Trends in Mathematics

Richard M. Aron
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Editors

Operator and Norm Inequalities and Related Topics

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Operator and Norm Inequalities and Related Topics

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Preface

Inequalities play a central role in mathematics with various applications in other disciplines. The main goal of this contributed volume is to present several important matrix, operator, and norm inequalities in a systematic and self-contained fashion.

The volume includes contributions by a number of the world's leading specialists in functional analysis and operator theory. It contains the latest developments of significant mathematical inequalities in numerous fields in the last decades that are of interest to a wide audience of pure and applied mathematicians.

This book consists of 5 parts and includes a total of 23 chapters. The chapters are written in a reader-friendly style and can be read independently. Each chapter contains a rich bibliography.

Part I: Matrix and Operator Inequalities

Whenever we see an inequality concerning real or complex numbers, an interesting question is to ask ourselves whether it is true for matrices or bounded linear operators on a Hilbert space. This is based on the fact that the real linear space of self-adjoint operators (Hermitian matrices) can be regarded as a generalization of the real line. One of the most significant notions in this part is the concept of operator monotone function, which was first studied by C. Löwner [Math. Z. 38 (1934), 177–216], and its connection with operator means was introduced by F. Kubo and T. Ando [Math. Ann. 246 (1980), no. 3, 205–224]; [cf. Simon, Barry. Loewner's theorem on monotone matrix functions. Grundlehren der mathematischen Wissenschaften, 354. Springer, Cham, 2019].

Chapter "Log-majorization Type Inequalities" is devoted to studying the link between majorization theory and several matrix inequalities such as Araki's log majorization, the Löwner–Heinz, the Furuta, the Golden-Thompson, the von Neumann trace, and their extensions.

In Chapter "Ando-Hiai Inequality: Extensions and Applications", extensions and applications of the Ando-Hiai inequality are investigated and the Furuta, the Bebiano–Lemos–Providência, and the grand Furuta inequalities are explored.

Chapter "Relative Operator Entropy" demonstrates the relative operator entropy that is the tangent vector of the geodesic in the manifold of positive invertible operators. Tsallis relative entropy is studied as the secant of a path of geometric matrix means.

Chapter "Matrix Inequalities and Characterizations of Operator Monotone Functions" includes various characterizations of operator monotone functions using matrix inequalities involving matrix means. A trace monotonicity inequality and the Powers-Størmer inequality are used to characterize operator monotone functions.

Chapter "Perspectives, Means and Their Inequalities" focuses on the operator perspective and its extensions including operator means and the Pusz–Woronowicz functional calculus.

Chapter "Cauchy–Schwarz Operator and Norm Inequalities for Inner Product Type Transformers in Norm Ideals of Compact Operators, with Applications" provides an overview of operator and norm inequalities of Cauchy–Schwarz type for strongly square integrable operator families and symmetrically norming functions. Some applications to the Aczél–Bellman, Grüss–Landau, arithmetic–geometric, Young, Heinz, Heron inequalities are presented.

Chapter "Norm Estimations for the Moore-Penrose Inverse of the Weak Perturbation of Hilbert C^*-module Operators" defines multiplicative perturbations and studies representations and norm estimations for the Moore-Penrose inverse associated with the multiplicative perturbation.

Part II: Orthogonality and Inequalities

There are several ways to extend the notion of orthogonality from inner product spaces to the framework of normed spaces. The most developed one is the Birkhoff–James orthogonality. It was introduced by Birkhoff [Duke Math. J. 1 (1935), 169–172] and extensively studied by R.C. James [Duke Math. J. 12 (1945), 291–302]; cf. C. Alsina, J. Sikorska, M.S. Tomás [Norm derivatives and characterizations of inner product spaces. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2010].

Chapter "Birkhoff–James Orthogonality: Characterizations, Preservers, and Orthogonality Graphs" reviews the Birkhoff–James orthogonality starting from historical perspectives throughout the current development and presents several characterizations of Birkhoff–James orthogonality in classical Banach spaces, C^*-algebras, and Hilbert C^*-modules. In addition, some characterizations of preservers of Birkhoff–James orthogonality are given.

Chapter "Approximate Birkhoff-James Orthogonality in Normed Linear Spaces and Related Topics" is an introduction to approximate Birkhoff–James orthogonality in real normed spaces and its characterizations.

Chapter “Orthogonally Additive Operators on Vector Lattices” focuses on the vector lattice structure of different partial subclasses of the vector space of all orthogonally additive operators, certain domination problems, representation theorems, and Banach lattice structure of orthogonally additive operators.

Part III: Inequalities Related to Types of Operators

This part mainly studies inequalities concerning closed range, normal, and Toeplitz operators (cf. A. Böttcher and B. Silbermann [Analysis of Toeplitz operators. Second edition. Prepared jointly with Alexei Karlovich. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2006] and I. Gohberg, S. Goldberg, and M. A. Kaashoek [Basic classes of linear operators. Birkhäuser Verlag, Basel, 2003]).

In Chapter “Normal Operators and Their Generalizations”, some aspects of local spectral theory and Fredholm theory of certain classes of operators that generalize normal operators on Hilbert spaces are studied.

Chapter “On Wold Type Decomposition for Closed Range Operators” surveys Wold-type decomposition for closed range operators satisfying certain operator inequalities. Several results on left invertible operators close to isometries are listed and extended to the case of regular operators.

Chapter “(Asymmetric) Dual Truncated Toeplitz Operators” considers properties of asymmetric dual truncated Toeplitz operators acting between the orthogonal complements of two model spaces.

Chapter “Boundedness of Toeplitz Operators in Bergman-Type Spaces” is devoted to the open problem of characterization of the bounded Toeplitz operators T_a in Bergman spaces. Based on the structure of the Bergman spaces, a characterization of the boundedness and compactness is presented in the case of operators in spaces with weighted sup-norms.

Part IV: Inequalities in Various Banach Spaces

This part deals with miscellaneous inequalities concerning topological and geometrical properties of various Banach spaces and operator algebras (cf. W.B. Johnson and J. Lindenstrauss (ed.) [Handbook of the geometry of Banach spaces. Vol. I. North-Holland Publishing Co., Amsterdam, 2001]).

In Chapter “Disjointness Preservers and Banach-Stone Theorems”, the so-called weak and strong Banach-Stone theorems are given. In addition, it is proved that in many cases lattice isomorphisms (Kaplansky’s Theorem), ring isomorphisms (Gelfand-Kolmogorov Theorem), multiplicative isomorphisms (Milgram’s Theorem), isometries (Banach-Stone Theorem), and nonvanishing preservers are $\perp$-isomorphisms.

Chapter “The Bishop–Phelps–Bollobás Theorem: An Overview” provides a comprehensive survey of the Bishop–Phelps–Bollobás theorem from 2008 to 2021.

Chapter “A New Proof of the Power Weighted Birman–Hardy–Rellich Inequalities” introduces a new proof of the optimal version of the power-weighted Birman–Hardy–Rellich integral inequalities. Extensions to homogeneous Sobolev spaces and the vector-valued case are also discussed.

Chapter “An Excursion to Multiplications and Convolutions on Modulation Spaces” is devoted to reviewing results on boundedness for multiplications and convolutions in (quasi-)Banach modulation spaces of ultradistributions. Furthermore, the Gabor product is investigated.

Chapter “The Hardy-Littlewood Inequalities in Sequence Spaces” presents modern proofs of m-linear versions of the results of Hardy and Littlewood and the state of the art of the subject, as well as an application to the combinatorial Gale-Berlekamp switching game.

Chapter “Symmetries of C^*-algebras and Jordan Morphisms” illustrates interrelations between symmetries of various structures attached to C^*-algebras and von Neumann algebras and Jordan $*$-isomorphisms. In this direction, one-dimensional projections in a Hilbert space with transition probability, projection lattices of von Neumann algebras, and measures on state spaces endowed with the Choquet order are extensively studied.

Part V: Inequalities in Commutative and Noncommutative Probability Spaces

This part includes generalizations of Doob’s maximal inequality, the Burkholder–Davis–Gundy inequality, and several inequalities related to Markov processes and noncommutative free probability (cf. P. Jorgensen and F. Tian [Non-commutative analysis. With a foreword by Wayne Polyzou. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017]).

In Chapter “Mixed Norm Martingale Hardy Spaces and Applications in Fourier Analysis”, martingale Hardy spaces defined with the help of mixed $L_{\vec{p}}$ -norm are investigated. Two different generalizations of Doob’s maximal inequality for mixed-norm $L_{\vec{p}}$ spaces and two versions of atomic decompositions are given. Several martingale inequalities and a generalization of the Burkholder–Davis–Gundy inequality are also presented. As an application in Fourier analysis, the boundedness of the Fejér maximal operator from $H_{\vec{p}}$ to $L_{\vec{p}}$, whenever $1/2 < \vec{p} < \infty$ is obtained.

Chapter “The First Eigenvalue for Nonlocal Operators” presents some results concerning the first eigenvalue for a nonlocal operator in convolution form with a smooth kernel and gives information on the asymptotic behavior of some natural Markov processes.

Chapter “Comparing Banach Spaces for Systems of Free Random Variables Followed by the Semicircular Law” studies certain Banach-space operators from noncommutative free probability, acting on systems of free random variables whose free distributions are followed by the semicircular law.

The editors are grateful for the hard work of numerous mathematicians who carefully reviewed the chapters and gave insightful comments to improve them.

The book can be used as an introduction to several active research areas within operator theory. It is intended for use by both researchers and graduate students in mathematics, scientific computing, physics, statistics, and engineering who have a basic grasp of the fundamentals in functional analysis and operator theory.

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Contents

Part I
Matrix and Operator Inequalities

Log-majorization Type Inequalities

N. Bebiano, R. Lemos, and G. Soares

Abstract Several inequalities have been established in the context of Hilbert spaces operators or operator algebras. Our discussion will be limited to matrices. Important inequalities in mathematics and other sciences, such as Golden-Thompson inequality or von Neumann trace inequality, and extensions, are revisited. Our main goal is to emphasize the link between majorization theory and other relevant inequalities.

Keywords Eigenvalues · Singular values · Majorization · Log-majorization · Norm inequalities · Determinant and trace inequalities · Operator connections · Ando-Hiai inequality · Araki's log-majorization · Löwner-Heinz inequality · Furuta inequality · Golden-Thompson inequality · von Neumann trace inequality

Notation

$\mathbb{N}$	Set of natural numbers
$\mathbb{N}_0$	Set of nonnegative integer numbers
$\mathbb{R}$	Set of real numbers
$\mathbb{R}_0^+$	Set of nonnegative real numbers
$\mathbb{C}$	Set of complex numbers
$\mathbb{R}^n$	Vector space of real n-tuples
$\mathbb{C}^n$	Vector space of complex n-tuples
$\|\cdot\|$	Euclidean norm; spectral norm or operator norm

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$\|\|\| \cdot \|\|\|$	Unitarily invariant norm
$\\| \cdot \\|_{(k)}$	Ky Fan k-norm
$\\| \cdot \\|_p$	Schatten p-norm
$\\| \cdot \\|_2$	Frobenius norm, Hilbert-Schmidt norm or Schur norm
$M_n(\mathbb{C})$	Algebra of $n \times n$ complex matrices
$M_{m\times n}(\mathbb{C})$	Vector space of $m \times n$ complex matrices
Ω_n	Set of $n \times n$ doubly stochastic matrices
$A = (a_{ij})$	Matrix A with entries a_{ij}
A^*	Adjoint of a matrix A
A^T	Transpose of a matrix A
$\overline{A}$	Entrywise conjugate of A
$\|A\|$	Unique positive semidefinite square root of A^*A
$A^{\wedge k}$	kth compound or kth antisymmetric tensor power of A
$\rho(A)$	Spectral radius of A
$f(A)$	Functional calculus applied to a function f
$A \geq 0$	Positive semidefinite matrix A
$A > 0$	Positive definite matrix A
$A \geq B$	$A - B \geq 0$
ReA (ImA)	Real (imaginary) part of A
tr(A)	Trace of a matrix A
det(A)	Determinant of a matrix A
$\lambda_i(A)$	Eigenvalue of A
$\lambda_1(A)$	Largest eigenvalue of A if A is Hermitian
$s_i(A)$	Singular value of A
$s_1(A)$	Largest singular value of A
I_n	Identity matrix of order n
$A \circ B$	Hadamard product of matrices A and B
$\|x\|$	Absolute value vector $(\|x_1\|, \ldots, \|x_n\|)$
$x \prec y$	x is majorized by y
$x \prec_w y$	x is weakly majorized by y
$x \prec_{\log} y$	x is log-majorized by y
$x \prec_{w\log} y$	x is weakly log-majorized by y
$\sharp_\alpha$	α-weighted geometric mean for $\alpha \in [0, 1]$
$\sharp$	Geometric mean
σ	Operator connection
$\sigma^\perp$	Dual of an operator connection σ
f_σ	Representing function of an operator connection σ
S_n	Symmetric group of degree n
$S(A, B)$	Umegaki relative entropy
$X^\sim$	X or X^T
H_n	Set of $n \times n$ Hermitian matrices
H_n^T	Set of $n \times n$ symmetric matrices
per A	Permanent of a matrix A
J	Hermitian involutive matrix

$\sigma_J^{\pm}(A)$	Set of eigenvalues with eigenvectors x, such that $x^*Jx = \pm 1$
$A \geq^J B$	$J(A - B) \geq 0$

1 Introduction

The concept of majorization was introduced by Hardy, Littlewood and Pólya [43]. Since then various majorizations were obtained for the eigenvalues and singular values of matrices and compact operators [72]. These majorizations are powerful devices for the derivation of several norm inequalities, as well as trace or determinant inequalities for matrices or operators. In this section, we review in a concise way the majorization theory used throughout this chapter.

Any vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is assumed to have its components sorted in non-increasing order, that is, $x_1 \geq \cdots \geq x_n$.

Let $x, y \in \mathbb{R}^n$. We say that x is *majorized* by y and write $x \prec y$ if

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i, \qquad k = 1, \ldots, n, \tag{1}$$

and equality occurs in (1) for $k = n$. Further, if (1) holds, then x is said to be *weakly majorized* or *submajorized* by y and the notation $x \prec_w y$ is used. We remark that $x \prec y$ is equivalent to

$$\sum_{i=k}^{n} x_i \geq \sum_{i=k}^{n} y_n, \qquad k = 1, \ldots, n, \tag{2}$$

with equality in (2) for $k = 1$. If (2) holds, then x is said to be *supermajorized* by y and we write $x \prec^w y$.

Naively, vector majorization means that one vector is more *disordered* than the other. For instance, a physics interpretation may be that x describes a more chaotic state than y, thinking of x_i as the probability of the system described by x being in state i.

Two important resources on the topic of majorization are [21, 72].

A square matrix with non-negative entries is called *doubly stochastic* if all its row and column sums are one. The class Ω_n of doubly stochastic matrices of order n is a convex set, whose extreme points are the permutation matrices as stated by the famous Birkhoff's Theorem [24]. In fact, there is a close relation between majorization and doubly stochastic matrices [72].

Proposition 1.1 *A matrix $A \in \Omega_n$ if and only if $Ax \prec x$ for all $x \in \mathbb{R}^n$.*

Proposition 1.2 *For $x, y \in \mathbb{R}^n$, the following statements are equivalent:*

i. $x \prec y$;
ii. *x is in the convex hull of all the vectors obtained by permutating the coordinates of y;*
iii. $x = Ay$ *for some* $A \in \Omega_n$.

For any real valued function f defined on an interval, containing all the components of the real n-tuple x, we adopt the notation

$$f(x) = \big(f(x_1), \dots, f(x_n)\big).$$

Proposition 1.3 *Let $x, y \in \mathbb{R}^n$ and f be a convex function on an interval containing all the components of x and y. Then*

i. *If $x \prec y$, then $f(x) \prec_w f(y)$.*
ii. *If $x \prec_w y$ and f is also non-decreasing, then $f(x) \prec_w f(y)$.*

Log-majorization can be defined as a multiplicative version of majorization. If $x, y \in \mathbb{R}^n$ have nonnegative components, $x \prec_{\log} y$ means that

$$\prod_{i=1}^{k} x_i \le \prod_{i=1}^{k} y_i, \qquad k = 1, \dots, n, \tag{3}$$

and equality occurs in (3) for $k = n$. If $x, y > 0$, i.e., all the components of x, y are positive, this is clearly equivalent to

$$\prod_{i=k}^{n} x_i \ge \prod_{i=k}^{n} y_n, \qquad k = 1, \dots, n, \tag{4}$$

with equality in (4) for $k = 1$. If $x, y > 0$, then

$$x \prec_{\log} y \quad \Leftrightarrow \quad \log x \prec \log y,$$

this justifying the log-majorization terminology. When equality between the products of all the components of x and y is not required, the following parallel notations are used:

$$x \prec_{w\log} y \quad \text{for} \quad (3) \qquad \text{and} \qquad x \prec^{w\log} y \quad \text{for} \quad (4).$$

Proposition 1.4 *Let $x, y \in \mathbb{R}^n$ have all the components positive and f be a non-decreasing continuous function on an interval containing all the components of x, y, such that $f(\mathrm{e}^t)$ is convex. Then*

$$x \prec_{w\log} y \quad \Rightarrow \quad f(x) \prec_w f(y).$$

In particular, $f(t) = t$ in the previous proposition shows that the weak log-majorization $\prec_{w\log}$ is stronger than the weak majorization $\prec_w$.

2 Matrix Majorization

If $A = (a_{ij})$, $B = (b_{ij})$ are $m \times n$ complex matrices, let $A \circ B = (a_{ij}b_{ij})$ be the Hadamard product of A and B. Let $M_n(\mathbb{C})$ be the algebra of n-square complex matrices and I_n be the identity matrix of order n. If $A \in M_n(\mathbb{C})$, then its eigenvalues are denoted by $\lambda_1(A), \ldots, \lambda_n(A)$ and

$$\rho(A) = \max_{i=1,\ldots,n} |\lambda_i(A)|$$

is the *spectral radius* of A. Further, considering the Euclidean norm $\|x\|$ of a vector $x \in \mathbb{C}^n$, let

$$\|A\| = \max_{\|x\|=1} \|Ax\|$$

be the *spectral norm* or *operator norm* of A. It is clear that

$$\rho(A) \leq \|A\|. \tag{5}$$

If $A \in M_n(\mathbb{C})$ has real eigenvalues, denote by $\lambda(A)$ the n-tuple of eigenvalues of A arranged in non-increasing order:

$$\lambda_1(A) \geq \cdots \geq \lambda_n(A).$$

For $A \in M_n(\mathbb{C})$, the unique positive semidefinite square root of A^*A is denoted by $|A|$. The eigenvalues of $|A|$ are the singular values of A, which are arranged in the vector $s(A)$ as follows:

$$s_1(A) \geq \cdots \geq s_n(A).$$

A norm $|||\cdot|||$ in $M_n(\mathbb{C})$ is said to be *unitarily invariant* if $|||UAV||| = |||A|||$ for any $A, U, V \in M_n(\mathbb{C})$ with U, V unitary. Examples of unitarily invariant norms are the *Schatten p-norms* given by

$$\|A\|_p = \left(\sum_{i=1}^{n} s_i^p(A)\right)^{\frac{1}{p}} = \left(\operatorname{tr}|A|^p\right)^{\frac{1}{p}}, \qquad p \geq 1,$$

and the *Ky Fan k-norms* defined by

$$\|A\|_{(k)} = \sum_{i=1}^{k} s_i(A), \qquad k = 1, \ldots, n,$$

including $\|A\| = s_1(A)$. The Schatten 2-norm

$$\|A\|_2 = \sqrt{\operatorname{tr}(A^*A)},$$

also called *Frobenius norm*, *Hilbert-Schmidt norm* or *Schur norm*, is the norm induced by to the *Frobenius* or *Hilbert-Schmidt inner product* in $M_n(\mathbb{C})$:

$$\langle A, B\rangle = \operatorname{tr}(B^*A).$$

The notion of majorization gives a mean for comparing two probability distributions or two density matrices, that is positive semidefinite matrices of trace one, using the eigenvalues, in an elegant way. It arises in fields like computer science, economics or quantum mechanics.

Important sources on majorization for eigenvalues and singular values of matrices are [21, 46, 47, 55, 72] and two survey articles of T. Ando [2, 3].

For simplicity, if $A, B \in M_n(\mathbb{C})$ have real eigenvalues, then $\lambda(A) \prec \lambda(B)$ and $\lambda(A) \prec_w \lambda(B)$ are abreviated to $A \prec B$ and $A \prec_w B$, respectively.

The main diagonal entries and the eigenvalues of a Hermitian matrix are related through majorization. This classical result due to I. Schur [84] can be briefly stated as follows.

Theorem 2.1 (Schur Majorization Theorem, 1923) *If $A \in M_n(\mathbb{C})$ is Hermitian, then $I_n \circ A \prec A$.*

In 1954, A. Horn [51] proved the converse, giving rise to the next fundamental result, which received considerable attention and led to generalizations in several directions.

Theorem 2.2 (Schur-Horn Theorem) *Let $x, y \in \mathbb{R}^n$. There exists a Hermitian matrix with prescribed diagonal entries and prescribed eigenvalues arranged, respectively, in x and y if and only if $x \prec y$.*

After this, Horn's subsequent work on the eigenvalues of sums of Hermitian matrices culminated in the inequalities conjectured in [53]. The solution to Horn's conjecture appeared in two papers, one by A. Klyachko [60] and the other one by A. Knutson and T. Tao [61].

Another relevant result in matrix majorization is due to Ky Fan [28].

Theorem 2.3 (Ky Fan Dominance Theorem, 1951) *Let $A, B \in M_n(\mathbb{C})$. Then the following are equivalent statements:*

i. $|A| \prec_w |B|$;

ii. $\|\|A\|\| \leq \|\|B\|\|$ *for any unitarily invariant norm* $\|\|\cdot\|\|$ *in* $M_n(\mathbb{C})$.

If $A, B \in M_n(\mathbb{C})$ have nonnegative eigenvalues, $A \prec_{\log} B$ stands for

$$\lambda(A) \prec_{\log} \lambda(B).$$

Abbreviated notations for the weaker versions, involving either $\prec_{w\log}$ or $\prec^{w\log}$, are analogously used. Clearly, if A, B have positive eigenvalues, then

$$A \prec_{w\log} B \quad \Leftrightarrow \quad B^{-1} \prec^{w\log} A^{-1}.$$

Matrix log-majorization is a powerful tool for establishing trace, determinantal and matrix norm inequalities. For instance,

$$A \prec_{\log} B \quad \Rightarrow \quad \det(I_n + A) \leq \det(I_n + B).$$

On the other hand, some classical determinantal inequalities can find their majorization counterparts.

As usual, $A > 0$ means that A is a positive definite matrix and $A \geq B$ means that $A - B$ is a positive semidefinite matrix. Real-valued continuous functions f defined on a real interval Γ, such that

$$A \geq B \quad \Rightarrow \quad f(A) \geq f(B)$$

for all Hermitian $A, B \in M_n(\mathbb{C})$ with spectra in Γ and all $n \in \mathbb{N}$, are said to be *operator monotone* on Γ. A useful and fundamental tool for treating operator inequalities is *Löwner-Heinz inequality*. Löwner's original proof [69] used an integral representation for operator monotone functions and an alternative proof was given by Heinz [44]. It states that

$$A \geq B \geq 0 \quad \Rightarrow \quad A^\alpha \geq B^\alpha, \tag{6}$$

that is, $f(t) = t^\alpha$ is operator monotone on $\mathbb{R}_0^+$, for $\alpha \in [0, 1]$. In general, (6) is not true for $\alpha > 1$.

For $k = 1, \ldots, n$ and $n_k = \binom{n}{k}$, the *kth compound* or *kth antisymmetric tensor power* of $A \in M_n(\mathbb{C})$ is the matrix $A^{\wedge k} \in M_{n_k}(\mathbb{C})$ with entries given by the minors $\det A(\mathbf{i}, \mathbf{j})$, where the index sets $\mathbf{i}, \mathbf{j} \subset \{1, \ldots, n\}$ have cardinality k and are lexicographically ordered. As usual, $A(\mathbf{i}, \mathbf{j})$ denotes the submatrix of A that lies in rows and columns indexed, respectively, by $\mathbf{i}, \mathbf{j}$. Some essential properties of these matrices [21] are listed below:

P1. $(AB)^{\wedge k} = A^{\wedge k} B^{\wedge k}$ (Binet-Cauchy formula);
P2. $\left(A^{\wedge k}\right)^* = (A^*)^{\wedge k}$;
P3. $\left(A^{\wedge k}\right)^r = (A^r)^{\wedge k}$, $r > 0$;
P4. $\left(A^{\wedge k}\right)^{-1} = (A^{-1})^{\wedge k}$ if A is invertible;

P5. $\lambda_{\mathbf{i}}\left(A^{\wedge k}\right) = \prod_{j=1}^{k} \lambda_{i_j}(A)$, where $\mathbf{i} = (i_1, \dots, i_k)$ and $1 \le i_1 < \cdots < i_k \le n$;

P6. $\|A^{\wedge k}\| = s_1\left(A^{\wedge k}\right) = \prod_{j=1}^{k} s_j\,(A),\ k = 1, \dots, n.$

Thus, a useful tool in log-majorization is provided by the next lemma.

Lemma 2.4 *Let $A, B \in M_n(\mathbb{C})$ have nonnegative eigenvalues. The following are equivalent:*

i. $A \prec_{\log} B$;
ii. $\lambda_1\left(A^{\wedge k}\right) \le \lambda_1\left(B^{\wedge k}\right)$, $k = 1, \dots, n$, *and* $\det(A) = \det(B)$.

A basic log-majorization in matrix theory is Weyl's relation between eigenvalues and singular values [98]. Let $|\lambda(A)|$ be the vector of the absolute values of the eigenvalues of $A \in M_n(\mathbb{C})$ arranged in non-increasing order of magnitude:

$$|\lambda_1(A)| \ge \cdots \ge |\lambda_n(A)|.$$

Theorem 2.5 (Weyl's Majorant Theorem, 1949) *If $A \in M_n(\mathbb{C})$, then*

$$|\lambda(A)| \prec_{\log} s(A). \tag{7}$$

Proof Use properties **P5** and **P6**, after applying 5, that is,

$$\rho(A) = |\lambda_1(A)| \le s_1(A).$$

to the kth antisymmetric tensor power of A, $k = 1, \dots, n$, and observe that

$$\left|\prod_{i=1}^{n} \lambda_i(A)\right| = |\det(A)| = \left(\overline{\det(A)}\,\det(A)\right)^{\frac{1}{2}} = \left(\det(A^*A)\right)^{\frac{1}{2}} = \det|A|$$

is the product of all the singular values of A. □

In 1954, A. Horn proved the converse [52], that is, there exists a square matrix with prescribed eigenvalues and singular values arranged in vectors x and y if the log-majorization $|x| \prec_{\log} y$ is satisfied.

In the sequel, we illustrate the potential of using the previous antisymmetric tensor power technique, also called Weyl trick, by using Lemma 2.4 to derive some other log-majorization for expressions involving products and fractional matrix powers, having in mind that these "commute" with the kth antisymmetric tensor power. As a consequence, some known results will be revisited. Some classical inequalities for the trace and the determinant are meanwhile surveyed in the next section.

3 Trace and Determinantal Inequalities

The von Neumann's trace inequality was first published in 1937 by von Neumann [96] with a complicated proof. Other proofs were given in 1959 and subsequently in 1975, based on doubly stochastic matrices, by Mirsky [76, 77]. However, these proofs only work in the finite dimensional case. A simple proof, which also extends to the infinite dimensional setting, was finally obtained in 1991 by R. D. Grigorieff [41].

Theorem 3.1 (von Neumann's Inequality, 1937) *Let $A, B \in M_n(\mathbb{C})$. Then*

$$|\mathrm{tr}(AB)| \le \sum_{i=1}^{n} s_i(A)s_i(B)$$

and equality occurs if A, B share a joint set of singular vectors.

This result is an important tool with various applications in pure and applied mathematics. For instance, just to mention a few, it is useful in Schatten's theory of cross spaces and in Ball's approach of the equations of nonlinear elasticity. Inspired by this famous inequality, further singular value inequalities have been meanwhile derived, among them Horn's multiplicative inequalities (see, e.g. [72]).

Theorem 3.2 (Horn, 1950) *If $A, B \in M_n(\mathbb{C})$, then*

$$s(AB) \prec_{\log} s(A) \circ s(B).$$

Proof By the submultiplicativity of the operator norm, we have

$$s_1(AB) = \|AB\| \le \|A\|\|B\| = s_1(A)s_1(B).$$

Apply the antisymmetric tensor power technique to the previous inequality, that is, replace A, B by their kth compounds, $k = 1, \ldots, n$, and use **P6**. Equality for $k = n$ is immediate by properties of the determinants. □

Remark 3.3 For $A, B \in M_n(\mathbb{C})$, Horn and Weyl's log-majorizations stated before, the second applied to the product AB, imply the corresponding weak majorizations, so we easily find for $k = 1, \ldots, n$ that

$$\left|\sum_{i=1}^{k} \lambda_i(AB)\right| \le \sum_{i=1}^{k} |\lambda_i(AB)| \le \sum_{i=1}^{k} s_i(AB) \le \sum_{i=1}^{k} s_i(A)s_i(B). \tag{8}$$

In particular, von Neumann trace inequality is obtained when $k = n$ in (8).

In 1958, Richter [81] proved a related trace inequality for the product of two Hermitian matrices. Other contributions in this vein are due to Marcus [70], Mirsky [76] and Theobald [88]. Ruhe [82] reobtained it under the more restrictive assumption of positive semidefiniteness of both matrices.

Theorem 3.4 *If $A, B \in M_n(\mathbb{C})$ are Hermitian, then*

$$\sum_{i=1}^{n} \lambda_i(A)\lambda_{n-i+1}(B) \leq \operatorname{tr}(AB) \leq \sum_{i=1}^{n} \lambda_i(A)\lambda_i(B). \tag{9}$$

We remark that the lower bound is an immediate consequence of the upper bound in (9) with the Hermitian matrix B replaced by $-B$, since $\lambda_i(-B) = -\lambda_{n-i+1}(B)$, $i = 1, \ldots, n$. Note that the previous inequality is a matrix version of the following classical rearrangement inequality [43]. Let S_n be the symmetric group of degree n of all permutations of $\{1, \ldots, n\}$.

Theorem 3.5 (Hardy-Littlewood-Pólya Rearrangement Inequality, 1929) *If $x, y \in \mathbb{R}^n$, then*

$$\sum_{i=1}^{n} x_i y_{n-i+1} \leq \sum_{i=1}^{n} x_i y_{\sigma(i)} \leq \sum_{i=1}^{n} x_i y_i$$

for any permutation $\sigma \in S_n$.

Having in mind that the trace of a matrix is the sum of the eigenvalues while the determinant is the product, we can think in "dual inequalities" in the sense of replacing sums by products and products by sums. In fact, the determinant of the sum of matrices has no simple relation with the determinants of the summands. We recall some inequalities in this avenue. We start with a remarkable result due to Fiedler [29], after the following remark.

Remark 3.6 A continuity argument will be repeatedly used, when possible, along the proof of some of the results, involving eigenvalues of Hermitian matrices. In such cases, we only need to prove the results for nonsingular matrices. Otherwise, we may replace in the inequalities each nonsingular matrix A by $A + \epsilon I_n$ for $\epsilon > 0$ and then take the limit as ϵ converges to 0.

Theorem 3.7 *If $A, B \in M_n(\mathbb{C})$ are Hermitian with eigenvalues $\alpha_1 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \cdots \geq \beta_n$, respectively, then*

$$\min_{\sigma \in S_n} \prod_{j=1}^{n} \left(\alpha_j + \beta_{\sigma(j)}\right) \leq \det(A + B) \leq \max_{\sigma \in S_n} \prod_{j=1}^{n} \left(\alpha_j + \beta_{\sigma(j)}\right).$$

If $\alpha_n + \beta_n \geq 0$, then the minimum is attained when σ is the identity permutation and the maximum is attained when $\sigma(j) = n - j + 1$, $j = 1, \ldots, n$.

Proof If A and B commute, they are simultaneously unitarily diagonalizable and the result easily follows. Otherwise, there exists $U \in M_n(\mathbb{C})$ unitary, such that

$$\det(A + B) = \det(A_0 + U^* B_0 U),$$

where A_0, B_0 are the diagonal forms of A, B. Since the unitary group is compact and the determinant is continuous, $\det(A_0 + V^* B_0 V)$ attains its maximum and minimum values for some unitary matrix $V \in M_n(\mathbb{C})$. Take

$$U_\epsilon = \mathrm{e}^{\mathrm{i}\epsilon S} = I_n + \mathrm{i}\epsilon S + O(\epsilon^2),$$

where ϵ is a small quantity and $S \in M_n(\mathbb{C})$ is Hermitian. Assuming that $A_0 + V^* B_0 V$ is nonsingular and calculating

$$\det(A_0 + U_\epsilon^* V^* B_0 V U_\epsilon)$$

to the first order in ϵ, it can be easily shown that $V^* B_0 V$ commutes with the inverse of $A_0 + V^* B_0 V$. Thus $V^* B_0 V$ commutes with A_0 and the theorem follows. If $A_0 + V^* B_0 V$ is singular, then the result follows by a limiting argument. □

Remark 3.8 A natural generalization of Fiedler's Theorem would be the following. If $A, B \in M_n(\mathbb{C})$ are normal matrices with eigenvalues $\alpha_1, \ldots, \alpha_n$, $\beta_1, \ldots, \beta_n$, respectively, then $\det(A + B)$ lies in the convex hull of the products

$$\prod_{j=1}^{n} \left(\alpha_j + \beta_{\sigma(j)}\right), \quad \sigma \in S_n.$$

This is *Marcus-de Oliveira conjecture* [71, 79], a longstanding open problem.

Concerning more general multiplicative inequalities, involving singular values of matrices, we state Gel,fand-Naimark Theorem (see, e.g. [47, 72]).

Theorem 3.9 (Gelfand-Naimark, 1950) *For $A, B \in M_n(\mathbb{C})$,*

$$\prod_{j=1}^{k} s_{i_j}(A)\, s_{n-i_j+1}(B) \le \prod_{j=1}^{k} s_j(AB), \qquad k = 1, \ldots, n,$$

equivalently,

$$\prod_{j=1}^{k} s_{i_j}(AB) \le \prod_{j=1}^{k} s_j(A)\, s_{i_j}(B), \qquad k = 1, \ldots, n,$$

for $1 \le i_1 < i_2 < \cdots < i_k \le n$, with equality for $k = n$.

The next result [9, 21] has a simple proof, using majorization theory.

Theorem 3.10 *If* $A, B \geq 0$ *have eigenvalues* $a_1 \geq \cdots \geq a_n$ *and* $b_1 \geq \cdots \geq b_n$*, respectively, then*

$$\prod_{j=1}^{n} \left(a_j^2 + b_j^2\right) \leq |\det(A + iB)|^2 \leq \prod_{j=1}^{n} \left(a_j^2 + b_{n-j+1}^2\right).$$

Proof We may assume $A, B > 0$. We easily find that

$$\begin{aligned} |\det(A + iB)|^2 &= \det(A)^2 \det\left(I_n + (A^{-1}B)^2\right) \\ &= \prod_{j=1}^{n} a_j^2 \prod_{j=1}^{n} \left(1 + \lambda_j^2(A^{-1}B)\right) \end{aligned}$$

and

$$\lambda_j(A^{-1}B) = s_j^2\left(A^{-\frac{1}{2}}B^{\frac{1}{2}}\right), \qquad j = 1, \ldots, n.$$

By Gelfand-Naimark Theorem with A, B replaced by $B^{\frac{1}{2}}, A^{-\frac{1}{2}}$, respectively,

$$\prod_{j=1}^{k} s_{n-j+1}(A^{-\frac{1}{2}})s_j(B^{\frac{1}{2}}) \leq \prod_{j=1}^{k} s_j(A^{-\frac{1}{2}}B^{\frac{1}{2}}) \leq \prod_{j=1}^{k} s_j(A^{-\frac{1}{2}})s_j(B^{\frac{1}{2}})$$

hold for $k = 1, \ldots, n$, with equality for $k = n$. Clearly,

$$s_{n-j+1}^2(A^{-\frac{1}{2}})s_j^2(B^{\frac{1}{2}}) = \frac{b_j}{a_j}, \quad s_j^2(A^{-\frac{1}{2}})s_j^2(B^{\frac{1}{2}}) = \frac{b_j}{a_{n-j+1}}, \quad j = 1, \ldots, n.$$

Thus, the previous singular values inequalities are equivalent to

$$\left(\frac{b_1}{a_1}, \ldots, \frac{b_n}{a_n}\right) \prec_{\log} \lambda(A^{-1}B) \prec_{\log} \left(\frac{b_1}{a_n}, \ldots, \frac{b_n}{a_1}\right).$$

Since the function $f(x) = \log(1 + x^2)$ is a continuous increasing function on $(0, \infty)$, such that $f(e^t)$ is convex in t, by Proposition 1.4 applied to the previous log-majorization, we obtain

$$\sum_{j=1}^{n} \log\left(1 + \frac{b_j^2}{a_j^2}\right) \leq \sum_{j=1}^{n} \log\left(1 + \lambda_j^2(A^{-1}B)\right) \leq \sum_{j=1}^{n} \log\left(1 + \frac{b_{n-j+1}^2}{a_j^2}\right).$$

Thus,

$$\prod_{j=1}^{n}\left(1+\frac{b_j^2}{a_j^2}\right) \le \prod_{j=1}^{n}\left(1+\lambda_j^2(A^{-1}B)\right) \le \prod_{j=1}^{n}\left(1+\frac{b_{n-j+1}^2}{a_j^2}\right)$$

and the result easily follows. □

Remark 3.11 If one of the two matrices in Theorem 3.10 is not positive semidefinite, the lower bound is not necessarily true. Indeed, let A, B be Hermitian matrices with eigenvalues $a_1 = 1, a_2 = -1$ and $b_1 = 2, b_2 = 1$, respectively. As Marcus-de Oliveira conjecture holds for $n = 2$, then $\det(A + iB)$ is in the line segment with endpoints $-3 - i$ and $-3 + i$, consequently, we have

$$3 \le |\det(A + iB)| \le \sqrt{10}.$$

If Theorem 3.10 was true, the lower bound of $|\det(A + iB)|$ would be $\sqrt{10}$, but in the previous case the lower bound 3 is attained.

Now, considering the Cartesian decomposition $A = \operatorname{Re} A + i \operatorname{Im} A$ of $A \in M_n(\mathbb{C})$, where

$$\operatorname{Re} A = \frac{A + A^*}{2} \qquad \text{and} \qquad \operatorname{Im} A = \frac{A - A^*}{2i}$$

are Hermitian matrices, the next corollary is easy to derive.

Corollary 3.12 *If* $A \in M_n(\mathbb{C})$ *is such that* $\operatorname{Re} A > 0$ *then*

$$\det |\operatorname{Re} A| \le |\det A| .$$

We remark that related inequalities are surveyed in [101, Section 3.4].

4 Golden-Thompson Inequality and Araki's Log-majorization

For matrices A, B that commute, the following identity holds:

$$e^{A+B} = e^A e^B .$$

In the noncommutative case, the situation is not so simple.

Let H, K be Hermitian matrices of the same order. It is obvious that

$$\det\left(e^{H+K}\right) = \det\left(e^H\right) \det\left(e^K\right).$$

Furthermore, the following remarkable trace inequality, motivated by considerations in statistical mechanics, states a relation between e^{H+K} and $e^H e^K$, even when these matrices do not commute.

Theorem 4.1 (Golden-Thompson Inequality, 1965) *If $H, K \in M_n(\mathbb{C})$ are Hermitian, then*

$$\operatorname{tr}\big(e^{H+K}\big) \leq \operatorname{tr}\big(e^H e^K\big). \tag{10}$$

Nowadays, (10) is a basic tool in quantum statistical mechanics. Some historical aspects and its applications in random matrix theory are collected in [30], including a not previously published proof due to Dyson. Golden [40] proved (10) for positive semidefinite matrices and observed that it may be used to get lower bounds for the Helmholtz free energy by partitioning the Hamiltonian. C. J. Thompson [89] showed (10) for Hermitian matrices, independently of the semidefiniteness condition, and applied it to obtain an upper bound for the *partition function* of an antiferromagnetic chain, that is, for $z = \operatorname{tr}\big(e^{-\beta H}\big)$, where H is the Hamiltonian of the physical system and $\beta = 1/(k_B T)$, with k_B the Boltzmann constant and T the absolute temperature. Symanzik [86] obtained (10) for particular selfadjoint Hilbert space operators, in that context showing that the classical partition function is an upper bound for the corresponding quantum partition function. Some discussion on Golden-Thompson inequality can also be found in the expository blog post by Terence Tao [87].

The direct extension of the Golden-Thompson inequality to three or more matrices fails. Then if any two of the Hermitian matrices H, K, L commute,

$$\operatorname{tr}\left(e^{H+K+L}\right) \leq \left|\operatorname{tr}\left(e^H e^K e^L\right)\right|$$

obviously holds, but this is not true in general as the next counter-example, due to C. J. Thompson [90], shows.

Example Consider the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

the real vector $a = (a_1, a_2, a_3)$ and the matrix $A = a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3$. Then

$$e^A = \cosh \|a\|\, I_2 + \frac{\sinh \|a\|}{\|a\|}\, I_2.$$

Here $\|a\|$ is the Euclidean norm of a. For $\epsilon \in \mathbb{R} \setminus \{0\}$, let

$$H = \epsilon\sigma_1, \qquad K = \epsilon\sigma_2, \qquad L = \epsilon(\sigma_3 - \sigma_2 - \sigma_1).$$

In this case,

$$\operatorname{tr}\left(e^{H+K+L}\right) = 2\cosh\epsilon$$

and, after some calculations, we find

$$\left|\operatorname{tr}\left(e^{H}e^{K}e^{L}\right)\right| = 2\cosh\epsilon\left(1-\frac{\epsilon^4}{12}+O(\epsilon^6)\right).$$

Therefore, for ϵ small enough, we have

$$\left|\operatorname{tr}\left(e^{H}e^{K}e^{L}\right)\right| < \operatorname{tr}\left(e^{H+K+L}\right).$$

Nevertheless, there is a nontrivial generalization of Golden-Thompson inequality to a triple of Hermitian matrices by Lieb [67], as well as recent multivariate versions [42, 85].

We notice in passing the interesting related result due to R. C. Thompson [91]: if H, K are Hermitian matrices, then

$$e^{iH}e^{iK} = e^{i(UHU^*+VKV^*)},$$

for some unitary matrices U, V. This result has application in quantum computing. We observe that Thompson's result was obtained before the long-standing Horn's conjecture on eigenvalues of sums of Hermitian matrices has been solved (see [20] for more details).

Several trace inequalities may be strengthened in the set up of majorization. This is the case of the Golden-Thompson inequality. In fact, it was proved by Lenard [66] and Thompson [90] that

$$e^{H+K} \prec_w e^{\frac{H}{2}}e^{K}e^{\frac{H}{2}}$$

holds for Hermitian matrices H, K.

Closely related, Araki [7] obtained a log-majorization presented in the next theorem that extends the *Lieb-Thirring trace inequality*:

$$\operatorname{tr}(AB)^r \le \operatorname{tr}\left(A^rB^r\right), \qquad r \in \mathbb{N},$$

for $A, B \ge 0$, firstly used to derive inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian [68].

Theorem 4.2 (Araki's Log-majorization, 1990) *Let* $A, B \ge 0$. *Then*

$$(A^{\frac{1}{2}}BA^{\frac{1}{2}})^r \prec_{\log} A^{\frac{r}{2}}B^rA^{\frac{r}{2}}, \qquad r \ge 1, \tag{11}$$

or equivalently

$$(A^{\frac{q}{2}} B^q A^{\frac{q}{2}})^{\frac{1}{q}} \prec_{\log} \left(A^{\frac{p}{2}} B^p A^{\frac{p}{2}}\right)^{\frac{1}{p}}, \quad 0 < q \le p. \tag{12}$$

Proof It is clear that $(A^{\frac{1}{2}} B A^{\frac{1}{2}})^r$ and $A^{\frac{r}{2}} B^r A^{\frac{r}{2}}$ have the same determinant. Assuming A invertible, let us prove that

$$\lambda_1((A^{\frac{1}{2}} B A^{\frac{1}{2}})^r) \le \lambda_1(A^{\frac{r}{2}} B^r A^{\frac{r}{2}}). \tag{13}$$

To do so we may prove that $A^{\frac{r}{2}} B^r A^{\frac{r}{2}} \le I_n$ implies that $A^{\frac{1}{2}} B A^{\frac{1}{2}} \le I_n$, because both sides of (13) have the same order of homogeneity for A and B, so that we can multiply A, B by a positive constant. Since $B^r \le A^{-r}$, for $r \ge 1$, then Löwner-Heinz inequality implies $B \le A^{-1}$. If A is not invertible, by a continuity argument, we obtain (13). By properties **P1** and **P3**, then

$$(A^{\wedge k})^{\frac{r}{2}} (B^{\wedge k})^r (A^{\wedge k})^{\frac{r}{2}} = (A^{\frac{r}{2}} B^r A^{\frac{r}{2}})^{\wedge k},$$

$$\left((A^{\wedge k})^{\frac{1}{2}} (B^{\wedge k}) (A^{\wedge k})^{\frac{1}{2}}\right)^r = \left((A^{\frac{1}{2}} B A^{\frac{1}{2}})^r\right)^{\wedge k}.$$

Hence, (11) follows from (13) applied to the matrices $A^{\wedge k}$, $B^{\wedge k}$, $k = 1, \ldots, n$, using Lemma 2.4.

For $0 < p \le q$, we may replace A and B by A^q, B^q and take $r = p/q$ in (11) so that (12) follows. □

Araki's log-majorization readily implies the next trace inequality.

Corollary 4.3 (Araki-Lieb-Thirring Inequality) *If $A, B \ge 0$, $r \ge 1$ and $s > 0$, then*

$$\mathrm{tr}\left(A^{\frac{1}{2}} B A^{\frac{1}{2}}\right)^{rs} \le \mathrm{tr}\left(A^{\frac{r}{2}} B^r A^{\frac{r}{2}}\right)^s.$$

The next extension of Golden-Thompson inequality is now easy to derive, as observed by Ando and Hiai [5].

Corollary 4.4 *If $H, K \in M_n(\mathbb{C})$ are Hermitian and $p > 0$, then*

$$\mathrm{e}^{H+K} \prec_{\log} \left(\mathrm{e}^{\frac{pH}{2}} \mathrm{e}^{pK} \mathrm{e}^{\frac{pH}{2}}\right)^{\frac{1}{p}}. \tag{14}$$

Proof Consider $A = \mathrm{e}^H$ and $B = \mathrm{e}^K$ in (12). Further, have in mind the continuous parameter version of Lie-Trotter formula [5, Lemma 1.6]:

$$\lim_{q \to 0} \left(\mathrm{e}^{\frac{qH}{2}} \mathrm{e}^{qK} \mathrm{e}^{\frac{qH}{2}}\right)^{\frac{1}{q}} = \mathrm{e}^{H+K}$$

and the result follows. □

Let H, K be Hermitian matrices. If $p = 1$, then (14) can be written as

$$e^{H+K} \prec_{\log} e^{H} e^{K}, \tag{15}$$

since $e^{\frac{H}{2}} e^{K} e^{\frac{H}{2}}$ and $e^{H} e^{K}$ have the same eigenvalues. From the previous results, we can see that Golden-Thompson inequality is strengthened to

$$|||e^{H+K}||| \leq |||(e^{\frac{pH}{2}} e^{pK} e^{\frac{pH}{2}})^{\frac{1}{p}}|||, \qquad p > 0,$$

for any unitarily invariant norm, and the right hand side decreases to the left hand side as p converges to 0.

5 Ando-Hiai Inequality

The axiomatic theory of operator connections was developed by F. Kubo and T. Ando [62]. A *matrix connection of order* n is a binary operation σ on the cone of positive semidefinite matrices in $M_n(\mathbb{C})$, satisfying for any A, B, C, D, $A_k, B_k \geq 0$:

C1. (*joint monotonicity*) $A \leq C$ and $B \leq D \Rightarrow A \,\sigma\, B \leq C \,\sigma\, D$;
C2. (*transformer inequality*) $X^*(A \,\sigma\, B)X \leq (X^*AX) \,\sigma\, (X^*BX)$ for any $X \in M_n(\mathbb{C})$;
C3. (*joint continuity from above*) $A_k \downarrow A$ and $B_k \downarrow B \Rightarrow A_k \,\sigma\, B_k \downarrow A \,\sigma\, B$,

where $A_k \downarrow A$ means that $A_1 \geq A_2 \geq \cdots A_k \geq \cdots$ and A_k converges strongly to A as $k \to \infty$. An *operator connection* is a matrix connection of every order $n \in \mathbb{N}$. An *operator mean* is an operator connection σ, satisfying the *normalization property* $I_n \,\sigma\, I_n = I_n$. For instance, for $\alpha \in [0, 1]$,

$$A \nabla_\alpha B = (1-\alpha)A + \alpha B \qquad \text{and} \qquad A !_\alpha B = \left((1-\alpha)A^{-1} + \alpha B^{-1}\right)^{-1}$$

are the *weighted arithmetic* and *harmonic means*, respectively; $A w_l B = A$ and $A w_r B = B$ are the left and right *trivial operator means*, respectively.

Kubo and Ando proved that there is a one-to-one correspondence between operator connections and operator monotone functions on $\mathbb{R}_0^+$.

Theorem 5.1 ([62]) *For each operator connection σ, there exists a unique operator monotone function $f : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, satisfying*

$$f(t)\, I_n = I_n \,\sigma\, (t I_n), \qquad t > 0,$$

and for $A, B > 0$ *the formula*

$$A \,\sigma\, B \;=\; A^{\frac{1}{2}} \, f\big(A^{-\frac{1}{2}} B \, A^{-\frac{1}{2}}\big) \, A^{\frac{1}{2}}$$

holds, with the right hand side defined via functional calculus, and extended to $A, B \geq 0$ *as follows*

$$A \,\sigma\, B \;=\; \lim_{\epsilon \to 0^+} \, (A + \epsilon I_n) \,\sigma\, (B + \epsilon I_n).$$

Let $\alpha \in [0, 1]$. In particular, associated with the operator monotone function $f(t) = t^{\alpha}$, the α-*weighted geometric mean* is

$$A \,\sharp_{\alpha} B \;=\; A^{\frac{1}{2}} \big(A^{-\frac{1}{2}} B \, A^{-\frac{1}{2}}\big)^{\alpha} A^{\frac{1}{2}}.$$

It is easy to see that $A \sharp_{\alpha} B = B \sharp_{1-\alpha} A$ and when A commutes with B, then $A \sharp_{\alpha} B = A^{1-\alpha} B^{\alpha}$. The *geometric mean*, simply denoted by $\sharp$, is the mean corresponding to $f(t) = t^{\frac{1}{2}}$. It is the unique positive semidefinite solution of the Riccati equation $X A^{-1} X = B$, also characterized [80] as

$$A \,\sharp\, B \;=\; \max\left\{ X \in H_n : \begin{bmatrix} A & X \\ X & B \end{bmatrix} \geq 0 \right\}. \tag{16}$$

Further, there is a unitary matrix U such that $A \sharp B = A^{\frac{1}{2}} U B^{\frac{1}{2}}$.

Ando and Hiai [5] proved the following interesting result, concerning the weighted geometric mean.

Theorem 5.2 (Ando-Hiai Inequality, 1994) *For* $A, B \geq 0$ *and* $\alpha \in [0, 1]$,

$$A^r \sharp_{\alpha} B^r \prec_{\log} (A \sharp_{\alpha} B)^r, \qquad r \geq 1, \tag{17}$$

or, equivalently,

$$\big(A^p \sharp_{\alpha} B^p\big)^{\frac{1}{p}} \prec_{\log} \big(A^q \sharp_{\alpha} B^q\big)^{\frac{1}{q}}, \qquad 0 < q \leq p. \tag{18}$$

Proof If $1 \leq r \leq 2$, then $r = 1 + \epsilon$ with $\epsilon \in [0, 1]$. Suppose

$$A \sharp_{\alpha} B \leq I_n. \tag{19}$$

Let $C = A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$. By continuity, we may assume that A, B are invertible. It follows from (19) that $A \leq C^{-\alpha}$. By Löwner-Heinz inequality, we have

$A^{\epsilon} \le C^{-\alpha\epsilon}$. In this case,

$$\begin{aligned} A^r \sharp_\alpha B^r &= A^{\frac{1}{2}} \big(A^{\epsilon} \sharp_\alpha \big((CAC) \sharp_{1-\epsilon} C \big) \big) A^{\frac{1}{2}} \\ &\le A^{\frac{1}{2}} \big(C^{-\alpha\epsilon} \sharp_\alpha \big((C^{2-\alpha} \sharp_{1-\epsilon} C \big) \big) A^{\frac{1}{2}} \\ &= A^{\frac{1}{2}} C^{\alpha} A^{\frac{1}{2}} = A \sharp_\alpha B, \end{aligned}$$

by the joint monotonicity of the weighted geometric means $\sharp_\alpha$ and $\sharp_{1-\epsilon}$. Therefore, $A^r \sharp_\alpha B^r \le I_n$. We have just proved that $\lambda_1(A \sharp_\alpha B) \le 1$ implies $\lambda_1(A^r \sharp_\alpha B^r) \le 1$. Thus,

$$\lambda_1(A^r \sharp_\alpha B^r) \le \lambda_1(A \sharp_\alpha B)^r$$

and applying the antisymmetric tensor power trick, having also in mind that

$$\det \big(A^r \sharp_\alpha B^r \big) = \det(A \sharp_\alpha B)^r,$$

then (17) holds for $1 \le r \le 2$. If $r > 2$, then $r = 2^m s$ for $m \in \mathbb{N}$ and $1 \le s \le 2$. By repeated use of the above case, we find that

$$A^{2^m s} \sharp_\alpha B^{2^m s} \prec_{\log} \Big(A^{2^{m-1}s} \sharp_\alpha B^{2^{m-1}s} \Big)^2 \prec_{\log} \cdots \prec_{\log} \big(A^s \sharp_\alpha B^s \big)^{2^m} \prec_{\log} (A \sharp_\alpha B)^{2^m s}.$$

This proves that (17) also holds for $r > 2$. Now, for $0 < q \le p$, the result easily follows. □

The following corollary complements the previous log-majorizations of Golden-Thompson type [5].

Corollary 5.3 *If* H, K *are Hermitian matrices and* $\alpha \in [0, 1]$*, then*

$$\big(e^{pH} \sharp_\alpha e^{pK} \big)^{\frac{1}{p}} \prec_{\log} e^{(1-\alpha)H + \alpha K}, \quad p > 0.$$

Proof Consider (18) applied to $A = e^H$ and $B = e^K$, then use

$$\lim_{q \to 0} \big(e^{qH} \sharp_\alpha e^{qK} \big)^{\frac{1}{q}} = e^{(1-\alpha)H + \alpha K},$$

that is, the Lie-Trotter like formula for the weighted geometric mean obtained in [45, Lemma 3.3]. □

The corresponding inequality for unitarily invariant norms holds, with the left hand side norm decreasing to the right hand side as p converges to 0.

A celebrated development of Löwner-Heinz inequality established by T. Furuta [35] is the next order preserving operator inequality, which was motivated by a previous conjecture by Chan and Kwong [25].

Theorem 5.4 (Furuta Inequality, 1987) *Let* $A \geq B \geq 0$. *Then*

$$A^{\frac{p+r}{q}} \geq \left(A^{\frac{r}{2}} B^{p} A^{\frac{r}{2}}\right)^{\frac{1}{q}} \quad \textit{and} \quad \left(B^{\frac{r}{2}} A^{p} B^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq B^{\frac{p+r}{q}} \tag{20}$$

hold for $r \geq 0$, $p \geq 0$ *and* $q \geq 1$ *with* $(1+r)q \geq p+r$.

The case of p, q, r all equal to 2 in (20) affirmatively answers Chan and Kwong's conjecture:

$$A \geq B \geq 0 \quad \Rightarrow \quad A^2 \geq \left(A\, B^2 A\right)^{\frac{1}{2}}.$$

Furuta and many other researchers refined and generalized (20) and applied these results to produce new inequalities [36].

The *essential part* of Furuta inequality is the case $q = \frac{p+r}{1+r}$, which can be formulated for invertible A, using the weighted geometric mean, as follows:

$$A \geq B \geq 0 \quad \Rightarrow \quad A^{-r} \sharp_{\frac{1+r}{p+r}} B^{p} \leq A, \qquad p \geq 1,\ r \geq 0. \tag{21}$$

Fujii and Kamei [33] showed that Ando-Hiai inequality is equivalent to Furuta inequality. Next, we show this direct implication. Indeed, let $A \geq B > 0$ and $p \geq 1$. Firstly, if $0 \leq r \leq 1$, then $A^r \geq B^r$ by Löwner-Heinz inequality. Consequently,

$$A^{-r} \sharp_{\frac{r}{p+r}} B^{p} \leq B^{-r} \sharp_{\frac{r}{p+r}} B^{p} = I_n.$$

On the other hand, for $r > 1$, observe that $A^{-1} \leq B^{-1}$ yields

$$A^{-1} \sharp_{\frac{r}{p+r}} B^{\frac{p}{r}} \leq B^{-1} \sharp_{\frac{r}{p+r}} B^{\frac{p}{r}} = I_n$$

and so Ando-Hiai inequality implies that

$$A^{-r} \sharp_{\frac{r}{p+r}} B^{p} \leq I_n.$$

Therefore, for any $r \geq 0$ we find that

$$A^{-r} \sharp_{\frac{1+r}{p+r}} B^{p} = \left(A^{-r} \sharp_{\frac{r}{p+r}} B^{p}\right) \sharp_{\frac{1}{p}} B^{p} \leq I_n \sharp_{\frac{1}{p}} B^{p} = B \leq A,$$

that is, the *essential part* of Furuta inequality (21) holds. The remaining part follows readily from Löwner-Heinz inequality.

Extensions of Furuta inequality and Ando-Hiai log-majorization were given by Furuta [37] and afterwards by other authors. Nowadays, the multivariate geometric mean as settled in [22, 78], following a Riemannian geometric approach, is often called the *Karcher mean* [63]. It is also called *Cartan mean* and *Riemannian mean*.

Extensions of Ando-Hiai inequality to the Karcher mean and generalized Karcher mean are due to Yamazaki [99, 100]. Other Ando-Hiai type inequalities have meanwhile been obtained. See the recent works by Hiai, Seo and Wada [49, 50], Kian, Moslehian and Seo [57–59] and references therein.

6 BLP and Matharu-Aujla Inequalities

Using the previous techniques of Ando and Hiai, Bebiano, Lemos and Providência [12, Theorem 2.1] obtained the next log-majorization of Araki's type.

Theorem 6.1 (BLP Inequality, 2005) *For $A, B \geq 0$,*

$$A^{\frac{1+q}{2}} B^q A^{\frac{1+q}{2}} \prec_{\log} A^{\frac{1}{2}} \big(A^{\frac{r}{2}} B^r A^{\frac{r}{2}}\big)^{\frac{q}{r}} A^{\frac{1}{2}}, \qquad 0 < q \leq r. \tag{22}$$

Proof The equality of the determinants is clear. It is enough to prove that

$$\lambda_1\Big(A^{\frac{1+q}{2}} B^q A^{\frac{1+q}{2}}\Big) \leq \lambda_1\Big(A^{\frac{1}{2}} \big(A^{\frac{r}{2}} B^r A^{\frac{r}{2}}\big)^{\frac{q}{r}} A^{\frac{1}{2}}\Big) \tag{23}$$

when A is invertible, otherwise a continuity argument is used. Assume that

$$A^{\frac{1}{2}} \big(A^{\frac{r}{2}} B^r A^{\frac{r}{2}}\big)^{\frac{q}{r}} A^{\frac{1}{2}} \leq I_n,$$

that is,

$$A^{-1} \geq \big(A^{\frac{r}{2}} B^r A^{\frac{r}{2}}\big)^{\frac{q}{r}} \geq 0.$$

By Furuta inequality, since $r > 0$, $\frac{r}{q} \geq 1$ and $(1+r)\frac{r}{q} \geq \frac{r}{q} + r$, we find

$$A^{-(1+q)} = A^{-\frac{r/q+r}{r/q}} \geq \big(A^{-\frac{r}{2}} \big(A^{\frac{r}{2}} B^r A^{\frac{r}{2}}\big)^{\frac{q}{r}\frac{r}{q}} A^{-\frac{r}{2}}\big)^{\frac{q}{r}} = B^q,$$

that is,

$$A^{\frac{1+q}{2}} B^q A^{\frac{1+q}{2}} \leq I_n$$

and then (23) holds. Using Lemma 2.4, the result follows from (23) replacing A, B by $A^{\wedge k}, B^{\wedge k}$, respectively, for $k = 1, \ldots, n$, by properties **P1**, **P3**, **P5**. □

For convenience of notation, for $\alpha \in \mathbb{R}$, let

$$A \natural_\alpha B = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$$

which is just the α-weighted geometric mean of $A, B \geq 0$ when $\alpha \in [0, 1]$.

Corollary 6.2 *If* $A, B > 0$ *and* $1 \le \alpha \le 2$*, then*

$$A^{1-\alpha} B^{\alpha} \prec_{\log} A \natural_{\alpha} B.$$

Proof The case $\alpha = 1$ is trivial. For $\alpha > 1$, let $q = \alpha - 1$, $r = 1$, replace A, B by B, A^{-1}, respectively, in Theorem 6.1 and note that $B \natural_{1-\alpha} A = A \natural_{\alpha} B$. □

A. Matsumoto, R. Nakamoto and M. Fujii [75, Theorem 1] proved that

$$\left\| A^{\frac{s+q}{2}} B^q A^{\frac{s+q}{2}} \right\| \le \left\| A^{\frac{s}{2}} (A^{\frac{p}{2}} B^p A^{\frac{p}{2}})^{\frac{q}{p}} A^{\frac{s}{2}} \right\|, \qquad 0 < q \le p, \quad s \ge 0, \tag{24}$$

for $A, B \ge 0$, which reduces to Araki-Cordes inequality [31] if $s = 0$. They also proved (24) with the reverse inequality sign if $0 \le s \le p \le q$ and $p > 0$ [75, Theorem 2]. Furuta [39, Corollary 3.1 iii.] obtained a norm inequality, that yields the reverse of (24) for $0 \le q \le p$ and $-s \ge q$ (see [64, p.28]). These norm inequalities can be restated as follows.

Theorem 6.3 *Let* $A, B \ge 0$. *If* $0 < q \le p$ *and* $s \ge 0$*, then*

$$A^{\frac{s+q}{2}} B^q A^{\frac{s+q}{2}} \prec_{\log} A^{\frac{s}{2}} \left(A^{\frac{p}{2}} B^p A^{\frac{p}{2}}\right)^{\frac{q}{p}} A^{\frac{s}{2}}. \tag{25}$$

If either $0 \le s \le p \le q$ *and* $p > 0$ *or* $0 \le q \le p$ *and* $-s \ge q$*, then (25) holds with reversed log-majorization.*

Araki's log-majorization is obtained if $s = 0$ and BLP inequality if $s = 1$.

Corollary 6.4 *If* $A > 0$, $B \ge 0$ *and* $\alpha \ge 2$*, then*

$$A \natural_{\alpha} B \prec_{\log} A^{1-\alpha} B^{\alpha}.$$

Proof Let $q = \alpha - 1$, $p = s = 1$ and replace A, B by B, A^{-1}, respectively, in Theorem 6.3. □

Clearly, if $\alpha = 2$, the matrices in both hand sides of the log-majorizations given in Corollary 6.2 and Corollary 6.4 have the same eigenvalues.

The *Umegaki relative entropy* [92] of the density matrices A, B is

$$S(A, B) = \operatorname{tr}(A(\log A - \log B)).$$

Fujii and Kamei [32] introduced the variant

$$\hat{S}(A|B) = A^{\frac{1}{2}} \log(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}.$$

A logarithmic trace inequality [64] is now presented.

Theorem 6.5 *Let $A, B > 0$. If $q, s \geq 0$, then*

$$\operatorname{tr}(A^s(\log A^q + \log B^q)) \leq \operatorname{tr}\left(A^s \log(A^{\frac{p}{2}} B^p A^{\frac{p}{2}})^{\frac{q}{p}}\right), \qquad p > 0, \tag{26}$$

and the left hand side converges to the right hand side as p converges to 0.

Proof The log-majorization (25) implies the trace inequality

$$\operatorname{tr}(A^s A^q B^q) \leq \operatorname{tr}\left(A^s (A^{\frac{p}{2}} B^p A^{\frac{p}{2}})^{\frac{q}{p}}\right), \qquad 0 \leq q \leq p, \quad s \geq 0,$$

occuring trace equality when $q = 0$. Taking the derivatives of the right and left hand sides of the previous inequality at $q = 0$, observing that

$$\frac{d}{dq}\left(A^q B^q\right)\Big|_{q=0} = \log A + \log B, \tag{27}$$

$$\frac{d}{dq}\left((A^{\frac{p}{2}} B^p A^{\frac{p}{2}})^{\frac{q}{p}}\right)\Big|_{q=0} = \log\left(A^{\frac{p}{2}} B^p A^{\frac{p}{2}}\right)^{\frac{1}{p}}, \quad p > 0, \tag{28}$$

yields a trace inequality. Multiplying both hand sides of the obtained trace inequality by q provides (26). By the parametric Lie-Trotter formula, we may see that (28) converges to (27) as p converges to 0. □

The case $q = s = 1$ in Theorem 6.5 is due to Hiai and Petz [45]. It was later complemented in [5]. Using relative entropy terminology, Theorem 6.5 for $q = s$, replacing B by B^{-1}, may be written in the condensed form

$$S(A^s, B^s) \leq -\frac{s}{p}\operatorname{tr}\left(\hat{S}(A^p|B^p)A^{s-p}\right), \qquad s \geq 0, \quad p > 0,$$

this providing an upper bound for the relative entropy $S(A, B)$ when $s = 1$.

Fujii, Nakamoto and Tominaga [34] improved BLP inequality as follows.

Theorem 6.6 *If $A, B \geq 0$, $p \geq 1$, $q \geq 0$, then*

$$\left\| A^{\frac{1+q}{2}} B^{1+q} A^{\frac{1+q}{2}} \right\|^{\frac{p+q}{p(1+q)}} \leq \left\| A^{\frac{1}{2}} \left(A^{\frac{q}{2}} B^{q+p} A^{\frac{q}{2}}\right)^{\frac{1}{p}} A^{\frac{1}{2}} \right\|.$$

The next log-majorization [73] is obtained from Furuta inequality too.

Theorem 6.7 (Matharu-Aujla Inequality, 2012) *Let $A, B \geq 0$ and $0 \leq \alpha \leq 1$, then*

$$A \natural_\alpha B \prec_{\log} A^{1-\alpha} B^\alpha.$$

Proof If $\alpha = 0$ or $\alpha = 1$, the result is trivial. Let $0 < \alpha < 1$. Clearly, $A \sharp_\alpha B$ and $A^{1-\alpha} B^\alpha$ have the same determinant. Let us prove that

$$\lambda_1(A \sharp_\alpha B) \leq \lambda_1(A^{1-\alpha} B^\alpha). \tag{29}$$

If A is invertible and $\lambda_1(A^{1-\alpha} B^\alpha) \leq 1$, then

$$A^{-(1-\alpha)} \geq B^\alpha.$$

By Furuta inequality with $p = q = \frac{1}{\alpha} \geq 1$ and $r \geq 0$, we find

$$\left(A^{-(1-\alpha)}\right)^{1+\alpha r} \geq \left(\left(A^{-(1-\alpha)}\right)^{\frac{r}{2}} B \left(A^{-(1-\alpha)}\right)^{\frac{r}{2}}\right)^\alpha .$$

Taking $r = \frac{1}{1-\alpha}$ yields

$$A^{-1} \geq \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^\alpha,$$

so that $\lambda_1(A \sharp_\alpha B) \leq 1$ holds. If A is not invertible, by a continuity argument, (29) is obtained. Using Lemma 2.4, the result follows from (29) replacing A, B by $A^{\wedge k}$, $B^{\wedge k}$, respectively, for $k = 1, \ldots, n$, by properties **P1**, **P3**, **P5**. □

Furuta considered other operator inequalities implying the generalized BLP inequality [38] as well as Matharu-Aujla inequality [39].

7 Inequalities for Operator Connections

In this section, some inequalities involving operators connections are presented. For that purpose, we recall that the *dual* of a nonzero operator connection σ is the operator connection $\sigma^\perp$ defined by

$$A \,\sigma^\perp B = \left(B^{-1} \sigma \, A^{-1}\right)^{-1}$$

for $A, B > 0$ and extended by continuity to $A, B \geq 0$ as usual. Its representing function satisfies

$$f_{\sigma^\perp}(t) = t / f_\sigma(t), \qquad t > 0.$$

In the sequel, for $C, X \in M_n(\mathbb{C})$ the condensed notation $X^\sim$ stands for X or X^T, whereas $C \in H_n^\sim$ means that if the symbol $^\sim$ is omitted along the stated result, then C is assumed Hermitian, and if $^\sim$ acts as the transpose along the result, then C is assumed symmetric.

Theorem 7.1 *Let $A, B \geq 0$ and $C \in H_n^{\sim}$. If the representing functions of the nonzero operator connections σ, τ, ρ satisfy $f_\sigma^2 \leq f_\tau f_\rho$, then*

$$s_1\big((A\,\tau^\perp B)^{\frac{1}{2}} C^*(A\,\sigma\,B)^\sim C\,(A\,\rho^\perp B)^{\frac{1}{2}}\big) \leq \lambda_1(A\,C^*B^\sim C). \tag{30}$$

Proof For C Hermitian, there exists U unitary, such that $U^*CU = D$ is real diagonal and it is enough to prove that

$$s_1\big((A\,\tau^\perp B)^{\frac{1}{2}} D(A\,\sigma\,B)D\,(A\,\rho^\perp B)^{\frac{1}{2}}\big) \leq \lambda_1(ADBD),$$

since we may replace A, B by U^*AU, U^*BU, respectively, and apply the transformer inequality. If C is symmetric, by Takagi's factorization [54, Corollary 4.4.4], there exist V unitary and D diagonal with the singular values of C in its main diagonal, such that $C = V D V^T$. In this case, we need to show that

$$s_1\big((A\,\tau^\perp B)^{\frac{1}{2}} D(A\,\sigma\,B)^T D\,(A\,\rho^\perp B)^{\frac{1}{2}}\big) \leq \lambda_1(ADB^TD)$$

from which the result follows, replacing A, B by $V^TA\overline{V}$, $V^TB\overline{V}$, respectively.

Thus, assuming D to be a real diagonal matrix, we will check that

$$\lambda_1(ADB^\sim D) \leq 1 \quad \Rightarrow \quad s_1\big((A\,\tau^\perp B)^{\frac{1}{2}} D\,(A\,\sigma\,B)^\sim D\,(A\,\rho^\perp B)^{\frac{1}{2}}\big) \leq 1. \tag{31}$$

Firstly, let $A, B > 0$. If $\lambda_1(ADB^\sim D) \leq 1$, equivalently, $\lambda_1(DA^\sim DB) \leq 1$, then $DA^\sim D \leq B^{-1}$ and $DB^\sim D \leq A^{-1}$. By the transformer inequality **C2** and the joint monotonicity **C1**, we find

$$\begin{aligned} D\,(A\,\rho^\perp B)^\sim D = D\left(A^\sim \rho^\perp B^\sim\right) D &\leq (DA^\sim D)\,\rho^\perp (DB^\sim D) \\ &\leq B^{-1}\rho^\perp A^{-1} \\ &= (A\,\rho\,B)^{-1}. \end{aligned}$$

Analogously, $D\,(A\,\tau^\perp B)^\sim D \leq (A\,\tau\,B)^{-1}$. Under the hypothesis, we see that

$$A^{\frac{1}{2}} f_\sigma(M)\big(f_\rho(M)\big)^{-1} f_\sigma(M)\,A^{\frac{1}{2}} \leq A^{\frac{1}{2}} f_\tau(M)\,A^{\frac{1}{2}} = A\,\tau\,B, \tag{32}$$

where $M = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$. Therefore

$$\lambda_1\big((A\,\sigma\,B)(A\,\rho\,B)^{-1}(A\,\sigma\,B)(A\,\tau\,B)^{-1}\big) \leq 1. \tag{33}$$

Moreover,

$$s_1\big((A\,\tau^\perp B)^{\frac{1}{2}} D\,(A\,\sigma\,B)^\sim D\,(A\,\rho^\perp B)^{\frac{1}{2}}\big)$$

is equal to the square root of

$$\lambda_1\big((A\,\sigma\,B)\,D\,(A\,\rho^{\perp}B)^{\sim}D\,(A\,\sigma\,B)\,D\,(A\,\tau^{\perp}B)^{\sim}D\big). \tag{34}$$

Now, it is clear that (34) is not greater than (33) and the implication (31) holds. If $A, B \geq 0$, by a continuity argument, the result is obtained. □

If the nonzero operator connections σ, τ, ρ satisfy $f_\sigma^2 \geq f_\tau f_\rho$, then (30) holds with each connection replaced by its dual [65].

Applying Theorem 7.1 for $A, B \geq 0$, $C \in H_n^{\sim}$ and $\sigma \leq \tau = \rho$ yields

$$\lambda_1\big((A\,\tau^{\perp}B)\,C^*(A\,\sigma\,B)^{\sim}C\big) \;\leq\; \lambda_1(A\,C^*B^{\sim}C). \tag{35}$$

If $\tau = \sigma$ and $\sim$ is absent, this was observed in [64, Theorem 2.1] for $C \geq 0$.

Corollary 7.2 *If $A, B \geq 0$, $C \in H_n^{\sim}$ and σ is a nonzero operator connection, then*

$$(A\,\sharp\,B)\,C^*(A\,\sharp\,B)^{\sim}\,C \prec_{\log} \left|(A\,\sigma^{\perp}B)^{\frac{1}{2}}\,C^*(A\,\sharp\,B)^{\sim}\,C\,(A\,\sigma\,B)^{\frac{1}{2}}\right|$$
$$\prec_{\log} (A\,\sigma^{\perp}B)\,C^*(A\,\sigma\,B)^{\sim}\,C.$$

Proof We can see that

$$\lambda_1\big((A\,\sharp\,B)\,C^*(A\,\sharp\,B)^{\sim}C\big) \leq s_1\big(A^{\frac{1}{2}}\,C^*(A\,\sharp\,B)^{\sim}C\,B^{\frac{1}{2}}\big) \leq \lambda_1(A\,C^*B^{\sim}C). \tag{36}$$

The last inequality in (36) is the case τ, ρ as the trivial operator means w_l, w_r and $\sigma = \sharp$ in Theorem 7.1. The first inequality in (36) follows after taking square roots of the obtained eigenvalues from the case $\tau = \sigma = \sharp$ with $^{\sim}$ deleted in (35), then replacing C by $C^*(A\,\sharp\,B)^{\sim}C$. Applying Weyl's trick to (36) and observing the equality of the determinants of the matrices involved, a log-majorization is obtained. Next, replace A by $A\,\sigma^{\perp}B$, B by $A\,\sigma\,B$ in that log-majorization and use the identity $(A\,\sigma^{\perp}B)\,\sharp\,(A\,\sigma\,B) = A\,\sharp\,B$. □

Corollary 7.3 *If $A, B \geq 0$, $C \in H_n^{\sim}$ and $0 \leq \alpha \leq \beta \leq 1$, then*

$$\left|(A\sharp_{1-\alpha}B)^{\frac{1}{2}}\,C^*(A\,\sharp_{\frac{\beta}{2}}\,B)^{\sim}\,C\,(A\,\sharp_{1+\alpha-\beta}B)^{\frac{1}{2}}\right| \prec_{\log} A\,C^*B^{\sim}\,C. \tag{37}$$

Proof If $\sigma = \sharp_{\frac{\beta}{2}}$, $\tau = \sharp_\alpha$, $\rho = \sharp_{\beta-\alpha}$ in Theorem 7.1, since $\beta - \alpha \in [0, 1]$, then

$$s_1\big((A\,\sharp_{1-\alpha}B)^{\frac{1}{2}}\;C^*(A\,\sharp_{\frac{\beta}{2}}\,B)^{\sim}\,C\,(A\,\sharp_{1+\alpha-\beta}B)^{\frac{1}{2}}\big) \;\leq\; \lambda_1(A\,C^*B^{\sim}\,C).$$

Replace A, B, C by their kth compounds and apply properties **P1–P6**. The determinants of the matrices in both hand sides of (37) are equal. □

Remark 7.4 Let $A, B \geq 0$, $C \in H_n^{\sim}$ and $\alpha \in [0, 1]$. If $\sigma = \sharp_\alpha$ in Corollary 7.2 and $\beta = 1$ in Corollary 7.3, then

$$\left|(A\sharp_{1-\alpha}B)^{\frac{1}{2}}\, C^*(A \sharp B)^{\sim}\, C\, (A \sharp_\alpha B)^{\frac{1}{2}}\right|$$

is log-majorized by $(A \sharp_{1-\alpha} B)\, C^*(A \sharp_\alpha B)^{\sim}\, C$ and $A\, C^* B^{\sim}\, C$, respectively, being these two matrices related as follows:

$$(A \sharp_{1-\alpha} B)\, C^*(A \sharp_\alpha B)^{\sim}\, C \;\prec_{\log}\; A\, C^* B^{\sim} C \tag{38}$$

as a consequence of applying Weyl's trick to (35) when $\sigma = \tau = \sharp_\alpha$. In particular, this implies the next trace inequality observed by Bhatia, Lim and Yamazaki [23]:

$$\operatorname{tr}\left((A\sharp_{1-\alpha}B)(A\sharp_\alpha B)\right) \leq \operatorname{tr}(AB).$$

The question on whether it is possible to extend (38) to

$$(A\, \sigma^{\perp} B)\, C^*(A\, \sigma\, B)^{\sim}\, C \;\prec_{\log}\; A\, C^* B^{\sim} C$$

for other operator connections σ naturally arises.

Theorem 7.5 *If $A, B \geq 0$ and $r \in \mathbb{N}_0$, then*

$$(A\sharp B)^{r+1} \prec_{\log} |A^{\frac{1}{2}}(A\sharp B)^r B^{\frac{1}{2}}| \prec_{\log} (AB)^{\frac{r+1}{2}}.$$

Proof We can check that

$$\lambda_1(A \sharp B)^{2(r+1)} \leq \lambda_1\left(A\,(A \sharp B)^r B\,(A \sharp B)^r\right) \leq \lambda_1(AB)^{r+1}. \tag{39}$$

The first inequality in (39) follows from (36) when the symbol $\sim$ is absent and $C = (A\sharp B)^r$. Concerning the second, if $A > 0$ and $\lambda_1(AB)^{r+1} \leq 1$, then $B \leq A^{-1}$ implies

$$(A \sharp B)\, B\, (A \sharp B) \leq (A\sharp B)\, A^{-1}(A \sharp B) = B \leq A^{-1}.$$

By induction on $r \in \mathbb{N}_0$, we easily prove that $(A \sharp B)^r B\, (A \sharp B)^r \leq A^{-1}$, so

$$\lambda_1\left(A\,(A \sharp B)^r B\,(A \sharp B)^r\right) \leq 1.$$

Thus, the last inequality in (39) is true. By continuity, it remains valid for $A \geq 0$. After applying Weyl's trick to (39), the obtained log-majorization implies the log-majorization between the corresponding square roots. □

If $A, B \geq 0$, the last log-majorization in Theorem 7.5 and $AB \prec_{\log} |AB|$ which follows readily from Weyl's Majorant Theorem yield

$$\prod_{i=1}^{k} s_i\left(A^{\frac{1}{2}}(A \sharp B)^r B^{\frac{1}{2}}\right) \leq \prod_{i=1}^{k} s_i^{\frac{r+1}{2}}(AB), \qquad k = 1, \ldots, n,$$

for $r \in \mathbb{N}_0$. If $r = 1$, these inequalities were obtained by Zou [102].

Conjecture 7.6 *If $A, B \geq 0$ then*

$$\left|A^{\alpha}(A \sharp_{\alpha} B)B^{1-\alpha}\right| \prec_{\log} |AB|$$

for all $\alpha \in [0, 1]$.

8 Ando and Visick's Inequalities for the Hadamard Product

In this section, Ando and Visick's inequalities [4, 94] for the Hadamard product of positive definite matrices, which settled affirmatively Bapat and Johnson's conjecture [56], are revisited and weighted interpolations are presented.

We recall that a map $\Phi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ is called *positive* if $A \geq 0$ implies $\Phi(A) \geq 0$ and it is called *unital* if $\Phi(I_m) = I_n$.

Lemma 8.1 ([1]) *If $\Phi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ is a unital positive linear map and f is operator monotone on $\mathbb{R}_0^+$, then*

$$f(\Phi(A)) \geq \Phi(f(A)), \qquad A \geq 0.$$

The proof presented below of Ando and Visick's inequalities follows Ando's arguments [4]. We state these results in the following condensed form, where $^\sim$ is either omitted or acts as the transpose.

Theorem 8.2 *If $A, B > 0$, then $A \circ B \prec^{w\log} AB^{\sim}$, that is,*

$$\prod_{i=k}^{n} \lambda_i(A \circ B) \geq \prod_{i=k}^{n} \lambda_i(AB^{\sim}), \qquad k = 1, \ldots, n. \tag{40}$$

Proof There exits a unital positive linear map Φ such that $\Phi(X \otimes Y) = X \circ Y$ for all $X, Y \in M_n(\mathbb{C})$. For $A, B > 0$,

$$\log(A \otimes B) = \log A \otimes I_n + I_n \otimes \log B.$$

Then $H = \log A$, $K = \log B$ are Hermitian and

$$\Phi\big(\log(A \otimes B)\big) = H \circ I_n + I_n \circ K = I_n \circ (H + K).$$

Using Lemma 8.1 with $f(t) = \log t$, $t > 0$, we have

$$\log\big(\Phi(A \otimes B)\big) = \log(A \circ B) \geq I_n \circ (H + K).$$

By Schur Majorization Theorem, $I_n \circ (H + K) \prec H + K$ holds, as $H + K$ is Hermitian. Therefore,

$$\sum_{i=k}^{n} \lambda_i(\log(A \circ B)) \geq \sum_{i=k}^{n} \lambda_i(I_n \circ (H + K)) \geq \sum_{i=k}^{n} \lambda_i(H + K)$$

for $k = 1, \ldots, n$. It follows that

$$\begin{aligned} \prod_{i=k}^{n} \lambda_i(A \circ B) &\geq \prod_{i=k}^{n} e^{\lambda_i(I_n \circ (H+K))} \geq \prod_{i=k}^{n} e^{\lambda_i(H+K)} = \prod_{i=k}^{n} \lambda_i\left(e^{H+K}\right) \\ &\geq \prod_{i=k}^{n} \lambda_i(e^H e^K) = \prod_{i=k}^{n} \lambda_i(AB), \qquad k = 1, \ldots, n. \end{aligned}$$

The last inequality is a consequence of the Golden-Thompson type log-majorization (15). Then (40) with $\sim$ deleted is proved.

Since K and K^T have the same diagonal entries, $I_n \circ (H + K)$ may be replaced by $I_n \circ (H + K^T)$ in the proof above. In such case, $e^H e^K$ is replaced by $e^H e^{K^T} = AB^T$ and (40) with $\sim$ acting as the transpose is fullfilled. □

Remark 8.3 For $A, B > 0$, by the Lie-Trotter formula, we have

$$\lim_{p \to 0} \left(A^p B^p\right)^{\frac{1}{p}} = e^{\log A + \log B}$$

and a Lie-Trotter type formula for the Hadamard product [97] is

$$\lim_{p \to 0} \left(A^p \circ B^p\right)^{\frac{1}{p}} = e^{I_n \circ (\log A + \log B)}$$

(see also [95, Theorem 1]). According to the previous proof, we can write

$$A \circ B \prec^{w\log} \lim_{p \to 0} \left(A^p \circ B^p\right)^{\frac{1}{p}} \prec^{w\log} \lim_{p \to 0} \left(A^p (B^{\sim})^p\right)^{\frac{1}{p}} \prec_{\log} AB^{\sim}.$$

Moreover, for $A, B > 0$ and $r > 0$, Visick [94] obtained

$$\sum_{i=k}^{n} \lambda_i (A \circ B)^{-r} \leq \sum_{i=k}^{n} \lambda_i \big(AB^{\sim}\big)^{-r}, \qquad k = 1, \ldots, n,$$

and deduced Theorem 8.2 from it. In fact, this is equivalent to Theorem 8.2 as shown by Bebiano and Perdigão [10]. One of the implications is a trivial consequence of the following limit:

$$\lim_{r \to 0} \frac{\lambda^{-r} - 1}{r} = -\log \lambda, \qquad \lambda > 0.$$

To prove the other, note that (40) implies

$$(A \circ B)^{-1} \prec_{w\log} (AB^{\sim})^{-1}.$$

Considering the function $f(t) = \log(1 + \epsilon\, \mathrm{e}^{rt})$, which is convex and increasing for $t > 0$, with $\epsilon, r > 0$, by Proposition 1.3 ii, we obtain

$$\prod_{i=k}^{n} \big(1 + \epsilon\, \lambda_i (A \circ B)^{-r}\big) \leq \prod_{i=k}^{n} \big(1 + \epsilon\, \lambda_i (AB^{\sim})^{-r}\big), \quad k = 1, \ldots, n.$$

The implication follows, because

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\prod_{i=k}^{n} \big(1 + \epsilon\, \lambda_i^{-r}\big) - 1 \right) = \sum_{i=k}^{n} \lambda_i^{-r}.$$

Theorem 8.4 *Let $A, B > 0$ and $D \in M_n(\mathbb{C})$ be a diagonal matrix, assumed real when $^{\sim}$ is absent. If $\alpha \in [0, 1]$, then*

$$\begin{aligned} \prod_{i=k}^{n} \lambda_i \big((A \circ B)\, |D|^2\big) &\geq \prod_{i=k}^{n} \lambda_i \big((A \sharp B)\, \overline{D}\, (A \sharp B)^{\sim} D\big) \\ &\geq \prod_{i=k}^{n} \lambda_i \big((A \sharp_{1-\alpha} B)\, \overline{D}\, (A \sharp_{\alpha} B)^{\sim} D\big) \\ &\geq \prod_{i=k}^{n} \lambda_i \big(A\, \overline{D}\, B^{\sim} D\big), \qquad k = 1, \ldots, n, \end{aligned} \tag{41}$$

equality occurring for $k = 1$ in the last two inequalities.

Proof Let $D \in M_n(\mathbb{C})$ be a diagonal matrix. Then

$$D(A \circ B)\overline{D} \geq D\big((A \sharp B) \circ (A \sharp B)\big)\overline{D} = \big(D(A \sharp B)\overline{D}\big) \circ (A \sharp B)$$

and replacing A, B in Theorem 8.2 by $D(A \sharp B)\overline{D}$, $A \sharp B$, respectively, yields

$$(A \circ B)|D|^2 \prec^{w\log} (D(A \sharp B)\overline{D}) \circ (A \sharp B) \prec^{w\log} (A \sharp B)\overline{D}(A \sharp B)^{\sim} D.$$

Further, if $C = D \in H_n^{\sim}$ in Corollary 7.2 with $\sigma = \sharp_\alpha$, $\alpha \in [0, 1]$, and in (38), the result is obtained. □

If $^{\sim}$ is deleted and $D = I_n$, then (41) was previously given by Ando [4, Theorem 2] and, in this case, the remaining inequalities were obtained by Hiai and Lin [48]. The complete version in Theorem 8.4 is derived in [65]. The inequalities in (41) hold for $A, B > 0$, $k = 1, \ldots, n$ and D diagonal, but

$$\prod_{i=k}^{n} \lambda_i\big((A \circ B)\,|D|^2\big) \geq \prod_{i=k}^{n} \lambda_i\big((A\sharp B)\,D^*(A\sharp B)^{\sim} D\big) \tag{42}$$

does not remain true, in general, when D is replaced by any Hermitian matrix.

Example Consider

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 1+i \\ 1-i & -3 \end{bmatrix}.$$

In this case, $A\sharp B = B^{\frac{1}{2}}$ and (42) with $\sim$ absent does not hold, because

$$\lambda_2((A \circ B)\,D^2) \approx 3.783 \leq \lambda_2\big(B^{\frac{1}{2}} D\big)^2 \approx 4.095.$$

9 Indefinite Inequalities

The permanent of $A = (a_{ij}) \in M_n(\mathbb{C})$ is denoted and defined as

$$\operatorname{per} A = \sum_{\sigma \in S_n} \prod_{j=1}^{n} a_{j\sigma(j)}, \quad \sigma \in S_n.$$

Although permanents and determinants have similar definitions and share some common properties, they exhibit substancial differences, such as the non-multiplicativity of the permanent.

In 1926, van der Waerden raised a question [93] and motivated a conjecture: the minimum of the permanent of a n-square doubly stochastic matrix is $\frac{n!}{n^n}$ and equality occurs when every entry of the matrix equals $\frac{1}{n}$. This conjecture attracted the attention of mathematicians all over the world, although it remained open for more than fifty years. The proof of this famous conjecture by G. P. Egoritjev [26] in 1981, also proved by Falikman [27], is based on an inequality for permanents, which is a special case of a result of A. D. Alexandroff on positive definite quadratic forms. In what follows we write $\operatorname{per} A = \operatorname{per}(a_1, \ldots, a_n)$ with a_i the ith column of A.

Theorem 9.1 (Alexandroff Permanental Inequality) *For $a_1, \ldots, a_n \in \mathbb{R}^n$,*

$$\operatorname{per}(a_1, \ldots, a_{n-1}, b)^2 \geq \operatorname{per}(a_1, \ldots, a_{n-1}, a_{n-1})\operatorname{per}(a_1, \ldots, b, b),$$

with equality if and only if $b = \lambda a_{n-1}$ for some constant λ.

This inequality resembles Schwartz inequality, but the direction of the inequality sign is reversed. The reason is the following. Taking the permanent as the inner product in $\mathbb{R}^n$:

$$\langle x, y\rangle = \operatorname{per}(a_1, \ldots, a_{n-2}, x, y),$$

the space $\mathbb{R}^n$ is no longer *Euclidean* but *Lorentzian*, accordingly as the length of the vector $(x_1, \ldots, x_n)$ is

$$x_1^2 + \cdots + x_n^2 \qquad \text{or} \qquad x_1^2 - x_2^2 \cdots - x_{n-1}^2 - x_n^2.$$

That is, we are dealing now with a so called *indefinite inner product space*.

In this section, we present miscellaneous indefinite inequalities obtained in this set up. First, we introduce some definitions and notations.

Let J be a Hermitian involutive matrix, that is, $J^* = J$ and $J^2 = I_n$. Consider $\mathbb{C}^n$ endowed with the indefinite inner product induced by J, given by $[x, y] = y^*Jx$ for all $x, y \in \mathbb{C}^n$. Let $A^\sharp = JA^*J$. A matrix $A \in M_n(\mathbb{C})$ is said to be *J-Hermitian* if $A = A^\sharp$, that is, if JA is Hermitian. These matrices appear in several problems of relativistic quantum mechanics and quantum physics. Let $A, B \in M_n(\mathbb{C})$ be J-Hermitian and consider $A \geq^J B$ defined by

$$[Ax, x] \geq^J [Bx, x], \quad x \in \mathbb{C}^n,$$

which means that $J(A - B) \geq 0$. A matrix $A \in M_n(\mathbb{C})$ is called a *J-contraction* if $I_n \geq^J A^\sharp A$. It is well known that the eigenvalues of a J-Hermitian matrix $A \in M_n(\mathbb{C})$ may not be real, nevertheless its spectrum is symmetric relative to the real axis. If A is J-Hermitian and $I_n \geq^J A$, then all the eigenvalues of A are real. In fact, in this case, $I_n - A$ is the product of the Hermitian matrix J and a positive semidefinite matrix. If A is a J-contraction, by a Theorem of Potapov-Ginzburg [8, Chapter 2, Section 4], then all the eigenvalues of $A^\sharp A$ are nonnegative. Sano [83, Theorem 2.6] obtained the next indefinite version of Löwner-Heinz inequality.

Theorem 9.2 (Löwner Inequality of Indefinite Type, 2007) *If $A, B \in M_n(\mathbb{C})$ are J-Hermitian matrices with nonnegative eigenvalues, $I_n \geq^J A \geq^J B$ and $0 < \alpha < 1$, then the J-Hermitian powers A^α, B^α are well defined and*

$$I_n \geq^J A^\alpha \geq^J B^\alpha.$$

The case $\alpha = \frac{1}{2}$ in Theorem 9.2 is due to Ando [6, Theorem 6], being the cases $\alpha = 0$ and $\alpha = 1$ trivially satisfied. Motivated by these results, the Furuta inequality of indefinite type in (43) and (44) was established by Sano [83, Theorem 3.4] and Bebiano et al. [14, Theorem 2.1], respectively.

Theorem 9.3 (Furuta Inequality of Indefinite Type) *Let $A, B \in M_n(\mathbb{C})$ be J-Hermitian with nonnegative spectra, $\mu I_n \geq^J A \geq^J B$ (or $A \geq^J B \geq^J \mu I_n$) for some $\mu > 0$. Then for each $r \geq 0$,*

$$A^{\frac{p+r}{q}} \geq^J \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{1}{q}} \tag{43}$$

and

$$\left(B^{\frac{r}{2}} A^p B^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq^J B^{\frac{p+r}{q}} \tag{44}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1 + r)q \geq p + r$.

In particular, Löwner-Heinz inequality of indefinite type is recovered by Theorem 9.3 for $r = 0$.

In order to present the indefinite version of Theorem 3.4 obtained in [13, Corollary 1.2], assume $(r, n - r)$ to be the inertia of J and $0 < r < n$. Without loss of generality, we may consider

$$J = I_r \oplus -I_{n-r}, \qquad 0 < r < n.$$

For an arbitrary J-Hermitian matrix $A \in M_n(\mathbb{C})$, we denote by $\sigma_J^{\pm}(A)$ the set of eigenvalues of A with eigenvectors x, such that $x^* J x = \pm 1$. We say that A is *J-unitarily diagonalizable* if every eigenvalue of A belongs to either $\sigma_J^{+}(A)$ or to $\sigma_J^{-}(A)$. In this case, $\sigma_J^{+}(A)$ and $\sigma_J^{-}(A)$ have r and $n - r$ eigenvalues, respectively. Consider a J-Hermitian matrix A, whose eigenvalues $\alpha_1 \geq \cdots \geq \alpha_r$ belong to $\sigma_J^{+}(A)$ and $\alpha_{r+1} \geq \cdots \geq \alpha_n$ belong to $\sigma_J^{-}(A)$. In this case, the eigenvalues of A are said to *not interlace* if either $\alpha_r > \alpha_{r+1}$ or $\alpha_n > \alpha_1$, otherwise, they are said to *interlace.*

Theorem 9.4 *Let $J = I_r \oplus -I_{n-r}$, $0 < r < n$, and $A, C \in M_n(\mathbb{C})$ be non-scalar J-Hermitian and J-unitarily diagonalizable matrices with eigenvalues $\alpha_1 \geq \cdots \geq \alpha_r$ ($c_1 \geq \cdots \geq c_r$) in $\sigma_J^{+}(A)$ ($\sigma_J^{+}(C)$) and $\alpha_{r+1} \geq \cdots \geq \alpha_n$ ($c_{r+1} \geq \cdots \geq c_n$) in $\sigma_J^{-}(A)$ ($\sigma_J^{-}(C)$). If the eigenvalues of A and C do not interlace, then statements i. and ii. hold.*

i. *If* $(\alpha_k - \alpha_l)(c_{k'} - c_{l'}) < 0$ *for all* $1 \leq k, k' \leq r$, $r + 1 \leq l, l' \leq n$, *then*

$$\mathrm{tr}(CA) \leq \sum_{i=1}^{n} c_i \alpha_i .$$

ii. *If* $(\alpha_k - \alpha_l)(c_{k'} - c_{l'}) > 0$ *for all* $1 \leq k, k' \leq r$, $r + 1 \leq l, l' \leq n$, *then*

$$\sum_{i=1}^{r} c_i \alpha_{r-i+1} + \sum_{i=r+1}^{n} c_i \alpha_{n+r-i+1} \leq \mathrm{tr}(CA).$$

Several other inequalities of indefinite type have been studied. For instance, just to mention a few, we refer some spectral inequalities for the trace of the exponential or the logarithmic of J-Hermitian matrices [15], operator inequalities associated with Furuta inequality of indefinite type [16], a reversed Heinz-Kato-Furuta inequality [17] and indefinite versions of some determinantal inequalities [19], including a Fiedler-type theorem for the determinant of J-positive matrices [18].

