continuous map from $\ell^{p_1}_{(\omega_1)}(\Lambda) \times \ell^{p_2}_{(\omega_2)}(\Lambda)$ to $\ell^{p_0}_{(\omega_0)}(\Lambda)$, and*

$$\|a_1 * a_2\|_{\ell^{p_0}_{(\omega_0)}} \le \|a_1\|_{\ell^{p_1}_{(\omega_1)}} \|a_2\|_{\ell^{p_2}_{(\omega_2)}}, \qquad a_j \in \ell^{p_j}_{(\omega_j)}(\Lambda),\ j = 1, 2. \tag{56}$$

The assertion 1. in Proposition 2.5 is the standard Hölder's inequality for discrete Lebesgue spaces. The assertion 2. in that proposition is the usual Young's inequality for Lebesgue spaces on lattices in the case when $p_0, p_1, p_2 \in [1, \infty]$. A proof of Proposition 2.5 is given in Appendix A in [54].

3 Modulation Spaces, Multiplications and Convolutions

In this section we introduce modulation spaces, and recall their basic properties, in particular in the context of Gelfand-Shilov spaces. Notice that we permit the Lebesgue exponents to belong to the full interval $(0, \infty]$ instead of the most common choice $[1, \infty]$, and general moderate weights which may have a (sub)exponential growth. Here we also recall some facts on Gabor expansions for modulation spaces.

Then we deduce multiplication and convolution estimates on modulation spaces. There are several approaches to multiplication and convolution in the case when the involved Lebesgue exponents belong to $[1, \infty]$ (see [9, 17, 19, 32, 43, 48]). Here we consider the case when these exponents belong to $(0, \infty)$ (see also [1, 2, 25, 41, 42, 50]). In addition, and in order to keep the survey style of our exposition, we focus on the bilinear case, and refer to [54] for extension of these results to multi-linear products.

3.1 Modulation Spaces

The (classical) modulation spaces, essentially introduced in [17] by Feichtinger are given in the following. (See e.g. [18] for definition of more general modulation spaces.)

Definition 3.1 Let $p, q \in (0, \infty]$, $\omega \in \mathscr{P}_E(\mathbb{R}^{2d})$ and $\phi \in \Sigma_1(\mathbb{R}^d) \setminus \{0\}$.

1. The *modulation space* $M^{p,q}_{(\omega)}(\mathbb{R}^d)$ consists of all $f \in \Sigma_1'(\mathbb{R}^d)$ such that

$$\|f\|_{M^{p,q}_{(\omega)}} \equiv \|V_\phi f\|_{L^{p,q}_{(\omega)}}$$

is finite. The topology of $M^{p,q}_{(\omega)}(\mathbb{R}^d)$ is defined by the (quasi-)norm $\|\cdot\|_{M^{p,q}_{(\omega)}}$;

2. The *modulation space (of Wiener amalgam type)* $W^{p,q}_{(\omega)}(\mathbb{R}^d)$ consists of all $f \in \Sigma_1'(\mathbb{R}^d)$ such that

$$\|f\|_{W^{p,q}_{(\omega)}} \equiv \|V_\phi f\|_{L^{p,q}_{*,(\omega)}}$$

is finite. The topology of $W^{p,q}_{(\omega)}(\mathbb{R}^d)$ is defined by the (quasi-)norm $\|\cdot\|_{W^{p,q}_{(\omega)}}$.

For convenience we set $M^{p,q} = M^{p,q}_{(\omega)}$ and $W^{p,q} = W^{p,q}_{(\omega)}$ when the weight ω is trivial, i.e. when $\omega(x, \xi) = 1$ for every $x, \xi \in \mathbb{R}^d$. We also set

$$M^p_{(\omega)} \equiv M^{p,p}_{(\omega)} \; (= W^{p,p}_{(\omega)}) \quad \text{and} \quad M^p \equiv M^{p,p} \; (= W^{p,p}).$$

Remark 3.2 Modulation spaces possess several convenient properties. Let $p, q \in (0, \infty]$, $\omega \in \mathscr{P}_E(\mathbb{R}^{2d})$ and $\phi \in \Sigma_1(\mathbb{R}^d) \setminus \{0\}$. Then the following is true (see [17–20, 25, 28] and their analyses for verifications):

- the definitions of $M^{p,q}_{(\omega)}(\mathbb{R}^d)$ and $W^{p,q}_{(\omega)}(\mathbb{R}^d)$ are independent of the choices of $\phi \in \Sigma_1(\mathbb{R}^d) \setminus \{0\}$, and different choices give rise to equivalent quasi-norms;
- the spaces $M^{p,q}_{(\omega)}(\mathbb{R}^d)$ and $W^{p,q}_{(\omega)}(\mathbb{R}^d)$ are quasi-Banach spaces which increase with p and q, and decrease with ω. If in addition $p, q \geq 1$, then they are Banach spaces;
- If $p, q \geq 1$, then the $L^2(\mathbb{R}^d)$ scalar product, $(\,\cdot\,,\,\cdot\,)_{L^2(\mathbb{R}^d)}$, on $\Sigma_1(\mathbb{R}^d) \times \Sigma_1(\mathbb{R}^d)$ is uniquely extendable to dualities between $M^{p,q}_{(\omega)}(\mathbb{R}^d)$ and $M^{p',q'}_{(1/\omega)}(\mathbb{R}^d)$, and between $W^{p,q}_{(\omega)}(\mathbb{R}^d)$ and $W^{p',q'}_{(1/\omega)}(\mathbb{R}^d)$. If in addition $p, q < \infty$, then the dual spaces of $M^{p,q}_{(\omega)}(\mathbb{R}^d)$ and $W^{p,q}_{(\omega)}(\mathbb{R}^d)$ can be identified with $M^{p',q'}_{(1/\omega)}(\mathbb{R}^d)$ respectively $W^{p',q'}_{(1/\omega)}(\mathbb{R}^d)$, through the form $(\,\cdot\,,\,\cdot\,)_{L^2(\mathbb{R}^d)}$;
- if $\omega_0(x, \xi) = \omega(-\xi, x)$, then $\mathscr{F}$ on $\Sigma_1'(\mathbb{R}^d)$ restricts to a homeomorphism from $M^{p,q}_{(\omega)}(\mathbb{R}^d)$ to $W^{q,p}_{(\omega_0)}(\mathbb{R}^d)$.
- The inclusions

$$\Sigma_1(\mathbb{R}^d) \subseteq M^{p,q}_{(\omega)}(\mathbb{R}^d), W^{p,q}_{(\omega)}(\mathbb{R}^d) \subseteq \Sigma_1'(\mathbb{R}^d) \quad \text{when} \quad \omega \in \mathscr{P}_E(\mathbb{R}^{2d}), \tag{57}$$

$$\mathcal{S}_1(\mathbb{R}^d) \subseteq M^{p,q}_{(\omega)}(\mathbb{R}^d), W^{p,q}_{(\omega)}(\mathbb{R}^d) \subseteq \mathcal{S}_1'(\mathbb{R}^d) \quad \text{when} \quad \omega \in \mathscr{P}^0_E(\mathbb{R}^{2d}) \tag{58}$$

and

$$\mathscr{S}(\mathbb{R}^d) \subseteq M^{p,q}_{(\omega)}(\mathbb{R}^d), W^{p,q}_{(\omega)}(\mathbb{R}^d) \subseteq \mathscr{S}'(\mathbb{R}^d) \quad \text{when} \quad \omega \in \mathscr{P}(\mathbb{R}^{2d}) \tag{59}$$

are continuous. If in addition $p, q < \infty$, then these inclusions are dense.