Recently, Matharu, Malhotra and Moslehian [74] defined a J-mean associated with a positive matrix monotone function f on $(0, \infty)$, such that $f(1) = 1$, for J-Hermitian matrices with spectra in $(0, \infty)$. Fundamental properties of this J-mean, such as the power monotonicity and an indefinite version of Ando-Hiai inequality [74, Theorem 3.11] were obtained.

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Ando-Hiai Inequality: Extensions and Applications

Masatoshi Fujii and Ritsuo Nakamoto

Abstract The Ando-Hiai inequality says that if $A\#_\alpha B \le I$ for a fixed $\alpha \in [0, 1]$ and positive invertible operators A, B on a Hilbert space, then $A^r\#_\alpha B^r \le I$ for $r \ge 1$, where $\#_\alpha$ is the α-geometric mean defined by $A\#_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$. This chapter is devoted by extensions and applications of Ando-Hiai inequality. It is closely related to Furuta inequality, Bebiano-Lemos-Providência inequality and grand Furuta inequality. Consequently they are given useful extensions.

Keywords Ando-Hiai inequality · Furuta inequality · Grand Furuta ineqiality · Bebiano-Lemos-Providência inequality

1 Introduction

Throughout this chapter, an operator A means a bounded linear operator acting on a complex Hilbert space $\mathcal{H}$. An operator A is positive, denoted by $A \ge 0$, if $\langle Ax, x\rangle \ge 0$ for all $x \in \mathcal{H}$. We denote $A > 0$ if A is positive and invertible. The α-geometric mean $\#_\alpha$ is defined by $A\#_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$ for $A > 0$ and $B \ge 0$.

A log-majorization theorem due to Ando-Hiai [1] is expressed as follows: For $\alpha \in [0, 1]$ and positive definite matrices A and B,

$$(A\#_\alpha B)^r \succ_{(\log)} A^r\#_\alpha B^r \quad (r \ge 1).$$

The core in the proof is that

$$A\#_\alpha B \le I \;\Rightarrow\; A^r\#_\alpha B^r \le I \quad (r \ge 1).$$

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It holds for positive operators A, B on a Hilbert space, and is called the Ando-Hiai inequality, denoted by (AH) simply .

Now the original proof of (AH) is a typical application of Löwner-Heniz inequality (LH), i.e.,

$$A \geq B \geq 0 \;\Rightarrow\; A^r \geq B^r \quad (0 \leq r \leq 1).$$

See [18, 25] and [27] for (LH).

Related to (AH), we should mention the Furuta inequality. Because it is a beautiful extension of (LH). It is presented as follows:

Furuta Inequality (FI)

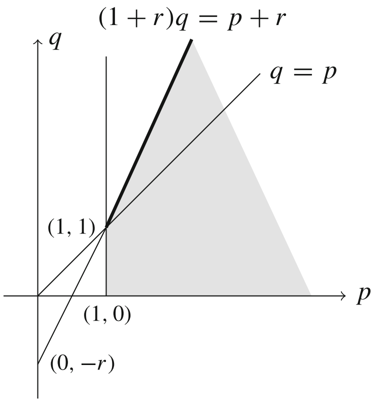
If $A \geq B \geq 0$, then for each $r \geq 0$,

$$\text{(i)} \qquad (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$$

and

$$\text{(ii)} \qquad (A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with
$(1+r)q \geq p+r$.



Related to Furuta inequality, see [4, 5, 15, 16, 31] and [9].

After publishing (AH), Furuta himself [17] presented so-called “grand Furuta inequality” which interpolates (AH) to his inequality (FI), see also [8] and [9].

Grand Furuta Inequality (GFI) If $A \geq B > 0$ and $t \in [0, 1]$, then

$$[A^{\frac{r}{2}} (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^s A^{\frac{r}{2}}]^{\frac{1-t+r}{(p-t)s+r}} \leq A^{1-t+r}$$

holds for $r \geq t$ and $p, s \geq 1$.

The relations among (AH), (FI) and (GFI) are as follows:
(GFI) for $t = 1,\ r = s \iff$ (AH)
(GFI) for $t = 0,\ (s = 1) \iff$ (FI).

On the other hand, we discussed the equivalence between (AH) and (FI) in [10]. Moreover we gave two variable version (GAH):

If $A\#_\alpha B \leq I$ for $\alpha \in [0, 1]$ and $A, B \geq 0$, then $A^r\#_\beta B^s \leq I$ for $r, s \geq 1$, where $\beta = \frac{\alpha r}{\alpha r+(1-\alpha)s}$.

The one-sided versions are considerable:

$$A \#_\alpha B \leq I \ \Rightarrow\ A^r \#_{\frac{\alpha r}{\alpha r+(1-\alpha)}} B \leq I \quad (r \geq 1);$$

$$A \#_\alpha B \leq I \ \Rightarrow\ A \#_{\frac{\alpha}{\alpha+(1-\alpha)s}} B^s \leq I \quad (s \geq 1).$$

It is known that both one-sided versions are equivalent, and that they are alternative expressions of (FI).

2 Extensions

This section is based on our paper [11]. First of all, a binary operation $\natural_\alpha$ is defined by the same formula as the α-geometric mean for $\alpha \notin [0, 1]$, that is,

$$A\natural_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}} \quad \text{for } A, B > 0.$$

Recently (AH) is extended by Seo [29] and [23] as follows: For $\alpha \in [-1, 0]$, $A\natural_\alpha B \leq I$ for $A, B > 0$ implies $A^r\natural_\alpha B^r \leq I$ for $r \in [0, 1]$.

So, following our previous work, we present two variable version of it. For this, we mention the following useful identity on the binary operation $\natural$: For $\beta \in \mathbb{R}$ and positive invertible operators X and Y,

$$X\natural_\beta Y = X(X^{-1}\natural_{-\beta} Y^{-1})X. \tag{1}$$

Lemma 2.1 *If $A\natural_\alpha B \leq I$ for $\alpha \in [-1, 0]$ and $A, B > 0$, then $A^r\natural_\beta B \leq I$ for $r \in [0, 1]$, where $\beta = \frac{\alpha r}{\alpha r+(1-\alpha)}$.*

Proof For convenience, we show that if $A^{-1}\natural_\alpha B \leq I$, then $A^{-r}\natural_\beta B \leq I$ for $r \in [0, 1]$. Thus the assumption ensures that $C^\alpha \leq A$, where $C = A^{\frac{1}{2}}BA^{\frac{1}{2}}$. Note that $\beta \in [-1, 0]$.

Now we first assume that $r = 1 - \epsilon \in [\frac{1}{2}, 1]$, i.e., $\epsilon \in [0, \frac{1}{2}]$. Then we have

$$\begin{aligned} A^{\epsilon} \natural_{\beta} C &= A^{\epsilon}(A^{-\epsilon} \#_{-\beta} C^{-1}) A^{\epsilon} \\ &\leq A^{\epsilon}(C^{-\alpha\epsilon} \#_{-\beta} C^{-1}) A^{\epsilon} \\ &= A^{\epsilon} C^{\alpha(1-2\epsilon)} A^{\epsilon} \\ &\leq A^{\epsilon} A^{1-2\epsilon} A^{\epsilon} = A. \end{aligned}$$

Hence it follows that

$$A^{-r} \natural_{\beta} B = A^{-\frac{1}{2}}(A^{\epsilon} \natural_{\beta} C) A^{-\frac{1}{2}} \leq A^{-\frac{1}{2}} A A^{-\frac{1}{2}} = I.$$

In particular, we note that $A^r \natural_{\beta} B \leq I$ for $r = \frac{1}{2}$, that is, $A^{-\frac{1}{2}} \natural_{\alpha_1} B \leq I$ holds for $\alpha_1 = \frac{\alpha}{2-\alpha}$. Hence it follows from the preceding paragraph that for $r \in [\frac{1}{2}, 1]$,

$$I \geq (A^{-\frac{1}{2}})^r \natural_{\beta_1} B = A^{-\frac{r}{2}} \natural_{\beta_1} B,$$

where $\beta_1 = \frac{\alpha_1 r}{\alpha_1 r + (1-\alpha_1)} = \frac{\alpha r/2}{\alpha r/2 + (1-\alpha)}$. This means that the desired inequality holds for $r \in [\frac{1}{4}, \frac{1}{2}]$. Finally we have the conclusion by the induction. □

Lemma 2.2 *If* $A \natural_{\alpha} B \leq I$ *for* $\alpha \in [-1, 0]$ *and* $A, B > 0$, *then* $A \natural_{\beta} B^s \leq I$ *for* $s \in [\frac{-2\alpha}{1-\alpha}, 1]$, *where* $\beta = \frac{\alpha}{\alpha + (1-\alpha)s}$.

Proof For convenience, we show that if $A \natural_{\alpha} B^{-1} \leq I$, then $A \natural_{\beta} B^{-s} \leq I$ for $s \in [\frac{-2\alpha}{1-\alpha}, 1]$. Thus the assumption is understood as $D^{1-\alpha} \leq B$, where $D = B^{\frac{1}{2}} A B^{\frac{1}{2}}$. We first note that $\beta \in [-1, 0]$ by $s \in [\frac{-2\alpha}{1-\alpha}, 1]$. So we put $s = 1 - \epsilon$ for some $\epsilon \in [0, 1 - \frac{-2\alpha}{1-\alpha}]$. Then we have

$$D \natural_{\beta} B^{\epsilon} = D(D^{-1} \#_{-\beta} B^{-\epsilon}) D \leq D(D^{-1} \#_{-\beta} D^{-\epsilon(1-\alpha)}) D = D^{1-\alpha} \leq B,$$

so that

$$A \natural_{\beta} B^{-s} = B^{-\frac{1}{2}}(D \natural_{\beta} B^{\epsilon}) B^{-\frac{1}{2}} \leq B^{-\frac{1}{2}} B B^{-\frac{1}{2}} = I.$$

□

Theorem 2.3 *If* $A \natural_{\alpha} B \leq I$ *for* $\alpha \in [-1, 0]$ *and* $A, B > 0$, *then* $A^r \natural_{\beta} B^s \leq I$ *for* $r \in [0, 1]$ *and* $s \in [\frac{-2\alpha r}{1-\alpha}, 1]$, *where* $\beta = \frac{\alpha r}{\alpha r + (1-\alpha)s}$.

Proof Suppose that $A \natural_{\alpha} B \leq 1$. Then Lemma 2.1 says that $A^r \natural_{\gamma} B \leq 1$ for $r \in [0, 1]$, where $\gamma = \frac{\alpha r}{\alpha r + (1-\alpha)}$. Next we apply Lemma 2.2 to this obtained inequality.

Then we have

$$1 \geq A^r \natural_{\frac{\gamma}{\gamma+(1-\gamma)s}} B^s = A^r \natural_{\frac{\alpha r}{\alpha r+(1-\alpha)s}} B^s$$

for $s \in [\frac{-2\gamma}{1-\gamma}, 1] = [\frac{-2\alpha r}{1-\alpha}, 1]$. □

As a special case $s = r$ in the above, we obtain Seo's original extension of (AH) because $\beta = \alpha$ (by $s = r$) and $r \in [\frac{-2\alpha r}{1-\alpha}, 1]$.

Corollary 2.4 *If* $A \natural_\alpha B \leq I$ *for* $\alpha \in [-1, 0]$ *and* $A, B > 0$*, then* $A^r \natural_\alpha B^r \leq I$ *for* $r \in [0, 1]$.

Remark 2.5 We here consider the condition $s \in [\frac{-2\alpha}{1-\alpha}, 1]$ in Lemma 2.2. In particular, take $\alpha = -1$. Then the assumption $A \natural_\alpha B \leq 1$ means that $B \geq A^2$, and $\beta = \frac{\alpha}{\alpha+(1-\alpha)s} = \frac{1}{1-2s}$. Though $s = 1$ in this case by $s \in [\frac{-2\alpha}{1-\alpha}, 1]$, the inequality in Lemma 2.2 still holds for $s \in [\frac{3}{4}, 1]$. We use the formula $X \natural_\gamma Y = Y \natural_{1-\gamma} X = Y(Y^{-1} \natural_{\gamma-1} X^{-1})Y$. Note that $-\beta \in [1, 2]$. Therefore we have

$$\begin{aligned} A \natural_\beta B^s &= A(A^{-1} \natural_\beta B^{-s})A = AB^{-s}(B^s \#_{\beta-1} A)B^{-s}A \\ &\leq AB^{-s}(B^s \#_{-\beta-1} B^{\frac{1}{2}})B^{-s}A = AB^{-1}A \leq AA^{-2}A = I. \end{aligned}$$

On the other hand, it is false for $s \in [0, \frac{1}{4}]$. Note that $\beta = \frac{1}{1-2s} \in [1, 2]$. Suppose to the contrary that $A \natural_\beta B^s \leq 1$ holds under the assumption $B \geq A^2$. Then it follows that $1 \geq A \natural_\beta B^s = B^s(B^{-s} \#_{\beta-1} A^{-1})B^s$ and so

$$B^{-2s} \geq B^{-s} \#_{\beta-1} A^{-1} \geq B^{-s} \#_{\beta-1} B^{-\frac{1}{2}} = B^{-2s},$$

so that $B = A^2$ follows, which is impossible in general.

Next we consider representations of Furuta type associated with extensions of (AH) obtained in the preceding discussion.

We here remark that the optimal case $(1+r)q = p+r$ is essential in (FI), which is realized as a beautiful formula by the use of the α-geometric mean:

If $A \geq B \geq 0$, *then for each* $r \geq 0$

$$A^{-r} \#_{\frac{1+r}{p+r}} B^p \leq A$$

holds for $p \geq 1$.

More precisely, the conclusion in above is improved by

$$A^{-r} \#_{\frac{1+r}{p+r}} B^p \leq B \ (\leq A)$$

holds for $p \geq 1$, due to Kamei [21].

The following result is also led by Lemma 2.1.

Theorem 2.6 *If* $A \geq B > 0$, *then*

$$A^{-r} \natural_{\frac{1+r}{p+r}} B^p \leq A$$

holds for $p \leq -1$ *and* $r \in [-1, 0]$.

Proof As in the proof of Lemma 2.1, it says that if $A^{-1} \natural_\alpha B \leq I$, then $A^{-r} \natural_\beta B \leq I$ for $r \in [0, 1]$, where $\beta = \frac{\alpha r}{\alpha r + (1-\alpha)}$. Thus the assumption is that $C^\alpha \leq A$, where $C = A^{\frac{1}{2}} B A^{\frac{1}{2}}$. So we put $B_1 = C^\alpha \leq A$, and $p = \frac{1}{\alpha}$, $r_1 = r - 1$. Then $p \leq -1$ and $r_1 \in [-1, 0]$ and $\beta = \frac{1+r_1}{p+r_1}$. Moreover the conclusion is rephrased as

$$A^{-r+1} \natural_\beta C \leq A, \text{ or } A^{-r_1} \natural_{\frac{1+r_1}{p+r_1}} B_1{}^p \leq A.$$

□

As well as (FI), (GFI) has also mean theoretic expression as follows:
If $A \geq B > 0$ *and* $t \in [0, 1]$, *then*

$$A^{-r+t} \#_{\frac{1-t+r}{(p-t)s+r}} (A^t \#_s B^p) \leq A$$

holds for $r \geq t$ and $p, s \geq 1$.

In succession with the above discussion, Theorem 2.3 gives us the following inequality of (GFI)-type:

Theorem 2.7 *If* $A \geq B > 0$, *then*

$$A^{-r+1} \natural_{\frac{r}{r+(p-1)s}} (A \#_s B^p) \leq A$$

holds for $p \leq -1$, $r \in [0, 1]$ *and* $s \in [\frac{-2r}{p-1}, 1]$.

Proof Theorem 2.3 says that if $A^{-1} \natural_\alpha B \leq I$, then $A^{-r} \natural_\beta B^s \leq I$ for $r \in [0, 1]$ and $s \in [\frac{-2\alpha r}{1-\alpha}, 1]$, where $\beta = \frac{\alpha r}{\alpha r + (1-\alpha)s}$. So the assumption is that $B_1 = C^\alpha \leq A$, where $C = A^{\frac{1}{2}} B A^{\frac{1}{2}}$. On the other hand, putting $\alpha = \frac{1}{p}$, it follows that

$$I \geq A^{-r} \natural_{\frac{\alpha r}{\alpha r + (1-\alpha)s}} B^s = A^{-r} \natural_{\frac{r}{r+(p-1)s}} (A^{-\frac{1}{2}} B_1{}^p A^{-\frac{1}{2}})^s$$

or equivalently

$$A \geq A^{-r+1} \natural_{\frac{r}{r+(p-1)s}} (A \#_s B_1{}^p),$$

which is the conclusion by $B_1^p = C = A^{\frac{1}{2}} B A^{\frac{1}{2}}$. □

From the viewpoint of (GFI), the following extension might be expected:

Conjecture If $A \geq B > 0$ and $t \in [0, 1]$, then

$$A^{-r+t} \natural_{\frac{1-t+r}{r+(p-t)s}} (A^t \#_s B^p) \leq A$$

holds for $p \leq -1$, $r \in [0, t]$ and $s \in [\frac{-2r}{p-t}, 1]$.

We can prove it under a restriction as follows:

Theorem 2.8 *If $A \geq B > 0$ and $t \in [0, 1]$, then*

$$A^{-r+t} \natural_{\frac{1-t+r}{r+(p-t)s}} (A^t \#_s B^p) \leq A$$

holds for $p \leq -1$, $r \in [0, t]$ and $s \in [\max\{\frac{-t}{p-t}, \frac{-2r-(1-t)}{p-t}\}, 1]$.

Proof First of all, we note that $r + (p - t)s \leq t + (p - t)s \leq 0$ by $s \geq \frac{-t}{p-t}$. So we have $\frac{1-t+r}{r+(p-t)s} \leq 0$. On the other hand, $-1 \leq \frac{1-t+r}{r+(p-t)s}$ is obtained by $-(r + (p - t)s) \geq 1 - t + r$ since the assumption $s \geq \frac{-2r-(1-t)}{p-t}$. Namely $\frac{-(1-t+r)}{r+(p-t)s} \in [0, 1]$. Hence we have

$$A^{r-t} \#_{\frac{-(1-t+r)}{r+(p-t)s}} (A^{-t} \#_s B^{-p}) \leq A^{r-t} \#_{\frac{-(1-t+r)}{r+(p-t)s}} B^{-(p-t)s-t} \leq A^{2(r-t)+1}.$$

The second inequality in above is shown as follows: The exponent $-(p - t)s - t$ of B is nonnegative as mentioned first. Thus, if $-(p - t)s - t \leq 1$, the second inequality holds. On the other hand, if $-(p-t)s-t \geq 1$, then the Furuta inequality assures that

$$(A^{\frac{t-r}{2}} B^{-(p-t)s-t} A^{\frac{t-r}{2}})^{\frac{1-t+r}{-(p-t)s-r}} \leq A^{1-t+r},$$

or equivalently

$$A^{r-t} \#_{\frac{1-t+r}{-(p-t)s-r}} B^{-(p-t)s-t} \leq A^{2(r-t)+1}.$$

Hence, noting that $X \natural_{-q} Y = X(X^{-1} \natural_q Y^{-1})X$, it follows that

$$\begin{aligned} A^{-r+t} \natural_{\frac{1-t+r}{r+(p-t)s}} (A^t \#_s B^p) &= A^{-r+t} \{A^{r-t} \#_{\frac{-(1-t+r)}{r+(p-t)s}} (A^{-t} \#_s B^{-p})\} A^{-r+t} \\ &\leq A^{-r+t} A^{2(r-t)+1} A^{-r+t} = A. \end{aligned}$$

□

The following theorems show that Theorem 2.8 holds for the critical points $s = \frac{-t}{p-t}, \frac{-2r-(1-t)}{p-t}$.

Theorem 2.9 *If* $A \geq B > 0$ *and* $t \in [0, 1]$, *then*

$$A^{-r+t} \natural_{\frac{1-t+r}{r+(p-t)s}} (A^t \#_s B^p) \leq A$$

holds for $p \leq -1$, $r \in [0, t]$ *and* $s = \frac{-2r-(1-t)}{p-t}$.

Proof First of all, we note that $\frac{1-t+r}{r+(p-t)s} = -1$ and $X \natural_{-1} Y = XY^{-1}X$. Therefore the conclusion is arranged as

$$A^{-r+t} \natural_{-1} (A^t \#_s B^p) \leq A,$$

$$A^{-r+t}(A^{-t} \#_s B^{-p})A^{-r+t} \leq A$$

and so

$$A^{-t} \#_s B^{-p} \leq A^{1+2r-2t}. \quad (*)$$

To prove this, we recall the Furuta inequality, i.e., if $A \geq B \geq 0$, then

$$(A^{\frac{t}{2}} B^P A^{\frac{t}{2}})^{\frac{1}{q}} \leq A^{\frac{P+t}{q}}$$

holds for t, $P \geq 0$ and $q \geq 1$ with $(1+t)q \geq P+t$. Taking $P = -p$ and $q = \frac{1}{s}$, the required condition $(1+t)q \geq P+t$ is enjoyed and we obtain

$$(A^{\frac{t}{2}} B^{-p} A^{\frac{t}{2}})^s \leq A^{1+2r-t},$$

which is equivalent to (*). □

In succession to the preceding theorem, the other case $s = \frac{-t}{p-t}$ can be proved as in the below discussion:

Theorem 2.10 *If* $A \geq B > 0$ *and* $t \in [0, 1]$, *then*

$$A^{-r+t} \natural_{\frac{1-t+r}{r+(p-t)s}} (A^t \#_s B^p) \leq A$$

holds for $p \leq -1$, $r \in [0, t]$ *and* $s = \frac{-t}{p-t}$.

By Theorem 2.8, we have to consider the case $\frac{-t}{p-t} < \frac{-2r-(1-t)}{p-t}$, that is, $0 \leq t - r < \frac{1}{2}$ can be assumed. Hence we have

$$\frac{1-t+r}{r+(p-t)s} = 1 - \frac{1}{t-r} < -1.$$

As a special case, we take $t = \frac{2}{3}$, $r = \frac{1}{3}$ and $p = -2$. Then $s = \frac{1}{4}$ and $\frac{1-t+r}{r+(p-t)s} = -2$, so that the statement in this case is arranged as follows:

If $A \geq B > 0$, then

$$A^{\frac{1}{3}} \natural_{-2} (A^{\frac{2}{3}} \#_{\frac{1}{4}} B^{-2}) \leq A$$

holds? It is proved by using Furuta inequality twice: First of all, since $A \geq B > 0$, (FI) ensures that

$$(A^{\frac{1}{3}} B^2 A^{\frac{1}{3}})^{\frac{5}{8}} \leq A^{\frac{5}{3}}; \ (A^{\frac{1}{3}} B^2 A^{\frac{1}{3}})^{\frac{1}{8}} \leq A^{\frac{1}{3}}.$$

So we have

$$\begin{aligned}
A^{\frac{1}{3}} \natural_{-2} (A^{\frac{2}{3}} \#_{\frac{1}{4}} B^{-2}) &= A^{\frac{1}{6}} (A^{-\frac{1}{6}} (A^{\frac{2}{3}} \#_{\frac{1}{4}} B^{-2}) A^{-\frac{1}{6}})^{-2} A^{\frac{1}{6}} \\
&= A^{\frac{1}{6}} (A^{\frac{1}{6}} (A^{-\frac{2}{3}} \#_{\frac{1}{4}} B^{2}) A^{\frac{1}{6}})^{2} A^{\frac{1}{6}} \\
&= A^{\frac{1}{6}} (A^{-\frac{1}{3}} \#_{\frac{1}{4}} A^{\frac{1}{6}} B^{2} A^{\frac{1}{6}})^{2} A^{\frac{1}{6}} \\
&= A^{\frac{1}{6}} (A^{-\frac{1}{6}} (A^{\frac{1}{3}} B^{2} A^{\frac{1}{3}})^{\frac{1}{4}} A^{-\frac{1}{6}})^{2} A^{\frac{1}{6}} \\
&= (A^{\frac{1}{3}} B^{2} A^{\frac{1}{3}})^{\frac{1}{4}} A^{-\frac{1}{3}} (A^{\frac{1}{3}} B^{2} A^{\frac{1}{3}})^{\frac{1}{4}} \\
&\leq (A^{\frac{1}{3}} B^{2} A^{\frac{1}{3}})^{\frac{1}{4}-\frac{1}{8}+\frac{1}{4}} \\
&\leq (A^{\frac{1}{3}} B^{2} A^{\frac{1}{3}})^{\frac{3}{8}} \\
&\leq A,
\end{aligned}$$

as desired.

To prove Theorem 2.10, we cite a lemma obtained by the Furuta inequality.

Lemma 2.11 *If $A \geq B > 0$, $t \geq 0$ and $p \leq -1$, then*

$$(A^{\frac{t}{2}} B^{-p} A^{\frac{t}{2}})^{\frac{1+t}{-p+t}} \leq A^{1+t};$$

in particular, $(A^{\frac{t}{2}} B^{-p} A^{\frac{t}{2}})^{s} \leq A^{t}$ holds for $s = \frac{t}{-p+t}$.

To show Theorem 2.10, we reformulate it as follows:

Theorem 2.12 *If $A \geq B > 0$, $t \geq \frac{c-1}{c+1}$ for some $c \geq 2$, $1 \geq t > r \geq 0$ with $t - r = \frac{1}{c+1}$ and $p \leq -1$, then*

$$A^{\frac{1}{c+1}} \natural_{-c} (A^{t} \#_{s} B^{p}) \leq A$$

holds for $s = \frac{t}{-p+t}$.

Proof Put $\alpha = t - r$. Then $\alpha = \frac{1}{c+1} < \frac{1}{2}$, $c = \frac{1-\alpha}{\alpha}$ and the assumption $t \geq \frac{c-1}{c+1}$ means $\alpha(c-1) \leq t$, which plays a role when we use the Löwner-Heinz inequality in the below. We put $X = A^{\frac{t}{2}} B^{-p} A^{\frac{t}{2}}$ and $Y = A^{-\frac{r}{2}} X^s A^{-\frac{r}{2}}$. Then $A^{\frac{1}{c+1}} \natural_{-c} (A^t \#_s B^p) = A^{\frac{\alpha}{2}} Y^c A^{\frac{\alpha}{2}}$, and $X^{\frac{s}{t}} = X^{\frac{1}{-p+t}} \leq A$, in particular, $X^s \leq A^t$ and $X^{\frac{st'}{t}} \leq A^{t'}$ for $0 \leq t' \leq 1 + t$ by Lemma 2.11.

(1) First we suppose that $2n \leq c < 2n + 1$ for some n, i.e., $c = 2n + \epsilon$ for some $\epsilon \in [0, 1)$. Since $\alpha(c-2) \leq t - \alpha = r$ by $\alpha(c-1) \leq t$, we have $\alpha\epsilon \leq \alpha(2(n-1) + \epsilon) = \alpha(c-2) \leq r$ and so

$$-1 \leq \frac{\alpha\epsilon - r}{t} \leq \frac{\alpha(2(n-k) + \epsilon) - r}{t} \leq 0$$

for $k = 1, 2, \cdots, n$. Noting that $0 \leq 2s + [\alpha(2(n-1) + \epsilon) - r]\frac{s}{t} \leq \frac{1+t}{-p+t}$ by $\frac{c-1}{c+1} \leq 1$, it follows that

$$\begin{aligned}
Y^c &= Y^n Y^\epsilon Y^n = Y^n (A^{-\frac{r}{2}} X^s A^{-\frac{r}{2}})^\epsilon Y^n \\
&\leq Y^n (A^{-\frac{r}{2}} A^t A^{-\frac{r}{2}})^\epsilon Y^n = Y^n A^{\alpha\epsilon} Y^n \quad \text{by } X^s \leq A^t \text{ and (LH)} \\
&= Y^{n-1} A^{-\frac{r}{2}} X^s A^{\alpha\epsilon - r} X^s A^{-\frac{r}{2}} Y^{n-1} \\
&\leq Y^{n-1} A^{-\frac{r}{2}} X^{2s + (\alpha\epsilon - r)\frac{s}{t}} A^{-\frac{r}{2}} Y^{n-1} \quad \text{by } X^s \leq A^t,\ \frac{\alpha\epsilon - r}{t} \in [-1, 0] \\
&\leq Y^{n-1} A^{2t + \alpha\epsilon - 2r} Y^{n-1} \quad \text{by putting } t' = 2t + \alpha\epsilon - r \leq 1 + t \\
&= Y^{n-1} A^{\alpha(2+\epsilon)} Y^{n-1} \\
&\leq Y^{n-2} A^{\alpha(4+\epsilon)} Y^{n-2} \\
&\cdots \\
&\leq Y A^{\alpha(2(n-1)+\epsilon)} Y \\
&\leq A^{\alpha(2n+\epsilon)} \\
&= A^{\alpha c}.
\end{aligned}$$

Hence we have

$$A^{\frac{1}{c+1}} \natural_{-c} (A^t \#_s B^p) = A^{\frac{\alpha}{2}} Y^c A^{\frac{\alpha}{2}} \leq A^{\alpha c + \alpha} = A,$$

as desired.

(2) Next we suppose that $2n+1 \le c < 2n+2$ for some n, i.e., $c = 2n+1+\epsilon$ for some $\epsilon \in [0,1)$. For this case, we prepare the inequality

$$Y^{1+\epsilon} \le A^{\alpha(1+\epsilon)}.$$

It is proved as follows:

$$\begin{aligned} Y^{1+\epsilon} &= (A^{-\frac{r}{2}} X^s A^{-\frac{r}{2}})^{1+\epsilon} \\ &= A^{-\frac{r}{2}} X^{\frac{s}{2}} (X^{\frac{s}{2}} A^{-r} X^{\frac{s}{2}})^{\epsilon} X^{\frac{s}{2}} A^{-\frac{r}{2}} \\ &\le A^{-\frac{r}{2}} X^{\frac{s}{2}} (X^{\frac{s}{2}} X^{-\frac{sr}{t}} X^{\frac{s}{2}})^{\epsilon} X^{\frac{s}{2}} A^{-\frac{r}{2}} \\ &= A^{-\frac{r}{2}} X^{s+(s-\frac{sr}{t})\epsilon} A^{-\frac{r}{2}} \\ &\le A^{-\frac{r}{2}} A^{t+\alpha\epsilon} A^{-\frac{r}{2}} = A^{\alpha(1+\epsilon)}. \end{aligned}$$

Now, if $n = 0$, i.e., $c = 1+\epsilon$, then

$$A^{\frac{\alpha}{2}} Y^{1+\epsilon} A^{\frac{\alpha}{2}} \le A^{\frac{\alpha}{2}} A^{\alpha(1+\epsilon)} A^{\frac{\alpha}{2}} = A^{\alpha(2+\epsilon)} = A.$$

Next, if $c = 2n+1+\epsilon$ for some $\epsilon \in [0,1)$ with $n \ne 0$, then

$$\begin{aligned} Y^c &= Y^n Y^{1+\epsilon} Y^n \le Y^n A^{\alpha(1+\epsilon)} Y^n \\ &= Y^{n-1} A^{-\frac{r}{2}} X^s A^{\alpha(1+\epsilon)-r} X^s A^{-\frac{r}{2}} Y^{n-1} \\ &\le Y^{n-1} A^{-\frac{r}{2}} X^{2s+(\alpha(1+\epsilon)-r)\frac{s}{t}} A^{-\frac{r}{2}} Y^{n-1} \\ &\le Y^{n-1} A^{2t+\alpha(1+\epsilon)-2r} Y^{n-1} \\ &= Y^{n-1} A^{\alpha(3+\epsilon)} Y^{n-1} \\ &\le Y^{n-2} A^{\alpha(5+\epsilon)} Y^{n-2} \\ &\cdots \\ &\le Y A^{\alpha(2(n-1)+1+\epsilon)} Y \\ &\le A^{\alpha(2n+1+\epsilon)} = A^{\alpha c}, \end{aligned}$$

in which $(-1 \le -r \le)\ \alpha(2(n-1)+1+\epsilon) - r \le 0$ is required in order to use the Löwner-Heinz inequality. (Fortunately it is assured by the assumption $t \ge \frac{c-1}{c+1}$.) Hence we have

$$A^{\frac{1}{c+1}} \natural_{-c} (A^t \#_s B^p) = A^{\frac{\alpha}{2}} Y^c A^{\frac{\alpha}{2}} \le A^{\alpha c+\alpha} = A,$$

as desired. □

Recently Ito and Kamei [20] give an improvement to the above results. For this, they rewrite it as follows:

$$A \geq B > 0 \quad \text{implies} \quad A^{-r} \#_{\frac{1-r}{p+r}} B^p \leq A^{1-2r} \quad \text{for } p \geq 1 \text{ and } 0 \leq r \leq 1. \tag{FN}$$

Moreover we recall a stellite of (FI) due to Kamei [21]:

$$A \geq B > 0 \Rightarrow A^{-r} \#_{\frac{1+r}{p+r}} B^p \leq B \leq A \leq B^{-r} \#_{\frac{1+r}{p+r}} A^p \ \text{ for } p \geq 1 \text{ and } r \geq 0. \tag{SF}$$

Under such preparation, the following improvement of Theorem 2.6 is proposed:

Theorem 2.13 *Let* $A \geq B > 0$ *and* $r > 0$. *Then for* $p \geq 1$, *the following inequalities hold.*

$$A^{-r} \#_{\frac{1-r}{p+r}} B^p \begin{cases} \leq B^{1-2r} \leq A^{1-2r} & \textit{if } 0 \leq r \leq \frac{1}{2}, \\ \leq A^{1-2r} \leq B^{1-2r} & \textit{if } \frac{1}{2} \leq r \leq 1, \end{cases} \tag{2}$$

$$A^{-r} \natural_{\frac{1-r}{p+r}} B^p \geq A^{1-2r} \ \textit{if } r > 1. \tag{3}$$

Proof If $0 \leq r \leq 1$, then we have

$$A^{-r} \#_{\frac{1-r}{p+r}} B^p \leq B^{-r} \#_{\frac{1-r}{p+r}} B^p = B^{1-2r}$$

and

$$\begin{aligned} A^{-r} \#_{\frac{1-r}{p+r}} B^p &= A^{-r} \#_{\frac{1-r}{1+r}} (A^{-r} \#_{\frac{1+r}{p+r}} B^p) \\ &\leq A^{-r} \#_{\frac{1-r}{1+r}} (B^{-r} \#_{\frac{1+r}{p+r}} B^p) \\ &= A^{-r} \#_{\frac{1-r}{1+r}} B \\ &\leq A^{-r} \#_{\frac{1-r}{1+r}} A = A^{1-2r}. \end{aligned}$$

Therefore we obtain (2) since $B^{1-2r} \leq A^{1-2r}$ holds if $0 \leq r \leq \frac{1}{2}$ and $A^{1-2r} \leq B^{1-2r}$ holds if $\frac{1}{2} \leq r \leq 1$.

If $r > 1$, then we have (3) because

$$\begin{aligned} A^{-r} \natural_{\frac{1-r}{p+r}} B^p &= A^{-r} (A^r \#_{\frac{r-1}{p+r}} B^{-p}) A^{-r} \\ &= A^{-r} (B^{-p} \#_{\frac{1+p}{r+p}} A^r) A^{-r} \geq A^{-r} A A^{-r} = A^{1-2r}. \end{aligned}$$

by (SF). ◻

An improvement of Theorem 2.8 is given as follows:

Theorem 2.14 *Let* $A \geq B > 0$ *and* $0 \leq r \leq t \leq 1$. *Then*

$$A^{r-t} \#_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \#_s B^p) \begin{cases} \leq B^{1-2(t-r)} \leq A^{1-2(t-r)} & \text{if } 0 \leq t-r \leq \frac{1}{2}, \\ \leq A^{1-2(t-r)} \leq B^{1-2(t-r)} & \text{if } \frac{1}{2} \leq t-r \leq 1 \end{cases}$$

holds for $p \geq 1$ *and* $\frac{1-t+2r}{p+t} \leq s \leq 1$.

Proof Noting that $0 \leq \frac{1-t+r}{(p+t)s-r} \leq 1$ and $0 \leq t-r \leq 1$ hold, we have

$$A^{r-t} \#_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \#_s B^p) \leq B^{r-t} \#_{\frac{1-t+r}{(p+t)s-r}} (B^{-t} \#_s B^p) = B^{1-2(t-r)}.$$

Next we show $A^{r-t} \#_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \#_s B^p) \leq A^{1-2(t-r)}$ by dividing into three cases:

(i) If $(p+t)s-t \geq 1$ holds, then

$$\begin{aligned} & A^{r-t} \#_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \#_s B^p) \\ & \leq A^{r-t} \#_{\frac{1-t+r}{(p+t)s-r}} (B^{-t} \#_s B^p) \\ & = A^{r-t} \#_{\frac{1-t+r}{(p+t)s-r}} B^{(p+t)s-t} \\ & = A^{r-t} \#_{\frac{1-t+r}{1+t-r}} (A^{r-t} \#_{\frac{1+(t-r)}{(p+t)s-t+(t-r)}} B^{(p+t)s-t}) \\ & \leq A^{r-t} \#_{\frac{1-t+r}{1+t-r}} (B^{r-t} \#_{\frac{1+(t-r)}{(p+t)s-t+(t-r)}} B^{(p+t)s-t}) \\ & = A^{r-t} \#_{\frac{1-t+r}{1+t-r}} B \\ & \leq A^{r-t} \#_{\frac{1-t+r}{1+t-r}} A = A^{1-2(t-r)}. \end{aligned}$$

(ii) If $0 \leq (p+t)s-t \leq 1$ holds, then

$$\begin{aligned} & A^{r-t} \#_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \#_s B^p) \\ & \leq A^{r-t} \#_{\frac{1-t+r}{(p+t)s-r}} (B^{-t} \#_s B^p) \\ & = A^{r-t} \#_{\frac{1-t+r}{(p+t)s-r}} B^{(p+t)s-t} \\ & \leq A^{r-t} \#_{\frac{1-t+r}{(p+t)s-r}} A^{(p+t)s-t} = A^{1-2(t-r)}. \end{aligned}$$

(iii) If $(p+t)s - t \leq 0$ holds, then

$$\begin{aligned} A^{-t} \#_s B^p &= A^{-t} \#_{\frac{(p+t)s}{t}} (A^{-t} \#_{\frac{t}{p+t}} B^p) \\ &\leq A^{-t} \#_{\frac{(p+t)s}{t}} (B^{-t} \#_{\frac{t}{p+t}} B^p) \\ &= A^{-t} \#_{\frac{(p+t)s}{t}} I = A^{(p+t)s-t}, \end{aligned}$$

so that we have

$$A^{r-t} \#_{\frac{1-t+r}{(p+t)s-r}} (A^{-t} \#_s B^p) \leq A^{r-t} \#_{\frac{1-t+r}{(p+t)s-r}} A^{(p+t)s-t} = A^{1-2(t-r)}.$$

Therefore we obtain the desired result since $B^{1-2(t-r)} \leq A^{1-2(t-r)}$ holds if $0 \leq t - r \leq \frac{1}{2}$ and $A^{1-2(t-r)} \leq B^{1-2(t-r)}$ holds if $\frac{1}{2} \leq t - r \leq 1$. □

3 Applications

This section is based on our recent paper [12]. Now Bebiano-Lemos-Providência [2] proposed the following norm inequality:

$$\| A^{\frac{1+t}{2}} B^t A^{\frac{1+t}{2}} \| \leq \| A^{\frac{1}{2}} (A^{\frac{s}{2}} B^s A^{\frac{s}{2}})^{\frac{t}{s}} A^{\frac{1}{2}} \|$$

holds for $A, B \geq 0$ and $s \geq t \geq 0$. We call it BLP inequality. For this, we generalized it in [13] from the viewpoint of Furuta inequality. In this section, we discuss further generalizations of BLP inequality as applications of the results in the preceding section.

We first mention the useful identity on the binary operation $\natural_\beta$ again: For $\beta \in \mathbb{R}$ and positive invertible operators X and Y,

$$X \natural_\beta Y = X(X^{-1} \natural_{-\beta} Y^{-1})X.$$

This means that if $\beta \in [-1, 0]$, then $\natural_\beta$ looks like an operator mean in some sense. We also rewrite Lemma 2.1 and Theorem 2.6 for convenience.

Lemma 3.1 *If $A \natural_\alpha B \leq 1$ for $\alpha \in [-1, 0]$ and $A, B > 0$, then $A^r \natural_\beta B \leq 1$ for $r \in [0, 1]$, where $\beta = \frac{\alpha r}{\alpha r + (1-\alpha)}$.*

It is reformulated as Furuta type as follows:

Theorem 3.2 *If $A \geq B > 0$, then*

$$A^{-r} \natural_{\frac{1+r}{p+r}} B^p \leq A$$

holds for $p \le -1$ *and* $r \in [-1, 0]$.

We remark that they are equivalent. Moreover it suggests us that the domain in which (FI) holds is extendable.

Theorem 3.3 *If* $A \ge B > 0$*, then*

$$A^{-r} \#_{\frac{1+r}{p+r}} B^p \le A$$

holds for $p \in [0, 1]$ *and* $r \le -1$.

Proof It is easily checked:

$$A^{-r} \#_{\frac{1+r}{p+r}} B^p \le A^{-r} \#_{\frac{1+r}{p+r}} A^p = A$$

by (LH). □

As an application, we show a generalization of BLP inequality in the below. For this, we need the following lemma:

Lemma 3.4 *Suppose that* $A, B > 0$.

(1) *If* $A^r \natural_{\frac{1}{p}} B^{p+r} \le A^{1+r}$ *for some* $p \le -1$ *and* $r \in [-1, 0]$*, then* $B^{1+r} \le A^{1+r}$.
(2) *If* $A^r \natural_{\frac{1}{p}} B^{p+r} \le A^{1+r}$ *for some* $p \in [0, 1]$ *and* $r \le -1$*, then* $B^{1+r} \le A^{1+r}$.

Proof (1) Since the assumption is rephrased as $B_1 = (A^{-\frac{r}{2}} B^{p+r} A^{-\frac{r}{2}})^{\frac{1}{p}} \le A$, it follows from Theorem 3.2 that

$$A \ge A^{-r} \#_{\frac{1+r}{p+r}} B_1^p = A^{-r} \#_{\frac{1+r}{p+r}} A^{-\frac{r}{2}} B^{p+r} A^{-\frac{r}{2}} = A^{-\frac{r}{2}} B^{1+r} A^{-\frac{r}{2}},$$

so that we have the conclusion $B^{1+r} \le A^{1+r}$.
(2) is proved by the same way as (1) with the use of Theorem 3.3.

□

By the use of (LH), we have the following.

Corollary 3.5 *Suppose that* $A, B > 0$.

(1) If $A^r \natural_{\frac{1}{p}} B^{p+r} \le A^{1+r}$ *for some* $p \le -1$ *and* $r \in [-1, 0]$*, then* $B^{1+s} \le A^{1+s}$ *for* $-1 \le s \le r$.
(2) If $A^r \natural_{\frac{1}{p}} B^{p+r} \le A^{1+r}$ *for some* $p \in [0, 1]$ *and* $r \le -1$*, then* $B^{1+s} \le A^{1+s}$ *for* $r \le s \le -1$.

Consequently we have a generalized BLP inequality as follows:

Theorem 3.6 *If* $A, B > 0$*, then*

$$\|A^{\frac{1+r}{2}} B^{1+r} A^{\frac{1+r}{2}}\|^{\frac{p+r}{p(1+r)}} \le \|A^{\frac{1}{2}} (A^{\frac{r}{2}} B^{p+r} A^{\frac{r}{2}})^{\frac{1}{p}} A^{\frac{1}{2}}\|$$

holds for either $p \le -1$ *and* $r \in [-1, 0]$ *or* $p \in [0, 1]$ *and* $r \le -1$.

Proof Lemma 3.4 implies that if $A^{-r} \natural_{\frac{1}{p}} B^{p+r} \le A^{-(1+r)}$, then $B^{1+r} \le A^{-(1+r)}$. It says that if $A^{\frac{1}{2}}(A^{\frac{r}{2}}B^{p+r}A^{\frac{r}{2}})^{\frac{1}{p}}A^{\frac{1}{2}} \le I$, then $A^{\frac{1+r}{2}}B^{1+r}A^{\frac{1+r}{2}} \le I$. Consequently we have the desired norm inequality. □

In addition, we also obtain norm inequalities corresponding to Corollary 3.5:

Theorem 3.7 *If* $A, B > 0$, *then*

$$\|A^{\frac{1+s}{2}}B^{1+s}A^{\frac{1+r}{2}}\|^{\frac{p+r}{p(1+s)}} \le \|A^{\frac{1}{2}}(A^{\frac{r}{2}}B^{p+r}A^{\frac{r}{2}})^{\frac{1}{p}}A^{\frac{1}{2}}\|$$

holds for either $p \le -1$ *and* $-1 \le s \le r \le 0$ *or* $p \in [0, 1]$ *and* $r \le s \le -1$.

Next we consider a reverse inequality of generalized BLP inequality. For this, we cite a reverse inequality of Araki-Cordes inequality (AC), i.e.,

$$\|ABA\|^p \le \|A^pB^pA^p\| \quad \text{for } A, B \ge 0.$$

Theorem R-AC ([14]) *If* $A \ge 0$, $0 < mI \le B \le MI$ *for some* $M > m > 0$ *and* $h = \frac{M}{m}$, *then*

$$(\|ABA\|^p \le)\ \|A^pB^pA^p\| \le K(h, p)\|ABA\|^p$$

holds for $p \ge 1$, *where* $K(h, p)$ *is the generalized Kantorovich constant defined by*

$$K(h, p) = \frac{1}{h-1}\frac{h^p - h}{p-1}\left(\frac{p-1}{h^p - h}\frac{h^p - 1}{p}\right)^p.$$

Theorem 3.8 *Suppose that* $A \ge 0$, $0 < mI \le B \le MI$ *for some* $M > m > 0$ *and* $h = \frac{M}{m}$. *Then*

$$\|A^{\frac{1}{2}}(A^{\frac{r}{2}}B^{p+r}A^{\frac{r}{2}})^{\frac{1}{p}}A^{\frac{1}{2}}\| \le K(h^{1+s}, -\frac{p+r}{1+s})^{-\frac{1}{p}}\|A^{\frac{1+s}{2}}B^{1+s}A^{\frac{1+s}{2}}\|^{\frac{p+r}{p(1+s)}}$$

holds for $p \le -1$, $-1 < s$ *and* $r \le -(p + 1 + s)$.

Proof It is proved by (AC) and Theorem R-AC. As a matter of fact, since $p \le -1$ and $-1 < s \le r \le 0$, we have $-p \ge 1$ and $\frac{-(p+r)}{1+s} \ge 1$, and so

$$\begin{aligned}&\|A^{\frac{1}{2}}(A^{\frac{r}{2}}B^{p+r}A^{\frac{r}{2}})^{\frac{1}{p}}A^{\frac{1}{2}}\|\\ &\le \|A^{-\frac{p}{2}}(A^{\frac{r}{2}}B^{p+r}A^{\frac{r}{2}})^{-1}A^{-\frac{p}{2}}\|^{-\frac{1}{p}}\end{aligned}$$

$$= \|A^{-\frac{p+r}{2}} B^{(1+s)\frac{-(p+r)}{1+s}} A^{-\frac{p+r}{2}}\|^{-\frac{1}{p}}$$
$$\leq K(h^{1+s}, \frac{-(p+r)}{1+s})^{-\frac{1}{p}} \|A^{\frac{1+s}{2}} B^{1+s} A^{\frac{1+s}{2}}\|^{\frac{p+r}{p(1+s)}}.$$

□

We note that Theorem 3.8 can be expressed as an operator inequality:

Corollary 3.9 *Suppose that* $A > 0$, $0 < mI \leq B \leq MI$ *for some* $M > m > 0$ *and* $h = \frac{M}{m}$. *If* $B^{1+s} \leq A^{1+s}$ *for some* $-1 < s \leq 0$, *then*

$$A^r \natural_{\frac{1}{p}} B^{p+r} \leq K(h^{1+s}, -\frac{p+r}{1+s})^{-\frac{1}{p}} A^{1+r}$$

for $p \leq -1$ *and* $r \leq -(p+1+s)$.

Taking $s = 0$ in Corollary 3.9, we have the following.

Corollary 3.10 *Suppose that* $0 < mI \leq B \leq MI$ *for some* $M > m > 0$ *and* $h = \frac{M}{m}$. *If* $A \geq B$, *then*

$$A^r \natural_{\frac{1}{p}} B^{p+r} \leq K(h, -(p+r))^{-\frac{1}{p}} A^{1+r}$$

for $p \leq -1$ *and* $r \leq -(p+1)$.

We remark that if we take $p = -1$ in the above, then we obtain a well-known result:

$$A \geq B > 0 \Rightarrow K(h, 1+r)A^{1+r} \geq B^{1+r} \quad \text{for } r \geq 0.$$

It is a complementary inequality related to (LH). As a matter of fact, we know that

$$A \geq B > 0 \not\Rightarrow A^{1+r} \geq B^{1+r} \quad \text{for } r > 0$$

in general.

By the way, Theorem 3.7 is expressed that

$$\|A^{\frac{1+s}{2}} B^{1+s} A^{\frac{1+r}{2}}\| \leq \|A^{\frac{1}{2}} (A^{\frac{r}{2}} B^{p+r} A^{\frac{r}{2}})^{\frac{1}{p}} A^{\frac{1}{2}}\|^{\frac{p(1+s)}{p+r}}$$

holds for $p \leq -1$ and $-1 \leq s \leq r \leq 0$

Next we discuss reverse inequalities with respect to difference, precisely we estimate a upper bound of the difference

$$\|A^{\frac{1}{2}} (A^{\frac{r}{2}} B^{p+r} A^{\frac{r}{2}})^{\frac{1}{p}} A^{\frac{1}{2}}\|^{\frac{p(1+s)}{p+r}} - \lambda \|A^{\frac{1+s}{2}} B^{1+s} A^{\frac{1+s}{2}}\|$$

for a given $\lambda > 0$.

For this, we put, for $q > 1$, $M > m > 0$ and $h = \frac{M}{m}$,

$$j_1 = j_{q,h}^{(1)} = \frac{h^q - 1}{q(h^q - h^{q-1})}, \quad j_2 = j_{q,h}^{(2)} = \frac{h^q - 1}{q(h-1)}$$

and

$$\beta(m, M, q; \lambda) = \begin{cases} (1-\lambda)M & \text{if } 0 < \lambda < j_1 \\ \frac{q-1}{q}\left(\frac{M^q - m^q}{\lambda q(M-m)}\right)^{\frac{1}{q-1}} + \frac{\lambda(Mm^q - mM^q)}{M^q - m^q} & \text{if } \lambda \in [j_1, j_2] \\ (1-\lambda)m & \text{if } \lambda > j_2. \end{cases}$$

Theorem 3.11 *Suppose that $A \geq 0$, $0 < mI \leq B \leq MI$ for some $M > m > 0$ and $h = \frac{M}{m}$. If $p \leq -1$, $-1 < s$ and $r \leq -(p+1+s)$, then for each $\lambda > 0$*

$$\begin{aligned} &\|A^{\frac{1}{2}}(A^{\frac{r}{2}}B^{p+r}A^{\frac{r}{2}})^{\frac{1}{p}}A^{\frac{1}{2}}\|^{\frac{p(1+s)}{p+r}} \\ &\leq \lambda\|A^{\frac{1+s}{2}}B^{1+s}A^{\frac{1+s}{2}}\| + \beta(m^{1+s}, M^{1+s}, -\frac{p+r}{1+s}, \lambda)\|A\|^{1+s}. \end{aligned}$$

Proof We first refer [14, Theorem 6]: If $A_1 > 0$, $0 < m_1 I \leq B_1 \leq M_1 I$ for some $M_1 > m_1 > 0$ and $q > 1$, then for each $\lambda > 0$

$$\|A_1^q B_1^q A_1^q\|^{\frac{1}{q}} \leq \lambda\|A_1 B_1 A_1\| + \beta(m_1, M_1, q; \lambda)\|A_1\|^2.$$

We apply it for $A_1 = A^{\frac{1+s}{2}}$, $B_1 = B^{1+s}$ and $q = -\frac{p+r}{1+s}$. Then it follows that

$$\begin{aligned} &\|A^{\frac{1}{2}}(A^{\frac{r}{2}}B^{p+r}A^{\frac{r}{2}})^{\frac{1}{p}}A^{\frac{1}{2}}\|^{\frac{p(1+s)}{p+r}} \\ &= \|A^{\frac{1}{2}}(A^{\frac{r}{2}}B^{p+r}A^{\frac{r}{2}})^{\frac{1}{p}}A^{\frac{1}{2}}\|^{\frac{-p}{q}} \\ &\leq \|A^{-\frac{p}{2}}(A^{\frac{r}{2}}B^{p+r}A^{\frac{r}{2}})^{-1}A^{-\frac{p}{2}}\|^{\frac{1}{q}} \quad \text{by } (AC) \\ &= \|A^{-\frac{p+r}{2}}B^{-(p+r)}A^{-\frac{p+r}{2}}\|^{\frac{1}{q}} \\ &= \|A_1^q B_1^q A_1^q\|^{\frac{1}{q}} \\ &\leq \lambda\|A_1 B_1 A_1\| + \beta(m^{1+s}, M^{1+s}, q, \lambda)\|A_1\|^2 \\ &= \lambda\|A^{\frac{1+s}{2}}B^{1+s}A^{\frac{1+s}{2}}\| + \beta(m^{1+s}, M^{1+s}, -\frac{p+r}{1+s}, \lambda)\|A\|^{1+s}. \end{aligned}$$

□

For convenience, we cite the original form of "grand Furuta inequality"

Grand Furuta Inequality If $A \geq B > 0$ and $t \in [0, 1]$, then

$$[A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^{p}A^{-\frac{t}{2}})^{s}A^{\frac{r}{2}}]^{\frac{1}{q}} \leq A^{1-t+r}$$

holds for $r \geq t$, $p \geq 0$, $q \geq 1$ and $s \geq 1$ with $(1-t+r)q \geq (p-t)s+r$.

The core of the grand Furuta inequality is the case $(1-t+r)q = (p-t)s+r$ by virtue of (LH). So we call it (GFI). That is,

Grand Furuta Inequality (GFI) If $A \geq B > 0$ and $t \in [0, 1]$, then

$$[A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^{p}A^{-\frac{t}{2}})^{s}A^{\frac{r}{2}}]^{\frac{1-t+r}{(p-t)s+r}} \leq A^{1-t+r}$$

holds for $r \geq t$ and $p, s \geq 1$.

As in the case of Furuta inequality, a mean theoreteic expression of (GFI) is given as follows:

If $A \geq B > 0$ *and* $t \in [0, 1]$ *is given, then*

$$A^{-r+t} \#_{\frac{1-t+r}{(p-t)s+r}} (A^{t} \natural_{s} B^{p}) \leq A$$

holds for $r \geq t$ and $p, s \geq 1$.

In the below, we discuss a generalization of BLP inequality corresponding to (GFI). To do this, we prepare the following operator inequality:

Theorem 3.12 *Suppose that* $A, B > 0$ *and* $t \in [0, 1]$ *satisfy*

$$A^{r-t} \#_{\frac{1}{p}} (A^{r} \#_{\frac{1}{s}} B^{(p-t)s+r}) \leq A^{1-t+r}$$

for some $p, s \geq 1$ *and* $r \geq t$. *Then* $B^{1-t+r} \leq A^{1-t+r}$.

Proof Multiplying $A^{\frac{t-r}{2}}$ on both sides of the assumption, we have

$$B_2 = [A^{\frac{t-r}{2}}(A^{r} \#_{\frac{1}{s}} B^{(p-t)s+r})A^{\frac{t-r}{2}}]^{\frac{1}{p}} \leq A.$$

Applying (GFI) for $B_2 \leq A$, it follows that

$$A^{t-r} \#_{\frac{1-t+r}{(p-t)s+r}} (A^{t} \natural_{s} B_2^{p}) \leq A.$$

Moreover we have

$$\begin{aligned} A^{t} \natural_{s} B_2^{p} &= A^{t} \natural_{s} [A^{\frac{t-r}{2}}(A^{r} \#_{\frac{1}{s}} B^{(p-t)s+r})A^{\frac{t-r}{2}}] \\ &= A^{\frac{t}{2}}[A^{-\frac{r}{2}}(A^{r} \#_{\frac{1}{s}} B^{(p-t)s+r})A^{-\frac{r}{2}}]^{s}A^{\frac{t}{2}} \end{aligned}$$

$$= A^{\frac{t}{2}}[A^{-\frac{r}{2}}B^{(p-t)s+r})A^{-\frac{r}{2}}]^{\frac{1}{s}\times s}A^{\frac{t}{2}}$$
$$= A^{\frac{t-r}{2}}B^{(p-t)s+r}A^{\frac{t-r}{2}},$$

so that

$$A \geq A^{t-r} \#_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B_2^p)$$
$$= A^{t-r} \#_{\frac{1-t+r}{(p-t)s+r}} A^{\frac{t-r}{2}}B^{(p-t)s+r}A^{\frac{t-r}{2}}$$
$$= A^{\frac{t-r}{2}}B^{1-t+r}A^{\frac{t-r}{2}},$$

as desired. □

As a corollary, we have a norm inequality of BLP type corresponding to (GFI):

Corollary 3.13 *Suppose that* $A, B > 0$ *and* $t \in [0, 1]$. *Then*

$$\|A^{\frac{1-t+r}{2}}B^{1-t+r}A^{\frac{1-t+r}{2}}\|^{\frac{(p-t)s+r}{ps(1-t+r)}} \leq \|A^{\frac{1}{2}}\{A^{-\frac{t}{2}}(A^{\frac{r}{2}}B^{(p-t)s+r}A^{\frac{r}{2}})^{\frac{1}{s}}A^{-\frac{t}{2}}\}^{\frac{1}{p}}A^{\frac{1}{2}}\|$$

holds for $p, s \geq 1$ *and* $r \geq t$.

Proof It suffices to show that $A^{\frac{1}{2}}\{A^{-\frac{t}{2}}(A^{\frac{r}{2}}B^{(p-t)s+r}A^{\frac{r}{2}})^{\frac{1}{s}}A^{-\frac{t}{2}}\}^{\frac{1}{p}}A^{\frac{1}{2}} \leq I$ implies $A^{\frac{1-t+r}{2}}B^{1-t+r}A^{\frac{1-t+r}{2}} \leq I$. Since the assumption is equivalent to

$$A^{-(r-t)} \#_{\frac{1}{p}} (A^{-r} \#_{\frac{1}{s}} B^{(p-t)s+r}) \leq A^{-(1-t+r)},$$

the conclusion is ensured by Theorem 3.12 with replacing A to A^{-1}. □

In Theorem 2.8, we extended domain where (GFI) holds:
If $A \geq B > 0$ *and* $t \in [0, 1]$, *then*

$$A^{-r+t} \natural_{\frac{1-t+r}{r+(p-t)s}} (A^t \#_s B^p) \leq A$$

holds for $p \leq -1$, $r \in [0, t]$ *and* $s \in [\max\{\frac{-t}{p-t}, \frac{-2r-(1-t)}{p-t}\}, 1]$.

Consequently, as similar to (GFI) itself, we have the following operator inequality and norm inequality.

Theorem 3.14 *Suppose that* $A, B > 0$ *and* $t \in [0, 1]$ *satisfy*

$$A^{r-t} \natural_{\frac{1}{p}} (A^r \natural_{\frac{1}{s}} B^{(p-t)s+r}) \leq A^{1-t+r}$$

for some $p \leq -1$*,* $r \in [0, t]$ *and* $s \in [\max\{\frac{-t}{p-t}, \frac{-2r-(1-t)}{p-t}\}, 1]$*. Then* $B^{1-t+r} \leq A^{1-t+r}$*.*

Corollary 3.15 *Suppose that* $A, B > 0$ *and* $t \in [0, 1]$*. Then*

$$\|A^{\frac{1-t+r}{2}} B^{1-t+r} A^{\frac{1-t+r}{2}}\|^{\frac{(p-t)s+r}{ps(1-t+r)}} \leq \|A^{\frac{1}{2}}\{A^{-\frac{t}{2}}(A^{\frac{r}{2}} B^{(p-t)s+r} A^{\frac{r}{2}})^{\frac{1}{s}} A^{-\frac{t}{2}}\}^{\frac{1}{p}} A^{\frac{1}{2}}\|$$

holds for $p \leq -1$*,* $r \in [0, t]$ *and* $s \in [\max\{\frac{-t}{p-t}, \frac{-2r-(1-t)}{p-t}\}, 1]$*.*

Proof of Theorem 3.14 The proof is similar to that of Theorem 3.12. As in the proof of it, we have

$$B_2 = [A^{\frac{t-r}{2}}(A^r \natural_{\frac{1}{s}} B^{(p-t)s+r})A^{\frac{t-r}{2}}]^{\frac{1}{p}} \leq A.$$

Applying Theorem 2.8 for $B_2 \leq A$, it follows that

$$A^{t-r} \#_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B_2^p) \leq A.$$

Moreover, since we have $A^t \natural_s B_2^p = A^{\frac{t-r}{2}} B^{(p-t)s+r} A^{\frac{t-r}{2}}$, it follows that

$$A \geq A^{t-r} \#_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B_2^p) = A^{t-r} \#_{\frac{1-t+r}{(p-t)s+r}} A^{\frac{t-r}{2}} B^{(p-t)s+r} A^{\frac{t-r}{2}}.$$

By multiplying $A^{-\frac{t-r}{2}}$ on both sides, we obtain the conclusion. □ □

In succession, we discuss some inequalities on the logarithm. The chaotic order $A \gg B$ for $A, B > 0$ is defined by $\log A \geq \log B$. It is weaker than the usual Löwner order $A \geq B$. The Furuta inequality for chaotic order is known in [6]:

Chaotic Furuta Inequality (CFI)

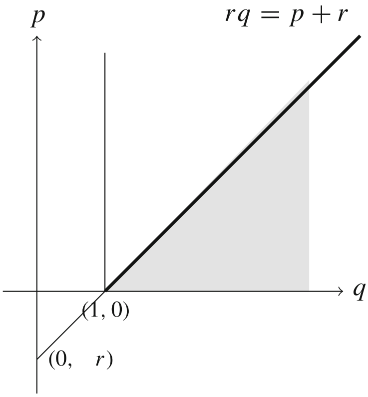
If $A \gg B$ for $A, B > 0$, then for each $r \geq 0$,

(i) $$(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with
$rq \geq p + r$.



As in (FI), the optimal case $rq = p + r$ is the most important, which is expressed as the following way by the use of the α-geometric mean:

If $A \gg B$ *for* $A, B > 0$, *then for each* $r \geq 0$

$$A^{-r} \#_{\frac{r}{p+r}} B^p \leq I \quad \text{and} \quad B^p \#_{\frac{p}{p+r}} A^{-r} \leq I$$

hold for $p \geq 0$.

As an application of (CFI), we obtain that if $A \gg B$ for $A, B > 0$, then

$$A^{-r} \#_{\frac{1+r}{p+r}} B^p = B^p \#_{\frac{p-1}{p+r}} A^{-r} = B^p \#_{\frac{p-1}{p}} (B^p \#_{\frac{p}{p+r}} A^{-r}) \leq B^p \#_{\frac{p-1}{p}} I = B.$$

Namely, satellite of (FI) is refined as follows:

If $A \gg B$ for $A, B > 0$, *then*

$$A^{-r} \#_{\frac{1+r}{p+r}} B^p \leq B$$

holds for $p \geq 1$ and $r \geq 0$.

Now, based on the theory of operator means, the relative operator entropy was introduced by Fujii-Kamei [3]:

$$S(A|B) = A^{\frac{1}{2}} \log(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}} \quad \text{for } A, B > 0.$$

As an application of (CFI), we show the following inequality, which is regarded as a chaotic order version of Corollary 3.5.

Theorem 3.16 *Let* $A, B > 0$ *and* $r > 0$ *be given. Then, if* $S(A^r|A^{p+r}) \geq S(A^r|B^{p+r})$ *for some* $p > 0$, *then* $A^r \geq B^r$.

Proof Since $\frac{1}{p}S(A^r|B^{p+r}) = A^{\frac{r}{2}}\log(A^{-\frac{r}{2}}B^{p+r}A^{-\frac{r}{2}})^{\frac{1}{p}}A^{\frac{r}{2}}$, the assumption is rephrased as

$$\log A \geq \log(A^{-\frac{r}{2}}B^{p+r}A^{-\frac{r}{2}})^{\frac{1}{p}}.$$

Putting $B_1 = (A^{-\frac{r}{2}}B^{p+r}A^{-\frac{r}{2}})^{\frac{1}{p}}$, we have $A \gg B_1$. Hence it follows from (CFI) that

$$I \geq A^{-r} \#_{\frac{r}{p+r}} B_1^p = A^{-r} \#_{\frac{r}{p+r}} A^{-\frac{r}{2}}B^{p+r}A^{-\frac{r}{2}},$$

so that $A^r \geq I \#_{\frac{r}{p+r}} B^{p+r} = B^r$. □

Next we show a generalization of the above, which is type of (GFI).

Theorem 3.17 *Let $A, B > 0$ and $t, r \geq 0$ be given. Then, if*

$$S(A^{t+r}|A^{p+t+r}) \geq S(A^{t+r}|A^r \natural_{\frac{1}{s}} B^{(p+t)s+r})$$

holds for some $p, s > 0$ with $(p+t)s \geq t$, then $A^{t+r} \geq B^{t+r}$.

Proof We first note that the assumption is rephrased as

$$\log A \geq \log[A^{-\frac{t+r}{2}}(A^r \natural_{\frac{1}{s}} B^{(p+t)s+r})A^{-\frac{t+r}{2}}]^{\frac{1}{p}}.$$

We here recall that the following operator inequality of type of (GFI), see [9, Theorem 3.16]: If $A \gg X$ for $A, X > 0$, then

$$A^{\frac{(p+t)s+r}{q}} \geq [A^{\frac{r}{2}}(A^{\frac{t}{2}}X^pA^{\frac{t}{2}})^sA^{\frac{r}{2}}]^{\frac{1}{q}}$$

holds for $p, t, r, s \geq 0$, $q \geq 1$ with $(t+r)q \geq (p+t)s+r$. Hence, taking $q = \frac{(p+t)s+r}{t+r}$ and $X = [A^{-\frac{t+r}{2}}(A^r \natural_{\frac{1}{s}} B^{(p+t)s+r})A^{-\frac{t+r}{2}}]^{\frac{1}{p}}$, it follows that

$$\begin{aligned}
A^{t+r} &\geq [A^{\frac{r}{2}}(A^{\frac{t}{2}}X^pA^{\frac{t}{2}})^sA^{\frac{r}{2}}]^{\frac{t+r}{(p+t)s+r}} \\
&= [A^{\frac{r}{2}}(A^{\frac{t}{2}}[A^{-\frac{t+r}{2}}(A^r \natural_{\frac{1}{s}} B^{(p+t)s+r})A^{-\frac{t+r}{2}}]A^{\frac{t}{2}})^sA^{\frac{r}{2}}]^{\frac{t+r}{(p+t)s+r}} \\
&= [A^{\frac{r}{2}}(A^{-\frac{r}{2}}(A^r \natural_{\frac{1}{s}} B^{(p+t)s+r})A^{-\frac{r}{2}})^sA^{\frac{r}{2}}]^{\frac{t+r}{(p+t)s+r}} \\
&= [A^{\frac{r}{2}}(A^{-\frac{r}{2}}B^{(p+t)s+r}A^{-\frac{r}{2}}A^{\frac{r}{2}}]^{\frac{t+r}{(p+t)s+r}} \\
&= B^{t+r},
\end{aligned}$$

which completes the proof. □

Finally we discuss log-majorization related to an operator inequality obtained in [11]. In the below, A and B are positive definite $n \times n$ matrices, denoted by $A, B > 0$. We denote the order of log-majorization by $A \succ_{(\log)} B$, i.e., A and B satisfies

$$\prod_{i=1}^{k} \lambda_i(A) \geq \prod_{i=1}^{k} \lambda_i(B) \quad \text{for } k = 1, \cdots, n-1$$

and

$$\prod_{i=1}^{n} \lambda_i(A) = \prod_{i=1}^{n} \lambda_i(B),$$

where $\{\lambda_i(X); i = 1, \cdots, n\}$ is the eigenvalues of $X > 0$, arranged in decreasing order.

For convenience, we briefly explain a relation between log-majorization and the k-fold antisymmetric tensor power of matrices. For an $n \times n$ matrix X, let $C_k(X)$ for $k = 1, \cdots, n$ be the k-fold antisymmetric tensor power of X. Then it has the following properties;

(1) $C_k(X^*) = C_k(X)^*$ for $k = 1, \cdots, n$.
(2) $C_k(XY) = C_k(X)C_k(Y)$ for $k = 1, \cdots, n$.
(3) $C_k(X^{-1}) = C_k(X)^{-1}$ for $k = 1, \cdots, n$ if X is invertible.
(4) $C_k(X^p) = C_k(X)^p$ for $k = 1, \cdots, n$ if $X > 0$ and $p \neq 0$.
(5) $\prod_{i=1}^{k} \lambda_i(A) = \lambda_1(C_k(A))$ $k = 1, \cdots, n$ if $A > 0$.

Cosequently, for $A, B > 0$, $A \succ_{(\log)} B$ if and only if $\det A = \det B$ and $\lambda_1(C_k(A)) \geq \lambda_1(C_k(B))$ for $k = 1, \cdots, n$. Incidentally we note that $C_k(A \natural_\alpha B) = C_k(A) \natural_\alpha C_k(B)$ for $A, B > 0$ by (2)-(4), so that matrix inequalities of Ando-Hiai type implies log-majorization inequalities corresponding to them.

The following log-majorization inequality corresponds to Theorem 2.3:

If $A \natural_\alpha B \leq 1$ *for* $\alpha \in [-1, 0]$ *and positive invertible operators* A and B, *then* $A^r \natural_\beta B^s \leq 1$ *for* $r \in [0, 1]$ and $s \in [\frac{-2\alpha r}{1-\alpha}, 1]$, *where* $\beta = \frac{\alpha r}{\alpha r + (1-\alpha)s}$.

Theorem 3.18 *For* $\alpha \in [-1, 0]$ *and* $A, B > 0$,

$$(A \natural_\alpha B)^{\frac{rs}{\alpha r + (1-\alpha)s}} \succ_{(\log)} A^r \natural_\beta B^s$$

holds for $r \in [0, 1]$ *and* $s \in [\frac{-2\alpha r}{1-\alpha}, 1]$, *where* $\beta = \frac{\alpha r}{\alpha r + (1-\alpha)s}$.

Moreover we obtain an extension of [17, Theorem 2.1], which corresponds to Theorem 2.8:

Theorem 3.19 *For* $\alpha \in [-1, 0]$ *and* $A, B > 0$,

$$(A \natural_\alpha B)^{\frac{(1-t+r)s}{\alpha r + (1-\alpha t)s}} \succ_{(\log)} A^{1-t+r} \natural_\beta (A^{1-t} \#_s B)$$

holds for $r \in [0,t]$ *and* $s \in [\max\{\frac{-t}{p-t}, \frac{-2r-(1-t)}{p-t}\}, 1]$*, where* $\beta = \frac{\alpha(1-t+r)}{\alpha r+(1-\alpha t)s}$.

Proof Theorem 2.8 says that if $A \geq B > 0$ and $t \in [0,1]$, then

$$A^{-r+t} \natural_{\frac{1-t+r}{r+(p-t)s}} (A^t \#_s B^p) \leq A$$

holds for $p \leq -1$, $r \in [0,t]$ and $s \in [\max\{\frac{-t}{p-t}, \frac{-2r-(1-t)}{p-t}\}, 1]$.

Putting $A_1 = A^{-1}$, $B_1 = A^{-\frac{1}{2}} B^p A^{-\frac{1}{2}}$ and $p = \frac{1}{\alpha}$, it implies that

$$A_1 \natural_\alpha B_1 \leq I \implies A_1^{1-t+r} \natural_\beta (A_1^{1-t} \natural_s B_1) \leq I.$$

Then we have the conclusion. □

4 Concluding Remarks

We mention remarkable results to Ando-Hiai inequality. Wada [33] gave the completion to (AH) in the sense that $r \geq 1$ if and only if (AH) holds, i.e., for a fixed $\alpha \in (0,1)$

$$A, B > 0,\ A\#_\alpha B \geq I \Rightarrow A^r \#_\alpha B^r \geq I.$$

He also generalized it as follows: For a fixed $\alpha \in (0,1)$ and an operator convex function f on $[0,\infty)$ with $f(0) = 0$ and $f(1) = 1$, $f \geq \psi_\alpha(f) := f(t^{\frac{-\alpha}{1-\alpha}})^{\frac{1-\alpha}{-\alpha}}$ if and only if

$$A, B > 0,\ A\#_\alpha B \geq I \Rightarrow f(A)\#_\alpha f(B) \geq I.$$

We here note that t^r is operator convex for $1 \leq r \leq 2$. In [32], he proposed a mean theoretic generalization. For a nonnegative operator monotone function f on $[0,\infty)$ with $f(1) = 1$, $f(t)^r \leq f(t^r)$ for $t > 0$ and $r \geq 1$ if and only if

$$A, B > 0,\ A\,\sigma_f\,B \geq I \Rightarrow A^r\,\sigma_f\,B^r \geq I \quad \text{for } r \geq 1,$$

where σ_f is the operator mean corresponding to the function f, i.e., $A\,\sigma_f\,B = A^{\frac{1}{2}} f(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}$, see [24].

Another improvement of (AH) is posed by Seo [28]: For a fixed $\alpha \in (0,1)$,

$$A, B > 0,\ A\#_\alpha B \geq I \Rightarrow A^r \#_\alpha B^r \leq (\|(A\#_\alpha B)^{-1}\|^{-1})^{r-1} I \quad \text{for } r \geq 1.$$

See also [7] and [19].

Reverse inequalities for Ando-Hiai inequality are presented by several authors, e.g. [26], [30] and [28]. A basic result is as follows: If $M \geq A, B \geq m > 0$ and $h = \frac{M}{m}$, then for each $\alpha \in (0, 1)$

$$K(h^{2r}, \alpha)\|A\#_\alpha B\|^r \leq \|A^r \#_\alpha B^r\| \leq \|A\#_\alpha B\|^r \quad \text{for } r \geq 1.$$

There are deep discussion on Ando-Hiai inequality for n-variable operator means in [34] and [22].

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Relative Operator Entropy

Jun Ichi Fujii and Yuki Seo

Abstract The relative operator entropy $\mathsf{S}(A|B)$ is an operator version of (the minis of) the Kullback-Leibler divergence in the information theory. It is introduced an extension, called solidarities $A \,\mathsf{s}\, B$, of the Kubo-Ando operator means $A \,\mathsf{m}\, B$ and moreover it is a tangent vector of the path of geometric means $A \#_t B$. So we discuss mean-like properties and geometric ones in the manifold of the positive invertible operators. In fact, this path $A \#_t B$ is a geodesic and $\mathsf{S}(A|B)$ is the initial tangent vector in this manifold with the principal fiber bundle, say the CPR geometry. The former defines the multivariate power mean and the latter the Karcher mean. Related to the quantum information theory, we discuss the Tsallis operator entropy and its trace as the secant vector between A and $A \#_t B$.

Keywords Relative operator entropy · Solidarity · CPR geometry · Tsallis entropy

1 Introduction

The relative operator entropy is derived from the Kubo-Ando operator means and the Uhlmann relative operator entropy [18, 19]. So we begin with the Kubo-Ando means [35] and the solidarities as their generalization in Sect. 2. We note that a solidarity can be defined among the positive invertible operators, while it is not always defined for non-invertible ones.

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Typical solidarities are given as derivatives at $t = 0$ (i.e., the tangent vectors at the initial point) of differentiable paths $P(t)$ of operator means. In particular, the relative operator entropy is the tangent vector of a path of the geometric operator means. We discuss in Sect. 3 an existence condition and related ones.

It is noteworthy that the relative operator entropy is indeed the tangent vector of the geodesic in the manifold of positive invertible operators, which we call the CPR geometry named after Corach et al. [9] who pointed the above fact first. Their geometric theory is based on that of fiber bundles which makes us free from complicated coordinate calculations. This geometric view might be due to E.Cartan and it is discussed in the famous standard text by Kobayashi–Nomizu [33]. Unfortunately this geometry might not be familiar, so that we discuss it in Sect. 4 in detail including a general framework of this geometric view. Consequently this consideration gives us a new meaningful representation of the Karcher equation which defines a multivariate geometric operator mean, the Karcher mean.

In the final section, as an application, we show the importance of the concept of relative operator entropy in the framework of the matrix theory. We discuss the Tsallis relative entropy as the secant of a path of geometric matrix means, while the relative operator entropy as the tangent of it. Here we concentrate on the numerical version of it since it is now related to various relative entropies in the quantum information theory.

2 Operator Means and Solidarities

The theory of operator means is started at Ando's lecture note [4] and established as the Kubo-Ando theory [35]. For positive operators on a Hilbert space, the theory of operator means is defined axiomatically: An *(operator) connection* m is a binary operation on positive operators satisfying the following axioms:

- *monotonicity:* $A_1 \le A_2$ and $B_1 \le B_2$ imply $A_1 \,\mathsf{m}\, B_1 \le A_2 \,\mathsf{m}\, B_2$.
- *semi-continuity:* $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n \,\mathsf{m}\, B_n \downarrow A \,\mathsf{m}\, B$.
- *transformer inequality:* $T^*(A \,\mathsf{m}\, B)T \le (T^*AT) \,\mathsf{m}\, (T^*BT)$.

An *operator mean* is a connection m satisfying

- *normalization:* $A \,\mathsf{m}\, A = A$.

It is easy to show that the transformer inequality becomes equality if T is invertible. For an operator mean m, the *representing function* $f_{\mathsf{m}}(x) = 1 \,\mathsf{m}\, x$ is *operator monotone*:

$$0 \le A \le B \qquad \text{implies} \qquad f_{\mathsf{m}}(A) \le f_{\mathsf{m}}(B).$$

This correspondence $\mathsf{m} \mapsto f_{\mathsf{m}}$ is bijective. In fact, if f is a continuous nonnegative operator monotone functional on $[0, \infty)$ with $f(1) = 1$, then a binary operation m defined by

$$A\,\mathsf{m}\,B = A^{1/2} f\left(A^{-1/2} B A^{-1/2}\right) A^{1/2}$$

for positive invertible operators A and B induces an operator mean $A\,\mathsf{m}\,B$. They also introduced the three operations in operator means:

The *transpose* $^\circ$, the *adjoint* * and the *dual* $^\perp$ are defined by:

$$A\,\mathsf{m}^\circ B = B\,\mathsf{m}\,A, \qquad f^\circ(x) = xf\left(\frac{1}{x}\right)$$

$$A\,\mathsf{m}^* B = (A^{-1}\,\mathsf{m}\,B^{-1})^{-1}, \qquad f^*(x) = \frac{1}{f(1/x)}$$

$$A\,\mathsf{m}^\perp B = (B^{-1}\,\mathsf{m}\,A^{-1})^{-1}, \qquad f^\perp(x) = \frac{x}{f(x)}.$$

An operation in the above is the composition of the other two. Self-transpose means are called *symmetric* and the geometric (operator) mean # (i.e., the mean corresponding to $f(x) = \sqrt{x}$) is invariant for all the above operations. The arithmetic and the harmonic ones, other typical symmetric means, are adjoint (or dual) each other.

As an extension of this class of binary operations on the positive invertible operators, we define a *solidarity* s by the following axioms:

- *right monotonicity:* $B_1 \le B_2$ implies $A\,\mathsf{s}\,B_1 \le A\,\mathsf{s}\,B_2$.
- *right semi-continuity:* $B_n \downarrow B$ implies $A\,\mathsf{s}\,B_n \downarrow A\,\mathsf{s}\,B$.
- *transformer inequality:* $T^*(A\,\mathsf{s}\,B)T \le (T^*AT)\,\mathsf{s}\,(T^*BT)$.

The corresponding normalized condition for solidarities is the following one which is often assumed:

- *normalization:* $I\,\mathsf{s}\,I = 0$.

This condition follows from the subclass of solidarities, say *derivative solidarity*: For a differentiable path of operator means $A\,\mathsf{m}_t B$ for positive invertible operators A and B and $t \in [0, 1]$ with $A = A\,\mathsf{m}_0 B$ and $B = A\,\mathsf{m}_1 B$, the derivative at $t = 0$ defines a solidarity s_{m}

$$A\,\mathsf{s}_{\mathsf{m}} B = \lim_{t \searrow 0} \frac{A\,\mathsf{m}_t B - A}{t}.$$

Then it is clear that s_{m} satisfies the above normalization.

Though it is extended for noninvertible cases, note that it does not always exists as a bounded operator (Later we discuss it in detail for the relative operator entropy) which is different from operator means. For a solidarity s, the representing function

$f(x) = 1 \mathrel{\mathsf{s}} x$ is operator monotone on $(0, \infty)$ and its transpose $F(x) = f^\circ(x) = x \mathrel{\mathsf{s}} 1 = xf(1/x)$ is operator concave as we will show later. The normalization implies $f(1) = 0$. Putting $f_\varepsilon(x) = f(x + \varepsilon)$, we have $f_\varepsilon(x)$ is operator monotone on $[0, \infty)$ and $\tilde{f}_\varepsilon(x) = f_\varepsilon(x) - f_\varepsilon(0)$ is nonnegative operator monotone and hence defines an operator mean. So $\tilde{f}_\varepsilon$ is also operator concave and hence so is f. Note that $F(0)$ is a nonnegative number by

$$\lim_{\varepsilon\to 0} F(\varepsilon) = \lim_{\varepsilon\to 0} \frac{f(1/\varepsilon)}{1/\varepsilon} = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} f'(x)$$

by the l'Hospital theorem where the limit always exists since f' is monotone nonincreasing. In general, f is not defined at 0, but this fact shows F can be always defined.

Lemma 2.1 *If f is operator monotone on* $(0, \infty)$*, then the corresponding solidarity* s *is defined by* $f(x) = 1 \mathrel{\mathsf{s}} x$ *and the transpose* $F(x) = x \mathrel{\mathsf{s}} 1$ *is operator concave on* $[0, \infty)$*. Conversely, if F is operator concave on* $[0, \infty)$ *and* $F(0) \geqq 0$*, then* $F(x) = x \mathrel{\mathsf{s}} 1$ *defines a solidarity., i.e.,* $f(x) = 1 \mathrel{\mathsf{s}} x$ *is operator monotone on* $(0, \infty)$.

Proof Suppose f is operator monotone. We may assume A and B are positive invertible operators. Then $(A+B)^{-\frac{1}{2}} A^{\frac{1}{2}}$ and $(A+B)^{-\frac{1}{2}} B^{\frac{1}{2}}$ are contractions. Since f is also operator concave, the Jensen inequality of Davis-Hansen-Pedersen type [10, 28, 29] shows

$$\begin{aligned}(A+B)^{-\frac{1}{2}} &\left(A^{\frac{1}{2}} f(A^{-1}) A^{\frac{1}{2}} + B^{\frac{1}{2}} f(B^{-1}) B^{\frac{1}{2}}\right) (A+B)^{-\frac{1}{2}} \\ &\le f\left((A+B)^{-1} + (A+B)^{-1}\right) = f\left(2(A+B)^{-1}\right).\end{aligned}$$

Multiplying $(A+B)^{\frac{1}{2}}$ from both sides in the above inequality, we have

$$\frac{Af(A^{-1}) + Bf(B^{-1})}{2} \le \frac{A+B}{2} f(2(A+B)^{-1}),$$

that is, $\frac{F(A)+F(B)}{2} \le F\left(\frac{A+B}{2}\right)$, which shows the operator concavity of F.

Conversely suppose F is operator concave and $F(0) \geqq 0$. Let $A \leqq B$ for positive invertible operators A and B. Then $B^{-\frac{1}{2}} A^{\frac{1}{2}}$ is contractive and then we also have

$$\begin{aligned}B^{-\frac{1}{2}} f(A) B^{-\frac{1}{2}} &= B^{-\frac{1}{2}} A^{\frac{1}{2}} F(A^{-1}) A^{\frac{1}{2}} B^{-\frac{1}{2}} \\ &\le B^{-\frac{1}{2}} \left(A^{\frac{1}{2}} F(A^{-1}) A^{\frac{1}{2}} + (B-A)^{\frac{1}{2}} F(0) (B-A)^{\frac{1}{2}}\right) B^{-\frac{1}{2}} \\ &\le F(B^{-1} + 0) = F(B^{-1}),\end{aligned}$$

so that

$$f(A) \le BF(B^{-1}) = f(B),$$

which completes the proof. □

To see the existence condition of s, we consider the tangent function G_α at $x = \alpha$ for F. Since

$$F'(x) = f(1/x) - \frac{1}{x}f'(1/x), \quad or \quad f'(x) = F(1/x) - \frac{1}{x}F'(1/x),$$

then we obtain

$$G_\alpha(x) = F'(\alpha)(x-\alpha) + F(\alpha) = F'(\alpha)x - \alpha F'(x) + F(\alpha) = f'(1/\alpha).$$

So we have the following existence condition:

Theorem 2.2 *For a solidarity* s*, let* $f(x) = 1\,\mathsf{s}\,x$ *and* $F(x) = x\,\mathsf{s}\,1$*. Then* $A\,\mathsf{s}\,B$ *exists if and only if* $G_\alpha(x) = F'(\alpha)x + f'(1/\alpha)$ *is bonded below for all* $\alpha > 0$*, which is precisely expressed as the existence of a selfadjoint operator C with*

$$F'(\alpha)A + f'(1/\alpha)B \ge C$$

for all $\alpha > 0$*.*

Proof For $\varepsilon > 0$, put $B_\varepsilon = B + \varepsilon$ and $X_\varepsilon = B_\varepsilon^{-\frac{1}{2}} A B_\varepsilon^{-\frac{1}{2}}$. Suppose $A\,\mathsf{s}\,B$ exists. Then

$$A\,\mathsf{s}\,B \le A\,\mathsf{s}\,B_\varepsilon = B_\varepsilon^{\frac{1}{2}} F(X_\varepsilon) B_\varepsilon^{\frac{1}{2}} \le B_\varepsilon^{\frac{1}{2}} G_\alpha(X_\varepsilon) B_\varepsilon^{\frac{1}{2}} = F'(\alpha)A + f'(1/\alpha)B_\varepsilon$$

for all $\alpha > 0$. Tending $\varepsilon \searrow 0$, we have by $f'(x) \ge 0$

$$A\,\mathsf{s}\,B \le F'(\alpha)A + f'(1/\alpha)B,$$

and hence it is bounded below.

Conversely suppose

$$F'(\alpha)A + f'(1/\alpha)B \ge C$$

for all $\alpha > 0$. Since $f'(1/\alpha) > 0$, we have

$$F'(\alpha)A + f'(1/\alpha)B_\varepsilon \ge C, \quad \text{that is,} \quad F'(\alpha)X_\varepsilon + f'(1/\alpha) \ge B_\varepsilon^{-\frac{1}{2}} C B_\varepsilon^{-\frac{1}{2}}.$$

The left hand of the latter inequality in the above attains $\eta(X_\varepsilon) = -X_\varepsilon \log X_\varepsilon$ as the infimum, that is, $A\,\mathsf{s}\,B_\varepsilon \ge C$. Therefore $A\,\mathsf{s}\,B_\varepsilon$ is bounded below and $A\,\mathsf{s}\,B$ exists with $A\,\mathsf{s}\,B \ge C$. □

Remark 2.3 In the proof of the above in [17], the only if part is a little ambiguous. The above proof is a complete version of [17, Theorem 1]. In the case of relative operator entropy, we showed it in [22].

If F is nonnegative, then F is also operator monotone, so that F defines the transpose of some operator mean. When s is operator mean, the above property is clear, so that we may assume that $F(x_0) = 0$ for some $x_0 \geqq 0$. If the set of zero points $\{x_0\}$ consists of 0 only, then F is nonpositive and so is f. In the case $x_0 > 0$,

$$f(1/x_0) = \frac{F(x_0)}{x_0} = 0.$$

Then we can normalize

$$\tilde{F}(x) = F(xx_0) \qquad \text{and} \qquad \tilde{f}(x) = f(x/x_0),$$

so that $\tilde{F}(1) = F(x_0) = 0$ and $\tilde{f}(1) = f(1/x) = 0$ like derivative solidarities: For a differentiable path of operator means m_t, we may define the corresponding solidarity s_m by

$$A \,\mathsf{s}_\mathsf{m}\, B = \lim_{t \searrow 0} \frac{A \,\mathsf{m}_t\, B - A}{t} = \frac{\partial}{\partial t} A \,\mathsf{m}_t\, B \Big|_{t=0}$$

if the limit exists. Since $f_\mathsf{m}(1) = 1$, we have $f(1) = F(1) = 0$ for the corresponding solidarity s.

3 Relative Operator Entropy

A typical and important example of solidarities is the relative operator entropy. First we review the *relative operator entropy* $\mathsf{S}(A|B)$ for positive (bounded linear) operators A, B on a Hilbert space, see [13, 14, 18–20]. If B is invertible, then it is defined by $\mathsf{S}(A|B) = B^{\frac{1}{2}} \eta \left(B^{-\frac{1}{2}} A B^{-\frac{1}{2}} \right) B^{\frac{1}{2}}$, where η is the entropy function:

$$\eta(x) = -x \log x \quad \text{if } x > 0, \qquad \eta(0) = 0.$$

In addition, if A is invertible, then $\mathsf{S}(A|B) = A^{\frac{1}{2}} \log \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}}$. Since $\mathsf{S}(A|B)$ has the right-term monotone decreasing property of $\mathsf{S}(A|B + \varepsilon)$ as $\varepsilon \downarrow 0$, we define for non-invertible A and B

$$\mathsf{S}(A|B) = \operatorname*{s-lim}_{\varepsilon \downarrow 0} \mathsf{S}(A|B + \varepsilon)$$

if the limit (in the strong operator topology) exists as a bounded operator. But, in general, $\mathsf{S}(A|B)$ does not always exist. On the other hand, based on the fact that $\dfrac{x^t - 1}{t} \searrow \log t$ as $t \downarrow 0$, it follows that $\frac{A\#_t B - A}{t}$ is monotone-decreasing as $t \downarrow 0$, so that another equivalent definition of Uhlmann's type is the derivative one for the path of geometric means $A\#_t B$:

$$\mathsf{S}(A|B) = \operatorname*{s-lim}_{t \downarrow 0} \frac{A\#_t B - A}{t}$$

if the limit exists. If A and B are commuting and $\mathsf{S}(A|B)$ is defined, then

$$\mathsf{S}(A|B) = A \log B - A \log A,$$

in particular, $\mathsf{S}(0|B) = 0$ for all positive operators $B \geq 0$. Though we often use unbounded expressions like $\log A$ from now on, these conventions are surely based on the total boundedness of $A \log A$. Under the existence, we have the following properties of $\mathsf{S}(A|B)$ for positive operators A and B by those for operator means:

Lemma 3.1 *Under the existence of relative operator entropies, the following properties like those of operator means hold:*

(1) *right monotonicity:* *If* $B \leq B'$, *then* $\mathsf{S}(A|B) \leq \mathsf{S}(A|B')$.
(2) *transformer inequality:* $T^*\mathsf{S}(A|B)T \leq \mathsf{S}(T^*AT|T^*BT)$ *for all* T
(*the equality holds for invertible* T).
(2′) *informational monotonicity:* $\Phi(\mathsf{S}(A|B)) \leq \mathsf{S}(\Phi(A)|\Phi(B))$
for all normal positive linear maps Φ.
(3) *sub-additivity:* $\mathsf{S}(A_1|B_1) + \mathsf{S}(A_2|B_2) \leq \mathsf{S}(A_1 + A_2|B_1 + B_2)$.
(3′) *joint concavity:* *For all* $t \in [0, 1]$,
$(1-t)\mathsf{S}(A_1|B_1) + t\mathsf{S}(A_2|B_2) \leq \mathsf{S}((1-t)A_1 + tA_2|(1-t)B_1 + tB_2)$.
(4) *upper bound:* $\mathsf{S}(A|B) \leq B - A$.
(5) *kernel inclusion:* $\ker \mathsf{S}(A|B) \supset \ker A$.
(6) *orthogonality:* $\mathsf{S}\left(\bigoplus_k A_k \mid \bigoplus_k B_k\right) = \bigoplus_k \mathsf{S}(A_k|B_k)$.
(7) *affine parametrization:* $\mathsf{S}(A|A\#_t B) = t\,\mathsf{S}(A|B)$ *for all* $t \in [0, 1]$.

Here we recall the equality condition in the transformer inequality (2) of Lemma 3.1 [12, Theorem 3]: If $\ker T^* \subset \ker A \cap \ker B$ for an operator T, then $T^*(A\,\mathsf{m}\,B)T = (T^*AT)\mathsf{m}(T^*BT)$ holds for all operator means m. Moreover this equality holds for in $\mathsf{S}(A|B)$ since $\mathsf{S}(A|B) = \operatorname{s-lim}_{t\downarrow 0} \frac{A\#_t B - A}{t}$:

Theorem 3.2 *Let* A *and* B *be positive operators. If* $\mathsf{S}(A|B)$ *exists and* $\ker T^* \subset \ker A \cap \ker B$ *for an operator* T, *then*

$$T^*\mathsf{S}(A|B)T = \mathsf{S}(T^*AT|T^*BT).$$

Then we have one of the (sufficient) conditions that $\mathsf{S}(A|B)$ exists;

Lemma 3.3 *If A is majorized by B, i.e., $A \le \alpha B$ for some $\alpha > 0$, then $\mathsf{S}(A|B)$ exists.*

In fact, by Douglas' majorization theorem [10], we have $A^{\frac{1}{2}} = DB^{\frac{1}{2}}$ for some 'derivative' operator D with $\ker D = \ker A \supset \ker B$ and so $\ker B = \ker A \cap \ker B$. Then, for the support projection P_B for B, we have $P_B A P_B = A$ and $P_B D^* D P_B = D^* D$. Hence it follows from Theorem 3.2 that

$$\mathsf{S}(A|B) = \mathsf{S}(B^{\frac{1}{2}} D^* D B^{\frac{1}{2}}|B) = B^{\frac{1}{2}} \mathsf{S}(D^* D|P_B) B^{\frac{1}{2}} = B^{\frac{1}{2}} \eta \left(D^* D\right) B^{\frac{1}{2}}$$

and so $\mathsf{S}(A|B)$ exists.

It is also shown that the majorization $A \le \alpha B$ is equivalent to the condition for the range inclusion;

$$\mathrm{ran} A^{\frac{1}{2}} \subset \mathrm{ran} B^{\frac{1}{2}}.$$

But it is too strong for the existence of $\mathsf{S}(A|B)$. In fact, A is not majorized by A^2 if $\sigma(A) = [0, 1]$, while we easily see $\mathsf{S}(A|A^2) = A \log A$.

Another candidate is the kernel inclusion

$$\ker A \supset \ker B,$$

which is weaker than the range inclusion. In fact, the kernel condition does not guarantee the existence: For B with $\sigma(B) = [0, 1]$ where 0 is not an eigenvalue, it follows that $\mathsf{S}(I|B) = \log B$ diverges while both kernels are trivial.

The third condition between the above ones is B-*absolute continuity* in the sense of Ando's Lebesgue decomposition [3]:

$$A = [B]A \equiv \underset{n\to\infty}{\text{s-lim}}\, A : nB$$

where $A : B$ defined by

$$\langle A : Bz, z\rangle = \inf_{x+y=z} [\langle Ax, x\rangle + \langle gBy, y\rangle] \qquad (\dagger)$$

is the *parallel addition* [2], which is the half of the *harmonic mean* $A \,\mathrm{h}\, B$ [4]. Kosaki [34] showed that

$$[B]A = A^{\frac{1}{2}} P_M A^{\frac{1}{2}}$$

for the projection P_M on the closed subspace

$$M = \overline{\{y \mid A^{\frac{1}{2}} y \in \mathrm{ran} B\}}.$$

This result implies $A = [B]A = \lim_{t\downarrow 0} A\#_t B$ and hence B-absolute continuity guarantees the continuity of $A\#_t B$ at $t = 0$ and it is a necessary condition for the existence of $\mathsf{S}(A|B)$ as the above derivative. In fact, this continuity is in the norm topology:

Lemma 3.4 *If $\mathsf{S}(A|B)$ exists, then $A\#_t B$ converges uniformly to A for $t \downarrow 0$.*

Proof Since there are scalars c_1 and c_2 with

$$c_1 \le \mathsf{S}(A|B) \le \frac{A\#_t B - A}{t} \le \frac{\|A\|\#_t\|B\| - A}{t} \le \|B\| - A \le c_2$$

for all $t \in (0, 1)$, we have $tc_1 \le A\#_t B - A \le tc_2$, so that the required convergence yields. □

Since $\ker A\#_t B \supset \ker A \vee \ker B$ for all $t \in (0, 1)$ as in [14] (as we will see later, these are equal indeed) and it is related to the ranges, it is a stronger condition than the kernel inclusion. But it is weaker than the existence condition: If A is the range projection P_B for B with $\sigma(B) = [0, 1]$, then $\mathsf{S}(P_B|B) = P_B \log B$ is not bounded.

The existence condition for solidarity shows that of $\mathsf{S}(A|B)$:

Corollary 3.5 *The relative operator entropy $\mathsf{S}(A|B)$ exists if and only if there exists a real number c with $\frac{1}{\alpha}B + (\log\alpha)A \ge c$ for all $\alpha > 0$.*

Summing up, we have the following relations around the existence condition:

Theorem 3.6 *The implications* (1) ⇒ (2) ⇒ (3) ⇒ (4) *hold in the following conditions for a pair of $A, B \ge 0$ and each converse does not always hold.*

(1) *majorization or range inclusion:* $\exists\alpha > 0;\ A \le \alpha B$, *i.e., ran*$A^{\frac{1}{2}} \subset$ *ran*$B^{\frac{1}{2}}$.
(2) *existence condition:* $\mathsf{S}(A|B)$ *exists as a bounded operator, i.e.,*

$$\frac{1}{\alpha}B + (\log\alpha)A > \exists c \in \mathbb{R} \qquad (\forall\alpha > 0)$$

(3) *B-absolute continuity:* $A = [B]A \left(= A^{\frac{1}{2}}P_M A^{\frac{1}{2}} = \lim_{t\downarrow 0} A\#_t B\right)$.
(4) *kernel inclusion:* $\ker A \supset \ker B$.

Remark 3.7 If both ranges of A and B are closed, in particular, for the case of matrices, the above conditions in Theorem 3.6 are mutually equivalent since the relation $\mathrm{ran}A^{\frac{1}{2}} = \overline{\mathrm{ran}A} = (\ker A)^{\perp}$ holds for all positive operators A.

4 CPR Geometry

The path of geometric means m_t and the relative operator entropy are important concepts in the geometric structure of the manifold of the positive invertible operators discussed by Corach et al. [8, 9] which we call it *CPR geometry*. In this section, we mention the general structure of the geometry of fiber bundles to read easily for readers not familiar with it. Here the *CPR geometry* represents the one on the Finsler manifold $\mathscr{A}^+$, the positive invertible elements in a unital C*-algebra $\mathscr{A}$, which we review in the below. Corach himself reformulated it in [8]: The *base manifold* is $\mathscr{A}^+$ with the *tangent vector bundle* $\mathscr{A}^h$ (the selfadjoint operators in $\mathscr{A}$). As we confirm later, the invertible elements $\mathcal{G} = \mathcal{G}(\mathscr{A})$ is the *principal fiber bundle* (of $\mathscr{A}^+$). Thus the *total space* $\mathcal{P} = \{\mathcal{G}, \mathscr{A}^+, \mathcal{U}, \pi\}$ is defined by

projection $\pi : \mathcal{G} \to \mathscr{A}^+$, $g \mapsto gg^*$
structure group the unitary group $\mathcal{U} = \mathcal{U}(\mathscr{A})$
principal fiber $\pi^{-1}(A) = A^{\frac{1}{2}}\mathcal{U}$

This definition of this fiber homeomorphic to $\mathcal{U}$ is consistent. In fact, for each unitary U, we obtain

$$\pi(A^{\frac{1}{2}}U) = A^{\frac{1}{2}}U(A^{\frac{1}{2}}U)^* = A^{\frac{1}{2}}UU^*A^{\frac{1}{2}} = A.$$

Conversely, take $g \in \pi^{-1}(A)$ for some $A \in \mathscr{A}^+$. Then, the polar decomposition of the adjoint g^*, which is the fundamental idea of the above correspondence, is $g^* = U^*(gg^*)^{\frac{1}{2}} = U^*A^{\frac{1}{2}}$ and hence $g = A^{\frac{1}{2}}U$. So the principal fiber bundle is the invertible operators $\mathcal{G}$ by

$$\mathcal{G} = \{A^{\frac{1}{2}}U \,|A \in \mathscr{A}^+,\ U \in \mathcal{U}\}.$$

As in Fig. 1, the projection $\pi : \mathcal{G} \to \mathscr{A}^+$ naturally induces the *derivative* π_* from the tangent bundle $\mathcal{T}(\mathcal{G})$ to the tangent vector bundle $\mathcal{T}(\mathscr{A}^+) = \mathscr{A}^h$. When

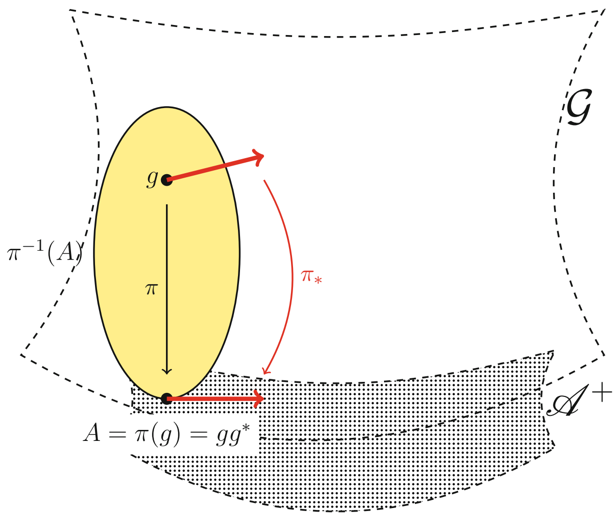


Fig. 1 Basic concept of the principal fiber bundle $\mathcal{P} = \{\mathcal{G}, \mathscr{A}^+, \mathcal{U}, \pi\}$

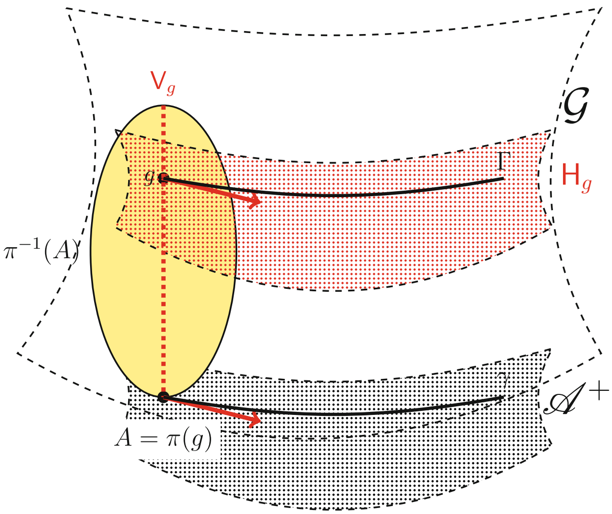


Fig. 2 Connection and the horizontal lift Γ of γ

we regard $\mathcal{G}$ as an upper structure of the base space $\mathscr{A}^+$, the kernel $\ker \pi_*$ is the vectors along π_* 'vertically'. So the subspace $\ker \pi_* \cap \mathcal{T}_g$ of the tangent space $\mathcal{T}_g(\mathcal{G})$ at $g \in \mathcal{G}$ is called the *vertical space* V_g. In this case, since the Lie algebra of a unitary group is the skew-hermitians, we have

$$\mathsf{V}_g = \{gX \mid X^* = -X\}.$$

In $\mathcal{T}_g(\mathcal{G})$, if a 'horizontal space' H_g is given (and it is compatible for the right action $\mathcal{U}$; $\mathsf{H}_{gU} = \mathsf{H}_g U$), then it is called the principal fiber $\mathcal{G}$ has a *connection* (in the sense of manifold). In CPR geometry, it is naturally determined by $\mathsf{H}_g = g\mathscr{A}^h$ considering the vertical one V_g. In fact, the compatibility of the right action is shown: Take $gH \in \mathsf{H}_g$ with $H = H^*$. Then, for all $U \in \mathcal{U}$, $gHU = gUU^*HU$ with $U^*HU \in \mathscr{A}^h$, which shows $\mathsf{H}_g U = \mathsf{H}_{gU}$.

Now, for a (differentiable) curve $\gamma(t)$ on $\mathscr{A}^+$, consider a *horizontal lift* $\Gamma(t) \in \pi^{-1}(\gamma(t))$.

The term 'lift' means

$$\gamma(t) = \pi(\Gamma(t)) = \Gamma(t)\Gamma(t)^*$$

and the term 'horizontal' means the tangent vector $\dot{\Gamma}(t) \in H_{\Gamma(t)}$. In other words, $\Gamma(t)^{-1}\dot{\Gamma}(t)$ is hermitian, i.e.,

$$\dot{\Gamma}^*(t)(\Gamma^*)^{-1}(t) = (\Gamma(t)^{-1}\dot{\Gamma}(t))^* = \Gamma(t)^{-1}\dot{\Gamma}(t).$$

This is equivalent to $\Gamma\dot{\Gamma}^* = \dot{\Gamma}\Gamma^*$. Then we have

$$\dot{\gamma} = \dot{\Gamma}\Gamma^* + \Gamma\dot{\Gamma}^* = 2\dot{\Gamma}\Gamma^*$$

and hence

$$\dot{\gamma}\gamma^{-1} = (2\dot{\Gamma}\Gamma^*)(\Gamma^*)^{-1}\Gamma^{-1} = 2\dot{\Gamma}\Gamma^{-1},$$

which is called the *transport equation* which characterizes a horizontal lift.

As we will see later, the geodesic from A to B is given by the path $\gamma(t) = A \#_t B$ of the geometric means. One of the horizontal lifts of the geodesic is given by the following simple form, which is our unpublished result:

Lemma 4.1 *The curve* $\Gamma(t) = A^{\frac{1}{2}} C^{\frac{t}{2}}$ *for* $C = A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$ *is a horizontal lift of* $\gamma(t) = A \#_t B$.

Proof The transport equation follows from:

$$\begin{aligned} 2\dot{\Gamma}(t)\Gamma(t)^{-1} &= A^{\frac{1}{2}}(\log C)C^{\frac{t}{2}}C^{-\frac{t}{2}}A^{-\frac{1}{2}} = A^{\frac{1}{2}}(\log C)A^{-\frac{1}{2}} \\ \dot{\gamma}(t)\gamma(t)^{-1} &= A^{\frac{1}{2}}(\log C)C^t A^{\frac{1}{2}} A^{-\frac{1}{2}} C^{-t} A^{-\frac{1}{2}} = A^{\frac{1}{2}}(\log C)A^{-\frac{1}{2}}. \end{aligned}$$ □

In order to obtain further results in this geometry, we identify the tangent vector bundle, which is $\mathscr{A}^h$, as the associated vector bundle for $\mathcal{G}$. Note that $\mathcal{G}$ acts on $A \in \mathscr{A}^+$ by $A \mapsto gAg^*$ and hence $X \to gXg^* \equiv \rho(g)X$. Regard $\mathscr{A}^h$ as the tangent vector space, consider the *associated bundle*: it is the quotient bundle $\mathcal{G} \times_\rho \mathscr{A}^h$ of $\mathcal{G} \times \mathscr{A}^h$ with the equivalence relation of the right action by $f \in \mathcal{G}$

$$(g, X)f = (gf, \rho(f)X) \sim (g, X) \sim (I_{\mathscr{A}}, \rho(g^{-1})X).$$

Roughly speaking, at the point $\pi(g)$, we see $\rho(g^{-1})X = g^{-1}X(g^*)^{-1}$ as a tangent vector. Considering the connection of $\mathcal{G}$, we reflects it on the tangent bundle via a horizontal lift: If γ is a path on $\mathscr{A}^+$ and Γ is a horizontal lift of it, then we observe $\Gamma^{-1}(t)\dot{\gamma}(t)(\Gamma(t)^{-1})^*$ instead of the tangent vector $\dot{\gamma}(t)$, and thereby the translation $\Gamma(t)\Gamma(0)^{-1} : \Gamma(0) \mapsto \Gamma(t)$ yield the *parallel displacement* of a tangent vector X along γ from 0 to t is

$$P_t X = \Gamma(t)\Gamma(0)^{-1}X(\Gamma(0)^*)^{-1}\Gamma(t)^*.$$

Note that

$$\Gamma(\dot{\Gamma^{-1}}) = \dot{\Gamma}\Gamma^{-1} = \frac{1}{2}\dot{\gamma}\gamma^{-1}$$

by the transport equation and the noncommutative inverse differential formula

$$(\dot{\Gamma^{-1}}) = -\Gamma^{-1}\dot{\Gamma}\Gamma^{-1}.$$

So the *covariant derivative* D_t of a tangent field $X(t)$ along the curve $\gamma(t)$ in $\mathscr{A}^+$ is given by

$$\begin{aligned}
D_t X &= \lim_{\varepsilon\to 0} \frac{\Gamma(t)\Gamma(t+\varepsilon)^{-1}X(t+\varepsilon)(\Gamma(t+\varepsilon)^*)^{-1}\Gamma(t)^* - X(t)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \Gamma(t)\Gamma(t+\varepsilon)^{-1} \cdot \frac{X(t+\varepsilon) - X(t)}{\varepsilon} \cdot (\Gamma(t+\varepsilon)^*)^{-1}\Gamma(t)^* \\
&\quad + \Gamma(t) \cdot \frac{\Gamma(t+\varepsilon)^{-1} - \Gamma(t)^{-1}}{\varepsilon} \cdot X(t)(\Gamma(t+\varepsilon)^*)^{-1}\Gamma(t)^* \\
&\quad + X(t) \cdot \frac{(\Gamma(t+\varepsilon)^*)^{-1} - (\Gamma(t)^*)^{-1}}{\varepsilon} \cdot \Gamma(t)^* \\
&= \dot{X} - (\dot{\Gamma}\Gamma^{-1}X + X(\dot{\Gamma^*})^{-1}\Gamma(t)^* = \dot{X} - (\dot{\Gamma}\Gamma^{-1}X + X(\Gamma\dot{\Gamma^{-1}})^*)(t) \\
&= \dot{X} - (\dot{\Gamma}\Gamma^{-1}X + X(\dot{\Gamma}\Gamma^{-1})^*)(t) = \dot{X} - \frac{1}{2}(\dot{\gamma}\gamma^{-1}X + X\gamma^{-1}\dot{\gamma})(t).
\end{aligned}$$

Then, as the self-parallel curve, the *geodesic equation* is

$$O = D_t\dot{\gamma} = \ddot{\gamma} - \dot{\gamma}\gamma^{-1}\dot{\gamma}$$

and we can obtain the geodesic:

Theorem 4.2 (Corach-Porta-Lecht) *The geodesic from A to B in the above geometry in $\mathscr{A}^+$ is the path of geometric Kubo-Ando means:*

$$\gamma(t) \equiv A\#_t B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}}.$$

To see this, we confirm the following approximation:

Lemma 4.3 *Let f be a (norm differentiable) path in $\mathcal{A}^+$. If $f(t)$ commutes with $f'(t)$ for each $t \in [0, 1]$, then*

$$\frac{d}{dt}\log f(t) = f'(t)f^{-1}(t).$$

Proof For arbitrary $\varepsilon > 0$, there is $\delta_\varepsilon > 0$ with

$$\left\| \frac{f(t+\delta_\varepsilon) - f(t)}{\delta_\varepsilon} - f'(t) \right\| < \varepsilon,$$

or equivalently

$$\delta_\varepsilon(f'(t)-\varepsilon) \le f(t+\delta_\varepsilon) - f(t) \le \delta_\varepsilon(f'(t)+\varepsilon).$$

We can choose δ_ε such that $\delta_\varepsilon \searrow 0$ if $\varepsilon \searrow 0$. Putting $h = \delta_\varepsilon(f'(t)+\varepsilon)$, we have h commutes with $f(t)$ and hence

$$\begin{aligned}\frac{\log f(t+\delta_\varepsilon) - \log f(t)}{\delta_\varepsilon} \le \frac{\log(f(t)+h) - \log f(t)}{\delta_\varepsilon} &= \frac{\log(f(t)+h) - \log f(t)}{h}\frac{h}{\delta_\varepsilon} \\ &= \frac{\log(f(t)+h) - \log f(t)}{h}(f'(t)+\varepsilon) \longrightarrow f(t)^{-1}f'(t)\end{aligned}$$

as $\varepsilon \to 0$. Thereby $\delta_\varepsilon \to 0$ and $h \to 0$. Thus $(\log f(t))' \le f'(t)f(t)^{-1}$. Similarly, considering the left hand in (4), we have $(\log f(t))' \ge f'(t)f(t)^{-1}$, so that $(\log f(t))' = f'(t)f(t)^{-1}$. □

Then we give a proof of Theorem 4.2:

Proof of Theorem 4.2 By the geodesic equation, for

$$f(t) = \gamma(0)^{-\frac{1}{2}}\gamma(t)\gamma(0)^{-\frac{1}{2}},$$

we have $f(0) = I$ and f also satisfies the geodesic equation (Indeed, it is a curve from I to $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$). By $f''f^{-1} = (f'f^{-1})^2$, we have

$$(f'f^{-1})' = f''f^{-1} + f'(f^{-1})' = f''f^{-1} - (f'f^{-1})^2 = 0$$

and then there exists C with $f'f^{-1} = C$. Since $C = f'(0)f(0)^{-1} = f'(0)$, we have $C = C^*$ and hence $f(t)$ and $f'(t)$ (also C) are commuting for all $t \in [0, 1]$. Note that $f'f^{-1} = C$ by the above lemma, so that there exists an operator D with $\log f(t) = tC + D$. Moreover, the case $t = 0$ shows $D = 0$: $f(t) = e^{tC}$. Thereby

$$\gamma(t) = \gamma(0)^{\frac{1}{2}}e^{tC}\gamma(0)^{\frac{1}{2}}.$$

Considering the terminal conditions $A = \gamma(0)$ and $B = \gamma(1) = A^{\frac{1}{2}}e^{C}A^{\frac{1}{2}}$ shows

$$\gamma(t) = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^tA^{\frac{1}{2}} = A\#_tB.$$

□

The *parallel transport* of the tangent vector X at $\gamma(t_1)$ to that at $\gamma(t_2)$ along the geodesic γ is described as $\mathsf{P}_{t_2}^{t_1}X$. Now by Lemma 4.1, we can obtain the parallel transform along the geodesic:

Theorem 4.4 *The parallel transform of the tangent vector X on A to that on B along the geodesic is given by*

$$\mathsf{P}_1^0 X = A^{\frac{1}{2}} C^{\frac{1}{2}} A^{-\frac{1}{2}} X A^{-\frac{1}{2}} C^{\frac{1}{2}} A^{\frac{1}{2}}$$

for $C = A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$.

In particular, the case $B = I$, we have $A^{-\frac{1}{2}} X A^{-\frac{1}{2}} = \rho^{-1}(A^{\frac{1}{2}})X$. This manifold is a symmetric space and then a homogeneous space; $\mathscr{A}^+ = \mathcal{G}/\mathcal{U}$. Thus properties around the identity I reflect on those around other points. Moreover, every symmetric space is *geodesic complete*, that is, the domain [0, 1] is extended to $\mathbb{R}$:

$$\gamma(t) = A \natural_t B \equiv A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^t A^{\frac{1}{2}}$$

for $t \in \mathbb{R}$.

From this viewpoint, the definition

$$L(X; A) = \|X\|_A = \|A^{-\frac{1}{2}} X A^{-\frac{1}{2}}\|$$

at each point $A \in \mathscr{A}$ makes the above manifold $\mathscr{A}^+$ a *Finsler space* with a Finsler metric as in the CPR main result. Since $\|X\|_A$ is equivalent to the operator norm $\|X\|$, it is a *Finsler metric* if

Finsler condition: $$L(P_t X; \gamma(t)) = \|P_t X\|_{\gamma(t)} = \|X\|_{\gamma(0)} = L(X; \gamma(0))$$

holds for all curves γ and parallel transports P_t along γ. The CPR geometry does not always determine a unique Finsler metric. In fact, we show each unitarily invariant norm $|||\ |||$ also gives a Finsler metric for the CPR geometry:

Theorem 4.5 *For a unitarily invariant norm* $|||\ |||$ *on* $\mathscr{A}$, *a function*

$$L_{|||\ |||}(X; A) = |||X|||_A = |||A^{-\frac{1}{2}} X A^{-\frac{1}{2}}|||$$

determines a Finsler metric on $\mathscr{A}^+$ *for the CPR geometry.*

Proof Since $U_t = \gamma(t)^{-\frac{1}{2}} \Gamma(t)$ defines a unitary for each t by $\gamma = \Gamma\Gamma^*$, the Finsler condition is satisfied by

$$\begin{aligned} |||P_t X|||_{\gamma(t)} &= |||U_t U_0^* \gamma(0)^{-\frac{1}{2}} X \gamma(0)^{-\frac{1}{2}} U_0 U_t^*||| \\ &= |||\gamma(0)^{-\frac{1}{2}} X \gamma(0)^{-\frac{1}{2}}||| = |||X|||_{\gamma(0)}. \end{aligned}$$

□

A Finsler metric $L_{|||\ |||}(X;A)$, which is called a *unitarily invariant Finsler one*, is homogeneous like $L(X;A)$:

Theorem 4.6 *For any invertible operator Y,*

$$L_{|||\ |||}(Y^*XY;Y^*AY)=L_{|||\ |||}(X;A).$$

Proof Since $|||Z|||=|||\,|Z|\,|||=|||\sqrt{Z^*Z}|||=|||\sqrt{ZZ^*}|||$, we have

$$\begin{aligned}
L_{|||\ |||}(Y^*XY;Y^*AY)&=|||(Y^*AY)^{-\frac{1}{2}}Y^*XY(Y^*AY)^{-\frac{1}{2}}|||\\
&=|||\sqrt{(Y^*AY)^{-\frac{1}{2}}Y^*XY(Y^*AY)^{-1}Y^*XY(Y^*AY)^{-\frac{1}{2}}}|||\\
&=|||\sqrt{(Y^*AY)^{-\frac{1}{2}}Y^*XA^{-1}XY(Y^*AY)^{-\frac{1}{2}}}|||\\
&=|||\sqrt{A^{-\frac{1}{2}}XY(Y^*AY)^{-1}Y^*XA^{-\frac{1}{2}}}|||\\
&=|||\sqrt{A^{-\frac{1}{2}}XA^{-1}XA^{-\frac{1}{2}}}|||=|||A^{-\frac{1}{2}}XA^{-\frac{1}{2}}|||\\
&=L_{|||\ |||}(X;A).
\end{aligned}$$

□

In particular, the case of the Hilbert-Schmidt norm shows it is a Riemannian manifold, which is discussed by Bhatia-Holbrook [7].

Finally in this section, we emphasise that the relative operator entropy $\mathsf{S}(A|B)$ is the (initial) tangent vector of the geodesic $A\,\#_t\,B$, and adding the initial point A, we reconstruct the geodesic by the exponential map Exp of this manifold:

$$\mathrm{Exp}_A t\mathsf{S}(A|B)\equiv\rho(A^{\frac{1}{2}})\exp t\left(\rho^{-1}(A^{\frac{1}{2}})(\mathsf{S}(A|B))\right)=A\,\#_t\,B.$$

Thus this geometric consideration says that the Karcher equation should be

$$\sum_n w(n)\mathsf{S}(A_n|X)=O$$

as a barycenter of the terminal tangent vectors, which is the geometric meaning for the Karcher equation, cf. [37, 38]. It is also consistent considering the power mean equation

$$X=w(n)\left(A_n\,\#_t\,X\right),$$

or

$$\sum_n w(n)\frac{A_n\,\#_t\,X-X}{t}=O.$$

5 Tsallis Relative Entropy

In this section, we consider quantum relative entropies in the framework of the matrix theory. Let $\mathbb{M}_n(\mathbb{C}) = \mathbb{M}_n$ be the algebra of $n \times n$ complex matrices, $\mathbb{P}_n$ the set of positive definite matrices in $\mathbb{M}_n$ and Tr the usual trace. We denote the set of all density matrices (positive definite matrices with trace one) by $\mathbb{S}_n(\mathbb{C}) = \mathbb{S}_n$.

As a quantum extension of the Shannon entropy [43], von Neumann [46] defined the entropy of the density matrix A in $\mathbb{S}_n$ by the formula

$$\mathrm{S}(A) = \mathrm{Tr}\,[\eta(A)],$$

where the entropy function $\eta(t) = -t \log t$. As for the Shannon entropy, it is extremely useful to define a quantum version of the relative entropy. Suppose that A and B are density matrices in $\mathbb{S}_n$. As a quantum generalization of the relative entropy due to Kullback and Leibler [36], Umegaki firstly introduced in the setting of von Neumann algebra [45] in 1962 the quantum relative entropy of A with respect to B, which is defined by

$$\mathrm{S}_U(A|B) = \begin{cases} \mathrm{Tr}\,[A(\log A - \log B)] & \text{if } \mathrm{supp}\, A \subset \mathrm{supp}\, B, \\ +\infty & \text{otherwise.} \end{cases} \tag{1}$$

We call (1) the Umegaki relative entropy. In [1], for any A and B in $\mathbb{S}_n$, the Tsallis relative entropy of A to B is defined by

$$\mathrm{D}_\alpha(A|B) = \frac{1 - \mathrm{Tr}\,[A^{1-\alpha}B^\alpha]}{\alpha} = \mathrm{Tr}\,[A^{1-\alpha}(\ln_\alpha A - \ln_\alpha B)] \tag{2}$$

for any $0 < \alpha \leq 1$, where $\ln_\alpha t = \frac{t^\alpha - 1}{\alpha}$ is the α-logarithmic function. The Tsallis relative entropy (2) is a 1-parameter extension of (1), and Ruskai and Stillinger in [41] showed the following relation between the Tsallis relative entropy and the Umegaki relative entropy:

$$\mathrm{D}_\alpha(A|B) \leq \mathrm{S}_U(A|B) \leq D_{-\alpha}(A|B)$$

for all $0 < \alpha \leq 1$, and $\lim_{\alpha \to 0} \mathrm{D}_\alpha(A|B) = \mathrm{S}_U(A|B)$.

On the other hand, there are another formulation of the quantum relative entropy. By Theorem 4.2, the path $\gamma(t) = A\#_t B$ for $t \in \mathbb{R}$ is the geodesic from A to B with $\gamma(0) = A$ and $\gamma(1) = B$, and the relative operator entropy $\mathsf{S}(A|B)$ is the velocity vector of the geodesic $A\#_t B$ at $t = 0$. By virtue of the relative operator entropy, we define the quantum relative entropy as

$$\mathrm{S}_{FK}(A|B) = -\mathrm{Tr}\,[\mathsf{S}(A|B)]. \tag{3}$$

The quantum quantity $\mathrm{Tr}\,[A(\log A^{1/2}B^{-1}A^{1/2})]$ is firstly proposed by Belavkin and Staszewski [6] in the framework of C*-algebra. Since we treat $\mathrm{S}_{FK}(A|B)$ as the minus of the trace of the relative operator entropy $\mathsf{S}(A|B)$, we call (3) the FK relative entropy, or the BS relative entropy in [39, pp125]. If A and B commute, then we have $\mathrm{S}_U(A|B) = \mathrm{S}_{FK}(A|B)$. Generally, two quantum formulations of the relative entropy are different. In fact, Hiai and Petz [30] showed the following relation:

$$\mathrm{S}_U(A|B) \le \mathrm{S}_{FK}(A|B) \tag{4}$$

and a 1-parameter extension of (4):

$$\mathrm{S}_U(A|B) \le -\frac{1}{q}\mathrm{Tr}\,[A^{1-q}\mathsf{S}(A^q|B^q)] \qquad \text{for all } q > 0. \tag{5}$$

In fact, if $q = 1$ in (5), then we have the Hiai-Petz inequality (4) and as $q \to 0$ the right-hand side of (5) converges to the Umegaki relative entropy $\mathrm{S}_U(A|B)$.

Moreover, Yanagi et al. [47] have been advancing research on the Tsallis relative operator entropy as an operator generalization of the Tsallis relative entropy, which is regarded as a 1-parameter extension of the relative operator entropy: For positive definite matrices A and B in $\mathbb{P}_n$, the Tsallis relative operator entropy is defined by

$$\mathsf{T}_\alpha(A|B) = \frac{A \sharp_\alpha B - A}{\alpha} \qquad \text{for } 0 < \alpha \le 1.$$

We recall the notation $\natural_\alpha$ for the binary operation

$$A \natural_\alpha B = A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2} \qquad \text{for } \alpha \notin [0, 1],$$

that have formula in common with $\sharp_\alpha$. Then the Tsallis relative entropy of negative order is defined by

$$\mathsf{T}_\alpha(A|B) = \frac{A \natural_\alpha B - A}{\alpha} \qquad \text{for } \alpha \in [-1, 0).$$

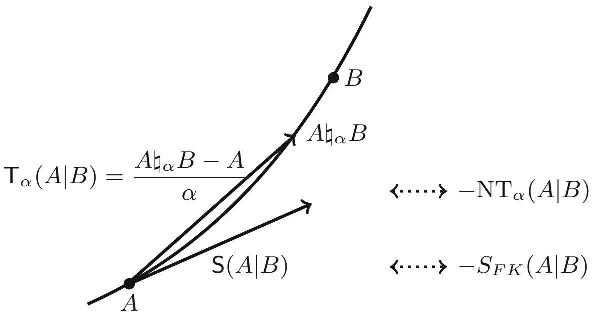
For convenience, we denote another quantum Tsallis relative entropy of order $\alpha \in [-1, 1]\backslash\{0\}$ by

$$\mathrm{NT}_\alpha(A|B) = -\mathrm{Tr}\left[\mathsf{T}_\alpha(A|B)\right]$$

for positive definite matrices A and B in $\mathbb{P}_n$. For two relative entropies, we know the following two relations:

$$\mathrm{NT}_\alpha(A|B) \le \mathrm{S}_{FK}(A|B) \le \mathrm{NT}_{-\alpha}(A|B) \qquad \text{for all } \alpha \in (0, 1]$$

Fig. 3 Geometric structure of Tsallis relative entoropy



(see Fig. 3) and

$$\mathrm{D}_\alpha(A|B) \le \mathrm{NT}_\alpha(A|B) \qquad \text{for all } \alpha \in [-1, 1]. \tag{6}$$

We have the following properties of the quantum relative entropy NT_α of order $\alpha \in [-1, 1]\backslash\{0\}$, also see [21]:

Theorem 5.1 *Let A and B be positive definite matrices in $\mathbb{P}_n$ and $\alpha \in [-1, 1]\backslash\{0\}$. Then the following properties of the quantum Tsallis relative entropy NT_α hold:*

(1) *(Non-negativity) $NT_\alpha(A|B) \ge 0$ if $A \ge B$.*
(2) *(Pseudoadditivity)*

$$\mathrm{NT}_\alpha(A_1 \otimes A_2|B_1 \otimes B_2) = \mathrm{NT}_\alpha(A_1|B_1) + \mathrm{NT}_\alpha(A_2|B_2) + \alpha\mathrm{NT}_\alpha(A_1|B_1)\mathrm{NT}_\alpha(A_2|B_2).$$

(3) *(Joint convexity)* $\mathrm{NT}_\alpha(\sum_j \lambda_j A_j | \sum_j \lambda_j B_j) \le \sum_j \lambda_j \mathrm{NT}_\alpha(A_j|B_j)$.
(4) *(Monotonicity) For any trace-preserving positive linear map Φ*

$$\mathrm{NT}_\alpha(\Phi(A)|\Phi(B)) \le \mathrm{NT}_\alpha(A|B).$$

Proof For (1), if $\alpha \in [-1, 0)$, then it follows that $(1-\alpha)A + \alpha B \le A \natural_\alpha B$ and so

$$\begin{aligned}\mathrm{NT}_\alpha(A|B) &= -\mathrm{Tr}\left[\frac{A \natural_\alpha B - A}{\alpha}\right] \ge -\mathrm{Tr}\left[\frac{(1-\alpha)A + \alpha B - A}{\alpha}\right] \\ &= -\mathrm{Tr}\,[B - A] \ge 0.\end{aligned}$$

If $\alpha \in (0, 1]$, then it follows that $(1-\alpha)A + \alpha B \ge A \sharp_\alpha B$ and so

$$\begin{aligned}\mathrm{NT}_\alpha(A|B) &= -\mathrm{Tr}\left[\frac{A \sharp_\alpha B - A}{\alpha}\right] \ge -\mathrm{Tr}\left[\frac{(1-\alpha)A + \alpha B - A}{\alpha}\right] \\ &= -\mathrm{Tr}\,[B - A] \ge 0.\end{aligned}$$

For (2), suppose that $\alpha \in [-1, 1]\backslash\{0\}$. Then we have

$$\begin{aligned}
&\mathsf{T}_\alpha(A_1 \otimes A_2|B_1 \otimes B_2)\\
&= \frac{1}{\alpha}\left[(A_1 \otimes A_2)\natural_\alpha(B_1 \otimes B_2) - A_1 \otimes A_2\right]\\
&= \frac{1}{2}\mathsf{T}_\alpha(A_1|B_1) \otimes (A_2\natural_\alpha B_2) + \frac{1}{2}(A_1\natural_\alpha B_1) \otimes \mathsf{T}_\alpha(A_2|B_2) + \frac{1}{2}A_1 \otimes \mathsf{T}_\alpha(A_2|B_2)\\
&\quad + \frac{1}{2}\mathsf{T}_\alpha(A_1|B_1) \otimes A_2\\
&= \frac{1}{2}\mathsf{T}_\alpha(A_1|B_1) \otimes (A_2\natural_\alpha B_2) - \frac{1}{2}\mathsf{T}_\alpha(A_1|B_1) \otimes A_2 + \mathsf{T}_\alpha(A_1|B_1) \otimes A_2\\
&\quad + \frac{1}{2}(A_1\natural_\alpha B_1) \otimes \mathsf{T}_\alpha(A_2|B_2) - \frac{1}{2}A_1 \otimes \mathsf{T}_\alpha(A_2|B_2) + A_1 \otimes \mathsf{T}_\alpha(A_2|B_2)\\
&= \alpha\mathsf{T}_\alpha(A_1|B_1) \otimes \mathsf{T}_\alpha(A_2|B_2) + \mathsf{T}_\alpha(A_1|B_1) \otimes A_2 + A_1 \otimes \mathsf{T}_\alpha(A_2|B_2).
\end{aligned}$$

Hence it follows that

$$\begin{aligned}
\mathsf{T}_\alpha(A_1 \otimes A_2|B_1 \otimes B_2) = \alpha\mathsf{T}_\alpha(A_1|B_1) \otimes \mathsf{T}_\alpha(A_2|B_2) + \mathsf{T}_\alpha(A_1|B_1) \otimes A_2&\\
+ A_1 \otimes \mathsf{T}_\alpha(A_2|B_2)&
\end{aligned}$$

and we have the desired equality (2).

For (3), it follows from jointly convexity of $\natural_\alpha$ for $\alpha \in [-1, 0)$ and jointly concavity of $\sharp_\alpha$ for $\alpha \in (0, 1]$ that

$$\begin{aligned}
\mathrm{NT}_\alpha(\sum_j \lambda_j A_j|\sum_j \lambda_j B_j) &= -\mathrm{Tr}\left[\frac{(\sum\lambda_j A_j)\ \natural_\alpha\ (\sum\lambda_j B_j) - \sum\lambda_j A_j}{\alpha}\right]\\
&\le -\mathrm{Tr}\left[\frac{\sum\lambda_j(A_j\ \natural_\alpha\ B_j) - \sum\lambda_j A_j}{\alpha}\right]\\
&= -\mathrm{Tr}\left[\sum_j \lambda_j \mathsf{T}_\alpha(A_j|B_j)\right] = \sum_j \lambda_j \mathrm{NT}_\alpha(A_j|B_j)
\end{aligned}$$

and thus we have (3).

For (4), it follows from the information monotonicity of $\natural_\alpha$ for $\alpha \in [-1, 1]\backslash\{0\}$ that

$$\begin{aligned}
\mathrm{NT}_\alpha(\Phi(A)|\Phi(B)) = -\mathrm{Tr}\left[\mathsf{T}_\alpha(\Phi(A)|\Phi(B))\right] &= -\mathrm{Tr}\left[\frac{\Phi(A)\ \natural_\alpha\ \Phi(B) - \Phi(A)}{\alpha}\right]\\
\le -\mathrm{Tr}\left[\frac{\Phi(A\ \natural_\alpha\ B) - \Phi(A)}{\alpha}\right] &= -\mathrm{Tr}\left[\Phi(\frac{A\ \natural_\alpha\ B - A}{\alpha})\right]
\end{aligned}$$

$$= -\mathrm{Tr}\left[\frac{A \natural_\alpha B - A}{\alpha}\right]$$

$$= \mathrm{NT}_\alpha(A|B)$$

and hence we have (4). □

To prove the main theorem, we need some preliminaries. Bebiano et al. in [5, Theorem 2.1] showed the following norm inequality, say BLP inequality: If A and B are positive definite matrices in $\mathbb{P}_n$, then

$$\left|\left|\left| A^{\frac{1+t}{2}} B^t A^{\frac{1+t}{2}} \right|\right|\right| \le \left|\left|\left| A^{1/2}(A^{s/2}B^s A^{s/2})^{\frac{t}{s}} A^{1/2} \right|\right|\right| \tag{7}$$

for all $s \ge t \ge 0$ and any unitarily invariant norm $|||\cdot|||$.

The Furuta inequality [24, pp49] says that if $A \ge B \ge 0$, then

$$A^{-r} \sharp_{\frac{1+r}{p+r}} B^p \le A \qquad \text{for } p \ge 1 \text{ and } r > 0. \tag{8}$$

We show the following variant of the BLP inequality (7):

Lemma 5.2 *[23, Theorem 3.2] Let A and B be positive definite matrices in $\mathbb{P}_n$. Then*

$$\left|\left|\left| A^{\frac{1}{2}} \left(A^{-\frac{p}{2}} B^p A^{-\frac{p}{2}}\right)^{\frac{q}{p}} A^{\frac{1}{2}} \right|\right|\right| \le \left|\left|\left| A^{\frac{1-q}{2}} B^q A^{\frac{1-q}{2}} \right|\right|\right| \tag{9}$$

for all $p \ge q > 0$ and $0 < q \le 1$, and any unitarily invariant norm $|||\cdot|||$.

Proof By the antisymmetric tensor technique, in order to prove (9), it suffices to show that

$$\lambda_1(A^{\frac{1}{2}} \left(A^{-\frac{p}{2}} B^p A^{-\frac{p}{2}}\right)^{\frac{q}{p}} A^{\frac{1}{2}}) \le \lambda_1(A^{\frac{1-q}{2}} B^q A^{\frac{1-q}{2}}) \tag{10}$$

for all $0 < q \le p$, where $\lambda_1(A)$ is the maximal eigenvalue of A.

For this purpose we may prove that

$$A^{\frac{1-q}{2}} B^q A^{\frac{1-q}{2}} \le I \quad \text{implies} \quad A^{\frac{1}{2}} \left(A^{-\frac{p}{2}} B^p A^{-\frac{p}{2}}\right)^{\frac{q}{p}} A^{\frac{1}{2}} \le I,$$

or equivalently, replacing A and B by $A^{\frac{1}{q-1}}$ and $B^{\frac{1}{q}}$ respectively

$$B \le A \qquad \Longrightarrow \qquad A^{\frac{p}{q-1}} \sharp_{\frac{q}{p}} B^{\frac{p}{q}} \le A^{\frac{p-1}{q-1}}$$

for all $0 < q \le p$ and $0 < q \le 1$, because both sides of (10) have the same order of homogeneity for A, B, so that we can multiply A, B by a positive constant.

Put $r = \frac{p}{1-q} > 0$ and $p' = \frac{p}{q} \geq 1$. Then $\frac{p-1}{q-1} = \frac{p'+r-p'r}{p'}$. It follows from the Furuta inequality (8) that

$$\begin{aligned} A^{\frac{p}{q-1}} \sharp_{\frac{q}{p}} B^{\frac{p}{q}} &= A^{-r} \sharp_{\frac{1}{p'}} B^{p'} \\ &= A^{-r} \sharp_{\frac{p'+r}{p'(1+r)}} \left(A^{-r} \sharp_{\frac{1+r}{p'+r}} B^{p'} \right) \qquad \text{by the multiplicity of } \sharp_\alpha \\ &\leq A^{-r} \sharp_{\frac{p+r}{p(1+r)}} A \qquad \text{by the Furuta inequality (8)} \\ &= A^{\frac{p'+r-p'r}{p'}} = A^{\frac{p-1}{q-1}} \end{aligned}$$

and so the proof is complete. □

We show a 1-parameter extension of the inequality (6), which is a generalization of the inequality (5) due to Hiai-Petz:

Theorem 5.3 *Let A and B be positive definite matrices in $\mathbb{P}_n$. Then*

$$\mathrm{D}_\alpha(A|B) \leq -\frac{1}{q} Tr\,[A^{1-q} \mathsf{T}_{\frac{\alpha}{q}}(A^q|B^q)]$$

for all $0 < \alpha \leq 1$ *and* $q \geq \alpha > 0$, *or* $-1 \leq \alpha < 0$ *and* $q \geq -\alpha > 0$.

Proof Suppose that $0 < \alpha \leq 1$. Since $\mathrm{Tr}\,[|A|]$ is unitarily invariant norm, it follows from Lemma 5.2 that

$$\mathrm{Tr}\,[A^{1-\alpha} B^\alpha] = \mathrm{Tr}\,[A^{\frac{1-\alpha}{2}} B^\alpha A^{\frac{1-\alpha}{2}}] \geq \mathrm{Tr}\,[A^{\frac{1}{2}} (A^{-\frac{q}{2}} B^q A^{-\frac{q}{2}})^{\alpha/q} A^{\frac{1}{2}}]$$

for all $q \geq \alpha > 0$. Hence for each $\alpha \in (0, 1]$

$$\begin{aligned} \mathrm{D}_\alpha(A|B) &= -\mathrm{Tr}\left[\frac{A^{1-\alpha} B^\alpha - A}{\alpha}\right] \\ &\leq -\mathrm{Tr}\left[\frac{A^{\frac{1}{2}} (A^{-\frac{q}{2}} B^q A^{-\frac{q}{2}})^{\frac{\alpha}{q}} A^{\frac{1}{2}} - A}{\alpha}\right] \\ &= -\mathrm{Tr}\left[\frac{A^{1-q}}{q} \left(\frac{A^{\frac{q}{2}} (A^{-\frac{q}{2}} B^q A^{-\frac{q}{2}})^{\frac{\alpha}{q}} A^{\frac{q}{2}} - A^q}{\frac{\alpha}{q}}\right)\right] \\ &= -\mathrm{Tr}\left[\frac{A^{1-q}}{q} \mathsf{T}_{\frac{\alpha}{q}}(A^q|B^q)\right] \end{aligned}$$

for all $q \geq \alpha > 0$.

Suppose that $-1 \le \alpha < 0$. If we put $s = q$ and $t = -\alpha$ in (7), then we have

$$\begin{aligned}\mathrm{Tr}\,[A^{1-\alpha}B^{\alpha}] &= \mathrm{Tr}\,[A^{\frac{1-\alpha}{2}}B^{\alpha}A^{\frac{1-\alpha}{2}}] \\ &\le \mathrm{Tr}\,[A^{1/2}(A^{q/2}B^{-q}A^{q/2})^{-\alpha/q}A^{1/2}] \\ &= \mathrm{Tr}\,[A^{1/2}(A^{-q/2}B^{q}A^{-q/2})^{\alpha/q}A^{1/2}]\end{aligned}$$

for all $q \ge -\alpha > 0$. Hence for each $\alpha \in [-1, 0)$

$$\begin{aligned}\mathsf{D}_{\alpha}(A|B) &= -\mathrm{Tr}\,[\frac{A^{1-\alpha}B^{\alpha} - A}{\alpha}] \\ &\le -\mathrm{Tr}\,[\frac{A^{1/2}(A^{-q/2}B^{q}A^{-q/2})^{\alpha/q}A^{1/2} - A}{\alpha}] \\ &= -\mathrm{Tr}\left[\frac{A^{1-q}}{q}\mathsf{T}_{\frac{\alpha}{q}}(A^{q}|B^{q})\right]\end{aligned}$$

for all $q \ge -\alpha > 0$. Hence the proof of Theorem 5.3 is complete. □

Remark 5.4 If we put $q = 1$ in Theorem 5.3, then we have (6). If we put $\alpha \to 0$ in Theorem 5.3, then we have (5).

6 Concluding Remarks

There are many related topics on the relative operator entropy and the Tsallis relative operator entropy, see [32, 40] and [11]. Among others, we present generalizations of the relative operator entropy and the Tsallis relative operator entropy for positive invertible operators on a Hilbert space due to Isa et al. [31]. We treat $\mathsf{T}_{\alpha}(A|B)$ in which the range of α is extended from $[-1, 1]$ to $\mathbb{R}$, that is, $\mathsf{T}_{x}(A|B) \equiv \dfrac{A \natural_x B - A}{x}$ $(x \in \mathbb{R}\backslash\{0\})$ and $\mathsf{T}_0(A|B) \equiv \lim_{x\to 0} \mathsf{T}_x(A|B) = \mathsf{S}(A|B)$.

We regard the Tsallis relative operator entropy $\mathsf{T}_x(A|B)$ as the average rate of change of the path $A \natural_t B$ over the interval $[0, x]$ and relative operator entropy $\mathsf{S}(A|B)$ as the rate of change of the path at $t = 0$. Based on this viewpoint, we define the n-th Tsallis relative operator entropy and the n-th relative operator entropy.

We begin by defining the first relative operator entropy $\mathsf{S}^{[1]}(A|B)$ and the first Tsallis relative operator entropy $\mathsf{T}_x^{[1]}(A|B)$ as $\mathsf{S}(A|B)$ and $\mathsf{T}_x(A|B)$, respectively, that is,

$$\mathsf{S}^{[1]}(A|B) \equiv \mathsf{S}(A|B) = A^{\frac{1}{2}}(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} \quad \text{and}$$

$$\mathsf{T}_x^{[1]}(A|B) \equiv \mathsf{T}_x(A|B) = A^{\frac{1}{2}}\frac{(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^x - I}{x}A^{\frac{1}{2}} \quad (x \in \mathbb{R}\backslash\{0\}).$$

The corresponding functions to $\mathsf{S}^{[1]}(A|B)$ and $\mathsf{T}_x^{[1]}(A|B)$ are $\log a$ and $\dfrac{a^x-1}{x}$ for $a>0$, respectively. Since $\lim_{x\to 0}\dfrac{a^x-1}{x}=\log a$, it follows that $\mathsf{T}_0^{[1]}(A|B)\equiv \lim_{x\to 0}\mathsf{T}_x^{[1]}(A|B)=\mathsf{S}^{[1]}(A|B)$. Next, we define the second Tsallis relative operator entropy $\mathsf{T}_x^{[2]}(A|B)$ as the average rate of change of $\mathsf{T}_x^{[1]}(A|B)$ over the interval $[0,x]$, that is,

$$\begin{aligned}\mathsf{T}_x^{[2]}(A|B) &\equiv \frac{\mathsf{T}_x^{[1]}(A|B)-\mathsf{S}^{[1]}(A|B)}{x}\\ &= A^{\frac12}\frac{(A^{-\frac12}BA^{-\frac12})^x-I-x(\log A^{-\frac12}BA^{-\frac12})}{x^2}A^{\frac12}.\end{aligned}$$

Since its corresponding function is $\dfrac{a^x-1-x\log a}{x^2}$ and $\lim_{x\to 0}\dfrac{a^x-1-x\log a}{x^2}=\dfrac{1}{2}(\log a)^2$, we define the second relative operator entropy as

$$\mathsf{S}^{[2]}(A|B)\equiv \frac{1}{2}A^{\frac12}(\log A^{-\frac12}BA^{-\frac12})^2A^{\frac12}=\frac{1}{2}\mathsf{S}(A|B)A^{-1}\mathsf{S}(A|B),$$

and then $\mathsf{T}_0^{[2]}(A|B)\equiv \lim_{x\to 0}\mathsf{T}_x^{[2]}(A|B)=\mathsf{S}^{[2]}(A|B)$.

Based on this consideration, it is natural to define the n-th relative operator entropy $\mathsf{S}^{[n]}(A|B)$ by using $\dfrac{1}{n!}(\log a)^n$ as its corresponding function, and the n-th Tsallis relative operator entropy $\mathsf{T}_x^{[n]}(A|B)$.

Definition 6.1 Let A and B be positive invertible operators and $n\in\mathbb{N}$. The n-th relative operator entropy $\mathsf{S}^{[n]}(A|B)$ are defined by

$$\mathsf{S}^{[n]}(A|B)\equiv \frac{1}{n!}A^{\frac12}(\log A^{-\frac12}BA^{-\frac12})^nA^{\frac12}=\frac{1}{n!}A(A^{-1}\mathsf{S}(A|B))^n$$

and the n-th Tsallis relative operator entropy $\mathsf{T}_x^{[n]}(A|B)$ are inductively defined by

$$\mathsf{T}_x^{[1]}(A|B)\equiv \mathsf{T}_x(A|B)$$

and for $n\geq 2$,

$$\mathsf{T}_x^{[n]}(A|B)\equiv \frac{\mathsf{T}_x^{[n-1]}(A|B)-\mathsf{S}^{[n-1]}(A|B)}{x}.$$

Since $\dfrac{d^n}{dx^n}\, a^x = a^x(\log a)^n$ holds for $a > 0$, we have

$$\frac{d^n}{dx^n}\, A \natural_x B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^x(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^n A^{\frac{1}{2}} = (A \natural_x B)(A^{-1}\mathsf{S}(A|B))^n$$

and so $\mathsf{S}^{[n]}(A|B) = \dfrac{1}{n!}\,\dfrac{d^n}{dx^n}\, A \natural_x B\,\Big|_{x=0}$.

Then we have

$$A \natural_x B = A + \left(\sum_{k=1}^{n-1} x^k \mathsf{S}^{[k]}(A|B)\right) + x^n \mathsf{T}_x^{[n]}(A|B),$$

which is the Taylor's expansion of $A \natural_x B$ around 0. We remark that the k-th relative operator entropy $\mathsf{S}^{[k]}(A|B)$ appears as the coefficient of the x^k-term and the n-th Tsallis relative operator entropy $\mathsf{T}_x^{[n]}(A|B)$ appears in the residual term. So we can call $\mathsf{T}_x^{[n]}(A|B)$ the n-th residual relative operator entropy.

There are deep discussion on the n-th relative operator entropy and the n-th Tsallis relative operator entropy in [31] and [44].

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Matrix Inequalities and Characterizations of Operator Monotone Functions

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Abstract In this chapter, we give a series of new characterizations of operator monotone functions using matrix inequalities involving different Kubo-Ando matrix means. We also use a trace monotonicity inequality and the Powers-Størmer inequality to characterize operator monotone functions.

Keywords Operator monotone functions · Kubo-Ando matrix means · Powers-Stormer's inequality · Matrix Heinz mean · Matrix Heron mean · Matrix geometric means · Matrix power means · Matrix inequalities

1 Introduction

Throughout this chapter, $\mathbb{M}_n$ stands for the algebra of $n \times n$ matrices over $\mathbb{C}$ and $\mathcal{P}_n$ denotes the cone of positive definite elements in $\mathbb{M}_n$. Denote by I_n the identity matrix of $\mathbb{M}_n$. For a Hermitian matrix A with eigenvalues in the domain of a function f, the matrix $f(A)$ is defined by means of the functional calculus.

Definition 1.1 A continuous function f on I $(\subset \mathbb{R})$ is called *n-monotone*, if

$$A \le B \quad \Longrightarrow \quad f(A) \le f(B)$$

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for any pair of self-adjoint matrices $A, B \in \mathbb{M}_n$ with $\sigma(A), \sigma(B) \subset I$, where $\sigma(A)$ stands for the spectrum of A.

Definition 1.2 A continuous function f on I ($\subset \mathbb{R}$) is called *n-convex* if the inequality

$$f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda) f(B)$$

holds for all self-adjoint matrices $A, B \in \mathbb{M}_n$ with $\sigma(A), \sigma(B) \subset I$ and for all $\lambda \in [0, 1]$. Also, f is called a *n-concave* on I if $(-f)$ is n-convex on I.

Let $f : I(\subset \mathbb{R}) \to \mathbb{R}$. We call a function f *operator convex* if f is n-convex for any $n \in \mathbb{N}$, and *operator monotone* if f is n-monotone for any $n \in \mathbb{N}$.

Operator monotone functions were firstly introduced and studied by Löwner in 1934 [35]. He has completely characterized all operator monotone functions as the class of Pick functions. These functions play an essential role in the theory of analytic functions. He has also proved that if f is operator monotone on $[0, \infty)$, then there exists a positive measure μ on $[0, \infty)$ such that

$$f(t) = a + bt + \int_0^\infty \frac{st}{t+s} d\mu(s),$$

where a is a real number and $b \geq 0$. According to this representation, for $r \in [0, 1]$, the function $f(t) = t^r$ is operator monotone on $[0, \infty)$. This is the well-known Löwner-Heinz's inequality [42, Theorem 1.1] for positive semidefinite matrices which states that for any $0 \leq A \leq B$ and $r \in [0, 1]$, $A^r \leq B^r$.

In [31], Hansen and Perdesen considered some basic equivalent assertions for operator monotone functions and operator convex functions. They showed that for a strictly positive, continuous function f on $(0, \infty)$, the following statements are equivalent:

(a) f is operator concave.
(b) $\dfrac{t}{f(t)}$ is operator monotone.

In [37], Osaka and Tomiyama discussed some similar assertions at each level n in order to see clearly the inside of the double piling structure of matrix monotone functions and of matrix convex functions. More precisely, the main results in [37] are in the following.

Theorem 1.3 *Let $n \in \mathbb{N}$ and $f : [0, \alpha) \to \mathbb{R}$. Let us consider the following assertions:*

(i) $f(0) \le 0$ *and* f *is* n*-convex in* $[0, \alpha)$.
(ii) *For each self-adjoint matrix* A *with its spectrum in* $[0, \alpha)$ *and a contraction* C *in* $\mathbb{M}_n$,

$$f(C^\star AC) \le C^\star f(A)C.$$

(iii) *The function* $g(t) = \frac{f(t)}{t}$ *is* n*-monotone on* $(0, \alpha)$.

Then

$$(i)_{n+1} \Longrightarrow (ii)_n \Longrightarrow (iii)_n \Longrightarrow (i)_{[\frac{n}{2}]},$$

where notation $(A)_m \Rightarrow (B)_n$ *means that "if* (A) *holds for the matrix algebra* $\mathbb{M}_m$, *then* (B) *holds for the matrix algebra* $\mathbb{M}_n$*".*

In 1980, Kubo and Ando [34] introduced the theory of operator means. Let $\mathcal{B}(\mathcal{H})^+$ be the set of positive invertible operators in a Hilbert space $\mathcal{H}$. A binary operation $\sigma : \mathcal{B}(\mathcal{H})^+ \times \mathcal{B}(\mathcal{H})^+ \to \mathcal{B}(\mathcal{H})^+$, $(A, B) \mapsto A\sigma B$, is called a *connection* if the following requirements are fulfilled:

(I) If $A \le C$ and $B \le D$, then $A\sigma B \le C\sigma D$;
(II) $C^*(A\sigma B)C \le (C^*AC)\sigma(C^*BC)$;
(III) If $A_n \searrow A$ and $B_n \searrow B$, then $A_n\sigma B_n \searrow A\sigma B$.
Further, a *mean* is a connection satisfying the normalized condition:
(IV) $1\sigma 1 = 1$.

Kubo and Ando showed that there exists an affine order-isomorphism from the class of connections to the class of positive operator monotone functions, which is given by $\sigma \mapsto f_\sigma(t) = 1\sigma t$. Let f be an operator monotone function, then for positive definite matrices A and B,

$$A\sigma B := A^{1/2} f(A^{-1/2}BA^{-1/2})A^{1/2}. \tag{1}$$

In 1996, Petz [38] introduced the theory of monotone metric in quantum information theory which was based on operator monotone functions. Therefore, such functions are important in matrix analysis, quantum information and other areas as well. The authors refer readers to the books of William Donoghue [29], Barry Simon [41] and Bhatia [5] for more details about operator monotone functions.

It is well-known that if σ is a symmetric matrix Kubo-Ando mean, i.e., $A\sigma B = B\sigma A$, then the representing function f_σ satisfies the following inequalities

$$\frac{2x}{1+x} \le f_\sigma(x) \le \frac{1+x}{2}.$$

Consequently, for any positive operators A and B,

$$A!B \le A\sigma B \le A\nabla B, \tag{2}$$

where $A!B = 2(A^{-1} + B^{-1})^{-1}$ is the harmonic mean of A and B, and $A\nabla B = (A + B)/2$ is the arithmetic mean of A and B. Obviously, if $f : [0, \infty) \to [0, \infty)$ is operator monotone, we have

$$f(A!B) \le f(A\sigma B) \le f(A\nabla B). \tag{3}$$

Interestingly, if a continuous function f satisfies either of the inequalities:

$$f(a!b) \le f(a\sigma b) \le f(a\nabla b). \tag{4}$$

for any positive numbers a and b, then f is monotonically increasing. Matrix generalizations of this observation for Kubo-Ando means were discussed by Hiai and Ando in [1, Proposition 4.1]. Namely, they showed that a continuous function f on $(0, \infty)$ is operator monotone if and only if one of the following conditions holds:

(A) $f(A\nabla B) \ge f(A\sigma B)$ for all positive definite matrices A, B and for some symmetric operator mean $\sigma \ne \nabla$;
(B) $f(A!B) \le f(A\sigma B)$ for all positive definite matrices A, B and for some symmetric operator mean $\sigma \ne !$.

One of the most important matrix means is the geometric mean,

$$A\#B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$$

as the mid-point of the geodesic,

$$A\#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2} \quad (t \in [0, 1])$$

connecting two matrices A and B in the Riemannian manifold of positive matrices. It is natural to consider a similar characterization using this mid-point. This importance becomes more evident when one considers that # is not only symmetric but also self-adjoint i.e. $(A\#B)^{-1} = A^{-1}\#B^{-1}$, so it seems as a natural candidate to extend this characterization to other classes of means.

Now assume that $\mathcal{H}$ is an infinite-dimensional Hilbert space. Let Tr be the canonical trace on the algebra $\mathcal{B}(\mathcal{H})$. It is well-known that for a monotone function f on $[0, \infty)$,

$$0 \le A \le B \quad \Longrightarrow \quad \mathrm{Tr}\,(f(A)) \le \mathrm{Tr}\,(f(B)).$$

For any normal state ϕ on $\mathcal{B}(\mathcal{H})$, a positive kernel operator S_ϕ with $\mathrm{Tr}\,(S_\phi) = 1$ such that $\phi(A) = \mathrm{Tr}\,(S_\phi A)$ $(A \in \mathcal{B}(\mathcal{H}))$ is uniquely defined. If f is an operator monotone function, and ϕ is a positive linear functional on $\mathcal{B}(\mathcal{H})$, then for any $0 \leq A \leq B$,

$$\phi(f(A)) \leq \phi(f(B)). \tag{5}$$

In this chapter, we give a series of new characterizations of operator monotone functions using matrix inequalities involving Kubo-Ando matrix means. In addition, we also use trace inequality (5) and the Powers-Størmer inequality in quantum hypothesis testing theory [3] to characterize operator monotone functions.

2 Matrix Inequalities and Characterizations of Operator Functions

2.1 Heinz Mean, Heron Mean, and Operator Monotone Functions

2.1.1 Scalar Inequality for Heinz Mean and Heron Mean

For two non-negative numbers x and y let us denote by

$$G_s(x, y) = \frac{x^s y^{1-s} + x^{1-s} y^s}{2}$$

the Heinz means and by

$$H_s(x, y) = s\frac{x + y}{2} + (1 - s)x^{1/2}y^{1/2}$$

the Heron means.

The family of Heron means and Heinz means are clearly interpolations between the arithmetic and the geometric means. In [6], Bhatia obtained a relation between the Heinz mean and the Heron mean which states that for $t \in [0, 1]$,

$$G_t(a, b) \leq H_{(2t-1)^2}(a, b). \tag{6}$$

Therefore, for any $t \in [0, 1]$, we have

$$\sqrt{ab} \leq G_t(a, b) \leq H_{(2t-1)^2} \leq H_{|2t-1|} \leq \frac{a + b}{2}. \tag{7}$$

Now, we show that the inequality between the Heinz mean and the Heron mean of scalars also characterizes monotonicity.

Theorem 2.1 *A continuous function f on $[0, \infty)$ is monotone increasing if and only if for any pair of positive numbers x, y and $s \in (0, 1/2) \cup (1/2, 1)$,*

$$f\left(\frac{x^s y^{1-s} + x^{1-s} y^s}{2}\right) \le f\left(\alpha(s)^2 \frac{x+y}{2} + (1 - \alpha(s)^2)\sqrt{xy}\right), \tag{8}$$

where $\alpha(s) = 2s - 1$.

Proof The implication follows from (7) and monotonicity, so we only need to show the converse. Given two positive numbers $a \le b$, it suffices to show that there exist positive numbers x and y such that

$$a = \frac{x^s y^{1-s} + x^{1-s} y^s}{2}, \quad b = \alpha(s)^2 \frac{x+y}{2} + (1 - \alpha(s)^2)\sqrt{xy}, \tag{9}$$

as this would imply $f(a) \le f(b)$ showing the desired monotonicity. If such x and y exist, from (9) we would have

$$\begin{aligned} \frac{a}{b} &= \frac{x^s y^{1-s} + x^{1-s} y^s}{\alpha(s)^2 (x+y) + 2(1 - \alpha(s)^2)\sqrt{xy}} \\ &= \frac{(y/x)^{\alpha(s)/2} + (y/x)^{-\alpha(s)/2}}{\alpha(s)^2((y/x)^{1/2} + (y/x)^{-1/2}) + 2(1 - \alpha(s)^2)} \\ &= \frac{\cosh(\alpha(s)c)}{\alpha(s)^2 \cosh(c) + (1 - \alpha(s)^2)}, \end{aligned}$$

where $e^{2c} = y/x$. We define

$$f_\alpha(c) = \frac{\cosh(\alpha c)}{\alpha^2 \cosh(c) + (1 - \alpha^2)}$$

and show that $f_\alpha : [0, \infty) \to (0, 1]$ is bijective. Indeed, notice that

$$f_\alpha(0) = 1 \quad \text{and} \quad \lim_{c \to \infty} f_\alpha(c) = 0.$$

Continuity and the Intermediate Value Theorem imply that the function $f_\alpha : [0, \infty) \to (0, 1]$ is surjective. Moreover, we can show that the function $f_\alpha : [0, \infty) \to (0, 1]$ is also injective. To do this, it is enough to show that the function is monotonic on $[0, \infty)$. So, note that

$$\frac{d}{dc} f_\alpha(c) \le 0$$

if and only if,

$$g_\alpha(c) := \alpha \sinh(\alpha c)(\alpha^2 \cosh(c) + (1-\alpha^2)) - \alpha^2 \sinh(c)\cosh(\alpha c) \le 0.$$

Since, $g_\alpha(0) = 0$, it suffices to show that g_α is monotonically decreasing on $[0, \infty)$. Taking a derivative with respect to c we obtain,

$$\frac{d}{dc} g_\alpha(c) = 2\alpha(-1+\alpha^2)\cosh(c\alpha)\sinh(c/2)^2$$

which is clearly non-positive when $c \ge 0$. Hence, the function $f_\alpha : [0, \infty) \to (0, 1]$ is bijective. To obtain a solution for (9), fix $s \in (0, 1/2) \cup (1/2, 1)$ and set $c = f_{\alpha(s)}^{-1}(a/b)$. With this, we can obtain the desired x and y satisfying (9). □

Let $0 \le p \le 1 \le q$. It is well-known that for non-negative numbers a and b,

$$\sqrt{ab} \le \left(\frac{a^p + b^p}{2}\right)^{1/p} \le \frac{a+b}{2} \le \left(\frac{a^q + b^q}{2}\right)^{1/q},$$

or,

$$\sqrt{ab} \le \mu(p, a, b) \le \mu(1, a, b) \le \mu(q, a, b), \tag{10}$$

where $\mu(p, a, b) = \left(\frac{a^p+b^p}{2}\right)^{1/p}$ is the power mean (or, binomial mean). Using similar arguments as in the proof of Theorem 2.1 one can obtain the following theorem.

Theorem 2.2 *Let f be a continuous function on $[0, \infty)$. For $0 \le p \le 1 \le q$, suppose that one of the following inequalities holds for all any non-negative numbers $a \le b$:*

(1) $f(a) \le f(\sqrt{ab})$;
(2) $f(\mu(1, a, b)) \le f(b)$,
(3) $f\left(\frac{a^s b^{1-s} + a^{1-s} b^s}{2}\right) \le f\left(|2s-1|\frac{a+b}{2} + (1-|2s-1|)\sqrt{ab}\right)$.
(4) $f(\sqrt{ab}) \le f(\mu(p, a, b))$;
(5) $f(\mu(p, a, b)) \le f(\mu(1, a, b))$;
(6) $f(\mu(1, a, b)) \le f(\mu(q, a, b))$.

Then the function f is increasingly monotone on $[0, \infty)$.

2.1.2 Matrix Inequalities and Operator Monotone Functions

Notice that from (7) we have the following inequalities for matrix means:

$$A\#B \le \frac{A\sharp_s B + A\sharp_{1-s} B}{2} \le \alpha(s)^2 \frac{A+B}{2} + (1-\alpha(s)^2)A\sharp B \le \frac{A+B}{2},$$

In this section, using above inequalities we establish new characterizations of operator monotone functions.

Theorem 2.3 *Let f be a continuous function on $[0, \infty)$, $s \in (0, 1/2) \cup (1/2, 1)$ and $\alpha = 1 - 2s$. The following statements are equivalent:*

(i) *f is operator monotone on $[0, \infty)$;*

(ii) *For any positive definite matrices A and B,*

$$f(A\sharp B) \le f\left(\frac{A\sharp_s B + A\sharp_{1-s} B}{2}\right); \tag{11}$$

(iii) *For any positive definite matrices A and B,*

$$f\left(\frac{A\sharp_s B + A\sharp_{1-s} B}{2}\right) \le f\left(\alpha(s)^2 \frac{A+B}{2} + (1-\alpha(s)^2)A\sharp B\right); \tag{12}$$

(iv) *For any positive definite matrices A and B,*

$$f\left(\alpha(s)^2 \frac{A+B}{2} + (1-\alpha(s)^2)A\sharp B\right) \le f\left(\frac{A+B}{2}\right).$$

Proof It is obvious that (i) implies (ii), (iii), and (iv). Let us show that (iii) implies (i) first and then we show (ii) implies (i). That would complete the proof since (iv) implies (i) follows from [1, Proposition 4.1] since the matrix Heron mean is symmetric.

Suppose (12) holds for any positive definite matrices A and B. We need to show that for any $0 < X \le Y$,

$$f(X) \le f(Y).$$

Firstly, let us consider the case when $Y = I_n$. We now show that there exist positive definite matrices A_0, B_0 such that

$$\frac{A_0\sharp_s B_0 + A_0\sharp_{1-s} B_0}{2} = A_0^{1/2}\left(\frac{C_0^s + C_0^{1-s}}{2}\right)A_0^{1/2} = X \tag{13}$$

and

$$A_0^{1/2}\left(\alpha(s)^2\frac{I_n + C_0}{2} + (1-\alpha(s)^2)C_0^{1/2}\right)A_0^{1/2} = I_n, \tag{14}$$

where $C_0 = A_0^{-1/2}B_0A_0^{-1/2}$. From (14), we get

$$A_0^{1/2} = \left(\alpha(s)^2\frac{I_n + C_0}{2} + (1-\alpha(s)^2)C_0^{1/2}\right)^{-1/2}.$$

Substituting the last identity to (13), we get

$$X = \left(\frac{C_0^s + C_0^{1-s}}{2}\right)\left(\alpha(s)^2\frac{I_n + C_0}{2} + (1-\alpha(s)^2)C_0^{1/2}\right)^{-1} \tag{15}$$

From the proof of Theorem 2.1 the function

$$f(x) = \left(\frac{x^s + x^{1-s}}{2}\right)\left(\alpha(s)^2\frac{1+x}{2} + (1-\alpha(s)^2)\sqrt{x}\right)^{-1}$$

is bijective and takes values in (0, 1]. Therefore, for any $0 < X \le I_n$ there exists a unique matrix C_0 satisfying (15). Hence, the matrix A_0 is obtained from (14) and the matrix B_0 equals $A_0^{1/2}C_0A_0^{1/2}$.

In general, for $0 < X \le Y$ we have $0 < Y^{-1/2}XY^{-1/2} \le I_n$. By the above arguments, we can find $A_0, B_0 \in \mathbb{M}_n^+$ such that

$$\frac{A_0\sharp_s B_0 + A_0\sharp_{1-s}B_0}{2} = Y^{-1/2}XY^{-1/2}$$

and

$$\alpha(s)^2\frac{A_0 + B_0}{2} + (1-\alpha(s)^2)A_0\sharp B_0 = I_n.$$

Consequently, applying (12) to matrices $A = Y^{1/2}A_0Y^{1/2}$, $B = Y^{1/2}B_0Y^{1/2}$ we obtain that $f(X) \le f(Y)$. Finally, by the continuity of f we conclude that the function f is operator monotone on $[0, \infty)$.

To show that (ii) implies (i), following the same argument, it suffices to show that the function $k_s(x) : (0, 1] \to (0, 1]$ defined by

$$k_s(x) = \frac{2\sqrt{x}}{x^s + x^{1-s}}$$

is bijective. However, by realizing k_s as a hyperbolic secant, this is obvious. □

Remark 2.4 There are numerous papers on the matrix Heron mean and the matrix Heinz mean and related questions. We refer the readers to [8, 16, 20, 22, 24–26] and the references therein.

2.2 *Symmetric and Self-adjoint Means via Integral Representations*

In this section we use characterizations of symmetric means given in [4] and of self-adjoint means given in [30] to characterize operator monotone functions via the geometric mean and a mean σ under certain constraints. More precisely, we use

$$g(A\#B) \le g(A\sigma B),$$

or

$$g(A\#B) \ge g(A\sigma B),$$

to characterize operator monotone functions.

2.2.1 Symmetric Means

Definition 2.5 ([4]) Let $f : \mathbb{R}_+ \to \mathbb{R}_+$, where $\mathbb{R}^+ = (0, \infty)$. We say that $f \in \mathcal{F}_{op}$ if it satisfies the following conditions:

1. f is operator monotone,
2. $tf(t^{-1}) = f(t)$ for all $t \in \mathbb{R}_+$, and
3. $f(1) = 1$.

Notice, that functions in $f \in \mathcal{F}_{op}$ are in one-to-one correspondence with symmetric means.

Definition 2.6 For $f, g \in \mathcal{F}_{op}$, define

$$\psi(t) = \frac{t+1}{2}\frac{f(t)}{g(t)}, \quad t > 0.$$

We say $f \preceq g$ if and only if $\psi \in \mathcal{F}_{op}$.

It is clear that if $f \in \mathcal{F}_{op}$, $\frac{2t}{1+t} \preceq f(t) \preceq \frac{1+t}{2}$ as $\psi(t) = t^{-1}f(t)$ or $\psi(t) = f(t)$ in these particular cases, both of which are operator monotone. It is shown in [4] that $\mathcal{F}_{op}$ forms a lattice under $\preceq$. It is worth noting that this order is stronger than the regular point-wise order $\le$. That is, if $f \preceq g$ then $f \le g$ because $\psi(t) \le \frac{1+t}{2}$ ($t \in \mathbb{R}_+$).

It is shown in [4, Proposition 2.1] that $f \in \mathcal{F}_{op}$ implies that f has an integral representation of the form

$$f(t) = \frac{1+t}{2} e^{H(t)} \tag{16}$$

where

$$H(t) = \int_0^1 \frac{(\lambda^2 - 1)(1-t)^2}{(t+\lambda)(1+t\lambda)(\lambda+1)^2} h(\lambda)\, d\lambda$$

and $h : [0, 1] \to [0, 1]$ is a measurable function that is uniquely determined by f *a.e.* We notice that if $h(\lambda) = \frac{1}{2}$, then $f(t) = \sqrt{t}$.

In [4, Theorem 2.4], they showed that $f \preceq g$ implies the following *a.e.* inequality at the level of the corresponding measurable functions: $h_f \geq h_g$. If $f \preceq g$ and $h_f \neq h_g$ on a set of non-zero measure, we will say $f \prec g$.

Lemma 2.7 *Let $f \in \mathcal{F}_{op}$ and define*

$$\varphi(t) = t^{-1} f(t^2).$$

Then,

1. *If $\sqrt{\cdot} \prec f$ then, as a real function, φ is monotonically decreasing on* $(0, 1)$ *and monotonically increasing on* $(1, \infty)$.
2. *If $\sqrt{\cdot} \succ f$ then φ is monotonically increasing on* $(0, 1)$ *and monotonically decreasing on* $(1, \infty)$.

Proof Consider the derivative

$$\varphi'(t) = -t^{-2} f(t^2) + 2f'(t^2).$$

To show monotonicity as a real function, it suffices to show

$$2tf'(t) \lesseqgtr f(t)$$

depending on the interval and the order relationship considered. Based on (16) we consider

$$2tf'(t) = te^{H(t)}(1 + (1+t)H'(t)) \lesseqgtr f(t), \text{ respectively,}$$

if and only if

$$H'(t) \lesseqgtr \frac{1-t}{2t(1+t)}, \text{ respectively.}$$

Explicitly calculating $H'(t)$ we obtain

$$H'(t) = \int_0^1 \left(\frac{1}{(t+\lambda)^2} - \frac{1}{(1+t\lambda)^2} \right) h(\lambda)\, d\lambda = \int_0^1 \frac{(1-\lambda^2)(1-t^2)}{(t+\lambda)^2(1+t\lambda)^2} h(\lambda)\, d\lambda.$$

An easy calculation shows that when $h(\lambda)$ is substituted by the constant function $1/2$, the integral becomes

$$\frac{1}{2} \int_0^1 \frac{(1-\lambda^2)(1-t^2)}{(t+\lambda)^2(1+t\lambda)^2}\, d\lambda = \frac{1-t}{2t(1+t)}.$$

So now we apply [4, Theorem 2.4] to determine the monotonicity of φ in each case. So, let $\sqrt{\cdot} \prec f$ and $t \in (0, 1)$. In this case $h(\lambda) \leq 1/2$ and the integrand,

$$\frac{(1-\lambda^2)(1-t^2)}{(t+\lambda)^2(1+t\lambda)^2} h(\lambda) \geq 0$$

for all $(t, \lambda) \in (0, 1) \times [0, 1]$. Therefore,

$$H'(t) \leq \frac{1-t}{2t(1+t)},$$

which implies that φ is monotonically decreasing on $(0, 1)$. When $t \in (1, \infty)$ the integrand is non-positive and the inequality is reversed, yielding that φ is monotonically increasing on that interval. The analysis for $\sqrt{\cdot} \succ f$ is similar, but in this case $h(\lambda) \geq 1/2$. □

Remark 2.8 Another way to obtain the previous result would be by using the monotonicity on one interval and using the fact that $\varphi(t) = \varphi(t^{-1})$ by the symmetry condition 2 in the definition of the class $\mathcal{F}_{op}$. As a corollary, φ has an absolute minimum/maximum at the point $(1, 1)$.

Suppose that $\sqrt{\cdot} \prec f$. Then, $\sqrt{t} < f(t)$ for some $t \in (1, \infty)$. By the preceding lemma φ is monotonically increasing on this interval, so

$$\gamma := \lim_{t\to\infty} \varphi(t) = \lim_{t\to\infty} t^{-1} f(t^2) > 1.$$

As a result the interval $(1, \gamma)$ is non-empty.

On the other hand, suppose that $\sqrt{\cdot} \succ f$. Then, $\sqrt{t} > f(t)$ for some $t \in (1, \infty)$. In this case, however, φ is monotonically decreasing on this interval, so

$$\gamma := \lim_{t\to\infty} \varphi(t) < 1$$

and $(1, \gamma)$ is non-empty.

Lemma 2.9 *Let σ be some symmetric operator mean on $\mathbb{R}^+$ with representing function f such that $\sqrt{\cdot} \prec f$ (resp. $\sqrt{\cdot} \succ f$) and let $\gamma = \lim_{t\to\infty} f(t^2)/t$. Then, if X and Y are positive definite operators such that $X \le Y < \gamma X$ (resp. $\gamma X < Y \le X$), then there exist positive operators A and B such that*

$$X = A\#B \quad \text{and} \quad Y = A\sigma B.$$

Proof Note that if we show that for $I_n \le X^{-1/2}YX^{-1/2} := Y_0 \le \gamma I_n$ we can find positive operators A_0 and B_0 such that:

$$I_n = A_0\#B_0 \quad \text{and} \quad Y_0 = A_0\sigma B_0,$$

we can obtain the desired result by choosing $A := X^{1/2}A_0X^{1/2}$ and $B := X^{1/2}B_0X^{1/2}$. This is equivalent to the following problem: Given $I_n \le Y_0 \le \gamma I_n$ find $A_0 \ge 0$ such that

$$Y_0 = A_0\sigma A_0^{-1}.$$

So, define $\varphi(t) := t\sigma t^{-1} = tf(t^{-2})$. By symmetry, we have that $\varphi(t) = t^{-1}f(t^2)$. Since $\varphi(t)$ is continuous on $[1,\infty)$, $\varphi(1) = f(1) = 1$, the function is bijective from $[1,\infty)$ onto $[1,\gamma)$ and so we can define $A_0 = \varphi^{-1}(Y_0)$. This gives the desired result. The proof for the case when $\sqrt{\cdot} \succ f$ is identical, but uses the fact that in this case $\varphi : [1,\infty) \to (\gamma, 1]$ is bijective instead. □

Theorem 2.10 *Let σ be some symmetric operator mean on $\mathbb{R}^+$ with representing function f such that $\sqrt{\cdot} \prec f$. Then, if*

$$g(A\#B) \le g(A\sigma B) \tag{17}$$

for any positive operators A and B, then the function g is operator monotone on $\mathbb{R}^+$. If, on the other hand, $\sqrt{\cdot} \succ f$ and

$$g(A\#B) \ge g(A\sigma B), \tag{18}$$

then g is operator monotone on $\mathbb{R}^+$.

Proof First we prove (17). Let f and φ be as in the proof of Lemma 2.9. Assume that $f \succ \sqrt{\cdot}$ and choose $\gamma_0 \in (1,\gamma)$. Let $0 < X \le Y$ and consider the sequence:

$$0 < X \le \gamma_0 X \le \gamma_0^2 X \le \ldots$$

Clearly, there exists $k \in \mathbb{N}$ such that:

$$0 \le X \le \gamma_0 X \le \gamma_0^2 X \le \ldots \le \gamma_0^k X \le Y \le \gamma_0^{k+1} X.$$

Since $\gamma_0^i X \leq \gamma_0^{i+1} X \leq \gamma\gamma_0^i X$, Lemma 2.9 implies that there exist positive operators A_i and B_i such that:

$$\gamma_0^i X = A_i \# B_i \qquad \text{and} \qquad \gamma_0^{i+1} X = A_i \sigma B_i$$

and so

$$g(X) \leq g(\gamma_0 X) \leq g(\gamma_0^2 X) \leq ... \leq g(\gamma_0^k X).$$

Since $\gamma_0^k X \leq Y \leq \gamma\gamma_0^k X$, the lemma gives

$$g(X) \leq g(\gamma_0 X) \leq g(\gamma_0^2 X) \leq ... \leq g(\gamma_0^k X) \leq g(Y).$$

The proof of (18) is similar, hence omitted. □

2.2.2 Self-adjoint Means

A mean σ is said to be self-adjoint if it satisfies

$$(A\sigma B)^{-1} = A^{-1}\sigma B^{-1}.$$

There exists a one-to-one correspondence between self-adjoint means and the class of operator monotone functions $\mathcal{E}$ defined below. This correspondence was considered by Hansen in [30] and a characterization was given in terms of the exponential of an integral.

Definition 2.11 Let $f : \mathbb{R}_+ \to \mathbb{R}_+$. We say that $f \in \mathcal{E}$ if it satisfies the following conditions:

1. f is operator monotone and
2. $f(t^{-1}) = f(t)^{-1}$ for all $t \in \mathbb{R}_+$.

The aforementioned characterization is proved in [30, Theorem 1.1] and it states that

$$f(t) = \exp \int_{-1}^{0} \left(\frac{1}{\lambda - t} + \frac{t}{1 - \lambda t} \right) h(\lambda)\, d\lambda,$$

where $h : [-1, 0] \to [0, 1]$ is a measurable function. We notice that if $h(\lambda) = 1/2$, then $f(t) = \sqrt{t}$.

Definition 2.12 For $f, g \in \mathcal{E}$, we say $f \succeq_{sa} g$ if and only if fg^{-1} is operator monotone.

In the following, we show that this so defined relation satisfies the same properties as the order defined in [4] on $\mathcal{F}_{op}$ that we introduced earlier in this section.

Note that $f, g \in \mathcal{E}$ implies that $(f/g)(t^{-1}) = ((f/g)(t))^{-1}$. So, requiring fg^{-1} be operator monotone is equivalent to requiring $fg^{-1} \in \mathcal{E}$. Therefore, there exists a class of measurable functions $h_{fg^{-1}} : [-1, 0] \to [0, 1]$ such that

$$(fg^{-1})(t) = \exp \int_{-1}^{0} \left(\frac{1}{\lambda - t} + \frac{t}{1 - \lambda t} \right) h_{fg^{-1}}(\lambda)\, d\lambda,$$

and

$$h_{fg^{-1}}(\lambda) = h_f(\lambda) - h_g(\lambda) \quad a.e.$$

So, clearly $f \succeq_{sa} g$ if and only if $h_f \geq h_g$ *a.e.* Therefore, $\succeq_{sa}$ defines an order relation on $\mathcal{E}$. Moreover, it is easy to see that for $f \in \mathcal{E}$ implies that $1 \preceq_{sa} f(t) \preceq_{sa} t$. Indeed, $1 \preceq_{sa} f(t)$ follows from the monotonicity of f and $f(t) \preceq_{sa} t$ follows from the monotonicity of $\frac{t}{f(t)}$.

We can also define the meet and join of any two elements in a similar fashion as in [4]. Let $f, g \in \mathcal{E}$ then define:

$$f \wedge g = \exp \int_{-1}^{0} \left(\frac{1}{\lambda - t} + \frac{t}{1 - \lambda t} \right) \min\{h_f(\lambda), h_g(\lambda)\}\, d\lambda,$$

$$f \vee g = \exp \int_{-1}^{0} \left(\frac{1}{\lambda - t} + \frac{t}{1 - \lambda t} \right) \max\{h_f(\lambda), h_g(\lambda)\}\, d\lambda.$$

As an immediate result we obtain that $\mathcal{E}$ with $\preceq_{sa}$ is a lattice. That is, for any two $f, g \in \mathcal{E}$,

$$f \wedge g \preceq_{sa} f \preceq_{sa} f \vee g.$$

Moreover, the map

$$f(t) \to f^{\dagger}(t) = \frac{t}{f(t)}$$

is an involutive order reversing map on $\mathcal{E}$. Indeed, it is easy to see $f^{\dagger\dagger} = f$,

$$f^{\dagger}(t^{-1}) = \frac{1}{tf(t^{-1})} = \frac{f(t)}{t} = (f^{\dagger}(t))^{-1},$$

and

$$f \preceq_{sa} g \quad \Longrightarrow \quad g^{\dagger} \preceq_{sa} f^{\dagger}.$$

Now we turn into a characterization of operator monotone functions using self-adjoint means. As before, if $f \preceq_{sa} g$ and $h_f \neq h_g$ on a set of non-zero measure, we will say $f \prec_{sa} g$.

Lemma 2.13 *Let $f \in \mathcal{E}$ and define*

$$\varphi(t) = t^{-1} f(t^2).$$

Then,

1. *If $\sqrt{\cdot} \preceq_{sa} f$ then, as a real function, φ is monotonically increasing on $\mathbb{R}_+$.*
2. *If $\sqrt{\cdot} \succeq_{sa} f$ then φ is monotonically decreasing on $\mathbb{R}_+$.*

Proof As before, to show the monotonicity of φ as a real function, it suffices to show

$$2tf'(t) \lesseqgtr f(t) \tag{19}$$

depending on the interval and the order relationship considered. With the integral expression of f, (19) becomes:

$$2t \int_{-1}^{0} \left(\frac{1}{(\lambda - t)^2} + \frac{1}{(1 - \lambda t)^2} \right) h(\lambda)\, d\lambda \lesseqgtr 1.$$

The result now follows from the fact that the integrand is non-negative and for $h(\lambda) = 1/2$, $f(t) = \sqrt{t}$ and

$$\int_{-1}^{0} \left(\frac{1}{(\lambda - t)^2} + \frac{1}{(1 - \lambda t)^2} \right) d\lambda = \frac{1}{t}.$$

□

Using the same arguments as in Lemma 2.9 and Theorem 2.10 we can show the following result.

Theorem 2.14 *Let σ be some self-adjoint operator mean on $\mathbb{R}^+$ with representing function f such that $\sqrt{\cdot} \prec_{sa} f$ and let A and B be positive operators such that $A < B$. Then, if*

$$g(A\#B) \leq g(A\sigma B), \tag{20}$$

then the function g is operator monotone on $\mathbb{R}^+$. If, on the other hand, $\sqrt{\cdot} \prec_{sa} f^\dagger$ and

$$g(A\#B) \geq g(A\sigma B), \tag{21}$$

then g is operator monotone on $\mathbb{R}^+$.

2.2.3 Kubo-Ando Condition

There is yet another class of means to consider. Let τ and $\tau^{\perp}$ be the means represented by operator monotone functions g and $g^{\dagger}$, respectively. Kubo and Ando showed in [34, Theorem 5.4] that if an operator mean σ with representing function f satisfies

$$(A\tau B)\sigma(A\tau^{\perp}B) \leq A\sigma B \tag{22}$$

for a non-trivial mean τ and all positive operators A and B then $f \geq \sqrt{\cdot}$. Moreover, in [34, Theorem 5.7], they showed that whenever σ satisfies (22) for every operator mean τ its representing function f satisfies $t^{-1}f(t^2)$ is non-increasing on $(0, 1)$ and non-decreasing on $(1, \infty)$. Moreover, in subsequent corollaries, they showed that if the inequality (22) is reversed then $f \leq \sqrt{\cdot}$ and $t^{-1}f(t^2)$ is non-decreasing on $(0, 1)$ and non-increasing on $(1, \infty)$.

These are precisely the behaviors needed in the proof of Lemma 2.9 and consequently Theorem 2.10. Therefore, this allows us to follow the same arguments to show a similar result for this particular class of means.

Theorem 2.15 *Let A and B be positive operators and σ be an operator mean on $\mathbb{R}^+$ satisfying (22) for every operator mean τ. Assume further that the representing function f satisfies $f(x) > \sqrt{x}$ for all $x \in (1, \infty)$. Then, if*

$$g(A\#B) \leq g(A\sigma B), \tag{23}$$

then the function g is operator monotone on $\mathbb{R}^+$. If, on the other hand, the reversed inequality is satisfied in (22) for every operator mean σ, $f(x) < \sqrt{x}$ for all $x \in (0, 1)$, and

$$g(A\#B) \geq g(A\sigma B), \tag{24}$$

then g is operator monotone on $\mathbb{R}^+$.

2.2.4 General Symmetric Means

In this section, we prove Theorem 2.10 for general symmetric Kubo-Ando means. We used monotonicity of the function φ on certain intervals to obtain bijectivity, thus obtaining a well-defined φ^{-1} when restricted to the appropriate intervals. With this function, we were able to solve the problem in Lemma 2.9, which then allowed us to obtain the desired characterization. With a little care, it is possible to obtain the same result when φ is only surjective on the prescribed intervals.

We recall some of our notation from Sect. 2.2. Suppose that $\sqrt{\cdot} \lesseqgtr f$, respectively, as before define $\varphi(t) = t^{-1} f(t^2)$. Then, we have

$$\gamma := \lim_{t\to\infty} \varphi(t) = \lim_{t\to\infty} t^{-1} f(t^2) \lesseqgtr 1, \text{ respectively.}$$

With this we can show a lemma which is stronger than Lemma 2.9.

Lemma 2.16 *Let σ be some symmetric operator mean on $\mathbb{R}^+$ with representing function f such that $\sqrt{\cdot} < f$ (resp. $\sqrt{\cdot} > f$) and let $\gamma = \lim_{t\to\infty} f(t^2)/t$. Then, if X and Y are positive definite matrices such that $X \le Y < \gamma X$ (resp. $\gamma X < Y \le X$), then there exist positive matrices A and B such that*

$$X = A\#B \quad \text{and} \quad Y = A\sigma B.$$

Proof As in the proof of Lemma 2.9, we show the lemma when $\sqrt{\cdot} < f$. In this case, it suffices to show that given $I_n \le Y_0 = U \operatorname{diag}(\{\lambda_i(Y_0)\})\, U^* \le \gamma I_n$, we can find $A_0 \ge 0$ such that

$$Y_0 = A_0 \sigma A_0^{-1} = \varphi(A_0).$$

While $\varphi(t)$ is not necessarily bijective in this case, it is continuous on $[1, \infty)$ and $\varphi(1) = f(1) = 1$. Therefore, the restriction of φ to some subset of $[1, \infty)$ is surjective onto $[1, \gamma)$.

Since $\sigma(Y_0) \subset [1, \gamma)$, surjectivity of the restriction of φ implies that the set

$$\varphi^{-1}(\lambda_i(Y_0)) := \{x \in [1, \infty) \mid \varphi(x) = \lambda_i(Y_0)\} \ne \emptyset.$$

In particular, if we choose $\delta_i(Y_0) \in \varphi^{-1}(\lambda_i(Y_0))$ for each i, the matrix

$$A_0 := U \operatorname{diag}(\{\delta_i(Y_0)\}) U^*$$

satisfies

$$\varphi(A_0) = U \operatorname{diag}(\{\varphi(\delta_i(Y_0))\}) U^* = U \operatorname{diag}(\{\lambda_i(Y_0)\}) U^*,$$

and the result follows as in Lemma 2.9. □

Now using the same argument as in the proof of Theorem 2.10 we can show the following theorem.

Theorem 2.17 ([19]) *Let f be a continuous function on $(0, \infty)$. Then, f is operator monotone if and only either one of the following holds:*

1. *If $f(A\#B) \le f(A\sigma B)$ for all positive definite A and B and some symmetric operator mean $\# < \sigma$.*
2. *If $f(A\#B) \ge f(A\sigma B)$ for all positive definite A and B and some symmetric operator mean $\# > \sigma$.*

2.3 Matrix Power Means and Operator Monotone Functions

Now, let p be a real number, and a, b be positive. According to the relation (1) the power mean $\mu(p, a, b)$ is corresponding to the monotone function f as

$$f_{\mu,p}(t) = \left(\frac{1+t^p}{2}\right)^{1/p}.$$

Then for positive definite matrices A and B, the Kubo-Ando matrix power mean is defined as

$$P_\mu(p, A, B) = A^{1/2}\left(\frac{1+(A^{-1/2}BA^{-1/2})^p}{2}\right)^{1/p} A^{1/2}.$$

Therefore, from the chain of inequalities (10) for an operator monotone function f on $(0, \infty)$ we have

$$\begin{aligned} f(A\sharp B) &\leq f(A^{1/2} f_{\mu,p}(A^{-1/2}BA^{-1/2})A^{1/2}) \\ &\leq f(A^{1/2} f_{\mu,1}(A^{-1/2}BA^{-1/2})A^{1/2}) \\ &\leq f(A^{1/2} f_{\mu,q}(A^{-1/2}BA^{-1/2})A^{1/2}) \end{aligned} \tag{25}$$

whenever A, B are positive definite matrices.

Notice that the matrix power mean $P_\mu(p, A, B)$ is not either a symmetric or self-adjoint Kubo-Ando mean. In this section, we investigate new characterizations of operator monotone functions by using inequalities in (25). We show that if one of the inequalities in (25) holds for any positive definite matrices A and B, then the function f is operator monotone on $(0, \infty)$.

The more difficult case is the one involving the naive matrix extension of the power means. Let $1/2 \leq p \leq 1 \leq q$. The function $t^{1/q}$ is operator concave, while the function $t^{1/p}$ is operator convex. Then we have

$$\left(\frac{A^p + B^p}{2}\right)^{1/p} \leq \frac{A+B}{2} \leq \left(\frac{A^q + B^q}{2}\right)^{1/q} \tag{26}$$

whenever A and B are positive semidefinite. It is worth noting that the inequalities in (26) were discussed by Audenaert and Hiai [2], where they obtained conditions on p and q such that (26) holds true. In [10, 18] the authors also studied new types of operator convex functions and related inequalities.

2.3.1 Kubo-Ando Matrix Power Means and Characterizations

In this section we study the problem of characterization of operator monotone functions using Kubo-Ando matrix power means.