We recall from [49] that the embeddings (57)–(59), are essentially special cases of certain characterizations of the Schwartz space, Gelfand-Shilov spaces and their distribution spaces in terms of suitable unions and intersections of modulation spaces. In fact, let $p, q \in (0, \infty]$ and $s \geq 1$ be fixed and set

$$v_{r,t}(x, \xi) = \begin{cases} e^{r(|x|^{\frac{1}{t}} + |\xi|^{\frac{1}{t}})}, & t \in \mathbb{R}_+ \\ (1 + |x| + |\xi|)^r, & t = \infty, \end{cases} \tag{60}$$

where $r > 0$. Then

$$\Sigma_s(\mathbb{R}^d) = \bigcap_{r>0} M^{p,q}_{(v_{r,s})}(\mathbb{R}^d) = \bigcap_{r>0} W^{p,q}_{(v_{r,s})}(\mathbb{R}^d), \tag{61}$$

$$\mathcal{S}_s(\mathbb{R}^d) = \bigcup_{r>0} M^{p,q}_{(v_{r,s})}(\mathbb{R}^d) = \bigcup_{r>0} W^{p,q}_{(v_{r,s})}(\mathbb{R}^d), \tag{62}$$

$$\mathscr{S}(\mathbb{R}^d) = \bigcap_{r>0} M^{p,q}_{(v_{r,\infty})}(\mathbb{R}^d) \quad = \bigcap_{r>0} W^{p,q}_{(v_{r,\infty})}(\mathbb{R}^d), \tag{63}$$

$$\mathscr{S}'(\mathbb{R}^d) = \bigcup_{r>0} M^{p,q}_{(1/v_{r,\infty})}(\mathbb{R}^d) = \bigcup_{r>0} W^{p,q}_{(1/v_{r,\infty})}(\mathbb{R}^d), \tag{64}$$

$$\mathcal{S}'_s(\mathbb{R}^d) = \bigcap_{r>0} M^{p,q}_{(1/v_{r,s})}(\mathbb{R}^d) \quad = \bigcap_{r>0} W^{p,q}_{(1/v_{r,s})}(\mathbb{R}^d) \tag{65}$$

and

$$\Sigma'_s(\mathbb{R}^d) = \bigcup_{r>0} M^{p,q}_{(1/v_{r,s})}(\mathbb{R}^d) \quad = \bigcup_{r>0} W^{p,q}_{(1/v_{r,s})}(\mathbb{R}^d). \tag{66}$$

The topologies of the spaces on the left-hand sides of (61)–(66) are obtained by replacing each intersection by projective limit with respect to $r > 0$ and each union with inductive limit with respect to $r > 0$.

The relations (61)–(66) are essentially special cases of [49, Theorem 3.9], see also [31, 45, 46]. In order to be self-contained we here give a proof of (62).

Proof of (62) Since

$$M^\infty_{(v_{2r,s})}(\mathbb{R}^d) \subseteq M^{p,q}_{(v_{r,s})}(\mathbb{R}^d),\ W^{p,q}_{(v_{r,s})}(\mathbb{R}^d) \subseteq M^\infty_{(v_{r,s})}(\mathbb{R}^d),$$

it suffices to prove the result for $p = q = \infty$. Let $\phi \in \Sigma_1(\mathbb{R}^d) \setminus \{0\}$ be fixed. First suppose that

$$f \in M^\infty_{(v_{r,s})}(\mathbb{R}^d) = W^\infty_{(v_{r,s})}(\mathbb{R}^d).$$

Then it follows from the definition of modulation space norm that (39) holds for some $r > 0$. By Remark 2.4 it follows that $f \in \mathcal{S}_s(\mathbb{R}^d)$, and we have proved

$$\bigcup_{r>0} M^\infty_{(v_{r,s})}(\mathbb{R}^d) \subseteq \mathcal{S}_s(\mathbb{R}^d). \tag{67}$$

Suppose instead that $f \in \mathcal{S}_s(\mathbb{R}^d)$. Then (39) holds for some $r > 0$, giving that $f \in M^\infty_{(v_{r,s})}(\mathbb{R}^d)$. Hence (67) holds with reversed inclusion, and the result follows. □

Example 3.3 Let $p = q = 1$ and $\omega = 1$. Then $M^{1,1}_{(\omega)}(\mathbb{R}^d) = M^1(\mathbb{R}^d)$ is *the Feichtinger algebra*, probably the most prominent example of a modulation space. We refer to a recent survey [34] for a detailed account on $M^1(\mathbb{R}^d)$, and to [14, Lemma 11] for a list of its basic properties.

Familiar examples arise when $p = q = 2$ and $\omega = 1$. Then $M^{2,2}_{(\omega)}(\mathbb{R}^d) = M^2(\mathbb{R}^d) = L^2(\mathbb{R}^d)$, and

$$M^{2,2}_{(\omega_s)}(\mathbb{R}^d) = H^s(\mathbb{R}^d), \qquad s \in \mathbb{R},$$

where $\omega_s(\xi) = \langle \xi \rangle^s$, and $H^s(\mathbb{R}^d)$ is the Sobolev space (also known as the Bessel potential space) of distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|^2_{H^s} := \int_{\mathbb{R}^d} \langle \xi \rangle^{2s} |\widehat{f}(\xi)|^2 d\xi < \infty,$$

cf. [28, Proposition 11.3.1]. Furthermore, if $v_s(x, \xi) = \langle (x, \xi) \rangle^s$, then $M^{2,2}_{(v_s)}(\mathbb{R}^d) = Q_s(\mathbb{R}^d)$, $s \in \mathbb{R}$, [7, Lemma 2.3]. Here Q_s denotes the Shubin-Sobolev space, [44].

Finally we remark that modulation spaces can be conveniently discretized in terms of Gabor expansions. In order for explaining some basic issues on this, in a similar way as in Subsection 1.5 in [54], we limit ourself to the case when the involved weights are moderated by subexponential functions. That is, we suppose that ω in $M^{p,q}_{(\omega)}(\mathbb{R}^d)$ satisfies

$$\omega(x + y, \xi + \eta) \lesssim \omega(x, \xi) e^{r(|x|^{\frac{1}{s}} + |\xi|^{\frac{1}{s}})}, \tag{68}$$

for some $s > 1$ and $r > 0$. We observe that this implies that

$$\Sigma_s(\mathbb{R}^d) \subseteq M^{p,q}_{(\omega)}(\mathbb{R}^d) \subseteq \Sigma_s'(\mathbb{R}^d), \tag{69}$$

in vew of (42), (61) and (66). For more general approaches we refer to [19, 27, 28, 42, 50].