Theorem 2.18 ([28]) *Let* $0 < p \leq 1$*, and* f *be a continuous function on* $[0, \infty)$ *that satisfies the following inequality*

$$f\,(A\sharp B) \leq f\left(A^{1/2} f_{\mu,p}\left(A^{-1/2} B A^{-1/2}\right) A^{1/2}\right), \tag{27}$$

for any positive definite matrices A *and* B. *Then* f *is operator monotone on* $(0, \infty)$.

Proof Suppose that the inequality (27) holds, it suffices to show that for any $0 \leq X \leq Y$, there exist two positive semidefinite matrices A and B such that

$$X = A\sharp B, \quad Y = A^{1/2} f_{\mu,p}\left(A^{-1/2} B A^{-1/2}\right) A^{1/2}.$$

Firstly, let us consider the case when $X = I_n$. We now show that there exist positive definite matrices A_0 and B_0 such that $A_0 \sharp B_0 = I_n$ and

$$Y = A_0^{1/2} f_{\mu,p}\left(A_0^{-1/2} B_0 A^{-1/2}\right) A_0^{1/2} = A_0 f_{\mu,p}\left(A_0^{-2}\right). \tag{28}$$

Since the function $h(x) = x f_{\mu,p}(x^{-2}) = \left(\dfrac{x^p + x^{-p}}{2}\right)^{1/p}$ is surjective from $(0, \infty)$ to $[1, \infty)$, we obtain that for any $I_n \leq Y$ there exists a matrix $A_0 > 0$ satisfying (28). The matrix B_0 is equal to A_0^{-1}.

In general, for $0 < X \leq Y$ we have $I_n \leq X^{-1/2} Y X^{-1/2}$. By the above arguments, we can find positive semidefinite matrices A_0 and B_0 such that $A_0 \sharp B_0 = I_n$ and

$$X^{-1/2} Y X^{-1/2} = A_0^{1/2} f_{\mu,p}\left(A_0^{-1/2} B_0 A^{-1/2}\right) A_0^{1/2} = P_\mu\left(p, A_0, B_0\right).$$

Consequently, applying (27) to matrices $A = X^{1/2} A_0 X^{1/2}$ and $B = X^{1/2} B_0 X^{1/2}$, we obtain that $f(X) \leq f(Y)$. In other words, f is operator monotone. □

Theorem 2.19 *Let* $0 < p \leq 1 \leq q$*, and* f *be a continuous function on* $[0, \infty)$ *that satisfies one of the following inequalities*

$$f\left(\frac{A+B}{2}\right) \leq f\left(A^{1/2} f_{\mu,q}\left(A^{-1/2} B A^{-1/2}\right) A^{1/2}\right), \tag{29}$$

$$f\left(A^{1/2} f_{\mu,p}\left(A^{-1/2} B A^{-1/2}\right) A^{1/2}\right) \leq f\left(\frac{A+B}{2}\right), \tag{30}$$

for any positive definite matrices A *and* B. *Then* f *is operator monotone on* $(0, \infty)$.

To prove Theorem 2.19, we will need the following lemma which is the inverse problem for the arithmetic mean and the power mean. Recently, in [27] we also study the inverse problem for the generalized contraharmonic means.

Lemma 2.20 *Suppose X and Y are positive definite matrices satisfying $X \leq Y < \gamma X$ (resp. $\gamma X < Y \leq X$) where $\gamma = 2^{1-1/q}$ (resp. $\gamma = 2^{1-1/p}$), then there exist positive matrices A and B such that*

$$X = \frac{A+B}{2} \quad \textit{and} \quad Y = P_\mu(q, A, B) \ (resp.\ Y = P_\mu(p, A, B)),$$

where $0 < p \leq 1 \leq q$.

Proof We show the lemma when $X \leq Y < \gamma X$, the remaining case can be obtained similarly. Firstly, let us consider the case when $X = I_n$. Then it suffices to show that given $I_n \leq Y = U\mathrm{diag}(\{\lambda_i(Y)\})U^* \leq \gamma I_n$, we can find $A_0, B_0 \geq 0$ such that

$$I_n = \frac{A_0 + B_0}{2} \quad \text{and} \quad Y = P_\mu(q, A_0, B_0).$$

Or, equivalently, there exists $0 < A_0 \leq 2I_n$ such that $Y = P_\mu(q, A_0, 2I_n - A_0) = \varphi(A_0)$, where

$$\varphi(x) = x^{1/2} f_{\mu,q}\left(x^{-1/2}(2-x)x^{-1/2}\right)x^{1/2} = \left(\frac{x^q + (2-x)^q}{2}\right)^{1/q}.$$

Note that φ is continuous on $[0, 2]$ and surjective from $[0, 2]$ onto $[1, \gamma]$. Since $\lambda_i(Y) \in [1, \gamma]$, for each i, we can choose $\delta_i(Y) \in [0, 2]$ such that $\varphi(\delta_i(Y)) = \lambda_i(Y)$. The matrix

$$A_0 := U\mathrm{diag}(\{\delta_i(Y)\})U^*$$

satisfies

$$\varphi(A_0) = U\mathrm{diag}(\{\varphi(\delta_i(Y))\})U^* = U\mathrm{diag}(\{\lambda_i(Y)\})U^*.$$

In general, for $0 < X \leq Y < \gamma X$, we have $I_n \leq X^{-1/2}YX^{-1/2} < \gamma I_n$. By the above arguments, we can find positive definite matrices A_0 and B_0 such that $(A_0 + B_0)/2 = I_n$ and $X^{-1/2}YX^{-1/2} = P_\mu(q, A_0, B_0)$. Now, let $A = X^{1/2}A_0X^{1/2}$ and $B = X^{1/2}B_0X^{1/2}$, we have $(A + B)/2 = X$ and

$$Y = X^{1/2}P_\mu(q, A_0, B_0)X^{1/2} = P_\mu(q, A, B),$$

which completes the proof of Lemma 2.20. □

We are now ready to prove Theorem 2.19.

Proof of Theorem 2.19 First we prove the case when f satisfies the inequality (29). Let $0 \le X \le Y$ and $Y_0 = X^{-1/2}YX^{-1/2}$, and choose $\gamma_0 \in (1, 2^{1-1/q})$. Consider the spectral decomposition, $Y_0 = \sum_{i=1}^{r} \lambda_i E_i$ with the eigenvalues λ_i listed in the decreasing order. Then, there exists a set of non-ascending integers $\{m_i \mid 1 \le i \le r\}$ such that

$$\gamma_0^{m_i} < \lambda_i \le \gamma_0^{m_i+1}.$$

Let $\ell_1 < \ell_2 < \cdots < \ell_t = r$ be the sequence of indexes such that

$$m_1 = \cdots = m_{\ell_1} > m_{\ell_1+1} = \cdots = m_{\ell_2} > m_{\ell_2+1} = \cdots = m_{\ell_3} \\ > \cdots > m_{\ell_{t-1}+1} = \cdots = m_{\ell_t} = m_r.$$

We have

$$\gamma_0^{m_{\ell_t}} < \lambda_r < \cdots < \lambda_{\ell_{t-1}+1} \le \gamma_0^{m_{\ell_t}+1} \le \gamma_0^{\ell_{t-1}} \\ < \lambda_{\ell_{t-1}} < \cdots < \lambda_{\ell_{t-2}+1} \le \gamma_0^{\ell_{t-1}+1} < \cdots < \lambda_1 \le \gamma_0^{\ell_1+1}.$$

It follows that

$$\begin{aligned}
&I < \gamma_0 I < \gamma_0^2 I < \cdots < \gamma_0^{m_{\ell_t}} I = \gamma_0^{m_{\ell_t}} (E_1 + E_2 + \cdots + E_r) \\
&\le \lambda_r E_r + \cdots + \lambda_{\ell_{t-1}+1} E_{\ell_{t-1}+1} + \gamma_0^{m_{\ell_t}} \left(E_{\ell_{t-1}} + E_{\ell_{t-1}-1} + \cdots + E_1\right) \\
&\le \lambda_r E_r + \cdots + \lambda_{\ell_{t-1}+1} E_{\ell_{t-1}+1} + \gamma_0^{m_{\ell_t}+1} \left(E_{\ell_{t-1}} + E_{\ell_{t-1}-1} + \cdots + E_1\right) \\
&\cdots \\
&\le \lambda_r E_r + \cdots + \lambda_{\ell_{t-1}+1} E_{\ell_{t-1}+1} + \gamma_0^{m_{\ell_{t-1}}} \left(E_{\ell_{t-1}} + E_{\ell_{t-1}-1} + \cdots + E_1\right) \\
&\le \lambda_r E_r + \cdots + \lambda_{\ell_{t-2}+1} E_{\ell_{t-2}+1} + \gamma_0^{m_{\ell_{t-1}}} \left(E_{\ell_{t-2}} + E_{\ell_{t-2}-1} + \cdots + E_1\right) \\
&\le \lambda_r E_r + \cdots + \lambda_{\ell_{t-2}+1} E_{\ell_{t-2}+1} + \gamma_0^{m_{\ell_{t-1}}+1} \left(E_{\ell_{t-2}} + E_{\ell_{t-2}-1} + \cdots + E_1\right) \\
&\cdots \\
&\le \lambda_r E_r + \cdots + \lambda_{\ell_2+1} E_{\ell_2+1} + \gamma_0^{m_{\ell_2}} \left(E_{\ell_1} + E_{\ell_1-1} + \cdots + E_1\right) \\
&\le \lambda_r E_r + \cdots + \lambda_{\ell_2+1} E_{\ell_2+1} + \gamma_0^{m_{\ell_2}+1} \left(E_{\ell_1} + E_{\ell_1-1} + \cdots + E_1\right) \\
&\cdots \\
&\le \lambda_r E_r + \cdots + \lambda_{\ell_2+1} E_{\ell_2+1} + \gamma_0^{m_{\ell_1}} \left(E_{\ell_1} + E_{\ell_1-1} + \cdots + E_1\right) \\
&\le \lambda_r E_r + \ldots \lambda_1 E_1 = Y_0 \le \gamma_0^{m_{\ell_1}+1} I.
\end{aligned}$$

After multiplying each term of the chain of inequalities on both sides by $X^{1/2}$, let Z_k be the k-th expression of the chain, we obtain the following chain inequalities

$$0 \le X = Z_1 \le Z_2 \le \cdots \le Z_{m-1} \le Y \le Z_m = \gamma_0^{m_{\ell_1}+1} X.$$

where m is a positive integer. The previous calculation gives us $Z_k \le Z_{k+1} \le \gamma Z_k$. Therefore, it follows from Lemma 2.20 that there exist positive definite matrices A and B such that

$$Z_k = (A+B)/2 \quad \text{and} \quad Z_{k+1} = P_\mu(q, A, B).$$

Consequently,

$$f(X) \le f(Z_1) \le f(Z_2) \le \cdots \le f(Z_{m-1}) \le f(Y).$$

In other words, f is operator monotone.

The proof in the case when (30) holds is similar. □

2.3.2 The Inverse Problem for Non-Kubo-Ando Matrix Power Means

In [2] Audenaert and Hiai determined values of p and q such that the following inequality holds true

$$\left(\frac{A^p + B^p}{2}\right)^{1/p} \le \left(\frac{A^q + B^q}{2}\right)^{1/q}$$

whenever A, B are positive semidefinite matrices. When $1/2 \le p \le 1 \le q$, according to the operator convexity of $t^{1/p}$ and operator concavity of $t^{1/q}$ we have

$$\left(\frac{A^p + B^p}{2}\right)^{1/p} \le \frac{A+B}{2} \le \left(\frac{A^q + B^q}{2}\right)^{1/q} \tag{31}$$

Suppose that $0 \le X \le Y$. Solving the inverse mean problem is to find positive definite matrices A and B such that

$$X = \left(\frac{A^p + B^p}{2}\right)^{1/p}, \quad Y = \left(\frac{A^q + B^q}{2}\right)^{1/q}. \tag{32}$$

If this system has a positive solution, then we may use the result to characterize operator monotone function. Unfortunately, inequalities in (31) do not characterize operator monotone functions, in general.

Proposition 2.21 *For any $q > 1$, there exists a non-monotone operator function satisfying*

$$f\left(\frac{A+B}{2}\right) \leq f\left(\left(\frac{A^q+B^q}{2}\right)^{1/q}\right)$$

for all positive definite matrices A and B.

Proof For a fixed number $1 < r \leq \min\{2, q\}$, we consider the function $f(x) = x^r$ which is not an operator monotone function. It follows from the operator convexity of x^r and the operator concavity of $x^{r/q}$ that

$$\left(\frac{A+B}{2}\right)^r \leq \frac{A^r+B^r}{2} \leq \left(\frac{A^q+B^q}{2}\right)^{r/q} \leq \left[\left(\frac{A^q+B^q}{2}\right)^{1/q}\right]^r.$$

Hence, the non-operator monotone function f satisfies

$$f\left(\frac{A+B}{2}\right) \leq f\left(\left(\frac{A^q+B^q}{2}\right)^{1/q}\right),$$

which completes the proof of Lemma 2.21. □

Similarly, one would expect to have the same conclusion for the first inequality, namely, for $1/2 \leq p \leq 1$, there exists a non-monotone operator satisfying

$$f\left(\left(\frac{A^p+B^p}{2}\right)^{1/p}\right) \leq f\left(\frac{A+B}{2}\right).$$

However, when $p = 1/2$ we were able to solve the inverse problem and establish a new characterization of operator monotone functions.

Theorem 2.22 *Let f be a continuous function on $[0, \infty)$ that satisfies the following inequality*

$$f\left(\left(\frac{A^{1/2}+B^{1/2}}{2}\right)^2\right) \leq f\left(\frac{A+B}{2}\right),$$

for any positive semidefinite matrices A and B. Then f is operator monotone.

Proof Firstly, we show that $f(X) \leq f(Y)$ for any positive semidefinite matrices X, Y with $0 \leq X \leq Y \leq 2X$. Indeed, we need to solve the following system

$$\begin{cases} \left(\dfrac{A^{1/2}+B^{1/2}}{2}\right)^2 = X \\ \dfrac{A+B}{2} = Y. \end{cases} \tag{33}$$

Subtracting the first equation from the second, we obtain

$$Y - X = \left(\frac{A^{1/2}-B^{1/2}}{2}\right)^2.$$

Therefore, system (33) is equivalent to

$$\begin{cases} \dfrac{A^{1/2}+B^{1/2}}{2} = X^{1/2} \\ \dfrac{A^{1/2}-B^{1/2}}{2} = (Y-X)^{1/2}. \end{cases}$$

The last system has a unique positive solution as

$$A = \left(X^{1/2} + (Y-X)^{1/2}\right)^2, \quad B = \left(X^{1/2} - (Y-X)^{1/2}\right)^2.$$

Notice that the condition $0 \leq X \leq Y < 2X$ guarantees the semidefinite positivity of A and B. Thus, $f(X) \leq f(Y)$.

Now, for any positive semidefinite matrices $0 \leq X \leq Y$, we apply the same arguments as in the proof of Theorem 2.19 to obtain a positive integer m and positive semidefinite matrices $Z_1, \dots, Z_m$ such that

$$0 \leq X = Z_1 \leq Z_2 \leq Z_3 \leq \cdots \leq Z_{m-1} \leq Y \leq Z_m$$

with $Z_k \leq Z_{k+1} < 2Z_k$ for all $k = 1, 2, \dots, m$. Therefore, combining with the previous arguments, we get

$$f(X) = f(Z_1) \leq f(Z_2) \leq \cdots \leq f(Z_{m-1}) \leq f(Y).$$

□

3 Powers-Størmer's Inequality and Characterizations of Operator Monotone Functions

Powers-Størmer's inequality [3, 39] asserts that for $s \in [0, 1]$ the following inequality

$$2\mathrm{Tr}\,(A^s B^{1-s}) \geq \mathrm{Tr}\,(A + B - |A - B|) \tag{34}$$

holds for any pair of positive matrices A, B. This is a key inequality to prove the upper bound of Chernoff bound, in quantum hypothesis testing theory [3]. This inequality was first proved in [3], using an integral representation of the function t^s. After that, N. Ozawa [32, Proposition 1.1] gave a much simpler proof for the same inequality, using fact that, for $s \in [0, 1]$, the function $f(t) = t^s$ $(t \in [0, +\infty))$ is an operator monotone. Recently, Ogata in [36] extended this inequality to standard von Neumann algebras. The motivation of this section is that if the function $f(t) = t^s$ is replaced by another operator monotone function (this class is intensively studied) then $\mathrm{Tr}\,(A + B - |A - B|)$ may get smaller upper bound that is used in quantum hypothesis testing. Based on Ozawa's proof we formulate Powers-Størmer's inequality for an arbitrary operator monotone function on $(0, +\infty)$.

Lemma 3.1 ([31, Theorem 2.5]) *Let f be a strictly positive, continuous function on $[0, \infty)$. If f is $2n$-monotone, then for any positive semidefinite A and a contraction C in $\mathbb{M}_n$*

$$C^* f(A)C \leq f(C^* AC).$$

Lemma 3.2 ([14]) *Let f be a strictly positive, continuous function on $(0, \infty)$. Then f is n-monotone if and only if $-\frac{1}{f(t)}$ is n-monotone.*

Proof For any $t_1, t_2, \cdots, t_n \in (0, \infty)$ we have

$$\frac{\frac{1}{f(t_i)} - \frac{1}{f(t_j)}}{t_i - t_j} = \frac{\frac{f(t_j)-f(t_i)}{f(t_i)f(t_j)}}{t_i - t_j} = -\frac{1}{f(t_i)f(t_j)} \frac{f(t_i) - f(t_j)}{t_i - t_j}.$$

Since f is n-monotone, $[\frac{f(t_i)-f(t_j)}{t_i-t_j}]$ is positive semidefinite by Löwner [35], hence, we have

$$\left(\frac{(-\frac{1}{f(t_i)}) - (-\frac{1}{f(t_j)})}{t_i - t_j}\right) = -\left(\frac{\frac{1}{f(t_i)} - \frac{1}{f(t_j)}}{t_i - t_j}\right)$$
$$= -\left(-\left(\frac{1}{f(t_i)f(t_j)} \frac{f(t_i) - f(t_j)}{t_i - t_j}\right)\right)$$

$$= \left([\frac{1}{f(t_i)f(t_j)}] \circ [\frac{f(t_i) - f(t_j)}{t_i - t_j} \right)$$
$$\geq 0,$$

where $\circ$ means the Hadamard product.

Therefore, the function $-\frac{1}{f(t)}$ is n-monotone by Löwner [35]. □

Proposition 3.3 *Let f be a strictly positive, continuous function on $(0, \infty)$. If f is $2n$-monotone, the function $g(t) = \frac{t}{f(t)}$ is n-monotone.*

Proof Let A, B be positive matrix in $\mathbb{M}_n$ such that $0 < A \leq B$.

Let $C = B^{-\frac{1}{2}}A^{\frac{1}{2}}$. Then $\|C\| \leq 1$. Since f is $2n$-monotone, $-f$ satisfies the Jensen type inequality from Lemma 3.1, that is,

$$-f(A) = -f(C^*BC) \leq -C^*f(B)C$$
$$-f(A) \leq -A^{\frac{1}{2}}B^{-\frac{1}{2}}f(B)B^{-\frac{1}{2}}A^{\frac{1}{2}}$$
$$-A^{-\frac{1}{2}}f(A)A^{-\frac{1}{2}} \leq -B^{-\frac{1}{2}}f(B)B^{-\frac{1}{2}}$$
$$-A^{-1}f(A) \leq -B^{-1}f(B).$$

Hence, the function $-\frac{f(t)}{t}$ is n-monotone. Therefore, from Lemma 3.2 we conclude that

$$-\frac{1}{-\frac{f(t)}{t}} = \frac{t}{f(t)}$$

is n-monotone. □

Theorem 3.4 ([12]) *Let f be a strictly positive, $2n$-monotone function on $(0, \infty)$. Then for any pair of positive definite matrices $A, B \in \mathbb{M}_n$ such that $A \leq B$*

$$\mathrm{Tr}\,(A) + \mathrm{Tr}\,(B) - \mathrm{Tr}\,(|A - B|) \leq 2\mathrm{Tr}\,(f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}}), \tag{35}$$

where $g(t) = \frac{t}{f(t)}$.

Proof Let A, B be positive matrices in $\mathbb{M}_n$ such that $A \leq B$. We may assume that A and B are invertible. For operator $A - B$ let us denote by $P = (A - B)^+$ and $Q = (A - B)^-$ its positive and negative part, respectively. Then we have

$$A - B = P - Q \quad \text{and} \quad |A - B| = P + Q, \tag{36}$$

from that it follows that

$$A + Q = B + P. \tag{37}$$

On account of (37) the inequality (35) is equivalent to the following

$$\mathrm{Tr}\,(A) - \mathrm{Tr}\,(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) \leq \mathrm{Tr}\,(P).$$

Since $B + P \geq B \geq 0$ and $B + P = A + Q \geq A \geq 0$ we have $g(A) \leq g(B + P)$ by Proposition 3.3 and

$$\begin{aligned}
&\mathrm{Tr}\,(A) - \mathrm{Tr}\,(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) \\
&\quad = \mathrm{Tr}\,(f(A)^{\frac{1}{2}} g(A) f(A)^{\frac{1}{2}}) - \mathrm{Tr}\,(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) \\
&\quad \leq \mathrm{Tr}\,(f(A)^{\frac{1}{2}} g(B+P) f(A)^{\frac{1}{2}}) - \mathrm{Tr}\,(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) \\
&\quad = \mathrm{Tr}\,(f(A)^{\frac{1}{2}} (g(B+P) - g(B)) f(A)^{\frac{1}{2}}) \\
&\quad \leq \mathrm{Tr}\,(f(B+P)^{\frac{1}{2}} (g(B+P) - g(B)) f(B+P)^{\frac{1}{2}}) \\
&\quad = \mathrm{Tr}\,(f(B+P)^{\frac{1}{2}} g(B+P) f(B+P)^{\frac{1}{2}}) \\
&\qquad - \mathrm{Tr}\,(f(B+P)^{\frac{1}{2}} g(B) f(B+P)^{\frac{1}{2}}) \\
&\quad = \mathrm{Tr}\,(B+P) - \mathrm{Tr}\,(f(B)^{\frac{1}{2}} g(B) f(B)^{\frac{1}{2}}) \\
&\quad = \mathrm{Tr}\,(B+P) - \mathrm{Tr}\,(B) \\
&\quad = \mathrm{Tr}\,(P).
\end{aligned}$$

Thus, we have the conclusion. □

Remark 3.5 In [9, 13] we studied the interpolation classes and matrix means, and proved the Powers-Stormer's inequality for interpolation functions. Some new characterizations of operator monotone functions using the arithmetic-geometric means inequality were obtained in [17].

Now let φ be a normal state on the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on a Hilbert space $\mathcal{H}$, f a strictly positive, continuous function on $(0, \infty)$, and g a function on $(0, \infty)$ defined by $g(t) = \frac{t}{f(t)}$. In the following theorem, we show that Powers-Størmer type inequality

$$\varphi(A + B) - \varphi(|A - B|) \leq 2\varphi(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}). \tag{38}$$

characterizes the matrix monotonicity.

The following two lemmas are obvious.

Lemma 3.6 *Let A and B be positive semidefinite matrices in $\mathbb{M}_n$ such that $A \not\leq B$. Then there is a unitary U in $\mathbb{M}_n$ such that $a_{11} > b_{11}$ for $[a_{ij}] = UAU^*$ and $[b_{ij}] = UBU^*$.*

Lemma 3.7 *Let $A = (a_{ij})$, $B = (b_{ij})$ be positive invertible in $\mathbb{M}_n$ and S a non-finite rank density operator on an infinite dimensional, separable Hilbert space $\mathcal{H}$. Suppose that $a_{11} > b_{11}$. Then there exist an orthogonal system $\{\xi_i\}_{i=1}^{\infty} \subset \mathcal{H}$ and $\{\lambda_i\}_{i=1}^{\infty} \subset [0, 1)$ such that $\sum_{i=1}^{\infty} \lambda_i = 1$, $S\xi_i = \lambda_i \xi_i$, and $\sum_{i=1}^{n} a_{ii}\lambda_i > \sum_{i=1}^{n} b_{ii}\lambda_i$.*

Theorem 3.8 *Let $\mathcal{H}$ be an infinite dimensional, separable Hilbert space and φ a normal state on $\mathcal{B}(\mathcal{H})$ such that its corresponding density operator is not finite rank. Let f be a strictly positive, continuous function on $(0, \infty)$, and g be a function on $(0, \infty)$ defined by $g(t) = \frac{t}{f(t)}$. Suppose that*

$$\varphi(A + B) - \varphi(|A - B|) \leq 2\varphi(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) \tag{39}$$

for any positive invertible $A, B \in \mathcal{B}(\mathcal{H})$. Then both functions f and g on $(0, \infty)$ are operator monotone.

Proof Suppose that g is not operator monotone. Then there exist $n \in \mathbb{N}$ and invertible positive matrices A, B in $\mathbb{M}_n$ with $A \leq B$ such that $g(A) \not\leq g(B)$. Hence,

$$A \not\leq f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}.$$

Put $A = [a_{ij}]$ and $f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}} = [b_{ij}] = B'$. Note that for any unitary U in $\mathbb{M}_n$

$$\begin{aligned} UAU^* \not\leq Uf(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}} U^* &= Uf(A)^{\frac{1}{2}} U^* U g(B) U^* U f(A)^{\frac{1}{2}} U^* \\ &= f(UAU^*)^{\frac{1}{2}} g(UBU^*) f(UAU^*)^{\frac{1}{2}}. \end{aligned}$$

Hence without loss of generality, we can assume that $a_{11} > b_{11}$ by Lemma 3.6.

Let S_φ be a density operator on $\mathcal{H}$ such that $\varphi(X) = \mathrm{Tr}(S_\varphi X)$ for all $X \in \mathcal{B}(\mathcal{H})$. By Lemma 3.7, there exists an orthogonal system $\{\xi_i\}_{i=1}^{\infty} \subset \mathcal{H}$ and $\{\lambda_i\}_{i=1}^{\infty} \subset [0, 1)$ such that $\sum_{i=1}^{\infty} \lambda_i = 1$ and $\sum_{i=1}^{n} a_{ii}\lambda_i > \sum_{i=1}^{n} b_{ii}\lambda_i$.

Let consider the following canonical inclusion:

$$\rho : \mathbb{M}_n \longrightarrow \left(\sum_{i=1}^{n} |\xi_i\rangle\langle\xi_i|\right) \mathcal{B}(\mathcal{H}) \left(\sum_{i=1}^{n} |\xi_i\rangle\langle\xi_i|\right)$$

$$\rho([x_{ij}]) = \sum_{i,j=1}^{n} x_{ij} |\xi_i\rangle\langle\xi_j|.$$

Put

$$C = \rho(A) + \sum_{i=n+1}^{\infty} |\xi_i\rangle\langle\xi_i| \quad \text{and} \quad D = \rho(B) + \sum_{i=n+1}^{\infty} |\xi_i\rangle\langle\xi_i|.$$

Then both of operators C and D are invertible on $\mathcal{H}$ and $C \leq D$. That means, inequality (39) holds true for selected C, D, that is,

$$\varphi(C) \leq \varphi(f(C)^{\frac{1}{2}} g(D) f(C)^{\frac{1}{2}}).$$

On the other hand, note that

$$\rho(f(A)^{\frac{1}{2}})\rho(g(B))\rho(f(A)^{\frac{1}{2}}) = \rho(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) = \sum_{i=1}^{n} b_{ij} |\xi_i\rangle\langle\xi_j|.$$

Then by straightforward calculations, we obtain

$$f(C) = \rho(f(A)) + f(1) \sum_{i=n+1}^{\infty} |\xi_i\rangle\langle\xi_i|,$$

$$\begin{aligned} f(C)^{\frac{1}{2}} g(D) f(C)^{\frac{1}{2}} &= \rho(f(A))^{\frac{1}{2}} \rho(g(B))\rho(f(A))^{\frac{1}{2}} + \sum_{i=n+1}^{\infty} |\xi_i\rangle\langle\xi_i| \\ &= \rho(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}) + \sum_{i=n+1}^{\infty} |\xi_i\rangle\langle\xi_i| \\ &= \sum_{i,j=1}^{n} b_{ij} |\xi_i\rangle\langle\xi_j| + \sum_{i=n+1}^{\infty} |\xi_i\rangle\langle\xi_i|. \end{aligned}$$

Consequently,

$$\begin{aligned} \varphi(C) &= \mathrm{Tr}(S_\varphi(\rho(A) + \sum_{i=n+1}^{\infty} |\xi_i\rangle\langle\xi_i|)) \\ &= \sum_{i=1}^{n} (S_\varphi \rho(A)\xi_i|\xi_i) + \sum_{i=n+1}^{\infty} \lambda_i \\ &= \sum_{i=1}^{n} a_{ii}\lambda_i + \sum_{i=n+1}^{\infty} \lambda_i \\ &> \sum_{i=1}^{n} b_{ii}\lambda_i + \sum_{i=n+1}^{\infty} \lambda_i \\ &= \sum_{i=1}^{n} (S_\varphi \rho(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}})\xi_i|\xi_i) + \sum_{i=n+1}^{\infty} \lambda_i \end{aligned}$$

$$= \mathrm{Tr}(S_\varphi(\rho(f(A)^{\frac{1}{2}}g(B)f(A)^{\frac{1}{2}}) + \sum_{i=n+1}^{\infty} |\xi_i\rangle\langle\xi_i|))$$

$$= \varphi(f(C)^{\frac{1}{2}}g(D)f(C)^{\frac{1}{2}}).$$

The last inequality contradicts (39). Therefore, the function g is operator monotone.

Moreover, the monotonicity of f follows from [31, Corollary 6]. □

Remark 3.9 The Powers-Størmer inequality for interpolation functions was proved in [9]. The matrix Powers-Størmer inequality was proved in [15]. Namely, we showed that for a strictly positive operator monotone function f on $(0, \infty)$ with $f((0, \infty)) \subset (0, \infty)$ and $f(1) = 1$,

$$\frac{A+B}{2} - A\sigma_f B \leq \frac{1}{2}|A - B|$$

for any positive semidefinite matrices A and B satisfying the condition $AB + BA \geq 0$. Using the same method as in Sect. 2, the first author [7] also obtained a new characterizations of operator monotone functions using the matrix Powers-Størmer inequality. It was showed that for a nonnegative function f on $[0, \infty)$, if

$$f\left(\frac{A+B}{2}\right) \leq f\left(A\sharp B + \frac{1}{2}A^{1/2}|I_n - A^{-1/2}BA^{-1/2}|A^{1/2}\right)$$

for any positive definite matrices A and B, then f is operator monotone on $[0, \infty)$.

Finally, we show that if the monotonicity inequality holds for at least one normal state on the algebra $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is some infinite-dimensional Hilbert space, then we also have a new characterization of operator monotone functions.

Theorem 3.10 ([11]) *Let φ be a normal state on $\mathcal{B}(\mathcal{H})$. The following condition is sufficient (and, evidently, necessary) for a continuous function $f : \Omega \to \mathbb{R}$ (where Ω is a subset of $\mathbb{R}$) to be operator monotone function:*

$(*)$*: for any $A, B \in \mathcal{B}(\mathcal{H})^{sa}$ such that $\sigma(A), \sigma(B) \subset \Omega$,*

$$A \leq B \quad \Longrightarrow \quad \varphi(f(A)) \leq \varphi(f(A)).$$

Proof Let f be a continuous function that satisfies the condition $(*)$, and suppose that f is not an operator monotone function on Ω. Therefore, there exists a natural number n, Hermitian matrices $A' = [a_{ij}]_{i,j=1}^n$ and $B' = [b_{ij}]_{i,j=1}^n$ such that $\sigma(A'), \sigma(B') \subset \Omega$, $A' \leq B'$, and $\alpha_{11} > \beta_{11}$, where $[\alpha_{ij}]_{i,j=1}^n = f(A'), [\beta_{ij}]_{i,j=1}^n = f(B')$.

Let ξ_k be the eigenvectors of the density operator S_φ corresponding to eigenvalues (possibly, zero ones) λ_k. In the space $\mathcal{H}$ we choose an orthonormal system $\{\xi_k\}_{k=1}^n$ of eigenvectors of the operator S_φ such that $\sum_{k=1}^n \alpha_{kk}\lambda_k > \sum_{k=1}^n \beta_{kk}\lambda_k$.

We can always do this, because the sum of all eigenvalues of the operator S_φ, taking into account their multiplicities, equals one, therefore we can choose λ_1 sufficiently large, and λ_k $(k = 2, 3, \cdots, n)$ arbitrarily small. Let us complement the system $\{\xi_k\}_{k=1}^n$ to the orthonormal basis $\{\xi_k\}_{k=1}^n \cup \{\xi_k\}_{k\in K}$ (where K is a set of indexes) consisting of eigenvectors of S_φ. Consider $\mathcal{H}$ as the direct sum $\mathcal{H}_1 \bigoplus \mathcal{H}_2$ of the n-dimensional Hilbert space $\mathcal{H}_1$ with the basis $\{\xi_k\}_{k=1}^n$ and the Hilbert space $\mathcal{H}_2$ with the basis $\{\xi_k\}_{k\in K}$. Choose some $\eta_0 \in \Omega$ and put $A = A' \bigoplus \eta_0 E''$, $B = B' \bigoplus \xi_0 E''$, where E'' is the unit operator in the space $\mathcal{H}_2$. Then $A, B \in \mathcal{B}(\mathcal{H})^{sa}$; $\sigma(A), \sigma(B) \subset \Omega$ and $A \leq B$; but

$$\begin{aligned}\varphi(f(A)) = \mathrm{Tr}\,(S_\varphi f(A)) &= \sum_{k=1}^n \alpha_{kk}\lambda_k + \sum_{k\in K} f(\xi_0)\lambda_k \\ &> \sum_{k=1}^n \beta_{kk}\lambda_k + \sum_{k\in K} f(\xi_0)\lambda_k = \varphi(f(B))\end{aligned}$$

which contradicts the condition $(*)$. □

Remark 3.11 In [21] we studied monotonicity inequality for extended part of a von Newmann algebra that involve operator monotone functions. In a recent paper [23] we studied functions preserving operator means, and also established new characterizations of operator monotone functions.

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Perspectives, Means and their Inequalities

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Abstract The perspective function is useful in convex analysis. In the study of operator theory, an operator mean can be realized as the operator perspective and its limits. On the other hand, an operator mean can be regarded as a two-variable functional calculus for positive operators. In this chapter, we study the operator perspective and its extensions including operator means and the Pusz–Woronowicz functional calculus. We also discuss about the related operator inequalities.

Keywords Operator perspective · Operator connection · Functional calculus · Operator mean · Operator convex · Operator monotone · Ando–Hiai inequality · Furuta inequality

1 Introduction

The perspective method is the standard operation to transform a convex function of n-variable into one of $n+1$-variable [12, Lemma 2]. Applying the same method for a given operator function, a binary operation on the set of invertible positive operators that inherits the properties of the original function, can be obtained.

We could identify the operator geometric mean which was introduced by Pusz and Woronowicz in 1975 as the operator perspective for the root function. This

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observation was applied to more general operator monotone functions, and as a result, operator means and related operations are defined by several authors [1, 18, 39].

The convexity of the scalar function is inherited by the perspective function. This is extended in the framework of operator functions by Effros [13] and Ebadian et al. [15]. That is, for a given real valued continuous function f on $(0, \infty)$, the operator perspective P_f is jointly convex if and only if f is operator convex. This observation is related to the joint convexity of Quantum f-divergence in quantum information theory [28, 29]. Recently, the perspective theory has been further discussed in [23, 32, 33, 37].

A perspective function is defined for two invertible positive operators. Under some conditions, it can be extended to a binary operation on the set of non-negative operators like an operator mean. We shall discuss about such conditions in Sect. 3.

2 Perspectives for Invertible Operators

Through this chapter, H is a Hilbert space. $B(H)_{sa}$ is the set of bounded self-adjoint operators, $B(H)_+$ is the set of bounded positive operators on H, and $B(H)_{++}$ is the set of invertible elements in $B(H)_+$. We also write $A \geq 0$ when $A \in B(H)_+$, and $A > 0$ when $A \in B(H)_{++}$.

To form the perspective function of a convex map is a useful trick in convex analysis [34, 35]. For a real valued continuous function f on $(0, \infty)$, a two-variable operator function P_f defined by

$$P_f(A, B) = B^{1/2} f(B^{-1/2} A B^{-1/2}) B^{1/2} \quad (A, B > 0)$$

is called the operator perspective of f [13–15, 32, 33]. In this section, some properties of operator perspectives are given.

Example 2.1 Consider the operator convex function $f(t) = -\log t$ defined on $(0, \infty)$. Then the perspective function P_f is given by

$$P_f(B, A) = -A^{1/2} \log(A^{-1/2} B A^{-1/2}) A^{1/2}$$

for $A, B > 0$.

Note that when A and B are commuting positive definite matrices, then

$$\mathrm{Tr}(P_f(B, A)) = S(A, B),$$

where $S(A, B)$ is relative entropy defined by $S(A, B) = \mathrm{Tr}(A \log A) - \mathrm{Tr}(A \log B)$.

2.1 Homogeneity

We first note the homogeneity of the operator perspective. If f is a polynomial function such as $f(t) = \sum_k \alpha_k t^k$, it is clear that

$$\begin{aligned} B^{1/2} f(B^{-1/2} A B^{-1/2}) B^{-1/2} &= B^{1/2} \sum_k \alpha_k (B^{-1/2} A B^{-1/2})^k B^{-1/2} \\ &= \sum_k \alpha_k (AB^{-1})^k. \end{aligned}$$

In the following, we conventionally denote the last term $f(AB^{-1})$.

The following is obtained from the similar argument.

Proposition 2.1 *Let C be an invertible operator and f be a real valued continuous function on $(0, \infty)$. Then $C^* P_f(A, B)C = P_f(C^*AC, C^*BC)$ for all $A, B > 0$.*

Proof It is enough to show the case if f is a polynomial function. Using the above argument, we have

$$\begin{aligned} C^* P_f(A, B)C &= C^* f(AB^{-1})BC \\ &= C^* f(AB^{-1})(C^*)^{-1} C^* BC \\ &= P_f(C^*AC, C^*BC). \end{aligned}$$

□

Corollary 2.1

$$C > 0 \Rightarrow CP_f(A, B)C = P_f(CAC, CBC).$$

Remark 2.1 Let f be a continuous function on $(0, \infty)$ which has the analytic continuation. If we regard $f(AB^{-1})$ as the holomorphic function calculus for AB^{-1}, a function $(A, B) \mapsto f(AB^{-1})B$ has the homogeneity property. Some authors define the operator perspective of non-self-adjoint operators in this way (cf. [8]).

2.2 Convexity

In convex analysis, the convexity of a given function is equivalent to one of its perspective function. Similar results are known for the convexity of a two-variable operator function [33, Theorem 10.1].

Definition 2.1 Let Ω be a convex subset of $B(H)_{sa} \times B(H)_{sa}$ and let P be a function from Ω to $B(H)_{sa}$. The function P is said to be jointly convex if

$$P(\alpha A_1 + (1-\alpha)A_2, \alpha B_1 + (1-\alpha)B_2) \le \alpha P(A_1, B_1) + (1-\alpha)P(A_2, B_2)$$

holds for all $(A_i, B_i) \in \Omega$ $(i = 1, 2)$ and $\alpha \in [0, 1]$.

In the above definition, if the function P satisfies the reverse inequality, P is said to be jointly concave.

Definition 2.2 Let $\mathcal{J}$ be an interval in $\mathbb{R}$ and f be a real valued continuous function on $\mathcal{J}$. The function f is said to be operator convex if

$$f(\alpha A + (1-\alpha)B) \le \alpha f(A) + (1-\alpha)f(B)$$

holds for all $A, B \in B(H)_{sa}$ with $\sigma(A), \sigma(B) \subset \mathcal{J}$ and $\alpha \in [0, 1]$.

The function f is said to be operator concave if the reverse inequality holds. It is known that a positive valued function f on $(0, \infty)$ is operator concave if and only if f is operator monotone (i.e., $f(A) \le f(B)$ holds if $0 < A \le B$) [6, Theorem V.2.5].

The following result is well-known [24].

Proposition 2.2 *Let f be a real valued continuous function on $[0, \infty)$ with $f(0) \le 0$. Then f is operator convex if and only if the inequality*

$$f(D^*AD + E^*BE) \le D^*f(A)D + E^*f(B)E$$

holds for all $A, B \ge 0$ and for all $D, E \in B(H)$ with $D^*D + E^*E \le I$.

Using this observation, we have the following:

Proposition 2.3 *Let f be an operator convex function on $[0, \infty)$ with $f(0) \le 0$. Then the perspective function P_f is jointly convex on $B(H)_{++} \times B(H)_{++}$.*

Proof Let $(A_1, B_1), (A_2, B_2)$ be in $B(H)_{++} \times B(H)_{++}$ and let $\alpha \in [0, 1]$. Put $C := \alpha B_1 + (1-\alpha)B_2$, $D := (\alpha B_1)^{1/2}C^{-1/2}$ and $E := ((1-\alpha)B_2)^{1/2}C^{-1/2}$. Since $D^*D + E^*E = I$,

$$\begin{aligned}
&C^{-1/2}P_f(\alpha A_1 + (1-\alpha)A_2, \alpha B_1 + (1-\alpha)B_2)C^{-1/2} \\
&= f(C^{-1/2}(\alpha A_1 + (1-\alpha)A_2)C^{-1/2}) \\
&= f(C^{-1/2}(\alpha A_1)C^{-1/2} + C^{-1/2}((1-\alpha)A_2)C^{-1/2}) \\
&= f\left(D^*B_1^{-1/2}A_1B_1^{-1/2}D + E^*B_2^{-1/2}A_2B_2^{-1/2}E\right) \\
&\le D^*f(B_1^{-1/2}A_1B_1^{-1/2})D + E^*f(B_2^{-1/2}A_2B_2^{-1/2})E \\
&= \alpha C^{-1/2}P_f(A_1, B_1)C^{-1/2} + (1-\alpha)C^{-1/2}P_f(A_2, B_2)C^{-1/2}.
\end{aligned}$$

□

Let f be an operator convex function on $(0, \infty)$. For an arbitrary $\epsilon > 0$, $f_\epsilon(t)(:= f(t+\epsilon) - f(\epsilon))$ satisfies the assumption of the above result. So the following is obtained.

Corollary 2.2 *Let f be an operator convex function on* $(0, \infty)$. *Then the perspective function P_f is jointly convex on $B(H)_{++} \times B(H)_{++}$.*

Proof Let $(A_1, B_1), (A_2, B_2)$ be in $B(H)_{++} \times B(H)_{++}$ and let $\alpha \in [0, 1]$. Put $A := \alpha A_1 + (1-\alpha)A_2$ and $B := \alpha B_1 + (1-\alpha)B_2$, then we have

$$\begin{aligned}
&P_{f_\epsilon}(\alpha A_1 + (1-\alpha)A_2, \alpha B_1 + (1-\alpha)B_2)\\
&= B^{1/2} f(B^{-1/2}AB^{-1/2} + \epsilon I)B^{1/2} - f(\epsilon)B\\
&\leq \alpha P_{f_\epsilon}(A_1, B_1) + (1-\alpha)P_{f_\epsilon}(A_2, B_2)\\
&= \alpha B_1^{1/2} f(B_1^{-1/2}A_1B_1^{-1/2} + \epsilon I)B_1^{1/2} - \alpha f(\epsilon)B_1\\
&\quad + (1-\alpha)B_2^{1/2} f(B_2^{-1/2}A_2B_2^{-1/2} + \epsilon I)B_2^{1/2} - (1-\alpha)f(\epsilon)B_2.
\end{aligned}$$

Thus

$$\begin{aligned}
&B^{1/2} f(B^{-1/2}AB^{-1/2} + \epsilon I)B^{1/2}\\
&\leq \alpha B_1^{1/2} f(B_1^{-1/2}A_1B_1^{-1/2} + \epsilon I)B_1^{1/2}\\
&\quad + (1-\alpha)B_2^{1/2} f(B_2^{-1/2}A_2B_2^{-1/2} + \epsilon I)B_2^{1/2},
\end{aligned}$$

which implies the desired result as ϵ goes to 0. □

The following is a fundamental result in the study of operator perspectives that is an easy consequence of the above [13, 15].

Theorem 2.1 *Let f be a real valued continuous function on* $(0, \infty)$. *Then the operator perspective P_f of f is jointly convex* (*resp. jointly concave*) *on $B(H)_{++} \times B(H)_{++}$ if and only if f is operator convex* (*resp. operator concave*).

Corollary 2.3 *Let f be a real valued operator convex function on* $(0, \infty)$. *Then*

$$P_f(A_1 + A_2, B_1 + B_2) \leq P_f(A_1, B_1) + P_f(A_2, B_2),$$

for all $(A_i, B_i) \in B(H)_{++} \times B(H)_{++} \quad (i = 1, 2)$.

Remark 2.2 For a real valued continuous function f, we define $\tilde{f}$ by $\tilde{f}(t) := tf(1/t)$. If f is a polynomial function, the equation

$$P_{\tilde{f}}(A, B) = \tilde{f}(AB^{-1})B = f(BA^{-1})A = P_f(B, A)$$

holds. So the equation $P_{\tilde{f}}(A, B) = P_f(B, A)$ always holds for an arbitrary f.

We next consider the case if f is an operator convex function on $(0, \infty)$. Using the last theorem, the map

$$(A, B) \mapsto P_f(B, A)(= P_{\tilde{f}}(A, B))$$

is jointly convex, which signifies that $\tilde{f}$ is also an operator convex function.

2.3 Monotonicity and Convergence

In the following, OC, OC_0, and OM denote, respectively, all the (real valued) operator convex functions on $(0, \infty)$, all the functions in OC that take the value 0 at 1, and all the (real valued) operator monotone functions on $(0, \infty)$.

2.3.1 Monotonicity for One Direction

Let $f \in OC_0$ and $A, B > 0$. Corollary 2.3 implies that

$$\begin{aligned} P_f(A + tI, B + tI) &= P_f(A + t'I + (t - t')I, B + t'I + (t - t')I) \\ &\le P_f(A + t'I, B + t'I) + P_f((t - t')I, (t - t')I) \\ &= P_f(A + t'I, B + t'I) + 0 \cdot I \qquad (0 \le t' < t). \end{aligned}$$

This can be extended as follows:

Proposition 2.4 *Let $f \in OC_0$ and let $A, B \in B(H)_{++}$ and $X, X' \in B(H)_+$. If $X' \le X$, then $P_f(A + X, B + X) \le P_f(A + X', B + X')$.*

Proof Note that $P_f(X - X' + \epsilon I, X - X' + \epsilon I) = 0$ for all $\epsilon > 0$. Thus

$$\begin{aligned} &\langle P_f(A + X, B + X)x \mid x\rangle \\ &= \sup_{\epsilon} \langle P_f(A + X + \epsilon I, B + X + \epsilon I)x \mid x\rangle \\ &\le \sup_{\epsilon} \langle \big(P_f(A + X', B + X') + P_f(X - X' + \epsilon I, X - X' + \epsilon I)\big)x \mid x\rangle \\ &= \langle P_f(A + X', B + X')x \mid x\rangle \end{aligned}$$

hold for all $x \in H$. □

Corollary 2.4 *Let $\alpha, \beta > 0$ and let $f \in OC$ with $f(\alpha/\beta) = 0$. Then the map $X \ge 0 \mapsto P_f(A + \alpha X, B + \beta X)$ is decreasing.*

Example 2.1 Take either $f(t) = -\log t$ or $f(t) = t\log t$. From the proposition above, the sequence $P_f(A + (1/n)I, B + (1/n)I)$ is increasing for all $A, B \geq 0$. Moreover, if A, B are invertible, $P_f(A + (1/n)I, B + (1/n)I) \nearrow P_f(A, B)$.

2.3.2 Monotonicity for Each Variable

Let $f \in OC$ with $|f(0+)| < \infty$. Then $f_0(:= f - f(0+))$ is operator convex and $f_0(0+) = 0$. Thus there exists $h \in OM$ such that $f_0(t) = th(t)$ and $\tilde{f_0} = h(1/t)$ [1], which implies

$$P_f(B, A_1) \geq P_f(B, A_2) - f(0+)(A_2 - A_1) \quad (A_1 \leq A_2), \tag{1}$$

because that

$$\begin{aligned} P_{f_0}(B, A_2) &= B^{\frac{1}{2}} h(B^{\frac{1}{2}} A_2^{-1} B^{\frac{1}{2}}) B^{\frac{1}{2}} \\ &\leq B^{\frac{1}{2}} h(B^{\frac{1}{2}} A_1^{-1} B^{\frac{1}{2}}) B^{\frac{1}{2}} \\ &= P_{f_0}(B, A_1). \end{aligned}$$

For an arbitrary positive operator monotone function k, the operator convex function $-k$ satisfies the condition above, so, we have

$$\begin{aligned} P_k(B_1, A_1) &\leq P_k(B_2, A_1) \\ &\leq P_k(B_2, A_2) - k(0+)(A_2 - A_1) \\ &\leq P_k(B_2, A_2) \qquad (A_1 \leq A_2, B_1 \leq B_2). \end{aligned}$$

Proposition 2.5 *Let f be a positive continuous function on* $(0, \infty)$. *Then f is operator monotone if and only if P_f is monotone for each variable.*

Corollary 2.5 *Let a_n and b_n be (strictly) positive decreasing sequences and let $f \in OM$ be positive function and $A, B \geq 0$. Then, $P_f(A+a_nI, B+b_nI)$ converges strongly.*

We next consider the convergence of the sequence

$$P_f(A + \epsilon I, B + \epsilon I)$$

for $A, B \geq 0$.

From the above discussion, if an operator convex function f on $(0, \infty)$ satisfies $|f(0+)| < \infty$, then the function $-\tilde{f_0}$ is operator monotone. For an arbitrary $\alpha \in (0, 1)$, we put g and g_α by $g(t) := -\tilde{f_0}(t)$ and $g_\alpha(t) := g(t + \alpha) - g(\alpha)$. Then g_α is a non-negative operator monotone function. The perspective function P_{g_α} is calculated as follows:

$$\begin{aligned}P_{g_\alpha}(A, B) &= B^{1/2}g(B^{-1/2}AB^{-1/2}+\alpha)B^{1/2} - g(\alpha)B \\ &= P_g(A+\alpha B, B) - g(\alpha)B \\ &= -P_f(B, A+\alpha B) + f(0+)(A+\alpha B) - g(\alpha)B\end{aligned}$$

for $A, B > 0$. Using this, the condition for $P_f(A+\epsilon I, B+\epsilon I)$ to converge is obtained.

Proposition 2.6 ([32, Theorem 6.2]) *Let f be an operator convex function on $(0, \infty)$. Then the followings are equivalent:*

(1) *$P_f(A+\epsilon I, B+\epsilon I)$ converges strongly as $\epsilon \downarrow 0$ for every $A, B \geq 0$ such that $\alpha A \leq B$ for some $\alpha > 0$;*
(2) $|f(0+)| < \infty$.

Proof (1) $\Rightarrow$ (2). Taking $A = 0$, $B = I$, then $P_f(A+\epsilon I, B+\epsilon I) = (1+\epsilon)f\left(\frac{\epsilon}{1+\epsilon}\right)I$ converges to $f(0+)I$ as $\epsilon \downarrow 0$.

(2) $\Rightarrow$ (1). We use the notation g and g_α in the above argument and put $A_\epsilon := A+\epsilon I$. Since $\beta A \leq \alpha A \leq B$ hold for $0 < \beta \leq \alpha$, we can assume that $\alpha \in (0, 1)$. It follows from the above argument and Corollary 2.5 that

$$\begin{aligned}P_f(A_\epsilon, B_\epsilon) &= -P_{g_\alpha}(B_\epsilon - \alpha A_\epsilon, A_\epsilon) - g(\alpha)A_\epsilon + f(0+)B_\epsilon \\ &= -P_{g_\alpha}(B - \alpha A + (1-\alpha)\epsilon, A_\epsilon) - g(\alpha)A_\epsilon + f(0+)B_\epsilon\end{aligned}$$

converges strongly. □

Corollary 2.6 ([32, Corollary 6.4]) *Let f be an operator convex function on $(0, \infty)$. Then the followings are equivalent:*

(1) *$P_f(A+\epsilon I, B+\epsilon I)$ converges strongly as $\epsilon \downarrow 0$ for every $A, B \geq 0$ such that $\alpha B \leq A$ for some $\alpha > 0$;*
(2) $|\tilde{f}(0+)| < \infty$.

As we said above, the function g defined as $g(t)(:= -\tilde{f}_0(t))$ is operator monotone. If $|g(0+)| = |\tilde{f}(0+)| < \infty$, then $g - g(0+)$ is a non-negative operator monotone function. Thus

$$P_f(A_\epsilon, B_\epsilon) = -P_g(B_\epsilon, A_\epsilon) - g(0+)A_\epsilon + f(0+)B_\epsilon$$

converges strongly as $\epsilon \downarrow 0$.

In the next statement, we say that a binary operation σ on $B(H)_{++}$ is an operator connection if it can be described as $A\sigma B = P_g(B, A)$ for some non-negative function $g \in OM$.

Corollary 2.7 ([32, Proposition 6.1]) *Let f be an operator convex function on $(0,\infty)$. Then the followings are equivalent:*

(1) $P_f(A+\epsilon I, B+\epsilon I)$ *converges strongly as* $\epsilon \downarrow 0$ *for every* $A, B \geq 0$*;*
(2) $|f(0+)| < \infty$ *and* $|\tilde{f}(0+)| < \infty$*;*
(3) *There exist* $\lambda, \mu \in \mathbb{R}$ *and an operator connection* σ *such that*

$$P_f(A,B) = -A\sigma B + \lambda A + \mu B$$

for all $A, B > 0$.

Put $f(t) := \frac{-t}{t+1}$ for $t > 0$. Then f is an operator convex function with $f(0+) = \tilde{f}(0+) = 0$. So, from Corollary 2.7, the strong limit of $-P_f(A+\epsilon I, B+\epsilon I)$ as $\epsilon \downarrow 0$ exists for $A, B \geq 0$ and we denote this limit by $A : B$.

Remark 2.3 If A, B are invertible, $A : B$ can be written as $(A^{-1} + B^{-1})^{-1}$.

Remark 2.4 In Sect. 3, an operator connection is defined as a binary operation on $B(H)_+$.

Remark 2.5 For a function $f \in OC$ having (2) of the above result, we can define a continuous homogeneous function Φ on $[0,\infty)^2$ by

$$\Phi(x,y) = \begin{cases} yf(x/y), & \text{if } x, y \in (0,\infty) \\ 0, & \text{if } x \cdot y = 0. \end{cases}$$

Remark 2.6 It is obvious that the perspective function is continuous w.r.t. operator norm topology (i.e., if $\|A_n - A\| \to 0$, $\|B_n - B\| \to 0$, then $\|P_f(A_n,B_n) - P_f(A,B)\| \to 0$). Moreover, if f is operator convex, then, from [33, Theorem 6.1], the following property (upper continuity) holds:

$$A_n \searrow A > 0 \text{ and } B_n \searrow B > 0 \Rightarrow P_f(A_n,B_n) \to P_f(A,B)$$

in the strong operator topology.

The proof will be given in the next section.

3 An Extension of the Perspective Function

3.1 A Functional Calculus for Commuting Positive Operators

In [49], Pusz and Woronowicz defined a two-variable functional calculus $f(A,B)$ for $A, B \in B(H)_+$ and for a homogeneous Borel measurable locally bounded function f on $[0,\infty)^2$. In this section, we first introduce Pusz–Woronowicz functional calculus (PW-functional calculus, for short). Then we will show the properties of this functional calculus in the case when the function f and the pair (A, B) are restricted.

Let A, B be positive operators with $AB = BA$. Then $N(:= A + iB)$ is a normal operator. So there is an isometric *-isomorphism φ_N from the function space $C(\sigma(N))$ of continuous functions on $\sigma(N)$ onto the C*-algebra $C^*(N)$ generated by N (see [11, Corollary I.3.3]).

Note that $\sigma(N) \subset \sigma(A) + i\sigma(B)$ and $C(\sigma(A) \times \sigma(B)) \simeq C(\sigma(A) + i\sigma(B))$. So, for $\Phi \in C(\sigma(A) \times \sigma(B))$, we can define a functional calculus $\Phi(A, B)$ by

$$\Phi(A, B) := \varphi_N \left(\hat{\Phi}|_{\sigma(N)} \right),$$

where $\hat{\Phi}(z) := \Phi(Rez, Imz)$.

The map φ_N can be extended to a norm-decreasing unital *-homomorphism on the set $B_b(\sigma(N))$ of bounded Borel functions as

$$\varphi_N \ : \ B_b(\sigma(N)) \to B(H)$$

having the following property: if f_n is a bounded increasing sequence and $f = \sup f_n$, then $\varphi_N(f_n) \nearrow \varphi_N(f)$, (cf. [48, Theorem 4.5.4]). As stated in [27], the monotone class theorem guarantees such homomorphisms take the same value on $B_b(\sigma(N))$ and then, for all $\Phi \in B_b(\sigma(N))$, the bounded Borel functional calculus $h(A, B)$ is defined by

$$h(A, B) := \varphi_N \left(\Phi|_{\sigma(N)} \right).$$

Remark 3.1 We shall mainly treat the bounded Borel function $\Phi(= \Phi_f)$ on $[0, \infty)^2$ defined by using a real valued continuous function f on $(0, \infty)$ as

$$\Phi_f(x, y) = \begin{cases} yf(x/y), & \text{if } x, y \in (0, \infty) \\ 0, & \text{if } x \cdot y = 0. \end{cases}$$

It is easy to verify that Φ_f is homogeneous.

3.2 Pusz–Woronowicz Functional Calculus

3.2.1 The Commuting Pair (R, S)

Let $A, B \geq 0$. We note that the obvious relation $A \leq A + B$ holds. Using Dougla's range inclusion theorem [10, Theorem 17.1], there exists a bounded operator $C \ : \ H \to \ker(A + B)^{\perp}$ such that $A^{1/2} = (A + B)^{1/2}C = C^*(A + B)^{1/2}$, which implies

$$A = (A + B)^{1/2} C C^* (A + B)^{1/2}.$$

Thus we have the following.

Proposition 3.1 *Let* $A, B \geq 0$ *such that* $A + B \neq 0$. *Then there uniquely exists the pair* (R, S) *of positive operators on the Hilbert space* $\ker(A + B)^{\perp}$ *such that*

$$A = (A + B)^{1/2} R (A + B)^{1/2}, \quad B = (A + B)^{1/2} S (A + B)^{1/2}. \tag{2}$$

Proof It is enough to show the uniqueness. If (R, S) and (R', S') are such pairs, then

$$0 = A - A = (A + B)^{1/2} (R - R') (A + B)^{1/2}.$$

So, the operator $R - R'$ must be 0. The relation $S = S'$ can be proved in a similar fashion. □

The pair (R, S) in the above proposition satisfies

$$A + B = (A + B)^{1/2} (R + S) (A + B)^{1/2}.$$

So we have $R + S = I_{\ker(A+B)^{\perp}}$ and $RS = SR$.

If $A > 0$, then the positive operator R is invertible and is written as $R = (A + B)^{-1/2} A (A + B)^{-1/2}$. So, we have

$$SR^{-1} = (A + B)^{-1/2} B A^{-1} (A + B)^{1/2}$$

and

$$\begin{aligned} p(SR^{-1}) &= (A + B)^{-1/2} p(BA^{-1}) (A + B)^{1/2} \\ &= (A + B)^{-1/2} A^{1/2} p(A^{-1/2} B A^{-1/2}) A^{-1/2} (A + B)^{1/2} \end{aligned}$$

for all polynomial p. Thus, for every continuous function f on $[0, \infty)$,

$$f(SR^{-1}) = (A + B)^{-1/2} A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{-1/2} (A + B)^{1/2}$$

and

$$f(SR^{-1}) R = (A + B)^{-1/2} A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2} (A + B)^{-1/2}.$$

Proposition 3.2 *If* $A > 0$, *then for every continuous function* f *on* $[0, \infty)$,

$$(A + B)^{1/2} f(SR^{-1}) R (A + B)^{1/2} = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2}.$$

Corollary 3.1 *If $B > 0$, then for every continuous function f on $[0, \infty)$,*

$$(A+B)^{1/2} f(RS^{-1}) S (A+B)^{1/2} = B^{1/2} f(B^{-1/2} A B^{-1/2}) B^{1/2}.$$

Furthermore, if $A, B > 0$, then

$$(A+B)^{1/2} P_f(R, S)(A+B)^{1/2} = P_f(A, B)$$

for every continuous function f on $(0, \infty)$.

Example 3.1 Take $f(t) = \frac{t}{1+t}$. If either $A > 0$ or $B > 0$, then

$$(A+B)^{1/2} \frac{RS}{R+S} (A+B)^{1/2} = B - B(A+B)^{-1} B = (A : B).$$

Example 3.2 Take $f(t) = \log t$. If $A, B > 0$, then

$$(A+B)^{1/2} S(\log R - \log S)(A+B)^{1/2} = B^{1/2} \log(B^{-1/2} A B^{-1/2}) B^{1/2}.$$

Remark 3.2 Suppose $0 \le cA \le B$ for some $c > 0$. Then it is clear that $cR \le S$ and $cI = c(R+S) \le (c+1)S$ hold. Thus we can define a bounded self-adjoint operator

$$(A+B)^{1/2} f(RS^{-1}) S (A+B)^{1/2}$$

for every continuous function f on $[0, \infty)$. We shall discuss later the corresponding mapping F of (A, B), namely

$$F(A, B) = (A+B)^{1/2} f(RS^{-1}) S (A+B)^{1/2}.$$

3.2.2 Variational Expression

To define the Pusz–Woronowicz functional calculus, we show the following variational expression. Let $A, B \in B(H)_{++}$ with $AB = BA$ and let $z \in H$. If $x, y \in H$ satisfy $z = x + y$, then by a simple calculation,

$$\langle \frac{AB}{A+B} z \mid z \rangle + \langle (A+B)u \mid u \rangle = \langle Ax \mid x \rangle + \langle By \mid y \rangle$$

holds, where $u := \frac{B}{A+B} z - x$. So we have

$$\inf_{x+y=z} (\langle Ax \mid x \rangle + \langle By \mid y \rangle) = \langle \frac{AB}{A+B} z \mid z \rangle. \quad (3)$$

This can be generalized for any pair of positive operators.

Proposition 3.3 *Let $A, B \in B(H)_+$. For $z \in H$,*

$$\inf_{x+y=z} (\langle Ax \mid x\rangle + \langle By \mid y\rangle) = \langle (A : B)z \mid z\rangle.$$

Proof We first consider the case if A, B are invertible. Since $(A : B) = B - B(A + B)^{-1}B$,

$$\begin{aligned}
&\langle Ax \mid x\rangle + \langle B(z - x) \mid (z - x)\rangle - \langle (A : B)z \mid z\rangle \\
&= \|(A + B)^{-1/2}Bz\|^2 + \|(A + B)^{1/2}x\|^2 \\
&\qquad - 2Re\langle (A + B)^{-1/2}Bz \mid (A + B)^{1/2}x\rangle \\
&\geq 0.
\end{aligned}$$

The equality attains if $x = (A + B)^{-1}Bz$.

For $A, B \geq 0$ and for $x \in H$,

$$\begin{aligned}
\langle (A : B)z \mid z\rangle &= \inf_{\epsilon} \langle (A_\epsilon : B_\epsilon)z \mid z\rangle \\
&= \inf_{\epsilon} \inf_{x} (\langle A_\epsilon x \mid x\rangle + \langle B_\epsilon (z - x) \mid (z - x)\rangle) \\
&\geq \inf_{x} (\langle Ax \mid x\rangle + \langle B(z - x) \mid (z - x)\rangle) \\
&\geq \langle (A : B)z \mid z\rangle.
\end{aligned}$$

□

As stated in the last subsection, for every $(A, B) \neq (0, 0)$, there uniquely exists the commuting pair (R, S) of positive contractive operators such that (2) holds. This pair (R, S) satisfy the following:

Corollary 3.2 *Let $H_0 = ran(A + B)^{1/2}$. For $z \in H_0$,*

$$\inf_{x,y \in H_0,\ x+y=z} (\langle Rx \mid x\rangle + \langle Sy \mid y\rangle) = \langle (R : S)z \mid z\rangle.$$

Proof From the preceding proposition, for every $\epsilon \in (0, 1)$, there exists $x_\epsilon, y_\epsilon \in \overline{H_0}$ such that $x_\epsilon + y_\epsilon = z$ and

$$\langle Rx_\epsilon \mid x_\epsilon\rangle + \langle Sy_\epsilon \mid y_\epsilon\rangle \ < \ \langle (R : S)z \mid z\rangle + \epsilon.$$

Let $\delta := \langle (R : S)z \mid z\rangle + \epsilon - (\langle Rx_\epsilon \mid x_\epsilon\rangle + \langle Sy_\epsilon \mid y_\epsilon\rangle)$ and let $K > \epsilon$ be a positive number such that

$$(2\epsilon/\delta)\,(2 \max\{\|x_\epsilon\|, \|y_\epsilon\|\} + 1) < K.$$

Then there exists $x'_\epsilon \in H_0$ such that $\|x_\epsilon - x'_\epsilon\| < \epsilon/K$ and so,

$$\begin{aligned}
&\langle Rx'_\epsilon \mid x'_\epsilon\rangle \\
&= \langle R(x'_\epsilon - x_\epsilon) \mid x'_\epsilon\rangle + \langle Rx_\epsilon \mid x'_\epsilon - x_\epsilon\rangle + \langle Rx_\epsilon \mid x_\epsilon\rangle \\
&\le (\epsilon/K)\|R\|(\|x_\epsilon\| + \|x'_\epsilon\|) + \langle Rx_\epsilon \mid x_\epsilon\rangle \\
&\le (\epsilon/K)(2\|x_\epsilon\| + \epsilon/K) + \langle Rx_\epsilon \mid x_\epsilon\rangle \\
&< \delta/2 + \langle Rx_\epsilon \mid x_\epsilon\rangle.
\end{aligned}$$

We next put $y'_\epsilon := z - x'_\epsilon$. Then $\|y_\epsilon - y'_\epsilon\| = \|y_\epsilon - (z - x'_\epsilon)\| = \|-x_\epsilon + x'_\epsilon\| < \epsilon/K$. The similar argument implies $\langle Sy'_\epsilon \mid y'_\epsilon\rangle \le \delta/2 + \langle Sy_\epsilon \mid y_\epsilon\rangle$. Thus we have

$$\begin{aligned}
\langle (R:S)z \mid z\rangle &\le \langle Rx'_\epsilon \mid x'_\epsilon\rangle + \langle Sy'_\epsilon \mid y'_\epsilon\rangle \\
&< \langle Rx_\epsilon \mid x_\epsilon\rangle + \langle Sy_\epsilon \mid y_\epsilon\rangle + \delta \\
&= \langle (R:S)z \mid z\rangle + \epsilon.
\end{aligned}$$

□

3.2.3 Pusz–Woronowicz Functional Calculus

Proposition 3.4 *For $A, B \in B(H)_+$,*

$$(A:B) = (A+B)^{1/2}\frac{RS}{R+S}(A+B)^{1/2}.$$

Proof Let $z \in H$. Put $w := (A+B)^{1/2}z$, then

$$\begin{aligned}
&\langle (A:B)z \mid z\rangle \\
&= \inf_{x+y=z} (\langle Ax \mid x\rangle + \langle By \mid y\rangle) \\
&= \inf_{x+y=z} \Big(\langle (A+B)^{1/2}R(A+B)^{1/2}x \mid x\rangle + \langle (A+B)^{1/2}S(A+B)^{1/2}y \mid y\rangle\Big) \\
&= \inf_{x,y\in ran(A+B)^{1/2},\ \ x+y=w} (\langle Rx \mid x\rangle + \langle Sy \mid y\rangle) \\
&= \inf_{x,y\in \overline{ran}(A+B)^{1/2},\ \ x+y=w} (\langle Rx \mid x\rangle + \langle Sy \mid y\rangle) \\
&= \langle \frac{RS}{R+S}w \mid w\rangle \\
&= \langle (A+B)^{1/2}\frac{RS}{R+S}(A+B)^{1/2}z \mid z\rangle.
\end{aligned}$$

□

From this, we have

$$A - (A : B) = (A + B)^{1/2} \left(R - \frac{RS}{R + S} \right) (A + B)^{1/2} \tag{4}$$

$$= (A + B)^{1/2} R^2 (A + B)^{1/2}. \tag{5}$$

Similarly, put

$$e_0 := R + S (= I), \quad e_1 := R, \quad e_{k+1} := e_k - \frac{e_k(e_{k-1} - e_k)}{e_k + (e_{k-1} - e_k)} \quad (k = 1, 2, \ldots).$$

By a simple calculation, $e_k = R^k$, and so, for any polynomial p,

$$(A + B)^{1/2} p(R) (A + B)^{1/2}$$

can be written as a two-variable function of (A, B).

Definition 3.3 Let Φ be a real valued homogeneous continuous function on $[0, \infty)^2$, i.e., $\Phi(\lambda r, \lambda s) = \lambda \Phi(r, s)$ for all $\lambda, r, s \geq 0$. The two-variable map

$$(A, B) \in B(H)_+ \times B(H)_+ \mapsto (A + B)^{1/2} \Phi(R, S) (A + B)^{1/2} \in B(H)_{sa}$$

is said to be the Pusz–Woronowicz functional calculus (PW-functional calculus, for short) associated with Φ. We write

$$\Phi(A, B) = (A + B)^{1/2} \Phi(R, S) (A + B)^{1/2}.$$

Put $f_\Phi(t) := \Phi(t, 1)$ and $\tilde{f}_\Phi(t) := \Phi(1, t)$. It is obvious that $\tilde{f}_\Phi(t) = \Phi(1, t) = t\Phi(1/t, 1) = t f_\Phi(1/t)$ for $t > 0$. From the continuity of Φ, f_Φ and $\tilde{f}_\Phi(t)$ are continuous function on $(0, \infty)$ and

$$|f_\Phi(0+)| < \infty, \quad |\tilde{f}_\Phi(0+)| < \infty.$$

Conversely, if a continuous function f on $(0, \infty)$ has the properties

$$|f(0+)| < \infty, \quad |\tilde{f}(0+)| \left(= \lim_{t \searrow 0} |t f(1/t)| \right) < \infty,$$

then a two variable function Φ_f defined by

$$\Phi_f(r, s) := \begin{cases} s f(r/s), & \text{if } r, s \in (0, \infty) \\ 0, & \text{if } r \cdot s = 0 \end{cases}$$

is a continuous function on $[0, \infty)^2$. As a conclusion, Φ_f is continuous function on $[0, \infty)^2$ if and only if

$$|f_\Phi(0+)| < \infty \text{ and } |\tilde{f}_\Phi(0+)| < \infty. \tag{6}$$

Let A, B are invertible positive operators. Then R, S are also invertible, and so, from Corollary 3.1,

$$\begin{aligned}\Phi_f(A, B) &= (A+B)^{1/2}\Phi_f(R, S)(A+B)^{1/2}\\ &= (A+B)^{1/2} S\, f(R/S)(A+B)^{1/2}\\ &= (A+B)^{1/2} P_f(R, S)(A+B)^{1/2}\\ &= P_f(A, B).\end{aligned}$$

Proposition 3.5 *Let f be a continuous function on $(0, \infty)$ with (6). If $A, B > 0$, then $\Phi_f(A, B) = P_f(A, B)$.*

In the following, we show some properties of PW-functional calculus.

3.2.4 Homogeneity, Upper Continuity and Convexity

Proposition 3.6 *Let H and K be Hilbert spaces and $A, B \in B(H)_+$. Let Φ be a real valued homogeneous continuous function and $C : K \to H$ be a bounded operator. If $\overline{ran}(A+B) \subset \overline{ran}(C)$, then*

$$\Phi(C^*AC, C^*BC) = C^*\Phi(A, B)C.$$

Proof The operator $(A+B)^{1/2}C$ has the polar decomposition as follows:

$$(A+B)^{1/2}C = U(C^*(A+B)C)^{1/2},$$

where U is a bounded operator from K into H with

$$U^*U = P_{\overline{ran}|(A+B)^{1/2}C|}$$

and

$$UU^* = P_{\overline{ran}((A+B)^{1/2}C)} = P_{\overline{ran}(A+B)^{1/2}}.$$

The last equality comes from the assumption. Note that by Proposition 3.1 there exists a commuting pair (R, S) of positive contractive operators on $\ker(A+B)^\perp$ such that

$$A = (A+B)^{1/2}R(A+B)^{1/2},\ \ B = (A+B)^{1/2}S(A+B)^{1/2}.$$

So, we have

$$\begin{aligned} C^*AC &= C^*(A+B)^{1/2}R(A+B)^{1/2}C \\ &= (C^*AC+C^*BC)^{1/2}U^*RU(C^*AC+C^*BC)^{1/2} \end{aligned}$$

and similarly

$$C^*BC = (C^*AC+C^*BC)^{1/2}U^*SU(C^*AC+C^*BC)^{1/2}.$$

Thus

$$\begin{aligned} \Phi(C^*AC, C^*BC) &= (C^*AC+C^*BC)^{1/2}\Phi(U^*RU, U^*SU)(C^*AC+C^*BC)^{1/2} \\ &= (C^*AC+C^*BC)^{1/2}U^*\Phi(R,S)U(C^*AC+C^*BC)^{1/2} \\ &= C^*(A+B)^{1/2}\Phi(R,S)(A+B)^{1/2}C = C^*\Phi(A,B)C. \end{aligned}$$

□

The next theorem plays an important role in the study of PW-functional calculus.

Theorem 3.1 ([33, Theorem 6.1]) *Let* Φ *be a real valued homogeneous continuous function on* $[0,\infty)^2$ *and let* A_n, B_n, A, B *are in* $B(H)_+$. *If* $A_n \searrow A$ *and* $B_n \searrow B$, *then* $\Phi(A_n, B_n)$ *strongly converges to* $\Phi(A, B)$.

We need some lemmas. Here, (R_n, S_n) is the pair of positive contractive operators on $\overline{ran}(A+B)^{1/2}$ corresponding to (A_n, B_n), namely,

$$A_n = (A_n+B_n)^{1/2}R_n(A_n+B_n)^{1/2},\ B_n = (A_n+B_n)^{1/2}S_n(A_n+B_n)^{1/2}.$$

Lemma 3.1 $\langle R_n\xi \mid \eta\rangle \to \langle R\xi \mid \eta\rangle$ *for* $\xi, \eta \in \overline{ran}(A+B)^{1/2}$.

Proof Put $T_n := (A_n+B_n)^{1/2}$ and $T := (A+B)^{1/2}$. Since ξ, η ($\in \overline{ran}\ T$) can be approximated by some elements in $ran\ T$, it is enough to show the case when $\xi, \eta \in ran\ T$.

From the assumption, it is clear that

$$A_n = T_nR_nT_n, \quad A = TRT, \quad T_n \searrow T$$

and for every $x, y \in H$,

$$\begin{aligned} &\langle R_nTx \mid Ty\rangle \\ &= \langle R_n(T-T_n)x \mid Ty\rangle + \langle R_nT_nx \mid (T-T_n)y\rangle + \langle R_nT_nx \mid T_ny\rangle \end{aligned}$$

holds. Using the fact that R_n and R are contractions,

$$
\begin{aligned}
&|\langle R_n T x \mid T y\rangle - \langle R T x \mid T y\rangle| \\
&\leq \|(T - T_n)x\|\|Ty\| + \|T_n x\|\|(T - T_n)y\| + \|(T_n R_n T_n - TRT)x\|\|y\| \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}
$$

□

Using the fact that $T_n R_n^2 T_n = A_n - (A_n : B_n)$ strongly converges to $A - (A : B) = TR^2T$ (see (5)), we have the following.

Lemma 3.2 $\langle R_n^2 \xi \mid \eta\rangle \to \langle R^2 \xi \mid \eta\rangle$ *for* $\xi, \eta \in \overline{ran}(A + B)^{1/2}$.

Proof It is enough to show the case if $\xi, \eta \in ran(A + B)^{1/2}$. By the similar calculation in the above proof, we have

$$
\begin{aligned}
&|\langle R_n^2 T x \mid T y\rangle - \langle R^2 T x \mid T y\rangle| \\
&\leq \|(T - T_n)x\|\|Ty\| + \|T_n x\|\|(T - T_n)y\| \\
&\qquad + \|(T_n R_n^2 T_n - TR^2T)x\|\|y\| \\
&= \|(T - T_n)x\|\|Ty\| + \|T_n x\|\|(T - T_n)y\| \\
&\qquad + \|(A_n - (A_n : B_n) - (A - (A : B)))x\|\|y\|.
\end{aligned}
$$

Thus $|\langle R_n^2 T x \mid T y\rangle - \langle R^2 T x \mid T y\rangle|$ tends to 0 as $n \to \infty$. □

Lemma 3.3 *For* $k \in \mathbb{N}$, $R_n^k \xi \to R^k \xi$ *for all* $\xi \in \overline{ran}(A + B)^{1/2}$.

Proof Let us prove this by induction on k. We first show the case if $k = 1$. By using the above two lemmas, for $\xi \in \overline{ran}(A + B)^{1/2}$,

$$
\|R_n \xi - R\xi\|^2 = \langle R_n^2 \xi \mid \xi\rangle - 2Re\langle R_n \xi \mid R\xi\rangle + \langle R^2 \xi \mid \xi\rangle
$$

tends to 0 as $n \to \infty$.

Assume the statement holds for some k. Then for $\xi \in \overline{ran}(A + B)^{1/2}$,

$$
\begin{aligned}
\|(R_n^{k+1} - R^{k+1})\xi\| &\leq \|(R_n^{k+1} - R_n R^k)\xi\| + \|(R_n - R)R^k \xi\| \\
&\leq \|(R_n^k - R^k)\xi\| + \|(R_n - R)R^k \xi\| \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

□

Proof of Theorem 3.1 We use the notations T_n and T defined in the proof of Lemma 3.1. Put $f(t) := \Phi(t, 1 - t)$. Then, f is a continuous function on [0, 1] and

$$
\Phi(A_n, B_n) = T_n f(R_n) T_n, \quad \Phi(A, B) = T f(R) T.
$$

Since f is uniformly approximated by a polynomial, the above lemma implies

$$f(R_n)\xi \to f(R)\xi \quad \text{for all } \xi \in \overline{ran}(A+B)^{1/2}.$$

Using the property $T_n \searrow T$,

$$\begin{aligned}
&\|\Phi(A_n, B_n)\xi - \Phi(A, B)\xi\| \\
&= \|T_n f(R_n) T_n \xi - T f(R) T \xi\| \\
&\leq \|T_n f(R_n)(T_n - T)\xi\| + \|T_n(f(R_n) - f(R))T\xi\| + \|(T_n - T) f(R) T\xi\| \\
&\leq \|T_1\| \|f\|_\infty \|(T_n - T)\xi\| + \|T_1\| \|(f(R_n) - f(R))T\xi\| + \|(T_n - T) f(R) T\xi\|
\end{aligned}$$

converges to 0 as $n \to \infty$, where $\|f\|_\infty := \max\{|f(t)| : t \in [0, 1]\}$. □

In the following, we denote the operator $A + \epsilon I$ by A_ϵ.

Corollary 3.3 *Let f be a real valued continuous function on $(0, \infty)$. Then the followings are equivalent:*

(1) *The strong limit of $P_f(A_\epsilon, B_\epsilon)$ exists for all $(A, B) \in B(H)_+ \times B(H)_+$;*
(2) *f satisfies (6).*

Proof (2) $\Rightarrow$ (1). Immediate from Theorem 3.1.

(1) $\Rightarrow$ (2). Take $(A, B) = (I, 0)$ (resp. $(A, B) = (0, I)$), then $P_f(A_\epsilon, B_\epsilon)$ converges to $\tilde{f}(0+) \cdot I$ (resp. $f(0+) \cdot I$) as $\epsilon \searrow 0$. □

Remark 3.3 The statement in Corollary 3.3 is similar to Corollary 2.7 except that the function f is operator convex.

Proposition 3.7 *Let f be a real valued continuous function on $(0, \infty)$ with (6). Then the map $(A, B) \mapsto \Phi_f(A, B)$ is jointly convex if and only if f is operator convex on $(0, \infty)$.*

Proof It is enough to show the "if" part. Assume f is operator convex on $(0, \infty)$ with (6). We fix positive operators A_i, B_i $(i = 1, 2)$ and $x \in H$. Put

$$\begin{aligned}
x_\alpha :=& \alpha \Phi_f(A_1, B_1)x + (1-\alpha)\Phi(A_2, B_2)x \\
&- \Phi_f(\alpha A_1 + (1-\alpha)B_1, \alpha A_2 + (1-\alpha)B_2)x
\end{aligned}$$

and

$$\begin{aligned}
x_\alpha(\epsilon) :=& \alpha \Phi_f(A_{1\epsilon}, B_{1\epsilon})x + (1-\alpha)\Phi(A_{2\epsilon}, B_{2\epsilon})x \\
&- \Phi_f(\alpha A_{1\epsilon} + (1-\alpha)B_{1\epsilon}, \alpha A_{2\epsilon} + (1-\alpha)B_{2\epsilon})x.
\end{aligned}$$

By using Proposition 3.5 and Theorem 2.1, we have $\langle x_\alpha(\epsilon) \mid x\rangle \geq 0$ for all $\alpha \in [0, 1]$. Thanks to Theorem 3.1, this sequence converges to $\langle x_\alpha \mid x\rangle \geq 0$ as $\epsilon \searrow 0$.

□

From Theorem 2.1, it is clear that the former properties of an operator perspective are inherited by PW-functional calculus.

Corollary 3.4 *Let f be a positive valued continuous function on $(0,\infty)$ with (6). Then the map $(A,B)\mapsto\Phi_f(A,B)$ is jointly concave if and only if f is operator monotone on $(0,\infty)$.*

Corollary 3.5 *Let f be a continuous function on $(0,\infty)$ with (6). Then $\Phi_f(A,B)=\Phi_{\tilde{f}}(B,A)$ holds for all $(A,B)\in B(H)_+\times B(H)_+$.*

Corollary 3.6 *Let f, g be continuous functions on $(0,\infty)$ with (6). If $f\leq g$, then $\Phi_f(A,B)\leq\Phi_g(A,B)$ holds for all $(A,B)\in B(H)_+\times B(H)_+$.*

Corollary 3.7 *Let $\alpha,\beta>0$ and let $f\in OC$ with (6). If $f(\alpha/\beta)=0$, then the map $X\geq 0\mapsto\Phi_f(A+\alpha X, B+\beta X)$ is decreasing.*

We denote by OM_+ the set of all positive (non-negative) operator monotone functions on $(0,\infty)$. If $f\in OM_+$, then $\tilde{f}$ is also in OM_+, so f satisfies (6).

Corollary 3.8 *Let $f\in OM_+$. If $0\leq A\leq C,\ 0\leq B\leq D$, then $0\leq\Phi_f(A,B)\leq\Phi_f(C,D)$.*

Proof Let $\epsilon>0$. The facts $0<A_\epsilon\leq C_\epsilon,\ 0<B_\epsilon\leq D_\epsilon$ and Proposition 2.5 imply the following :

$$0\leq P_f(A_\epsilon,B_\epsilon)=\Phi_f(A_\epsilon,B_\epsilon),$$

and

$$0\leq\Phi_f(C_\epsilon,D_\epsilon)-\Phi_f(A_\epsilon,B_\epsilon),$$

which implies the desired result by Theorem 3.1. □

Corollary 3.9 *Let $f\in OM_+$. Then, for $A,B,C\geq 0$,*

$$C\Phi_f(A,B)C\leq\Phi_f(CAC,CBC)$$

holds.

Proof For $\epsilon_1,\epsilon_2,\epsilon_3>0$,

$$\begin{aligned}C_{\epsilon_3}\Phi_f(A,B)C_{\epsilon_3}&\leq C_{\epsilon_3}\Phi_f(A_{\epsilon_1},B_{\epsilon_1})C_{\epsilon_3}\ \text{(Corollary 3.7)}\\&=\Phi_f(C_{\epsilon_3}A_{\epsilon_1}C_{\epsilon_3},C_{\epsilon_3}B_{\epsilon_1}C_{\epsilon_3})\ \text{(Proposition 3.6)}\\&\leq\Phi_f(C_{\epsilon_3}A_{\epsilon_1}C_{\epsilon_3}+\epsilon_2 I,C_{\epsilon_3}B_{\epsilon_1}C_{\epsilon_3}+\epsilon_2 I)\\&=P_f(C_{\epsilon_3}A_{\epsilon_1}C_{\epsilon_3}+\epsilon_2 I,C_{\epsilon_3}B_{\epsilon_1}C_{\epsilon_3}+\epsilon_2 I)\ \text{(Proposition 3.5)}\end{aligned}$$

hold. So, we have

$$\langle C_{\epsilon_3}\Phi_f(A,B)C_{\epsilon_3}\xi\mid\xi\rangle\leq\langle P_f(C_{\epsilon_3}A_{\epsilon_1}C_{\epsilon_3}+\epsilon_2 I,C_{\epsilon_3}B_{\epsilon_1}C_{\epsilon_3}+\epsilon_2 I)\xi\mid\xi\rangle,$$

for all $\xi \in H$. By letting $\epsilon_1 \searrow 0$,

$$\langle C_{\epsilon_3}\Phi_f(A, B)C_{\epsilon_3}\xi \mid \xi\rangle \le \langle P_f(C_{\epsilon_3}AC_{\epsilon_3} + \epsilon_2 I, C_{\epsilon_3}BC_{\epsilon_3} + \epsilon_2 I)\xi \mid \xi\rangle,$$

and then letting $\epsilon_3 \searrow 0$, we have

$$\langle C\Phi_f(A, B)C\xi \mid \xi\rangle \le \langle P_f(CAC + \epsilon_2 I, CBC + \epsilon_2 I)\xi \mid \xi\rangle,$$

which implies the desired result. □

3.2.5 Restricted Domain

We denote by $(B(H)_+ \times B(H)_+)_\le$ (resp. $(B(H)_+ \times B(H)_+)_\ge$) the set of all pairs of positive operators (A, B) such that $cA \le B$ (resp. $A \ge cB$) for some $c > 0$. These domains have appeared several times in this chapter. We show some properties of PW-functional calculus of a pair of positive operators in these domain.

Let (A, B) be in $B(H)_+ \times B(H)_+$ and let (R, S) be a pair of positive operators that correspond to (A, B).

Lemma 3.4 *Let $c > 0$ be a positive number. Then the followings are equivalent:*

(1) $cA \le B$ (*resp.* $cB \le A$) *;*
(2) $cR \le S$ (*resp.* $cS \le R$) *;*
(3) $\frac{c}{c+1}I \le S$ (*resp.* $\frac{c}{c+1}I \le R$).

Proof Immediate (see Remark 3.2). □

By using this lemma, if the pair (A, B) is in $(B(H)_+ \times B(H)_+)_\le$, then PW-functional calculus $\Phi(A, B)$ is written as follows:

Proposition 3.8 *Let f be a continuous function on* $(0, \infty)$ *with* (6)*. Then*

$$\Phi_f(A, B) = (A + B)^{1/2} f(RS^{-1})S(A + B)^{1/2}.$$

We next treat the domain $(B(H)_+ \times B(H)_+)_\ge$. Let f be a real valued continuous function on $(0, \infty)$ with $|\tilde{f}(0+)| < \infty$ and let $\alpha, c > 0$. Assume that positive operators A, B satisfy $A \ge cB$. Then, from the above lemma, we have $R \ge cS$ and $R > 0$, which implies $0 \le \frac{S}{R} \le (1/c)I$.

The functions $\tilde{f}$ and $\tilde{f}_\alpha$ extend to continuous functions on $[0, 1/c]$, where $\tilde{f}(0) := \tilde{f}(0+)$ and $\tilde{f}_\alpha(0) := \tilde{f}_\alpha(0+) \left(= \tilde{f}(0+)\right)$. Here, we define a two variable function $\varphi(\alpha, t)$ on $[0, 1] \times [0, 1/c]$ by

$$\varphi(\alpha, t) := \begin{cases} \tilde{f}_\alpha(t) \left(= tf\left(\frac{1}{t} + \alpha\right)\right), & \text{if } \alpha \in [0, 1], t \in (0, 1/c], \\ \tilde{f}_\alpha(0+), & \text{if } \alpha \in [0, 1], t = 0. \end{cases}$$

This function is continuous on the compact set $[0, 1] \times [0, 1/c]$. So, that is uniformly continuous, i.e., for every $\epsilon > 0$, there exists $\delta > 0$ such that if $d((\alpha, t), (\alpha', t')) = |(\alpha - \alpha')^2 + (t - t')^2|^{1/2} < \delta$, then $|\varphi(\alpha, t) - \varphi(\alpha', t')| < \epsilon$. If $\alpha (= d((\alpha, t), (0, t))) < \delta$, then

$$|\tilde{f}_\alpha(t) - \tilde{f}(t)| = |\varphi(\alpha, t) - \varphi(0, t)| < \epsilon \quad (t \in [0, 1/c]),$$

which signifies the sequence $\tilde{f}_\alpha$ uniformly converges to $\tilde{f}$ as $\alpha \searrow 0$.

Put $X := (A + B)^{1/2} \tilde{f}(S/R) R (A + B)^{1/2}$, then

$$\begin{aligned}
&\|X - \Phi_{f_\alpha}(A, B)\| \\
&= \|X - \Phi_{\tilde{f}_\alpha}(B, A)\| \\
&\le \|(A + B)^{1/2}\| \|\tilde{f}(S/R) - \tilde{f}_\alpha(S/R)\| \|R(A + B)^{1/2}\| \\
&\le \|\tilde{f} - \tilde{f}_\alpha\|_\infty \|A + B\|,
\end{aligned}$$

where $\|\tilde{f} - \tilde{f}_\alpha\|_\infty := \max\{|\tilde{f}(t) - \tilde{f}_\alpha(t)| \ : \ t \in [0, 1/c]\}$.

Proposition 3.9 *Let f be a continuous function on $(0, \infty)$ with $|\tilde{f}(0+)| < \infty$ and let $(A, B) \in (B(H)_+ \times B(H)_+)_\ge$. Then the sequence $\Phi_{f_\alpha}(A, B)$ converges to $(A + B)^{1/2} \tilde{f}(S/R) R (A + B)^{1/2}$ in the operator norm topology.*

Corollary 3.10 *Let f be a continuous function on $(0, \infty)$ with $|\tilde{f}(0+)| < \infty$ and let $(A, B) \in (B(H)_+ \times B(H)_+)_\ge$. Then $P_f(A_\epsilon, B_\epsilon)$ strongly converges to $(A + B)^{1/2} \tilde{f}(S/R) R (A + B)^{1/2}$ as $\epsilon \searrow 0$.*

Proof Assume that $c \in (0, 1)$, $\alpha > 0$ and $A \ge cB$. Then, for every $\epsilon > 0$, we have $A_\epsilon \ge cB_\epsilon$, which implies

$$0 \le \frac{S(\epsilon)}{R(\epsilon)} \le (1/c)I$$

holds, where $(R(\epsilon), S(\epsilon))$ is the pair of positive operators corresponding to (A_ϵ, B_ϵ).

Put $X := (A + B)^{1/2} \tilde{f}(S/R) R (A + B)^{1/2}$. Using the argument before the above proposition,

$$\|X - \Phi_{f_\alpha}(A, B)\| \le \|\tilde{f} - \tilde{f}_\alpha\|_\infty \|A + B\|,$$

$$\begin{aligned}
\|P_{f_\alpha}(A_\epsilon, B_\epsilon) - P_f(A_\epsilon, B_\epsilon)\| &\le \|\tilde{f} - \tilde{f}_\alpha\|_\infty \|A_\epsilon + B_\epsilon\| \\
&\le \|\tilde{f} - \tilde{f}_\alpha\|_\infty (\|A + B\| + 2) \quad (1 > \epsilon > 0).
\end{aligned}$$

So, for every $\xi \in H$ and for every $\epsilon \in (0, 1)$,

$$\begin{aligned}\|X\xi - P_f(A_\epsilon, B_\epsilon)\xi\| &\le \|X\xi - \Phi_{f_\alpha}(A, B)\xi\| + \|\Phi_{f_\alpha}(A, B)\xi - P_{f_\alpha}(A_\epsilon, B_\epsilon)\xi\| \\ &\quad + \|P_{f_\alpha}(A_\epsilon, B_\epsilon)\xi - P_f(A_\epsilon, B_\epsilon)\xi\| \\ &\le \|\tilde{f} - \tilde{f}_\alpha\|_\infty (2\|A + B\| + 2)\|\xi\| \\ &\quad + \|\Phi_{f_\alpha}(A, B)\xi - P_{f_\alpha}(A_\epsilon, B_\epsilon)\xi\|,\end{aligned}$$

where $\|\tilde{f} - \tilde{f}_\alpha\|_\infty := \max\{|\tilde{f}(t) - \tilde{f}_\alpha(t)| \ : \ t \in [0, 1/c]\}$. Thus the desired result is obtained by Theorem 3.1 and Proposition 3.9. □

Corollary 3.11 *Let f be a continuous function on $(0, \infty)$. Then the followings are equivalent:*

(1) *$P_f(A_\epsilon, B_\epsilon)$ converges strongly as $\epsilon \searrow 0$ for all $(A, B) \in (B(H)_+ \times B(H)_+)_\ge$;*
(2) $|\tilde{f}(0+)| < \infty$.

Proof (2)⇒(1). Immediate from the above proposition. (1)⇒(2). Take $(A, B) = (I, 0)$. □

Corollary 3.12 *Let f be a continuous function on $(0, \infty)$. Then the followings are equivalent:*

(1) *$P_f(A_\epsilon, B_\epsilon)$ converges strongly as $\epsilon \searrow 0$ for all $(A, B) \in (B(H)_+ \times B(H)_+)_\le$;*
(2) $|f(0+)| < \infty$.

Example 3.3 Take $f(t) = t^\alpha$ $(\alpha \in \mathbb{R})$. Then the followings are equivalent:

(1) $P_f(A_\epsilon, B_\epsilon)$ converges strongly as $\epsilon \searrow 0$ for all $(A, B) \in (B(H)_+ \times B(H)_+)_\le$ (resp. for all $(A, B) \in (B(H)_+ \times B(H)_+)_\ge$) ;
(2) $\alpha \ge 0$ (resp. $\alpha \le 1$).

Remark 3.4 When the Hilbert space H is finite dimensional, $(A, B) \in (B(H)_+ \times B(H)_+)_\ge$ if and only if $P_{(\ker A)^\perp} \ge P_{(\ker B)^\perp}$. Since the "only if" part is obvious, we show the "if" part. The positive operators A, B have the following spectral decomposition:

$$A = \sum_{i=1}^{n} \alpha_i P_i \quad (\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_n > 0),$$

$$B = \sum_{j=1}^{m} \beta_j Q_j \quad (\beta_1 \ge \beta_2 \ge \cdots \ge \beta_m > 0).$$

So if $P_{(\ker A)^\perp} \ge P_{(\ker B)^\perp}$, then

$$A \ge \alpha_n \sum_{i=1}^{n} P_i \ge \alpha_n \sum_{j=1}^{m} Q_j = \left(\frac{\alpha_n}{\beta_1}\right)\beta_1 \sum_{j=1}^{m} Q_j \ge \left(\frac{\alpha_n}{\beta_1}\right) \sum_{j=1}^{m} \beta_j Q_j = \left(\frac{\alpha_n}{\beta_1}\right) B.$$

4 Theory of Operator Means

4.1 Kubo-Ando's Axiomatization

When a function f is in OM_+, f satisfies (6) and $\Phi_{\tilde{f}}(A, B)$ is well-defined for all $(A, B) \in B(H)_+ \times B(H)_+$. As stated before, the binary operation $(A, B) \mapsto A\sigma B (:= \Phi_{\tilde{f}}(A, B))$ satisfies the following statements :

(i) $A \le C, B \le D \Rightarrow A\sigma B \le C\sigma D$,
(ii) $C(A\sigma B)C \le (CAC)\sigma(CBC)$ for all $C \ge 0$,
(iii) $A_n \searrow A \ge 0, B_n \searrow B \ge 0 \Rightarrow A_n\sigma B_n \searrow A\sigma B$.

We call such a binary operation σ an operator connection and denote the set of operator connections by Σ. In [39], Kubo and Ando show that the above three statements characterize the class $\{\Phi_{\tilde{f}} \mid f \in OM_+\}$.

Theorem 4.1 *For every $\sigma \in \Sigma$, there uniquely exists $f_\sigma \in OM_+$ such that $A\sigma B = \Phi_{\tilde{f}_\sigma}(A, B)$ for all $A, B \in B(H)_+$. The map $\sigma \mapsto f_\sigma$ is an affine order isomorphism from Σ onto OM_+.*

Lemma 4.1 *For $C \in B(H)_{++}$, $C(A\sigma B)C = (CAC)\sigma(CBC)$.*

Proof From the statement (ii), we have

$$\begin{aligned} A\sigma B &= (C^{-1}CACC^{-1})\sigma(C^{-1}CBCC^{-1}) \\ &\ge C^{-1}\left((CAC)\sigma(CBC)\right)C^{-1} \\ &\ge C^{-1}C(A\sigma B)CC^{-1} = A\sigma B. \end{aligned}$$

□

Lemma 4.2 *Let $\sigma \in \Sigma$ and $A, B \in B(H)_+$. If an orthogonal projection P commutes with A, B, then $((AP)\sigma(BP))P = (A\sigma B)P = P(A\sigma B)$ holds.*

Proof Since the condition (i) and (ii) hold, we have

$$P(A\sigma B)P \le (PAP)\sigma(PBP) \le A\sigma B.$$

This implies

$$\begin{aligned} &\|(A\sigma B)P - P(A\sigma B)P\|^2 \\ &= \|((A\sigma B) - P(A\sigma B)P)P\|^2 \\ &\le \|(A\sigma B) - P(A\sigma B)P\|\|P(A\sigma B - P(A\sigma B)P)P\| = 0. \end{aligned}$$

Thus $(A\sigma B)P = P(A\sigma B)P = (P(A\sigma B)P)^* = P(A\sigma B)$. □

Lemma 4.3 *For $t \ge 0$, there exists $\alpha_t \ge 0$ such that $I\sigma(tI) = \alpha_t I$.*

Proof Let P be an orthogonal projection. It follows from the fact that P commutes with I and tI and the preceding lemma that P commutes with $I\sigma(tI)$. Thus $I\sigma(tI)$ reduces all closed subspaces in H. □

Put $f(t) := I\sigma(tI)$. From the statement (i), $f(t)$ is right continuous on $[0,\infty)$. On the other hand, from Lemma 4.1, $f(t)/t(= ((1/t)I)\sigma I)$ is left continuous on $(0,\infty)$. Combining them, f is continuous on $[0,\infty)$.

Lemma 4.4 *$f(t)(:= I\sigma(tI))$ is an operator monotone function on* $[0,\infty)$.

Proof We first show the case that there exist orthogonal projections $\{P_i\}$ such that $A = \sum_i \alpha_i P_i$, $\sum_i P_i = I$ and $P_i P_j = 0$ $(i \neq j)$. From Lemma 4.2,

$$\begin{aligned} I\sigma A &= (I\sigma A)(\sum_i P_i) = \sum_i (I\sigma A)P_i \\ &= \sum_i (P_i\sigma(\alpha_i P_i)) = \sum_i (I\sigma(\alpha_i I))P_i \\ &= \sum_i f(\alpha_i)P_i = f(A). \end{aligned}$$

For general A, there exist a sequence A_n of the above form such that $A_n \searrow A$. So, $I\sigma A_n = f(A_n)$ converges to $I\sigma A(= f(A))$ strongly. From the statement (i), it follows that f is operator monotone. □

Proof of Theorem 4.1 Let $\sigma \in \Sigma$. From the above lemmas, $f(t)(:= I\sigma(tI))$ is operator monotone and $f(A) = I\sigma A$ for all $A \geq 0$. Thus we have

$$\begin{aligned} A_\epsilon \sigma B_\epsilon &= A_\epsilon^{1/2}(I\sigma(A_\epsilon^{-1/2}B_\epsilon A_\epsilon^{-1/2}))A_\epsilon^{1/2} \\ &= A_\epsilon^{1/2} f(A_\epsilon^{-1/2}B_\epsilon A_\epsilon^{-1/2})A_\epsilon^{1/2} \\ &= \Phi_{\tilde{f}}(A_\epsilon, B_\epsilon) \end{aligned}$$

hold for all $A, B \in B(H)_+$ and $\epsilon > 0$. Taking the strong limit of each side, $A\sigma B = \Phi_{\tilde{f}}(A, B)$ holds for all $A, B \in B(H)_+$.

Let us prove the second half of the theorem. The equivalences

$$\sigma = \alpha\sigma_1 + (1-\alpha)\sigma_2 \iff f_\sigma = \alpha f_{\sigma_1} + (1-\alpha) f_{\sigma_2},$$

$$\sigma = 0 \iff f_\sigma = 0$$

and

$$\sigma_1 \leq \sigma_2 \iff f_{\sigma_1} \leq f_{\sigma_2}$$

are obvious. So, it is enough to show that the map $\sigma \mapsto f$ is surjective. For every $f \in OM_+$, put $A\sigma B := \Phi_{\tilde{f}}(A, B)$. This binary operation σ satisfies the statements (i) and (ii) by Corollary 3.8 and Corollary 3.9, respectively. The proof of the statement (iii) comes from Theorem 3.1 and Corollary 3.7 This implies $\sigma \in \Sigma$. □

From the above argument, a normalized positive valued operator monotone function on $(0, \infty)$ is identified with an operator mean. The binary operation σ satisfying $A\sigma B = A$ (resp. $A\sigma B = B$) is a trivial example for an operator mean and is denoted by l (resp. r).

4.2 *Operator Means*

An operator connection σ having $I\sigma I = I$ is called an operator mean. From the theorem in the last subsection, there exists an affine order isomorphism $\sigma \leftrightarrow f_\sigma$ from the set Σ^1 of all operator means onto the set $OM_+^1 := \{f \in OM_+ \mid f(1) = 1\}$.

4.2.1 Integral Representation

It is known that a necessary and sufficient condition for a positive continuous function f on $(0, \infty)$ to be operator monotone is that f has an integral representation as follows :

$$f(t) = \int_{[0,\infty]} \left(\frac{x : t}{x : 1} \right) dm(x) \qquad (t > 0), \tag{7}$$

where m is a positive finite Borel measure on $[0, \infty]$ (See [6, V.53, p144]).

Example 4.1 We give some examples of the correspondence $m \leftrightarrow f$.

$$(1-\alpha)\delta_{\{0\}} + \alpha\delta_{\{\infty\}} \quad \longleftrightarrow \quad f_{\nabla_\alpha}(t) := (1-\alpha) + \alpha t \quad (0 \le \alpha \le 1)$$

$$\frac{\sin \alpha\pi}{\pi} \cdot \frac{x^{\alpha-1}}{1+x}\, dx \quad \longleftrightarrow \quad f_{\#_\alpha}(t) := t^\alpha \quad (0 < \alpha < 1),$$

$$\delta_{\{\alpha/(1-\alpha)\}} \quad \longleftrightarrow \quad f_{!_\alpha}(t) := ((1-\alpha) + \alpha t^{-1})^{-1} \quad (0 < \alpha < 1).$$

Example 4.2 We above introduced f_{∇_α}, $f_{\#_\alpha}$ and $f_{!_\alpha}$. The corresponding operator means can be written as follows:

$$\begin{aligned} A\nabla_\alpha B &= (1-\alpha)A + \alpha B \\ &= (A+B)^{1/2}((1-\alpha)R + \alpha S)(A+B)^{1/2}, \end{aligned}$$

$$A\#_\alpha B = s - \lim_\epsilon A_\epsilon^{1/2}(A_\epsilon^{-1/2}B_\epsilon^{1/2}A_\epsilon^{-1/2})^\alpha A_\epsilon^{1/2}$$
$$= (A+B)^{1/2}R^{1-\alpha}S^\alpha(A+B)^{1/2},$$

where $s - \lim_\epsilon$ denotes the strong limit. Moreover,

$$A!_\alpha B = s - \lim_\epsilon((1-\alpha)A_\epsilon^{-1} + \alpha B_\epsilon^{-1})^{-1}$$
$$= (A+B)^{1/2}\left(\frac{RS}{\alpha R + (1-\alpha)S}\right)(A+B)^{1/2}.$$

Remark 4.1 In the following, we write $f_{!_0}(t) := 1$, $f_{\#_0}(t) := 1$, $f_{!_1}(t) := t$ and $f_{\#_1}(t) := t$.

The operator means ∇_α, $\#_\alpha$ and $!_\alpha$ are called *the arithmetic mean , the geometric mean* and *the harmonic mean*. By a simple calculation, we have $f_{!_\alpha} \le f_{\#_\alpha} \le f_{\nabla_\alpha}$, which implies $!_\alpha \le \#_\alpha \le \nabla_\alpha$ for all $\alpha \in [0, 1]$.

In the following, we denote $\nabla := \nabla_{1/2}$, $\# := \#_{1/2}$ and $! :=!_{1/2}$.

Example 4.3 Since the power functions t^α $(0 \le \alpha \le 1)$ are in OM_+^1, the integral $\int_0^1 t^\alpha d\alpha = (t-1)/\log t$ is also in OM_+^1. The corresponding operator mean is denoted by λ and is called the logarithmic mean. The relation between the logarithmic mean and the operator means stated above is $\# \le \lambda \le \nabla$.

Remark 4.2 The upper and lower bounds for the logarithmic mean have been studied [18, 57]. The following is curious in this respect [38, 42]:

$$\min\{r \ge 0 \mid \lambda(a,b) \le p_{r,1/2}(a,b) \quad (\forall a, b \in (0,\infty))\} = 1/3,$$

where $p_{r,1/2}(a,b) := ((a^r + b^r)/2)^{1/r}$.

4.2.2 Geometric Mean

Let's talk about statements that characterize the geometric mean. Recall the fundamental formula for the operator geometric mean:

$$A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2} \qquad (A > 0, B \ge 0).$$

We first show that it is the unique positive solution of the Riccati equation.

Proposition 4.1 *Let A, B, X be positive operators. If A is invertible, then $X = A\#B$ if and only if $XA^{-1}X = B$.*

Proof

$$\begin{aligned}X = A\#B &\iff A^{-1/2}XA^{-1/2} = (A^{-1/2}BA^{-1/2})^{1/2}\\&\iff A^{-1/2}XA^{-1}XA^{-1/2} = (A^{-1/2}XA^{-1/2})^2 = A^{-1/2}BA^{-1/2}\\&\iff XA^{-1}X = B.\end{aligned}$$

□

Proposition 4.2 *Let A, B, X be positive operators. Consider the following statements*

(1) $\begin{bmatrix} A & X \\ X & B \end{bmatrix} \geq 0$;
(2) $XA_\epsilon^{-1}X \leq B \quad (\forall \epsilon > 0)$;
(3) $X \leq A\#B$.

Then (1) $\iff$ (2) $\Rightarrow$ (3) hold.

Proof (1) $\iff$ (2). Let $\delta > 0$. Put $S := A_\epsilon^{-1/2}XB_\delta^{-1/2}$. Since

$$\begin{bmatrix} I & S \\ S^* & I \end{bmatrix} = \begin{bmatrix} A_\epsilon^{-1/2} & \\ & B_\delta^{-1/2} \end{bmatrix} \begin{bmatrix} A_\epsilon & X \\ X & B_\delta \end{bmatrix} \begin{bmatrix} A_\epsilon^{-1/2} & \\ & B_\delta^{-1/2} \end{bmatrix} \geq 0,$$

we have

$$\left\langle \begin{bmatrix} I & -S \\ -S^* & I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \mid \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} I & S \\ S^* & I \end{bmatrix} \begin{bmatrix} x \\ -y \end{bmatrix} \mid \begin{bmatrix} x \\ -y \end{bmatrix} \right\rangle \geq 0$$

for $x, y \in H$. So, the inequalities

$$\begin{bmatrix} I & \\ & I \end{bmatrix} \geq \begin{bmatrix} & S \\ S^* & \end{bmatrix} \geq \begin{bmatrix} -I & \\ & -I \end{bmatrix}$$

and $\|S\| = \|A_\epsilon^{-1/2}XB_\delta^{-1/2}\| \leq 1$ hold, which is equivalent to

$$XA_\epsilon^{-1}X = B_\delta^{1/2}(B_\delta^{-1/2}XA_\epsilon^{-1}XB_\delta^{-1/2})B_\delta^{1/2} \leq B_\delta.$$

(2)$\Rightarrow$ (3).

$$\begin{aligned}&XA_\epsilon^{-1}X \leq B\\&\iff (A_\epsilon^{-1/2}XA_\epsilon^{-1/2})^2 = A_\epsilon^{-1/2}XA_\epsilon^{-1}XA_\epsilon^{-1/2} \leq A_\epsilon^{-1/2}BA_\epsilon^{-1/2}\\&\implies A_\epsilon^{-1/2}XA_\epsilon^{-1/2} \leq (A_\epsilon^{-1/2}BA_\epsilon^{-1/2})^{1/2}\\&\iff X \leq A_\epsilon\#B.\end{aligned}$$

□

Theorem 4.2 *Let A, B be positive operators. Then*

$$A\#B = \max\left\{X \geq 0 \mid \begin{bmatrix} A & X \\ X & B \end{bmatrix} \geq 0\right\}.$$

Proof From the above proposition,

$$\left\{X \geq 0 \mid \begin{bmatrix} A & X \\ X & B \end{bmatrix} \geq 0\right\} \subseteq \{X \geq 0 \mid X \leq A\#B\}.$$

So, to show this theorem, it is enough to prove

$$\begin{bmatrix} A & A\#B \\ A\#B & B \end{bmatrix} \geq 0.$$

Thanks to $(A_\epsilon \# B_\delta)A_\epsilon^{-1}(A_\epsilon \# B_\delta) = B_\delta$, we have

$$\begin{bmatrix} A_\epsilon & A_\epsilon \# B_\delta \\ A_\epsilon \# B_\delta & B_\delta \end{bmatrix} \geq 0,$$

which implies the desired result. □

4.2.3 Mean of Projections

Let P, Q be orthogonal projections with $P + Q \neq 0$ and let (R, S) be positive operators on $\ker(P+Q)^\perp$ such that $P = (P+Q)^{1/2}R(P+Q)^{1/2}$ and $Q = (P+Q)^{1/2}S(P+Q)^{1/2}$. On the direct sum $\ker(P+Q)^\perp = (ran(P) \cap ran(Q)) \oplus (ran(P) \cap ran(Q)^\perp) \oplus (ran(P)^\perp \cap ran(Q))$, the operators R, S can be denoted as

$$R = \begin{bmatrix} 2^{-1}I & & \\ & I & \\ & & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 2^{-1}I & & \\ & 0 & \\ & & I \end{bmatrix}.$$

Thus, for $f \in OM_+^1$,

$$\begin{aligned} \Phi_{\tilde{f}}(P, Q) &= (P+Q)^{1/2} \begin{bmatrix} 2^{-1}I & & \\ & \Phi_{\tilde{f}}(1,0)I & \\ & & \Phi_{\tilde{f}}(0,1)I \end{bmatrix} (P+Q)^{1/2} \\ &= P \wedge Q + \Phi_{\tilde{f}}(1,0)(P - P\wedge Q) + \Phi_{\tilde{f}}(0,1)(Q - P \wedge Q), \end{aligned}$$

where $P \wedge Q$ is the orthogonal projection onto $ran(P) \cap ran(Q)$.

Proposition 4.3 *Let P, Q be orthogonal projections.*

$$P\sigma Q = P \wedge Q + a(P - P \wedge Q) + b(Q - P \wedge Q),$$

where $a := f_\sigma(0+)$ and $b := \widetilde{f_\sigma}(0+)$.

4.2.4 Transforms on OM_+^1

Let σ be an operator mean. The following binary operations :

$$A\tilde{\sigma}B := B\sigma A, \quad A\sigma^* B := \text{s-}\lim_{\epsilon}(A_\epsilon^{-1}\sigma B_\epsilon^{-1})^{-1}, \quad A\sigma^\perp B := A\tilde{\sigma}^* B$$

correspond to operator monotone functions

$$\tilde{f_\sigma}(t) = tf_\sigma(1/t), \quad f_\sigma^*(t) = f_\sigma(1/t)^{-1}, \quad f_\sigma^\perp(t) = t/f_\sigma(t),$$

which implies the binary operations $(\tilde{\sigma}, \sigma^*, \sigma^\perp)$ are in Σ^1. We show some properties of these transforms ($\tilde{}$, *, $^\perp$) on Σ^1. It follows from the fact $\tilde{\tilde{\sigma}} = \sigma$, $(\sigma^*)^* = \sigma$ and $(\sigma^\perp)^\perp = \sigma$ that these transforms are bijective map on Σ^1. By a simple calculation, we have the following.

Proposition 4.4 *Let $\sigma_1, \sigma_2 \in \Sigma^1$. If $\sigma_1 \le \sigma_2$, then*

$$\tilde{\sigma_1} \le \tilde{\sigma_2}, \quad \sigma_2^* \le \sigma_1^* \quad \textit{and} \quad \sigma_2^\perp \le \sigma_1^\perp.$$

Corollary 4.1

$$! = \nabla^* \le \lambda^* = \lambda^\perp \le \#^* = \# \le \lambda \le \nabla.$$

Remark 4.3 The injective map $f \mapsto \hat{f} := \frac{t+f}{1+f}$ is called the Barbour transform on OM_+. This map plays an important role in the analysis of OM_+ ([40, 46]). The Barbour transform has the following properties:

$$\widehat{OM_+} = OM_+^1 \backslash \{1\}, \quad \widehat{OM_+^1} = \{f \in OM_+^1 \mid f_! \le f \le f_\nabla\},$$

$$\widehat{f^\perp} = (\hat{f})^\perp, \quad \widehat{(\tilde{f})} = (\hat{f})^*, \quad \widehat{(f^*)} = \tilde{\hat{f}} \quad (f \in OM_+^1).$$

4.2.5 Weight and Symmetricity

Let f be a positive function on $(0, \infty)$. It is known [6, Theorem V. 25] that f is operator monotone if and only if f is operator concave. Using this, the following is obtained.

Proposition 4.5 *Let σ be an operator mean. Then* $!_\alpha \le \sigma \le \nabla_\alpha$ *hold, where* $\alpha := \frac{df_\sigma}{dx}\big|_{x=1}$.

Proof Put $f := f_\sigma$. Note that the tangent line of $f(x)$ at $x = 1$ is written as $y = f'(1)x + (1 - f'(1))$. Since f is an operator monotone function, we have

$$(f'(1)x + (1 - f'(1))) \ge f(x)$$

for all $x > 0$, which implies $1 - f'(1) \ge f(0+) \ge 0$ and $f'(1) \ge 0$. So $\sigma \le \nabla_{f'(1)}$. Applying this argument to f^*, we have $\sigma^* = \sigma_{f^*} \le \nabla_{(f^*)'(1)} = \nabla_{f'(1)} = (!_{f'(1)})^*$, which implies $!_{f'(1)} \le \sigma$. □

Remark 4.4 We call the positive number $\alpha \left(:= \frac{df_\sigma}{dx}\big|_{x=1}\right)$ the weight of σ.

An operator mean σ having $\tilde{\sigma} = \sigma$ is called a symmetric operator mean. Since $(\widetilde{f_\sigma})'(1) = 1 - f'_\sigma(1) = f'_\sigma(1)$, we have $f'_\sigma(1) = \frac{1}{2}$ and $! \le \sigma \le \nabla$.

Proposition 4.6

$$\{\sigma \in \Sigma^1 \mid \tilde{\sigma} = \sigma\} \subsetneq \{\sigma \in \Sigma^1 \mid ! \le \sigma \le \nabla\}.$$

Proof Put $f(t) := \frac{3t+1}{t+3}$. Then the operator mean σ_f which corresponds f is in $\big(\{\sigma \in \Sigma^1 \mid ! \le \sigma \le \nabla\} \backslash \{\sigma \in \Sigma^1 \mid \tilde{\sigma} = \sigma\}\big)$. □

Remark 4.5 Recall the Barbour transform $f \mapsto \hat{f} = \frac{t+f}{1+f}$. For $f \in OM_+^1$ with $f^* \neq f$, $\sigma_{\hat{f}}$ is not symmetric, but $! \le \sigma_{\hat{f}} \le \nabla$.

Remark 4.6 An operator mean σ having $\sigma^* = \sigma$ is called a self-adjoint operator mean. The geometric mean is an easy-to-understand and important example for a self-adjoint operator mean. A non-trivial operator mean σ (i.e., $\sigma \neq l, \sigma \neq r$) is self-adjoint if and only if it can be written as the Barbour transform of a symmetric operator connection, namely

$$\{f \in OM_+^1 \backslash \{1\} \mid f = f^*\} = \{\hat{f} \mid f = \tilde{f}, f \in OM_+\}.$$

For example, for $r \in [0, 1]$, the function $\widehat{t^{1-r}} = \frac{t^{1-r}+t}{t^{1-r}+1}$ is symmetric [7, 43]. An operator mean which corresponds this function is called the Lehmar mean. The exact definition of this mean will be given.

Example 4.4 For an arbitrary operator mean σ, the operator mean $(\sigma + \tilde{\sigma})/2$ is a symmetric mean. *The Heinz mean* σ_{h_α} defined by

$$A\sigma_{h_\alpha}B := (A\#_\alpha B + A\#_{1-\alpha}B)/2 \quad (0 \le \alpha \le 1)$$

is a typical example.

Let $f \in OM_+^1$. The two variable positive function $\Phi_{\tilde{f}}(s,t) := sf(t/s)$ $(s,t>0)$ is homogeneous and is monotone for each variable. Furthermore, it satisfies

$$\min\{s,t\} \le \Phi_{\tilde{f}}(s,t) \le \max\{s,t\}.$$

So, the function $\Phi_{\tilde{f}}$ can be viewed as a numerical mean.

In what follows, we treat an operator mean as a numerical mean induced from an element of OM_+^1.

4.2.6 Power Means

Let $\alpha \in [0,1]$ and $r \in \mathbb{R}$. For $s,t>0$,

$$p_{r,\alpha}(s,t) := ((1-\alpha)s^r + \alpha t^r)^{1/r} \quad (r \neq 0)$$

and

$$p_{0,\alpha}(s,t) := \lim_{r\to 0} p_{r,\alpha}(s,t) = s^{1-\alpha}t^{\alpha}.$$

The function $p_{r,\alpha}(s,t)$ is increasing w.r.t. r and $(s,t) \mapsto p_{r,\alpha}(s,t)$ is an operator mean for all $\alpha \in [0,1]$ if and only if $r \in [-1,1]$. The map $r \mapsto p_{r,\alpha}$ is a path connecting familiar operator means. For example,

$$!_\alpha(r=-1),\ \ \#_\alpha(r=0) \quad \text{and} \quad \nabla_\alpha(r=1).$$

An operator mean $p_{r,\alpha}$ $(r \in [-1,1])$ is called *the power mean* [45].

Proposition 4.7 *Let A, B, X be positive invertible operators and let $(r,\alpha) \in [-1,1]\times[0,1]$. Then $X = A\,\sigma_{p_{r,\alpha}} B$ if and only if*

$$(1-\alpha)P_{f_r}(A,X) + \alpha P_{f_r}(B,X) = 0, \tag{8}$$

where $\sigma_{p_{r,\alpha}}$ is an operator mean which corresponds to the function $t \mapsto p_{r,\alpha}(1,t)$, $f_r(t) := (t^r-1)/r$ $(r \neq 0)$ and $f_0(t) := \log t$.

The proof is left to the reader.

Corollary 4.2 *Let A, B, X be positive operators and let $r > 0$. Then*

$$X = A\sigma_{p_{r,\alpha}}B \Rightarrow (1-\alpha)\Phi_{f_r}(A,X) + \alpha\Phi_{f_r}(B,X) = 0.$$

Proof Since $X(\epsilon)(:= A_\epsilon \sigma_{p_{r,\alpha}} B_\epsilon) \searrow X$ as $\epsilon \searrow 0$,

$$(1-\alpha)\Phi_{f_r}(A_\epsilon, X(\epsilon)) + \alpha\Phi_{f_r}(B_\epsilon, X(\epsilon))(=0)$$

strongly converges to $(1-\alpha)\Phi_{f_r}(A,X) + \alpha\Phi_{f_r}(B,X)(=0)$. □

Remark 4.7 Let X, $A_i \in B(H)_{++}$ $(i = 1, 2, \cdots n)$ and let $\alpha_i \in (0, 1)$ $(i = 1, 2, \cdots n)$ with $\sum_i \alpha_i = 1$. A generalized equation of (8)

$$\sum_i \alpha_i P_{f_r}(A_i, X) = 0$$

is discussed in [41, Theorem 5.6]. The existence and uniqueness of the positive solution X is not trivial in general, whereas the proof in the case when $n = 2$ is very easy. When $r = 0$, the solution X is a multivariate extension of the geometric mean and it is called the Karcher mean.

4.2.7 Stolarsky Means

For $\alpha \in \mathbb{R}\backslash\{0, 1\}$, *the Stolarsky mean* is defined by

$$S_\alpha(s, t) := \begin{cases} \left(\frac{s^\alpha - t^\alpha}{\alpha(s-t)}\right)^{\frac{1}{\alpha-1}}, & \text{if } s \neq t : \\ s, & \text{if } s = t. \end{cases}$$

and

$$S_0(s, t) := \lim_{\alpha \to 0} S_\alpha(s, t) = s\ \lambda\ t,$$

$$S_1(s, t) := \lim_{\alpha \to 1} S_\alpha(s, t) = \frac{1}{e}\left(\frac{s^s}{t^t}\right)^{1/(s-t)}.$$

The function $S_\alpha(s, t)$ is monotone increasing w.r.t. α ([50]) and is an operator mean if and only if $-2 \leq \alpha \leq 2$ ([45]). Thus we have

$$\#(= S_{-1}) \leq p_{1/2,1/2}(= S_{1/2}) \leq S_1 \leq \nabla(= S_2).$$

The operator mean S_1 is called *the identric mean*.

4.2.8 Means of Szabó Type

In [51], V.E. Szabó define the following function and discuss its operator monotonicity:

$$t \mapsto t^\gamma \frac{(t^{\alpha_1} - 1)(t^{\alpha_2} - 1) \cdots (t^{\alpha_n} - 1)}{(t^{\beta_1} - 1)(t^{\beta_2} - 1) \cdots (t^{\beta_n} - 1)},$$

where $\gamma \in \mathbb{R}$, $\alpha_i, \beta_j > 0$ with $\alpha_i \neq \beta_j$ $(i, j = 1, 2, \ldots, n)$. Several familiar operator means are induced from the function of this type as follows.

The function pd_p defined by

$$pd_p(s,t) := \begin{cases} \frac{p-1}{p}\frac{s^p-t^p}{s^{p-1}-t^{p-1}}, & \text{if } s \neq t: \\ s, & \text{if } s = t. \end{cases}$$

is increasing w.r.t. p and interpolates some familiar means. For example,

$$!(= pd_{-1}) \leq \#(t)(= pd_{1/2}) \leq \lambda(= pd_1) \leq \nabla(= pd_2).$$

This function is called *the power difference mean* and is an operator mean if and only if $p \in [-1, 2]$ [18, 30, 44, 52].

The Lehmer mean defined by

$$l_p(s,t) := \frac{s^p + t^p}{s^{p-1} + t^{p-1}} \left(= \frac{s^{2p} - t^{2p}}{s^{2p-2} - t^{2p-2}} \cdot \frac{s^{p-1} - t^{p-1}}{s^p - t^p} \right)$$

is an operator mean if and only if $p \in [0, 1]$ [45]. Clearly,

$$!(= l_0) \leq \#(= l_{1/2}) \leq \nabla(= l_1).$$

The following mean

$$ph_p(s,t) := \begin{cases} p(1-p)\frac{(s-t)^2}{(s^p-t^p)(s^{1-p}-t^{1-p})}, & \text{if } p \neq 0, p \neq 1, s \neq t, \\ \frac{s-t}{\log s - \log t}, & \text{if } p = 0 \text{ or } p = 1, s \neq t, \\ t, & \text{if } s = t. \end{cases}$$

is introduced in [25, 26]. We call this *the Petz-Hasegawa mean*. This function is an operator mean if and only if $p \in [-1, 2]$ ([5, 25]). An elementary calculus shows that this function equals the harmonic mean if $p = -1, 2$, the logarithmic mean if $p = 0, 1$ and the power mean $p_{1/2,1/2}$ if $p = 1/2$.

Remark 4.8 Let (a, b) be a pair of real numbers with $|a|, |b| \leq 2$. Put

$$m_{a,b}(t) := \frac{b}{a}\frac{t^a - 1}{t^b - 1}.$$

A necessary and sufficient condition for this function to be in OM_+^1 is that (a, b) is in the following set [44]:

$$\{(a,b) \mid 0 < a - b \leq 1, a \geq -1, b \leq 1\} \\ \cup ([0,1] \times [-1,0]) \backslash \{(0,0)\}.$$

5 Operator Inequalities

5.1 Positive Maps

In this section, we study some inequalities involving an operator mean. A linear map $\Phi : B(H) \to B(K)$ is said to be *positive* if $\Phi(A) \in B(K)_+$ for every $A \in B(H)_+$.

For a positive map Φ, by estimating $\Phi(I_H)$, some properties of Φ becomes clear. The continuity of a positive map Φ is guaranteed by the fact $\|\Phi\| = \|\Phi(I_H)\|$ (cf. [47]). A positive map Φ is said to be *strictly positive* (resp. *unital*) if $\Phi(I_H) \in B(K)_{++}$ (resp. $\Phi(I_H) = I_K$). Since a positive map Φ preserves the order relation, we have

$$\Phi(A) \geq \Phi(\alpha I_H) \geq \alpha \Phi(I_H)$$

for all $A \in B(H)_{++}$, where $\alpha := \min sp(A)$. So, a positive unital map is strictly positive.

In what follows, we denote $\mathcal{P}(H, K)$ (resp. $\mathcal{P}(H, K)_1$) the set of strictly positive (resp. unital positive) map from $B(H)$ to $B(K)$.

If A is a positive definite matrix having the spectral decomposition: $A = \sum_i \alpha_i P_i$, then, for $\Phi \in \mathcal{P}(H, K)_1$,

$$\begin{aligned}
&\begin{bmatrix} I_K & \Phi(A^{-1})^{-1} \\ 0 & I_K \end{bmatrix} \begin{bmatrix} \Phi(A) - \Phi(A^{-1})^{-1} & 0 \\ 0 & \Phi(A^{-1}) \end{bmatrix} \begin{bmatrix} I_K & 0 \\ \Phi(A^{-1})^{-1} & I_K \end{bmatrix} \\
&= \begin{bmatrix} \Phi(A) & I_K \\ I_K & \Phi(A^{-1}) \end{bmatrix} \\
&= \sum_i \begin{bmatrix} \alpha_i \Phi(P_i) & \Phi(P_i) \\ \Phi(P_i) & \alpha_i^{-1} \Phi(P_i) \end{bmatrix} = \sum_i \begin{bmatrix} \alpha_i & 1 \\ 1 & \alpha_i^{-1} \end{bmatrix} \otimes \Phi(P_i) \geq 0.
\end{aligned}$$

By the similar argument, the following two lemmas (Choi's inequality and Kadison's inequality) are obtained.

Lemma 5.1 ([9, Theorem 2.1]) *Let* $\Phi \in \mathcal{P}(H, K)_1$. *Then*

$$\Phi(A^{-1}) \geq \Phi(A)^{-1}$$

for all $A \in B(H)_{++}$.

Lemma 5.2 ([36, Theorem 1]) *Let* $\Phi \in \mathcal{P}(H, K)_1$. *Then*

$$\Phi(A^2) \geq \Phi(A)^2$$

for all $A \in B(H)_{sa}$.

Proposition 5.1 *Let $\Phi \in \mathcal{P}(H, K)_1$ and let f be an operator convex function on the open interval (α, β). Then,*

$$f(\Phi(A)) \leq \Phi(f(A))$$

for all $A \in B(H)_{sa}$ with $sp(A) \subseteq (\alpha, \beta)$.

Proof It is enough to assume that f is not linear function. We first show the case when $(\alpha, \beta) = (-1, 1)$. Note that

$$I_K \pm \Phi(A) = \Phi(I_H \pm A) > 0.$$

From the assumption, there exist $a, b \in \mathbb{R}$ such that

$$f(t) = a + bt + \int_{-1}^{1} \frac{t^2}{1 - xt} dm(x),$$

where m is a finite positive Borel measure on $[-1, 1]$. So, we have

$$\Phi(f(A)) = a + b\Phi(A) + \int_{-1}^{1} \Phi\left(\frac{A^2}{I_H - xA}\right) dm(x).$$

Here, for $x \neq 0$, using Choi's inequality,

$$\begin{aligned}\Phi\left(\frac{A^2}{I_H - xA}\right) &= \frac{-1}{x}\Phi(A) + \frac{-1}{x^2}\Phi(I_H) + \frac{1}{x^2}\Phi\left(\frac{1}{I_H - xA}\right) \\ &\geq \frac{-1}{x}\Phi(A) + \frac{-1}{x^2}\Phi(I_H) + \frac{1}{x^2}\left(\frac{1}{I_K - x\Phi(A)}\right) \\ &= \frac{\Phi(A)^2}{I_K - x\Phi(A)}.\end{aligned}$$

For $x = 0$, using Kadison's inequality,

$$\Phi\left(\frac{A^2}{I_H - xA}\right) = \Phi(A^2) \geq \Phi(A)^2 = \frac{\Phi(A)^2}{I_K - x\Phi(A)}.$$

Thus $\Phi(f(A)) \geq f(\Phi(A))$.

In the general case, if we put $g(t) := f\left(\frac{\beta-\alpha}{2}t + \frac{\alpha+\beta}{2}\right)$, then g is operator convex on $(-1, 1)$ and we have

$$\begin{aligned} f(\Phi(A)) &= g\left(\frac{2}{\beta-\alpha}\Phi(A) - \frac{\alpha+\beta}{\beta-\alpha}I_K\right) \\ &= g\left(\Phi\left(\frac{2}{\beta-\alpha}A - \frac{\alpha+\beta}{\beta-\alpha}I_H\right)\right) \\ &\leq \Phi\left(g\left(\frac{2}{\beta-\alpha}A - \frac{\alpha+\beta}{\beta-\alpha}I_H\right)\right) = \Phi(f(A)). \end{aligned}$$

□

Theorem 5.1 *Let $\Phi \in \mathcal{P}(H, K)$ and let f be a real valued operator convex function on $(0, \infty)$. Then*

$$\Phi(P_f(A, B)) \geq P_f(\Phi(A), \Phi(B))$$

holds for all $A, B \in B(H)_{++}$.

Proof Put

$$\Psi(X) := \Phi(B)^{-1/2}\Phi(B^{1/2}XB^{1/2})\Phi(B)^{-1/2}, \quad C := B^{-1/2}AB^{-1/2}.$$

Then Ψ is in $\mathcal{P}(H, K)_1$. So using the preceding proposition,

$$\begin{aligned} \Phi(P_f(A, B)) &= \Phi(B^{1/2}P_f(C, I_H)B^{1/2}) \\ &= \Phi(B)^{1/2}\Psi(P_f(C, I_H))\Phi(B)^{1/2} \\ &\geq \Phi(B)^{1/2}P_f(\Psi(C), I_K)\Phi(B)^{1/2} \\ &= P_f(\Phi(B)^{1/2}\Psi(C)\Phi(B)^{1/2}, \Phi(B)) = P_f(\Phi(A), \Phi(B)). \end{aligned}$$

□

Corollary 5.1 *Let σ be an operator mean and let $\Phi \in \mathcal{P}(H, K)$. Then*

$$\Phi(A\sigma B) \leq \Phi(A)\sigma\Phi(B)$$

for all $A, B \in B(H)_+$.

Proof Since $-f_\sigma$ is operator convex, for $A, B \geq 0$,

$$\begin{aligned} \Phi(A\sigma B) &\leq \Phi(A_\epsilon \sigma B_\epsilon) \\ &= -\Phi(P_{-f_\sigma}(B_\epsilon, A_\epsilon)) \end{aligned}$$

$$\leq -P_{-f_\sigma}(\Phi(B_\epsilon), \Phi(A_\epsilon))$$
$$= \Phi(A_\epsilon)\sigma\Phi(B_\epsilon)$$
$$= (\Phi(A) + \epsilon\Phi(I_H))\sigma(\Phi(B) + \epsilon\Phi(I_H)).$$

□

5.2 Power Monotonicity

Let t be a positive real number with $t \geq 1$ (resp. $t \leq 1$). The function $[1, \infty) \ni r \mapsto t^r$ is monotone increasing (resp. decreasing). A certain numerical mean $(a, b) \mapsto a\ \sigma_f\ b (:= af(b/a))$ also satisfies this property. If the positive function f satisfies $f(t)^r \leq f(t^r)\ \ (r \geq 1)$, then

$$a^r\ \sigma_f\ b^r \geq (a\ \sigma_f\ b)^r \quad (r \geq 1)$$

which is equivalent to

$$a\ \sigma_f\ b \geq 1 \Rightarrow a^r \sigma_f b^r \geq 1 \quad (r \geq 1).$$

In [4], Ando and Hiai prove that an operator version of this inequality holds for the weighted geometric mean. In this section, we study the similar statement

$$A, B > 0, \quad A\sigma B \geq I \Rightarrow A^r \sigma B^r \geq I \tag{9}$$

and

$$A, B > 0, \quad A\sigma B \leq I \Rightarrow A^r \sigma B^r \leq I. \tag{10}$$

5.2.1 Ando–Hiai Type Inequalities

For a positive operator X, we denote the minimum value of $sp(X)$ by $\lambda_{\min}(X)$.

The following inequality holds for an arbitrary operator mean.

Proposition 5.2 *Let σ be an operator mean and $A, B > 0$. Then*

$$A^r \sigma B^r \geq \lambda_{\min}\left(\frac{f_\sigma(C^r)}{f_\sigma(C)^r}\right)\lambda_{\min}(A\sigma B)^{r-1}(A\sigma B)\ \ \textit{for}\ \ 1 \leq r \leq 2,$$

where $C := A^{\frac{-1}{2}} B A^{\frac{-1}{2}}$ and $f_\sigma(t) := 1\sigma t\ (t > 0)$.

Proof We first show the case when $\lambda_{\min}(A\sigma B) = 1$. Since $A\sigma B \geq I$, we have $\frac{1}{f_\sigma(C)} \leq A$. Set $\epsilon := 2 - r$. Then

$$\begin{aligned}
A^r \sigma B^r &= A^{\frac{r}{2}} f_\sigma(A^{-\frac{r}{2}+\frac{1}{2}} C A^{\frac{1}{2}} B^{-\epsilon} A^{\frac{1}{2}} C A^{-\frac{r}{2}+\frac{1}{2}}) A^{\frac{r}{2}} \\
&= A^{\frac{r}{2}} f_\sigma(A^{-\frac{r}{2}+\frac{1}{2}} C A^{\frac{1}{2}} (A^{-\frac{1}{2}} C^{-1} A^{-\frac{1}{2}})^{\epsilon} A^{\frac{1}{2}} C A^{-\frac{r}{2}+\frac{1}{2}}) A^{\frac{r}{2}} \\
&= A^{\frac{1}{2}} A^{\frac{1-\epsilon}{2}} f_\sigma(A^{\frac{-1+\epsilon}{2}} C[A \#_\epsilon C^{-1}] C A^{\frac{-1+\epsilon}{2}}) A^{\frac{1-\epsilon}{2}} A^{\frac{1}{2}} \\
&= A^{\frac{1}{2}} \left[A^{1-\epsilon} \sigma \{ C[A \#_\epsilon C^{-1}] C \} \right] A^{\frac{1}{2}} \\
&\geq A^{\frac{1}{2}} \left[\frac{1}{f_\sigma(C)^{1-\epsilon}} \sigma \left\{ C \left(\frac{1}{f_\sigma(C)} \#_\epsilon C^{-1} \right) C \right\} \right] A^{\frac{1}{2}} \\
&= A^{\frac{1}{2}} \left(\frac{f_\sigma(C^{2-\epsilon})}{f_\sigma(C)^{1-\epsilon}} \right) A^{\frac{1}{2}} = A^{\frac{1}{2}} \left(\frac{f_\sigma(C^r)}{f_\sigma(C)^{r-1}} \right) A^{\frac{1}{2}} \\
&\geq \lambda_{\min} \left(\frac{f_\sigma(C^r)}{f_\sigma(C)^r} \right) A^{\frac{1}{2}} f_\sigma(C) A^{\frac{1}{2}} = \lambda_{\min} \left(\frac{f_\sigma(C^r)}{f_\sigma(C)^r} \right) (A\sigma B).
\end{aligned}$$

In the general case, we put $\alpha := \lambda_{\min}(A\sigma B)$. Then

$$\lambda_{\min}((A/\alpha)\sigma(B/\alpha)) = \lambda_{\min}((1/\alpha)A\sigma B) = 1.$$

Thus the above argument implies

$$(A/\alpha)^r \sigma (B/\alpha)^r \geq \lambda_{\min} \left(\frac{f_\sigma(D^r)}{f_\sigma(D)^r} \right) (A/\alpha)\sigma(B/\alpha),$$

where $D := (A/\alpha)^{-1/2}(B/\alpha)(A/\alpha)^{-1/2} = A^{-1/2} B A^{-1/2} = C$. The last inequality is equivalent to

$$A^r \sigma B^r \geq \lambda_{\min} \left(\frac{f_\sigma(C^r)}{f_\sigma(C)^r} \right) \alpha^{r-1} (A\sigma B).$$

□

Using this result, the condition $f_\sigma(t^r) \geq f_\sigma(t)^r \quad (t > 0, 2 \geq r \geq 1)$ clearly implies

$$A^r \sigma B^r \geq \lambda_{\min}(A\sigma B)^{r-1}(A\sigma B).$$

Corollary 5.2 *Let $f \in OM_+^1$ and σ_f be an operator mean which corresponds to f. The followings are equivalent:*

(i) $f(t^r) \geq f(t)^r \quad (r \geq 1)$;
(ii) $A^r \sigma_f B^r \geq \lambda_{\min}(A\sigma_f B)^{r-1}(A\sigma_f B) \qquad (A, B > 0,\ 1 \leq r \leq 2)$;
(iii) $A, B > 0$, $A\sigma_f B \geq I \Rightarrow A^r \sigma_f B^r \geq I \qquad (r \geq 1)$.

Proof $(i) \Rightarrow (ii)$. Immediate from the above proposition.

$(ii) \Rightarrow (iii)$. It is enough to show the case when $r > 2$. There exist a positive integer n and $1 \le r_0 \le 2$ such that $r = 2^n r_0$. Iterating (ii) gives (iii).

$(iii) \Rightarrow (i)$. Take $A = (1/f(t))I$ and $B = (t/f(t))I$. □

In the statement of the preceding proposition, replacing σ by σ^*, the following is obtained.

Corollary 5.3 *Let σ be an operator mean and $A, B > 0$. Then*

$$A^r \sigma B^r \le \left\| \frac{f_\sigma(C^r)}{f_\sigma(C)^r} \right\| \|A\sigma B\|^{r-1}(A\sigma B) \text{ for } 1 \le r \le 2.$$

Corollary 5.4 *Let $f \in OM_+^1$ and σ_f be an operator mean which corresponds to f. The followings are equivalent:*

(i) $f(t^r) \le f(t)^r \quad (r \ge 1)$;
(ii) $A^r \sigma_f B^r \le \|A\sigma_f B\|^{r-1}(A\sigma_f B) \qquad (A, B > 0,\ 1 \le r \le 2)$;
(iii) $A, B > 0,\ A\sigma_f B \le I \Rightarrow A^r \sigma_f B^r \le I \qquad (r \ge 1)$.

Remark 5.9 In the case when $\dim H < \infty$, for $A, B \ge 0$, $\lim_\epsilon \|A_\epsilon \sigma_f B_\epsilon\| = \|A\sigma_f B\|$. So, the statement (i)–(iii) above are equivalent to

$$A, B \ge 0,\ A\sigma_f B \le I \Rightarrow A^r \sigma_f B^r \le I \qquad (r \ge 1).$$

In what follows, we denote the set of all functions in OM_+^1 satisfying (i) in Corollary 5.2 (resp. (i) in Corollary 5.4) by PMI (resp. PMD). Note that

$$f \in PMI \iff f^* \in PMD.$$

Corollary 5.5 *For $f \in PMD$ and $A, B > 0$,*

$$\|(A^p \sigma_f B^p)^{1/p}\| \le \|(A^q \sigma_f B^q)^{1/q}\| \qquad (0 < q \le p)$$

Proof Using Corollary 5.3,

$$\|A^r \sigma_f B^r\| \le \|A\sigma_f B\|^r \quad (1 \le r).$$

So

$$\begin{aligned}\|A^q \sigma_f B^q\|^{1/q} &= \|(A^p)^{q/p} \sigma_f (B^p)^{q/p}\|^{1/q} \\ &\le \|A^p \sigma_f B^p\|^{(1/q)(q/p)} = \|A^p \sigma_f B^p\|^{1/p}\end{aligned}$$

□

Since the function $t \mapsto t^\alpha$ $(\alpha \in [0, 1])$ is in $PMI \cap PMD$, we have the following.

Corollary 5.6 (The Ando–Hiai Inequality [4]) *Let* $\alpha \in [0, 1]$. *Then the followings hold.*

$$A, B > 0, \quad A\#_\alpha B \geq I \Rightarrow A^r\#_\alpha B^r \geq I \qquad (r \geq 1), \tag{11}$$

$$A, B > 0, \quad A\#_\alpha B \leq I \Rightarrow A^r\#_\alpha B^r \leq I \qquad (r \geq 1). \tag{12}$$

Remark 5.10 Statements (11) and (12) are equivalent. Assume (11). Then, for $A, B > 0$ with $A\#_\alpha B \leq I$, we have $A^{-1}\#_\alpha B^{-1} \geq I$. So $A^{-r}\#_\alpha B^{-r} \geq I$ holds by (11). This implies $A^r\#_\alpha B^r \leq I$.

5.3 *Furuta Inequality*

By using the Ando–Hiai inequality, the essential part of *the Furuta inequality* [19] is obtained.

Proposition 5.3 *If* $A \geq B > 0$, *then*

$$A^{-r}\#_{\frac{1+r}{p+r}} B^p \leq B$$

for $p \geq 1$ *and* $r \geq 0$.

Proof It is enough to assume $r > 0$. We first show

$$A^{-r}\#_{\frac{r}{p+r}} B^p \leq I \qquad (p \geq 1, \; r > 0). \tag{13}$$

Note that there exists $s \in (0, 1]$ such that $r/s \geq 1$. Set $q := p/r$. It follows from $A \geq B > 0$ that

$$A^{-s}\#_{\frac{1}{1+q}} B^{sq} \leq B^{-s}\#_{\frac{1}{1+q}} B^{sq} = I$$

which implies

$$A^{-r}\#_{\frac{r}{p+r}} B^p = A^{-s(r/s)}\#_{\frac{1}{1+q}} B^{sq(r/s)} \leq I$$

by the Ando–Hiai inequality. So (13) is obtained. From this,

$$\begin{aligned} A^{-r}\#_{\frac{1+r}{p+r}} B^p &= B^p\#_{\frac{p-1}{p+r}} A^{-r} \\ &= B^p\#_{\frac{p-1}{p}} \left(B^p\#_{\frac{p}{p+r}} A^{-r}\right) \end{aligned}$$

$$= B^p \#_{\frac{p-1}{p}} \left(A^{-r} \#_{\frac{r}{p+r}} B^p\right)$$
$$\leq B^p \#_{\frac{p-1}{p}} I = B.$$

□

Proposition 5.4 (The Furuta Inequality) *If* $A \geq B \geq 0$*, then*

$$\left(A^{r/2} B^p A^{r/2}\right)^{1/q} \leq A^{\frac{p+r}{q}}$$

for $p \geq 0$*,* $r \geq 0$ *and* $q \geq 1$ *with* $q \geq (p+r)/(r+1)$*.*

Lemma 5.3 *If* $A \geq B \geq 0$*, then*

$$\left(A^{r/2} B^p A^{r/2}\right)^{1/q} \leq A^{\frac{p+r}{q}}$$

for $p \geq 1$*,* $r \geq 0$ *and* $q > 0$ *with* $q \geq (p+r)/(r+1)$*.*

Proof It is enough to prove the case when $A \geq B > 0$. Proposition 5.3 gives

$$\left(A^{r/2} B^p A^{r/2}\right)^{\frac{1+r}{p+r}} \leq A^{r/2} B A^{r/2} \leq A^{r+1}.$$

Thus, by taking $\frac{p+r}{q(1+r)}$ power of each side,

$$\left(A^{r/2} B^p A^{r/2}\right)^{1/q} = \left(A^{r/2} B^p A^{r/2}\right)^{\frac{p+r}{q(1+r)} \frac{1+r}{p+r}} \leq A^{\frac{p+r}{q}}.$$

□

Proof of Proposition 5.4 Suppose (p, q, r) satisfies the condition. If $p = 0$ or $r = 0$, then the desired inequality clearly holds. So we assume $p > 0$ and $r > 0$. Put $s := \min\{p, 1\} (\leq 1)$. Then $p' := (p/s) \geq 1$ and $r' := (r/s) > 0$. From the above lemma,

$$\left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{s+r}{p+r}} = \left((A^s)^{\frac{r'}{2}} (B^s)^{p'} (A^s)^{\frac{r'}{2}}\right)^{\frac{1+r'}{p'+r'}}$$
$$\leq (A^s)^{(p'+r')\frac{1+r'}{p'+r'}}$$
$$= A^{s+r}.$$

Since $q \geq \max\left\{\frac{p+r}{1+r}, 1\right\}$, we have $1 \geq \frac{p+r}{q(s+r)}$. Thus, by taking $\frac{p+r}{q(s+r)}$ power of each side, the desired inequality is obtained. □

5.4 Chaotic Order

Let $A, B > 0$. In [3], Ando proved the following equivalence:

$$\log A \geq \log B \iff A^{-p} \# B^p \leq I \quad (p \geq 0)$$
$$\iff A^{-p} \# B^p \geq A^{-q} \# B^q \quad (0 \leq p \leq q).$$

In this subsection, a generalization of this result will be given. We first introduce, as a preliminary, Lie-Trotter-type formula for an operator perspective.

Proposition 5.5 *[[32, Theorem 5.1]] Assume that f is a C^1 function on $(0, \infty)$ with $f > 0$, $f(1) = 1$. Then for every $A, B > 0$,*

$$\lim_{p \to 0} P_f(A^p, B^p)^{1/p} = \exp(f'(1) \log A + (1 - f'(1)) \log B)$$

in the operator norm topology.

The proof is omitted.

Let $f \in PMD$. The function $p \mapsto \|P_f(A^p, B^p)^{1/p}\|$ is decreasing (see Corollary 5.5). It follows from this fact and the preceding proposition that

$$\|(A^p \sigma_{\tilde{f}} B^p)^{1/p}\| = \|P_f(A^p, B^p)^{1/p}\| \leq \|\exp(f'(1) \log A + (1 - f'(1)) \log B)\|.$$

Proposition 5.6 ([32]) *Let $\alpha \in [0, 1]$ and PMD_α be the set of all $f \in OM_+^1$ such that $f'(1) = \alpha$. If $A, B > 0$. Then the followings are equivalent:*

(i) $(1 - \alpha) \log A + \alpha \log B \leq 0$*;*
(ii) $A^p \sigma_f B^p \leq I$ *for all* $p > 0$ *and for all* $f \in PMD_\alpha$*;*
(iii) $A^p \#_\alpha B^p \leq I$ *for all* $p > 0$*;*
(iv) $p \mapsto A^p \sigma_f B^p$ *is a decreasing map from* $[0, \infty)$ *into* $B(H)_{++}$ *for all* $f \in PMD_\alpha$*;*
(v) $p \mapsto A^p \#_\alpha B^p$ *is a decreasing map from* $[0, \infty)$ *into* $B(H)_{++}$.

Proof The equivalence of $(i) - (iii)$ is immediate from the above argument.

$(iii) \Rightarrow (iv)$. From Corollary 5.4, we have $A^r \sigma_f B^r \leq A \sigma_f B$ $(r \in [1, 2])$. Thus for all $r \geq 1$, $A^r \sigma_f B^r \leq A \sigma_f B$ holds. Using this,

$$A^p \sigma_f B^p = A^{q(p/q)} \sigma_f B^{q(p/q)} \leq A^q \sigma_f B^q \qquad (p \geq q > 0).$$

Combining this and the fact that $A^p \sigma_f B^p \leq I = A^0 \sigma_f B^0 \quad (p > 0)$, the desired result is obtained.

$(iv) \Rightarrow (v)$. Immediate.

$(v) \Rightarrow (i)$. From Proposition 5.5,

$$I \geq \lim_p (A^p \#_\alpha B^p)^{1/p} = \exp((1 - \alpha) \log A + \alpha \log B).$$

□

Replacing A, B by A^{-1}, B^{-1}, we have the following.

Corollary 5.7 *Let $\alpha \in [0,1]$ and $A, B > 0$. Then the followings are equivalent:*

(i) $(1-\alpha)\log A + \alpha \log B \geq 0$;
(ii) $A^p \sigma_{f^*} B^p \geq I$ *for all* $p > 0$ *and for all* $f \in PMD_\alpha$;
(iii) $A^p \#_\alpha B^p \geq I$ *for all* $p > 0$.
(iv) $p \mapsto A^p \sigma_{f^*} B^p$ *is a increasing map from* $[0, \infty)$ *into* $B(H)_{++}$ *for all* $f \in PMD_\alpha$;
(v) $p \mapsto A^p \#_\alpha B^p$ *is a increasing map from* $[0, \infty)$ *into* $B(H)_{++}$.

The next result is a generalization for the Ando–Hiai inequality.

Corollary 5.8 ([21]) *Let $A, B > 0$. Suppose that*

$$A \#_{\frac{\beta}{\alpha+\beta}} B \geq I \text{ for fixed } \alpha \geq 0 \text{ and } \beta \geq 0 \text{ with } \alpha + \beta > 0.$$

Then the following inequality holds

$$A^\mu \#_{\frac{\beta\mu}{\alpha\lambda+\beta\mu}} B^\lambda \geq I \text{ for } \lambda \geq 1 \text{ and } \mu \geq 1.$$

Proof Put $W := A\#_{\frac{\beta}{\alpha+\beta}} B (\geq I)$. Since

$$(W^{-1/2}AW^{-1/2}) \#_{\frac{\beta}{\alpha+\beta}} (W^{-1/2}BW^{-1/2}) = I,$$

$$\alpha\mu\lambda \log(W^{-1/2}AW^{-1/2}) + \beta\mu\lambda \log(W^{-1/2}BW^{-1/2}) = 0. \tag{14}$$

Then, there exists non-negative integer n such that $1 \leq \lambda_0 := \lambda/2^n \leq 2$, and

$$\begin{aligned} \beta\mu\lambda \log(W^{-1/2}BW^{-1/2}) &= \beta\mu 2^n \log(W^{-1/2}BW^{-1/2})^{\lambda_0} \\ &\leq \beta\mu 2^n \log(W^{-1/2}B^{\lambda_0}W^{-1/2}) \\ &\leq \beta\mu \log(W^{-1/2}B^{\lambda}W^{-1/2}). \end{aligned}$$

Thus, from (14),

$$\alpha\lambda \log(W^{-1/2}A^\mu W^{-1/2}) + \beta\mu \log(W^{-1/2}B^\lambda W^{-1/2}) \geq 0,$$

$$\frac{\alpha\lambda}{\alpha\lambda+\beta\mu} \log(W^{-1/2}A^\mu W^{-1/2}) + \frac{\beta\mu}{\alpha\lambda+\beta\mu} \log(W^{-1/2}B^\lambda W^{-1/2}) \geq 0,$$

which implies the desired result by using the preceding corollary. □

5.5 Notes and Remarks

The perspective function for a real valued continuous function is often appeared in the theory of convex analysis. The operator perspective of invertible positive operators is a non-commutative analogy of this. In Sect. 2, we state some fundamental properties of an operator perspective such as the equivalence between the operator convexity of the representation function and the joint convexity of the perspective [13, 15].

In Sect. 3, we study a two-variable functional calculus $\Phi(A, B)$ of positive operators A, B developed by W. Pusz and S. L. Woronowicz (PW-functional calculus for short)[49]. We introduce some important properties in [27, 33]. We defined the PW-functional calculus for a real valued continuous function with the condition (6). The set of such functions does not contain some important functions such as $\log t$ and $t \log t$, but $\Phi(A, B)$ can often be calculated in some restricted domain (Sect. 3.2.5).

The operator connection in the sense of Kubo and Ando [39] can be considered as the PW-functional calculus for a positive operator monotone function on $[0, \infty)$. Their axiomatization not only teaches us the essence of an operator connection, but also serves as a tool for checking whether a given binary operation is an operator connection. In Sect. 4, we introduce some fundamental properties of an operator mean and give some examples.

Thanks to the basic results of the operator perspective described in the previous section, some of the properties of the operator mean become apparent by taking limits.

Transformations $(\tilde{\ }, {}^*, {}^\perp)$ on the set of operator means are well-known [39]. In Sect. 4.2, the Barbour transform is introduced. Though it was first defined in [40], but before that, its sprouting can be seen in the papers of some authors [7, 43].

The fixed point with respect to the transformation $\sigma \mapsto \sigma^*$ is called a self-adjoint operator mean. F. Hansen's work concerning a characterization of that was groundbreaking [22]. We briefly mention the relations between a self-adjoint operator mean and a symmetric one proved by H. Osaka and S. Wada [46].

Inequalities involving operator perspectives and positive linear maps have been studied(e.g.,[1, 2, 9]). In Sect. 5.1, we discuss about how the operator convexity and the operator concavity of a representing function are reflected in the inequalities. Almost all statements are classical and fundamental.

For an operator mean σ, some statements comparing $A\sigma B$ and $A^r\sigma B^r$ are treated in Sect. 5.2. A central result (Proposition 5.2) is stated in [4, Theorem 2.1]. Functions satisfying the condition (i) in Corollary 5.2(resp. Corollary 5.4) is said to be power monotone increasing (resp. decreasing) [54] and has been studied in [31, 32, 55].

The Furuta inequality was developed as a generalization of the Löwner-Heinz inequality: $A \geq B \geq 0 \Rightarrow A^s \geq B^s \geq 0 \ \ (s \in [0, 1])$ [19]. Though the original proof is elegant and not very difficult, but we gave a slightly longer proof using the Ando–Hiai inequality in Sect. 5.3. The proof is given by M. Fujii and E. Kamei

[17]. They showed that the essential part of the Furuta inequality(Proposition 5.3) and the Ando–Hiai inequality (Corollary 5.6) imply each other.

For $A, B \in B(H)_{++}$, a (weaker) order defined by $\log A \geq \log B$ is called the chaotic order. Many equivalent conditions have been studied [16, 20, 21, 32, 53]. Among them, we introduce some statements that come from the Ando–Hiai inequality [32].

The last statement given here is developed by T. Furuta, M. Yanagida and T. Yamazaki [21]. We show this result using the method developed by T. Yamazaki in the theory of the multivariate Ando–Hiai type inequality [56].

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Cauchy–Schwarz Operator and Norm Inequalities for Inner Product Type Transformers in Norm Ideals of Compact Operators, with Applications

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Abstract In this survey paper we present operator and norm inequalities of Cauchy–Schwarz type:

$$\left\|\int_\Omega A_t X B_t \, d\mu(t)\right\|_\Psi \le \left\|\left(\int_\Omega A_t^* A_t \, d\mu(t)\right)^{1/2} X \left(\int_\Omega B_t B_t^* \, d\mu(t)\right)^{1/2}\right\|_\Psi,$$

for strongly square integrable operator families $\{A_t\}_{t\in\Omega}$, $\{B_t^*\}_{t\in\Omega}$ and symmetrically norming functions Ψ, such that the associated unitarily invariant norm is nuclear, Q* or arbitrary, under some additional commutativity conditions. The applications of this and complementary inequalities for Q and Schatten–von Neumann norms to Aczél–Bellman, Grüss–Landau, arithmetic–geometric, Young, Minkowski, Heinz, Zhan, Heron, and generalized derivation norm inequalities are also presented.

Keywords Norm inequalities · Elementary operators · i.p.t. transformers · Q and Q*-norms · Generalized derivations · Operator monotone functions · Subnormal · $\mathcal{N}$-hyper-accretive and $\mathcal{N}$-hyper-contractive operators

1 Introduction

In this paper we will denote respectively by $\mathbb{B}(\mathcal{H})$ and $\mathbb{K}(\mathcal{H})$ the spaces of all bounded and all compact linear operators on a separable, complex Hilbert space $\mathcal{H}$. Each symmetrically norming (s.n.) function Υ, defined on sequences of complex numbers, gives rise to the associated symmetric or a unitary invariant (u.i.) norm $\|\cdot\|_\Upsilon$ on operators, defined on the naturally associated norm ideal $\mathcal{C}_\Upsilon(\mathcal{H}) \subset \mathbb{K}(\mathcal{H})$. Most important example of s.n. functions is the trace s.n. function ℓ (denoted also by

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ℓ^1 or ℓ_1), defined by $\ell\big((\lambda_n)_{n=1}^{\infty}\big) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} |\lambda_n|$. For any s.n. function Υ, there is their adjoint s.n. function, which we denote by Υ^*. For any $p \geq 1$ a s.n. function Υ could be p modified and this p-modification $\Upsilon^{(p)}$also represents a new s.n. functions, for which the corresponding ideal of compact operators will be denoted by $\mathcal{C}_{\Upsilon^{(p)}}(\mathcal{H})$. Similarly, the ideal of compact operators, associated to dual s.n. function $\Upsilon^{(p)*}$ we will denote by $\mathcal{C}_{\Upsilon^{(p)*}}(\mathcal{H})$ and by $\mathcal{C}_{\Upsilon^{(p)}_{\circ}}(\mathcal{H})$ we will denote the closure of finite rank operators in the $\|\cdot\|_{\Upsilon^{(p)}}$ norm. The Schatten–von Neumann trace classes $\mathcal{C}_p(\mathcal{H}) \stackrel{\text{def}}{=} \mathcal{C}_{\ell^{(p)}}(\mathcal{H})$ are the most important and the best known examples of norm ideals associated to degree p modified (i.e. its s.n. function ℓ) norms. Amongst them, $\mathcal{C}_1(\mathcal{H})$ is also known as the class of nuclear operators, while $\mathcal{C}_2(\mathcal{H})$ is known as the Hilbert–Schmidt class. For the norm in $\mathcal{C}_p(\mathcal{H})$ we will use the simplified notation $\|\cdot\|_p$. For $p \geq 2$, all norms $\|\cdot\|_{\Upsilon^{(p)}}$ are also known as Q-norms, as $\Upsilon^{(p)} = (\Upsilon^{(\frac{p}{2})})^{(2)}$ and $\Upsilon^{(\frac{p}{2})}$ is also a s.n. function, while their associated dual norms $\|\cdot\|_{\Upsilon^{(p)*}}$ are also known as Q*-norms.

If $(\Omega, \mathfrak{M}, \mu)$ is a space Ω with a measure μ on σ-algebra $\mathfrak{M}$, consisting of (measurable) subsets of Ω, then we will refer to a function $A\colon \Omega \to \mathbb{B}(\mathcal{H})\colon t \mapsto A_t$ as to a ([$\mathfrak{M}$]) *weakly*-measurable* whenever $t \mapsto \langle A_t g, h\rangle$ is a ([$\mathfrak{M}$]) measurable for all $g, h \in \mathcal{H}$. If, in addition, those functions are [μ] integrable on Ω, then A is called ([μ]) *weakly*-integrable* on Ω, in which case there is the unique (known as Gel'fand or *weak*-integral* and denoted by $\int_\Omega A_t\, d\mu(t)$) operator in $\mathbb{B}(\mathcal{H})$, having the property

$$\left\langle \int_\Omega A_t\, d\mu(t)h, k\right\rangle = \int_\Omega \langle A_t h, k\rangle\, d\mu(t) \qquad \text{for all } h, k \in \mathcal{H}.$$

Note that $\int_\Omega A_t\, d\mu(t)$ also satisfies the definition of Pettis integral. For a more complete account about weak*-integrability of operator valued (o.v.) functions the interested reader is referred to [5, p.53], [15, p.320] and [20, lemma 1.2]. Let also $L^2(\Omega, \mu, \mathcal{H})$ denote the space of all (weakly) measurable functions $f\colon \Omega \to \mathcal{H}$ such that $\int_\Omega \|f(t)\|^2\, d\mu(t) < +\infty$, and similarly, let $L^2_G(\Omega, \mu, \mathbb{B}(\mathcal{H}))$ denote the space of all weak*-measurable functions $A\colon \Omega \to \mathbb{B}(\mathcal{H})$ such that $\int_\Omega \|A_t f\|^2\, d\mu(t) < +\infty$ for all $f \in \mathcal{H}$. In this case we say that A is ([μ]) *square integrable* (*s.i.*), and by analogy, a family $\{A_n\}_{n=1}^{\infty}$ in $\mathbb{B}(\mathcal{H})$ will be called a (*strongly*) square summable (*s.s.*) if $\sum_{n=1}^{\infty} \|A_n h\|^2 < +\infty$ for all $h \in \mathcal{H}$. Also, A^*A is Gel'fand integrable iff $A \in L^2_G(\Omega, \mu, \mathbb{B}(\mathcal{H}))$, as shown (amongst others) in [15, ex. 2]. If a family $\{A_t\}_{t\in\Omega}$ consists of mutually commuting normal operators, we will refer to it as to a *m.c.n.o.* family. If for $A \in L^2_G(\Omega, \mu, \mathbb{B}(\mathcal{H}))$ the associated family $\{A_t\}_{t\in\Omega}$ is a m.c.n.o. family, then we will refer to $\{A_t\}_{t\in\Omega}$ (and A) as to *s.i.* and *m.c.n.o.* family (o.v. function).

For $A, B \in \mathbb{B}(\mathcal{H})$ the *bilateral multiplier* $A \otimes B$ is defined by $A \otimes B\colon\ \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{H})\colon X \mapsto AXB$ and the *generalized derivation* $\Delta_{A,B}$ by $\Delta_{A,B} \stackrel{\text{def}}{=} A\otimes I + I\otimes B$. For sequences $\{A_n\}_{n=1}^{\infty}, \{B_n\}_{n=1}^{\infty}$ in $\mathbb{B}(\mathcal{H})$ and a norm ideal $\mathcal{C}_\Upsilon(\mathcal{H})$ the associated transformer $\sum_{n=1}^{\infty} A_n\otimes B_n$ is called *σ-elementary transformer* on $\mathcal{C}_\Upsilon(\mathcal{H})$ iff for every $X \in \mathcal{C}_\Upsilon(\mathcal{H})$ there is $Y \in \mathcal{C}_\Upsilon(\mathcal{H})$, such that $Y = {}^{[w]}\sum_{n=1}^{\infty} A_n X B_n \stackrel{\text{def}}{=} {}^{[w]}\lim_{n\to\infty} \sum_{k=1}^{n} A_k X B_k$, where ${}^{[w]}\lim_{n\to\infty}$ (resp. ${}^{[s]}\lim_{n\to\infty}$) denotes the weak (resp. strong) operator limit. More generally, if $A, B\colon \Omega \to \mathbb{B}(\mathcal{H})$ are weak*-measurable, such that $t \mapsto A_t X B_t$ is weak*-integrable on Ω for all $X \in \mathcal{C}_\Upsilon(\mathcal{H})$, then $\int_\Omega A_t\otimes B_t\,d\mu(t)\colon \mathcal{C}_\Upsilon(\mathcal{H}) \to \mathbb{B}(\mathcal{H})\colon X \mapsto \int_\Omega A_t X B_t\,d\mu(t)$ is called *inner product type (i.p.t.) transformer* on $\mathcal{C}_\Upsilon(\mathcal{H})$ if $\int_\Omega A_t X B_t\,d\mu(t) \in \mathcal{C}_\Upsilon(\mathcal{H})$ for all $X \in \mathcal{C}_\Upsilon(\mathcal{H})$. For related questions about the existence and different types of convergence for σ-elementary transformers, weak*-integrability, convergence properties for weak*-integrals and boundedness of i.p.t. transformers the interested reader is referred to [13, th. 2.2], [14, th. 2.1, cor. 2.1], [15, lemma 3.1, th. 3.2, th. 3.3, th. 3.4], [16, th. 2.1, th. 3.1, th. 3.2, cor. 3.1] and [26, lemma 2.1, th. 3.1].

Cauchy–Schwarz norm inequalities for i.p.t. transformers appear in different forms, depending on ideals of compact operators on which transformers act, as well as of normality and commutativity properties of s.i. families involved therein. By providing those new tools for investigation of i.p.t. transformers, this enables (amongst others) to threat derivation inequalities for different classes of operators, including N-hyper accretive, N-hyper contraction, quasinormal, hyponormal, subnormal and operators with the contractive real part. This gives new contributions to the perturbation theory for non necessarily normal operators in ideals of compact operators, well beyond scopes of the standard theory of double operator integrals (D.O.I.), developed earlier by Birman, Solomyak and its collaborators. For successful adaptation of D.O.I. and its applications to means inequalities for operators see [10, 11, 30].

Next, we recall definitions of some important subclasses of bounded operators on Hilbert spaces, which will be investigated in the sequel.

Definition 1.1 For operators $A, C \in \mathbb{B}(\mathcal{H})$ and $n, N \in \mathbb{N}$ we say that A is:

- *accretive* if $A^*+A \geq 0$;
- *strictly accretive* if there is a constant $c > 0$, such that $A^*+A \geq cI$, which will be denoted by $A^*+A > 0$;
- *N-hyper-accretive* if, and only if, $\sum_{k=0}^{n} \binom{n}{k} A^{*k}A^{n-k} \geq 0$ for all $1 \leq n \leq N$;
- *dissipative* if $\Im A \stackrel{\text{def}}{=} A_\Im \stackrel{\text{def}}{=} \frac{A-A^*}{2i} \geq 0$;
- *(co)hyponormal* if $(A^*A \leq AA^*)\, A^*A \geq AA^*$;
- *subnormal* if there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a normal operator N on $\mathcal{K}$ such that $N\mathcal{H} \subset \mathcal{H}$ and $A = N\restriction_{\mathcal{H}}$;
- *quasinormal* if A^*A commutes with A, i.e. $A^*AA = AA^*A$.

C is *N-hyper-contraction* if $\sum_{k=0}^{n} (-1)^k \binom{n}{k} C^{*k}C^k \geq 0$ for all $1 \leq n \leq N$.

Throughout this paper we use conventions $\mathbb{R}_+ \overset{\text{def}}{=} [0, +\infty)$, $\mathbb{C}_+ \overset{\text{def}}{=} \{z \in \mathbb{C} : \Im z \overset{\text{def}}{=} \frac{z-z^*}{2i} > 0\}$, $\mathbb{C}_- \overset{\text{def}}{=} \{z \in \mathbb{C} : \Im z < 0\}$ and $\ell^2_{\mathbb{Z}}(\mathcal{H})$ for the Hilbert space of all sequences $\{h_n\}_{n\in\mathbb{Z}}$ in $\mathcal{H}$, satisfying $\sum_{n\in\mathbb{Z}} \|h_n\|^2 < +\infty$.

We also emphasize that we will treat (address to) every unnumbered line in a multiline formula as (to) a part of the consequent numbered one.

2 Cauchy–Schwarz Operator and Norm Inequalities for i.p.t. Transformers

We start with different types of operator Cauchy–Schwarz inequalities.

Theorem 2.1 *Let $X \in \mathbb{B}(\mathcal{H})$ and $A^*, B \in L^2_G(\Omega, \mu, \mathbb{B}(\mathcal{H}))$.*

(a1) *Then $t \mapsto A_t X B_t$, acting on Ω, is weak*-integrable and for all $f, g \in \mathcal{H}$*

$$\Big(\int_\Omega \big|\langle A_t X B_t f, g\rangle\big|\, d\mu(t)\Big)^2 \le$$
$$\int_\Omega \langle A_t |X^*|^{2-2\theta} A_t^* g, g\rangle\, d\mu(t) \int_\Omega \langle B_t^* |X|^{2\theta} B_t f, f\rangle\, d\mu(t) \quad \text{for all } \theta \in [0, 1], \tag{1}$$

$$\Big|\int_\Omega A_t X B_t\, d\mu(t)\Big|^2 \le \Big\|\int_\Omega A_t A_t^*\, d\mu(t)\Big\| \int_\Omega B_t^* X^* X B_t\, d\mu(t). \tag{2}$$

(a2) *For every $\varepsilon > 0$*

$$\Big|\Big(\varepsilon I + \int_\Omega A_t A_t^*\, d\mu(t)\Big)^{-1/2} \int_\Omega A_t X B_t\, d\mu(t)\Big|^2 \le \int_\Omega B_t^* X^* X B_t\, d\mu(t). \tag{3}$$

(a3) *If $\int_\Omega A_t A_t^*\, d\mu(t)$ is additionally invertible, then εI appearing in the inequality (3) could be omitted.*

(b) *If, in addition, $\{A_t\}_{t\in\Omega}$ is a m.c.n.o. family, then*

$$\Big|\int_\Omega A_t X B_t\, d\mu(t)\Big|^2 \le \int_\Omega B_t^* X^* \Big(\int_\Omega A_t^* A_t\, d\mu(t)\Big) X B_t\, d\mu(t). \tag{4}$$

The inequality (1) is a special case $\mathcal{X}_t := X$, $\mathcal{C}_t := A_t$ and $\mathcal{D}_t := B_t$ for all $t \in \Omega$ in [15, th. 3.1(a)], inequalities (2), (3) and (a3) are proved in [26, lemma 2.1], while the inequality (4) is proved in [26, cor. 2.3].

For elementary operators the inequality (2) is further refined in [19, lemma 2.2], saying that

Lemma 2.2 *If $X \in \mathbb{B}(\mathcal{H})$ and $\{A_1, \ldots, A_N\}$, $\{B_1, \ldots, B_N\}$ are families in $\mathbb{B}(\mathcal{H})$ for $N \in \mathbb{N}$, then for all $c \geq \|\sum_{n=1}^{N} A_n^* A_n\|^{1/2} > 0$ we have the identity*

$$\Big|\sum_{n=1}^{N} A_n^* X B_n\Big|^2 + \sum_{n=1}^{N}\Big|A_n\Big(cI + \Big(c^2 I - \sum_{n=1}^{N} A_n^* A_n\Big)^{\frac{1}{2}}\Big)^{-1} \sum_{m=1}^{N} A_m^* X B_m - cXB_n\Big|^2$$

$$= c^2 \sum_{n=1}^{N} |XB_n|^2.$$

$$\text{Consequently,} \quad \Big|\sum_{n=1}^{N} A_n^* X B_n\Big|^2 \leq \Big\|\sum_{n=1}^{N} A_n^* A_n\Big\| \sum_{n=1}^{N} B_n^* X^* X B_n, \tag{5}$$

and thus necessary and sufficient conditions for equality to take place in (5) are that there exists $D \in \mathbb{B}(\mathcal{H})$ such that $A_n D = XB_n$ for all $n = 1, \ldots, N$ and $\big(\|\sum_{n=1}^{N} A_n^* A_n\| - \sum_{n=1}^{N} A_n^* A_n\big)D = 0$.

The fundamental role in investigation of i.p.t. transformers is played by the following list of Cauchy–Schwarz norm inequalities.

Theorem 2.3 *Let $p \geq 2$, Ψ, Υ be s.n. functions and $X \in \mathbb{B}(\mathcal{H})$, then $\int_\Omega A_t X B_t \, d\mu(t) \in \mathcal{C}_\Psi(\mathcal{H})$:*

(a) *if $A, B^* \in L^2_G(\Omega, \mu, \mathbb{B}(\mathcal{H}))$ and $\big(\int_\Omega A_t^* A_t \, d\mu(t)\big)^{\frac{1}{2}} X \big(\int_\Omega B_t B_t^* \, d\mu(t)\big)^{\frac{1}{2}} \in \mathcal{C}_\Psi(\mathcal{H})$, in which case*

$$\Big\|\int_\Omega A_t X B_t \, d\mu(t)\Big\|_\Psi \leq \Big\|\Big(\int_\Omega A_t^* A_t \, d\mu(t)\Big)^{\frac{1}{2}} X \Big(\int_\Omega B_t B_t^* \, d\mu(t)\Big)^{\frac{1}{2}}\Big\|_\Psi, \tag{6}$$

under any of the following conditions:

(a1) *μ is a σ-finite measure on Ω and $\|\cdot\|_\Psi := \|\cdot\|_1$,*

(a2) *$L^2(\Omega, \mu)$ is separable, $\Psi := \Upsilon^{(p)*}$ and (at least) one of families $\{A_t\}_{t\in\Omega}$ or $\{B_t\}_{t\in\Omega}$ is a m.c.n.o. family. If, in addition, also $X \in \mathcal{C}_{\Upsilon^{(p)*}}(\mathcal{H})$, then (6) remains valid if (at least) one of families $\{A_t\}_{t\in\Omega}$ or $\{B_t^*\}_{t\in\Omega}$ is a u.e.2s.i.m.c.n.o. family,*

(a3) *μ is a σ-finite measure on Ω, $\|\cdot\|_\Psi := |||\cdot|||$ and both $\{A_t\}_{t\in\Omega}$ and $\{B_t\}_{t\in\Omega}$ are m.c.n.o. families. If, in addition, $X \in \mathcal{C}_{|||\cdot|||}(\mathcal{H})$, then (6) remains valid if both families $\{A_t\}_{t\in\Omega}$ and $\{B_t^*\}_{t\in\Omega}$ are u.e.2s.i.m.c.n.o. families;*

(b) *if $A, B \in L^2_G(\Omega, \mu, \mathbb{B}(\mathcal{H}))$ and $\big(\int_\Omega A_t^* A_t \, d\mu(t)\big)^{\frac{1}{2}} X \in \mathcal{C}_\Psi(\mathcal{H})$, in which case*

$$\Big\|\int_\Omega A_t X B_t \, d\mu(t)\Big\|_\Psi \leq \Big\|\Big(\int_\Omega A_t^* A_t \, d\mu(t)\Big)^{\frac{1}{2}} X\Big\|_\Psi \Big\|\int_\Omega B_t^* B_t \, d\mu(t)\Big\|^{\frac{1}{2}}, \tag{7}$$

under any of the following conditions:

(b1) $\|\cdot\|_\Psi := \|\cdot\|_2$,

(b2) $\Psi := \Upsilon^{(p)}$ *and* $\{A_t\}_{t\in\Omega}$ *is additionally a m.c.n.o. family,*

(b3) $\Psi := \Upsilon^{(p)}$, $X \in \mathcal{C}_{\Upsilon^{(p)}}(\mathcal{H})$ (*additionally*) *and* $\{A_t\}_{t\in\Omega}$ *is additionally a u.e.2s.i.m.c.n.o. family;*

(c) *if under conditions of* (b) *operator* $\int_\Omega B_t^* B_t \, d\mu(t)$ *is invertible, then*

$$\left\| \int_\Omega A_t X B_t \, d\mu(t) \left(\int_\Omega B_t^* B_t \, d\mu(t) \right)^{-\frac{1}{2}} \right\|_\Psi \leq \left\| \left(\int_\Omega A_t^* A_t \, d\mu(t) \right)^{\frac{1}{2}} X \right\|_\Psi, \tag{8}$$

under any of the following conditions:

(c1) $\|\cdot\|_\Psi := \|\cdot\|_2$,

(c2) $\Psi := \Upsilon^{(p)}$ *and* $\{A_t\}_{t\in\Omega}$ *is additionally a m.c.n.o. family;*

(d) *if* $A^*, B^* \in L^2_G(\Omega, \mu, \mathbb{B}(\mathcal{H}))$ *and* $X\left(\int_\Omega B_t B_t^* \, d\mu(t)\right)^{\frac{1}{2}} \in \mathcal{C}_\Psi(\mathcal{H})$,

$$\left\| \int_\Omega A_t X B_t \, d\mu(t) \right\|_\Psi \leq \left\| \int_\Omega A_t A_t^* \, d\mu(t) \right\|^{\frac{1}{2}} \left\| X \left(\int_\Omega B_t B_t^* \, d\mu(t) \right)^{\frac{1}{2}} \right\|_\Psi, \tag{9}$$

under any of the following conditions:

(d1) $\|\cdot\|_\Psi := \|\cdot\|_2$,

(d2) $\Psi := \Upsilon^{(p)}$ *and* $\{B_t\}_{t\in\Omega}$ *is additionally a m.c.n.o. family,*

(d3) $\Psi := \Upsilon^{(p)}$, $X \in \mathcal{C}_{\Upsilon^{(p)}}(\mathcal{H})$ (*additionally*) *and* $\{B_t^*\}_{t\in\Omega}$ *is additionally a u.e.2s.i.m.c.n.o. family;*

(e) *if under conditions of* (d) *operator* $\int_\Omega A_t A_t^* \, d\mu(t)$ *is invertible, then*

$$\left\| \left(\int_\Omega A_t A_t^* \, d\mu(t) \right)^{-\frac{1}{2}} \int_\Omega A_t X B_t \, d\mu(t) \right\|_\Psi \leq \left\| X \left(\int_\Omega B_t B_t^* \, d\mu(t) \right)^{\frac{1}{2}} \right\|_\Psi, \tag{10}$$

under any of the following conditions:

(e1) $\|\cdot\|_\Psi := \|\cdot\|_2$,

(e2) $\Psi := \Upsilon^{(p)}$ *and* $\{B_t\}_{t\in\Omega}$ *is additionally a m.c.n.o. family.*

The inequality:

- (6) in the case (a1) of Theorem 2.3 is exactly the case $p := q := r := 1$ of the inequality (24) in [15, th. 3.3];
- (6) in the case (a2) of Theorem 2.3 is the inequality (32) in [26, th. 3.1(d)]. If $X \in \mathcal{C}_{\Upsilon^{(p)*}}(\mathcal{H})$, then (6) is the inequality (3.3) in [18, th. 3.1(c)];

- (6) in the case (a3) of Theorem 2.3 is the inequality (23) in [15, th. 3.2]. In the case of the counting measure μ of $\Omega := \mathbb{N}$, the inequality (6) was proved earlier in [13, th. 2.2]. If $X \in \mathcal{C}_{|||\cdot|||}(\mathcal{H})$, then (6) is the inequality (3.4) in [18, th. 3.1(d)] ;
- (7) in the case (b1) (resp. (b2)) of Theorem 2.3 is exactly the inequality (28) in [26, th. 3.1(b)] (resp. the special case $\mathscr{C}_t := A_t^*$ and $\mathscr{D}_t := B_t$ for all $t \in \Omega$ of the inequality (33) in [21, lemma 3.4]). In the case (b3) the inequality (7) is exactly the inequality (3.1) in [18, th. 3.1(a)];
- (9) in the case (d1) (resp. (d2)) of Theorem 2.3 is exactly the inequality (29) in [26, th. 3.1(b)] (resp. the special case $\mathscr{C}_t := A_t^*$ and $\mathscr{D}_t := B_t$ for all $t \in \Omega$ of the inequality (34) in [21, lemma 3.4]). In the case (d3) the inequality (7) is exactly the inequality (3.2) in [18, th. 3.1(b)];
- (8) in the case (c1) (resp. (c2)) in Theorem 2.3 derives similarly by applying the inequality (28) in [26, th. 3.1(b)] to $L^2_G(\Omega, \mu, \mathbb{B}(\mathcal{H}))$ families $\{A_t\}_{t\in\Omega}$ and $\big\{B_t\big(\int_\Omega B_t^* B_t \, d\mu(t)\big)^{-\frac{1}{2}}\big\}_{t\in\Omega}$ (instead of $\{B_t\}_{t\in\Omega}$) (resp. the inequality (33) in [21, lemma 3.4] to s.i. families $\{\mathscr{C}_t^*\}_{t\in\Omega}, \{\mathscr{D}_t\}_{t\in\Omega}$, given by $\mathscr{C}_t := A_t^*$ and $\mathscr{D}_t := B_t\big(\int_\Omega B_t^* B_t \, d\mu(t)\big)^{-\frac{1}{2}}$ for all $t \in \Omega$);
- (10) in the case (e1) (resp. (e2)) in Theorem 2.3 proves by analogy to the proof of the case (c1) (resp. (c2)).

  We conclude our list of Cauchy–Schwarz inequalities for i.p.t. transformers within the context of Schatten–von Neumann ideals.

Theorem 2.4 *If μ is a σ-finite measure on Ω and $A, A^*, B, B^* \in L^2_G(\Omega, \mu, \mathbb{B}(\mathcal{H}))$, then for all $1 \le p, q, r < +\infty$ satisfying $\frac{2}{p} = \frac{1}{q} + \frac{1}{r}$ and $X \in \mathcal{C}_p(\mathcal{H})$*

$$\left\| \int_\Omega A_t X B_t \, d\mu(t) \right\|_p \le \tag{11}$$

$$\left\| \sqrt[2q]{\int_\Omega A_t^* \left(\int_\Omega A_t A_t^* \, d\mu(t) \right)^{q-1} A_t \, d\mu(t)} \, X \sqrt[2r]{\int_\Omega B_t \left(\int_\Omega B_t^* B_t \, d\mu(t) \right)^{r-1} B_t^* \, d\mu(t)} \right\|_p ,$$

while for all $2 \le p < +\infty$

$$\left\| \int_\Omega A_t X B_t \, d\mu(t) \right\|_p \le$$

$$\left\| \int_\Omega A_t A_t^* \, d\mu(t) \right\|^{\frac{1}{2}} \left\| X \sqrt[p]{\int_\Omega B_t \left(\int_\Omega B_t^* B_t \, d\mu(t) \right)^{\frac{p}{2}-1} B_t^* \, d\mu(t)} \right\|_p , \tag{12}$$

$$\left\| \int_\Omega A_t X B_t \, d\mu(t) \right\|_p \le$$

$$\left\| \sqrt[p]{\int_\Omega A_t^* \left(\int_\Omega A_t A_t^* \, d\mu(t) \right)^{\frac{p}{2}-1} A_t \, d\mu(t)} \, X \right\|_p \left\| \int_\Omega B_t^* B_t \, d\mu(t) \right\|^{\frac{1}{2}} . \tag{13}$$

If, in addition, $\int_\Omega A_t A_t^* \, d\mu(t)$ *and* $\int_\Omega B_t^* B_t \, d\mu(t)$ *are invertible, then*

$$\left\| \left(\int_\Omega A_t A_t^* \, d\mu(t) \right)^{\frac{1}{2q} - \frac{1}{2}} \int_\Omega A_t X B_t \, d\mu(t) \left(\int_\Omega B_t^* B_t \, d\mu(t) \right)^{\frac{1}{2r} - \frac{1}{2}} \right\|_p \leq \left\| \left(\int_\Omega A_t^* A_t \, d\mu(t) \right)^{\frac{1}{2q}} X \left(\int_\Omega B_t B_t^* \, d\mu(t) \right)^{\frac{1}{2r}} \right\|_p, \tag{14}$$

while for all $2 \leq p < +\infty$

$$\left\| \left(\int_\Omega A_t A_t^* \, d\mu(t) \right)^{\frac{1}{p} - \frac{1}{2}} \int_\Omega A_t X B_t \, d\mu(t) \left(\int_\Omega B_t^* B_t \, d\mu(t) \right)^{-\frac{1}{2}} \right\|_p \leq \left\| \left(\int_\Omega A_t^* A_t \, d\mu(t) \right)^{\frac{1}{p}} X \right\|_p, \tag{15}$$

$$\left\| \left(\int_\Omega A_t A_t^* \, d\mu(t) \right)^{-\frac{1}{2}} \int_\Omega A_t X B_t \, d\mu(t) \left(\int_\Omega B_t^* B_t \, d\mu(t) \right)^{\frac{1}{p} - \frac{1}{2}} \right\|_p \leq \left\| X \left(\int_\Omega B_t B_t^* \, d\mu(t) \right)^{\frac{1}{p}} \right\|_p. \tag{16}$$

The inequality (11) was proved in [15, th. 3.3], while the inequality (12) (resp. (13)) represents the special case $q := +\infty$ (resp. $r := +\infty$), which actually follows from the proof of [15, th. 3.3]. In the case of the counting measure μ of $\Omega := \mathbb{N}$, the special case $p := q := r$ of the inequality (11) was proved earlier in [14, th. 2.1].

The inequality (14) follows directly by applying the inequality (11) to the s.i. family $\left\{ \left(\int_\Omega A_t A_t^* \, d\mu(t) \right)^{\frac{1}{2q} - \frac{1}{2}} A_t \right\}_{t \in \Omega}$ instead of $\{A_t\}_{t \in \Omega}$ and to the s.i. family $\left\{ B_t \left(\int_\Omega B_t^* B_t \, d\mu(t) \right)^{\frac{1}{2r} - \frac{1}{2}} \right\}_{t \in \Omega}$ instead of $\{B_t\}_{t \in \Omega}$.

Similarly, the inequality (15) (resp. (16)) follows immediately from the inequality (12) (resp. (13)) applied to s.i. families $\left\{ \left(\int_\Omega A_t A_t^* d\mu(t) \right)^{\frac{1}{p} - \frac{1}{2}} A_t \right\}_{t \in \Omega}$ and $\left\{ B_t \left(\int_\Omega B_t^* B_t \, d\mu(t) \right)^{-\frac{1}{2}} \right\}_{t \in \Omega}$ (resp. to s.i. families $\left\{ \left(\int_\Omega A_t A_t^* d\mu(t) \right)^{-\frac{1}{2}} A_t \right\}_{t \in \Omega}$ and $\left\{ B_t \left(\int_\Omega B_t^* B_t \, d\mu(t) \right)^{\frac{1}{p} - \frac{1}{2}} \right\}_{t \in \Omega}$.

3 Aczél–Bellman Type Norm Inequalities for i.p.t. Transformers

Aczél–Bellman u.i. norm inequality for i.p.t. transformers was presented in [15, th. 4.1], saying that

Theorem 3.1 *If μ is a σ-finite measure on Ω, $\{A_t\}_{t\in\Omega}$ and $\{B_t\}_{t\in\Omega}$ are m.c.n.o. families in $L^2_G(\Omega,\mu,\mathbb{B}(\mathcal{H}))$, such that $\int_\Omega A_t^* A_t \, d\mu(t)$ and $\int_\Omega B_t B_t^* d\mu(t)$ are contractions and $X-\int_\Omega A_t X B_t \, d\mu(t)\in \mathcal{C}_{|||\cdot|||}(\mathcal{H})$ for some $X\in\mathbb{B}(\mathcal{H})$, then*

$$\left|\left|\left|\left(I-\int_\Omega A_t^*A_t\,d\mu(t)\right)^{\frac{1}{2}} X\left(I-\int_\Omega B_t B_t^* d\mu(t)\right)^{\frac{1}{2}}\right|\right|\right| \le \left|\left|\left|X-\int_\Omega A_t X B_t\,d\mu(t)\right|\right|\right|. \tag{17}$$

This complements the Cauchy–Schwarz u.i. norm inequality [15, th. 3.2], also generalizing the previous inequality in [13, th. 2.3] in two directions.

In the sequel, we will need the following terminology.

Definition 3.2 For an analytic function f on some neighborhood of zero, let $R_f \overset{\text{def}}{=} \left(\limsup_{n\to\infty}\sqrt[n]{|c_n|}\,\right)^{-1}$ denotes the *radius of convergence* for its Taylor (power) series ($f(z)=$) $\sum_{n=0}^{\infty} c_n z^n$, with $0<R_f\le+\infty$. For $A\in L^2_G(\Omega,\mu,\mathbb{B}(\mathcal{H}))$ and for an analytic function f with non-negative Taylor coefficients, satisfying $r\big(\int_\Omega A_t^*\otimes A_t\,d\mu(t)\big) \overset{\text{def}}{=} \inf_{n\in\mathbb{N}} \big\|\big(\int_\Omega A_t^*\otimes A_t\,d\mu(t)\big)^n(I)\big\|^{\frac{1}{n}} \le R_f$ (when transformer $\int_\Omega A_t^*\otimes A_t\,d\mu(t)$ acts on $\mathbb{B}(\mathcal{H})$) and $f(0)>0$, we define its associated *generalized spectral defect operator* $\Delta_{f,\boldsymbol{A}}$ by

$$\Delta_{f,\boldsymbol{A}} \overset{\text{def}}{=} {}^{[s]}\lim_{\rho\nearrow 1}\Big(f\big(\rho^2\textstyle\int_\Omega A_t^*\otimes A_t\,d\mu(t)\big)(I)\Big)^{-1/2}.$$

For a function $f:\mathbb{D}\to\mathbb{C}\colon z\mapsto\sum_{n=0}^{\infty} z^n$ (in which case $R_f=1$) $\Delta_A:=\Delta_{f,\boldsymbol{A}}$ was introduced by [21, def. 2.1], while in the case that μ is the counting measure on $\mathbb{N}$ operator $\Delta_{f,\boldsymbol{A}}$ was given in [23, def. 2.3]. In both cases above it follows from [21, lemma 2.2] and [23, rem. 2, rem. 7] that $\Delta_{f,\boldsymbol{A}} = f\big(\int_\Omega A_t^*A_t\,d\mu(t)\big)^{-1/2}$ if $\boldsymbol{A}:=\{A_t\}_{t\in\Omega}$ is s.i. and m.c.n.o. family.

The above definition helps us in formulation of the following generalization of Theorem 3.1, which reformulates [21, th. 3.5]:

Theorem 3.3 *Let Υ be a s.n. function and $p\ge 2$. If $A^*, B\in L^2_G(\Omega,\mu,\mathbb{B}(\mathcal{H}))$ are such that $r\big(\int_\Omega A_t\otimes A_t^*\,d\mu(t)\big)\le 1$, $r\big(\int_\Omega B_t^*\otimes B_t\,d\mu(t)\big)\le 1$ and at least one of families $\{A_t\}_{t\in\Omega}$ or $\{B_t\}_{t\in\Omega}$ is a m.c.n.o. family, then for $X\in\mathcal{C}_{\Upsilon^{(p)}}(\mathcal{H})$*

$$\|\Delta_{A^*}X\Delta_B\|_{\Upsilon^{(p)}} \leqslant \left\|X-\int_\Omega A_t X B_t\,d\mu(t)\right\|_{\Upsilon^{(p)}}.$$

Theorem 3.4 *Let Υ be a s.n. function, $p \geq 2$, $X \in \mathcal{C}_{\Upsilon^{(p)*}}(\mathcal{H})$, $L^2(\Omega,\mu)$ be separable and let $A, B^* \in L^2_G(\Omega,\mu,\mathbb{B}(\mathcal{H}))$ be such that at least one of families $\{A_t\}_{t\in\Omega}$ or $\{B_t^*\}_{t\in\Omega}$ is a m.c.n.o. family.*

(a) *If, in addition, $\sum_{n=1}^{\infty} \int_{\Omega^n} \big(\|A_{t_1}\cdots A_{t_n}h\|^2 + \|B_{t_1}^*\cdots B_{t_n}^* h\|^2\big)\, d\mu^n(t_1,\ldots,t_n) < +\infty$ for all $h \in \mathcal{H}$, then $r\big(\int_\Omega A_t^*\otimes A_t\, d\mu(t)\big) \leq 1$, $r\big(\int_\Omega B_t\otimes B_t^*\, d\mu(t)\big) \leq 1$ and*

$$\|X\|_{\Upsilon^{(p)*}} \leq \Big\|\Delta_A^{-1}\Big(X - \int_\Omega A_t X B_t\, d\mu(t)\Big)\Delta_{B^*}^{-1}\Big\|_{\Upsilon^{(p)*}}.$$

(b) *Alternatively, if additionally $\int_\Omega A_t^* A_t\, d\mu(t) \leq I$ and $\int_\Omega B_t B_t^*\, d\mu(t) \leq I$, then the inequality (17) remains valid if $|||\cdot|||$ is replaced by $\|\cdot\|_{\Upsilon^{(p)*}}$.*

The case (a) (resp. (b)) of Theorem 3.4 is a reformulation of the case (a) (resp. (c)) of [26, th. 3.2].