Since $s > 1$, it follows from Sections 1.3 and 1.4 in [33] that there are $\phi, \psi \in \Sigma_s(\mathbb{R}^d)$ with values in $[0, 1]$ such that

$$\operatorname{supp} \phi \subseteq \Big[-\frac{3}{4}, \frac{3}{4} \Big]^d, \quad \phi(x) = 1 \quad \text{when} \quad x \in \Big[-\frac{1}{4}, \frac{1}{4} \Big]^d \tag{70}$$

$$\operatorname{supp} \psi \subseteq [-1, 1]^d, \qquad \psi(x) = 1 \quad \text{when} \quad x \in \Big[-\frac{3}{4}, \frac{3}{4} \Big]^d \tag{71}$$

and

$$\sum_{j \in \mathbb{Z}^d} \phi(\cdot - j) = 1. \tag{72}$$

Let $f \in \Sigma_s'(\mathbb{R}^d)$. Then $x \mapsto f(x)\phi(x-j)$ belongs to $\Sigma_s'(\mathbb{R}^d)$ and is supported in $j+[-\frac{3}{4},\frac{3}{4}]^d$. Hence, by periodization it follows from Fourier analysis that

$$f(x)\phi(x-j) = \sum_{\iota\in\pi\mathbb{Z}^d} c(j,\iota)e^{i\langle x,\iota\rangle}, \qquad x \in j+[-1,1]^d, \tag{73}$$

where

$$c(j,\iota) = 2^{-d}(f,\phi(\,\cdot\,-j)e^{i\langle\,\cdot\,,\iota\rangle}) = \left(\frac{\pi}{2}\right)^{\frac{d}{2}} V_\phi f(j,\iota), \qquad j \in \mathbb{Z}^d,\ \iota \in \pi\mathbb{Z}^d.$$

Since $\psi = 1$ on the support of ϕ, (73) gives

$$f(x)\phi(x-j) = \left(\frac{\pi}{2}\right)^{\frac{d}{2}} \sum_{\iota\in\pi\mathbb{Z}^d} V_\phi f(j,\iota)\psi(x-j)e^{i\langle x,\iota\rangle}, \qquad x \in \mathbb{R}^d, \tag{73$'$}$$

By (72) it now follows that

$$f(x) = \left(\frac{\pi}{2}\right)^{\frac{d}{2}} \sum_{(j,\iota)\in\Lambda} V_\phi f(j,\iota)\psi(x-j)e^{i\langle x,\iota\rangle}, \qquad x \in \mathbb{R}^d, \tag{74}$$

where

$$\Lambda = \mathbb{Z}^d \times (\pi\mathbb{Z}^d), \tag{75}$$

which is the *Gabor expansion* of f with respect to the *Gabor pair* (ϕ,ψ) and lattice Λ, i.e. with respect to the *Gabor atom* ϕ and the *dual Gabor atom* ψ. Here the series converges in $\Sigma_s'(\mathbb{R}^d)$. By duality and the fact that compactly supported elements in $\Sigma_s(\mathbb{R}^d)$ are dense in $\Sigma_s'(\mathbb{R}^d)$ we also have

$$f(x) = \left(\frac{\pi}{2}\right)^{\frac{d}{2}} \sum_{(j,\iota)\in\Lambda} V_\psi f(j,\iota)\phi(x-j)e^{i\langle x,\iota\rangle}, \qquad x \in \mathbb{R}^d, \tag{76}$$

with convergence in $\Sigma_s'(\mathbb{R}^d)$.

Let T be a linear continuous operator from $\Sigma_s(\mathbb{R}^d)$ to $\Sigma_s'(\mathbb{R}^d)$ and let $f \in \Sigma_s(\mathbb{R}^d)$. Then it follows from (74) that

$$(Tf)(x) = \left(\frac{\pi}{2}\right)^{\frac{d}{2}} \sum_{(j,\iota)\in\Lambda} V_\phi f(j,\iota) T(\psi(\,\cdot\,-j)e^{i\langle\,\cdot\,,\iota\rangle})(x)$$

and

$$T(\psi(\cdot - j)e^{i\langle \cdot ,\iota\rangle})(x) = \left(\frac{\pi}{2}\right)^{\frac{d}{2}} \sum_{(k,\kappa)\in\Lambda} (V_\phi(T(\psi(\cdot - j)e^{i\langle \cdot ,\iota\rangle})))(k,\kappa)\psi(x-k)e^{i\langle x,\kappa\rangle}.$$

A combination of these expansions show that

$$(Tf)(x) = \left(\frac{\pi}{2}\right)^{\frac{d}{2}} \sum_{(j,\iota)\in\Lambda} (A\cdot V_\phi f)(j,\iota)\psi(x-j)e^{i\langle x,\iota\rangle}, \tag{77}$$

where $A = (a(\boldsymbol{j},\boldsymbol{k}))_{\boldsymbol{j},\boldsymbol{k}\in\Lambda}$ is the $\Lambda\times\Lambda$-matrix, given by

$$a(\boldsymbol{j},\boldsymbol{k}) = \left(\frac{\pi}{2}\right)^{\frac{d}{2}} (T(\psi(\cdot - j)e^{i\langle \cdot ,\iota\rangle}), \phi(\cdot - k)e^{i\langle \cdot ,\kappa\rangle})_{L^2(\mathbf{R}^d)}$$

$$\text{when}\quad \boldsymbol{j} = (j,\iota) \text{ and } \boldsymbol{k} = (k,\kappa). \tag{78}$$

By the Gabor analysis for modulation spaces we get the following restatement of [54, Proposition 1.8]. We refer to [17, 19–21, 25, 27, 28, 50] for details.

Proposition 3.4 *Let $s > 1$, $p,q \in (0,\infty]$, $\omega \in \mathscr{P}_E(\mathbf{R}^{2d})$ be such that (68) holds for some $r > 0$, $\phi,\psi \in \Sigma_s(\mathbf{R}^d)$ with values in $[0,1]$ be such that (70), (71) and (72) hold true, and let $f \in \Sigma_s'(\mathbf{R}^d)$. Then the following is true:*

1. *$f \in M^{p,q}_{(\omega)}(\mathbf{R}^d)$, if and only if $\|V_\phi f\|_{\ell^{p,q}_{(\omega)}(\mathbf{Z}^d\times\pi\mathbf{Z}^d)} < \infty$;*
2. *$f \in M^{p,q}_{(\omega)}(\mathbf{R}^d)$, if and only if $\|V_\psi f\|_{\ell^{p,q}_{(\omega)}(\mathbf{Z}^d\times\pi\mathbf{Z}^d)} < \infty$;*
3. *the quasi-norms*

$$f \mapsto \|V_\phi f\|_{\ell^{p,q}_{(\omega)}(\mathbf{Z}^d\times\pi\mathbf{Z}^d)} \quad \text{and} \quad f \mapsto \|V_\psi f\|_{\ell^{p,q}_{(\omega)}(\mathbf{Z}^d\times\pi\mathbf{Z}^d)}$$

are equivalent to $\|\cdot\|_{M^{p,q}_{(\omega)}}$.

The same holds true with $W^{p,q}_{(\omega)}$ and $\ell^{p,q}_{*,(\omega)}$ in place of $M^{p,q}_{(\omega)}$ respectively $\ell^{p,q}_{(\omega)}$ at each occurrence.

3.2 Multiplications and Convolutions in Modulation Spaces

As a first step for approaching multiplications and convolutions for elements in modulation spaces, we reformulate such products in terms of short-time Fourier transforms. Let $\phi_0,\phi_1,\phi_2 \in \Sigma_1(\mathbf{R}^d)$ be fixed such that

$$\phi_0 = (2\pi)^{-\frac{d}{2}}\phi_1\phi_2 \tag{79}$$

and let $f_1, f_2 \in \Sigma_1(\mathbb{R}^d)$. Then the multiplication $f_0 = f_1 f_2$ can be expressed by

$$F_0(x,\xi) = \big(F_1(x,\,\cdot\,) * F_2(x,\,\cdot\,)\big)(\xi). \tag{80}$$

where

$$F_j = V_{\phi_j} f_j, \qquad j = 0, 1, 2. \tag{81}$$

In fact, by Fourier's inversion formula we get

$$\begin{aligned}
&\big((V_{\phi_1} f_1)(x,\,\cdot\,) * (V_{\phi_2} f_2)(x,\,\cdot\,)\big)(\xi)\\
&= (2\pi)^{-d} \iiint f_1(y_1)\overline{\phi_1(y_1 - x)} f_2(y_2)\overline{\phi_2(y_2 - x)} e^{-i\langle y_1, \xi-\eta\rangle} e^{-i\langle y_2,\eta\rangle}\, dy_1 dy_2 d\eta\\
&= \int f_1(y)\overline{\phi_1(y - x)} f_2(y)\overline{\phi_2(y - x)} e^{-i\langle y,\xi\rangle}\, dy = (2\pi)^{\frac{d}{2}} (V_{\phi_1\phi_2}(f_1 f_2))(x,\xi).
\end{aligned}$$

We also observe that we may extract $f_0 = f_1 f_2$ by the formula

$$f_0 = (\|\phi_0\|_{L^2})^{-1} V_{\phi_0}^* F_0, \tag{82}$$

provided ϕ_0 is not trivially equal to 0.

In the same way, let $\phi_0, \phi_1, \phi_2 \in \Sigma_1(\mathbb{R}^d)$ be fixed such that

$$\phi_0 = (2\pi)^{\frac{d}{2}} \phi_1 * \phi_2, \tag{83}$$

and let $f_1, f_2, g \in \Sigma_1(\mathbb{R}^d)$. Then the convolution $f_0 = f_1 * f_2$ can be expressed by

$$F_0(x,\xi) = \big(F_1(\,\cdot\,,\xi) * F_2(\,\cdot\,,\xi)\big)(x). \tag{84}$$

where F_j are given by (81), and that we may extract $f_0 = f_1 * f_2$ from (82).

Next we discuss convolutions and multiplications for modulation spaces, and start with the following convolution result for modulation spaces. For multiplications of elements in modulation spaces we need to swap the conditions for the involved Lebesgue exponents compared to (51) and (52). That is, these conditions become

$$\frac{1}{p_0} \le \frac{1}{p_1} + \frac{1}{p_2}, \qquad \frac{1}{q_0} \le \frac{1}{q_1} + \frac{1}{q_2} - \max\left(1, \frac{1}{p_0}, \frac{1}{q_1}, \frac{1}{q_2}\right), \tag{85}$$

or

$$\frac{1}{p_0} \le \frac{1}{p_1} + \frac{1}{p_2}, \qquad \frac{1}{q_0} \le \frac{1}{q_1} + \frac{1}{q_2} - \max\left(1, \frac{1}{q_1}, \frac{1}{q_2}\right). \tag{86}$$

The conditions on the weight functions are

$$\omega_0(x, \xi_1 + \xi_2) \le \omega_1(x, \xi_1)\omega_2(x, \xi_2), \qquad x, \xi_1, \xi_2 \in \mathbb{R}^d, \tag{87}$$

respectively

$$\omega_0(x_1 + x_2, \xi) \le \omega_1(x_1, \xi)\omega_2(x_2, \xi), \qquad x_1, x_2, \xi \in \mathbb{R}^d. \tag{88}$$

Theorem 3.5 *Let $p_j, q_j \in (0, \infty)$ and $\omega_j \in \mathscr{P}_E(\mathbb{R}^{2d})$, $j = 0, 1, 2$, be such that (85) and (87) hold. Then $(f_1, f_2) \mapsto f_1 f_2$ from $\Sigma_1(\mathbb{R}^d) \times \Sigma_1(\mathbb{R}^d)$ to $\Sigma_1(\mathbb{R}^d)$ is uniquely extendable to a continuous map from $M^{p_1,q_1}_{(\omega_1)}(\mathbb{R}^d) \times M^{p_2,q_2}_{(\omega_2)}(\mathbb{R}^d)$ to $M^{p_0,q_0}_{(\omega_0)}(\mathbb{R}^d)$, and*

$$\|f_1 f_2\|_{M^{p_0,q_0}_{(\omega_0)}} \lesssim \|f_1\|_{M^{p_1,q_1}_{(\omega_1)}} \|f_2\|_{M^{p_2,q_2}_{(\omega_2)}}, \quad f_j \in M^{p_j,q_j}_{(\omega_j)}(\mathbb{R}^d),\ j = 1, 2. \tag{89}$$

Theorem 3.6 *Let $p_j, q_j \in (0, \infty)$ and $\omega_j \in \mathscr{P}_E(\mathbb{R}^{2d})$, $j = 0, 1, 2$, be such that (86) and (87) hold. Then $(f_1, f_2) \mapsto f_1 f_2$ from $\Sigma_1(\mathbb{R}^d) \times \Sigma_1(\mathbb{R}^d)$ to $\Sigma_1(\mathbb{R}^d)$ is uniquely extendable to a continuous map from $W^{p_1,q_1}_{(\omega_1)}(\mathbb{R}^d) \times W^{p_2,q_2}_{(\omega_2)}(\mathbb{R}^d)$ to $W^{p_0,q_0}_{(\omega_0)}(\mathbb{R}^d)$, and*

$$\|f_1 f_2\|_{W^{p_0,q_0}_{(\omega_0)}} \lesssim \|f_1\|_{W^{p_1,q_1}_{(\omega_1)}} \|f_2\|_{W^{p_2,q_2}_{(\omega_2)}}, \quad f_j \in W^{p_j,q_j}_{(\omega_j)}(\mathbb{R}^d),\ j = 1, 2. \tag{90}$$

The corresponding results for convolutions are the following. Here the conditions on the involved Lebesgue exponents are swapped as

$$\frac{1}{p_0} \le \frac{1}{p_1} + \frac{1}{p_2} - \max\left(1, \frac{1}{q_0}, \frac{1}{p_1}, \frac{1}{p_2}\right), \qquad \frac{1}{q_0} \le \frac{1}{q_1} + \frac{1}{q_2} \tag{91}$$

or

$$\frac{1}{p_0} \le \frac{1}{p_1} + \frac{1}{p_2} - \max\left(1, \frac{1}{p_1}, \frac{1}{p_2}\right), \qquad \frac{1}{q_0} \le \frac{1}{q_1} + \frac{1}{q_2} \tag{92}$$

Theorem 3.7 *Let $p_j, q_j \in (0, \infty)$ and $\omega_j \in \mathscr{P}_E(\mathbb{R}^{2d})$, $j = 0, 1, 2$, be such that (88) and (92) hold. Then $(f_1, f_2) \mapsto f_1 * f_2$ from $\Sigma_1(\mathbb{R}^d) \times \Sigma_1(\mathbb{R}^d)$ to $\Sigma_1(\mathbb{R}^d)$ is uniquely extendable to a continuous map from $M^{p_1,q_1}_{(\omega_1)}(\mathbb{R}^d) \times M^{p_2,q_2}_{(\omega_2)}(\mathbb{R}^d)$ to $M^{p_0,q_0}_{(\omega_0)}(\mathbb{R}^d)$, and*

$$\|f_1 * f_2\|_{M^{p_0,q_0}_{(\omega_0)}} \lesssim \|f_1\|_{M^{p_1,q_1}_{(\omega_1)}} \|f_2\|_{M^{p_2,q_2}_{(\omega_2)}}, \quad f_j \in M^{p_j,q_j}_{(\omega_j)}(\mathbb{R}^d),\ j = 1, 2. \tag{93}$$