In the context of Schatten–von Neumann ideals the counterpart of Theorems 3.1, 3.3 and 3.4 says:

Theorem 3.5 *Let $A^*, B \in L^2_G(\Omega,\mu,\mathbb{B}(\mathcal{H}))$ be such that $r\big(\int_\Omega A_t\otimes A_t^*\, d\mu(t)\big) \leq 1$ and $r\big(\int_\Omega B_t^*\otimes B_t\, d\mu(t)\big) \leq 1$. Then for all $X \in \mathbb{B}(\mathcal{H})$*

$$\|\Delta_{A^*} X \Delta_B\| \leq \Big\|X - \int_\Omega A_t X B_t\, d\mu(t)\Big\|.$$

If additionally $p \geq 2$ and $\sum_{n=1}^{\infty} \int_{\Omega^n} \|A_{t_1}\cdots A_{t_n}h\|^2 d\mu^n(t_1,\ldots,t_n) < +\infty$ for all $h \in \mathcal{H}$, then $r\big(\int_\Omega A_t^*\otimes A_t\, d\mu(t)\big) \leq 1$ and for all $X \in \mathcal{C}_p(\mathcal{H})$

$$\Big\|\Delta_{A^*}^{1-\frac{2}{p}} X \Delta_B\Big\|_p \leq \Big\|\Delta_A^{-\frac{2}{p}}\Big(X - \int_\Omega A_t X B_t\, d\mu(t)\Big)\Big\|_p.$$

Similarly, if $p \geq 2$ and $\sum_{n=1}^{\infty} \int_{\Omega^n} \|B_{t_1}^*\cdots B_{t_n}^* h\|^2 d\mu^n(t_1,\ldots,t_n) < +\infty$ for all $h \in \mathcal{H}$, then $r\big(\int_\Omega B_t\otimes B_t^*\, d\mu(t)\big) \leq 1$ and for all $X \in \mathcal{C}_p(\mathcal{H})$

$$\Big\|\Delta_{A^*} X \Delta_B^{1-\frac{2}{p}}\Big\|_p \leq \Big\|\Big(X - \int_\Omega A_t X B_t\, d\mu(t)\Big)\Delta_{B^*}^{-\frac{2}{p}}\Big\|_p.$$

If $\sum_{n=1}^{\infty} \int_{\Omega^n} \big(\|A_{t_1}\cdots A_{t_n}h\|^2 + \|B_{t_1}^*\cdots B_{t_n}^* h\|^2\big)\, d\mu^n(t_1,\ldots,t_n) < +\infty$ for all $h \in \mathcal{H}$ and $p,q,r \geq 1$ are such that $\frac{2}{p} = \frac{1}{q} + \frac{1}{r}$, then $r\big(\int_\Omega A_t^*\otimes A_t\, d\mu(t)\big) \leq 1$, $r\big(\int_\Omega B_t\otimes B_t^*\, d\mu(t)\big) \leq 1$ and for all $X \in \mathcal{C}_p(\mathcal{H})$

$$\Big\|\Delta_{A^*}^{1-\frac{1}{q}} X \Delta_B^{1-\frac{1}{r}}\Big\|_p \leq \Big\|\Delta_A^{-\frac{1}{q}}\Big(X - \int_\Omega A_t X B_t\, d\mu(t)\Big)\Delta_{B^*}^{-\frac{1}{r}}\Big\|_p.$$

Theorem 3.5 is just a modest reformulation of [21, th. 3.1].

4 Norm Inequalities for Transformers Generated by Analytic Functions with Non-negative Taylor Coefficients

In this section we will show that Aczél–Bellman type norm inequalities in the previous section are just a part of a wider family of Cauchy–Schwarz norm inequalities.

Theorem 4.1 *Let* Ψ, Υ *be s.n. functions, let* $p \geqslant 2$, *let* f *be an analytic function with non-negative Taylor coefficients and* $X \in \mathcal{C}_\Psi(\mathcal{H})$. *Then*

$$\left\| f\Big(\sum_{n=1}^{\infty} A_n \otimes B_n\Big) X \right\|_\Psi \leqslant \left\| \sqrt{f\Big(\sum_{n=1}^{\infty} A_n^* \otimes A_n\Big)(I)}\, X \sqrt{f\Big(\sum_{n=1}^{\infty} B_n \otimes B_n^*\Big)(I)} \right\|_\Psi,$$

if both $\{A_n\}_{n=1}^\infty$ *and* $\{B_n^*\}_{n=1}^\infty$ *are s.s. families such that* $\left\|\sum_{n=1}^\infty A_n^* A_n\right\| < R_f$, $\left\|\sum_{n=1}^\infty B_n B_n^*\right\| < R_f$ *and one of additional pair of conditions are satisfied:*

(a) $\Psi := \Upsilon^{(p)*}$ *and at least one of* $\{A_n\}_{n=1}^\infty$ *or* $\{B_n^*\}_{n=1}^\infty$ *is a m.c.n.o. family,*
(b) $\|\cdot\|_\Psi := |||\cdot|||$ *and both* $\{A_n\}_{n=1}^\infty$ *and* $\{B_n^*\}_{n=1}^\infty$ *are m.c.n.o. families.*
(c) *If* $\Psi := \Upsilon^{(p)}$, $\{A_n^*\}_{n=1}^\infty$ *and* $\{B_n\}_{n=1}^\infty$ *are s.s. families, such that* $\left\|\sum_{n=1}^\infty A_n A_n^*\right\| < R_f$, $\left\|\sum_{n=1}^\infty B_n^* B_n\right\| < R_f$, *then:*

$$\left\| f\Big(\sum_{n=1}^{\infty} A_n \otimes B_n\Big) X \right\|_{\Upsilon^{(p)}} \leq \left\| \sqrt{f\Big(\sum_{n=1}^{\infty} A_n^* A_n\Big)} X \right\|_{\Upsilon^{(p)}} \left\| f\Big(\sum_{n=1}^{\infty} B_n^* \otimes B_n\Big)(I) \right\|^{\frac{1}{2}}$$

(c1) *if, in addition,* $\{A_n\}_{n=1}^\infty$ *is a m.c.n.o. family;*

$$\left\| f\Big(\sum_{n=1}^{\infty} A_n \otimes B_n\Big) X \right\|_{\Upsilon^{(p)}} \leq \left\| f\Big(\sum_{n=1}^{\infty} A_n \otimes A_n^*\Big)(I) \right\|^{\frac{1}{2}} \left\| X \sqrt{f\Big(\sum_{n=1}^{\infty} B_n^* B_n\Big)} \right\|_{\Upsilon^{(p)}}$$

(c2) *if, in addition,* $\{B_n\}_{n=1}^\infty$ *is a m.c.n.o. family.*

$$\left\| f\Big(\sum_{n=1}^{\infty} A_n \otimes B_n\Big)(X) \Big(f\Big(\sum_{n=1}^{\infty} B_n^* \otimes B_n\Big)(I)\Big)^{-\frac{1}{2}} \right\|_{\Upsilon^{(p)}} \leq \left\| \sqrt{f\Big(\sum_{n=1}^{\infty} A_n^* A_n\Big)} X \right\|_{\Upsilon^{(p)}}$$

(d1) *if* $f\Big(\sum_{n=1}^{\infty} B_n^* \otimes B_n\Big)(I)$ *is invertible in addition to conditions* (c) *and* (c1)*;*

$$\left\| \Big(f\Big(\sum_{n=1}^{\infty} A_n \otimes A_n^*\Big)(I)\Big)^{-\frac{1}{2}} f\Big(\sum_{n=1}^{\infty} A_n \otimes B_n\Big) X \right\|_{\Upsilon^{(p)}} \leq \left\| X \sqrt{f\Big(\sum_{n=1}^{\infty} B_n^* B_n\Big)} \right\|_{\Upsilon^{(p)}}$$

(d2) *if, in addition to conditions* (c) *and* (c2), $f\Big(\sum_{n=1}^{\infty} A_n \otimes A_n^*\Big)(I)$ *is invertible.*

Theorem 4.1(a) (resp. (b)) is a reformulation of [23, th. 2.4(a)] (resp. [23, th. 2.2]), while cases (c1) and (c2) are based on [23, th. 2.4(b)]. The case (d1) (resp. (d2)) follows by applying the inequality (8) in Theorem 2.3(c2) (resp. the inequality (10) in Theorem 2.3(e2)) to the counting measure μ on $\Omega := \{0\} \sqcup \bigsqcup_{m=1}^{\infty} \mathbb{N}^m$ and to families $\{\sqrt{c_0} I\} \cup \big\{\sqrt{c_m} A_{n_m^{(m)}} \cdots A_{n_1^{(m)}}\big\}_{\substack{1 \le k \le m \\ 1 \le n_k^{(m)}}}$ and $\{\sqrt{c_0} I\} \cup \big\{\sqrt{c_m} B_{n_1^{(m)}} \cdots B_{n_m^{(m)}}\big\}_{\substack{1 \le k \le m \\ 1 \le n_k^{(m)}}}$, combined with the arguments already used in the proof of [23, th. 2.2].

For two-side multipliers $A \otimes B$ Theorem 4.1 remains valid under some additional subnormality conditions for A and B^*, as stated in [18, th. 3.2]:

Theorem 4.2 *Let Ψ, Υ be s.n. functions, $p \ge 2$, $f(z) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} c_n z^n$ be analytic function with non-negative Taylor coefficients $c_n \ge 0$ and $A, B \in \mathbb{B}(\mathcal{H})$ have their spectra $\sigma(A) \cup \sigma(B) \subset \mathbb{D}\big(0, R_f^{1/2}\big)$. Then for all $X \in \mathcal{C}_\Psi(\mathcal{H})$*

$$\Big\| \sum_{n=0}^{\infty} c_n A^n X B^n \Big\|_\Psi = \| f(A \otimes B) X \|_\Psi \le \big\| \sqrt{f(A^* \otimes A)(I)}\, X \big\|_\Psi \| f(B^* \otimes B)(I) \|^{\frac{1}{2}}$$

$$= \Big\| \Big(\sum_{n=0}^{\infty} c_n A^{*n} A^n \Big)^{\frac{1}{2}} X \Big\|_\Psi \Big\| \sum_{n=0}^{\infty} c_n B^{*n} B^n \Big\|^{\frac{1}{2}}$$

(a1) *if $\Psi := \Upsilon^{(p)}$ and A is subnormal,*
(a2) *if $\|\cdot\|_\Psi := \|\cdot\|_2$;*

$$\Big\| \sum_{n=0}^{\infty} c_n A^n X B^n \Big\|_\Psi \le \big\| \sqrt{f(A^* A)}\, X \big\|_\Psi \Big\| \sum_{n=0}^{\infty} c_n B^{*n} B^n \Big\|^{\frac{1}{2}}$$

(b) *if $\Psi := \Upsilon^{(p)}$ and A is quasinormal;*

$$\Big\| \sum_{n=0}^{\infty} c_n A^n X B^n \Big\|_\Psi \le \| f(A \otimes A^*)(I) \|^{\frac{1}{2}} \big\| X \sqrt{f(B \otimes B^*)(I)} \big\|_\Psi$$

$$= \Big\| \sum_{n=0}^{\infty} c_n A^n A^{*n} \Big\|^{\frac{1}{2}} \Big\| X \Big(\sum_{n=0}^{\infty} c_n B^n B^{*n} \Big)^{\frac{1}{2}} \Big\|_\Psi$$

(c1) *if $\Psi := \Upsilon^{(p)}$ and B^* is subnormal,*
(c2) *if $\|\cdot\|_\Psi := \|\cdot\|_2$;*

$$\Big\| \sum_{n=0}^{\infty} c_n A^n X B^n \Big\|_\Psi \le \Big\| \sum_{n=0}^{\infty} c_n A^n A^{*n} \Big\|^{\frac{1}{2}} \big\| X \sqrt{f(B B^*)} \big\|_\Psi$$

(d) *if* $\Psi := \Upsilon^{(p)}$ *and* B^* *is quasinormal;*

$$\Big\|\sum_{n=0}^{\infty} c_n A^n X B^n\Big\|_\Psi \le \big\|\sqrt{f(A^*\otimes A)(I)}X\sqrt{f(B\otimes B^*)(I)}\big\|_\Psi$$

$$= \Big\|\Big(\sum_{n=0}^{\infty} c_n A^{*n}A^n\Big)^{\frac{1}{2}} X\Big(\sum_{n=0}^{\infty} c_n B^n B^{*n}\Big)^{\frac{1}{2}}\Big\|_\Psi$$

(e1) *if* $\Psi := \Upsilon^{(p)*}$ *and* A *or* B^* *is subnormal,*
(e2) *if* $\|\cdot\|_\Psi := |||\cdot|||$ *and both* A *and* B^* *are subnormal,*
(e3) *if* $\|\cdot\|_\Psi := \|\cdot\|_1$;

$$\Big\|\sum_{n=0}^{\infty} c_n A^n X B^n\Big\|_\Psi \le \Big\|\Big(\sum_{n=0}^{\infty} c_n A^{*n}A^n\Big)^{\frac{1}{2}} X\sqrt{f(BB^*)}\Big\|_\Psi$$

(f1) *if* $\Psi := \Upsilon^{(p)*}$ *and* B^* *is quasinormal,*
(f2) *if* $\|\cdot\|_\Psi := |||\cdot|||$, B^* *is quasinormal and* A *is subnormal;*

$$\Big\|\sum_{n=0}^{\infty} c_n A^n X B^n\Big\|_\Psi \le \Big\|\sqrt{f(A^*A)}X\Big(\sum_{n=0}^{\infty} c_n B^n B^{*n}\Big)^{\frac{1}{2}}\Big\|_\Psi$$

(g1) *if* $\Psi := \Upsilon^{(p)*}$ *and* A *is quasinormal,*
(g2) *if* $\|\cdot\|_\Psi := |||\cdot|||$, A *is quasinormal and* B^* *is subnormal;*

$$\Big\|\sum_{n=0}^{\infty} c_n A^n X B^n\Big\|_\Psi \le \big\|\sqrt{f(A^*A)}X\sqrt{f(BB^*)}\big\|_\Psi$$

(h) *if* $\|\cdot\|_\Psi := |||\cdot|||$ *and both* A *and* B^* *are quasinormal.*

For Schatten–von Neumann ideals Theorem 4.1 has the following counterpart:

Theorem 4.3 *Let* $p, q, r \geqslant 1$ *satisfy* $\frac{2}{p} = \frac{1}{q} + \frac{1}{r}$, $X \in \mathcal{C}_p(\mathcal{H})$, $\{A_n\}_{n=1}^{\infty}$, $\{A_n^*\}_{n=1}^{\infty}$, $\{B_n\}_{n=1}^{\infty}$ *and* $\{B_n^*\}_{n=1}^{\infty}$ *be s.s. families in* $\mathbb{B}(\mathcal{H})$ *and let* f *be an analytic function with non-negative Taylor coefficients.*

(a) *If* $\max\big\{\big\|\sum_{n=1}^{\infty} A_n^*A_n\big\|, \big\|\sum_{n=1}^{\infty} A_nA_n^*\big\|, \big\|\sum_{n=1}^{\infty} B_n^*B_n\big\|, \big\|\sum_{n=1}^{\infty} B_nB_n^*\big\|\big\| \le R_f$, $\big\|\sum_{n=1}^{\infty} A_nA_n^*\big\|\big\|\sum_{n=1}^{\infty} B_n^*B_n\big\| < R_f^2$, $f(0) > 0$ *and for all* $h \in \mathcal{H}$ $\sum_{m=1}^{\infty} c_m \sum_{n_1^{(m)},\dots,n_m^{(m)}=1}^{\infty} \big(\big\|A_{n_1^{(m)}}\cdots A_{n_m^{(m)}}h\big\|^2 + \big\|B^*_{n_1^{(m)}}\cdots B^*_{n_m^{(m)}}h\big\|^2\big) < +\infty$, *then*

$$\Big\|\Delta_{f,\boldsymbol{A}^*}^{1-\frac{1}{q}} f\Big(\sum_{n=1}^{\infty} A_n\otimes B_n\Big)(X)\Delta_{f,\boldsymbol{B}}^{1-\frac{1}{r}}\Big\|_p \le \big\|\Delta_{f,\boldsymbol{A}}^{-\frac{1}{q}} X\Delta_{f,\boldsymbol{B}^*}^{-\frac{1}{r}}\big\|_p. \tag{18}$$

(b) *If* $\max\left\{\left\|\sum_{n=1}^{\infty} A_n^* A_n\right\|, \left\|\sum_{n=1}^{\infty} A_n A_n^*\right\|, \left\|\sum_{n=1}^{\infty} B_n^* B_n\right\|, \left\|\sum_{n=1}^{\infty} B_n B_n^*\right\|\right\} < R_f$, *then*

$$\left\| f\Big(\sum_{n=1}^{\infty} A_n \otimes B_n\Big) X\right\|_p \le \left\| \left(f\Big(\sum_{n=1}^{\infty} A_n^* \otimes A_n\Big)\Big(f\Big(\sum_{n=1}^{\infty} A_n \otimes A_n^*\Big)(I)\Big)^{q-1}\right)^{\frac{1}{2q}} \right.$$

$$\left. \times X \left(f\Big(\sum_{n=1}^{\infty} B_n \otimes B_n^*\Big)\Big(f\Big(\sum_{n=1}^{\infty} B_n^* \otimes B_n\Big)(I)\Big)^{r-1}\right)^{\frac{1}{2r}} \right\|_p, \tag{19}$$

and if additionally $f(0) > 0$, *then also*

$$\left\| \Big(f\Big(\sum_{n=1}^{\infty} A_n \otimes A_n^*\Big)(I)\Big)^{\frac{1}{2q}-\frac{1}{2}} f\Big(\sum_{n=1}^{\infty} A_n \otimes B_n\Big)(X) \Big(f\Big(\sum_{n=1}^{\infty} B_n^* \otimes B_n\Big)(I)\Big)^{\frac{1}{2r}-\frac{1}{2}} \right\|_p$$

$$\le \left\| \Big(f\Big(\sum_{n=1}^{\infty} A_n^* \otimes A_n\Big)(I)\Big)^{\frac{1}{2q}} X \Big(f\Big(\sum_{n=1}^{\infty} B_n \otimes B_n^*\Big)(I)\Big)^{\frac{1}{2r}} \right\|_p. \tag{20}$$

The part (a), including the inequality (18), is a simple reformulation of [23, th. 2.9], while the inequality (19) in part (b) is also a reformulation of [23, th. 2.8]. According to [23, rem. 7] we can conclude that $\Delta_{f,\boldsymbol{A}} = \big(f\big(\sum_{n=1}^{\infty} A_n^* \otimes A_n\big)(I)\big)^{-1/2}$ based on the conditions given in part (b). With analogous formulas for $\Delta_{f,\boldsymbol{A}^*}$, $\Delta_{f,\boldsymbol{B}}$ and $\Delta_{f,\boldsymbol{B}^*}$ and by the fact that conditions in (b) provides the fulfilment of all requirements in (a), then the inequality (19) becomes the proclaimed inequality (20).

The next is a list of selected inequalities from [23, cor. 2.5, cor. 2.6]

Corollary 4.4 *Let* Υ *be a s.n. function, let* $p \ge 2$, $\alpha, \beta \in [-1, 1]$, $\gamma \leqslant 0$ *and let* $A, B \in \mathbb{B}(\mathcal{H})$, $X \in \mathcal{C}_{\Upsilon^{(p)}}(\mathcal{H})$ *and* $Y \in \mathcal{C}_{\Upsilon^{(p)*}}(\mathcal{H})$.

(a) *If* A *and* B *are strict contractions, i.e.,* $\max\{\|A\|, \|B\|\} < 1$, *with* B *being additionally normal, then*

$$\left\| (A \otimes B) X + (I - A \otimes B) \log\big(I - A \otimes B\big) X \right\|_{\Upsilon^{(p)}} \le$$

$$\left\| \sum_{n=2}^{\infty} \frac{A^n A^{*n}}{n(n-1)} \right\|^{1/2} \left\| X \sqrt{B^* B + (I - B^* B) \log\big(I - B^* B\big)} \right\|_{\Upsilon^{(p)}},$$

$$\left\| \big(I - A \otimes B\big)^{\gamma} Y + \alpha\big(I + \beta A \otimes B\big)^{\gamma} Y \right\|_{\Upsilon^{(p)*}} \le$$

$$\left\| \Big(\sum_{n=0}^{\infty} \big((-1)^n + \alpha \beta^n\big) \tbinom{\gamma}{n} A^{*n} A^n \Big)^{1/2} Y \sqrt{\big(I - B^* B\big)^{\gamma} + \alpha\big(I + \beta B^* B\big)^{\gamma}} \right\|_{\Upsilon^{(p)*}}.$$

(b) *If A is normal, then*

$$\left\|\exp(A\otimes B)X+\alpha\exp(\beta A\otimes B)X\right\|_{\Upsilon^{(p)}}\le$$
$$\left\|\sqrt{\exp(A^*A)+\alpha\exp(\beta A^*A)}X\right\|_{\Upsilon^{(p)}}\left\|\sum_{n=0}^{\infty}\tfrac{1+\alpha\beta^n}{n!}B^{*n}B^n\right\|^{1/2},$$

$$\left\|\exp(A\otimes B)Y+\alpha\exp(\beta A\otimes B)Y\right\|_{\Upsilon^{(p)*}}\le$$
$$\left\|\sqrt{\exp(A^*A)+\alpha\exp(\beta A^*A)}\,Y\Bigl(\sum_{n=0}^{\infty}\tfrac{1+\alpha\beta^n}{n!}B^nB^{*n}\Bigr)^{1/2}\right\|_{\Upsilon^{(p)*}},$$

$$\left\|X-(I-A\otimes B)\exp\Bigl(\sum_{k=1}^{n}\frac{A^k\otimes B^k}{k}\Bigr)X\right\|_{\Upsilon^{(p)}}$$
$$\le\left\|\sqrt{I-(I-A^*A)\exp\Bigl(\sum_{k=1}^{n}\frac{(A^*A)^k}{k}\Bigr)}X\right\|_{\Upsilon^{(p)}}$$
$$\times\left\|I-(I-B^*\otimes B)\exp\Bigl(\sum_{k=1}^{n}\frac{B^{*k}\otimes B^k}{k}\Bigr)(I)\right\|^{1/2},$$

$$\left\|Y-(I-A\otimes B)\exp\Bigl(\sum_{k=1}^{n}\frac{A^k\otimes B^k}{k}\Bigr)Y\right\|_{\Upsilon^{(p)*}}$$
$$\le\left\|\sqrt{I-(I-A^*A)\exp\Bigl(\sum_{k=1}^{n}\frac{(A^*A)^k}{k}\Bigr)}\,Y\right.$$
$$\left.\times\sqrt{I-(I-B\otimes B^*)\exp\Bigl(\sum_{k=1}^{n}\frac{B^k\otimes B^{*k}}{k}\Bigr)(I)}\right\|_{\Upsilon^{(p)*}}.$$

Also, the list of selected inequalities from [18, th. 3.4] is displayed in

Theorem 4.5 *Let Υ be a s.n. function, let $p\ge 2$, let $\alpha,\beta\in[-1,1]$ and let $A,B,C,D,T\in\mathbb{B}(\mathcal{H})$, with A,T^* being subnormal and C,D^* being quasinormal, such that their spectra satisfy $\sigma(A)\cup\sigma(B)\cup\sigma(C)\cup\sigma(D)\cup\sigma(T)\subset\mathbb{D}$. Then for all $X\in\mathcal{C}_{\Upsilon^{(p)}}(\mathcal{H})$, $Y\in\mathcal{C}_{\Upsilon^{(p)*}}(\mathcal{H})$ and $Z\in\mathcal{C}_{\Upsilon}(\mathcal{H})$*

$$\left\|\alpha\log\bigl(I+\beta C\otimes B\bigr)X-\log\bigl(I-C\otimes B\bigr)X\right\|_{\Upsilon^{(p)}}\le$$
$$\left\|\sqrt{\alpha\log(I+\beta C^*C)-\log\bigl(I-C^*C\bigr)}X\right\|_{\Upsilon^{(p)}}\left\|\sum_{n=1}^{\infty}\tfrac{1-\alpha(-\beta)^n}{n}B^{*n}B^n\right\|^{1/2},$$

$$\Big\|\arcsin(C\otimes B)\,X+\alpha\log\big(\beta C\otimes B+\sqrt{I+\beta^2C^2\otimes B^2}\big)X\Big\|_{\Upsilon^{(p)}}\le$$

$$\Big\|\sqrt{\arcsin(C^*C)+\alpha\log\big(\beta C^*C+\sqrt{I+\beta^2(C^*C)^2}\big)}X\Big\|_{\Upsilon^{(p)}}$$

$$\times\Big\|\sum_{n=0}^{\infty}\tfrac{(1+(-1)^n\alpha\beta^{2n+1})(2n)!}{2^{2n}(n!)^2(2n+1)}(B^*)^{2n+1}B^{2n+1}\Big\|^{1/2},$$

$$\Big\|\tan\Big(\frac{\pi}{2}C\otimes B\Big)X+\alpha\tanh\Big(\frac{\beta\pi}{2}C\otimes B\Big)X\Big\|_{\Upsilon^{(p)}}\le$$

$$\Big\|\sqrt{\tan\Big(\frac{\pi}{2}C^*C\Big)+\alpha\tanh\Big(\frac{\beta\pi}{2}C^*C\Big)}X\Big\|_{\Upsilon^{(p)}}$$

$$\times\Big\|\sum_{n=1}^{\infty}\tfrac{(2^{2n}-1)\pi^{2n-1}(|B_{2n}|+\alpha\beta^{2n-1}B_{2n})}{2(2n)!}(B^*)^{2n-1}B^{2n-1}\Big\|^{1/2},$$

$$\big\|\alpha\log\big(I+\beta A\otimes B\big)Y-\log\big(I-A\otimes B\big)Y\big\|_{\Upsilon^{(p)*}}\le$$

$$\Big\|\Big(\sum_{n=1}^{\infty}\tfrac{1-\alpha(-\beta)^n}{n}A^{*n}A^n\Big)^{1/2}Y\Big(\sum_{n=1}^{\infty}\tfrac{1-\alpha(-\beta)^n}{n}B^nB^{*n}\Big)^{1/2}\Big\|_{\Upsilon^{(p)*}},$$

$$|||\alpha\log\big(I+\beta A\otimes T\big)Z-\log\big(I-A\otimes T\big)Z|||\le$$

$$\Big|\Big|\Big|\Big(\sum_{n=1}^{\infty}\tfrac{1-\alpha(-\beta)^n}{n}A^{*n}A^n\Big)^{1/2}Z\Big(\sum_{n=1}^{\infty}\tfrac{1-\alpha(-\beta)^n}{n}T^nT^{*n}\Big)^{1/2}\Big|\Big|\Big|,$$

$$|||\arcsin(C\otimes D)Z+\alpha\log\big(\beta C\otimes D+\sqrt{I+\beta^2C^2\otimes D^2}\big)Z|||\le$$

$$\Big|\Big|\Big|\sqrt{\arcsin(C^*C)+\alpha\log\big(\beta C^*C+\sqrt{I+\beta^2(C^*C)^2}\big)}\,Z$$

$$\times\sqrt{\arcsin(DD^*)+\alpha\log\big(\beta DD^*+\sqrt{I+\beta^2(DD^*)^2}\big)}\Big|\Big|\Big|,$$

$$\Big|\Big|\Big|\tan\Big(\frac{\pi}{2}C\otimes D\Big)Z+\alpha\tanh\Big(\frac{\beta\pi}{2}C\otimes D\Big)Z\Big|\Big|\Big|\le$$

$$\Big|\Big|\Big|\sqrt{\tan\Big(\frac{\pi}{2}C^*C\Big)+\alpha\tanh\Big(\frac{\beta\pi}{2}C^*C\Big)}\,Z\sqrt{\tan\Big(\frac{\pi}{2}DD^*\Big)+\alpha\tanh\Big(\frac{\beta\pi}{2}DD^*\Big)}\Big|\Big|\Big|.$$

5 Grüss–Landau Type Norm Inequalities for i.p.t. Transformers

We begin this section by presenting [20, th. 2.4]:

Theorem 5.1 *Let μ be a probability measure on Ω, let $1 \le p, q, r < +\infty$ satisfy $\frac{2}{p} = \frac{1}{q} + \frac{1}{r}$ and let $A, A^*, B, B^* \in L^2_G(\Omega, \mu, \mathbb{B}(\mathcal{H}))$. Then for all $X \in \mathcal{C}_p(\mathcal{H})$*

$$\left\| \int_\Omega A_t X B_t \, d\mu(t) - \int_\Omega A_t \, d\mu(t) X \int_\Omega B_t \, d\mu(t) \right\|_p \le$$

$$\left\| \left(\int_\Omega \left| \left(\int_\Omega \left| A_t^* - \int_\Omega A_t^* d\mu(t) \right|^2 d\mu(t) \right)^{\frac{q-1}{2}} \left(A_t - \int_\Omega A_t \, d\mu(t) \right) \right|^2 d\mu(t) \right)^{\frac{1}{2q}} X \right.$$

$$\left. \times \left(\int_\Omega \left| \left(\int_\Omega \left| B_t - \int_\Omega B_t \, d\mu(t) \right|^2 d\mu(t) \right)^{\frac{r-1}{2}} \left(B_t^* - \int_\Omega B_t^* \, d\mu(t) \right) \right|^2 d\mu(t) \right\|^{\frac{1}{2r}} \right)_p .$$

For some other types of u.i. norms the counterpart of Theorem 5.1 says:

Theorem 5.2 *Let μ be a probability measure on Ω, $p \ge 2$, Ψ, Υ be s.n. functions and $X \in \mathbb{B}(\mathcal{H})$.*

(a) *If $A, B^* \in L^2_G(\Omega, \mu, \mathbb{B}(\mathcal{H}))$, then*

$$\left\| \int_\Omega A_t X B_t \, d\mu(t) - \int_\Omega A_t \, d\mu(t) X \int_\Omega B_t \, d\mu(t) \right\|_\Psi \le \tag{21}$$

$$\left\| \sqrt{\int_\Omega A_t^* A_t \, d\mu(t) - \left| \int_\Omega A_t \, d\mu(t) \right|^2} X \sqrt{\int_\Omega B_t B_t^* \, d\mu(t) - \left| \int_\Omega B_t \, d\mu(t) \right|^2} \right\|_\Psi ,$$

under any of the following conditions:

(a1) $\|\cdot\|_\Psi := \|\cdot\|_1$,

(a2) *$L^2(\Omega, \mu)$ is separable, $\Psi := \Upsilon^{(p)^*}$ and (at least) one of families $\{A_t\}_{t\in\Omega}$ or $\{B_t\}_{t\in\Omega}$ is a m.c.n.o. family,*

(a3) *$\|\cdot\|_\Psi := |||\cdot|||$ and both $\{A_t\}_{t\in\Omega}$ and $\{B_t\}_{t\in\Omega}$ are m.c.n.o. families.*

(b) *If $A, B \in L^2_G(\Omega, \mu, \mathbb{B}(\mathcal{H}))$, then*

$$\left\| \int_\Omega A_t X B_t d\mu(t) - \int_\Omega A_t \, d\mu(t) X \int_\Omega B_t \, d\mu(t) \right\|_\Psi \le \tag{22}$$

$$\left\| \sqrt{\int_\Omega A_t^* A_t \, d\mu(t) - \left| \int_\Omega A_t \, d\mu(t) \right|^2} X \right\|_\Psi \left\| \int_\Omega B_t^* B_t \, d\mu(t) - \left| \int_\Omega B_t \, d\mu(t) \right|^2 \right\|^{1/2} ,$$

under any of the following conditions:

(b1) $\|\cdot\|_\Psi := \|\cdot\|_2$,

(b2) $\Psi := \Upsilon^{(p)}$ *and* $\{A_t\}_{t\in\Omega}$ *is additionally a m.c.n.o. family.*

(c) *If* $A^*, B^* \in L^2_G(\Omega, \mu, \mathbb{B}(\mathcal{H}))$, *then*

$$\left\| \int_\Omega A_t X B_t d\mu(t) - \int_\Omega A_t\, d\mu(t) X \int_\Omega B_t\, d\mu(t) \right\|_\Psi \le \tag{23}$$
$$\left\| \int_\Omega A_t A_t^* d\mu(t) - \left|\int_\Omega A_t^* d\mu(t)\right|^2 \right\|^{1/2} \left\| X \sqrt{\int_\Omega B_t B_t^* d\mu(t) - \left|\int_\Omega B_t^* d\mu(t)\right|^2} \right\|_\Psi,$$

under any of the following conditions:

(c1) $\|\cdot\|_\Psi := \|\cdot\|_2$,

(c2) $\Psi := \Upsilon^{(p)}$ *and* $\{B_t\}_{t\in\Omega}$ *is additionally a m.c.n.o. family.*

As we can confine to the case in which the righthand side of the inequality (21) is finite, so (21) in the case (a1) is a special case $p := q := r := 1$ of Theorem 5.1. The proof of [20, th. 2.1] provides the proof for the inequality (21) in the case (a3), together with the proof for the case (a2), with the only difference that we now apply (to the same families) the Cauchy–Schwarz inequality [26, th. 3.1(d)] instead of [15, th. 3.2]. For σ-elementary transformers the case (a2) was proved earlier in [32, th. 2.10].

The case (b1) (resp. (c1)) was proved in [32, th. 2.6(21)] (resp. [32, th. 2.6(20)], while the case (b2) (resp. (c2)) was proved in [32, th. 2.7(28)] (resp. [32, th. 2.7(27)].

Definition 5.3 For a bounded field of operators $\boldsymbol{A} := \{A_t\}_{t\in\Omega}$ its **radius of its essential range** is given by $r_\infty(\boldsymbol{A}) \overset{\text{def}}{=} \inf_{B\in\mathbb{B}(\mathcal{H})} \sup\operatorname{ess}_{t\in\Omega}\|A_t - B\| = \inf_{B\in\mathbb{B}(\mathcal{H})} \|\boldsymbol{A} - B\|_\infty = \min_{B\in\mathbb{B}(\mathcal{H})} \|\boldsymbol{A} - B\|_\infty$, while its *essential diameter* is given by $\operatorname{diam}_\infty(\boldsymbol{A}) \overset{\text{def}}{=} \sup\operatorname{ess}_{s,t\in\Omega}\|A_s - A_t\|$.

This definition helps us to abbreviate the formulation of the following:

Theorem 5.4 *If* μ *is a* σ*-finite measure on* Ω, $\boldsymbol{A} := \{A_t\}_{t\in\Omega}$ *and* $\boldsymbol{B} := \{B_t\}_{t\in\Omega}$ *are* $[\mu]$ *a.e. bounded fields of operators and* $X \in \mathcal{C}_{|||\cdot|||}(\mathcal{H})$, *then*

$$\sup_{\substack{\delta\in\mathfrak{M}\\ 0<\mu(\delta)<+\infty}} \left|\left|\left| \frac{1}{\mu(\delta)} \int_\Omega A_t X B_t\, d\mu(t) - \frac{1}{\mu(\delta)} \int_\Omega A_t\, d\mu(t)\, X \frac{1}{\mu(\delta)} \int_\Omega B_t\, d\mu(t) \right|\right|\right|$$
$$\le \min\left\{ r_\infty(\boldsymbol{A}) r_\infty(\boldsymbol{B}), \frac{\operatorname{diam}_\infty(\boldsymbol{A}) \operatorname{diam}_\infty(\boldsymbol{B})}{2} \right\} |||X|||. \tag{24}$$

Specially, if μ *is a probability measure on* Ω *and if* $\{A_t\}_{t\in\Omega}$ *and* $\{B_t\}_{t\in\Omega}$ *are bounded self-adjoint fields satisfying* $C \le A_t \le D$ *and* $E \le B_t \le F$ *for all*

$t \in \Omega$ and some bounded self-adjoint operators C, D, E, F, then

$$\left|\left|\left|\int_\Omega A_t X B_t \, d\mu(t) - \int_\Omega A_t \, d\mu(t) X \int_\Omega B_t \, d\mu(t)\right|\right|\right| \leq \frac{\|D - C\| \cdot \|F - E\|}{4} |||X|||. \tag{25}$$

The inequality (24) (resp. (25)) was proved in [20, th. 2.8] (resp. [20, cor. 2.9]).

The operator Grüss–Landau inequality is given by:

Theorem 5.5 *Let μ be a probability measure on Ω, $A^*, B \in L^2_G(\Omega, \mu, \mathbb{B}(\mathcal{H}))$, $X \in \mathbb{B}(\mathcal{H})$ and $\eta \in [0, 1]$. Then*

$$\left|\int_\Omega A_t X B_t \, d\mu(t) - \int_\Omega A_t \, d\mu(t) X \int_\Omega B_t \, d\mu(t)\right|^{2\eta} \leq \tag{26}$$

$$\left\|\int_\Omega A_t A_t^* \, d\mu(t) - \left|\int_\Omega A_t^* \, d\mu(t)\right|^2\right\|^\eta \left(\int_\Omega B_t^* X^* X B_t \, d\mu(t) - \left|X \int_\Omega B_t \, d\mu(t)\right|^2\right)^\eta.$$

Specially, if $\boldsymbol{A} := \{A_t\}_{t\in\Omega}$ and $\boldsymbol{B} := \{B_t\}_{t\in\Omega}$ are (bounded) self-adjoint fields satisfying $C \leq A_t \leq D$ and $E \leq B_t \leq F$ for all $t \in \Omega$ and some bounded self-adjoint operators C, D, E, F, such that C, D (resp. E, F) commutes with A_t (resp. B_t) for all $t \in \Omega$, satisfying $CD = DC$ and $EF = FE$, then

$$\left|\int_\Omega A_t B_t \, d\mu(t) - \int_\Omega A_t \, d\mu(t) \int_\Omega B_t \, d\mu(t)\right|^{2\eta} \leq$$

$$\left\|\left(D - \int_\Omega A_t \, d\mu(t)\right)\left(\int_\Omega A_t \, d\mu(t) - C\right)\right\|^\eta \left(F - \int_\Omega B_t \, d\mu(t)\right)^\eta \left(\int_\Omega B_t \, d\mu(t) - E\right)^\eta$$

$$\leq \frac{1}{4^{2\eta}} \|D - C\|^{2\eta} (F - E)^{2\eta}. \tag{27}$$

The inequality (26) (resp. (27)) was proved in [32, th. 2.1] (resp. [32, th. 2.2]).
The refined Grüss–Landau operator and norm inequalities are presented in:

Theorem 5.6 *If $\{\alpha_1, \ldots, \alpha_N\}$ are in $(0, 1]$, satisfying $\sum_{n=1}^N \alpha_n = 1$ for some $N \in \mathbb{N}$, and if $X, \{A_1, \ldots, A_N\}$ and $\{B_1, \ldots, B_N\}$ are in $\mathbb{B}(\mathcal{H})$, then for all $c \geq \left\|\sum_{n=1}^N \alpha_n^{-1} A_n^* A_n - \left|\sum_{n=1}^N A_n\right|^2\right\|^{\frac{1}{2}} > 0$*

$$\left|\sum_{n=1}^N \alpha_n^{-1} A_n^* X B_n - \Big(\sum_{n=1}^N A_n^*\Big) X \Big(\sum_{n=1}^N B_n\Big)\right|^2 +$$

$$\sum_{1\leq m<n\leq N} \alpha_m \alpha_n \left|(\alpha_m^{-1} A_m - \alpha_n^{-1} A_n)\left(cI + \left(c^2 I - \sum_{n=1}^N \alpha_n^{-1} A_n^* A_n + \left|\sum_{n=1}^N A_n\right|^2\right)^{\frac{1}{2}}\right)^{-1}\right.$$

$$\times\Big(\sum_{n=1}^{N}\alpha_n^{-1}A_n^*XB_n-\Big(\sum_{n=1}^{N}A_n^*\Big)X\Big(\sum_{n=1}^{N}B_n\Big)\Big)-cX(\alpha_m^{-1}B_m-\alpha_n^{-1}B_n)\Big|^2$$

$$=c^2\Big(\sum_{n=1}^{N}\alpha_n^{-1}B_n^*X^*XB_n-\Big|X\sum_{n=1}^{N}B_n\Big|^2\Big). \tag{28}$$

If $2\le p\le q<+\infty$ *and* Υ *is a s.n. function, then*

$$\Big\|\sum_{n=1}^{N}\alpha_n^{-1}A_n^*XB_n-\Big(\sum_{n=1}^{N}A_n^*\Big)X\Big(\sum_{n=1}^{N}B_n\Big)\Big\|_{\Upsilon^{(q)}}^p\le \tag{29}$$

$$\Big\|\Big|\sum_{n=1}^{N}\alpha_n^{-1}A_n^*XB_n-\Big(\sum_{n=1}^{N}A_n^*\Big)X\Big(\sum_{n=1}^{N}B_n\Big)\Big|^p+$$

$$\sum_{1\le m<n\le N}\alpha_m^{\frac{p}{2}}\alpha_n^{\frac{p}{2}}\Big|(\alpha_m^{-1}A_m-\alpha_n^{-1}A_n)\Big(cI+\Big(c^2I-\sum_{n=1}^{N}\alpha_n^{-1}A_n^*A_n+\Big|\sum_{n=1}^{N}A_n\Big|^2\Big)^{\frac{1}{2}}\Big)^{-1}$$

$$\times\Big(\sum_{n=1}^{N}\alpha_n^{-1}A_n^*XB_n-\Big(\sum_{n=1}^{N}A_n^*\Big)X\Big(\sum_{n=1}^{N}B_n\Big)\Big)-cX(\alpha_m^{-1}B_m-\alpha_n^{-1}B_n)\Big|^p\Big\|_{\Upsilon^{(\frac{q}{p})}}$$

$$\le c^p\Big\|\sum_{n=1}^{N}\alpha_n^{-1}B_n^*X^*XB_n-\Big(\sum_{n=1}^{N}B_n^*\Big)X^*X\Big(\sum_{n=1}^{N}B_n\Big)\Big\|_{\Upsilon^{(\frac{q}{2})}}^{\frac{p}{2}}. \tag{30}$$

If $\{B_n\}_{n=1}^N$ *is additionally a m.c.n.o. family, then the expression appearing in* (30) *further estimates by* $c^p\big\|X\big(\sum_{n=1}^N\alpha_n^{-1}B_n^*B_n-\big|\sum_{n=1}^N B_n\big|^2\big)^{1/2}\big\|_{\Upsilon^{(q)}}^p$.

The identity (28) is a subject of [22, lemma 2.1], while the chain of inequalities (29)–(30) was proven in [22, th. 2.2] and was further generalized in [22, th. 2.3].

6 Norm Inequalities for Holomorphic Functions on Simply Connected Domains in the Complex Plane

Important applications of previous Cauchy–Schwarz norm inequalities to generalized derivations (which includes perturbations) are presented in:

Theorem 6.1 *Let* $A, B\in\mathbb{B}(\mathcal{H})$ *be contractions,* $p\ge 2$, Ψ,Υ *be s.n. functions,* $\sum_{n=0}^\infty c_n$ *be an absolutely summable complex series satisfying* $\sum_{n=0}^\infty|c_n|\le 1$ *and* $f(z)\overset{\text{def}}{=}\sum_{n=0}^\infty c_nz^n$ *for all* $|z|\le 1$. *If* $AX-XB\in\mathcal{C}_\Psi(\mathcal{H})$ *and* $Y-AYB\in\mathcal{C}_\Psi(\mathcal{H})$

for some $X, Y \in \mathbb{B}(\mathcal{H})$, *then*

(a) $\sqrt{I - A^*A}\,\big(f(A)X - Xf(B)\big)\sqrt{I - BB^*} \in \mathcal{C}_\Psi(\mathcal{H})$ *and*

$$\sqrt{I - A^*A}\,\big(f(I)Y - f(A\otimes B)(Y)\big)\sqrt{I - BB^*} \in \mathcal{C}_\Psi(\mathcal{H}),\ \textit{satisfying}$$

$$\big\|\sqrt{I - A^*A}\big(f(A)X - Xf(B)\big)\sqrt{I - BB^*}\,\big\|_\Psi \leq \big\|\sqrt{I - f(A)^*f(A)}\,(AX - XB)\sqrt{I - f(B)f(B)^*}\,\big\|_\Psi, \tag{31}$$

$$\big\|\sqrt{I - A^*A}\,\big(f(I)Y - f(A\otimes B)(Y)\big)\sqrt{I - BB^*}\,\big\|_\Psi \leq \big\|\sqrt{I - f(A)^*f(A)}(Y - AYB)\sqrt{I - f(B)f(B)^*}\,\big\|_\Psi, \tag{32}$$

under any of the following conditions:

(a1) $\|\cdot\|_\Psi := \|\cdot\|_1$,

(a2) $\Psi := \Upsilon^{(p)*}$ *and (at least) one of* A *or* B *is normal,*

(a3) $\|\cdot\|_\Psi := |||\cdot|||$ *and both* A *and* B *are normal;*

(b) *if* $\|A\| < 1$, *then* $\big(f(A)X - Xf(B)\big)\sqrt{I - BB^*} \in \mathcal{C}_\Psi(\mathcal{H})$ *and*

$$\big(f(I)Y - f(A\otimes B)(Y)\big)\sqrt{I - BB^*} \in \mathcal{C}_\Psi(\mathcal{H}),\ \textit{satisfying}$$

$$\big\|\big(f(A)X - Xf(B)\big)\sqrt{I - BB^*}\,\big\|_\Psi \leq \big\|I - f(A)f(A)^*\big\|^{1/2}\big\|(I - AA^*)^{-1/2}(AX - XB)\sqrt{I - f(B)f(B)^*}\,\big\|_\Psi, \tag{33}$$

$$\big\|\big(f(I)Y - f(A\otimes B)(Y)\big)\sqrt{I - BB^*}\,\big\|_\Psi \leq \big\|I - f(A)f(A)^*\big\|^{1/2}\big\|(I - AA^*)^{-1/2}(Y - AYB)\sqrt{I - f(B)f(B)^*}\,\big\|_\Psi, \tag{34}$$

under any of the following conditions:

(b1) $\|\cdot\|_\Psi := \|\cdot\|_2$,

(b2) $\Psi := \Upsilon^{(p)}$ *and* B *is normal;*

(c) *if* $\|B\| < 1$, *then* $\sqrt{I - A^*A}\,\big(f(A)X - Xf(B)\big) \in \mathcal{C}_\Psi(\mathcal{H})$ *and*

$$\sqrt{I - A^*A}\,\big(f(I)Y - f(A\otimes B)(Y)\big) \in \mathcal{C}_\Psi(\mathcal{H}),\ \textit{satisfying}$$

$$\big\|\sqrt{I - A^*A}\big(f(A)X - Xf(B)\big)\big\|_\Psi \leq \big\|\sqrt{I - f(A)^*f(A)}\,(AX - XB)(I - B^*B)^{-1/2}\big\|_\Psi\big\|I - f(B)^*f(B)\big\|^{1/2}, \tag{35}$$

$$\left\|\sqrt{I-A^*A}\,\big(f(I)Y-f(A\otimes B)(Y)\big)\right\|_{\Psi}\le$$
$$\left\|\sqrt{I-f(A)^*f(A)}(Y-AYB)(I-B^*B)^{-1/2}\right\|_{\Psi}\left\|I-f(B)^*f(B)\right\|^{1/2}, \tag{36}$$

under any of the following conditions:

(c1) $\|\cdot\|_{\Psi}:=\|\cdot\|_2$,

(c2) $\Psi:=\Upsilon^{(p)}$ *and A is normal.*

The inequality

- (31) in the case (a1) (resp. (a2) and (a3)) was proved in [24, th. 2.1(a3)] (resp. [24, th. 2.1(a2)] and [24, th. 2.1(a1)]);
- (32) in the case (a1) (resp. (a2) and (a3)) was proved in [24, th. 2.1(b3)] (resp. [24, th. 2.1(b2)] and [24, th. 2.1(b1)]);
- (33) in cases (b1) and (b2) is just the inequality (6) in [25, th. 2.1(a4′)];
- (34) in cases (b1) and (b2) is a reformulation of the inequality (9) in [25, th. 2.1(b4′)];
- (35) in cases (c1) and (c2) is just the inequality (7) in [25, th. 2.1(a4″)];
- (36) in cases (c1) and (c2) is a reformulation of the inequality (10) in [25, th. 2.1(b4″)].

Parts of Theorem 6.1 remains valid for the subclass of functions in disc algebra $\mathbf{A}(\overline{\mathbb{D}})$ possessing non-negative Taylor coefficients, if at least one of operators A or B^* is quasinormal, as presented in [18, th. 3.6]:

Theorem 6.2 *Let Υ be a s.n. function, $p\ge 2$, $f(z)\stackrel{\text{def}}{=}\sum_{n=0}^{\infty}c_nz^n$ be an analytic function with non-negative Taylor coefficients $c_n\ge 0$, $0<\sum_{n=0}^{\infty}c_n<+\infty$ and let $A,B,X,Y\in\mathbb{B}(\mathcal{H})$ be such that A, B^* are contractions and at least one of them is quasinormal. If $AY-YB\in\mathcal{C}_{\Upsilon^{(p)*}}(\mathcal{H})$, then*

$$\left\|\sqrt{I-A^*A}\big(f(A)Y-Yf(B)\big)\sqrt{I-BB^*}\right\|_{\Upsilon^{(p)*}}\le$$
$$\left\|\sqrt{f(1)I-|f(A)|^2/f(1)}(AY-YB)\sqrt{f(1)I-|f(B^*)|^2/f(1)}\right\|_{\Upsilon^{(p)*}}.$$

If A and B^* are both quasinormal and $AX-XB\in\mathcal{C}_{|||\cdot|||}(\mathcal{H})$, then

$$\left|\left|\left|\sqrt{I-A^*A}\big(f(A)X-Xf(B)\big)\sqrt{I-BB^*}\right|\right|\right|\le$$
$$\left|\left|\left|\sqrt{f(1)I-f(A^*A)}(AX-XB)\sqrt{f(1)I-f(BB^*)}\right|\right|\right|\le$$
$$\left|\left|\left|\sqrt{f(1)I-|f(A)|^2/f(1)}(AX-XB)\sqrt{f(1)I-|f(B^*)|^2/f(1)}\right|\right|\right|.$$

Definition 6.3 Let μ be a complex Borel measure on $\mathbb{R}$ and $A, B, X \in \mathbb{B}(\mathcal{H})$. If a function $t \mapsto e^{itA}$ is Gel'fand μ integrable on $\mathbb{R}$, then

$$\hat{\mu}(A) \stackrel{\text{def}}{=} \mathcal{F}\mu(A) \stackrel{\text{def}}{=} \int_{\mathbb{R}} e^{itA} d\mu(t),$$

will denote the *operator valued* (o.v.) *Fourier transform* of μ evaluated in A.

For dissipative operators we present the following Q* norm inequality for generalized derivations, given in [26, th. 4.2]:

Theorem 6.4 *Let $p \geq 2$, Υ be a s.n. function, μ be a complex Borel measure on $\mathbb{R}_+$, with its total variation $|\mu|(\mathbb{R}_+) \leq 1$ and $A, B, X \in \mathbb{B}(\mathcal{H})$ be such that A, B are dissipative and at least one of them is normal. If $AX - XB \in \mathcal{C}_{\Upsilon^{(p)*}}(\mathcal{H})$, then*

$$\left\|\sqrt{iA^* - iA}\big(\hat{\mu}(A)X - X\hat{\mu}(B)\big)\sqrt{iB^* - iB}\right\|_{\Upsilon^{(p)*}} \leq$$
$$\left\|\sqrt{I - \left|\hat{\mu}(A)\right|^2}(AX - XB)\sqrt{I - \left|\hat{\mu}(B)^*\right|^2}\right\|_{\Upsilon^{(p)*}}.$$

7 Mean Value Norm Inequalities for Operator Monotone Functions and Heinz Inequalities, with Applications

Definition 7.1 For an interval $I \subset \mathbb{R}$ and a Borel measurable function $\varphi \colon I \to \mathbb{R}$ we say that φ is *operator (increasingly) monotone* (OM) on I if $\varphi(A) \leq \varphi(B)$ for all $A, B \in \mathbb{B}(\mathcal{H})$ satisfying $\sigma(A) \cup \sigma(B) \subset I$ and $A \leq B$.

Every OM function φ on I uniquely extends to a Pick class $\mathcal{P}(I)$ function $\tilde{\varphi}$, which is analytic in $\mathbb{C}_+ \cup I \cup \mathbb{C}_-$, so it satisfies $\Im\tilde{\varphi}(z) > 0$ for all $\Im z > 0$ (see [9, th. 2.7.7]). Thus, we will also use a simplified notation φ for $\tilde{\varphi}$.

The first result in this section is the following extension of "mean value" norm inequalities (68) and (69) for operator monotone functions on $\mathbb{R}_+$ in [15, th. 4.4], from positive operators to normal strictly accretive operators.

Theorem 7.2 *Let Ψ, Υ be s.n. functions, $p \geq 2$ and $A, B, X \in \mathbb{B}(\mathcal{H})$ be such that $AX - XB \in \mathcal{C}_\Psi(\mathcal{H})$. If A and B have strictly contractive real parts and $\varphi \in \mathcal{P}(-1, 1)$ is a non-constant function or A and B are strictly accretive and $\varphi \in \mathcal{P}[0, +\infty)$ is a non-constant function, then $\varphi'\big(\frac{A^*+A}{2}\big)$ and $\varphi'\big(\frac{B+B^*}{2}\big)$ are invertible. Also, $\varphi(A)X - X\varphi(B) \in \mathcal{C}_\Psi(\mathcal{H})$:*
(a) *if A is cohyponormal and B is hyponormal, in which case*

$$\left\|\varphi(A)X - X\varphi(B)\right\|_\Psi \leq \left\|\sqrt{\varphi'\left(\frac{A^*+A}{2}\right)}(AX - XB)\sqrt{\varphi'\left(\frac{B+B^*}{2}\right)}\right\|_\Psi, \qquad (37)$$

under any of the following conditions:

(a1) $\|\cdot\|_\Psi := \|\cdot\|_1$,
(a2) $\Psi := \Upsilon^{(p)*}$ *and* (*at least*) *one of* A *or* B *is normal,*
(a3) $\|\cdot\|_\Psi := |||\cdot|||$ *and both* A *and* B *are normal.*
Moreover, the rightmost expression in (37) *can be further estimated by* $\big\|\big(\frac{A^*+A}{2}\big)^{-\frac{1}{2}}\varphi\big(\frac{A^*+A}{2}\big)^{\frac{1}{2}}(AX-XB)\big(\frac{B+B^*}{2}\big)^{-\frac{1}{2}}\varphi\big(\frac{B+B^*}{2}\big)^{\frac{1}{2}}\big\|_\Psi$ *if* $\varphi \in \mathcal{P}[0,+\infty)$ *and* $\varphi(0)=0$;

(b) *if* A *is hyponormal, in which case*

$$\left\|\varphi'\left(\frac{A+A^*}{2}\right)^{-\frac{1}{2}}\big(\varphi(A)X-X\varphi(B)\big)\right\|_\Psi \le \left\|(AX-XB)\varphi'\left(\frac{B+B^*}{2}\right)^{\frac{1}{2}}\right\|_\Psi, \tag{38}$$

under any of the following conditions:
(b1) $\|\cdot\|_\Psi := \|\cdot\|_2$ *and* B *is hyponormal,*
(b2) $\Psi := \Upsilon^{(p)}$ *and* B *is normal;*

(c) *if* B *is cohyponormal, in which case*

$$\left\|\big(\varphi(A)X-X\varphi(B)\big)\varphi'\left(\frac{B+B^*}{2}\right)^{-\frac{1}{2}}\right\|_\Psi \le \left\|\varphi'\left(\frac{A+A^*}{2}\right)^{\frac{1}{2}}(AX-XB)\right\|_\Psi, \tag{39}$$

under any of the following conditions:
(c1) $\|\cdot\|_\Psi := \|\cdot\|_2$ *and* A *is cohyponormal,*
(c2) $\Psi := \Upsilon^{(p)}$ *and* A *is normal;*

(d) *if* $\varphi \in \mathcal{P}[0,+\infty)$ *and* $\varphi(0)=0$, A *is hyponormal and* B *is cohyponormal, in which case*

$$\begin{aligned}&\left\|\left(\frac{A^*+A}{2}\right)^{\frac{1}{2}}\varphi\left(\frac{A^*+A}{2}\right)^{-\frac{1}{2}}\big(\varphi(A)X-X\varphi(B)\big)\left(\frac{B+B^*}{2}\right)^{\frac{1}{2}}\varphi\left(\frac{B+B^*}{2}\right)^{-\frac{1}{2}}\right\|_\Psi \le \\ &\left\|\varphi'\left(\frac{A^*+A}{2}\right)^{-\frac{1}{2}}\big(\varphi(A)X-X\varphi(B)\big)\varphi'\left(\frac{B+B^*}{2}\right)^{-\frac{1}{2}}\right\|_\Psi \le \|AX-XB\|_\Psi,\end{aligned} \tag{40}$$

under any of the following conditions:
(d1) $\|\cdot\|_\Psi$ *is the operator norm* $\|\cdot\|$,
(d2) $\Psi := \Upsilon^{(p)}$ *and* (*at least*) *one of* A *or* B *is normal,*
(d3) $\|\cdot\|_\Psi := |||\cdot|||$ *and both* A *and* B *are normal.*
Moreover, the last inequality in (40) *is valid for all* $\varphi \in \mathcal{P}[0,+\infty) \cup \mathcal{P}(-1,1)$.

If $J := (-1,1)$ or $J := (0,+\infty)$, then $\varphi \in \mathcal{P}(J)$ satisfies conditions of [9, th. 2.4.1], implying that $\varphi' > 0$ on J and φ' is continuous on J, so $\varphi' \ge m_A \stackrel{\text{def}}{=}$

$\min_{\sigma(\frac{A^*+A}{2})} \varphi' > 0$. Thus, for every $h \in \mathcal{H}$ there is a positive measure μ_h, such that $\langle \varphi'(\frac{A^*+A}{2})h, h\rangle = \int_{\sigma(\frac{A^*+A}{2})} \varphi'(\lambda)d\mu_h(\lambda) \geq m_A\|h\|^2$, implying that $\varphi'\big(\frac{A^*+A}{2}\big) \geq m_A I$, so $\varphi'\big(\frac{A^*+A}{2}\big)$ is invertible, as proclaimed. Similarly we also conclude that $\varphi'\big(\frac{B+B^*}{2}\big)$ is positive and invertible.

• The inequality (37) in the case (a1) (resp. (a2) and (a3)) is stated at the end of [24, th. 3.1(c)] (resp. is given by the inequality (37) in [24, th. 3.1(c)] and by the inequality (39) in [24, th. 3.1(d)]). The rest of the statement in the part (a) is based on estimates

$$\varphi'\big(\tfrac{A^*+A}{2}\big) \leq \big(\tfrac{A^*+A}{2}\big)^{-1}\varphi\big(\tfrac{A^*+A}{2}\big), \qquad \varphi'\big(\tfrac{B+B^*}{2}\big) \leq \big(\tfrac{B+B^*}{2}\big)^{-1}\varphi\big(\tfrac{B+B^*}{2}\big), \tag{41}$$

presented in the proof of [24, th. 3.1(c),(d)], additionally combined with the (double) monotonicity property [26, (1)].

- The inequality (38) in the case (b1) (resp. (b2)) is stated at the end of [24, th. 3.1(a)] (resp. is given by the inequality (35) in [24, th. 3.1(a)]).
- The inequality (39) in the case (c1) (resp. (c2)) is stated at the end of [24, th. 3.1(b)] (resp. is given by the inequality (36) in [24, th. 3.1(b)]).
- The first inequality in (40) in the case (d) is again based on the inequalities in (41), combined with application of the monotonicity property [26, (1)].

The proof of the second inequality in (40) in the case (d1) for $\varphi \in \mathcal{P}(-1, 1)$ relays on the estimate

$$\begin{aligned}
&\Big\|\varphi'\Big(\tfrac{A^*+A}{2}\Big)^{-\frac12}\big(\varphi(A)X - X\varphi(B)\big)\varphi'\Big(\tfrac{B+B^*}{2}\Big)^{-\frac12}\Big\| = \\
&\varphi'(0)\Big\|\int_{-1}^{1} \varphi'\Big(\tfrac{A^*+A}{2}\Big)^{-\frac12}(I - tA)^{-1}(AX - XB)(I - tB)^{-1}\varphi'\Big(\tfrac{B+B^*}{2}\Big)^{-\frac12} d\mu(t)\Big\| \\
&\leq \Big\|\varphi'\Big(\tfrac{A^*+A}{2}\Big)^{-\frac12}\varphi'(0)\int_{-1}^{1}(I - tA)^{-1}(I - tA^*)^{-1}d\mu(t)\varphi'\Big(\tfrac{A^*+A}{2}\Big)^{-\frac12}\Big\|^{\frac12} \\
&\times\|AX - XB\|\Big\|\varphi'\Big(\tfrac{B+B^*}{2}\Big)^{-\frac12}\varphi'(0)\int_{-1}^{1}(I - tB^*)^{-1}(I - tB)^{-1}d\mu(t)\varphi'\Big(\tfrac{B+B^*}{2}\Big)^{-\frac12}\Big\|^{\frac12},
\end{aligned} \tag{42}$$

where the inequality in (42) is based on application of the inequality (12) in [15, lemma 3.1(a1)], applied to s.i. families $\{\varphi'\big(\frac{A^*+A}{2}\big)^{-1/2}(I - tA)^{-1}\}_{t\in[-1,1]}$ and $\{(I - tB)^{-1}\varphi'\big(\frac{B+B^*}{2}\big)^{-1/2}\}_{t\in[-1,1]}$. With the norms of the expression appearing before (42) and the second expression appearing after (42) not exceeding 1, based on the same arguments used to justify inequalities (47)–(48) in the proof of [24, th. 3.1], this confirms the second inequality in (40) in the case (d1). The case $\varphi \in \mathcal{P}[0, +\infty)$ proves by analogy.

The second inequality in (40) in the case (d2) follows from the inequality (38) in the case (b2) (resp. (39) in the case (c2)), if applied to $X\varphi'\big(\frac{B+B^*}{2}\big)^{-1/2}$ (resp.

$\varphi'\big(\frac{A^*+A}{2}\big)^{-1/2}X)$ instead of X. The second inequality in (40) in the case (d3) is given by the inequality (42) in [24, th. 3.1(d)].

The previous Theorem 7.2 provides the following generalization of some well known norm inequality to normal operators, which are either accretive or they have strictly contractive real parts.

Corollary 7.3 *Let $A, B \in \mathbb{B}(\mathcal{H})$ be normal, such that $AX - XB \in \mathcal{C}_{|||\cdot|||}(\mathcal{H})$ for some $X \in \mathbb{B}(\mathcal{H})$.*

(a) *If A and B are accretive operators, then for all $\theta \in [0, 1]$*

$$\Big|\Big|\Big| \big(\tfrac{A^*+A}{2}\big)^{\frac{1-\theta}{2}}(A^\theta X - XB^\theta)\big(\tfrac{B+B^*}{2}\big)^{\frac{1-\theta}{2}} \Big|\Big|\Big| \le \theta |||AX - XB|||, \tag{43}$$

$$\Big|\Big|\Big|\sqrt{A^*+A}\big(\log(A)X - X\log(B)\big)\sqrt{B+B^*}\Big|\Big|\Big| \le 2|||AX - XB|||. \tag{44}$$

If, in addition, A and B are strictly accretive operators, then

$$\begin{aligned}&\Big|\Big|\Big| A\big(\log(I+A)\big)^{-1}X - XB\big(\log(I+B)\big)^{-1} \Big|\Big|\Big| \le \\ &\Big|\Big|\Big|\sqrt{\log\big(I+\tfrac{A^*+A}{2}\big) - \tfrac{A^*+A}{2}\big(I+\tfrac{A^*+A}{2}\big)^{-1}}\Big(\log\big(I+\tfrac{A^*+A}{2}\big)\Big)^{-1}(AX-XB) \\ &\times\sqrt{\log\big(I+\tfrac{B+B^*}{2}\big) - \tfrac{B+B^*}{2}\big(I+\tfrac{B+B^*}{2}\big)^{-1}}\Big(\log\big(I+\tfrac{B+B^*}{2}\big)\Big)^{-1}\Big|\Big|\Big|.\end{aligned}$$

(b) *If A and B have strictly contractive real parts and $0 \le \alpha, \beta \le 1$, then*

$$\begin{aligned}&\Big|\Big|\Big|\sqrt{I-\big|\tfrac{A^*+A}{2}\big|^2}\Big(\log\tfrac{I+A}{I-A}X - X\log\tfrac{I+B}{I-B}\Big)\sqrt{I-\big|\tfrac{B+B^*}{2}\big|^2}\Big|\Big|\Big| \le 2|||AX-XB|||, \\ &\tfrac{\pi}{2}\Big|\Big|\Big|\cos\tfrac{A^*+A}{\pi}\big(\tan\tfrac{2A}{\pi}X - X\tan\tfrac{2B}{\pi}\big)\cos\tfrac{B+B^*}{\pi}\Big|\Big|\Big| \le |||AX-XB|||, \\ &\Big|\Big|\Big|\big((I+A)^\alpha - (I-A)^\beta\big)X - X\big((I+B)^\alpha - (I-B)^\beta\big)\Big|\Big|\Big| \le \\ &\Big|\Big|\Big|\Big(\alpha\big(I+\tfrac{A^*+A}{2}\big)^{\alpha-1} + \beta\big(I-\tfrac{A^*+A}{2}\big)^{\beta-1}\Big)^{1/2}(AX-XB) \\ &\times\Big(\alpha\big(I+\tfrac{B+B^*}{2}\big)^{\alpha-1} + \beta\big(I-\tfrac{B+B^*}{2}\big)^{\beta-1}\Big)^{1/2}\Big|\Big|\Big|.\end{aligned}$$

Corollary 7.3 represents a reformulation of [24, cor. 3.2].

As noted in [24, rem. 3], the inequality (43) extends "difference" version of celebrated Heinz inequality in [8, Hilfssatz 3.] and its norm ideal version (9) in [3, th. 2], [9, (5.3.2)] and [30, (4.1)] to accretive normal operators. Its weakened version

$$|||H^{\frac12+\beta}XK^{\frac12-\beta} - H^{\frac12-\beta}XK^{\frac12+\beta}||| \le |||HX - XK||| \tag{45}$$

in [30, (4.1)] and another important generalization of the first inequality in [8, Hilfssatz 3.] and Heinz-type means inequality [9, (5.3.1)] from positive to self-adjoint operators is given by the special case $p := 1$ of [12, lemma 3.2]:

Lemma 7.4 *For self-adjoint* $A, B \in \mathbb{B}(\mathcal{H})$ *and* $X \in \mathbb{B}(\mathcal{H})$, $p \geq 1$ *and u.i. norms* $|||\cdot|||$, *the function* $f : [0, p] \to [0, +\infty]$, *defined by*

$$f(s) \stackrel{\text{def}}{=} \left|\left|\left| |A|^{s-1}AX|B|^{p-s} + |A|^{p-s}XB|B|^{s-1} \right|\right|\right| \quad \text{for all } s \in [0, p],$$

is convex and symmetric on $[0, p]$, *non-increasing on* $[0, p/2]$ *and non-decreasing on* $[p/2, p]$, *implying that for all* $s \in [0, p]$

$$\left|\left|\left||A|^{s-1}AX|B|^{p-s}+|A|^{p-s}XB|B|^{s-1}\right|\right|\right| \leq \left|\left|\left||A|^{p-1}AX+XB|B|^{p-1}\right|\right|\right|. \tag{46}$$

The inequality (46) follows from the first part of the above lemma, as $f(s) \leq \max_{[0,p]} f = f(0) = f(p) = \left|\left|\left| |A|^{p-1}AX + XB|B|^{p-1} \right|\right|\right|$ for all $s \in [0, p]$. The inequality (45) follows from inequality (46) by taking $p := 1$, $A := H^{\frac{1+2\beta}{2s}}$ and $B := -K^{\frac{1+2\beta}{2s}}$ for $s \in (0, 1)$. Inequality (46) plays the important role in the proofs of the norm inequality for self-adjoint derivations

$$\left|\left|\left| |AX + XB|^p \right|\right|\right| \leq 2^{p-1}\|X\|^{p-1}\left|\left|\left| |A|^{p-1}AX + XB|B|^{p-1} \right|\right|\right|$$

in [12, th. 3.1] and the perturbation norm inequality in [12, th. 3.3]:

$$\left|\left|\left| |A - B|^p \right|\right|\right| \leq 2^{p-1}\left|\left|\left| A|A|^{p-1} - B|B|^{p-1} \right|\right|\right|.$$

It is also noted in [24, rem. 3] that the special case of (44) for strictly positive operators A and B can also be derived from geometric-logarithmic mean inequality [30, (5.2)].

8 Laplace Transformers, Arithmetic-Geometric and Young Norm Inequalities

In this section we show some applications of Theorem 2.3 to Laplace transformers, which represent an important subclass of i.p.t. transformers.

Definition 8.1 Let $f : \mathbb{R}_+ \to \mathbb{C}$ be a Lebesque measurable function and $A, B, X \in \mathbb{B}(\mathcal{H})$. If a function $t \mapsto e^{-tA}f(t)$ is Gel'fand integrable on $\mathbb{R}_+$ (in respect to the Lebesque measure), then

$$\mathcal{L}_{\mathbb{B}(\mathcal{H})}f(A) \stackrel{\text{def}}{=} \int_{\mathbb{R}_+} e^{-tA}f(t)\,dt$$

will denote the *operator valued Laplace transform* of f (in A), and, similarly, if a function $t \mapsto e^{-tA}Xe^{-tB}f(t)$ is Gel'fand integrable on $\mathbb{R}_+$, then

$$\mathcal{L}_{\mathbb{B}(\mathbb{B}(\mathcal{H}))} f(A \otimes I + I \otimes B)(X) \stackrel{\text{def}}{=} \int_{\mathbb{R}_+} e^{-tA}Xe^{-tB} f(t)\, dt$$

will denote the *Laplace transformer* of f (in a generalized derivation $A \otimes I + I \otimes B$, evaluated in X).

In the sequel, we will use the simplified notation $\mathcal{L}f$ instead of both $\mathcal{L}_{\mathbb{B}(\mathcal{H})} f$ and $\mathcal{L}_{\mathbb{B}(\mathbb{B}(\mathcal{H}))} f$, as their (exact) meaning will be clear from the context. As it is usual for linear transformations, we will also often write $\mathcal{L}f(A \otimes I + I \otimes B)X$ instead of $\mathcal{L}f(A \otimes I + I \otimes B)(X)$, except in the case $X := I$ in which the brackets will not be omitted.

For Laplace transformers we have the following reformulation of [28, th. 2.2]:

Theorem 8.2 *Let Ψ, Υ be s.n. functions, $p \geq 2$, $A, B \in \mathbb{B}(\mathcal{H})$, $f, g \colon \mathbb{R}_+ \to \mathbb{C}$ be some Lebesgue measurable functions and $X \in \mathcal{C}_\Psi(\mathcal{H})$.*

(a) *If $\int_{\mathbb{R}_+} (\|e^{-tA}h\|^2 |f(t)|^2 + \|e^{-tB^*}h\|^2 |g(t)|^2)\, dt < +\infty$ for all $h \in \mathcal{H}$, then*

$$\big\|\mathcal{L}(fg)(\Delta_{A,B})X\big\|_\Psi \leq \big\|(\mathcal{L}|f|^2(\Delta_{A^*,A})(I))^{\frac{1}{2}} X (\mathcal{L}|g|^2(\Delta_{B,B^*})(I))^{\frac{1}{2}}\big\|_\Psi, \qquad (47)$$

under any of the following conditions:

(a1) $\|\cdot\|_\Psi := \|\cdot\|_1$,
(a2) $\Psi := \Upsilon^{(p)*}$ *and (at least) one of A or B is normal operator,*
(a3) $\|\cdot\|_\Psi := |||\cdot|||$ *and both A and B are normal operators;*

(b) *If $\int_{\mathbb{R}_+} (\|e^{-tA}h\|^2 |f(t)|^2 + \|e^{-tB}h\|^2 |g(t)|^2)\, dt < +\infty$ for all $h \in \mathcal{H}$, then*

$$\big\|\mathcal{L}(fg)(\Delta_{A,B})X\big\|_\Psi \leq \big\|(\mathcal{L}|f|^2(\Delta_{A^*,A})(I))^{\frac{1}{2}} X\big\|_\Psi \big\|\mathcal{L}|g|^2(\Delta_{B^*,B})(I)\big\|^{\frac{1}{2}}, \qquad (48)$$

under any of the following conditions:

(b1) $\|\cdot\|_\Psi := \|\cdot\|_2$,
(b2) $\Psi := \Upsilon^{(p)}$ *and A is normal operator;*

(c) *If $\int_{\mathbb{R}_+} (\|e^{-tA^*}h\|^2 |f(t)|^2 + \|e^{-tB^*}h\|^2 |g(t)|^2)\, dt < +\infty$ for all $h \in \mathcal{H}$, then*

$$\big\|\mathcal{L}(fg)(\Delta_{A,B})X\big\|_\Psi \leq \big\|\mathcal{L}|f|^2(\Delta_{A,A^*})(I)\big\|^{\frac{1}{2}} \big\|X(\mathcal{L}|g|^2(\Delta_{B,B^*})(I))^{\frac{1}{2}}\big\|_\Psi, \qquad (49)$$

under any of the following conditions:

(c1) $\|\cdot\|_\Psi := \|\cdot\|_2$,
(c2) $\Psi := \Upsilon^{(p)}$ *and B is normal operator.*

As $\mathcal{L}_{\mathbb{B}(\mathbb{B}(\mathcal{H}))}|f|^2\,(\Delta_{A^*,A})\;\;(I) \;=\; \mathcal{L}_{\mathbb{B}(\mathcal{H})}|f|^2(A^*+A)$ if A is normal and $\mathcal{L}_{\mathbb{B}(\mathbb{B}(\mathcal{H}))}|g|^2(\Delta_{B,B^*})\;(I) \;=\; \mathcal{L}_{\mathbb{B}(\mathcal{H})}|g|^2(B+B^*)$ if B is normal, let us note that inequalities (47), (48) and (49) can be simplified in those cases.

The following Aczél–Bellman type norm inequality for Laplace transformers reformulates [28, th. 2.3]:

Theorem 8.3 *Let $p \geq 2$, Ψ, Υ be s.n. functions, $f, g\colon \mathbb{R}_+ \to \mathbb{C}$ be Lebesgue measurable functions and A, $B \in \mathbb{B}(\mathcal{H})$, such that*

$$\int_{\mathbb{R}_+} \|e^{-tA}h\|^2|f(t)|^2dt \leq 1 \qquad \text{and} \qquad \int_{\mathbb{R}_+} \|e^{-tB^*}h\|^2|g(t)|^2dt \leq 1$$

for all $h \in \mathcal{H}$ satisfying $\|h\| \leq 1$. Then for all $X \in \mathcal{C}_\Psi(\mathcal{H})$

$$\begin{aligned}&\big\|X - \mathcal{L}(fg)(\Delta_{A,B})X\big\|_\Psi \geq\\ &\big\|(I - \mathcal{L}|f|^2(\Delta_{A^*,A})(I))^{1/2}X(I - \mathcal{L}|g|^2(\Delta_{B,B^*})(I))^{1/2}\big\|_\Psi,\end{aligned} \tag{50}$$

under any of the following conditions:

(a) $\Psi := \Upsilon^{(p)*}$ *and (at least) one of A or B is normal operator,*
(b) $\|\cdot\|_\Psi := |\!|\!|\cdot|\!|\!|$ *and both A and B are normal operators.*
More precisely, $\mathcal{L}|f|^2(\Delta_{A^*,A})(I)$ (resp. $\mathcal{L}|g|^2(\Delta_{B,B^*})(I)$) in (50) can be replaced by $\mathcal{L}|f|^2(A^*+A)(I)$ (resp. $\mathcal{L}|g|^2(B+B^*)$) if A (resp. B) is normal.

Jocić et al. [28, th. 2.9] for convolutional norm inequalities for Laplace transformers can be reformulated as:

Theorem 8.4 *Let Ψ, Υ be s.n. functions, $p \geq 2$, $\alpha, \beta \in [0,1]$, A, $B \in \mathbb{B}(\mathcal{H})$, $f, g\colon \mathbb{R}_+ \to \mathbb{C}$ be some Lebesgue measurable functions and $X \in \mathcal{C}_\Psi(\mathcal{H})$.*

(a) *If $\int_{\mathbb{R}_+}(\|e^{-tA}h\|^2|f|^{2-2\alpha} * |g|^{2-2\beta}(t) + \|e^{-tB^*}h\|^2|f|^{2\alpha} * |g|^{2\beta}(t))dt < +\infty$ for all $h \in \mathcal{H}$, then*

$$\begin{aligned}&\|\mathcal{L}(f*g)(\Delta_{A,B})X\|_\Psi \leq\\ &\big\|\big(\mathcal{L}\big(|f|^{2-2\alpha} * |g|^{2-2\beta}\big)(\Delta_{A^*,A})(I)\big)^{\frac{1}{2}}X\big(\mathcal{L}\big(|f|^{2\alpha} * |g|^{2\beta}\big)(\Delta_{B,B^*})(I)\big)^{\frac{1}{2}}\big\|_\Psi,\end{aligned} \tag{51}$$

under any of the following conditions:

(a1) $\|\cdot\|_\Psi := \|\cdot\|_1$,
(a2) $\Psi := \Upsilon^{(p)*}$ *and (at least) one of A or B is normal operator,*
(a3) $\|\cdot\|_\Psi := |\!|\!|\cdot|\!|\!|$ *and both A and B are normal operators;*

(b) *If* $\int_{\mathbb{R}_+}(\|e^{-tA}h\|^2|f|^{2-2\alpha}*|g|^{2-2\beta}(t)+\|e^{-tB}h\|^2|f|^{2\alpha}*|g|^{2\beta}(t))\,dt<+\infty$ *for all* $h\in\mathcal{H}$, *then*

$$\|\mathcal{L}(f*g)(\Delta_{A,B})X\|_\Psi \le \tag{52}$$
$$\Big\|\Big(\mathcal{L}\big(|f|^{2-2\alpha}*|g|^{2-2\beta}\big)(\Delta_{A^*,A})(I)\Big)^{\frac12}X\Big\|_\Psi\Big\|\mathcal{L}\big(|f|^{2\alpha}*|g|^{2\beta}\big)(\Delta_{B^*,B})(I)\Big\|^{\frac12}.$$

under any of the following conditions:

(b1) $\|\cdot\|_\Psi := \|\cdot\|_2$,

(b2) $\Psi := \Upsilon^{(p)}$ *and* A *is normal operator;*

(c) *If* $\int_{\mathbb{R}_+}(\|e^{-tA^*}h\|^2|f|^{2-2\alpha}*|g|^{2-2\beta}(t)+\|e^{-tB^*}h\|^2|f|^{2\alpha}*|g|^{2\beta}(t))\,dt<+\infty$ *for all* $h\in\mathcal{H}$, *then*

$$\|\mathcal{L}(f*g)(\Delta_{A,B})X\|_\Psi \le \tag{53}$$
$$\Big\|\mathcal{L}\big(|f|^{2-2\alpha}*|g|^{2-2\beta}\big)(\Delta_{A,A^*})(I)\Big\|^{\frac12}\Big\|X\Big(\mathcal{L}\big(|f|^{2\alpha}*|g|^{2\beta}\big)(\Delta_{B,B^*})(I)\Big)^{\frac12}\Big\|_\Psi.$$

under any of the following conditions:

(c1) $\|\cdot\|_\Psi := \|\cdot\|_2$,

(c2) $\Psi := \Upsilon^{(p)}$ *and* B *is normal operator.*

As $\mathcal{L}\big(|f|^{2-2\alpha}*|g|^{2-2\beta}\big)(\Delta_{A^*,A})(I)=\mathcal{L}\big(|f|^{2-2\alpha}*|g|^{2-2\beta}\big)(A^*+A)$ if A is normal and $\mathcal{L}\big(|f|^{2\alpha}*|g|^{2\beta}\big)(\Delta_{B,B^*})(I)=\mathcal{L}\big(|f|^{2\alpha}*|g|^{2\beta}\big)(B+B^*)$ if B is normal, inequalities (51), (52) and (53) can be simplified in those cases.

Jocić et al. [27, lemma 2.5] guarantees the correctness of the following

Definition 8.5 For accretive operators $A, B\in\mathbb{B}(\mathcal{H})$ and $X\in\mathbb{B}(\mathcal{H})$ we define

$$V^-(A,B)X \stackrel{\text{def}}{=} {}^{[s]}\lim_{T\to+\infty}\sqrt{A^*+A}\,e^{-TA}Xe^{-TB}\sqrt{B+B^*},$$
$$\mho^-(A,B)X \stackrel{\text{def}}{=} {}^{[s]}\lim_{T\to+\infty}e^{-TA}Xe^{-TB}.$$

The previous Definition 8.5 is useful for the formulation of the next:

Theorem 8.6 *Let* $\alpha\in(0,1)$, $p\ge2$, Υ *be a s.n. function and* $A,B,X\in\mathbb{B}(\mathcal{H})$, *such that A and B are accretive.*

(a) *If* $AX+XB\in\mathcal{C}_1(\mathcal{H})$, *then*

$$\big\|\sqrt{A^*+A}\,X\sqrt{B+B^*}-V^-(A,B)X\big\|_1 \le \tag{54}$$
$$\big\|\big(I-\mho^-(A^*,A)I\big)^{1/2}(AX+XB)\big(I-\mho^-(B,B^*)I\big)^{1/2}\big\|_1\le\|AX+XB\|_1.$$

Consequently, the left side of (54) *can be replaced by* $\big\|\sqrt{A^*+A}\,X\sqrt{B+B^*}\big\|_1$ *under additional conditions required in* (b1) *or* (b2) *of* [27, *lemma 2.7*].

(b1) *If, in addition, A is strictly accretive, B is normal and $AX+XB \in \mathcal{C}_{\Upsilon^{(p)}}(\mathcal{H})$, then*

$$\left\| X(B+B^*)^{1/2} \right\|_{\Upsilon^{(p)}} \le \left\| (A^*+A)^{-1/2}(AX+XB)P_{\overline{\mathcal{R}(B+B^*)}} \right\|_{\Upsilon^{(p)}} \le \left\| (A^*+A)^{-1/2}(AX+XB) \right\|_{\Upsilon^{(p)}}. \tag{55}$$

(b2) *Alternatively, if A is normal, B is strictly accretive and $AX+XB \in \mathcal{C}_{\Upsilon^{(p)}}(\mathcal{H})$, then*

$$\left\| (A^*+A)^{1/2}X \right\|_{\Upsilon^{(p)}} \le \left\| P_{\overline{\mathcal{R}(A^*+A)}}(AX+XB)(B+B^*)^{-1/2} \right\|_{\Upsilon^{(p)}} \le \left\| (AX+XB)(B+B^*)^{-1/2} \right\|_{\Upsilon^{(p)}}. \tag{56}$$

(c) *Specially, in the case of the Hilbert-Schmidt class $\mathcal{C}_2(\mathcal{H})$ (i.e. if $\Phi := \ell$ and $p := 2$), the requirement of normality for A or B can be omitted, with* (55) *and* (56) *still remaining valid, if $P_{\overline{\mathcal{R}(B+B^*)}}$ (resp. $P_{\overline{\mathcal{R}(A^*+A)}}$) is replaced by $\big(I-\mho^-(B,B^*)I\big)^{1/2}$ (resp. $\big(I-\mho^-(A^*,A)I\big)^{1/2}$).*

(d) *If at least one of operators A or B is additionally normal and $AX+XB \in \mathcal{C}_{\Upsilon^{(p)*}}(\mathcal{H})$, then*

$$\left\| \sqrt{A^*+A}\,X\sqrt{B+B^*} \right\|_{\Upsilon^{(p)*}} \le$$
$$\left\| \big(I-\mho^-(A^*,A)I\big)^{\frac12}(AX+XB)\big(I-\mho^-(B,B^*)I\big)^{\frac12} \right\|_{\Upsilon^{(p)*}} \le \|AX+XB\|_{\Upsilon^{(p)*}}.$$

(e) *If A and B are both normal, then*

$$\left| (A^*+A)^{1-\alpha}X(B+B^*)^{\alpha} \right|^2 \le \tag{57}$$
$$\Gamma(2-2\alpha)\int_{\mathbb{R}_+} e^{-tB^*}(B^*+B)^{\alpha}|AX+XB|^2(B^*+B)^{\alpha}e^{-tB}t^{2\alpha-1}dt,$$

and, if additionally $AX+XB \in \mathcal{C}_{|||\cdot|||}(\mathcal{H})$, then also

$$|||(A^*+A)^{1-\alpha}X(B+B^*)^{\alpha}||| \le \sqrt{\Gamma(2-2\alpha)\Gamma(2\alpha)}\,|||P_{\overline{\mathcal{R}(A^*+A)}}(AX+XB)P_{\overline{\mathcal{R}(B+B^*)}}|||$$
$$\le \sqrt{\Gamma(2-2\alpha)\Gamma(2\alpha)}\,|||AX+XB||| = \sqrt{\tfrac{(1-2\alpha)\pi}{\sin 2\alpha\pi}}\,|||AX+XB|||. \tag{58}$$

(f) *If $p>1$ and $p' \overset{\text{def}}{=} \frac{p}{p-1}$, then for all normal accretive operators A and B, such that $AX+XB \in \mathcal{C}_{|||\cdot|||}(\mathcal{H})$,*

$$|||(A^*+A)^{1/p}X(B+B^*)^{1/p'}||| \le \sqrt[p]{p}\,\sqrt[p']{p'}\sqrt{\Gamma(2-2\alpha)\Gamma(2\alpha)}\,\left|\left|\left| \frac{A}{p}X + X\frac{B}{p'} \right|\right|\right|.$$

For the proof see [27, th. 2.9] and its proof, as well as [27, rem. 2.2]. In addition, for the inequality (57) see [27, th. 2.8(d)] and its proof.

Thus, Theorem 8.6 provides extensions of the Young's norm inequality [15, cor. 4.1] in various direction, where the norm inequality (58) in part (e) extends it from positive definite to normal accretive operators.

The next Theorem 8.7 represents the main part of [27, th. 2.10].

Theorem 8.7 *Let $X \in \mathcal{C}_{|||\cdot|||}(\mathcal{H})$, $(\alpha_n)_{n=1}^{\infty}$ be a sequence in $(0,1)$, $(t_n)_{n=1}^{\infty}$ be a summable sequence in $(0,+\infty)$, $f\colon \mathbb{C} \to \mathbb{C}\colon z \mapsto \prod_{n=1}^{\infty}(1+t_n z)$ and let each of the absolutely summable (in $\mathbb{B}(\mathcal{H})$) families $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ consists of normal, accretive and mutually commuting operators. Then*

$$|||X||| \le \left|\left|\left|\left(\prod_{n=1}^{\infty}(I+A_n^*+A_n)\right)^{1/2} X \left(\prod_{n=1}^{\infty}(I+B_n+B_n^*)\right)^{1/2}\right|\right|\right|$$
$$\le \left|\left|\left|\prod_{n=1}^{\infty}(I\otimes I+A_n\otimes I+I\otimes B_n)X\right|\right|\right|.$$

Specially, if A and B are normal, accretive operators, then

$$|||X||| \le \left|\left|\left|\sqrt{f(A^*+A)}\,X\sqrt{f(B+B^*)}\right|\right|\right|$$
$$\le \left|\left|\left| f(A\otimes I+I\otimes B)X\right|\right|\right| \le \left|\left|\left|\sqrt{f(I+A^*A)}\,X\sqrt{f(I+BB^*)}\right|\right|\right|.$$

The entire function $s\colon \mathbb{C}\to\mathbb{C}\colon z\mapsto \sum_{n=0}^{\infty}\frac{z^n}{(2n+1)!}$ satisfies a functional relation $\sinh(z) = zs(z^2)$ for all $z \in \mathbb{C}$ and it admits an infinite product representation $s(z) = \prod_{n=1}^{\infty}\left(1+\frac{z}{n^2\pi^2}\right)$, playing the prominent role in

Corollary 8.8 *If $X \in \mathcal{C}_{|||\cdot|||}(\mathcal{H})$ and $A, B, C, D \in \mathbb{B}(\mathcal{H})$ are such that A and B are normal and accretive, $C = C^*$ and $D = D^*$, then*

$$|||X||| \le \left|\left|\left|\sqrt{s(A^*+A)}\,X\sqrt{s(B+B^*)}\right|\right|\right| \le$$
$$|||s(A\otimes I+I\otimes B)X||| \le \left|\left|\left|\sqrt{s(I+A^*A)}\,X\sqrt{s(I+BB^*)}\right|\right|\right|, \tag{59}$$
$$|||CX+XD||| \le |||\sinh(C\otimes I+I\otimes D)X||| = \tfrac{1}{2}\left|\left|\left|e^C X e^D - e^{-C} X e^{-D}\right|\right|\right|. \tag{60}$$

Corollary 8.8 was earlier presented in [27, cor. 2.13]. For its connections with [31, prop. 21(1)] see [27, rem. 2.5].

For 2-hyper-accretive and 2-hyper-contractive operators [28, th. 3.6] says:

Theorem 8.9 *Let* $A, B, C, D, X \in \mathbb{B}(\mathcal{H})$, Ψ, Υ *be s.n. functions and* $p \geq 2$. *If* A, B^* *are* 2*-hyper-accretive and* $A^2X + 2AXB + XB^2 \in \mathcal{C}_\Psi(\mathcal{H})$, *then*

$$\left\|(A^{*2} + 2A^*A + A^2)^{1/2} X (B^2 + 2BB^* + B^{*2})^{1/2}\right\|_\Psi \leq \left\| A^2X + 2AXB + XB^2 \right\|_\Psi,$$

under any of the following conditions:

(a1) $\|\cdot\|_\Psi := \|\cdot\|_1$ *and* $X \in \mathbb{K}(\mathcal{H})$,
(a2) $\Psi := \Upsilon^{(p)^*}$ *and (at least) one of operators* A *or* B *is normal,*
(a3) $\|\cdot\|_\Psi := |||\cdot|||$ *and both* A *and* B *are normal operators.*

If C, D^* *are* 2*-hyper-contractions and* $X - 2CXD + C^2XD^2 \in \mathcal{C}_\Psi(\mathcal{H})$, *then*

$$\left\|(I - 2C^*C + C^{*2}C^2)^{1/2} X (I - 2DD^* + D^2D^{*2})^{1/2}\right\|_\Psi \leq \left\| X - 2CXD + C^2XD^2 \right\|_\Psi,$$

under any of the following conditions:

(b1) $\|\cdot\|_\Psi := \|\cdot\|_1$ *and* $X \in \mathbb{K}(\mathcal{H})$,
(b2) $\Psi := \Upsilon^{(p)^*}$ *and (at least) one of operators* C *or* D *is normal,*
(b3) $\|\cdot\|_\Psi := |||\cdot|||$ *and both* C *and* D *are normal operators.*

Under stronger conditions $X \in \mathcal{C}_\Psi(\mathcal{H})$ we have the following reformulation of [27, th. 2.15]:

Theorem 8.10 *If* Ψ, Υ *are s.n. functions,* $p \geq 2$, $M, N \in \mathbb{N}$ *are such that* $(M + N)/2 \in \mathbb{N}$ *and* $A, B \in \mathbb{B}(\mathcal{H})$ *are such that* A *is* M*-hyper-accretive and* B^* *is* N*-hyper-accretive, then for all* $X \in \mathcal{C}_\Psi(\mathcal{H})$

$$\left\|\left(\sum_{m=0}^{M} \binom{M}{m} A^{*m}A^{M-m}\right)^{1/2} X \left(\sum_{n=0}^{N} \binom{N}{n} B^{N-n}B^{*n}\right)^{1/2}\right\|_\Psi \leq \frac{\sqrt{(M-1)!(N-1)!}}{\left(\frac{M+N}{2}-1\right)!} \left\| \sum_{k=0}^{(M+N)/2} \binom{\frac{M+N}{2}}{k} A^{\frac{M+N}{2}-k} X B^k \right\|_\Psi,$$

under any of the following conditions:

(a1) $\|\cdot\|_\Psi := \|\cdot\|_1$,
(a2) $\Psi := \Upsilon^{(p)^*}$ *and (at least) one of operators* A *or* B *is normal,*
(a3) $\|\cdot\|_\Psi := |||\cdot|||$ *and both* A *and* B *are normal operators.*

9 Applications to Refinements and Generalizations of Minkowski, Zhan and Heron Inequalities

We start this section with the shortened version of [17, th. 2.5], which provides Minkowski inequality for p-modified u.i. norms.

Theorem 9.1 *Let* $p \geq 2, \{B_n\}_{n\in\mathbb{N}}$ *be a sequence in* $\mathbb{B}(\mathcal{H})$, *such that* $\sum_{n\in\mathbb{N}} B_n$ *converges in the strong operator topology and let* $\{\beta_n\}_{n\in\mathbb{N}}$ *be a sequence in* $(0, +\infty)$ *satisfying* $\sum_{n\in\mathbb{N}} \beta_n < +\infty$. *Then*

$$\left|\!\left|\!\left|\left|\sum_{n\in\mathbb{N}} B_n\right|^p + \frac{1}{2}\sum_{m,n\in\mathbb{N}} \beta_m^{\frac{p}{2}}\beta_n^{\frac{p}{2}}\left|\frac{B_m}{\beta_m}-\frac{B_n}{\beta_n}\right|^p\right|\!\right|\!\right|^{\frac{1}{p}} \leq \left(\sum_{n\in\mathbb{N}}\beta_n\right)^{1-\frac{1}{p}}\left|\!\left|\!\left|\sum_{n\in\mathbb{N}}\beta_n^{1-p}|B_n|^p\right|\!\right|\!\right|^{\frac{1}{p}}.$$

If $0 \neq B_n \in \mathcal{C}_{|||\cdot|||_{(p)}}(\mathcal{H})$ *for all* $n \in \mathbb{N}$ *and* $\sum_{n\in\mathbb{N}} |||B_n|||_{(p)} < +\infty$, *then*

$$\left|\!\left|\!\left|\sum_{n\in\mathbb{N}} B_n\right|\!\right|\!\right|_{(p)} \leq \left|\!\left|\!\left|\left|\sum_{n\in\mathbb{N}} B_n\right|^p + \frac{1}{2}\sum_{m,n\in\mathbb{N}} |||B_m|||_{(p)}^{\frac{p}{2}}|||B_n|||_{(p)}^{\frac{p}{2}}\left|\frac{B_m}{|||B_m|||_{(p)}}-\frac{B_n}{|||B_n|||_{(p)}}\right|^p\right|\!\right|\!\right|^{\frac{1}{p}}$$

$$\leq \left(\sum_{n\in\mathbb{N}}|||B_n|||_{(p)}\right)^{1-\frac{1}{p}}\left|\!\left|\!\left|\sum_{n\in\mathbb{N}}|||B_n|||_{(p)}^{1-p}|B_n|^p\right|\!\right|\!\right|^{\frac{1}{p}} \leq \sum_{n\in\mathbb{N}}|||B_n|||_{(p)}. \tag{61}$$

Here we used the notation $|||\cdot|||_{(p)} \stackrel{\text{def}}{=} \|\cdot\|_{\Upsilon^{(p)}}$, where s.n. function Υ is uniquely determined by $\|\cdot\|_{\Upsilon} = |||\cdot|||$, as well as the monotonicity property for u.i. norms to get the first (of three) inequality in (61).