Theorem 3.8 *Let $p_j, q_j \in (0, \infty)$ and $\omega_j \in \mathscr{P}_E(\mathbb{R}^{2d})$, $j = 0, 1, 2$, be such that (88) and (91) hold. Then $(f_1, f_2) \mapsto f_1 * f_2$ from $\Sigma_1(\mathbb{R}^d) \times \Sigma_1(\mathbb{R}^d)$ to $\Sigma_1(\mathbb{R}^d)$ is uniquely extendable to a continuous map from $W^{p_1,q_1}_{(\omega_1)}(\mathbb{R}^d) \times W^{p_2,q_2}_{(\omega_2)}(\mathbb{R}^d)$ to $W^{p_0,q_0}_{(\omega_0)}(\mathbb{R}^d)$, and*

$$\|f_1 * f_2\|_{W^{p_0,q_0}_{(\omega_0)}} \lesssim \|f_1\|_{W^{p_1,q_1}_{(\omega_1)}} \|f_2\|_{W^{p_2,q_2}_{(\omega_2)}}, \qquad f_j \in W^{p_j,q_j}_{(\omega_j)}(\mathbb{R}^d),\ j = 1, 2. \tag{94}$$

We observe that Theorems 3.2–3.5 in [54] are multi-linear versions of the previous results. In particular, Theorems 3.5 and 3.6 are Fourier transformations of Theorems 3.7 and 3.8. Hence it suffices to prove the last two theorems, cf. [54]. To shed some ideas of the arguments, we give a proof in the unweighted case of Theorem 3.7. We will use Proposition A.1 from Appendix A, which is a special case of [54, Proposition 3.6].

Proof of Theorem 3.7 Suppose $f_j \in \mathscr{S}(\mathbb{R}^d)$, $\phi_j \in \mathscr{S}(\mathbb{R}^d)$, $j = 0, 1, 2$ be such that

$$f_0 = f_1 * f_2 \quad \text{and} \quad \phi_0 = (2\pi)^{\frac{d}{2}} \phi_1 * \phi_2 \neq 0,$$

and let F_j be the same as in (81). Then

$$F_0(x, \xi) = (V_{\phi_1} f_1(\,\cdot\,, \xi) * V_{\phi_2} f_2(\,\cdot\,, \xi))(x),$$

in view of (84).

We have

$$0 \le \chi_{k_1+Q} * \chi_{k_2+Q} \le \chi_{k_1+k_2+Q_{d,2}}, \qquad k_1, k_2 \in \mathbb{Z}^d,$$

where $Q_{d,r}$ is the cube

$$Q_{d,r} = [0, r]^d \quad \text{and} \quad Q = Q_{d,1} = [0, 1]^d,$$

and χ_E is the characteristic function with respect to the set E.

Set

$$G(x, \xi) = (|V_{\phi_1} f_1(\,\cdot\,, \xi)| * |V_{\phi_2} f_2(\,\cdot\,, \xi)|)(x),$$

$$a_j(k, \kappa) = \|V_{\phi_j} f_j\|_{L^\infty((k,\kappa)+Q_{2d,1})}, \quad j = 1, 2,$$

and

$$b(k, \kappa) = \|G\|_{L^\infty((k,\kappa)+Q_{2d,1})}$$

Then

$$\begin{aligned}\|V_{\phi_0}^* F_0\|_{M^{p_0,q_0}} &\asymp \|P_{\phi_0} F_0\|_{\mathsf{W}(\ell^{p_0,q_0})} \lesssim \|F_0\|_{\mathsf{W}(\ell^{p_0,q_0})} \\ &\le \|G\|_{\mathsf{W}(\ell^{p_0,q_0})} \asymp \|b\|_{\ell^{p_0,q_0}}, \end{aligned} \tag{95}$$

and

$$\|f_j\|_{M^{p_j,q_j}} \asymp \|a_j\|_{\ell^{p_j,q_j}} \tag{96}$$

in view of (A.5) and Proposition A.1 in Appendix A (see also [25, Theorem 3.3])).

By (84) we have

$$\begin{aligned} G(x,\lambda) &\le \sum_{k_1,k_2\in\mathbb{Z}^d} a_1(k_1,\lambda)a_2(k_2,\lambda)(\chi_{k_1+Q} * \chi_{k_2+Q})(x) \\ &\le \sum_{k_1,k_2\in\mathbb{Z}^d} a_1(k_1,\lambda)a_2(k_2,\lambda)\chi_{k_1+k_2+Q_{d,2}}(x). \end{aligned} \tag{97}$$

We observe that

$$\chi_{k_1+k_2+Q_{d,2}}(x) = 0 \quad \text{when} \quad x \notin l + Q_d,\ (k_1,k_2) \notin \Omega_l,$$

where

$$\Omega_l = \{\, (k_1,k_2) \in \mathbb{Z}^{2d}\,;\ l_j - 2 \le k_{1,j} + k_{2,j} \le l_j + 1\,\},$$

and

$$k_j = (k_{j,1},\dots,k_{j,d}) \in \mathbb{Z}^d, \qquad j = 1,2, \quad \text{and} \quad l = (l_1,\dots,l_d) \in \mathbb{Z}^d.$$

Hence, if $x = l$ in (97), we get

$$\begin{aligned} b(l,\lambda) &\le \sum_{(k_1,k_2)\in\Omega_l} a_1(k_1,\lambda)a_2(k_2,\lambda) \\ &\le \sum_{m\in I} (a_1(\,\cdot\,,\lambda) * a_2(\,\cdot\,,\lambda))(l - 2e_0 + m), \end{aligned} \tag{98}$$

where $e_0 = (1,\dots,1) \in \mathbb{Z}^d$ and $I = \{0,1,2,3\}^d$.

If we apply the ℓ^{p_0} quasi-norm on (98) with respect to the l variable, then Proposition 2.5 (2) and the fact that I is finite set give

$$\begin{aligned}\|b(\,\cdot\,,\lambda)\|_{\ell^{p_0}} &\le \Big\| \sum_{m\in I} (a_1(\,\cdot\,,\lambda) * a_2(\,\cdot\,,\lambda))(\,\cdot\, - 2e_0 + m) \Big\|_{\ell^{p_0}} \\ &\le \sum_{m\in I} \|(a_1(\,\cdot\,,\lambda) * a_2(\,\cdot\,,\lambda))(\,\cdot\, - 2e_0 + m)\|_{\ell^{p_0}} \\ &\asymp \|a_1(\,\cdot\,,\lambda) * a_2(\,\cdot\,,\lambda)\|_{\ell^{p_0}} \\ &\le \|a_1(\,\cdot\,,\lambda)\|_{\ell^{p_1}} \|a_2(\,\cdot\,,\lambda)\|_{\ell^{p_2}}.\end{aligned}$$

By applying the ℓ^{q_0} quasi-norm and using Proposition 2.5 (1) we now get

$$\|b\|_{\ell^{p_0,q_0}} \lesssim \|a_1\|_{\ell^{p_1,q_1}} \|a_2\|_{\ell^{p_2,q_2}}.$$

This is the same as

$$\|G\|_{L^{p_0,q_0}} \lesssim \|F_1\|_{L^{p_1,q_1}} \|F_2\|_{L^{p_2,q_2}}.$$

A combination of this estimate with (95) and (96) gives that $f_1 * f_2$ is well-defined and that (93) holds.

The uniqueness now follows from that (93) holds for $f_1, f_2 \in \mathscr{S}(\mathbb{R}^d)$, and that $\mathscr{S}(\mathbb{R}^d)$ is dense in $M^{p,q}(\mathbb{R}^d)$ when $p, q < \infty$. □

4 Gabor Products and Modulation Spaces

In this section we give an illustration how the multiplication properties for modulation spaces can be used when treating certain nonlinear problems. We consider the Gabor product which is connected to such multiplication properties. It is introduced in [14] in order to derive a phase space analogue to the usual convolution identity for the Fourier transform. We will prove a formula related to (80), and then use results from previous section to extend the Gabor product initially defined on $M^1(\mathbb{R}^{2d}) \times M^1(\mathbb{R}^{2d})$ to some other spaces. Finally, we show how the Gabor product gives rise to a phase-space formulation of the qubic Schrödinger equation.

Definition 4.1 Let $\phi \in M^1(\mathbb{R}^d) \setminus \{0\}$, and let $F_1, F_2 \in M^1(\mathbb{R}^{2d})$. Then the Gabor product $\natural_\phi$ is given by

$$\begin{aligned}&\big(F_1 \natural_\phi F_2\big)(x,\xi) \\ &= (2\pi)^{-d} e^{-i\langle x,\xi\rangle} \iiint_{\mathbb{R}^{3d}} \overline{\widehat{\phi}(\zeta - \xi)} e^{i\langle x,\zeta\rangle} F_1(y,\eta) F_2(y,\zeta-\eta)\, dy d\eta d\zeta. \qquad (99)\end{aligned}$$

In the proof of [14, Lemma 13] it is justified that the Gabor product in (99) is well-defined, and that

$$\natural_\phi : M^1(\mathbb{R}^{2d}) \times M^1(\mathbb{R}^{2d}) \to M^1(\mathbb{R}^{2d})$$

is a continuous map.

The Gabor product is particularly well-suited in the context of the STFT.

Theorem 4.2 *Let* $\phi, \phi_1, \phi_2 \in M^1(\mathbb{R}^d) \setminus \{0\}$. *Then*

$$(\phi_2, \phi_1)_{L^2(\mathbb{R}^d)} V_\phi(f_1 \cdot f_2) = (V_{\phi_1} f_1) \natural_\phi (V_{\overline{\phi}_2} f_2), \quad f_1, f_2 \in M^1(\mathbb{R}^d). \tag{100}$$

Moreover, $V_\phi(f_1 \cdot f_2) \in M^1(\mathbb{R}^{2d})$.

Proof We have

$$((V_{\phi_1} f_1) \natural_\phi (V_{\overline{\phi}_2} f_2))(x, \xi) \tag{101}$$

$$= (2\pi)^{-d} e^{-i\langle x,\xi\rangle} \iint_{\mathbb{R}^{2d}} \overline{\widehat{\phi}(\zeta - \xi)} e^{i\langle x,\zeta\rangle} G(y, \zeta)\, dy d\zeta, \tag{102}$$

where

$$G(y, \zeta) = \int_{\mathbb{R}^d} (V_{\phi_1} f_1)(y, \eta)(V_{\overline{\phi}_2} f_2)(y, \zeta - \eta)\, d\eta.$$

By Parseval's formula we get

$$\begin{aligned} G(y, \zeta) &= \int_{\mathbb{R}^d} (V_{\phi_1} f_1)(y, \eta)(V_{\overline{\phi}_2} f_2)(y, \zeta - \eta)\, d\eta \\ &= \int_{\mathbb{R}^d} \mathscr{F}(f_1 \overline{\phi_1(\cdot - y)})(\eta) \mathscr{F}(f_2 \phi_2(\cdot - y))(\zeta - \eta)\, d\eta \\ &= (\mathscr{F}(f_1 \overline{\phi_1(\cdot - y)}) , \mathscr{F}(\overline{f_2\, \phi_2(\cdot - y)} e^{i\langle \cdot ,\zeta\rangle}))_{L^2(\mathbb{R}^d)} \\ &= (f_1 \overline{\phi_1(\cdot - y)} , \overline{f_2\, \phi_2(\cdot - y)} e^{i\langle \cdot ,\zeta\rangle})_{L^2(\mathbb{R}^d)} \\ &= \int_{\mathbb{R}^d} f_1(z) \overline{\phi_1(z - y)} f_2(z) \phi_2(z - y) e^{-i\langle z,\zeta\rangle}\, dz. \end{aligned}$$

By inserting this into (102) and using Fubini's theorem we get

$$((V_{\phi_1} f_1)\natural_\phi (V_{\overline{\phi_2}} f_2))(x,\xi)$$
$$= (2\pi)^{-d} e^{-i\langle x,\xi\rangle} \iint_{\mathbb{R}^{2d}} \overline{\widehat{\phi}(\zeta-\xi)} e^{-i\langle z-x,\zeta\rangle} f_1(z) f_2(z) H(z)\, dz d\zeta,$$

where

$$H(z) = \int_{\mathbb{R}^d} \phi_2(z-y)\overline{\phi_1(z-y)}\, dy = (\phi_2,\phi_1)_{L^2}.$$

Hence, by evaluating the integral with respect to ζ, and using Fourier's inversion formula, we get

$$((V_{\phi_1} f_1)\natural_\phi ((V_{\overline{\phi_2}} f_2)))(x,\xi)$$
$$= (2\pi)^{-\frac{d}{2}} e^{-i\langle x,\xi\rangle} (\phi_2,\phi_1)_{L^2} \int_{\mathbb{R}^d} \overline{\phi(z-x)} e^{i\langle x-z,\xi\rangle} f_1(z) f_2(z)\, dz$$
$$= (\phi_2,\phi_1)_{L^2} V_\phi (f_1 f_2)(x,\xi),$$

which gives (100), and the result follows. □

The formula (100) is closely related to (80). In fact, the windows $\phi_j \in \Sigma_1(\mathbb{R}^d)$, $j = 0,1,2$, in (80) should satisfy the condition (79), while (100) is valid for arbitrary non-zero elements from $M^1(\mathbb{R}^d)$. For example, when $\phi = \phi_1 = \phi_2$ and $\|\phi\|_{L^2(\mathbb{R}^d)} = 1$, then (100) reduces to

$$V_\phi(f_1 \cdot f_2) = (V_\phi f_1)\natural_\phi (V_\phi f_2), \qquad f_1, f_2 \in M^1(\mathbb{R}^d), \tag{103}$$

while (80) does not allow such choice of windows.

One of the main goals of [14] are extensions of the Gabor product to some function spaces $\mathcal{F}_j(\mathbb{R}^{2d})$, $j = 0,1,2$, so that $\natural_\phi$ maps $\mathcal{F}_1 \times \mathcal{F}_2$ into $\mathcal{F}_0$, with:

$$\|F_1 \natural_\phi F_2\|_{\mathcal{F}_0} \le C \|F_1\|_{\mathcal{F}_1} \|F_2\|_{\mathcal{F}_2}. \tag{104}$$

This can be considered as a phase space form of the Young convolution inequality.