We also have the following refined norm inequality for bi-infinite operator matrices, given as a part of [17, th. 2.8]:

Theorem 9.2 *If* $2 \leq p < +\infty$, *then for all* $[A_{m,n}]_{m,n\in\mathbb{Z}} \in \mathcal{C}_{|||\cdot|||_{(p)}}(\ell^2_{\mathbb{Z}}(\mathcal{H}))$

$$|||[A_{m,n}]_{m,n\in\mathbb{Z}}|||_{(p)}^2 \leq \left|\!\left|\!\left|\,|[A_{m,n}]_{m,n\in\mathbb{Z}}|^p + \frac{1}{2}\sum_{k,l\in\mathbb{Z}} |||[A_{k,n}]_{n\in\mathbb{Z}}|||_{(p)}^{\frac{p}{2}}|||[A_{l,n}]_{n\in\mathbb{Z}}|||_{(p)}^{\frac{p}{2}}\right.\right.\right.$$

$$\left.\left.\left.\times\left|\frac{|[A_{k,n}]_{n\in\mathbb{Z}}|^2}{|||[A_{k,n}]_{n\in\mathbb{Z}}|||_{(p)}^2}-\frac{|[A_{l,n}]_{n\in\mathbb{Z}}|^2}{|||[A_{l,n}]_{n\in\mathbb{Z}}|||_{(p)}^2}\right|^p\right|\!\right|\!\right|^{\frac{2}{p}} \leq \sum_{m\in\mathbb{Z}}|||[A_{m,n}]_{n\in\mathbb{Z}}|||_{(p)}^2 \leq$$

$$\sum_{m\in\mathbb{Z}}\left|\!\left|\!\left|\,|[A^*_{n,m}]_{n\in\mathbb{Z}}|^p + \frac{1}{2}\sum_{k,l\in\mathbb{Z}}|||A^*_{k,m}|||_{(p)}^{\frac{p}{2}}|||A^*_{l,m}|||_{(p)}^{\frac{p}{2}}\left|\frac{|A^*_{k,m}|^2}{|||A^*_{k,m}|||_{(p)}^2}-\frac{|A^*_{l,m}|^2}{|||A^*_{l,m}|||_{(p)}^2}\right|^p\right|\!\right|\!\right|^{\frac{2}{p}}$$

$$\leq \sum_{m,n\in\mathbb{Z}}|||A_{m,n}|||_{(p)}^2.$$

If $q \geq p^\star \stackrel{\text{def}}{=} \frac{p}{p-1}$*, if* $\mathcal{C}_{|||\cdot|||_{(q)}}(\ell^2_{\mathbb{Z}}(\mathcal{H}))$ *is the associated ideal of compact operators and* $|||\cdot|||_{(q)*}$ *is a norm on its dual space* $\mathcal{C}^*_{|||\cdot|||_{(q)}}(\ell^2_{\mathbb{Z}}(\mathcal{H}))$*, then for all* $[A_{m,n}]_{m,n\in\mathbb{Z}} \in \mathcal{C}^*_{|||\cdot|||_{(q)}}(\ell^2_{\mathbb{Z}}(\mathcal{H}))$

$$|||[A_{m,n}]_{m,n\in\mathbb{Z}}|||^p_{(q)*} \geq \sum_{m\in\mathbb{Z}} |||[A_{m,n}]_{n\in\mathbb{Z}}|||^p_{(q)*} \geq \sum_{m,n\in\mathbb{Z}} |||A_{m,n}|||^p_{(q)*}.$$

We proceed with another generalization of the case $n := 1$ in Young norm inequality (60) in [15, th. 4.3] and its consequences to refinements and generalizations of Zhan inequality, provided in the wider form in [27, th. 2.11]:

Theorem 9.3 *Let* $A, B, X \in \mathbb{B}(\mathcal{H})$, $A \geq 0$, $B \geq 0$, $r_1, \dots, r_N \in (0, +\infty)$, $\eta, \theta, \theta_1, \dots, \theta_N \in [-\pi, \pi)$, $\alpha, \alpha_1, \dots, \alpha_N \in (0, 1)$ *for some* $N \in \mathbb{N}$ *and* $\beta \in [0, 1]$.

(a) *If* $e^{i\eta}AX + e^{i\theta}XB \in \mathcal{C}_{|||\cdot|||}(\mathcal{H})$*, then*

$$\left|e^{i\eta} + e^{i\theta}\right| |||A^{1-\alpha}XB^{\alpha}||| \leq \sqrt{\Gamma(2-2\alpha)\Gamma(2\alpha)}\, |||e^{i\eta}AX + e^{i\theta}XB|||.$$

(b) *If* $A^2X + 2\cos\theta\, AXB + XB^2 \in \mathcal{C}_{|||\cdot|||}(\mathcal{H})$*, then*

$$\begin{aligned}
&(2 + 2\cos\theta)|||A^{2-2\alpha}XB^{2\alpha}||| \leq \Gamma(2-2\alpha)\Gamma(2\alpha)\, |||A^2X + 2\cos\theta AXB + XB^2|||,\\
&(2 + 2\cos\theta)|||AXB||| \leq (1 + \cos\theta)\left|\left|\left|\sqrt{A}\left(A^{1-\beta}XB^{\beta} + A^{\beta}XB^{1-\beta}\right)\sqrt{B}\right|\right|\right|\\
&\leq \tfrac{1+\cos\theta}{2} |||A^{1-\beta}(AX + XB)B^{\beta} + A^{\beta}(AX + XB)B^{1-\beta}|||\\
&\leq \tfrac{1+\cos\theta}{2} |||A^2X + 2AXB + XB^2||| \leq |||A^2X + 2\cos\theta\, AXB + XB^2|||.
\end{aligned}$$

(c) *If* $c_0, \dots, c_N \in \mathbb{C}$ *satisfy* $c_N \neq 0$, $\sum_{n=0}^{N} c_n z^n \stackrel{\text{def}}{=} c_N \prod_{n=1}^{N}\left(z - r_n e^{i\theta_n}\right)$ *for all* $z \in \mathbb{C}$ *and* $\sum_{n=0}^{N} c_n A^n X B^{N-n} \in \mathcal{C}_{|||\cdot|||}(\mathcal{H})$*, then*

$$\begin{aligned}
2^N|c_N| \prod_{n=1}^{N} &\left(|\sin(\theta_n/2)| r_n^{\alpha_n}\right) |||A^{N-\sum_{n=1}^{N}\alpha_n} X B^{\sum_{n=1}^{N}\alpha_n}|||\\
&\leq \prod_{n=1}^{N}\sqrt{\Gamma(2-2\alpha_n)\Gamma(2\alpha_n)} \left|\left|\left|\sum_{n=0}^{N} c_n A^n X B^{N-n}\right|\right|\right|.
\end{aligned}$$

(d) *If a sequence* $\{c_n\}_{n=-N}^{N}$ *in* $\mathbb{C}$ *satisfies* $\sum\limits_{n=-N}^{N} c_n e^{int} \stackrel{\text{def}}{=} \prod\limits_{n=1}^{N} \left|e^{it} - r_n e^{i\theta_n}\right|^2$ *for all* $t \in \mathbb{R}$, $c_N \neq 0$ *and* $\sum_{n=-N}^{N} c_n A^{N+n} X B^{N-n} \in \mathcal{C}_{|||\cdot|||}(\mathcal{H})$*, then*

$$2^N|c_N| \prod_{n=1}^{N}(1 - \cos\theta_n)\, |||A^N X B^N||| \leq \left|\left|\left|\sum_{n=-N}^{N} c_n A^{N+n} X B^{N-n}\right|\right|\right|.$$

(e) *If* $\sum_{n=0}^{N}\binom{N}{n}e^{in\eta+i(N-n)\theta}A^nXB^{N-n}\in\mathcal{C}_{|||\cdot|||}(\mathcal{H})$, *then*

$$2^N\left|\cos\tfrac{\eta-\theta}{2}\right|^N\left|\left|\left|A^{N-\sum_{n=1}^N\alpha_n}XB^{\sum_{n=1}^N\alpha_n}\right|\right|\right|$$
$$\le\prod_{n=1}^{N}\sqrt{\Gamma(2-2\alpha_n)\Gamma(2\alpha_n)}\left|\left|\left|\sum_{n=0}^{N}\binom{N}{n}e^{in\eta+i(N-n)\theta}A^nXB^{N-n}\right|\right|\right|\quad\text{and}$$
$$\frac{\left|\sin\frac{N+1}{2}\eta_\circ\right|}{\sin\frac{\eta_\circ}{2}}\left|\left|\left|A^{N-\sum_{n=1}^N\alpha_n}XB^{\sum_{n=1}^N\alpha_n}\right|\right|\right|$$
$$\le\prod_{n=1}^{N}\sqrt{\Gamma(2-2\alpha_n)\Gamma(2\alpha_n)}\left|\left|\left|\sum_{n=0}^{N}e^{in\eta_\circ}A^nXB^{N-n}\right|\right|\right|,$$

if $\sum_{n=0}^{N}e^{in\eta_\circ}A^nXB^{N-n}\in\mathcal{C}_{|||\cdot|||}(\mathcal{H})$ *for some* $\eta_\circ\in(0,2\pi)$.

(f) *If* $\theta_\circ\in(-\pi,\pi)$ *and* $\sum_{n=0}^{2N}\binom{2N}{n}e^{i(N-n)\theta_\circ}A^nXB^{2N-n}-\tau A^NXB^N\in\mathcal{C}_{|||\cdot|||}(\mathcal{H})$ *for* $0\le\tau<2^{2N}\cos^{2N}\frac{\theta_\circ}{2}$, *then* $\sum_{n=0}^{2N}\binom{2N}{n}e^{i(2N-n)\theta_\circ}A^nXB^{2N-n}\in\mathcal{C}_{|||\cdot|||}(\mathcal{H})$ *and*

$$\left(\sum_{n=0}^{2N}\binom{2N}{n}e^{i(N-n)\theta_\circ}-\tau\right)\left|\left|\left|A^NXB^N\right|\right|\right|\le$$
$$\left(1-\frac{\tau}{2^{2N}\cos^{2N}\frac{\theta_0}{2}}\right)\left|\left|\left|\sum_{n=0}^{2N}\binom{2N}{n}e^{i(N-n)\theta_\circ}A^nXB^{2N-n}\right|\right|\right|\le$$
$$\left|\left|\left|\sum_{n=0}^{2N}\binom{2N}{n}e^{i(N-n)\theta_\circ}A^nXB^{2N-n}-\tau A^NXB^N\right|\right|\right|.$$

Some relations between Young and Heron means norm inequalities are investigated in [27, cor. 2.12], which we present here in the next reduced form:

Corollary 9.4 *If* $A,B,X\in\mathbb{B}(\mathcal{H})$, $A\ge0$, $B\ge0$, $0\le\beta<1/2\le\alpha\le\alpha'$, *such that* $(1-\alpha')\sqrt{A}X\sqrt{B}+\alpha'(AX+XB)/2\in\mathcal{C}_{|||\cdot|||}(\mathcal{H})$, *then*

$$\left|\left|\left|\sqrt{A}X\sqrt{B}\right|\right|\right|\le\frac{1}{2(1-2\beta)}\left|\left|\left|A^{\frac{\beta}{2}+\frac14}\int_{[\beta,1-\beta]}A^{\frac{1-\beta-t}{2}}XB^{\frac{t-\beta}{2}}+A^{\frac{t-\beta}{2}}XB^{\frac{1-t-\beta}{2}}dt\,B^{\frac{\beta}{2}+\frac14}\right|\right|\right|\le$$
$$\frac{1}{2(1-2\beta)}\int_{[\beta,1-\beta]}\left|\left|\left|A^{\frac34-\frac{t}{2}}XB^{\frac14+\frac{t}{2}}+A^{\frac14+\frac{t}{2}}XB^{\frac34-\frac{t}{2}}\right|\right|\right|dt\le$$
$$\frac12\left|\left|\left|A^{\frac34-\frac{\beta}{2}}XB^{\frac14+\frac{\beta}{2}}+A^{\frac14+\frac{\beta}{2}}XB^{\frac34-\frac{\beta}{2}}\right|\right|\right|\le$$
$$\frac14\left|\left|\left|A^{\frac{1-\beta}{2}}\left(A^{\frac12}X+XB^{\frac12}\right)B^{\frac{\beta}{2}}+A^{\frac{\beta}{2}}\left(A^{\frac12}X+XB^{\frac12}\right)B^{\frac{1-\beta}{2}}\right|\right|\right|\le$$

$$\tfrac{1}{4}\big\|\big| AX + 2\sqrt{A}X\sqrt{B} + XB \big\|\big| \leq$$
$$\big\|\big| (1-\alpha)\sqrt{A}X\sqrt{B} + \tfrac{\alpha}{2}(AX+XB) \big\|\big| \leq \big\|\big| (1-\alpha')\sqrt{A}X\sqrt{B} + \tfrac{\alpha'}{2}(AX+XB) \big\|\big|.$$

For some additional insight in this topics see also [4, 11], [27, rem. 2.3,rem. 2.4], [29, 34] and references therein.

10 Connections with Cauchy–Schwarz Inequalities for Hilbert Modules

I.p.t. transformers on $\mathbb{B}(\mathcal{H})$ belong to a more general class of transformers on C*-algebras in the framework of Hilbert C*-modules. Namely, the space $L^2_G(\Omega,\mu,\mathbb{B}(\mathcal{H}))$ of weak*-measurable, square integrable functions represents a Hilbert C*-module over C*-algebra $\mathbb{B}(\mathcal{H})$, with the inner product defined by

$$\langle A, B\rangle \stackrel{\text{def}}{=} {}^{w^*}\!\!\int_\Omega A_t^* B_t \, d\mu(t) \qquad \text{for all } A, B \in L^2_G(\Omega,\mu,\mathbb{B}(\mathcal{H})), \tag{62}$$

where ${}^{w^*}\!\!\int_\Omega A_t^* B_t \, d\mu(t)$ denotes w* or Gel'fand integral of $\mathbb{B}(\mathcal{H})$ valued w* (or Gel'fand) integrable function $A^*B\colon \Omega \to \mathbb{B}(\mathcal{H})\colon t \mapsto A_t^* B_t$. A wider class of semi-inner product modules is treated in [15, th. 2.1] and [20, lemma 1.3]. Thus, ${}^{w^*}\!\!\int_\Omega A_t^*\otimes B_t \, d\mu(t)\colon \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{H})\colon X \mapsto {}^{w^*}\!\!\int_\Omega A_t^* X B_t \, d\mu(t) = \langle A, X{\cdot}B\rangle$, where $X{\cdot}B\colon \Omega \to \mathbb{B}(\mathcal{H})\colon t \mapsto XB_t$, explaining the origin of the name for inner product type transformers considered in this and previous papers.

If $(\mathcal{M}, \langle\cdot,\cdot\rangle)$ is a semi-inner product module over a C*-algebra $\mathfrak{A}$, then Cauchy–Schwarz inequality asserts that

$$\big|\langle A, B\rangle\big|^2 = \langle B, A\rangle\langle A, B\rangle \leq \|\langle A, A\rangle\| \langle B, B\rangle \qquad \text{for all } A, B \in \mathcal{M}. \tag{63}$$

By applying (63) to $X{\cdot}B$ instead of B it follows

$$\big|\langle A, X{\cdot}B\rangle\big|^2 = \langle X{\cdot}B, A\rangle\langle A, X{\cdot}B\rangle \leq \|\langle A, A\rangle\| \langle X{\cdot}B, X{\cdot}B\rangle \text{for all } A, B \in \mathcal{M}, X \in \mathfrak{A},$$

which for $\mathcal{M} := L^2_G(\Omega,\mu,\mathbb{B}(\mathcal{H}))$ becomes the Cauchy–Schwarz inequality (2).

In [1] semi-inner product $\mathfrak{A}$-modules are considered with respect to alternated semi-inner products. For an arbitrary $C \in \mathcal{M}$ another semi-inner product on $\mathcal{M}$ defines correctly by $\langle\cdot,\cdot\rangle_C\colon \mathcal{M}\times\mathcal{M} \to \mathfrak{A}\colon (A, B) \mapsto \|\langle C, C\rangle\|\langle A, B\rangle - \langle A, C\rangle\langle C, B\rangle$. By applying (63) to $X{\cdot}B$ instead of B and to $\langle\cdot,\cdot\rangle_C$ instead of $\langle\cdot,\cdot\rangle$

gives

$$\left|\|\langle C,C\rangle\|\langle A,X\cdot B\rangle-\langle A,C\rangle\langle C,X\cdot B\rangle\right|^2 \le \left\|\|\langle C,C\rangle\|\langle A,A\rangle-\left|\langle C,A\rangle\right|^2\right\|\left(\|\langle C,C\rangle\|\langle X\cdot B,X\cdot B\rangle-\left|\langle C,X\cdot B\rangle\right|^2\right) \tag{64}$$

for all $A, B \in \mathcal{M}$ and $X \in \mathfrak{A}$. A special case $\mathcal{M} := L^2_G(\Omega, \mu, \mathbb{B}(\mathcal{H}))$ for a probability measure μ on Ω, $A^*, B \in L^2_G(\Omega, \mu, \mathbb{B}(\mathcal{H}))$ and $C_t := I$ for all $t \in \Omega$ provides the Grüss-Landau operator inequality (26) for $\eta := 1$, which then implies the validity of Theorem 5.5 for all $\eta \in [0, 1]$. If in addition $\int_\Omega A_t\,d\mu(t) = 0$ then we have the strengthened Cauchy–Schwarz inequality

$$\left|\int_\Omega A_t X B_t\,d\mu(t)\right|^2 \le \left\|\int_\Omega A_t A_t^*\,d\mu(t)\right\|\left(\int_\Omega B_t^* X^* X B_t\,d\mu(t)-\left|X\int_\Omega B_t\,d\mu(t)\right|^2\right),$$

which improves the inequality (2) in Theorem 2.1 if $\int_\Omega A_t\,d\mu(t) = 0$. Another special case $X := I$ coincides with the inequality [1, (2.7)]. More generally, in the special case when $\langle C, A\rangle = 0$ the inequality (64) provides i.t.p. transformer's Ostrowski inequality on a semi-inner product over C*-algebra

$$\left|\langle A,X\cdot B\rangle\right|^2 \le \tfrac{\|\langle A,A\rangle\|}{\|\langle C,C\rangle\|}\left(\|\langle C,C\rangle\|\langle X\cdot B,X\cdot B\rangle-\left|\langle C,X\cdot B\rangle\right|^2\right), \tag{65}$$

which also generalizes the inequality (63). Again, the special case $X := I$ coincides with the inequality [1, (2.8)]. Alternatively, if $\langle C, B\rangle = 0$ and $X := I$, then (64) implies another Ostrowski inequality on C*-modules

$$\left|\langle A,B\rangle\right|^2 \le \left\|\langle A,A\rangle-\tfrac{|\langle C,A\rangle|^2}{\|\langle C,C\rangle\|}\right\|\langle B,B\rangle, \tag{66}$$

which also improves the Cauchy–Schwarz inequality (63) if $\langle C, B\rangle = 0$.

Alternated semi-inner products also enables in [1] to improve an inequality related to the Gram matrix and to obtain a sequence of nested inequalities that emerges from Cauchy–Schwarz inequality.

The framework of Hilbert C*-modules and semi-inner product C*-modules over unital C*-algebras proves to be very useful for generalizing many other classical inequalities. In [7] polar decomposition and the operator geometric mean are used to present the Cauchy–Schwarz inequality and to give several additive and multiplicative type reverses of it in this setting. Further, this allows to authors to obtain various operator inequalities on a Hilbert C*-modules, including Kantorovich inequality, Pólya–Szegö inequality, the covariance-variance inequality, Ozeki–Izumino–Mori–Seo inequality, Wielandt inequality, Heinz–Kato–Furuta inequality and Malamud inequality.

Another alternated semi-inner product $\langle\cdot,\cdot\rangle_{\mathcal{T}}\colon \mathcal{M}\times\mathcal{M} \to \mathfrak{A}\colon (A, B) \mapsto \langle A, \mathcal{T}B\rangle$ for an arbitrary positive $\mathcal{T} \in \mathbb{B}(\mathcal{M})$ was considered in [6], where a polar decomposition $\langle A, \mathcal{T}B\rangle = V\left|\langle A, \mathcal{T}B\rangle\right|$ with a partial isometry $V \in \mathfrak{A}$ is used to prove generalized Cauchy–Schwarz inequality on semi-inner product over C*-algebra

$$\big|\langle A, \mathcal{T}B\rangle\big| \le V^*\langle A, \mathcal{T}A\rangle V\#\langle B, \mathcal{T}B\rangle. \tag{67}$$

Here, $C\#D \stackrel{\text{def}}{=} C^{1/2}(C^{-1/2}DC^{-1/2})^{1/2}C^{1/2}$ denotes the operator geometric mean for positive $C, D \in \mathfrak{A}$, if C is invertible. For the definition of $C\#D$ if C is not invertible see [9, ex. 3.3.1(3)]. The special case in which $\mathcal{T}$ is the identity operator $I_{\mathcal{M}}$ on $\mathcal{M}$ gives the Cauchy–Schwarz inequality [6, (3.1)] and Kantorovich inequality on Hilbert C*-modules is also given by [6, th. 4.4]. An application of (67) to $\mathcal{T} := I_{\mathcal{M}}$ and to the semi–inner product $\langle\cdot,\cdot\rangle_C$ gives

$$\begin{aligned}
&\big|\|\langle C, C\rangle\|\langle A, B\rangle - \langle A, C\rangle\langle C, B\rangle\big| \\
&\le \big(V^*\big(\|\langle C, C\rangle\|\langle A, A\rangle - \big|\langle C, A\rangle\big|^2\big)V\big)\#\big(\|\langle C, C\rangle\|\langle B, B\rangle - \big|\langle C, B\rangle\big|^2\big) \\
&\le \big\|\,\|\langle C, C\rangle\|\langle A, A\rangle - \big|\langle C, A\rangle\big|^2\big\|^{1/2}\big(\|\langle C, C\rangle\|\langle B, B\rangle - \big|\langle C, B\rangle\big|^2\big)^{1/2},
\end{aligned} \tag{68}$$

which applied to $X{\cdot}B$ instead of B for $X \in \mathfrak{A}$ leads to a refined version for the inequality (64).

Like all C*-algebras, $\mathbb{B}(\mathcal{H})$ itself represents a Hilbert C*-module over $\mathbb{B}(\mathcal{H})$ with inner products $\langle A, B\rangle \stackrel{\text{def}}{=} A^*B$, for all $A, B \in \mathbb{B}(\mathcal{H})$, so the special case $\mathcal{T} := I$ in (67) says $|A^*B| \le (V^*A^*AV)\#B^*B$, which generalizes the matrix inequality (4.2) in [33, lemma 4.2].

We conclude this paper with the following

Remark 10.1 The inequality $|||A^*XB|||^2 \le |||AA^*X|||\,|||XBB^*|||$ for all $A, B, X \in \mathbb{B}(\mathcal{H})$ and u.i. norms $|||\cdot|||$ in [3, (7)] is also known as the Cauchy–Schwarz inequality for u.i. norms, which for $X := I$ implies $|||A^*B|||^2 \le |||AA^*|||\,|||BB^*||| = |||A^*A|||\,|||B^*B|||$. In the framework of matrices this inequality was generalized in [2], saying that $|||A^*B|||^2 \le |||\eta AA^* + (1-\eta)BB^*|||\,|||(1-\eta)AA^* + \eta BB^*|||$ for all $\eta \in [0, 1]$.

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Norm Estimations for the Moore-Penrose Inverse of the Weak Perturbation of Hilbert C^*-Module Operators

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Abstract Let $\mathscr{A}$ be a C^*-algebra, H and K be Hilbert $\mathscr{A}$ modules and $\mathcal{L}(H, K)$ be the set of all adjointable operators from H to K. A multiplicative perturbation M of a Moore-Penrose invertible operator $T \in \mathcal{L}(H, K)$ has the form $M = ETF^*$ with $E \in \mathcal{L}(K)$ and $F \in \mathcal{L}(H)$, which can be expressed alternately as $M = ETT^\dagger \cdot T \cdot (FT^\dagger T)^* = L_{Z,T} \cdot T^\dagger \cdot R^*_{F,T}$, where $T^\dagger$ is the Moore-Penrose inverse of T and

$$L_{Z,T} = ETT^\dagger + I_K - TT^\dagger,\ R_{F,T} = FT^\dagger T + I_H - T^\dagger T.$$

In view of the above $ETT^\dagger$, $FT^\dagger T$, $L_{E,T}$ and $R_{F,T}$, the relationship between various types of multiplicative perturbations are investigated, and formulas for $M^\dagger$, $MM^\dagger$ and $M^\dagger M$ are derived in the case that M is a weak perturbation of T. Based on these derived formulas, some norm computations are carried out by using certain C^*-algebraic techniques, through which some norm estimations for the Moore-Penrose inverse are obtained.

Keywords Hilbert C^*-module · Moore–Penrose inverse · Multiplicative perturbation · Weak perturbation · Strong perturbation

1 Introduction

Let $M_{m\times n}(\mathbb{C})$ be the set of all $m \times n$ complex matrices, I_n and 0_n be the identity matrix and zero matrix in $M_n(\mathbb{C})$, respectively. For every $A \in M_{m\times n}(\mathbb{C})$, let $A^\dagger$ and $\|A\|$ denote the Moore-Penrose inverse and the spectral norm of A, respectively.

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Given any $M, T \in M_{m\times n}(\mathbb{C})$, if $\mathrm{ran}(M) = \mathrm{ran}(T)$, then clearly there exist two matrices $E \in M_m(\mathbb{C})$ and $F \in M_n(\mathbb{C})$ such that

$$M = ETF^*, \text{ where both } E \text{ and } F \text{ are nonsingular.} \tag{1}$$

The matrix M of the form (1) is called a multiplicative perturbation of T.

One research field associated to the multiplicative perturbation (1) is the study of representations for $M^\dagger$, which can be applied to the derivation of formulas for the Moore-Penrose inverse of certain 2 by 2 block matrices [3]. Another research field associated to (1) is the study of norm estimations for $M^\dagger - T^\dagger$, which can be carried out by directly using the SVD of M and T [8, 20] or by using the Halmos' two projections theorem [5] (a theorem also known for Hilbert space operators [1]).

One generalization of (1) is the case that both E and F may fail to be square. For instance, the rank-revealing decomposition of a matrix was considered in [2], which can be applied to the study of the accurate solutions of structured least squares problems. Another generalization of (1) is the case that E and F are still square matrices, whereas both of them may be singular [4].

Let M be a multiplicative perturbation of $T \in M_{m\times n}(\mathbb{C})$ given by (12), where both $E \in M_m(\mathbb{C})$ and $F \in M_n(\mathbb{C})$ may be singular, so it may happen that $\mathrm{ran}(M) \neq \mathrm{ran}(T)$. Inspired by [3], an alternative expression of M was given in [21] as

$$M = L_{E,T} \cdot T \cdot R^*_{F,T}, \tag{2}$$

where $L_{E,T}$ and $R_{F,T}$ are given by (13) below, in which $H = \mathbb{C}^n$ and $K = \mathbb{C}^m$. Based on the expression (2), the terms of the strong perturbation and the weak perturbation were introduced in [21], through which representations for $M^\dagger$ as well as norm estimations for $M^\dagger - T^\dagger$ were carried out therein. Recently, another expression of M was introduced in [17] as

$$M = ETT^\dagger \cdot T \cdot (FT^\dagger T)^*.$$

As a result, the representation theory for $M^\dagger$ obtained in [21] has been improved in [17], which leads to some new norm estimation for $M^\dagger - T^\dagger$ in [4].

The Moore-Penrose inverse associated to the rank-preserving perturbation and the stable perturbation is considered originally for the additive perturbation [7, 11, 13, 18, 19], which can also be dealt with in the multiplicative perturbation case. Given a multiplicative perturbation (12) in the matrix case, as shown in

[4, Lemma 2.1 and Corollary 2.1], the relationship between various types of multiplicative perturbations for matrices can be figured out as follows:

$$\begin{array}{c} M \text{ is a stable perturbation of } T \\ \Uparrow \\ M \text{ is a weak perturbation of } T \\ \Uparrow \\ \|M-T\|\cdot\|T^\dagger\|<1 \Longrightarrow M \text{ is a strong perturbation of } T \\ \Updownarrow \\ M \text{ is a weak perturbation of } T \text{ and } \operatorname{ran}(M)=\operatorname{ran}(T). \end{array}$$

The diagram above, together with [4, Examples 2.1 and 2.2], indicates that the weak perturbation is actually parallel to the rank-preserving perturbation. It is notable that much progress has been made on norm estimations for $M^\dagger - T^\dagger$ in the case of rank-preserving perturbation, yet little has been done up to now in the case of the weak perturbation. The purpose of this paper is, in the general setting of Hilbert C^*-module operators, to make some generalizations of the main results originally obtained in [4, 17] for matrices.

Let $\mathscr{A}$ be a C^*-algebra, H and K be two Hilbert $\mathscr{A}$-modules, and M be a weak perturbation of $T \in \mathcal{L}(H,K)$ given by (12). Formulas for $M^\dagger$, $MM^\dagger$ and $M^\dagger M$ are derived in Theorem 4.5, hence a generalization of [17, Corollary 3.6] is obtained from the matrix case to the case of Hilbert C^*-module operators. Based on these formulas, some norm computations of the associated operators can be carried out in Lemmas 5.1 and 5.2 by using certain C^*-algebraic techniques employed in [4]. Consequently in Theorem 5.3, an elegant estimation is derived in the weak perturbation case as

$$\max\left\{\|MM^\dagger - TT^\dagger\|,\ \|M^\dagger M - T^\dagger T\|\right\} \le \|T^\dagger\|\cdot\|M-T\|. \tag{3}$$

Thus, a generalization of [4, Theorem 3.1] is obtained from the matrix case to the case of Hilbert C^*-module operators. Note that [4, Theorem 3.2] indicates an interesting phenomenon that the similar estimation

$$\max\left\{\|MM^\dagger - TT^\dagger\|,\ \|M^\dagger M - T^\dagger T\|\right\} \le \|M^\dagger\|\cdot\|M-T\| \tag{4}$$

may be false for a general weak perturbation. It is gratifying that (4) is always true for every strong perturbation (see Theorem 5.4 for the details). So in the strong perturbation case, another elegant estimation for $\|M^\dagger - T^\dagger\|$ is derived in Theorem 5.6 by using both (3) and (4). This shows theoretically that some well-known results, associated to the stable additive perturbation satisfying a norm inequality (28) [11, 13, 18, 19], have been generalized in this paper to the strong perturbation case, since as is shown in Sect. 3.2, every stable additive perturbation is essentially a strong perturbation whenever this widely used norm inequality is satisfied. Furthermore, some new results concerning the strong perturbation of operators are also obtained; see Theorem 5.7 for the details.

The paper is organized as follows. In Sect. 2, we recall some basic knowledge about the Moore-Penrose inverse of Hilbert C^*-module operators. In Sect. 3, we study the relationship between various types of multiplicative perturbations. In Sects. 4 and 5, we focus on the study of representations and norm estimations for the Moore-Penrose inverse associated to the multiplicative perturbation, respectively.

2 Some Basic Knowledge About the Moore-Penrose Inverse

Let $\mathbb{C}$ be the complex field and $\mathscr{A}$ be a C^*-algebra [10]. An inner-product $\mathscr{A}$-module [6] is a linear space E which is a right $\mathscr{A}$-module, together with a map $E \times E \to \mathscr{A}, (x, y) \to \langle x, y\rangle$ such that for every $x, y, z \in E, \alpha, \beta \in \mathbb{C}$ and $a \in \mathscr{A}$, the following conditions hold:

(i) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$;
(ii) $\langle x, ya\rangle = \langle x, y\rangle a$;
(iii) $\langle y, x\rangle = \langle x, y\rangle^*$;
(iv) $\langle x, x\rangle \geq 0$, and $\langle x, x\rangle = 0 \Longleftrightarrow x = 0$.

An inner-product $\mathscr{A}$-module E which is complete with respect to the induced norm $\|x\| = \sqrt{\|\langle x, x\rangle\|}$ $(x \in E)$ is called a (right) Hilbert $\mathscr{A}$-module.

Throughout the rest of this paper, H and K are Hilbert $\mathscr{A}$-modules. Let $\mathcal{L}(H, K)$ be the set of operators $T : H \to K$ for which there is an operator $T^* : K \to H$ such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \text{ for every } x \in H \text{ and } y \in K.$$

It is known that each element T of $\mathcal{L}(H, K)$ is a bounded linear operator. We call $\mathcal{L}(H, K)$ the set of adjointable operators from H to K. For every $A \in \mathcal{L}(H, K)$, its range and null space are denoted by $\mathcal{R}(A)$ and $\mathcal{N}(A)$, respectively. In case $H = K$, $\mathcal{L}(H, H)$ which we abbreviate to $\mathcal{L}(H)$, is a C^*-algebra. Let $\mathcal{L}(H)_{sa}$ and $\mathcal{L}(H)_+$ denote the sets of self-adjoint elements and positive elements in $\mathcal{L}(H)$, respectively. The identity operator on H is denoted by I_H. An element M of $\mathcal{L}(H)$ is said to be positive definite or strictly positive [9], if M is positive and invertible in $\mathcal{L}(H)$. By a projection $P \in \mathcal{L}(H)$, we always mean that $P = P^*$ and $P^2 = P$. In the special case that H is a Hilbert space, $\mathcal{L}(H)$ consists of all bounded linear operators on H, and in this case we use the notation $\mathbb{B}(H)$ instead of $\mathcal{L}(H)$.

The notations of "$\oplus$" and "$\dot{+}$" are used in this paper with different meanings for the sake of reader's convenience. Given Hilbert $\mathscr{A}$-modules H_1 and H_2, let

$$H_1 \oplus H_2 = \left\{(h_1, h_2)^T : h_i \in H_i, i = 1, 2\right\},$$

which is also a Hilbert $\mathscr{A}$-module whose $\mathscr{A}$-valued inner product is given by

$$\left\langle (x_1, y_1)^T, (x_2, y_2)^T \right\rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle, \ \forall x_i \in H_1, y_i \in H_2, i = 1, 2.$$

On the other hand, if both H_1 and H_2 are closed submodules of a Hilbert $\mathscr{A}$-module H such that $H_1 \cap H_2 = \{0\}$, then we write

$$H_1 \dot{+} H_2 = \{h_1 + h_2 : h_i \in H_i, i = 1, 2\}.$$

Definition 2.1 ([12, 15]) Let $A \in \mathcal{L}(H, K)$. The Moore-Penrose inverse of A, written $A^\dagger$, is the unique element $X \in \mathcal{L}(K, H)$ which satisfies

$$AXA = A, XAX = X, (AX)^* = AX \text{ and } (XA)^* = XA. \tag{5}$$

Lemma 2.2 ([16, Theorem 1.3]) *For every $A \in \mathcal{L}(H, K)$, $A^\dagger$ exists if and only if $\mathcal{R}(A)$ is closed.*

Lemma 2.3 (cf. [6, Theorem 3.2] and [15, Remark 1.1]) *Let $A \in \mathcal{L}(H, K)$. Then the closedness of any one of the following sets implies the closedness of the remaining three sets:*

$$\mathcal{R}(A),\ \mathcal{R}(A^*),\ \mathcal{R}(AA^*) \text{ and } \mathcal{R}(A^*A).$$

If $\mathcal{R}(A)$ is closed, then $\mathcal{R}(A) = \mathcal{R}(AA^*)$, $\mathcal{R}(A^*) = \mathcal{R}(A^*A)$ and the following orthogonal decompositions hold:

$$H = \mathcal{N}(A) \dot{+} \mathcal{R}(A^*) \text{ and } K = \mathcal{R}(A) \dot{+} \mathcal{N}(A^*).$$

Remark 2.4 ([14, Section 1]) Let $A \in \mathcal{L}(H, K)$. If $A^\dagger$ exists, then A is said to be Moore-Penrose invertible (briefly, M-P invertible). In such case,

$$(A^\dagger)^* = (A^*)^\dagger, (A^*A)^\dagger = A^\dagger(A^*)^\dagger, \mathcal{R}(A^\dagger) = \mathcal{R}(A^*), \mathcal{N}(A^\dagger) = \mathcal{N}(A^*). \tag{6}$$

If $A \in \mathcal{L}(H)_{sa}$, then $AA^\dagger = A^\dagger A$. If furthermore $A \in \mathcal{L}(H)_+$, then $A^\dagger \in \mathcal{L}(H)_+$ such that $(A^\dagger)^{\frac{1}{2}} = (A^{\frac{1}{2}})^\dagger$, since

$$\left(A^{\frac{1}{2}}\right)^* = A^{\frac{1}{2}} \text{ and } A^\dagger = \left(A^{\frac{1}{2}} \cdot A^{\frac{1}{2}}\right)^\dagger = (A^{\frac{1}{2}})^\dagger \cdot (A^{\frac{1}{2}})^\dagger.$$

Lemma 2.5 (cf. [18, Lemma 4.1]) *For every $A \in \mathcal{L}(K,H)$, let*

$$\rho(A) = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} \in \mathcal{L}(H \oplus K)_{sa}. \tag{7}$$

Then $\rho(A)^\dagger$ exists if and only if $A^\dagger$ exists. In such case,

$$\rho(A)^\dagger = \begin{pmatrix} 0 & (A^\dagger)^* \\ A^\dagger & 0 \end{pmatrix}. \tag{8}$$

Proof Note that

$$\rho(A)\rho(A)^* = \begin{pmatrix} AA^* & 0 \\ 0 & A^*A \end{pmatrix}, \tag{9}$$

so $\mathcal{R}\big(\rho(A)\rho(A)^*\big)$ is closed if and only if both $\mathcal{R}(AA^*)$ and $\mathcal{R}(A^*A)$ are closed. Thus, from Lemmas 2.2 and 2.3 we conclude that $\rho(A)^\dagger$ exists if and only if $A^\dagger$ exists, and in which case the verification of (8) follows directly from four equations in (5). □

Lemma 2.6 (cf. [18, Lemma 4.1]) *For every $A \in \mathcal{L}(K,H)$, let $\rho(A)$ be defined by* (7)*. Then*

$$\|\rho(A)\| = \|A\|. \tag{10}$$

If furthermore A is M-P invertible, then

$$\|\rho(A)^\dagger\| = \|A^\dagger\| \text{ and } \|\rho(A)\rho(A)^\dagger\| = \max\big\{\|AA^\dagger\|, \|A^\dagger A\|\big\}. \tag{11}$$

Proof It follows from (9) that

$$\|\rho(A)\| = \sqrt{\|\rho(A)\rho(A)^*\|} = \max\{\sqrt{\|AA^*\|}, \sqrt{\|A^*A\|}\} = \|A\|.$$

If furthermore A is M-P invertible, then by (8), (7) and (10) we have

$$\|\rho(A)^\dagger\| = \|\rho\big((A^\dagger)^*\big)\| = \big\|(A^\dagger)^*\big\| = \|A^\dagger\|,$$
$$\|\rho(A)\rho(A)^\dagger\| = \max\big\{\|AA^\dagger\|, \|A^\dagger A\|\big\}.$$

□

3 Various Types of Perturbations

In this section, we study relationships between various perturbations. Throughout the rest of this paper, $T \in \mathcal{L}(H, K)$ is M-P invertible.

3.1 The Multiplicative Perturbation Case

Let M be a multiplicative perturbation of T given by

$$M = ETF^*, \text{ where } E \in \mathcal{L}(K) \text{ and } F \in \mathcal{L}(H). \tag{12}$$

Definition 3.1 ([21]) The operator M given by (12) is said to be semi-strong perturbation of T if both $L_{E,T}$ and $R_{F,T}$ are M-P invertible and injective, where

$$L_{E,T} = ETT^\dagger + I_K - TT^\dagger \text{ and } R_{F,T} = FT^\dagger T + I_H - T^\dagger T. \tag{13}$$

If furthermore $L_{E,T}$ and $R_{F,T}$ are both invertible, then M is said to be a strong perturbation of T.

Definition 3.2 ([4, 21]) The operator M given by (12) is said to be a weak perturbation of T if both $L_{E,T}$ and $R_{F,T}$ defined by (13) are M-P invertible, and the following three conditions are satisfied:

(i) $TT^\dagger L_{E,T}^\dagger (I_K - TT^\dagger) = 0$;
(ii) $T^\dagger T R_{F,T}^\dagger (I_H - T^\dagger T) = 0$;
(iii) $L_{E,T}^\dagger L_{E,T} T = T R_{F,T}^\dagger R_{F,T}$.

Definition 3.3 ([4, 18, 19]) The operator M given by (12) is said to be a stable perturbation of T if $\mathcal{R}(M) \cap \mathcal{R}(T)^\perp = \{0\}$.

Lemma 3.4 (cf. [4, Lemma 2.1]) *Let M be the multiplicative perturbation of $T \in \mathcal{L}(H, K)$ given by (12). Then the following statements are valid:*

(i) *If M is a semi-strong perturbation of T, then M is a weak perturbation of T;*
(ii) *If M is a weak perturbation of T, then M is a stable perturbation of T;*
(iii) *If $\|T^\dagger\| \cdot \|M - T\| < 1$, then both $L_{E,T}^*$ and $R_{F,T}^*$ are injective, where $L_{E,T}$ and $R_{F,T}$ are defined by (13).*

Proof

(i) Suppose that M is a semi-strong perturbation of T. Then

$$L_{E,T}^\dagger L_{E,T} = I_K \text{ and } R_{F,T}^\dagger R_{F,T} = I_H, \tag{14}$$

hence Definition 3.2 (iii) is satisfied. Since $L_{E,T}(I_K - TT^\dagger) = I_K - TT^\dagger$, by (14) we get $L_{E,T}^\dagger(I_K - TT^\dagger) = I_K - TT^\dagger$ and thus $TT^\dagger L_{E,T}^\dagger(I_K - TT^\dagger) = 0$. This shows the validity of Definition 3.2 (i). Similarly, Definition 3.2 (ii) is also satisfied. So M is a weak perturbation of T.

(ii) Suppose that M is a weak perturbation of T. By (13) M can be expressed alternately as (2). So given every $u \in \mathcal{R}(M) \cap \mathcal{R}(T)^\perp$, there exist $x \in H$ and $y \in K$ such that

$$u = L_{E,T} T R_{F,T}^* x = (I_K - TT^\dagger) y. \tag{15}$$

Then Definition 3.2 (iii) and (i) yield

$$\begin{aligned} TR_{F,T}^* x &= TT^\dagger \cdot TR_{F,T}^\dagger R_{F,T} \cdot R_{F,T}^* x = TT^\dagger \cdot L_{E,T}^\dagger L_{E,T} T \cdot R_{F,T}^* x \\ &= TT^\dagger L_{E,T}^\dagger u = TT^\dagger L_{E,T}^\dagger (I_K - TT^\dagger) y = 0, \end{aligned}$$

hence by (15) we conclude that $u = 0$. This completes the proof that $\mathcal{R}(M) \cap \mathcal{R}(T)^\perp = \{0\}$.

(iii) Suppose that $\|T^\dagger\| \cdot \|M - T\| < 1$. Let

$$\alpha = \|T^\dagger (L_{E,T} T R_{F,T}^* - T)\|.$$

Then by (2), we have

$$\alpha = \|T^\dagger (M - T)\| \le \|T^\dagger\| \cdot \|M - T\| < 1.$$

Given every $x \in \mathcal{N}(R_{F,T}^*)$, since $(I_H - T^\dagger T) R_{F,T}^* = I_H - T^\dagger T$, we have $x = T^\dagger T x$, hence

$$\|x\| = \left\| \left[T^\dagger (L_{E,T} T R_{F,T}^* - T) \right] x \right\| \le \alpha \|x\|,$$

which happens only if $x = 0$. This completes the proof that $\mathcal{N}(R_{F,T}^*) = \{0\}$. Similarly, due to $I_K - TT^\dagger = (I_K - TT^\dagger) L_{E,T}^*$ and

$$\|(T^\dagger)^* (R_{F,T} T^* L_{E,T}^* - T^*)\| = \|(T^\dagger)^* (M - T)^*\| \le \|T^\dagger\| \cdot \|M - T\| < 1,$$

we can also conclude that $\mathcal{N}(L_{E,T}^*) = \{0\}$. $\square$

An application of the preceding lemma is as follows.

Corollary 3.5 ([4, Corollary 2.1]) *Let M be the multiplicative perturbation of $T \in M_{m\times n}(\mathbb{C})$ given by (12), where $E \in M_m(\mathbb{C})$ and $F \in M_n(\mathbb{C})$. Then the following statements are valid:*

(i) *If $\|T^\dagger\| \cdot \|M - T\| < 1$, then M is a strong perturbation of T;*
(ii) *M is a strong perturbation of T if and only if* $\mathrm{ran}(M) = \mathrm{ran}(T)$ *and M is a weak perturbation of T.*

Proof

(i) The conclusion follows immediately from Lemma 3.4 (iii).
(ii) Suppose that M is a strong perturbation of T. Then by Lemma 3.4 (i) M is weak perturbation of T. Furthermore, since both $L_{E,T}$ and $R_{F,T}$ are nonsingular, we know from (2) that $\mathrm{ran}(M) = \mathrm{ran}(T)$.

Conversely, suppose that M is a weak perturbation of T such that $\mathrm{ran}(M) = \mathrm{ran}(T)$. Then clearly $\mathrm{ran}(TR_{F,T}^*) = \mathrm{ran}(T)$, which means by Definition 3.2 (iii) that

$$\mathcal{R}(T) = \mathcal{R}(TR_{F,T}^*) = \mathcal{R}(TR_{F,T}^\dagger R_{F,T}) = \mathcal{R}(L_{E,T}^\dagger L_{E,T}T),$$

hence $\mathcal{N}(T^\dagger) = \mathcal{N}(T^*) = \mathcal{N}(T^* L_{E,T}^\dagger L_{E,T})$. Now, given every $x \in \mathcal{N}(L_{E,T})$, we have $T^\dagger x = 0$, hence $x = ETT^\dagger x + (I_K - TT^\dagger)x = L_{E,T}x = 0$. Therefore, $L_{E,T}$ is nonsingular. The proof of the nonsingularity of $R_{F,T}$ is similar. □

Lemma 3.6 (cf. [17, Theorem 3.2]) *Let M be the multiplicative perturbation of $T \in \mathcal{L}(H, K)$ given by (12) such that both $L_{E,T}$ and $R_{F,T}$ defined by (13) are M-P invertible. Then M is a strong (weak) perturbation of T if and only if*

$$\rho(M) = H\rho(T)H^* \text{ with } H = \begin{pmatrix} E & 0 \\ 0 & F \end{pmatrix} \tag{16}$$

is a strong (weak) perturbation of $\rho(T)$, where $\rho(T)$ and $\rho(M)$ are defined by (7).

Proof Direct computation yields (16). Also, from (8) we can obtain

$$L_{H,\rho(T)} = H\rho(T)\rho(T)^\dagger + I_{K\oplus H} - \rho(T)\rho(T)^\dagger = \begin{pmatrix} L_{E,T} & 0 \\ 0 & R_{F,T} \end{pmatrix}, \tag{17}$$

which clearly indicates that M is a strong perturbation of T if and only if $\rho(M) = H\rho(T)H^*$ is a strong perturbation of $\rho(T)$. Furthermore, by (17) we can obtain $L^\dagger_{H,\rho(T)} = \begin{pmatrix} L^\dagger_{E,T} & 0 \\ 0 & R^\dagger_{F,T} \end{pmatrix}$, hence from (7) and (8) we know that Definition 3.2 (i)–(iii) can be rephrased as

$$\rho(T)\rho(T)^\dagger L^\dagger_{H,\rho(T)} \left(I_{K\oplus H} - \rho(T)\rho(T)^\dagger\right) = 0,$$

$$L^\dagger_{H,\rho(T)} L_{H,\rho(T)}\rho(T) = \rho(T)L^\dagger_{H,\rho(T)} L_{H,\rho(T)},$$

which means that M is a weak perturbation of T if and only if $\rho(M) = H\rho(T)H^*$ is a weak perturbation of $\rho(T)$. □

Before ending this section, we provide an interpretation of the weak (strong) perturbation by using operator block matrices. To this end, a lemma is stated as follows, whose proof is the same as that of the matrix case initiated in [21, Lemma 3.3].

Lemma 3.7 (cf. [21, Lemma 3.3]) *Let* $A = \begin{pmatrix} B & 0 \\ C & I_K \end{pmatrix} \in \mathcal{L}(H \oplus K)$*, where* $B \in \mathcal{L}(H)$*. Then the following statements are equivalent:*

(i) *B is M-P invertible such that $\mathcal{R}(C^*) \subseteq \mathcal{R}(B^*)$;*
(ii) *B is M-P invertible such that $C = CB^\dagger B$;*
(iii) *A is M-P invertible such that $A^\dagger$ has the form $\begin{pmatrix} Z_{11} & 0 \\ Z_{21} & Z_{22} \end{pmatrix}$, where $Z_{11} \in \mathcal{L}(H)$.*

In each case,

$$A^\dagger = \begin{pmatrix} B^\dagger & 0 \\ -CB^\dagger & I_K \end{pmatrix}. \tag{18}$$

Next, we recall some elementary results on operator block matrices. Let H_1 be a closed submodule of H. Then for every $A \in \mathcal{L}(H, K)$, we use the notation $A|_{H_1}$ to denote the restriction A on H_1.

Suppose that $P \in \mathcal{L}(H)$ and $Q \in \mathcal{L}(K)$ are two projections. Let

$$H_1 = PH, \quad H_2 = (I_H - P)H, \quad K_1 = QK, \quad K_2 = (I_K - Q)K,$$

and let $U_P : H \to H_1 \oplus H_2$ be the unitary operator defined by

$$U_P h = \big(Ph, (I_H - P)h\big)^T \text{ for every } h \in H. \tag{19}$$

Then $U_P^* \in \mathcal{L}(H_1 \oplus H_2, H)$ which is given by

$$U_P^*\big((h_1, h_2)^T\big) = h_1 + h_2 \text{ for every } h_i \in H_i, i = 1, 2.$$

The unitary operator $U_Q \in \mathcal{L}(K, K_1 \oplus K_2)$ can be defined similarly.

With the notations as above, it is easy to verify that for every $S \in \mathcal{L}(H, K)$, the operator matrix $(S_{ij})_{1\le i,j\le 2}$ corresponding to the operator $U_Q S U_P^*$ is formulated by

$$\begin{aligned} S_{11} &= QSP|_{H_1}, & S_{12} &= QS(I_H - P)|_{H_2}, \\ S_{21} &= (I_K - Q)SP|_{H_1}, & S_{22} &= (I_K - Q)S(I_H - P)|_{H_2}. \end{aligned} \tag{20}$$

Conversely, given every $X = (X_{ij})_{1\le i,j\le 2} \in \mathcal{L}(H_1 \oplus H_2, K_1 \oplus K_2)$ and every $h \in H$, we have

$$U_Q^* X U_P h = X_{11}Ph + X_{12}(I_H - P)h + X_{21}Ph + X_{22}(I_H - P)h.$$

It follows that

$$U_Q^* X U_P = (X_{11} + X_{21})P + (X_{12} + X_{22})(I_H - P). \tag{21}$$

An interpretation of the weak (strong) perturbation for matrices was given in [4, Remark 2.1], which can also be carried out for operators.

Remark 3.8 Let M be the multiplicative perturbation of $T \in \mathcal{L}(H, K)$ given by (12) such that both $L_{E,T}$ and $R_{F,T}$ defined by (13) are M-P invertible. Let $P_T \in \mathcal{L}(H)$ and $Q_T \in \mathcal{L}(K)$ be two projections defined by

$$P_T = T^\dagger T \text{ and } Q_T = TT^\dagger, \tag{22}$$

and put $H_1 = \mathcal{R}(P_T) = \mathcal{R}(T^*) \subseteq H$ and $K_1 = \mathcal{R}(Q_T) = \mathcal{R}(T) \subseteq K$. By (20) we have

$$U_{Q_T} T U_{P_T}^* = \begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix} \text{ and } U_{P_T} T^\dagger U_{Q_T}^* = \begin{pmatrix} T_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \tag{23}$$

where $T_{11} = T|_{H_1,K_1}$ is invertible such that $T_{11}^{-1} = T^\dagger|_{K_1,H_1}$. Then

$$U_{Q_T} T T^\dagger U_{Q_T}^* = \begin{pmatrix} I_{K_1} & 0 \\ 0 & 0 \end{pmatrix} \text{ and } U_{P_T} T^\dagger T U_{P_T}^* = \begin{pmatrix} I_{H_1} & 0 \\ 0 & 0 \end{pmatrix}, \tag{24}$$

which lead to

$$U_{Q_T} E T T^\dagger U_{Q_T}^* = \begin{pmatrix} B & 0 \\ C & 0 \end{pmatrix} \text{ and } U_{P_T} F T^\dagger T U_{P_T}^* = \begin{pmatrix} D & 0 \\ G & 0 \end{pmatrix} \tag{25}$$

for some $B \in \mathcal{L}(K_1)$, $C \in \mathcal{L}(K_1, K_1^\perp)$, $D \in \mathcal{L}(H_1)$ and $G \in \mathcal{L}(H_1, H_1^\perp)$. It follows that

$$U_{Q_T} L_{E,T} U_{Q_T}^* = \begin{pmatrix} B & 0 \\ C & I_{K_1^\perp} \end{pmatrix} \text{ and } U_{P_T} R_{F,T} U_{P_T}^* = \begin{pmatrix} D & 0 \\ G & I_{H_1^\perp} \end{pmatrix}. \tag{26}$$

Note that

$$\left(U_{Q_T} L_{E,T} U_{Q_T}^*\right)^\dagger = U_{Q_T} L_{E,T}^\dagger U_{Q_T}^* \text{ and } \left(U_{P_T} R_{F,T} U_{P_T}^*\right)^\dagger = U_{P_T} R_{F,T}^\dagger U_{P_T}^*,$$

so (26) and Lemma 3.7 indicate that

Definition 3.2 (i) is satisfied $\Longleftrightarrow$ B is M-P invertible such that $C = CB^\dagger B$,

Definition 3.2 (ii) is satisfied $\Longleftrightarrow$ D is M-P invertible such that $G = GD^\dagger D$.

In such case, by (18) and (26) we obtain

$$\begin{aligned} L_{E,T}^\dagger L_{E,T} T &= U_{Q_T}^* \begin{pmatrix} B^\dagger & 0 \\ -CB^\dagger & I_{K_1^\perp} \end{pmatrix} \begin{pmatrix} B & 0 \\ C & I_{K_1^\perp} \end{pmatrix} \begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix} U_{P_T} \\ &= U_{Q_T}^* \begin{pmatrix} B^\dagger B T_{11} & 0 \\ 0 & 0 \end{pmatrix} U_{P_T}. \end{aligned}$$

Similarly, we have

$$T R_{F,T}^\dagger R_{F,T} = U_{Q_T}^* \begin{pmatrix} T_{11} D^\dagger D & 0 \\ 0 & 0 \end{pmatrix} U_{P_T}.$$

Therefore, Definition 3.2 (iii) is furthermore satisfied if and only if

$$B^\dagger B T_{11} = T_{11} D^\dagger D. \tag{27}$$

Moreover, it easily follows from (26) that M is a strong perturbation of T if and only if both B and D are invertible.

It is remarkable that in the matrix case, a weak perturbation may fail to be rank-preserving, and vice visa; see [4, Examles 2.1 and 2.2] for such two examples. An example of the strong perturbation of an operator is as follows.

Example Let $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22} \end{pmatrix} \in \mathcal{L}(H \oplus K)_+$ be such that both $M_{11} \in \mathcal{L}(H)$ and $S \in \mathcal{L}(K)$ are M-P invertible, where S is the generalized Schur complement of M_{11} in M defined by

$$S = M_{22} - M_{12}^* M_{11}^\dagger M_{12}.$$

Put

$$E = \begin{pmatrix} I_H & 0 \\ M_{12}^* M_{11}^\dagger & I_K \end{pmatrix} \text{ and } T = \begin{pmatrix} M_{11} & 0 \\ 0 & S \end{pmatrix}.$$

By [15, Corollary 3.5] we know that

$$M_{11} \in \mathcal{L}(H)_+, \quad M_{12} = M_{11} M_{11}^\dagger M_{12}, \quad S \in \mathcal{L}(K)_+,$$

hence $M = ETE^*$. Direct computation yields $L_{E,T} = E$, which is invertible. Therefore, M is a strong perturbation of T.

3.2 The Stable Additive Perturbation and the Strong Perturbation

Let $M = T + \Delta$ be a stable additive perturbation of $T \in \mathcal{L}(H, K)$ such that

$$\|T^\dagger\| \cdot \|\Delta\| < 1. \tag{28}$$

With the conditions as above, norm estimations for $M^\dagger$ were considered in [19] and [18] for Hilbert space operators and Hilbert C^*-module operators, respectively. In this subsection, we will show that M can be in fact expressed as a strong perturbation of T.

Let P_T and Q_T be defined by (22) such that $U_{Q_T} T U_{P_T}^*$ and $U_{P_T} T^\dagger U_{Q_T}^*$ are given by (23), where $H_1 = \mathcal{R}(T^*)$, $K_1 = \mathcal{R}(T)$ and $T_{11} \in \mathcal{L}(H_1, K_1)$ is invertible. Put

$$U_{Q_T} M U_{P_T}^* = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \text{ and } \Delta_{11} = M_{11} - T_{11}.$$

Then

$$\|T_{11}^{-1}\| \cdot \|\Delta_{11}\| = \|T^\dagger\| \cdot \|\Delta_{11}\| \le \|T^\dagger\| \cdot \|U_{Q_T} \cdot \Delta \cdot U_{P_T}^*\| < 1,$$

therefore M_{11} is invertible such that $M_{11}^{-1} = \left(I_{H_1} + T_{11}^{-1}\Delta_{11}\right)^{-1} T_{11}^{-1}$, hence

$$U_{Q_T} M U_{P_T}^* = W_1 \begin{pmatrix} M_{11} & 0 \\ 0 & S \end{pmatrix} W_2, \tag{29}$$

where $S = M_{22} - M_{21}M_{11}^{-1}M_{12} \in \mathcal{L}(H_1^{\perp}, K_1^{\perp})$ and

$$W_1 = \begin{pmatrix} I_{K_1} & 0 \\ M_{21}M_{11}^{-1} & I_{K_1^{\perp}} \end{pmatrix} \text{ and } W_2 = \begin{pmatrix} I_{H_1} & M_{11}^{-1}M_{12} \\ 0 & I_{H_1^{\perp}} \end{pmatrix}. \tag{30}$$

By assumption M is a stable perturbation of T, so $\mathcal{R}(M) \cap \mathcal{R}(I_K - TT^{\dagger}) = \{0\}$, which clearly gives

$$\mathcal{R}(U_{Q_T} M U_{P_T}^* W_2^{-1}) \cap \mathcal{R}\big(U_{Q_T}(I_K - TT^{\dagger})U_{Q_T}^*\big) = \{0\}.$$

It follows from (24), (29) and (30) that

$$W_1 \begin{pmatrix} M_{11} & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} 0 \\ x \end{pmatrix} \neq \begin{pmatrix} 0 \\ y \end{pmatrix} \text{ for every } x \in H_1^{\perp} \text{ and } y \in K_1^{\perp} \setminus \{0\},$$

which happens only if $S = 0$. Accordingly, by (29) and (30) we have

$$\begin{aligned} U_{Q_T} M U_{P_T}^* &= W_1 \begin{pmatrix} M_{11} & 0 \\ 0 & 0 \end{pmatrix} W_2 \\ &= \begin{pmatrix} I_{K_1} & 0 \\ M_{21}M_{11}^{-1} & 0 \end{pmatrix} \begin{pmatrix} T_{11} + \Delta_{11} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_{H_1} & M_{11}^{-1}M_{12} \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} B & 0 \\ C & 0 \end{pmatrix} \begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} D & 0 \\ G & 0 \end{pmatrix}^*, \end{aligned}$$

where $B = I_{K_1}$ and $D = (I_{H_1} + T_{11}^{-1}\Delta_{11})^*$. It follows from (21) that $M = ETF^*$, where

$$\begin{aligned} E &= (I_{K_1} + M_{21}M_{11}^{-1})Q_T, \\ F &= \big[I_{H_1} + T_{11}^{-1}\Delta_{11} + (I_{H_1} + T_{11}^{-1}\Delta_{11})M_{11}^{-1}M_{12}\big]^* P_T. \end{aligned}$$

Furthermore, by Remark 3.8 we conclude that M is a strong perturbation of T, since both B and D are invertible.

4 Representations for the Moore-Penrose Inverse

In this section, we study representations for the Moore-Penrose inverse associated to the weak perturbation of operators.

Lemma 4.1 ([10, Proposition 1.3.5]) *Let x and y be two positive elements in a C^*-algebra. If $x \leq y$, then $\|x\| \leq \|y\|$.*

Definition 4.2 Given every $B \in \mathcal{L}(H)$ and $C \in \mathcal{L}(H, K)$, let $\Gamma \in \mathcal{L}(H)$ be defined by

$$\Gamma = \Gamma(B, C) = B^*B + C^*C. \tag{31}$$

Lemma 4.3 (cf. [17, Lemma 2.4]) *Given every $B \in \mathcal{L}(H)$ and $C \in \mathcal{L}(H, K)$, let $\Gamma = \Gamma(B, C)$ be defined by (31) and let Ω be defined by*

$$\Omega = \begin{pmatrix} B & 0 \\ C & 0 \end{pmatrix} \in \mathcal{L}(H \oplus K).$$

Then Ω is M-P invertible if and only if Γ is M-P invertible. In such case,

$$\Omega^\dagger = \begin{pmatrix} \Gamma^\dagger B^* & \Gamma^\dagger C^* \\ 0 & 0 \end{pmatrix}. \tag{32}$$

Proof Note that $\Omega^*\Omega = \begin{pmatrix} \Gamma & 0 \\ 0 & 0 \end{pmatrix}$, so by Lemmas 2.2 and 2.3 we know that

$$\Omega^\dagger \text{ exists} \Longleftrightarrow \mathcal{R}(\Omega^*\Omega) \text{ is closed} \Longleftrightarrow \mathcal{R}(\Gamma) \text{ is closed} \Longleftrightarrow \Gamma^\dagger \text{ exists.}$$

Now, suppose that $\Gamma^\dagger$ exists. Since Γ is self-adjoint, we have

$$(\Gamma^\dagger)^* = \Gamma^\dagger \text{ and } \Gamma^\dagger\,\Gamma = \Gamma\,\Gamma^\dagger.$$

Let $P = \Gamma\Gamma^\dagger$ and $H_1 = PH = \mathcal{R}(\Gamma)$. Then for every $x \in H$,

$$B^*x = u + v, \tag{33}$$

where $u = P(B^*x) \in H_1$ and $v = (I - P)(B^*x) \in H_1^\perp$. Note that

$$0 = \langle \Gamma v, v \rangle = \langle Bv, Bv \rangle + \langle Cv, Cv \rangle ,$$

which implies by Lemma 4.1 that $\langle Bv, Bv \rangle = 0$, hence $Bv = 0$. Therefore, by (33) we have

$$\langle v, v \rangle = \langle v, B^*x - u \rangle = \langle Bv, x \rangle - \langle v, u \rangle = 0,$$

hence $v = 0$ and thus $B^*x = u \in H_1$. Since $x \in H$ is arbitrary, we have $\mathcal{R}(B^*) \subseteq \mathcal{R}(\Gamma)$. Similarly, we have $\mathcal{R}(C^*) \subseteq \mathcal{R}(\Gamma)$. It follows that

$$\Gamma\Gamma^\dagger B^* = B^* \text{ and } \Gamma\Gamma^\dagger C^* = C^*. \tag{34}$$

Taking $*$-operation yields

$$B\Gamma^\dagger\Gamma = B \text{ and } C\Gamma^\dagger\Gamma = C. \tag{35}$$

In view of (34) and (35), the four Penrose equations for Ω and $\Omega^\dagger$ stated in (5) are satisfied. □

Lemma 4.4 *Let $B \in \mathcal{L}(H)$ and $C \in \mathcal{L}(H, K)$ be such that B is M-P invertible and $C = CB^\dagger B$. Then the operator $\Gamma = \Gamma(B, C)$ defined by (31) is also M-P invertible such that*

$$\Gamma^\dagger\Gamma = \Gamma\Gamma^\dagger = B^\dagger B.$$

Proof Let P_B and Q_B be two projections in $\mathcal{L}(H)$ defined by

$$P_B = BB^\dagger \text{ and } Q_B = B^\dagger B,$$

and put $H_1 = \mathcal{R}(P_B) = \mathcal{R}(B)$ and $H_2 = \mathcal{R}(Q_B) = \mathcal{R}(B^*)$. Then by (20) we have

$$U_{P_B} B U_{Q_B}^* = \begin{pmatrix} B_{11} & 0 \\ 0 & 0 \end{pmatrix} \text{ and } U_{Q_B} B^\dagger U_{P_B}^* = \begin{pmatrix} B_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \tag{36}$$

where $B_{11} = B|_{H_2, H_1}$ is invertible such that $B_{11}^{-1} = B^\dagger|_{H_1, H_2}$. It follows that

$$U_{Q_B} B^\dagger B U_{Q_B}^* = U_{Q_B} B^\dagger U_{P_B}^* \cdot U_{P_B} B U_{Q_B}^* = \begin{pmatrix} I_{H_2} & 0 \\ 0 & 0 \end{pmatrix}. \tag{37}$$

Since $C = CB^\dagger B$, we have $C^*C = B^\dagger BC^*CB^\dagger B$, and thus

$$C^*C = U_{Q_B}^* \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix} U_{Q_B}, \text{ where } W \in \mathcal{L}(H_2) \text{ is positive.} \tag{38}$$

Furthermore, from (36) we know that $B_{11}^* B_{11}$ is invertible in $\mathcal{L}(H_2)$, which obviously leads to the invertibility of $B_{11}^* B_{11} + W$ in $\mathcal{L}(H_2)$. In view of (31), (36) and (38), we have

$$\Gamma = U_{Q_B}^* \begin{pmatrix} B_{11}^* B_{11} + W & 0 \\ 0 & 0 \end{pmatrix} U_{Q_B}, \tag{39}$$

and thus Γ is M-P invertible such that

$$\Gamma^\dagger = U_{Q_B}^* \begin{pmatrix} (B_{11}^* B_{11} + W)^{-1} & 0 \\ 0 & 0 \end{pmatrix} U_{Q_B}. \tag{40}$$

It follows from (39), (40) and (37) that

$$\Gamma^\dagger\Gamma = \Gamma\Gamma^\dagger = U_{Q_B}^* \begin{pmatrix} I_{H_2} & 0 \\ 0 & 0 \end{pmatrix} U_{Q_B} = B^\dagger B. \qquad \square$$

Theorem 4.5 (cf. [17, Corollary 3.6]) *Suppose that $T \in \mathcal{L}(H, K)$ is M-P invertible. Let M be a weak perturbation of T given by (12). Then*

$$ETT^\dagger,\ FT^\dagger T,\ TT^\dagger ETT^\dagger \text{ and } T^\dagger TFT^\dagger T$$

are all M-P invertible, and

$$M^\dagger = \left((FT^\dagger T)^\dagger\right)^* \cdot T^\dagger \cdot (ETT^\dagger)^\dagger, \tag{41}$$

$$MM^\dagger = ETT^\dagger \cdot (ETT^\dagger)^\dagger \text{ and } M^\dagger M = FT^\dagger T \cdot (FT^\dagger T)^\dagger. \tag{42}$$

Proof Let $\Sigma_1 = TT^\dagger ETT^\dagger$ and $\Sigma_2 = T^\dagger TFT^\dagger T$. Following the notations in Remark 3.8, by (24) and (25) we have

$$\Sigma_1 = U_{Q_T}^* \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} U_{Q_T} \text{ and } \Sigma_2 = U_{P_T}^* \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} U_{P_T},$$

where both B and D are M-P invertible, hence

$$\Sigma_1^\dagger = U_{Q_T}^* \begin{pmatrix} B^\dagger & 0 \\ 0 & 0 \end{pmatrix} U_{Q_T} \text{ and } \Sigma_2^\dagger = U_{P_T}^* \begin{pmatrix} D^\dagger & 0 \\ 0 & 0 \end{pmatrix} U_{P_T}.$$

Furthermore, by Remark 3.8 we have

$$C = CB^\dagger B \text{ and } G = GD^\dagger D. \tag{43}$$

We may then combine (25), (43), Lemmas 4.3 and 4.4 to conclude that

$$(ETT^\dagger)^\dagger = U_{Q_T}^* \begin{pmatrix} \Gamma_E^\dagger B^* & \Gamma_E^\dagger C^* \\ 0 & 0 \end{pmatrix} U_{Q_T}, \tag{44}$$

$$(FT^\dagger T)^\dagger = U_{P_T}^* \begin{pmatrix} \Gamma_F^\dagger D^* & \Gamma_F^\dagger G^* \\ 0 & 0 \end{pmatrix} U_{P_T}, \tag{45}$$

where Γ_E and Γ_F are defined by

$$\Gamma_E = B^* B + C^* C \text{ and } \Gamma_F = D^* D + G^* G \tag{46}$$

such that

$$\Gamma_E^\dagger \Gamma_E = \Gamma_E \Gamma_E^\dagger = B^\dagger B \text{ and } \Gamma_F^\dagger \Gamma_F = \Gamma_F \Gamma_F^\dagger = D^\dagger D. \tag{47}$$

It follows from (25), (23), (41), (45) and (44) that

$$M = ETT^\dagger \cdot T \cdot (FT^\dagger T)^* \tag{48}$$

$$= U_{Q_T}^* \begin{pmatrix} B & 0 \\ C & 0 \end{pmatrix} \begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} D^* & G^* \\ 0 & 0 \end{pmatrix} U_{P_T}$$

$$= U_{Q_T}^* \begin{pmatrix} BT_{11}D^* & BT_{11}G^* \\ CT_{11}D^* & CT_{11}G^* \end{pmatrix} U_{P_T}, \tag{49}$$

$$M^\dagger = U_{P_T}^* \begin{pmatrix} D\Gamma_F^\dagger & 0 \\ G\Gamma_F^\dagger & 0 \end{pmatrix} \begin{pmatrix} T_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \Gamma_E^\dagger B^* & \Gamma_E^\dagger C^* \\ 0 & 0 \end{pmatrix} U_{Q_T}$$

$$= U_{P_T}^* \begin{pmatrix} D\Gamma_F^\dagger T_{11}^{-1} \Gamma_E^\dagger B^* & D\Gamma_F^\dagger T_{11}^{-1} \Gamma_E^\dagger C^* \\ G\Gamma_F^\dagger T_{11}^{-1} \Gamma_E^\dagger B^* & G\Gamma_F^\dagger T_{11}^{-1} \Gamma_E^\dagger C^* \end{pmatrix} U_{Q_T}. \tag{50}$$

Then we may combine (49), (50), (46), (47) and (27) to get an expression of $MM^\dagger$ as

$$MM^\dagger = U_{Q_T}^* \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix} U_{Q_T},$$

where

$$\Theta_{11} = BT_{11}(D^*D + G^*G)\Gamma_F^\dagger T_{11}^{-1} \Gamma_E^\dagger B^* = BT_{11}\Gamma_F \Gamma_F^\dagger T_{11}^{-1} \Gamma_E^\dagger B^*$$
$$= B \cdot T_{11} D^\dagger D \cdot T_{11}^{-1} \Gamma_E^\dagger B^* = B \cdot B^\dagger B T_{11} \cdot T_{11}^{-1} \Gamma_E^\dagger B^* = B\Gamma_E^\dagger B^*,$$
$$\Theta_{12} = B\Gamma_E^\dagger C^*, \Theta_{21} = CB^\dagger B \Gamma_E^\dagger B^* = C\Gamma_E^\dagger B^* \text{ and } \Theta_{22} = C\Gamma_F^\dagger C^*.$$

Therefore,

$$MM^\dagger = U_{Q_T}^* \begin{pmatrix} B\Gamma_E^\dagger B^* & B\Gamma_E^\dagger C^* \\ C\Gamma_E^\dagger B^* & C\Gamma_E^\dagger C^* \end{pmatrix} U_{Q_T} = ETT^\dagger \cdot (ETT^\dagger)^\dagger \tag{51}$$

by (25) and (32). As a result, $(MM^\dagger)^* = MM^\dagger$. Moreover, (51) together with (48) yields

$$MM^\dagger M = (ETT^\dagger)(ETT^\dagger)^\dagger (ETT^\dagger) T (FT^\dagger T)^* = (ETT^\dagger) T (FT^\dagger T)^* = M.$$

Similarly, we can prove that

$$M^\dagger M = U_{P_T}^* \begin{pmatrix} D\Gamma_F^\dagger D^* & D\Gamma_F^\dagger G^* \\ G\Gamma_F^\dagger D^* & G\Gamma_F^\dagger G^* \end{pmatrix} U_{P_T} = FT^\dagger T \cdot (FT^\dagger T)^\dagger,$$

hence $(M^\dagger M)^* = M^\dagger M$, and

$$\begin{aligned} M^\dagger M M^\dagger &= FT^\dagger T \cdot (FT^\dagger T)^\dagger \cdot \big((FT^\dagger T)^\dagger\big)^* \cdot T^\dagger \cdot (ETT^\dagger)^\dagger \\ &= \big((FT^\dagger T)^\dagger\big)^* \cdot T^\dagger \cdot (ETT^\dagger)^\dagger = M^\dagger. \end{aligned}$$

Therefore, the four equations in (5) are satisfied for M and $M^\dagger$. □

5 Norm Estimations for the Moore-Penrose Inverse

In this section, we study norm estimations for the Moore-Penrose inverse. First, we present a technical result, which was originally obtained in [4] for matrices.

Lemma 5.1 ([4, Lemma 3.3]) *Let $B \in \mathcal{L}(H)$ be M-P invertible and $C \in \mathcal{L}(H, K)$ be such that $C = CB^\dagger B$. Then*

$$\|C\Gamma^\dagger C^*\| = \frac{\|CB^\dagger\|^2}{1 + \|CB^\dagger\|^2} \text{ and } \|I_H - B\Gamma^\dagger B^*\| = \theta(B, C), \tag{52}$$

where $\Gamma = \Gamma(B, C)$ is defined by (31) and

$$\theta(B, C) = \begin{cases} 1, & \text{if } B \text{ is not surjective,} \\ \frac{\|CB^\dagger\|^2}{1+\|CB^\dagger\|^2}, & \text{if } B \text{ is surjective.} \end{cases} \tag{53}$$

Proof Following the notations as in the proof of Lemma 4.4, we know that Γ is M-P invertible such that $\Gamma^\dagger$ is given by (40), which can be expressed alternately as

$$\Gamma^\dagger = U_{Q_B}^* \begin{pmatrix} B_{11}^{-1}(I_{H_1} + Z)^{-1}(B_{11}^*)^{-1} & 0 \\ 0 & 0 \end{pmatrix} U_{Q_B}, \tag{54}$$

where $H_1 = \mathcal{R}(B)$, $H_2 = \mathcal{R}(B^*)$ and

$$S = (B_{11}^*)^{-1} W^{\frac{1}{2}} \in \mathcal{L}(H_2, H_1),\ Z = SS^* = (B_{11}^*)^{-1} W B_{11}^{-1} \in \mathcal{L}(H_1). \tag{55}$$

Meanwhile, by (36), (38) and (55) we obtain

$$(CB^\dagger)^*(CB^\dagger) = (B^\dagger)^*(C^*C)B^\dagger = U_{P_B}^* \begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix} U_{P_B},$$

which implies that

$$\|Z\| = \|(CB^\dagger)^*(CB^\dagger)\| = \|CB^\dagger\|^2. \tag{56}$$

Since Z is positive, we have $\|Z\| = \max\{t : t \in \sigma(Z)\}$, where $\sigma(Z)$ is the spectrum of Z in $\mathcal{L}(H_1)$. Note that the function from t to $\frac{t}{1+t}$ is monotonically increasing on $[0, +\infty)$, so by the spectral theory for normal elements in a C^*-algebra [10, Section 1], we have

$$\|Z(I_{H_1} + Z)^{-1}\| = \max\left\{\frac{t}{1+t} : t \in \sigma(Z)\right\} = \frac{\|Z\|}{1+\|Z\|}. \tag{57}$$

Now, we prove the first equation in (52). By (40), (38) and (54)–(56), we have

$$\begin{aligned}
\|C\Gamma^\dagger C^*\| &= \left\| C\big(\Gamma^\dagger\big)^{\frac{1}{2}} \cdot \left(C\big(\Gamma^\dagger\big)^{\frac{1}{2}}\right)^* \right\| = \left\| \left(C\big(\Gamma^\dagger\big)^{\frac{1}{2}}\right)^* \cdot C\big(\Gamma^\dagger\big)^{\frac{1}{2}} \right\| \\
&= \left\| \big(\Gamma^\dagger\big)^{\frac{1}{2}} C^*C \big(\Gamma^\dagger\big)^{\frac{1}{2}} \right\| = \left\| (B_{11}^*B_{11} + W)^{-\frac{1}{2}} W (B_{11}^*B_{11} + W)^{-\frac{1}{2}} \right\| \\
&= \left\| \left[\big(B_{11}^*B_{11} + W\big)^{-\frac{1}{2}} W^{\frac{1}{2}}\right] \cdot \left[\big(B_{11}^*B_{11} + W\big)^{-\frac{1}{2}} W^{\frac{1}{2}}\right]^* \right\| \\
&= \left\| \left[\big(B_{11}^*B_{11} + W\big)^{-\frac{1}{2}} W^{\frac{1}{2}}\right]^* \cdot \left[\big(B_{11}^*B_{11} + W\big)^{-\frac{1}{2}} W^{\frac{1}{2}}\right] \right\| \\
&= \left\| W^{\frac{1}{2}} (B_{11}^*B_{11} + W)^{-1} W^{\frac{1}{2}} \right\| = \left\| W^{\frac{1}{2}} B_{11}^{-1} (I_{H_1} + Z)^{-1} (B_{11}^*)^{-1} W^{\frac{1}{2}} \right\| \\
&= \|S^*(I_{H_1} + SS^*)^{-1} \cdot S\| = \|(I_{H_2} + S^*S)^{-1} S^* \cdot S\| = \frac{\|S^*S\|}{1 + \|S^*S\|} \\
&= \frac{\|SS^*\|}{1 + \|SS^*\|} = \frac{\|Z\|}{1 + \|Z\|} = \frac{\|CB^\dagger\|^2}{1 + \|CB^\dagger\|^2}.
\end{aligned}$$

Finally, we prove the second equation in (52). First, we consider the case that B is not surjective. In this case, $H_1 \neq H$ and thus $I_{H_1^\perp} \neq 0$. Therefore, by (36) and (54) we have

$$\begin{aligned} I_H - B\Gamma^\dagger B^* =& U_{P_B}^* \begin{pmatrix} I_{H_1} - (I_{H_1} + Z)^{-1} & 0 \\ 0 & I_{H_1^\perp} \end{pmatrix} U_{P_B} \\ =& U_{P_B}^* \begin{pmatrix} Z(I_{H_1} + Z)^{-1} & 0 \\ 0 & I_{H_1^\perp} \end{pmatrix} U_{P_B}. \end{aligned}$$

The equation above together with (57) yields

$$\|I_H - B\Gamma^\dagger B^*\| = \max\{\|Z(I_{H_1} + Z)^{-1}\|, \|I_{H_1^\perp}\|\} = 1.$$

Next, we consider the case that B is surjective. In this case $I_{H_1^\perp} = 0$, so the argument above indicates that

$$\|I_H - B\Gamma^\dagger B^*\| = \|Z(I_H + Z)^{-1}\| = \frac{\|Z\|}{1 + \|Z\|} = \frac{\|CB^\dagger\|^2}{1 + \|CB^\dagger\|^2}. \qquad \square$$

Our next technical result is as follows.

Lemma 5.2 *Let M be a weak perturbation of $T \in \mathcal{L}(H, K)$ given by (12) and let $L_{E,T}$ and $R_{F,T}$ be defined by (13). Put*

$$\pi_L = \|(I_K - TT^\dagger)L_{E,T}^\dagger TT^\dagger\|, \tag{58}$$

$$\pi_R = \|(I_H - T^\dagger T)R_{F,T}^\dagger T^\dagger T\|, \tag{59}$$

$$\theta_L = \begin{cases} 1, & \text{if } L_{E,T} \text{ is not surjective,} \\ \frac{\pi_L^2}{1+\pi_L^2}, & \text{if } L_{E,T} \text{ is surjective.} \end{cases} \tag{60}$$

$$\theta_R = \begin{cases} 1, & \text{if } R_{F,T} \text{ is not surjective,} \\ \frac{\pi_R^2}{1+\pi_R^2}, & \text{if } R_{F,T} \text{ is surjective.} \end{cases} \tag{61}$$

Then

$$\|MM^\dagger - TT^\dagger\| = \|TT^\dagger(I_K - MM^\dagger)\| = \sqrt{\theta_L}, \tag{62}$$

$$\|MM^\dagger(I_K - TT^\dagger)\| = \frac{\pi_L}{\sqrt{1 + \pi_L^2}}, \tag{63}$$

$$\|M^\dagger M - T^\dagger T\| = \|T^\dagger T(I_H - M^\dagger M)\| = \sqrt{\theta_R}, \tag{64}$$

$$\|M^\dagger M(I_H - T^\dagger T)\| = \frac{\pi_R}{\sqrt{1+\pi_R^2}}. \tag{65}$$

Proof For a proof, see [4, Lemma 3.5]. □

Theorem 5.3 *Let M be a weak perturbation of $T \in \mathcal{L}(H, K)$ given by (12). Then upper bound (3) is valid.*

Proof For a proof, see [4, Theorem 3.1]. □

Theorem 5.4 *Let M be a strong perturbation of $T \in \mathcal{L}(H, K)$ given by (12). Then upper bound (4) is valid.*

Proof We first consider the self-adjoint case that $H = K$, $T = T^*$ and $E = F$. In this case $L_{E,T} = R_{F,T}$, which is invertible by assumption, hence by (58)–(61) we have

$$\sqrt{\theta_L} = \frac{\pi_L}{\sqrt{1+\pi_L^2}} = \frac{\pi_R}{\sqrt{1+\pi_R^2}} = \sqrt{\theta_R}.$$

Then it follows from (62) and (65) that

$$\begin{aligned}\|MM^\dagger - TT^\dagger\| &= \|M^\dagger M(I_H - T^\dagger T)\| \\ &= \|M^\dagger (M-T)(I_H - T^\dagger T)\| \\ &\le \|M^\dagger\| \cdot \|M - T\| \cdot \|I_H - T^\dagger T\| \\ &\le \|M^\dagger\| \cdot \|M - T\|. \end{aligned} \tag{66}$$

Next, we consider the general case that M is given by (12). By Lemma 3.6 we know that $\rho(M) = H\rho(T)H^*$ is also a strong perturbation of $\rho(T)$, where $\rho(T)$, $\rho(M)$ and H are given by (7) and (16), respectively. Using (9), (11), (66) and (10), we obtain

$$\begin{aligned}\max\left\{\|MM^\dagger - TT^\dagger\|,\ \|M^\dagger M - T^\dagger T\|\right\} &= \|\rho(M)\rho(M)^\dagger - \rho(T)\rho(T)^\dagger\| \\ &\le \|\rho(M)^\dagger\| \cdot \|\rho(M) - \rho(T)\| \\ &= \|M^\dagger\| \cdot \|M - T\|. \end{aligned}$$

□

Remark 5.5 It is remarkable that upper bound (4) may be false for a general weak perturbation. For a counterexample, see [4, Theorem 3.2].

Theorem 5.6 *Let M be a strong perturbation of $T \in \mathcal{L}(H, K)$ given by* (12)*. Then*

$$\|M^\dagger - T^\dagger\| \le \frac{\sqrt{5}+1}{2}\|M^\dagger\| \cdot \|T^\dagger\| \cdot \|M - T\|. \tag{67}$$

Proof First we consider the case that $H = K$. Following the line in the proof of [19, Theorem 3.3.5], we give a detailed proof of (67) based on (3) and (4). Note that $T, E, M, T^\dagger, M^\dagger$ all belong to $\mathcal{L}(H)$, which is a unital C^*-algebra. So there exists a Hilbert space L and a C^*-morphism $\pi : \mathcal{L}(H) \to \mathbb{B}(L)$ such that π is faithful [10, Corollary 3.7.5]. Replacing L with $\pi(I_H)L$ if necessary, we may assume that π is unital.

Since $\pi : \mathcal{L}(H) \to \mathcal{R}(\pi)$ is a C^*-isomorphism, we have $\|\pi(X)\| = \|X\|$ for every $X \in \mathcal{L}(H)$ [10, Theorem 1.5.7], and from (5) we know that both $\pi(T)$ and $\pi(M)$ are M-P invertible such that $\pi(T)^\dagger = \pi(T^\dagger)$ and $\pi(M)^\dagger = \pi(M^\dagger)$. Direct computation yields

$$M^\dagger - T^\dagger = A + B = M^\dagger M A + (I_H - M^\dagger M)B, \tag{68}$$

where

$$A = -M^\dagger(M-T)T^\dagger + M^\dagger(I_K - TT^\dagger) \text{ and } B = -(I_H - M^\dagger M)T^\dagger. \tag{69}$$

Now, given any $x \in L$ with $\|x\| = 1$, we put

$$\sin\phi = \|\pi(T)\pi(T)^\dagger x\| \text{ and } \cos\phi = \|\big(I_L - \pi(T)\pi(T)^\dagger\big)x\|$$

for some $\phi \in [0, \frac{\pi}{2}]$. Let

$$\lambda = \|M^\dagger\| \cdot \|T^\dagger\| \cdot \|M - T\|.$$

Then by (69) we have

$$\begin{aligned}\pi(A)x =& -\pi(M^\dagger)\pi(M-T)\pi(T)^\dagger\pi(T)\pi(T)^\dagger x\\ &+ \pi(M^\dagger)\pi\big[MM^\dagger - TT^\dagger\big]\big[I_L - \pi(T)\pi(T)^\dagger\big]x,\\ \pi(B)x =& -\pi\big[T^\dagger T - M^\dagger M\big]\pi(T)^\dagger\pi(T)\pi(T)^\dagger x.\end{aligned}$$

It follows from (3) and (4) that

$$\begin{aligned}\|\pi(A)x\| \le& \|M^\dagger\| \cdot \|M-T\| \cdot \|T^\dagger\| \cdot \sin\phi\\ &+ \|M^\dagger\| \cdot \|MM^\dagger - TT^\dagger\| \cdot \cos\phi \qquad (70)\\ \le& \lambda \cdot \big[\sin\phi + \cos\phi\big],\end{aligned}$$

$$\|\pi(B)x\| \le \|T^\dagger T - M^\dagger M\| \cdot \|T^\dagger\| \cdot \sin\phi \le \lambda \cdot \sin\phi. \tag{71}$$

In view of (68), (70) and (71), we have

$$\begin{aligned}\|\pi(M^\dagger - T^\dagger)x\|^2 &= \|\pi(A)x\|^2 + \|\pi(B)x\|^2\\ &\le \Big[\big(\sin\phi + \cos\phi\big)^2 + \sin^2\phi\Big]\lambda^2\\ &= \Big[\frac{3}{2} + \sin(2\phi) - \frac{1}{2}\cos(2\phi)\Big]\lambda^2\\ &\le \frac{3+\sqrt{5}}{2}\lambda^2 = \left[\frac{\sqrt{5}+1}{2}\lambda\right]^2.\end{aligned}$$

Therefore,

$$\|M^\dagger - T^\dagger\| = \|\pi(M^\dagger - T^\dagger)\| \le \frac{\sqrt{5}+1}{2}\lambda.$$

This completes the proof of (67) in the case that $H = K$.

Next we consider the general case that $H \ne K$. Let $\widehat{M} = \widehat{E}\cdot\widehat{T}\cdot\widehat{F}^* \in \mathcal{L}(H \oplus K)$, where

$$\widehat{E} = \begin{pmatrix} 0 & 0\\ 0 & E\end{pmatrix}, \widehat{T} = \begin{pmatrix} 0 & 0\\ T & 0\end{pmatrix} \text{ and } \widehat{F} = \begin{pmatrix} F & 0\\ 0 & 0\end{pmatrix}. \tag{72}$$

Then

$$\widehat{T}^\dagger = \begin{pmatrix} 0 & T^\dagger\\ 0 & 0\end{pmatrix}, \widehat{M} = \begin{pmatrix} 0 & 0\\ M & 0\end{pmatrix}, \widehat{M}^\dagger = \begin{pmatrix} 0 & M^\dagger\\ 0 & 0\end{pmatrix}, \tag{73}$$

$$L_{\widehat{E},\widehat{T}} = \widehat{E}\widehat{T}\widehat{T}^\dagger + I_{H\oplus K} - \widehat{T}\widehat{T}^\dagger = \begin{pmatrix} I_H & 0\\ 0 & L_{E,T}\end{pmatrix}, \tag{74}$$

$$R_{\widehat{F},\widehat{T}} = \widehat{F}\widehat{T}^\dagger\widehat{T} + I_{H\oplus K} - \widehat{T}^\dagger\widehat{T} = \begin{pmatrix} R_{F,T} & 0\\ 0 & I_K\end{pmatrix}. \tag{75}$$

Therefore, both $L_{\widehat{E},\widehat{T}}$ and $R_{\widehat{F},\widehat{T}}$ are invertible in $\mathcal{L}(H \oplus K)$, hence $\widehat{M}$ is a strong perturbation of $\widehat{T}$. It follows that

$$\begin{aligned}\|M^\dagger - T^\dagger\| = \|\widehat{M}^\dagger - \widehat{T}^\dagger\| &\le \frac{\sqrt{5}+1}{2}\|\widehat{M}^\dagger\|\cdot\|\widehat{T}^\dagger\|\cdot\|\widehat{M} - \widehat{T}\|\\ &= \frac{\sqrt{5}+1}{2}\|M^\dagger\|\cdot\|T^\dagger\|\cdot\|M - T\|. \qquad \square\end{aligned}$$

In the rest of this section, we study further the perturbation estimation for the strong perturbation. In this case, a technical result can be provided as follows.

Theorem 5.7 *Let M be a strong perturbation of $T \in \mathcal{L}(H, K)$ given by (12). Then*

$$T = L_{E,T}^{-1} \cdot M \cdot \left(R_{F,T}^{-1}\right)^{*} \tag{76}$$

is a strong perturbation of M such that

$$\left(L_{L_{E,T}^{-1},M}\right)^{-1} = ETT^{\dagger} + \left(I_K - ETT^{\dagger}\right)(I_K - MM^{\dagger})\left(L_{E,T}^{-1}\right)^{*}(I_K - TT^{\dagger}), \tag{77}$$

$$\left(R_{R_{F,T}^{-1},M}\right)^{-1} = FT^{\dagger}T + \left(I_H - FT^{\dagger}T\right)(I_H - M^{\dagger}M)\left(R_{F,T}^{-1}\right)^{*}(I_H - T^{\dagger}T), \tag{78}$$

$$\left\|(I_K - MM^{\dagger})\left(L_{L_{E,T}^{-1},M}\right)^{-1}MM^{\dagger}\right\| = \pi_L, \tag{79}$$

$$\left\|L_{L_{E,T}^{-1},M}\right\| \le \sqrt{\frac{\pi_L}{\sqrt{1+\pi_L^2}} + \max\left\{\|(ETT^{\dagger})^{\dagger}\|, \|I_K - TT^{\dagger}\|\right\}^2}, \tag{80}$$

$$\left\|L_{L_{E,T}^{-1},M} - I_K\right\| \le \|L_{E,T} - I_K\| \cdot \|(ETT^{\dagger})^{\dagger}\|, \tag{81}$$

$$\left\|\left(L_{L_{E,T}^{-1},M}\right)^{-1}\right\| \le \sqrt{(1+\pi_L+\pi_L^2)\cdot \max\left\{\|ETT^{\dagger}\|^2, 1+\pi_L^2\right\}}, \tag{82}$$

$$\left\|\left(L_{L_{E,T}^{-1},M}\right)^{-1} - I_K\right\| \le \sqrt{1+\pi_L^2}\cdot\|L_{E,T} - I_K\|, \tag{83}$$

where $L_{E,T}$, $R_{F,T}$ and π_L are defined by (13) and (58) respectively, and

$$L_{L_{E,T}^{-1},M} = L_{E,T}^{-1}MM^{\dagger} + I_K - MM^{\dagger}, \tag{84}$$

$$R_{R_{F,T}^{-1},M} = R_{F,T}^{-1}M^{\dagger}M + I_H - M^{\dagger}M. \tag{85}$$

Proof

(1) Note that (76) can be derived directly from (2). Put

$$P_M = MM^{\dagger},\ K_M = \mathcal{R}(P_M) = \mathcal{R}(M)$$

and define the unitary $U_{P_M} : K \to K_M \oplus K_M^{\perp}$ as (19). By (42) and (13), we have

$$\begin{aligned}(I_K - P_M)L_{E,T} &= \left[I_K - ETT^{\dagger}(ETT^{\dagger})^{\dagger}\right](ETT^{\dagger} + I_K - TT^{\dagger})\\ &= \left[I_K - ETT^{\dagger}(ETT^{\dagger})^{\dagger}\right](I_K - TT^{\dagger})\\ &= (I_K - P_M)(I_K - TT^{\dagger}).\end{aligned} \tag{86}$$

Taking $*$-operation, we get

$$L_{E,T}^*(I_K - P_M) = (I_K - TT^\dagger)(I_K - P_M). \tag{87}$$

Let $A \in \mathcal{L}(K)$ be defined by

$$A = P_M + L_{E,T}(I_K - P_M). \tag{88}$$

Then from (88), (20) and (86) we have

$$U_{P_M} A U_{P_M}^* = \begin{pmatrix} I_{K_M} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \tag{89}$$

where $A_{12} = P_M L_{E,T}(I_K - P_M)|_{K_M^\perp}$ and

$$A_{22} = (I_K - P_M)(I_K - TT^\dagger)(I_K - P_M)|_{K_M^\perp} = SS^*|_{K_M^\perp} \in \mathcal{L}(K_M^\perp), \tag{90}$$

in which

$$S = (I_K - P_M)(I_K - TT^\dagger) \in \mathcal{L}(K). \tag{91}$$

We prove that A_{22} is invertible in $\mathcal{L}(K_M^\perp)$. Indeed, given any $\xi \in K_M^\perp$ such that $A_{22}\xi = 0$, by (90) we have

$$(I_K - TT^\dagger)(I_K - P_M)\xi = S^*\xi = 0,$$

which leads by (87) to $\xi = (I_K - P_M)\xi = 0$, since $\xi \in K_M^\perp$ and $L_{E,T}^*$ is invertible. Therefore, A_{22} is injective. Following the notations in Remark 3.8 and Theorem 4.5, we know from (24), (51), (26) and (18) that

$$\begin{aligned}
&U_{Q_T}(I_K - TT^\dagger)(I_K - P_M)\big(L_{E,T}^{-1}\big)^*(I_K - TT^\dagger)U_{Q_T}^* \\
&= \begin{pmatrix} 0 & 0 \\ 0 & I_{K_1^\perp} \end{pmatrix} \cdot \begin{pmatrix} I_{K_1} - B\Gamma^{-1}B^* & -B\Gamma^{-1}C^* \\ -C\Gamma^{-1}B^* & I_{K_1^\perp} - C\Gamma^{-1}C^* \end{pmatrix} \\
&\quad \cdot \begin{pmatrix} (B^*)^{-1} & -(B^*)^{-1}C^* \\ 0 & I_{K_1^\perp} \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & I_{K_1^\perp} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & I_{K_1^\perp} \end{pmatrix} = U_{Q_T}(I_K - TT^\dagger)U_{Q_T}^*,
\end{aligned}$$

where $\Gamma = B^*B + C^*C$. Therefore,

$$(I_K - TT^\dagger)(I_K - P_M)\big(L_{E,T}^{-1}\big)^*(I_K - TT^\dagger) = I_K - TT^\dagger. \tag{92}$$

It follows that

$$A_{22}(I_K - P_M)\big(L_{E,T}^{-1}\big)^*(I_K - TT^\dagger) = S, \tag{93}$$

which means that A_{22} is surjective, since by (86) and (91) we have $\mathcal{R}(S) = K_M^\perp$. Therefore, A_{22} is invertible in $\mathcal{L}(K_M^\perp)$, and then (93) gives

$$A_{22}^{-1}S = (I_K - P_M)\big(L_{E,T}^{-1}\big)^*(I_K - TT^\dagger). \tag{94}$$

It follows from (89) that the operator A defined by (88) is invertible such that

$$U_{P_M}A^{-1}U_{P_M}^* = \begin{pmatrix} I_{K_M} & -A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{pmatrix}. \tag{95}$$

In view of (84) and (88), we have $L_{L_{E,T}^{-1},M} = L_{E,T}^{-1}A$, hence $L_{L_{E,T}^{-1},M}$ is invertible such that

$$\big(L_{L_{E,T}^{-1},M}\big)^{-1} = A^{-1}L_{E,T}. \tag{96}$$

Similarly, we can prove that the operator $R_{R_{F,T}^{-1},M}$ defined by (85) is also invertible. Therefore, the operator T written in the form (76) is a strong perturbation of M.