Next we discuss continuity of the Gabor product on certain spaces involving superpositions of short-time Fourier transforms. In the end we deduce properties similar to [14, Theorem 29]. Instead of modulation spaces of the form $M^{p,q}_{(\omega)}(\mathbb{R}^d)$, $p,q \in [1,\infty)$, $\omega \in \mathscr{P}_E(\mathbb{R}^{2d})$, here we consider modulation spaces of Wiener amalgam types $W^{p,q}_{(\omega)}(\mathbb{R}^d)$, and allow the "quasi-Banach" choice for Lebesgue parameters, i.e. p and q are allowed to be smaller than one.

Thus, in what follows we assume that $p, q \in (0,\infty)$, $\omega \in \mathscr{P}_E(\mathbb{R}^{2d})$ is v-moderate, and consider $L^{p,q}_{*,(\omega)}(\mathbb{R}^{2d})$ spaces rather than $L^{p,q}_{(\omega)}(\mathbb{R}^{2d})$ which are treated in [14].

We need some additional notation. Let $s > 1$, $N \in \mathbb{N}$ be given, and let

$$\mathscr{G} = \left\{\phi_n = \overline{\phi_n}\, ;\ n \in \mathbb{N}\right\} \subseteq \Sigma_s(\mathbb{R}^d),$$

be an orthonormal basis of $L^2(\mathbb{R}^d)$. Then let $\mathcal{V}^{(N),p,q}_{\mathscr{G},\omega}(\mathbb{R}^{2d})$ be the closure of

$$\mathcal{V}^{(N)}_{\mathscr{G}}(\mathbb{R}^{2d}) = \left\{ \sum_{n=1}^{N} V_{\phi_n} f_n\ ;\ f_n \in \Sigma_1(\mathbb{R}^d) \right\} \tag{105}$$

with respect to the $L^{p,q}_{*,(\omega)}(\mathbb{R}^{2d})$ norm. In particular, if $N = 1$, $\phi = \phi_1$ and $p, q \geq 1$, then this reduces to the closure

$$P_\phi(L^{p,q}_{*,(\omega)}(\mathbb{R}^{2d})) = V_\phi(W^{p,q}_{(\omega)}(\mathbb{R}^d))$$

of

$$P_\phi(\Sigma_1(\mathbb{R}^{2d})) = V_\phi(\Sigma_1(\mathbb{R}^d))$$

with respect to the $L^{p,q}_{*,(\omega)}(\mathbb{R}^{2d})$ norm.

By [14, Theorem 26], it follows that for every $F \in \mathcal{V}^{(N),p,q}_{\mathscr{G},\omega}(\mathbb{R}^{2d})$ there exist $f_n \in W^{p,q}_{(\omega)}(\mathbb{R}^d)$, $n = 1, 2, \ldots, N$, and such that

$$F = \sum_{n=1}^{N} V_{\phi_n} f_n\ . \tag{106}$$

Theorem 4.3 *Let $p_j, q_j \in (0,\infty)$ and $\omega_j \in \mathscr{P}_E(\mathbb{R}^{2d})$ be v_j–moderate, $j = 0, 1, 2$, and such that (86) and (87) hold, and let $\phi \in \Sigma_s(\mathbb{R}^d)$, $s > 1$. Then the Gabor product $\natural_\phi$ from $\mathcal{V}^{(N)}_{\mathscr{G}}\left(\mathbb{R}^{2d}\right) \times \mathcal{V}^{(N)}_{\mathscr{G}}\left(\mathbb{R}^{2d}\right)$ to $W^{1,1}_{(v)}(\mathbb{R}^{2d})$, extends uniquely to a continuous map from $\mathcal{V}^{(N),p_1,q_1}_{\mathscr{G},\omega_1}(\mathbb{R}^{2d}) \times \mathcal{V}^{(N),p_2,q_2}_{\mathscr{G},\omega_2}(\mathbb{R}^{2d})$ to the closure of $P_\phi(L^{p_0,q_0}_{*,(\omega_0)}(\mathbb{R}^{2d}))$, and*

$$\|F_1 \natural_\phi F_2\|_{L^{p_0,q_0}_{*,(\omega_0)}} \lesssim \|F_1\|_{L^{p_1,q_1}_{*,(\omega_1)}} \|F_2\|_{L^{p_2,q_2}_{*,(\omega_2)}}\ , \tag{107}$$

for all $F_j \in \mathcal{V}^{(N),p_j,q_j}_{\mathscr{G},\omega_j}(\mathbb{R}^{2d})$, $j = 1, 2$.

In particular, if $F_j = V_\phi f_j$, $j = 1, 2$, and $\|\phi\|_{L^2} = 1$, then (107) reduces to

$$\|V_\phi f_1 \natural_\phi V_\phi f_2\|_{L^{p_0,q_0}_{*,(\omega_0)}} = \|f_1 f_2\|_{W^{p_0,q_0}_{(\omega_0)}} \lesssim \|f_1\|_{W^{p_1,q_1}_{(\omega_1)}} \|f_2\|_{W^{p_2,q_2}_{(\omega_2)}}\ . \tag{108}$$

We omit the proof which is a slight modification of the proof of Theorem 29 in [14].

We end the paper by formally demonstrating how the Gabor product arises in a phase space version of the cubic Schrödinger equation. Consider the elliptic nonlinear Schrödinger equation (NLSE) given by

$$i\frac{\partial \psi}{\partial t} + \Delta\psi + \lambda|\psi|^2\psi = 0, \tag{109}$$

subject to the initial condition:

$$\psi(x,0) = \varphi(x).$$

Here $\lambda = \pm 1$ stands for an attracting ($\lambda = +1$) or repulsive ($\lambda = -1$) power-law nonlinearity, and the Laplacian is given by

$$\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}.$$

Thus we consider $\psi = \phi(x,t)$ with $x \in \mathbb{R}^d$, and t in an open interval $I \subseteq \mathbb{R}$.

Using the following intertwining relations

$$V_\phi(x_j\psi) = -D_{\xi_j}V_\phi\psi, \qquad V_\phi(D_{x_j}\psi) = \left(\xi_j + D_{x_j}\right)V_\phi\psi,$$

$j = 1,\cdots,d$, and assuming that ϕ is a real-valued window, we obtain upon application of the STFT V_ϕ to (109) that

$$i\frac{\partial F}{\partial t} - \sum_{j=1}^{d}\left(\xi_j + D_{x_j}\right)^2 F + \lambda\widetilde{F}\natural_\phi F\natural_\phi F = 0. \tag{110}$$

Here, $D_{x_j} = -i\frac{\partial}{\partial x_j}$,

$$\begin{aligned} F(x,\xi,t) &= V_\phi(\psi(\,\cdot\,,t))(x,\xi) \\ &= (2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d}\psi(y,t)\overline{\phi(y-x)}e^{-i\langle y,\xi\rangle}\,dy, \quad x,\xi\in\mathbb{R}^d,\ t\in\mathbb{R}, \end{aligned}$$

and $\widetilde{F}$ is given by

$$\widetilde{F}(x,\xi) = \overline{F(x,-\xi)}. \tag{111}$$

By considering (110) the phase-space formulation of the initial value problem may be well-posed for more general initial distributions. This means that the phase-space formulation "contains" the solutions of the standard NLSE, but it is richer, as it admits other solutions. We refer to [11–13], where phase-space extensions are explored in several different contexts.

Let us conclude by noticing that (110) contains the triple product. Thus, its qualitative analysis calls for a multilinear extension of Theorems 3.6 and 4.3. Then the conditions (86) and (87) become more involved, see [54]. Such analysis demands a more technical tools and arguments and goes beyond the scope of this survey article.

Appendix A: Some Properties of Wiener Amalgam Spaces

There are convenient characterizations of modulation spaces in the framework of Gabor analysis.

Let $\omega_0 \in \mathscr{P}_E(\mathbb{R}^d)$, $\omega \in \mathscr{P}_E(\mathbb{R}^{2d})$, $p, q, r \in (0, \infty]$, $Q_d = [0, 1]^d$ be the unit cube, and set for measurable f on $\mathbb{R}^d$,

$$\|f\|_{\mathsf{W}^r(\omega_0,\ell^p)} \equiv \|a_0\|_{\ell^p(\mathbb{Z}^d)} \tag{A.1}$$

when

$$a_0(j) \equiv \|f \cdot \omega_0\|_{L^r(j+Q_d)}, \qquad j \in \mathbb{Z}^d,$$

and for measurable F on $\mathbb{R}^{2d}$,

$$\|F\|_{\mathsf{W}^r(\omega,\ell^{p,q})} \equiv \|a\|_{\ell^{p,q}(\mathbb{Z}^{2d})} \quad \text{and} \quad \|F\|_{\mathsf{W}(\omega,\ell^{p,q}_*)} \equiv \|a\|_{\ell^{p,q}_*(\mathbb{Z}^{2d})} \tag{A.2}$$

when

$$a(j, \iota) \equiv \|F \cdot \omega\|_{L^r((j,\iota)+Q_{2d})}, \qquad j, \iota \in \mathbb{Z}^d.$$

The Wiener amalgam space

$$\mathsf{W}^r(\omega_0, \ell^p) = \mathsf{W}^r(\omega_0, \ell^p(\mathbb{Z}^d))$$

consists of all measurable $f \in L^r_{loc}(\mathbb{R}^d)$ such that $\|F\|_{\mathsf{W}^r(\omega_0,\ell^p)}$ is finite, and the Wiener amalgam spaces

$$\mathsf{W}^r(\omega, \ell^{p,q}) = \mathsf{W}^r(\omega, \ell^{p,q}(\mathbb{Z}^{2d})) \quad \text{and} \quad \mathsf{W}^r(\omega, \ell^{p,q}_*) = \mathsf{W}^r(\omega, \ell^{p,q}_*(\mathbb{Z}^{2d}))$$

consist of all measurable $F \in L^r_{loc}(\mathbb{R}^{2d})$ such that $\|F\|_{\mathsf{W}^r(\omega,\ell^{p,q})}$ respectively $\|F\|_{\mathsf{W}^r(\omega,\ell^{p,q}_*)}$ are finite. We observe that $\mathsf{W}^r(\omega_0, \ell^p)$ is often denoted by $W(L^r, \ell^p_{(\omega)})$ in the literature (see e. e. [17, 19, 25, 41]).

The topologies are defined through their corresponding quasi-norms in (A.1) and (A.2). For conveniency we set

$$\mathsf{W}(\omega, \ell^{p,q}) = \mathsf{W}^\infty(\omega, \ell^{p,q}) \quad \text{and} \quad \mathsf{W}(\omega, \ell^{p,q}_*) = \mathsf{W}^\infty(\omega, \ell^{p,q}_*),$$

and if in addition $\omega = 1$, we set

$$\mathsf{W}(\ell^{p,q}) = \mathsf{W}(\omega, \ell^{p,q}) \quad \text{and} \quad \mathsf{W}(\ell^{p,q}_*) = \mathsf{W}(\omega, \ell^{p,q}_*).$$

Obviously, $\mathsf{W}^r(\omega_0, \ell^p)$ and $\mathsf{W}^r(\omega, \ell^{p,q})$ increase with p, q, decrease with r, and

$$\mathsf{W}(\omega, \ell^{p,q}) \hookrightarrow L^{p,q}_{(\omega)}(\mathbb{R}^{2d}) \cap \Sigma_1'(\mathbb{R}^{2d}) \hookrightarrow L^{p,q}_{(\omega)}(\mathbb{R}^{2d}) \hookrightarrow \mathsf{W}^r(\omega, \ell^{p,q}) \tag{A.3}$$

and

$$\|\cdot\|_{\mathsf{W}^r(\omega,\ell^{p,q})} \le \|\cdot\|_{L^{p,q}_{(\omega)}} \le \|\cdot\|_{\mathsf{W}(\omega,\ell^{p,q})}, \qquad r \le \min(1, p, q). \tag{A.4}$$

On the other hand, for modulation spaces we have

$$f \in M^{p,q}_{(\omega)}(\mathbb{R}^d) \quad \Leftrightarrow \quad V_\phi f \in L^{p,q}_{(\omega)}(\mathbb{R}^{2d}) \quad \Leftrightarrow \quad V_\phi f \in \mathsf{W}^r(\omega, \ell^{p,q}) \tag{A.5}$$

with

$$\|f\|_{M^{p,q}_{(\omega)}} = \|V_\phi f\|_{L^{p,q}_{(\omega)}} \asymp \|V_\phi f\|_{\mathsf{W}^r(\omega,\ell^{p,q})}. \tag{A.6}$$

The same holds true with $W^{p,q}_{(\omega)}$, $L^{p,q}_{*,(\omega)}$ and $\mathsf{W}(\omega, \ell^{p,q}_*)$ in place of $M^{p,q}_{(\omega)}$, $L^{p,q}_{(\omega)}$ and $\mathsf{W}(\omega, \ell^{p,q})$, respectively, at each occurrence. (For $r = \infty$, see [28] when $p, q \in [1, \infty]$, [25, 50] when $p, q \in (0, \infty]$, and for $r \in (0, \infty]$, see [53].)

We have now the following result on the projection operator P_ϕ in (20) when acting on Wiener amalgam spaces.

Proposition A.1 *Let $p, q \in (0, \infty]$ and $\phi \in \mathscr{S}(\mathbb{R}^d) \setminus \{0\}$. Then P_ϕ from $\mathscr{S}'(\mathbb{R}^{2d})$ to $\mathscr{S}'(\mathbb{R}^{2d})$, and V_ϕ^* from $\mathscr{S}'(\mathbb{R}^{2d})$ to $\mathscr{S}'(\mathbb{R}^d)$ restrict to continuous mappings*

$$P_\phi : \mathsf{W}(\ell^{p,q}(\mathbb{Z}^{2d})) \to V_\phi(M^{p,q}(\mathbb{R}^d)), \tag{A.7}$$

$$P_\phi : \mathsf{W}(\ell^{p,q}_*(\mathbb{Z}^{2d})) \to V_\phi(W^{p,q}(\mathbb{R}^d)), \tag{A.8}$$

$$V_\phi^* : \mathsf{W}(\ell^{p,q}(\mathbb{Z}^{2d})) \to M^{p,q}(\mathbb{R}^d) \tag{A.9}$$

and

$$V_\phi^* : \mathsf{W}(\ell^{p,q}_*(\mathbb{Z}^{2d})) \to W^{p,q}(\mathbb{R}^d). \tag{A.10}$$

We refer to [54, Proposition 3.6] for the proof of Proposition A.1 and to [19, 21, 28, 41, 42, 54] for some facts about the operators P_ϕ and V_ϕ^*,

For $p, q \geq 1$, i.e. the case when all spaces are Banach spaces, proofs of Proposition A.1 can be found in e.g. [28] as well as in abstract forms in [19]. In the general case when $p, q > 0$, we refer to [25, 42], since proofs of Proposition A.1 are essentially given there.

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