Now we are ready to prove the validity of (77). Indeed, from (42) and (92) we have

$$\begin{aligned} &P_M L_{E,T}(I_K - P_M)(L_{E,T}^{-1})^*(I_K - TT^\dagger) \\ &= P_M \cdot ETT^\dagger \cdot (I_K - P_M)(L_{E,T}^{-1})^*(I_K - TT^\dagger) \\ &\quad + P_M \cdot (I_K - TT^\dagger) \cdot (I_K - P_M)(L_{E,T}^{-1})^*(I_K - TT^\dagger) \\ &= ETT^\dagger \cdot (I_K - P_M)(L_{E,T}^{-1})^*(I_K - TT^\dagger) + P_M(I_K - TT^\dagger). \end{aligned} \tag{97}$$

Therefore, it can be deduced by (96), (95), (21), (86), (91), (94) and (97) that

$$\begin{aligned} \big(L_{L_{E,T}^{-1},M}\big)^{-1} &= \Big[P_M - A_{12}A_{22}^{-1}(I_K - P_M) + A_{22}^{-1}(I_K - P_M)\Big]L_{E,T} \\ &= P_M L_{E,T} - P_M L_{E,T}(I_K - P_M)A_{22}^{-1}S + A_{22}^{-1}S \\ &= P_M L_{E,T} + \big(I_K - P_M L_{E,T}\big)(I_K - P_M)\Big(L_{E,T}^{-1}\Big)^*(I_K - TT^\dagger) \\ &= ETT^\dagger + \Big(I_K - ETT^\dagger\Big)(I_K - P_M)\Big(L_{E,T}^{-1}\Big)^*(I_K - TT^\dagger). \end{aligned}$$

This completes the proof of (77). The proof of (78) is similar.

(2) We prove the norm equation (79). For simplicity, we put

$$X = (I_K - P_M)\big(L_{L_{E,T}^{-1},M}\big)^{-1} P_M,\ Y = A_{22}^{-1} S P_M|_{K_M} \in \mathcal{L}(K_M, K_M^{\perp}), \tag{98}$$

where A_{22} is formulated by (90) and S is defined by (91). Note that both $I_K - P_M$ and $I_K - TT^{\dagger}$ are idempotent, so by (90) we have

$$\begin{aligned} YY^* = {} & A_{22}^{-1}(I_K - P_M)(I_K - TT^{\dagger})\big[-(I_K - P_M) + I_K\big] \\ & (I_K - TT^{\dagger})(I_K - P_M)A_{22}^{-1} \\ = {} & -I_{K_M^{\perp}} + A_{22}^{-1}. \end{aligned} \tag{99}$$

Since the operator A_{22} defined by (90) is positive definite and $\|A_{22}\| \le 1$, we know that A_{22}^{-1} is also positive definite. Then (99) indicates that each spectral point λ of A_{22}^{-1} satisfies $\lambda \ge 1$, which means that

$$\|Y\|^2 = \|YY^*\| = \|A_{22}^{-1}\| - 1. \tag{100}$$

Once again from (90) we have

$$\|A_{22}^{-1}\| = \|(SS^*)^{\dagger}\| = \|(S^{\dagger})^* S^{\dagger}\| = \|S^{\dagger}(S^{\dagger})^*\| = \|(S^*S)^{\dagger}\|. \tag{101}$$

Following the notations as in the proof of Lemma 5.2, we have

$$S^*S = (I_K - TT^{\dagger})(I_K - P_M)(I_K - TT^{\dagger}) = U_{Q_T}^* \begin{pmatrix} 0 & 0 \\ 0 & I_{K_1^{\perp}} - C\Gamma^{\dagger}C^* \end{pmatrix} U_{Q_T}.$$

By (52) we know that the norm of $C\Gamma^{\dagger}C^*$ is less than 1, so $I_{K_1^{\perp}} - C\Gamma^{\dagger}C^*$ is invertible, hence

$$(S^*S)^{\dagger} = U_{Q_T}^* \begin{pmatrix} 0 & 0 \\ 0 & (I_{K_1^{\perp}} - C\Gamma^{\dagger}C^*)^{-1} \end{pmatrix} U_{Q_T}. \tag{102}$$

Furthermore, since $C\Gamma^{\dagger}C^*$ is positive and $\|C\Gamma^{\dagger}C^*\| < 1$, we have

$$\|(I_{K_1^{\perp}} - C\Gamma^{\dagger}C^*)^{-1}\| = \frac{1}{1 - \|C\Gamma^{\dagger}C^*\|}.$$

The equation above together with (52) and $\pi_L = \|CB^{\dagger}\|$ yields

$$\|(I_{K_1^{\perp}} - C\Gamma^{\dagger}C^*)^{-1}\| = 1 + \|CB^{\dagger}\|^2 = 1 + \pi_L^2. \tag{103}$$

Moreover, by (98), (96), (95) and (86), we have

$$\begin{aligned} U_{P_M}\cdot X\cdot U_{P_M}^* = &\begin{pmatrix} 0 & 0 \\ 0 & I_{K_M^\perp} \end{pmatrix}\begin{pmatrix} I_{K_M} & -A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{pmatrix} \\ &\begin{pmatrix} * & * \\ SP_M|_{K_M} & A_{22} \end{pmatrix}\begin{pmatrix} I_{K_M} & 0 \\ 0 & 0 \end{pmatrix} \\ = &\begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix}. \end{aligned}$$

So we may combine the equation above with (100), (101), (102) and (103) to conclude that

$$\|X\| = \|Y\| = \sqrt{\|A_{22}^{-1}\| - 1} = \sqrt{\|(S^*S)^\dagger\| - 1} = \sqrt{(1+\pi_L^2) - 1} = \pi_L.$$

This completes the proof of (79).

(3) We prove the norm estimations (80) and (81). Let $L_{L_{E,T}^{-1},M}$ be given by (84). Put

$$Z = U_{Q_T} L_{L_{E,T}^{-1},M} U_{Q_T}^* \text{ and } \theta = \max\left\{\|(ETT^\dagger)^\dagger\|, \|I_K - TT^\dagger\|\right\}.$$

Then direct computation yields

$$Z = \begin{pmatrix} I_{K_1} + (I_{K_1} - B)\Gamma^{-1}B^* & (I_{K_1} - B)\Gamma^{-1}C^* \\ -C\Gamma^{-1}B^* & I_{K_1^\perp} - C\Gamma^{-1}C^* \end{pmatrix},$$

which gives

$$\begin{aligned} ZZ^* = &\begin{pmatrix} I_{K_1} - B\Gamma^{-1}B^* + \Gamma^{-1} & -B\Gamma^{-1}C^* \\ -C\Gamma^{-1}B^* & I_{K_1^\perp} - C\Gamma^{-1}C^* \end{pmatrix} \\ = &\begin{pmatrix} I_{K_1} - B\Gamma^{-1}B^* & -B\Gamma^{-1}C^* \\ -C\Gamma^{-1}B^* & -C\Gamma^{-1}C^* \end{pmatrix} + \begin{pmatrix} \Gamma^{-1} & 0 \\ 0 & I_{K_1^\perp} \end{pmatrix} \\ = &U_{Q_T}(TT^\dagger - P_M)U_{Q_T}^* + \begin{pmatrix} \Gamma^{-1} & 0 \\ 0 & I_{K_1^\perp} \end{pmatrix}. \end{aligned}$$

Note that

$$\begin{pmatrix} \Gamma^{-1} & 0 \\ 0 & 0 \end{pmatrix} = U_{Q_T}(ETT^\dagger)^\dagger U_{Q_T}^* \cdot \left[U_{Q_T}(ETT^\dagger)^\dagger U_{Q_T}^*\right]^*,$$

hence by (62) and (60), we obtain

$$\|L_{L_{E,T}^{-1},M}\|^2 = \|ZZ^*\| \leq \|TT^\dagger - P_M\| + \theta^2 = \frac{\pi_L}{\sqrt{1+\pi_L^2}} + \theta^2.$$

Similarly,

$$\begin{aligned} Z - I_K &= \begin{pmatrix} (I_{K_1} - B)\Gamma^{-1}B^* & (I_{K_1} - B)\Gamma^{-1}C^* \\ -C\Gamma^{-1}B^* & -C\Gamma^{-1}C^* \end{pmatrix} \\ &= \begin{pmatrix} I_{K_1} - B & 0 \\ -C & 0 \end{pmatrix} \cdot \begin{pmatrix} \Gamma^{-1}B^* & \Gamma^{-1}C^* \\ 0 & 0 \end{pmatrix} \\ &= U_{Q_T}(I_K - L_{E,T})U_{Q_T}^* \cdot U_{Q_T}(ETT^\dagger)^\dagger U_{Q_T}^*. \end{aligned}$$

Therefore, $\|Z - I_K\| \leq \|I_K - L_{E,T}\| \cdot \|(ETT^\dagger)^\dagger\|$.

(4) We prove norm estimations (82) and (83). Let

$$W = U_{Q_T}\big(L_{L_{E,T}^{-1},M}\big)^{-1}U_{Q_T}^* \text{ and } W_1 = \begin{pmatrix} I_{K_1} & 0 \\ CB^{-1} & I_{K_1^\perp} \end{pmatrix}. \tag{104}$$

Then by direct computation, from (77) we can obtain

$$W = \begin{pmatrix} B & -(I_{K_1} - B)(CB^{-1})^* \\ C & I_{K_1^\perp} + C(B^*)^{-1}C^* \end{pmatrix}, \tag{105}$$

hence

$$\begin{aligned} W^*W &= \begin{pmatrix} \Gamma & \Gamma(CB^{-1})^* \\ CB^{-1}\Gamma & I_{K_1^\perp} + CB^{-1}(I_{K_1} + \Gamma)(CB^{-1})^* \end{pmatrix} \\ &= W_1 \begin{pmatrix} \Gamma & 0 \\ 0 & I_{K_1^\perp} + CB^{-1}(CB^{-1})^* \end{pmatrix} W_1^*. \end{aligned} \tag{106}$$

Note that $\|CB^{-1}\| = \pi_L$, so by (10) we can get $\|\rho\big((CB^{-1})^*\big)\| = \pi_L$, where $\rho\big((CB^{-1})^*\big)$ is defined by (7). It follows from (104) that

$$\begin{aligned} \|W_1W_1^*\| &= \left\| I_{K_1 \oplus K_1^\perp} + \rho\big((CB^{-1})^*\big) + \begin{pmatrix} 0 & 0 \\ 0 & CB^{-1}(CB^{-1})^* \end{pmatrix} \right\| \\ &\leq 1 + \pi_L + \pi_L^2. \end{aligned}$$

Therefore (106) gives

$$\begin{aligned}\left\|\left(L_{L_{E,T}^{-1},M}\right)^{-1}\right\|^2 =&\|W\|^2 \le \|W_1\|^2 \cdot \max\left\{\|\Gamma\|, 1+\pi_L^2\right\}\\ \le&(1+\pi_L+\pi_L^2)\cdot \max\left\{\|\Gamma\|, 1+\pi_L^2\right\}\\ =&(1+\pi_L+\pi_L^2)\cdot \max\left\{\|ETT^\dagger\|^2, 1+\pi_L^2\right\}.\end{aligned}$$

This finishes the proof of (82).

Finally, from (105) we can obtain

$$W - I_{K_1\oplus K_1^\perp} = \begin{pmatrix} B - I_{K_1} & 0 \\ C & 0 \end{pmatrix}\cdot W_2, \tag{107}$$

where $W_2 = \begin{pmatrix} I_{K_1} & (CB^{-1})^* \\ 0 & 0 \end{pmatrix}$. As is shown above, $\|W_2\| = \sqrt{\|W_2W_2^*\|} = \sqrt{1+\pi_L^2}$. Therefore (107) yields

$$\left\|\left(L_{L_{E,T}^{-1},M}\right)^{-1} - I_K\right\| = \|W - I_{K_1\oplus K_1^\perp}\| \le \|L_{E,T} - I_K\|\cdot\sqrt{1+\pi_L^2}.$$

This completes the proof of all the assertions. □

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Part II
Orthogonality and Inequalities

Birkhoff–James Orthogonality: Characterizations, Preservers, and Orthogonality Graphs

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Abstract We present Birkhoff–James orthogonality from historical perspectives to the current development. We compare it with some other orthogonalities, present its properties and its applications, and review the characterizations of Birkhoff–James orthogonality in classical Banach spaces like $\mathbb{B}(\mathcal{H})$, C^*-algebras, Hilbert C^*-modules, or the space of rectangular matrices normed with Schatten norms. We also present the results on characterizations of preservers of Birkhoff–James orthogonality and, by devising a directed graph of the relation, show that in smooth spaces it can completely determine the norm up to (conjugate) linear isometry.

Most, though not all, of the results that we state are supplied with (sketches of) the proof.

Keywords Normed vector space · Birkhoff–James orthogonality · C^*-algebra · Hilbert C^*-module · Preservers · Graph · Clique

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1 Introduction

In a real or complex inner product space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$, two vectors x, y are orthogonal (denoted by $x \perp y$) if $\langle x, y\rangle = 0$. The familiar, and mostly trivial, properties of orthogonality relation are

Homogeneity $x \perp y$ implies $(\alpha x) \perp (\beta y)$ for scalars α, β.
Left additivity $x_1 \perp y$ and $x_2 \perp y$ imply $(x_1 + x_2) \perp y$.
Right additivity $x \perp y_1$ and $x \perp y_2$ imply $x \perp (y_1 + y_2)$.
Symmetricity $x \perp y$ implies $y \perp x$.

Would it be possible to extend the orthogonality relation from inner product spaces to a more general normed space $(\mathcal{X}, \|\cdot\|)$? The lack of the inner product in such spaces suggests that, if there is any chance to succeed, one should better express orthogonality with the help of the norm alone. This is possible in many ways and it gives rise to several nonequivalent extensions of orthogonality. We present a few of them, each with its own virtues and vices. Unless explicitly stated otherwise, our claims will hold equally well for real and complex normed spaces. Thus, we will occasionally omit specifying the underlying field and will simply refer that something holds for all scalars or that there exists a scalar with a particular property, etc. Let $\mathbb{F}$ denote the real or the complex field.

1.1 Roberts Orthogonality

In 1934 Roberts [76] observed that in inner product spaces $x \perp y$ if and only if

$$\|x + \lambda y\| = \|x - \lambda y\| \tag{1}$$

for every scalar λ. Clearly, this condition can be verified in every normed space and gives rise to Roberts orthogonality: $x \perp_R y$ if (1) holds.

It is an easy exercise to see that Roberts orthogonality is homogeneous and symmetric. It does not have an additivity property in general—say, in three-dimensional real space equipped with supremum norm, $(\mathbb{R}^3, \|\cdot\|_\infty)$, the vector $x = (1, 1, 1)$ is Roberts orthogonal to $y_1 = (1, -1, 0)$ and to $y_2 = (0, 2, -2)$ but not to $y_1 + y_2$.

Further vices of Roberts orthogonality are that the condition (1) is very restrictive, so much that in some normed spaces $x \perp_R y$ only when either x or y is a zero vector. An example of such a (real, two-dimensional) space was given by James [45]; it equals the set of all real polynomials of degree at most two restricted to unit interval $[0, 1]$ and vanishing at 0, and equipped with the supremum norm.

In the same paper, James also proved the next proposition.

Proposition 1.1 (Cf. [45, Corollary 4.7]) *The following conditions are equivalent for a real and at least two-dimensional normed space $\mathcal{X}$:*

(i) *$\mathcal{X}$ is an inner product space.*
(ii) *In every two-dimensional plane $\Pi \subseteq \mathcal{X}$ and for every $x \in \Pi$ there exists a nonzero $y \in \Pi$ which is Roberts orthogonal to x.*

The proof will be given within Sect. 1.3.

Similar characterization holds also for complex normed spaces provided that we take for Π two-dimensional *real*-linear planes. Namely, each complex normed space $(\mathcal{X}, \|\cdot\|)$ can be considered as a real normed space (denoted temporarily as $(\mathcal{X}, \|\cdot\|_{\mathbb{R}})$) by restricting the scalars, and the following general result applies:

Proposition 1.2 (Cf. [31, Theorem 7.2]) *A complex normed space $(\mathcal{X}, \|\cdot\|)$ is an inner product space if and only if the associated real normed space $(\mathcal{X}, \|\cdot\|_{\mathbb{R}})$ is an inner product space. When this happens, the inner products $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ are related by the equations $\langle a, b\rangle_{\mathbb{R}} = \mathrm{Re}\langle a, b\rangle$ and $\langle a, b\rangle = \langle a, b\rangle_{\mathbb{R}} - i\langle ia, b\rangle_{\mathbb{R}}$.*

Sketch of the Proof If $\|\cdot\|_{\mathbb{R}}$ is induced by a real inner product $\langle\cdot,\cdot\rangle_{\mathbb{R}}$, then $2\|x\|^2 = \|(1+i)x\|^2 = \|x\|^2 + \|ix\|^2 + 2\langle ix, x\rangle_{\mathbb{R}}$, so $\langle ix, x\rangle_{\mathbb{R}} = 0$. Linearization gives further $\langle x, iy\rangle_{\mathbb{R}} + \langle ix, y\rangle_{\mathbb{R}} = 0$. From here it is straightforward that $\langle x, y\rangle := \langle x, y\rangle_{\mathbb{R}} - i\langle ix, y\rangle_{\mathbb{R}}$ is an inner product over $\mathbb{C}$ which induces the given norm. □

1.2 Isosceles Orthogonality

James in [45] observed that one does not need the full set of scalars in order that (1) be equivalent to the usual orthogonality in real inner product spaces; it suffices that

$$\|x + y\| = \|x - y\|. \tag{2}$$

This condition gives another orthogonality relation, denoted by $\perp_I$ and termed *isosceles* (also known as *James*, see [5]) orthogonality, by which $x \perp_I y$ if (2) holds. As for the terminology: isosceles means that the diagonals of a parallelogram spanned by x, y can form an isosceles triangle.

Isosceles orthogonality is clearly symmetric. Also, unlike Roberts orthogonality, it is always nontrivial. This was observed by James [45] who argued as follows: Given two vectors x, y, a positive number α, and a large enough integer n, we have, from $|\|a\| - \|b\|| \le \|a - b\|$, that

$$\left| n\|x + \tfrac{1}{n+\alpha}y\| - n\|x + \tfrac{1}{n}y\| \right| \le n\|(\tfrac{1}{n+\alpha} - \tfrac{1}{n})y\| = \|(\tfrac{n}{n+\alpha} - 1)y\| \xrightarrow{n\to+\infty} 0.$$

Therefore,

$$\|(n+\alpha)x + y\| - \|nx + y\| = (n+\alpha)\|x + \tfrac{1}{n+\alpha}y\| - n\|x + \tfrac{1}{n}y\| \xrightarrow{n\to+\infty} \alpha\|x\|.$$

This shows that the sequence

$$f(k) := \|x + (kx + y)\| - \|x - (kx + y)\|$$

satisfies $\lim_{k\to+\infty} f(k) = \lim_{k\to+\infty}(\|(k+1)x + y\| - \|(k-1)x + y\|) = \lim_{n\to+\infty}(\|(n+2)x+y\|-\|nx+y\|) = 2\|x\|$. Similarly, $\lim_{k\to-\infty} f(k) = -2\|x\|$. By continuity, the function f must take the zero value, so there is a real scalar λ with $x \perp_I (\lambda x + y)$.

However, isosceles orthogonality is neither homogeneous nor additive in general. In fact, James in [45] obtained the following characterization of inner product spaces:

Proposition 1.3 (Cf. [45, Theorem 4.7, 4.8]) *The following are equivalent for a normed space $\mathcal{X}$ over the field $\mathbb{F}$:*

(i) *$\mathcal{X}$ is an inner product space.*
(ii) *Isosceles orthogonality is real-homogeneous in $\mathcal{X}$.*
(iii) *Isosceles orthogonality is additive in $\mathcal{X}$.*

Sketch of the Proof (i) $\Longrightarrow$ (ii) and (iii). Given any $x, y \in \mathcal{X}$, the condition $x \perp_I y$ is equivalent to $\mathrm{Re}\langle x, y\rangle = 0$ and is clearly homogeneous under multiplication by real scalars and additive.

(ii) $\Longrightarrow$ (i). Take any two normalized vectors x, y. Then $(x+y) \perp_I (x-y)$, so, by real-homogeneity, also $(1+\alpha)(x+y) \perp_I (1-\alpha)(x-y)$, that is,

$$\begin{aligned} 2\|x+\alpha y\| &= \|(1+\alpha)(x+y) + (1-\alpha)(x-y)\| \\ &= \|(1+\alpha)(x+y) - (1-\alpha)(x-y)\| \\ &= 2\|\alpha x + y\|; \qquad \alpha \in \mathbb{R}. \end{aligned}$$

It now follows by Ficken's [36] characterization of (real or complex) inner product spaces as the ones where $\|x\| = \|y\|$ implies that $\|\alpha x + \beta y\| = \|\beta x + \alpha y\|$ for all scalars $\alpha, \beta \in \mathbb{R}$ that $\mathcal{X}$ is an inner product space.

(iii) $\Longrightarrow$ (ii). By additivity, $x \perp_I y$ implies $(nx) \perp_I (my)$ for all positive integers n, m. Therefore, $\|nx + my\| = \|nx - my\|$, and so $\|x + \frac{m}{n}y\| = \|x - \frac{m}{n}y\|$. By continuity, and as $x \perp_I y$ trivially implies $x \perp_I (-y)$, we see that $\|x + \lambda y\| = \|x - \lambda y\|$ for every $\lambda \in \mathbb{R}$. Then, by the implication (ii) $\Longrightarrow$ (i), $\mathcal{X}$ is an inner product space. □

1.3 Birkhoff–James Orthogonality

Birkhoff in [22], based on geometric considerations, introduced another orthogonality relation as follows: "A vector $\vec{pq}$ issuing from a point p is perpendicular to a second such vector $\vec{pr}$ if and only if there is no point on the extended line through

$\vec{pr}$ nearer to q than p." In modern terminology, $x \perp y$ (since we will consider only this orthogonality in the sequel, we write $\perp$ instead of $\perp_B$) if

$$\|x + \lambda y\| \geq \|x\|$$

for every scalar λ. This orthogonality was more thoroughly investigated by James in [46, 47], and because of this it is often known as Birkhoff–James (B-J for short) orthogonality. We remark, however, that B-J orthogonality can be traced back at least as far as Carathéodory (see [5]).

One can see from its very definition that B-J orthogonality is homogeneous, but we prefer to verify this from its equivalent geometrical reformulation (see Proposition 1.4 below), due to James [47, Corollary 2.2], which will reveal much more than just homogeneity. To do that we need to introduce some terminology. Recall that $\mathbb{F}$ denotes either the field of real or the field of complex numbers; also, given a vector $x \in (\mathcal{X}, \|\cdot\|)$ in a normed space $\mathcal{X}$ over $\mathbb{F}$, the linear functional $f : \mathcal{X} \to \mathbb{F}$ for which

$$\|f\| = 1 \quad \text{and} \quad f(x) = \|x\|$$

will be called a *supporting functional at a vector* x.

It is at least intuitively geometrically clear that, for a normalized vector x and a vector y, we have $x \perp y$ if and only if the line, passing through x in the direction of y, does not contain the interior points of the norm's unit ball (see Fig. 1). That is, this line must be a supporting line to the norm's unit ball. The intuition is correct:

Proposition 1.4 (Cf. [47, Corollary 2.2]) *Let $\mathcal{X}$ be a normed space over $\mathbb{F}$. Then $x \perp y$ if and only if there exists a supporting functional f_x at point x, which annihilates y.*

Sketch of the Proof If x is a nonzero vector and its supporting functional f_x annihilates y, then $\|x + \lambda y\| \geq |f_x(x + \lambda y)| = |f_x(x)| = \|x\|$, so $x \perp y$.

Conversely, by its definition, $x \perp y$ for nonzero $x, y \in \mathcal{X}$ implies that a linear functional $f\colon \operatorname{span}\{x, y\} \to \mathbb{F}$, defined by $f(x) = \|x\|$ and $f(y) = 0$, is norm-one and can be extended by the Hahn–Banach theorem to a supporting functional at x which annihilates y. □

Corollary 1.5 *In inner product spaces Birkhoff–James orthogonality is equivalent to the usual one.*

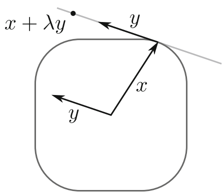

Fig. 1 Birkhoff–James orthogonality: $x \perp y$

Proof Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be an inner product space and $\overline{\mathcal{H}}$ its completion. By the Riesz representation theorem and the Bunyakovsky–Cauchy–Schwarz inequality—the part which claims that equality can hold only for linearly dependent vectors [42, p. 4], the only supporting functional at a normalized vector $x \in \mathcal{H} \subseteq \overline{\mathcal{H}}$ is given by $\langle\cdot, x\rangle$. □

From the equivalence stated in Proposition 1.4, the homogeneity of B-J orthogonality is evident; it is also clear that in every two-dimensional plane Π and for every $x \in \Pi$ there exists $y \in \Pi$ with $x \perp y$. That is, similarly to isosceles, B-J orthogonality is always nontrivial. It is easily seen that B-J orthogonality is in general nonsymmetric, that is, $x \perp y$ does not always imply $y \perp x$ (e.g., in the space $(\mathbb{F}^2, \|\cdot\|_\infty)$, where $\|\cdot\|_\infty$ is the maximum norm, we have $(1,1) \perp (1,0)$ but $(1,0) \not\perp (1,1)$). We will discuss the properties of B-J orthogonality more thoroughly in Sect. 2.

Proposition 1.4 yields a simple procedure to find all vectors y in a normed space which are B-J orthogonal to a given *normalized* x, namely:

$$N_x := \{y \in \mathcal{X};\quad x \perp y\} = \bigcup_{\substack{f\in\mathcal{X}^* \\ \|f\|=f(x)=1}} \ker f.$$

But one can do slightly better: Singer [82] (see also his monograph [84, Lemma 1.3, p. 169]) observed that if $\mathcal{M} \subseteq \mathcal{X}$ is an n-dimensional subspace of a real normed space $\mathcal{X}$, and f is a bounded linear functional on $\mathcal{X}$ such that $\|f|_{\mathcal{M}}\| = 1$, then f coincides on $\mathcal{M}$ with some convex combination of at most n extremal points of the dual norm. This works also for complex normed spaces, except that in this case one requires at most $2n - 1$ extremal points of the dual norm. By taking $\mathcal{M} = \operatorname{span}\{x, y\}$, one obtains the following result, which was the starting point of Li and Schneider's [60] investigation into B-J orthogonality on rectangular matrices equipped with Schatten p-norm (see also the next chapter).

Proposition 1.6 (Cf. [60, Proposition 2.1]) *Let $\mathcal{X}$ be a normed space over $\mathbb{F}$ and let $x, y \in \mathcal{X}$ be linearly independent. Then the following are equivalent:*

(i) $x \perp y$.
(ii) *There exist h ($h \le 2$ if $\mathbb{F} = \mathbb{R}$, respectively, $h \le 3$ if $\mathbb{F} = \mathbb{C}$) extremal points $f_1, \dots, f_h$ in unit sphere of the dual norm and h numbers $\lambda_1, \dots, \lambda_h > 0$ with $\sum_{i=1}^h \lambda_i = 1$ such that*

$$\sum_{i=1}^{h} \lambda_i f_i(y) = 0 \quad \textit{and} \quad f_1(x) = \cdots = f_h(x) = \|x\|.$$

In [13] a symmetrized version of B-J orthogonality was introduced. Two vectors $x, y \in \mathcal{X}$ are *mutually B-J orthogonal* (denoted by $x \perp\!\!\!\perp y$) if $x \perp y$ and $y \perp x$.

It is immediate that Roberts orthogonality is finer than isosceles. It is also finer than mutual B-J:

Proposition 1.7 *Let $\mathcal{X}$ be a normed space over $\mathbb{F}$. If $x, y \in \mathcal{X}$ are Roberts orthogonal, then they are also isosceles and mutual B-J orthogonal.*

Proof For any $\lambda \in \mathbb{F}$ it holds

$$2\|x\| = \|(x + \lambda y) + (x - \lambda y)\| \leq \|x + \lambda y\| + \|x - \lambda y\| = 2\|x + \lambda y\|,$$

so $x \perp y$. Since Roberts orthogonality is symmetric, we similarly obtain $y \perp x$, and thus $x \perp\!\!\!\perp y$. □

The proposition above is required to prove Proposition 1.1. We remark that James in his paper [45] mentions that this is a trivial consequence of Proposition 1.3.

Proof of Proposition 1.1 Choose any nonzero $x \in \mathcal{X}$ and a two-dimensional plane $\Pi \subseteq \mathcal{X}$, $x \in \Pi$. By the assumptions of Proposition 1.1, there exists a nonzero $y \in \Pi$ such that $x \perp_R y$. We now show that $z = \alpha x + \beta y \in \Pi \setminus \{0\}$ and $x \perp_I z$ imply $z \parallel y$. By definition of isosceles orthogonality,

$$\|\beta y + (1 + \alpha)x\| = \|x + z\| = \|x - z\| = \|\beta y - (1 - \alpha)x\|.$$

Since $x \perp_R y$ implies $\beta y \perp_R x$, we also have

$$\|\beta y - (1 + \alpha)x\| = \|\beta y + (1 + \alpha)x\| = \|\beta y - (1 - \alpha)x\| = \|\beta y + (1 - \alpha)x\|.$$

Assume that $\alpha \neq 0$. Then $\beta y \pm (1 + \alpha)x$ and $\beta y \pm (1 - \alpha)x$ are at least three distinct points on one line which have the same norm. Thus, the norm's unit sphere S contains a nontrivial line segment inside Π. Consider a maximal line segment I of $S \cap \Pi$ and let $w \in I$ be interior point which is not the middle of I. We show below that for each nonzero $u \in \Pi$ we have $w \not\perp_R u$. This violates the assumptions of Proposition 1.1. Thus, $\alpha = 0$ and hence $z \in \mathbb{R}y$. This shows that, on an arbitrary two-dimensional space Π, isosceles orthogonality implies Roberts orthogonality, so they are equivalent, and in particular, isosceles orthogonality is homogeneous. It remains to apply Proposition 1.3.

To finish the proof, assume that $w \perp_R u$. Then $w \perp u$ (in B-J sense), so, by Proposition 1.4, u is parallel to I. However, since w is not the middle of I, there exists $\lambda \in \mathbb{R}$ such that $w + \lambda u \in I$ but $w - \lambda u \notin I$. Therefore, $\|w + \lambda u\| \neq \|w - \lambda u\|$, so $w \not\perp_R u$, a contradiction. □

1.4 A Plethora of Orthogonalities

There are many additional possibilities to define orthogonality in general normed spaces which on real inner product spaces agree with the usual one. We list a few of them; more detailed study can be found in survey papers [3] and [5]. One possibility is Pythagorean orthogonality introduced in [47] by $x \perp_P y$ if

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

Day [31] obtained another nice characterization of inner product spaces based on the (non)equivalence between Pythagorean and isosceles orthogonalities.

Proposition 1.8 (Cf. [31, Theorems 5.1 and 5.2]) *The following are equivalent for a normed space $\mathcal{X}$.*

(i) *$\mathcal{X}$ is an inner product space.*
(ii) *Isosceles orthogonality implies Pythagorean orthogonality.*
(iii) *Pythagorean orthogonality implies isosceles orthogonality.*

Sketch of the Proof (i) $\Longrightarrow$ (ii) and (iii). Given any $x, y \in \mathcal{X}$, both $x \perp_I y$ and $x \perp_P y$ are equivalent to $\mathrm{Re}\langle x, y\rangle = 0$.

(ii) $\Longrightarrow$ (i). Choose any normalized $x, y \in \mathcal{X}$. Then $(x + y) \perp_I (x - y)$, so by the assumptions also $(x + y) \perp_P (x - y)$, that is,

$$4 = \|2x\|^2 = \|(x + y) + (x - y)\|^2 = \|x + y\|^2 + \|x - y\|^2.$$

We first assume that $\mathcal{X}$ is a real normed space. It can be shown [31, Theorem 2.2] that a two-dimensional real normed space $\mathcal{Y}$ is an inner product space (equivalently, the norm's unit sphere is an ellipse) if and only if $\|x + y\|^2 + \|x - y\|^2 = 4$ for all normalized vectors $x, y \in \mathcal{Y}$.

Consider now two arbitrary vectors $u, v \in \mathcal{X}$ and a two-dimensional subspace $\mathcal{Y} \subseteq \mathcal{X}$ which contains them. Then $\mathcal{Y}$ is an inner product space, so $\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2)$, and thus parallelogram identity holds in $\mathcal{X}$. Therefore, by Jordan–von Neumann characterization [49], $\mathcal{X}$ is an inner product space.

Assume now that $\mathcal{X}$ is a complex normed space. We have already proved that $\mathcal{X}$ is a real inner product space, so it remains to apply Proposition 1.2.

(iii) $\Longrightarrow$ (i). Assume that $x \perp_P y$, that is, $\|x+y\|^2 = \|x\|^2+\|y\|^2$. Then we also have $x \perp_I y$, so $\|x+y\| = \|x-y\|$, and thus $\|x-y\|^2 = \|x\|^2+\|y\|^2$, which means that $x \perp_P (-y)$. By [31, Lemma 5.3], $\mathcal{X}$ is a real inner product space whenever the conditions $x \perp_P y$ and $x \perp_P (-y)$ are equivalent. Since Pythagorean orthogonality does not depend on the field $\mathbb{F}$, for a complex normed space $\mathcal{X}$ it remains to apply Proposition 1.2. □

In 1957 Singer [83] introduced a homogeneous version of isosceles orthogonality by $x \perp_S y$ if either $\|x\| \cdot \|y\| = 0$ or $\frac{x}{\|x\|} \perp_I \frac{y}{\|y\|}$. It is additive in dimension two. The question of its additivity in general was finally put to the rest by Lin [61] who found that in higher dimensions Singer orthogonality is additive if and only if the norm is induced by the inner product.

In 1984 Alonso [2] introduced *area orthogonality* in real normed spaces by $x \perp_A y$ if either $\|x\| \cdot \|y\| = 0$ or x, y are linearly independent and the lines $\mathbb{R}x$, $\mathbb{R}y$ divide the norm's unit ball of the plane $\mathrm{span}\{x, y\} \simeq \mathbb{R}^2$ into four parts of equal area; together with Benítez they obtained yet another interesting characterization of real inner product spaces, whose proof we omit, as follows:

Proposition 1.9 (Cf. [4, Proposition 3]) *The following are equivalent for a real normed space $\mathcal{X}$:*

(i) *$\mathcal{X}$ is an inner product space.*
(ii) *B-J orthogonality implies area orthogonality.*
(iii) *Area orthogonality implies B-J orthogonality.*

An interesting equivalent description of orthogonality in inner product spaces is presented in [40]. It does not rely neither on the inner product nor on the norm but rather on isometries of the norm. To state it properly, we require the following terminology. Given a vector x, denote by $\vec{x} := \{\lambda x;\quad \lambda \geq 0\}$ the closed ray in direction of x. Observe that $\vec{0} = \{0\}$ and that, for x nonzero, $\vec{x} \setminus \{0\} = \{\lambda x;\quad \lambda > 0\}$ is an open ray.

Proposition 1.10 (Cf. [40]) *Two vectors x, y in a real inner product space $\mathcal{H}$ are orthogonal (in a classical sense) if and only if there exists a rigid motion, i.e., an isometry T of $\mathcal{H}$ such that*

$$\left(\overrightarrow{Tx} \cup \overrightarrow{Ty} \cup \overrightarrow{x}\right) \setminus (\overrightarrow{y} \setminus \{0\}) = \overrightarrow{x} \cup \overrightarrow{(-x)}. \tag{3}$$

Two vectors x, y in a complex inner product space $\mathcal{H}$ are orthogonal if and only if there exists an isometry T for which, simultaneously, (3) and

$$\left(\overrightarrow{T(ix)} \cup \overrightarrow{Ty} \cup \overrightarrow{(ix)}\right) \setminus (\overrightarrow{y} \setminus \{0\}) = \overrightarrow{(ix)} \cup \overrightarrow{(-ix)} \tag{3'}$$

are satisfied.

The condition in (3) is clearly homogeneous and a moment's thought reveals that an isometry T satisfies (3) if and only if the isometry $T' = -T$ satisfies (3) with x and y swapped. Hence, the relation based on (3) is always symmetric. However, in general normed spaces it might be trivial, i.e., if vectors x, y satisfy (3), then at least one of them must be zero. This can happen when the isometry group consists of scalar operators only. Such normed spaces do exist: Davis [29] was the first to construct separable Banach space over reals such that its only isometries (surjective or not) are $\pm I$. We also refer to Gordon and Loewy [38, Theorem 3.1] who, by answering a question of Lindenstrauss, showed much more: Any finite subgroup of (real) orthogonal n-by-n matrices which contains $-I$ is the isometry group of some norm on $\mathbb{R}^n$. Moreover, any real or complex Banach space can be renormed, so that its isometry group consists of scalar operators only, see Jarosz [48].

The following proposition presents several other nice criteria for a normed space to be an inner product space.

Proposition 1.11 *Let $\mathcal{X}$ be a real or complex normed space. Then the following conditions are equivalent:*

(i) *$\mathcal{X}$ is an inner product space.*
(ii) $\|x\| = \|y\| = 1$ *implies* $\|\alpha x+\beta y\| = \|\beta x+\alpha y\|$ *for all* $\alpha, \beta \in \mathbb{R}$ (Ficken [36]).
(iii) *There exists some* $\gamma \in \mathbb{R}$, $\gamma \neq 0, 1$, *such that* $\|x\| = \|y\| = 1$ *implies* $\|x + \gamma y\| = \|\gamma x + y\|$ (Lorch [62]).
(iv) $\|x\| = \|y\| = 1$ *and* $x \perp y$ *imply* $(x + y) \perp (x - y)$ (Oman [70, Theorem 5.21]).

The interested reader is referred to a monograph by Amir [7] for hundreds of additional conditions on the norm which force it to be induced by an inner product.

2 Properties of B-J Orthogonality

2.1 Gateaux Derivatives and Complex Case Related to Real Orthogonality

A normalized vector $x \in (\mathcal{X}, \|\cdot\|)$ is said to be a smooth point of the norm if it has a unique supporting functional; if every normalized vector is a smooth point then one says that the norm (or, more colloquial, the space) is smooth. Similarly to Proposition 1.2, each complex normed space $(\mathcal{X}, \|\cdot\|)$ can be regarded as a real normed space by restricting the scalars. There is an elegant relationship between smooth points in $(\mathcal{X}, \|\cdot\|)$ and the real counterpart $(\mathcal{X}, \|\cdot\|_{\mathbb{R}})$ akin Proposition 1.2:

Proposition 2.1 *A normalized vector x in a complex normed space $(\mathcal{X}, \|\cdot\|)$ is a smooth point if and only if it is a smooth point in $(\mathcal{X}, \|\cdot\|_{\mathbb{R}})$.*

Sketch of the Proof Each complex-linear functional $f\colon \mathcal{X} \to \mathbb{C}$ is uniquely determined from its real part $(\mathrm{Re}\, f)\colon \mathcal{X} \to \mathbb{R}$; the correspondence being $f(x) = (\mathrm{Re}\, f)(x) - i(\mathrm{Re}\, f)(ix)$; moreover, $\|\mathrm{Re}\, f\| = \|f\|$. Thus, if $x \in (\mathcal{X}, \|\cdot\|_{\mathbb{R}})$ has two complex-linear supporting functionals, then their real parts are two different real-linear supporting functionals for x. Conversely, let $x \in (\mathcal{X}, \|\cdot\|)$ have two different real-linear supporting functionals g_1, g_2, and let $f_k(z) := g_k(z) - ig_k(iz)$. Since $|g_k(x) - ig_k(ix)| \leq \|f_k\| = \|g_k\| = g_k(x)$, we have $g_1(ix) = g_2(ix) = 0$, and thus f_1, f_2 are two different complex-linear supporting functionals for x. □

It is immediate from Propositions 1.4 and 2.1 that, in smooth complex normed spaces, $x \perp y$ if and only if $x \perp^{\mathbb{R}} y$ and $x \perp^{\mathbb{R}} (iy)$.

It is well-known that supporting functionals are intimately related to the (one-sided) derivative of the norm. Let $x, y \in \mathcal{X}$. Notice that $t \mapsto \|x + ty\|$ is a convex function of a real variable t and as such has left and right derivatives at each point [77, Theorem 23.1]. In particular, there exists the norm's directional derivative at x in direction of y:

$$D_-(x; y) := \lim_{t \nearrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad \text{and} \quad D_+(x; y) := \lim_{t \searrow 0} \frac{\|x + ty\| - \|x\|}{t}.$$

Observe that $D_-(x; y) = -D_+(x; -y)$. We show next their geometric significance. It will be beneficial to denote by $J(x)$ the set of all supporting functionals at a vector x. Recall that $\phi \in J(x)$ if and only if $\phi(x) = \|x\|$ and $\|\phi\| = 1$.

Proposition 2.2 (Cf. [33, Lemma 1 and Theorem 15]) *Let x be a nonzero vector in $\mathcal{X}$. Then*

$$D_-(x; y) = \min_{\phi \in J(x)} \operatorname{Re} \phi(y),$$

$$D_+(x; y) = \max_{\phi \in J(x)} \operatorname{Re} \phi(y).$$

Sketch of the Proof If $\phi \in J(x)$, then $\|\phi\| = \|\operatorname{Re} \phi\| = 1$ and $\phi(x) = \|x\|$. So, for $t \in \mathbb{R}$ sufficiently close to zero, $\|x+ty\| \geq |\operatorname{Re} \phi(x+ty)| = \|x\|+t \operatorname{Re} \phi(y)$. After dividing by t and keeping an eye on its sign we get $\lim_{t \nearrow 0} \frac{\|x+ty\|-\|x\|}{t} \leq \operatorname{Re} \phi(y) \leq \lim_{t \searrow 0} \frac{\|x+ty\|-\|x\|}{t}$.

To show that both inequalities are achieved, define two real-linear functionals on $\operatorname{span}_{\mathbb{R}}\{x, y\}$ by $\psi_{\pm}(\alpha x+\beta y) := \alpha\|x\|+\beta D_{\pm}(x; y), (\alpha, \beta \in \mathbb{R})$. Since $t \mapsto \|x+ty\|$ is convex, the function $\beta \mapsto \frac{\|x+\beta y\|-\|x\|}{\beta}$ increases to $D_-(x; y)$ as $\beta \nearrow 0$ and decreases to $D_+(x; y)$ as $\beta \searrow 0$, respectively. Thus, with $\beta \geq 0$ we have

$$\psi_+(x + \beta y) = \|x\| + \beta D_+(x; y) \leq \|x\| + (\|x + \beta y\| - \|x\|) = \|x + \beta y\|$$

with a similar derivation and the same conclusion also for $\beta < 0$. Moreover, $\psi_+(-x - \beta y) = -\|x\| - \beta D_+(x; y) \leq -\|x\| + (\|x - \beta y\| - \|x\|) = \|x - \beta y\| - \|2x\| \leq \|-x - \beta y\|$. Thus, $\psi_+(\alpha x + \beta y) \leq \|\alpha x + \beta y\|$ $(\alpha, \beta \in \mathbb{R})$. Combined with $\psi_+(x) = \|x\|$, we get $\|\psi_+\| = 1$. By the Hahn–Banach theorem, we can enlarge the domain of ψ_+ to $\mathcal{X}$ without affecting its norm, and (if $\mathcal{X}$ is a complex space) make it into a complex-linear functional $\phi_+ \in J(x)$ with $\operatorname{Re} \phi_+ = \psi_+$.

One argues similarly with ψ_- to construct $\phi_- \in J(x)$ with $\operatorname{Re} \phi_-(y) = \psi_-(y) = D_-(x; y)$. □

Proposition 2.3 (Cf. [47, Theorem 3.1]) *Let $\mathcal{X}$ be a real normed space, $x, y \in \mathcal{X}$. Then $x \perp (y - \alpha x)$ if and only if $D_-(x; y) \leq \alpha\|x\| \leq D_+(x; y)$.*

Sketch of the Proof Apply Propositions 1.4 and 2.2. □

Definition 2.4 Let $(\mathcal{X}, \|\cdot\|)$ be a real or complex normed space. The norm is *Gateaux differentiable* at a point x if for all $y \in \mathcal{X}$ there exists $D(x; y) = \lim_{\mathbb{R} \ni t \to 0} \frac{\|x+ty\|-\|x\|}{t}$.

Clearly, the norm is Gateaux differentiable at $x \in \mathcal{X}$ if and only if $D_+(x; y) = D_-(x; y)$ for all $y \in \mathcal{X}$. In this case the Gateaux derivative induces a bounded linear functional $y \mapsto D(x; y)$, see [73, Corollary 4.7]. We present below another proof of this fact.

Proposition 2.5 (Cf. [18, part 3, ch. 1, §2, Proposition 2 and Remark 1]) *A normalized vector x in a real or complex normed space $\mathcal{X}$ is a smooth point if and only if the norm is Gateaux differentiable at x.*

Proof In view of Proposition 2.1, it is sufficient to consider only a real-linear normed space $\mathcal{X}$. Assume first that the norm is Gateaux differentiable at $x \in \mathcal{X}$, and consider any supporting functional ϕ at x. Then, by Proposition 2.2, for all $y \in \mathcal{X}$ we have

$$D(x; y) = D_-(x; y) \leq \phi(y) \leq D_+(x; y) = D(x; y),$$

so $\phi(y) = D(x; y)$. It follows that the Gateaux derivative $y \mapsto D(x; y)$ is the unique supporting functional at x, and thus x is a smooth point.

Assume now that x is not a smooth point, that is, there exist two different supporting functionals ϕ_1, ϕ_2 at x. Then there exists some $y \in \mathcal{X}$ such that $\phi_1(y) < \phi_2(y)$. Hence it follows from Proposition 2.2 that

$$D_-(x; y) \leq \phi_1(y) < \phi_2(y) \leq D_+(x; y),$$

so $D_-(x; y) \neq D_+(x; y)$, and the norm is not Gateaux differentiable at x. □

Since the norm is a convex function, it follows from [77, Theorem 25.2] that the norm on a real finite-dimensional normed space $\mathcal{X} = (\mathbb{R}^n, \|\cdot\|)$ is Gateaux differentiable if and only if it is differentiable at all nonzero points of $\mathcal{X}$. In this case the unique supporting functional f_x at a normalized vector $x = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ is given by

$$f_x\colon y \to \langle y, \partial\|x\|\rangle, \tag{4}$$

where $\langle u, v\rangle := uv^t$ is the standard scalar product of row vectors in $\mathbb{R}^n$ and $\partial\|x\| := \left(\frac{\partial\|\cdot\|}{\partial\xi_1}, \dots, \frac{\partial\|\cdot\|}{\partial\xi_n}\right)(x)$ is the norm's differential evaluated at vector x.

A complex normed space $(\mathbb{C}^n, \|\cdot\|)$ can be regarded as a real normed space $(\mathbb{R}^{2n}, \|\cdot\|)$ by restricting the scalars. We write $x = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$ and consider the real and complex components of $\xi_k = \operatorname{Re}\xi_k + i\operatorname{Im}\xi_k$ as independent real variables. The $\mathbb{R}$-linear supporting functional g_x which depends on $2n$ real variables $(\operatorname{Re}\xi_1, \operatorname{Im}\xi_1, \dots, \operatorname{Re}\xi_n, \operatorname{Im}\xi_n)$ is the real part of a $\mathbb{C}$-linear functional given by $f_x\colon z \mapsto g_x(z) - ig_x(iz)$. By the proof of Proposition 2.1, f_x is a $\mathbb{C}$-linear supporting functional at a normalized vector x. If the norm on $\mathcal{X}$ is smooth, then it follows easily that f_x is given by

$$f_x\colon z \mapsto \left\langle z, \left(\frac{\partial\|\cdot\|}{\partial\,\mathrm{Re}\,\xi_1} + i\frac{\partial\|\cdot\|}{\partial\,\mathrm{Im}\,\xi_1}, \dots, \frac{\partial\|\cdot\|}{\partial \mathrm{Re}\,\xi_n} + i\frac{\partial\|\cdot\|}{\partial\,\mathrm{Im}\,\xi_n}\right)(x)\right\rangle,$$

where $\langle u, v\rangle := uv^*$ is the standard scalar product of row vectors in $\mathbb{C}^n$. This can be simplified if one introduces complex partial derivatives

$$\frac{\partial\|\cdot\|}{\partial\xi_k} := \frac{\partial\|\cdot\|}{\partial\mathrm{Re}\,\xi_k} - i\frac{\partial\|\cdot\|}{\partial\,\mathrm{Im}\,\xi_k} \quad \text{and} \quad \frac{\partial\|\cdot\|}{\partial\overline{\xi_k}} := \frac{\partial\|\cdot\|}{\partial\mathrm{Re}\,\xi_k} + i\frac{\partial\|\cdot\|}{\partial\,\mathrm{Im}\,\xi_k};$$

then we can identify the $\mathbb{C}$-linear supporting functional f_x with the norm's complex conjugate differential

$$\begin{aligned}\partial_{\mathbb{C}}\|x\| &:= \left(\frac{\partial\|\cdot\|}{\partial\overline{\xi_1}}, \dots, \frac{\partial\|\cdot\|}{\partial\overline{\xi_n}}\right)(x)\\ &= \left(\frac{\partial\|\cdot\|}{\partial\,\mathrm{Re}\,\xi_1} + i\frac{\partial\|\cdot\|}{\partial\,\mathrm{Im}\,\xi_1}, \dots, \frac{\partial\|\cdot\|}{\partial\,\mathrm{Re}\,\xi_n} + i\frac{\partial\|\cdot\|}{\partial\,\mathrm{Im}\,\xi_n}\right)(x).\end{aligned} \tag{5}$$

We refer an interested reader to a monograph [6] for additional results based on (one sided) norm's derivative $D_\pm$. In particular, one can find characterizations of continuous maps $T\colon (\mathbb{R}^2, \|\cdot\|) \to (\mathbb{R}^2, \|\cdot\|)$ satisfying $D_\pm(Tx; Ty) = D_\pm(x; y)$ (a generalization to Wigner's unitary-antiunitary theorem [92], see also [37, 66]) as well as additional characterizations of inner product spaces in terms of specific triangle points.

2.2 *Symmetry, Additivity and Uniqueness of B-J Orthogonality*

We will need two additional properties of orthogonality which hold trivially for orthogonality in inner product spaces.

Left uniqueness For any $x, y \in \mathcal{X}$, $x \neq 0$, there exists at most one $\alpha \in \mathbb{F}$ such that $(\alpha x + y) \perp x$.

Right uniqueness For any $x, y \in \mathcal{X}$, $x \neq 0$, there exists at most one $\alpha \in \mathbb{F}$ such that $x \perp (\alpha x + y)$.

Theorem 2.6 (Cf. [46, Theorems 1 and 2], [47, Theorems 4.1–4.3 and 5.1]) *Let $\mathcal{X}$ be a real or complex normed space.*

(i) *B-J orthogonality in $\mathcal{X}$ is left unique if and only if the norm on $\mathcal{X}$ is strictly convex.*
(ii) *B-J orthogonality in $\mathcal{X}$ is right unique if and only if it is right additive if and only if the norm on $\mathcal{X}$ is smooth.*

(iii) *If* $\dim \mathcal{X} \geq 3$ *then B-J orthogonality in* $\mathcal{X}$ *is symmetric if and only if* $\mathcal{X}$ *is an inner product space.*
(iv) *If* $\dim \mathcal{X} = 2$ *then B-J orthogonality in* $\mathcal{X}$ *is left additive if and only if the norm on* $\mathcal{X}$ *is strictly convex.*
(v) *If* $\dim \mathcal{X} \geq 3$ *then B-J orthogonality in* $\mathcal{X}$ *is left additive if and only if* $\mathcal{X}$ *is an inner product space.*

Sketch of the Proof

(i) Assume first that the norm is not strictly convex, that is, the norm's unit sphere contains some nontrivial line segment. Then there exist two distinct normalized vectors $x, y \in \mathcal{X}$ such that $\|(1-\lambda)x + \lambda y\| = 1$ for all $\lambda \in (-\varepsilon, 1+\varepsilon)$, where $\varepsilon > 0$ is small enough. Consider an arbitrary supporting functional f at x, that is, $f(x) = \|f\| = 1$. Then

$$|1 + \lambda(f(y) - 1)| = |f((1-\lambda)x + \lambda y)| \leq \|f\| \cdot \|(1-\lambda)x + \lambda y\| = 1$$

for any $\lambda \in (-\varepsilon, \varepsilon)$, so $f(y) = 1$. Hence $f((1-\lambda)x + \lambda y) = 1$ for all $\lambda \in (-\varepsilon, 1+\varepsilon)$, and thus f is also supporting at $(1-\lambda)x+\lambda y$. Since $f(x-y) = 0$, we obtain from Proposition 1.4 that $((1-\lambda)x + \lambda y) \perp (x - y)$ for all $\lambda \in (-\varepsilon, 1+\varepsilon)$. Therefore, B-J orthogonality in $\mathcal{X}$ is not left unique.

Assume now that B-J orthogonality in $\mathcal{X}$ is not left unique. Then there exists a two-dimensional subspace $\mathcal{Y} \subseteq \mathcal{X}$, two linearly independent normalized vectors $x, y \in \mathcal{Y}$ and a normalized vector $z \in \mathcal{Y}$ such that $x \perp z$ and $y \perp z$. Consider two supporting functionals $f_x, f_y : \mathcal{Y} \to \mathbb{F}$ at x and y, respectively, such that $f_x(z) = f_y(z) = 0$. Since $\mathcal{Y}$ is two-dimensional, f_y is a scalar multiple of f_x, so we may assume that $f_x = f_y = f$. Hence

$$1 = f((1-\lambda)x + \lambda y) \leq \|f\| \cdot \|(1-\lambda)x + \lambda y\| \leq 1$$

for all $\lambda \in [0, 1]$. Therefore, the norm's unit sphere contains a line segment $[x, y]$, and the norm on $\mathcal{X}$ is not strictly convex.

(ii) We first show that if B-J orthogonality is right unique, then for any nonzero $x \in \mathcal{X}$ there exists a unique supporting functional f_x at x. Assume from the contrary that f_1 and f_2 are two distinct supporting functionals for some normalized vector $x \in \mathcal{X}$. Then there exists $y \in \mathcal{X}$ such that $f_1(y) \neq f_2(y)$. We have $f_1(y - f_1(y)x) = 0$, so $x \perp (y - f_1(y)x)$. Similarly, $f_2(y - f_2(y)x) = 0$, so $x \perp (y - f_2(y)x)$. Hence B-J orthogonality in $\mathcal{X}$ is not right unique.

It follows from Proposition 1.4 that uniqueness of the supporting functional f_x for any nonzero $x \in \mathcal{X}$ implies right additivity of B-J orthogonality in $\mathcal{X}$. We next show that right additivity implies right uniqueness.

Assume that B-J orthogonality is right additive and that $x \perp (\alpha x + y)$, $x \perp (\beta x + y)$ for some $\alpha, \beta \in \mathbb{F}$ and $x, y \in \mathcal{X}$. Then $x \perp -(\beta x + y)$, so $x \perp ((\alpha x + y) - (\beta x + y)) = (\alpha - \beta)x$. This is only possible for $\alpha - \beta = 0$, so B-J orthogonality in $\mathcal{X}$ is right unique.

(iii) Let $\mathcal{Y}$ be an arbitrary three-dimensional subspace of $\mathcal{X}$, and consider any two elements $x, y \in \mathcal{Y}$. Let f be a norm-one linear functional on $\mathcal{Y}$ such that $\mathcal{Z} = \operatorname{span}\{x, y\} \subseteq \ker f$. There exists a normalized vector $z \in \mathcal{Y}$ such that $f(z) = \|f\|$. By Proposition 1.4, $z \perp \mathcal{Z}$. If B-J orthogonality is symmetric, then we also have $\mathcal{Z} \perp z$.

Hence, if a projection $P: \mathcal{Y} \to \mathcal{Z}$ is defined by $w = P(w) + k_w z$, $k_w \in \mathbb{F}$, then $\|P(w)\| \leq \|w\|$ for all $w \in \mathcal{Y}$. Therefore, $\|P\| = 1$. Now, if $\mathcal{X}$ is a real normed space, then, by Kakutani's result [50, Theorem 4], the existence of such a normalized projection implies that the norm's unit ball in $\mathcal{Y}$ is an ellipsoid, so $\mathcal{Y}$ is an inner product space. The same conclusion holds true also if $\mathcal{X}$ is a complex normed space as shown by Bohnenblust [26, Theorem A] (but see also [7, §12] for a modern treatment). Since $\mathcal{Y}$ is arbitrary, it follows from Jordan–von Neumann characterization [49] that $\mathcal{X}$ is an inner product space.

(iv) If $\dim \mathcal{X} = 2$ then B-J orthogonality is left additive if and only if it is left unique. Then the statement follows from (i).

(v) Consider two arbitrary linearly independent elements $y_1, y_2 \in \mathcal{X}$. If f_i is any supporting functional at y_i and $\mathcal{Y}_i = \ker f_i$, then $y_i \perp \mathcal{Y}_i$, $i \in \{1, 2\}$. Let $\mathcal{Y} = \mathcal{Y}_1 \cap \mathcal{Y}_2$. If B-J orthogonality is left additive, then $\mathcal{Z} \perp \mathcal{Y}$, where $\mathcal{Z} = \operatorname{span}\{y_1, y_2\}$. Hence $\mathcal{Z} \cap \mathcal{Y} = 0$, so $\dim \mathcal{Z} = 2$ and $\operatorname{codim} \mathcal{Y} \leq 2$ imply that $\mathcal{X} = \mathcal{Z} \oplus \mathcal{Y}$. We can now define a norm-one projection $P: \mathcal{X} \to \mathcal{Z}$ along $\mathcal{Y}$. It follows again from Kakutani's [50, Theorem 4] and Bohnenblust's [26, Theorem A] results that $\mathcal{X}$ is an inner product space. □

Note that Theorem 2.6(iii) does not hold for $\dim \mathcal{X} = 2$. In other words, there exist two-dimensional normed spaces in which B-J orthogonality is symmetric, however, the norm is not induced by an inner product. Such spaces are called *Radon planes* (see [5]), and their first examples are due to Birkhoff [22] and James [46, p. 561]. We provide a complete characterization of real Radon planes which was obtained by Day [31].

Remark 2.7 (Cf. [31, pp. 330–333]) Let $\mathcal{X}$ be a two-dimensional real normed space. Then B-J orthogonality in $\mathcal{X}$ is symmetric if and only if modulo a linear transformation its unit sphere $S_{\mathcal{X}}$ can be obtained by the following procedure:

(1) Choose any (auxiliary) two-dimensional real normed space $\mathcal{Y}$.
(2) Find any two normalized vectors $x, y \in \mathcal{Y}$ with $x \perp\!\!\!\perp y$. They exist by Proposition 2.13(i) below.
(3) Find supporting functionals $f_x, f_y \in \mathcal{Y}^*$ with $f_x(x) = f_y(y) = 1$ and $f_x(y) = f_y(x) = 0$.
(4) Choose a coordinate system in $\mathcal{Y}$ with x and y at $(1, 0)$ and $(0, 1)$, correspondingly. Similarly, choose a coordinate system in $\mathcal{Y}^*$ with f_x and f_y at $(0, 1)$ and $(-1, 0)$.
(5) Set the first and the third quadrants of $S_{\mathcal{X}}$ equal to the first and the third quadrants of $S_{\mathcal{Y}}$, set the second and the fourth quadrants of $S_{\mathcal{X}}$ equal to the second and the fourth quadrants of $S_{\mathcal{Y}^*}$.

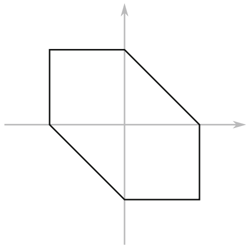

Fig. 2 Real Radon plane based on ℓ_1–ℓ_∞ norm

Example If one takes $\mathcal{Y} = (\mathbb{R}^2, \|\cdot\|_p)$ and $\mathcal{Y}^* = (\mathbb{R}^2, \|\cdot\|_q)$ with $p, q \in [1, \infty]$ and $\frac{1}{p} + \frac{1}{q} = 1$ in Remark 2.7, then the resulting Radon plane $\mathcal{X}$ is the one which was constructed by James (Fig. 2).

Some examples of complex Radon planes were given by Oman [70, pp. 43–48, Constructions III and V], though he did not obtain a complete characterization for them. His constructions use the following result:

Proposition 2.8 (Cf. [70, Theorem 3.4]) *A two-dimensional real or complex normed space $\mathcal{X}$ is a Radon plane if and only if there exists an isometry $\Phi\colon \mathcal{X} \to \mathcal{X}^*$ such that $\Phi(x)(x) = 0$, $x \in \mathcal{X}$.*

Consider a general real normed space. How far is isosceles orthogonality from satisfying B-J condition? To quantify this we might try by reformulating into: how deep inside a norm's unit sphere can a line go if it contains two points at the same distance through the origin and placed symmetrically relative to a given normalized vector x? The answer was given by James [45, Theorem 4.2]: If $\|x + y\| = \|x - y\|$, then $\frac{1}{2} < \inf_{\lambda\in\mathbb{R}} \frac{\|x+\lambda y\|}{\|x\|}$. For real Radon planes this estimate was improved by Mizuguchi, [65, Theorem 2.9] into $\frac{8}{9} \le \inf_{\lambda\in\mathbb{R}} \frac{\|x+\lambda y\|}{\|x\|}$; the equality holds for some nonzero $x \perp_I y$ if and only if the norm is hexagonal (see [65, Theorem 2.10]).

2.3 Mutual B-J Orthogonality

We have already mentioned that if $\dim \mathcal{X} \ge 2$, then for every nonzero $x \in \mathcal{X}$ there is a nonzero $y \in \mathcal{X}$ such that $x \perp y$. However, as the following example shows, it is possible that for some nonzero x there is no nonzero y with $x \perp\!\!\!\perp y$. By Corollary 2.10 below, this can happen only in two-dimensional spaces.

Example Let $\mathcal{X} = (\mathbb{F}^2, \|\cdot\|_1)$. Then the only supporting functional at $x = (\frac{1}{3}, \frac{2}{3}) \in \mathcal{X}$ is given by $f(\xi_1, \xi_2) = \xi_1 + \xi_2$. By Proposition 1.4, if $y = (\zeta_1, \zeta_2)$ is a normalized vector such that $x \perp y$, then $y \in \operatorname{Ker} f$, i.e., $\zeta_1 + \zeta_2 = 0$. We may assume that

$y = \left(\frac{1}{2}, -\frac{1}{2}\right)$. The only supporting functional at y is given by $g(\xi_1, \xi_2) = \xi_1 - \xi_2$, and since $x \notin \operatorname{Ker} g$, we conclude that $y \not\perp x$. Hence for this x there exists no nonzero vector y such that $x \perp\!\!\!\perp y$.

Theorem 2.9 (Cf. [13, Theorem 2.3]) *Let $n \in \mathbb{N}$, and let $\mathcal{X}$ be a real or complex normed space with $\dim \mathcal{X} \geq 2n + 1$. Then for any normalized vectors $x_1, \dots, x_n \in \mathcal{X}$ there exists a normalized vector $y \in \mathcal{X}$ such that $y \perp\!\!\!\perp x_k$ for all $k = 1, \dots, n$.*

Sketch of the Proof We are going to prove a stronger result, namely, we will find a normalized vector $y \in \mathcal{X}$ such that

$$\begin{cases} x_k \perp y & \text{for all } k = 1, \dots, n; \\ y \perp \sum_{k=1}^n \gamma_k x_k & \text{for all } \gamma_1, \dots, \gamma_n \in \mathbb{F}. \end{cases} \tag{6}$$

We can assume that $\dim \mathcal{X} = 2n + 1$, since we may pass from $\mathcal{X}$ to any $(2n+1)$-dimensional subspace of $\mathcal{X}$ which contains $x_1, \dots, x_n$ and look for y there. Moreover, it is enough to prove the theorem for the case when the norm on $\mathcal{X}$ is smooth. Indeed, if the norm on $\mathcal{X}$ is nonsmooth, we can construct a sequence of smooth norms $\|\cdot\|_{(m)}$ converging uniformly to $\|\cdot\|$ on compact subsets of $\mathcal{X}$ (see [32, p. 52–53] or [41, Theorem 2.10]), and for any $m \in \mathbb{N}$ choose a normalized (in $\|\cdot\|$ norm) vector $y_m \in \mathcal{X}$ which satisfies condition (6) with respect to the norm $\|\cdot\|_{(m)}$. Since the unit sphere in $\mathcal{X}$ is compact, passing to a subsequence if necessary, we may assume that $\lim_{m\to\infty} y_m = y$ for some $y \in \mathcal{X}$. This y satisfies condition (6) with respect to the original norm $\|\cdot\|$.

Let now $\mathcal{X}$ be a $(2n + 1)$-dimensional space with a smooth norm. For each $k \in \{1, \dots, n\}$ let f_k be a supporting functional at x_k, and

$$\mathcal{N} := \bigcap_{k=1}^{n} \ker f_k. \tag{7}$$

Then $\dim \mathcal{X} = 2n + 1$ implies $\dim \mathcal{N} \geq n + 1$, and each vector $z \in \mathcal{N}$ satisfies $x_k \perp z$ for all $k = 1, \dots, n$. In order to find $z \in \mathcal{N}$ which satisfies the second condition in (6), we split into two cases.

Case 1: $\mathcal{X}$ is a real normed space.
We identify $\mathbb{R}^{2n+1}$ with $\mathcal{X}$ and the standard basis vectors $e_1, \dots, e_{2n+1} \in \mathbb{R}^{2n+1}$ with a particular basis of $\mathcal{X}$ such that $e_1, \dots, e_{n+1} \in \mathcal{N}$. Hence each $z \in \mathbb{R}^{n+1} \oplus 0_n \subseteq \mathcal{X}$ (here 0_n denotes the zero vector in $\mathbb{R}^n$) satisfies $x_k \perp z$.

The gradient function of the norm $\|\cdot\|$

$$z \mapsto \partial \|z\|$$

exists everywhere on the slice of the norm's unit sphere

$$\Omega := \{z = (\zeta_1, \dots, \zeta_{n+1}, 0, \dots, 0) \in \mathbb{R}^{2n+1};\quad \|z\| = 1\}$$

and it is continuous there by [77, Theorem 25.5]. Also, $\|-z\| = \|z\|$ implies that

$$\partial\|-z\| = -\partial\|z\| \tag{8}$$

for all $z \in \Omega$. Note that the Euclidean unit n-sphere $\mathbf{S}^n \subset \mathbb{R}^{n+1}$ is homeomorphic to Ω (via the radial projection $r\colon (\zeta_1, \dots, \zeta_{n+1}) \mapsto \frac{(\zeta_1,\dots,\zeta_{n+1},0,\dots,0)}{\|(\zeta_1,\dots,\zeta_{n+1},0,\dots,0)\|}$). In particular, by Borsuk–Ulam's theorem applied to the function

$$z \mapsto \Big(\big\langle x_1, \partial\|r(z)\|\big\rangle, \dots, \big\langle x_n, \partial\|r(z)\|\big\rangle\Big) : \mathbf{S}^n \to \mathbb{R}^n, \tag{9}$$

which is odd (by (8)) and continuous, there exists a point $y \in \Omega \subseteq \mathbb{R}^{n+1} \oplus 0_n$ on the unit sphere where the function (9) is equal to 0_n. By Proposition 2.5, $g\colon x \mapsto \big\langle x, \partial\|y\|\big\rangle$ is a supporting functional at y, and for all $\gamma_1, \dots, \gamma_n \in \mathbb{R}$ we have $g(\sum_{k=1}^n \gamma_k x_k) = 0$, so $y \perp \sum_{k=1}^n \gamma_k x_k$. Since $y \in \mathcal{N}$, we finally obtain that y satisfies condition (6).

Case 2: $\mathcal{X}$ is a complex normed space.
Recall that $\dim_{\mathbb{C}} \mathcal{X} = 2n + 1$ and that each $z \in \mathcal{N}$ satisfies $x_k \perp z$ in a complex normed space $\mathcal{X}$. It remains to find an element in $\mathcal{N}$ which satisfies the second condition in (6). To do this, we consider $\mathcal{X}$ and $\mathcal{N}$ as real normed spaces. We may apply the construction from the real case to $2n$ vectors $x_1, ix_1, \dots, x_n, ix_n$, since $\dim_{\mathbb{R}} \mathcal{X} = 2(2n+1) \geq 2(2n) + 1$ and $\dim_{\mathbb{R}} \mathcal{N} \geq 2(n+1) \geq (2n) + 1$. This gives us a normalized vector $y \in \mathcal{N}$ such that, for all $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in \mathbb{R}$, it holds $y \perp \sum_{k=1}^n (\alpha_k x_k + \beta_k i x_k)$ in a real vector space $\mathcal{X}$. This is equivalent to $y \perp \sum_{k=1}^n \gamma_k x_k$ for all $\gamma_1, \dots, \gamma_n \in \mathbb{C}$ in a complex vector space $\mathcal{X}$, so y satisfies both conditions in (6). □

Example 2.4 in [13] demonstrates that the lower bound on $\dim \mathcal{X}$ in Theorem 2.9 is exact. We state two immediate corollaries.

Corollary 2.10 *Let $\mathcal{X}$ be a normed space over $\mathbb{F}$ with* $\dim \mathcal{X} \geq 3$. *Then for every normalized vector x there is a normalized vector y with $x \perp\!\!\!\perp y$.*

Corollary 2.11 *Let $\mathcal{X}$ be a normed space over $\mathbb{F}$ with* $\dim \mathcal{X} \geq 5$. *Then for every two normalized vectors $x, y \in \mathcal{X}$ there is a normalized vector $z \in \mathcal{X}$ with $x \perp\!\!\!\perp z \perp\!\!\!\perp y$.*

Another corollary to Theorem 2.9 is that in infinite-dimensional spaces we can find infinitely many pairwise B-J orthogonal nonzero vectors.

Corollary 2.12 *Let $\mathcal{X}$ be an infinite-dimensional normed space over $\mathbb{F}$. Then there exists an infinite number of nonzero vectors which are pairwise B-J orthogonal.*

Proof We construct a sequence $(x_n)_n$ of pairwise B-J orthogonal normalized vectors in $\mathcal{X}$ recursively. Let $x_1 \in \mathcal{X}$ be an arbitrary normalized vector. Assume now that we already have $x_1, \dots, x_n$. Since $\dim \mathcal{X} > 2n + 1$, Theorem 2.9 implies that there exists a normalized vector $x_{n+1} \in \mathcal{X}$ such that $x_{n+1} \perp\!\!\!\perp x_k$ for all $k = 1, \ldots, n$. □

In the case of finite-dimensional spaces, the lower bound on the number of pairwise B-J orthogonal nonzero vectors was obtained by Taylor [90, Theorem 2], see Proposition 2.13(i) below. This result is also known as Auerbach's lemma [16, 17]. The main idea of Taylor's proof [90, Theorem 1] was to maximize the volume of a parallelepiped inscribed into the norm's unit ball. It has also a nice geometrical consequence mentioned by Day [30, Theorem 4.1]. Namely, let **S** be a unit ball of some norm $\|\cdot\|$ in $\mathbb{R}^n$. Then **S** can be inscribed into an n-dimensional parallelepiped P centered at 0, so that the middle point of every hyperface of P belongs to **S**.

Proposition 2.13 (Cf. [90, Theorem 2], [13, Proposition 3.3, Theorem 3.5]) *Let $n \geq 2$ and let $\mathcal{X}$ be a real or complex n-dimensional normed space.*

(i) *There exist at least n pairwise B-J orthogonal normalized vectors in $\mathcal{X}$.*
(ii) *If the norm on $\mathcal{X}$ is smooth then there exist exactly n pairwise B-J orthogonal nonzero vectors in $\mathcal{X}$.*
(iii) *There exists a positive integer $\gamma_{n,\mathbb{F}}$ (it depends only on the dimension n and on the field $\mathbb{F}$ but not on the normed space $\mathcal{X}$) such that any set of pairwise B-J orthogonal nonzero vectors in $\mathcal{X}$ has at most $\gamma_{n,\mathbb{F}}$ members.*

3 Examples of B-J Orthogonality

3.1 B-J Orthogonality in $\mathbb{B}(\mathcal{H})$ and Beyond

How does B-J orthogonality look like in classical Banach spaces? For Hilbert spaces it coincides with the usual inner product orthogonality. The next important example we provide the answer for is $\mathbb{B}(\mathcal{H})$, the space of bounded linear operators on a complex Hilbert space $\mathcal{H}$ (equipped with the usual operator norm). Let us begin with two examples.

Example

(a) Let $A, B \in \mathbb{B}(\mathcal{H})$ be operators with orthogonal ranges, that is, such that $A^*B = 0$. Then, by Pythagorean identity, for all $\lambda \in \mathbb{C}$ it holds

$$\|A + \lambda B\|^2 = \sup_{\|x\|=1} \|(A + \lambda B)x\|^2 = \sup_{\|x\|=1} \|Ax\|^2 + \|\lambda Bx\|^2 \geq \|A\|^2,$$

so $A \perp B$.

(b) Let $A \in \mathbb{B}(\mathcal{H})$ be an operator which is not bounded from below. Let $(x_n)_n$ be a sequence of normalized vectors such that $\lim_{n\to\infty} Ax_n = 0$. Then $\lim_{n\to\infty} \|(I + \lambda A)x_n\| = 1 = \|I\|$ for all $\lambda \in \mathbb{C}$, so $\|I + \lambda A\| \geq \|I\|$ and $I \perp A$.

B-J orthogonality in $\mathbb{B}(\mathcal{H})$ was studied by several authors. A characterization in the special case when one of the operators is the identity was obtained by Stampfli [86, Theorem 2] in the study on derivations. It was stated in terms of the maximal numerical range of A defined as

$$W_0(A) = \{\lambda \in \mathbb{C};\ \ \exists (x_n)_n \in \mathcal{H},\ \|x_n\| = 1,\ \lim_{n\to\infty} \langle Ax_n, x_n\rangle = \lambda,\\ \lim_{n\to\infty} \|Ax_n\| = \|A\|\}.$$

Stampfli first proved that $W_0(A)$ is a closed convex subset of $\mathbb{C}$, and this was used in the proof of the equivalence $d(A, \mathbb{C}I) = \|A\| \Leftrightarrow 0 \in W_0(A)$.

Later, in his study on the distance to finite-dimensional subspaces in operator algebras, Magajna [63] introduced, for $A, B \in \mathbb{B}(\mathcal{H})$, the notion of the maximal numerical range of B^*A relative to A in the following way

$$W_A(B^*A) = \Big\{\mu \in \mathbb{C};\ \ \exists (x_n)_n \in \mathcal{H},\ \|x_n\| = 1,\ \lim_{n\to\infty} \langle B^*Ax_n, x_n\rangle = \mu,\\ \lim_{n\to\infty} \|Ax_n\| = \|A\|\Big\}.$$

Obviously, if $B = I$, this reduces to Stampfli's maximal numerical range of A. Magajna observed that Stampfli's results hold, with the same arguments, in the more general case, and this implies a complete characterization of B-J orthogonal operators in $\mathbb{B}(\mathcal{H})$. The same characterization was obtained with the help of different methods by Bhatia and Šemrl [20] in their study of the diameter of a unitary orbit of a matrix, and also by Roy, Bagchi, and Sain [78] within their directional approach to B-J orthogonality.

The following proof is an adaptation of Theorem 2 from [86].

Theorem 3.1 *Let $A, B \in \mathbb{B}(\mathcal{H})$. Then $A \perp B$ if and only if $0 \in W_A(B^*A)$. In other words, $A \perp B$ if and only if there is a sequence of normalized vectors $(x_n)_n$ in $\mathcal{H}$ such that*

$$\lim_{n\to\infty} \|Ax_n\| = \|A\| \text{ and } \lim_{n\to\infty} \langle B^*Ax_n, x_n\rangle = 0.$$

Proof We may assume that $\|B\| = 1$. By using the same arguments as in [86, Lemma 2], it can be shown that the set $W_A(B^*A)$ is a nonempty, closed and convex subset of $\mathbb{C}$.

Suppose that $0 \in W_A(B^*A)$. Let $(x_n)_n$ be a sequence of normalized vectors in $\mathcal{H}$ such that $\lim_{n\to\infty} \langle B^*Ax_n, x_n\rangle = 0$ and $\lim_{n\to\infty} \|Ax_n\| = \|A\|$. Then for each n we have

$$\begin{aligned}\|A + \lambda B\|^2 &\geq \|(A + \lambda B)x_n\|^2\\ &= \|Ax_n\|^2 + 2\,\mathrm{Re}(\overline{\lambda}\langle B^*Ax_n, x_n\rangle) + |\lambda|^2\|Bx_n\|^2\\ &\geq \|Ax_n\|^2 + 2\,\mathrm{Re}(\overline{\lambda}\langle B^*Ax_n, x_n\rangle) \xrightarrow{n\to\infty} \|A\|^2.\end{aligned}$$

Conversely, suppose that $0 \notin W_A(B^*A)$. Since $W_A(B^*A)$ is closed and convex, without loss of generality we may assume that there is $\tau > 0$ such that $\operatorname{Re} W_A(B^*A) \geq \tau$ (by changing B with $e^{i\phi}B$ for an appropriate ϕ). Let

$$S = \{x \in \mathcal{H}; \quad \|x\| = 1, \ \operatorname{Re}\langle B^*Ax, x\rangle \leq \tau/2\}$$

and $\eta = \sup\{\|Ax\|; \quad x \in S\}$. Then $\operatorname{Re} W_A(B^*A) \geq \tau$ implies that $\eta < \|A\|$. Let

$$\mu = \min\left\{\tfrac{\tau}{2}, \tfrac{\|A\|-\eta}{2}\right\} > 0.$$

We will show that $\|A - \mu B\| < \|A\|$ which will imply that $A \not\perp B$.

Let $x \in \mathcal{H}$ be a normalized vector. If $x \in S$ then

$$\|(A - \mu B)x\| \leq \|Ax\| + \mu\|Bx\| \leq \eta + \mu \leq \|A\| - \mu.$$

Suppose now that $x \notin S$. Let us write $Ax = (a + ib)Bx + y$ with $\langle Bx, y\rangle = 0$. Then $0 < \mu \leq \tau/2 < \operatorname{Re}\langle B^*Ax, x\rangle = a\|Bx\|^2 \leq a$. After some manipulation this implies $\mu(\mu - 2a)\|Bx\|^2 < -\mu^2$. Then

$$\begin{aligned}\|(A - \mu B)x\|^2 &= \|(a + ib - \mu)Bx + y\|^2 = ((a - \mu)^2 + b^2)\|Bx\|^2 + \|y\|^2 \\ &= ((a^2 + b^2)\|Bx\|^2 + \|y\|^2) + \mu(\mu - 2a)\|Bx\|^2 \\ &= \|Ax\|^2 + \mu(\mu - 2a)\|Bx\|^2 < \|A\|^2 - \mu^2.\end{aligned}$$

This proves that $\|A - \mu B\| < \|A\|$, so $A \not\perp B$. □

Directly from this characterization we obtain a family of B-J orthogonal pairs of operators on a complex Hilbert space.

Corollary 3.2 *For all $A, B \in \mathbb{B}(\mathcal{H})$ it holds $A \perp B(\|A\|^2 - A^*A)$.*

Another corollary is a simpler characterization of B-J orthogonality in the case of compact operators on a complex Hilbert space and, in particular, when $\dim \mathcal{H} < \infty$. B-J orthogonality of compact operators on a separable, complex Hilbert space was discussed by Kečkić, see [54, Corollary 2.8]. We present here a short proof based on Theorem 3.1. Also, in our proof only A needs to be compact.

Corollary 3.3 *Let $A, B \in \mathbb{B}(\mathcal{H})$, with A compact. The following are equivalent.*

(i) $A \perp B$.
(ii) *There exists a normalized vector $x \in \mathcal{H}$ such that $\|A\| = \|Ax\|$ and* $\langle Ax, Bx\rangle = 0$.

Proof Suppose that $A \perp B$. By the preceding result, then $0 \in W_A(B^*A)$, so there is a sequence $(x_n)_n$ of normalized vectors such that $\lim_{n\to\infty} \|Ax_n\| = \|A\|$ and $\lim_{n\to\infty}\langle B^*Ax_n, x_n\rangle = 0$. Since $|A| = \sqrt{A^*A}$ is compact and $(x_n)_n$ is bounded, we may assume that $(|A|x_n)_n$ is convergent. By [91, Lemma 2.1], it holds

$\lim_{n\to\infty}(|A|x_n - \|A\|x_n) = 0$, so $(x_n)_n$ is convergent as well. Then (ii) holds with $x := \lim x_n$. The converse is obvious. □

In the case when $\dim \mathcal{H} < \infty$, this corollary can also be stated in terms of matrices (see [20, Theorem 1.1]). Moreover, Li and Schneider used Proposition 1.6 to generalize it to rectangular (real or complex) matrices $M_{m\times n}(\mathbb{F})$ with Schatten p-norm

$$\|A\|_p := \sqrt[p]{\sum\nolimits_{i=1}^{n}(\sigma_i(A))^p} = \left(\mathrm{tr}\left(\left(\sqrt{A^*A}\right)^p\right)\right)^{1/p},$$

where $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_n(A) \geq 0$ are the singular values of A (the square roots of eigenvalues of A^*A). Note that $\|A\|_\infty = \sigma_1(A)$ is the spectral norm. By [79, Theorem 9], Schatten norms $\|\cdot\|_p$ and $\|\cdot\|_q$ on $M_{m\times n}(\mathbb{F})$ with $1/p + 1/q = 1$ are dual to each other; the duality is given by $(A, B) \mapsto \mathrm{tr}(B^*A)$.

Except for $p = 1$ and $p = \infty$, Schatten p-norm is differentiable, hence smooth. This can be seen directly for $p > 2$, because $X \mapsto \left(\mathrm{tr}\left(\left(\sqrt{X^*X}\right)^p\right)\right)^{1/p}$ is a composition of real differentiable functions: $X \mapsto H(X) = X^*X$ is clearly real differentiable, while

$$H = U\left(\sum\nolimits_{i=1}^{n}\lambda_i(H)E_{ii}\right)U^* \mapsto \sqrt[p]{\mathrm{tr}(|H|^{p/2})} := \sqrt[p]{\sum\nolimits_{i=1}^{n}|\lambda_i(H)|^{p/2}}$$

is also a differentiable map on the real space of Hermitian matrices by, e.g., Lewis's [59, Theorem 1.1]. One should mention that the differentiability of $H \mapsto \sqrt[p]{\mathrm{tr}(|H|^{p/2})}$ follows also from Rellich's result [75] (a simplified proof can be found in Kato's monograph [53, Theorem II.6.8]). Namely, for each Hermitian $H, H' \in M_n(\mathbb{C})$, there exist n continuously differentiable functions in a real variable t which represent n eigenvalues (counted with multiplicities) of $H + tH'$. It is then immediate that $H \mapsto |H|^{p/2}$ has a continuous directional derivative in direction H', so it must be differentiable. We remark that [59, Theorem 1.1] gives a much more general differentiability result for matrix functions. The differentiability for $1 < p < 2$ was proved in [15], but the reader is again referred to Lewis [58, Corollary 2.6] for a more general result.

Let us introduce a bit more notation: Given a matrix $A \in M_n(\mathbb{F})$, we denote by

$$W_{\mathbb{C}}(A) := \{x^*Ax;\ \ x \in \mathbb{C}^n,\ x^*x = 1\}$$

its numerical range. If $\mathbb{F} = \mathbb{R}$ we let

$$W_{\mathbb{R}}(A) := \{x^*Ax;\ \ x \in \mathbb{R}^n,\ x^*x = 1\}$$

be the restricted numerical range. Recall that $W_{\mathbb{C}}(A)$ is convex, and so is $W_{\mathbb{R}}(A)$ because it is an interval (it is an image of a continuous function $x \mapsto x^*Ax$ whose domain is the Euclidean $(n-1)$-sphere).

We proceed to prove Li and Schneider's extension for spectral norm. Recall that the dual of spectral norm is the trace norm (i.e., Schatten 1-norm). Also, as required by Proposition 1.6, the extreme points of the trace norm on $M_{m\times n}(\mathbb{F})$ are the operators yx^* for $x \in \mathbb{F}^n$ and $y \in \mathbb{F}^m$ with $\|x\| = \|y\| = 1$ (see Ziętak [95] or [58, Theorem 3.4]).

Proposition 3.4 (Cf. [60, Theorem 3.1]) *Let $m \le n$. The following are equivalent for $A, B \in (M_{m\times n}(\mathbb{F}), \|\cdot\|_\infty)$, equipped with Schatten ∞-norm (i.e., spectral norm):*

(i) $A \perp B$.
(ii) *There exists a normalized vector $x \in \mathbb{F}^n$ such that $\|A\|_\infty = \|Ax\|$ and $\langle Ax, Bx\rangle = 0$.*
(iii) *There exist normalized vectors $y \in \mathbb{F}^m$ and $x \in \mathbb{F}^n$ such that $\|A\|_\infty = y^*Ax$ and $y^*Bx = 0$.*
(iv) *For any $U \in M_{m\times r}(\mathbb{F})$ whose columns form an orthonormal basis for the eigenspace of AA^* corresponding to the largest eigenvalue, we have*

$$0 \in W_{\mathbb{F}}(U^*BA^*U).$$

Sketch of the Proof (i) $\Longrightarrow$ (iv). By Proposition 2.1, there exist extreme points $F_k = y_kx_k^* \in M_{m\times n}(\mathbb{F})$ ($\|x_k\| = \|y_k\| = 1$) for $1 \le k \le h$ and positive scalars λ_k summing to one, such that

$$\|A\|_\infty = \langle F_k, A\rangle := \mathrm{tr}(F_k^*A) = y_k^*Ax_k \quad \text{and} \quad \sum_{k=1}^{h}\langle \lambda_k F_k, B\rangle = 0. \tag{10}$$

Thus, A achieves its norm on x_k, and y_k is proportional to Ax_k. By the assumption on U in item (iv) (together with the singular value decomposition of A), there exist normalized vectors $v_k \in \mathbb{F}^r$ with

$$Uv_k = y_k \quad \text{and} \quad x_k = \frac{1}{\|A\|_\infty}A^*y_k = Vv_k; \qquad V = \frac{1}{\|A\|_\infty}A^*U \in M_{n\times r}(\mathbb{F}).$$

Thus, by the second condition in (10), $0 = \sum_{k=1}^{h} \lambda_k v_k^*U^*BVv_k$, which is a point in the convex hull of $W_{\mathbb{F}}(U^*BV)$. Due to its convexity, $0 \in W_{\mathbb{F}}(U^*BV)$, giving (iv).

(iv) $\Longrightarrow$ (iii). Choose a normalized vector $v \in \mathbb{F}^r$ with $0 = v^*U^*BA^*Uv$, then $y := Uv$ and $x := \frac{1}{\|A\|_\infty}A^*Uv$ are normalized vectors satisfying (ii).

The remaining implications (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) are straightforward. □

Proposition 3.5 (Cf. [60, Theorem 3.2]) *Let $m \le n$ and let $p \in (1,\infty)$. The following are equivalent for $A, B \in (M_{m\times n}(\mathbb{F}), \|\cdot\|_p)$, equipped with Schatten p-norm:*

(i) $A \perp B$.
(ii) $\mathrm{tr}(P^{p-1}UB^*) = 0$ *where $A = PU$ is the polar decomposition ($P \in M_m(\mathbb{F})$ is positive semidefinite and $U \in M_{m\times n}(\mathbb{F})$ satisfies $UU^* = I_m$).*

Sketch of the Proof Assume that $A \neq 0$ and write $P = \sum_{i=1}^{m} \sigma_i(A) x_i x_i^*$ for some orthonormal basis $(x_i)_i$. One can calculate that

$$T := \frac{1}{\|A\|_p^p} P^{p-1} U = \frac{1}{\|A\|_p^p} \sum_{i=1}^{m} \sigma_i(A)^{p-1} x_i x_i^* U$$

satisfies $\|T\|_q = \operatorname{tr}(T^* A) = 1$. Therefore, by smoothness of the norm, T induces the only supporting functional at A, and the result follows from Proposition 1.4. □

In the same paper [60], Li and Schneider also characterized B-J orthogonality of $M_{m\times n}(\mathbb{F})$ in trace norm. Note that it corresponds to Schatten 1-norm which is not differentiable.

Spectral and trace norms are the first and the last among Ky Fan norms. These are defined for any positive integer k, not larger than the size of a matrix, by (see [43, Theorem 3.4.1])

$$\|A\|_{(k)} := \sum_{i=1}^{k} \sigma_i(A) = \max_{\substack{U,V \in M_{n\times k}(\mathbb{C}) \\ U^*U = V^*V = I_k}} |\operatorname{tr}(U^* A V)|; \qquad A \in M_n(\mathbb{C}).$$

The special importance of Ky Fan norms is their dominance property: Given two matrices $A, B \in M_n(\mathbb{C})$, then $\|A\| \leq \|B\|$ for every unitarily invariant norm if and only if this inequality holds for all n Ky Fan norms (see [43, Corollary 3.5.9]). Grover in [39] gave the following characterization of B-J orthogonality in Ky Fan norms, which we state without a proof:

Proposition 3.6 (Cf. [39, Theorem 1.1 and Theorem 3.2]) *Let $1 \leq k \leq n$, let $A = U|A|$ be a polar decomposition of $A \in M_n(\mathbb{F})$, and let $B \in M_n(\mathbb{F})$. If there exist k orthonormal vectors $u_1, \ldots, u_k \in \mathbb{F}^n$ such that*

$$|A|u_i = \sigma_i(A) u_i \quad \textit{for } i = 1, \ldots, k \qquad \textit{and} \qquad \sum_{i=1}^{k} \langle u_i, U^* B u_i \rangle = 0,$$

then A is B-J orthogonal to B in $\|\cdot\|_{(k)}$. If in addition $\sigma_k(A) > 0$, then the converse is also true.

The condition (ii) from Corollary 3.3 can be restated as:

(ii′) There exists a normalized vector $x \in \mathbb{F}^n$ such that $\|Ax\| = \|A\|$ and $Ax \perp Bx$.

Now (ii′), unlike (ii), makes sense not only in inner product spaces but also in general normed spaces. It is easy to see that, in any norm $\|\cdot\|$ on $\mathbb{F}^n$, (ii′) implies (i) of Corollary 3.3 (i.e., $A \perp B$) provided that $M_n(\mathbb{F})$ is equipped with the induced operator norm $\|T\| := \sup_{\|x\|=1} \|Tx\|$. Bhatia and Šemrl conjectured in [20] that, conversely, (i) implies (ii′) in any *norm* on $\mathbb{F}^n$.

The first counterexample to this conjecture was provided by Li and Schneider [60, Example 4.3], in ℓ_p norm on $\mathbb{F}^n$ for $p \neq 2$, and with matrices $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right) \oplus 0_{n-2}$ and $B = I_2 \oplus 0_{n-2}$. Later, it was shown in Benítez, Fernández, and Soriano [19] that a counterexample to this conjecture exists in $\mathbb{R}^n$ whenever the regarded norm is not induced by an inner product, that is, the implication (i) of Corollary 3.3 $\Rightarrow$ (ii′) is a characterization of real inner product spaces of finite dimension.

The conditions on an individual operator A which force that (i) implies (ii′) were studied in a series of papers by Ghosh, Hait, Paul, Sain, and others. In real Banach spaces $\mathcal{X}$ of finite dimension one such neat sufficient condition is that the norm-attaining set $M_A := \{x \in \mathcal{X};\quad \|x\| = 1, \|Ax\| = \|A\|\}$ contains at most two connected components, see the survey paper [72, Theorem 8.2.1]. When $\dim \mathcal{X} = 2$, this condition is also necessary, see [72, Theorem 8.4.8]. Moreover, by [72, Theorem 8.8.2], for arbitrary operators A, B acting on a finite-dimensional real Banach space we have $A \perp B$ if and only if there exist $x, y \in M_A$ such that

$$\|Ax + \lambda Bx\| \geq \|Ax\| = \|A\| \quad \text{for } \lambda \geq 0 \qquad \text{and}$$
$$\|Ay + \lambda By\| \geq \|Ay\| = \|A\| \quad \text{for } \lambda \leq 0$$

(sufficiency is trivial: $\|A + \lambda B\| \geq \|Ax + \lambda Bx\| \geq \|A\|$ for $\lambda \geq 0$ and $\|A + \lambda B\| \geq \|Ay + \lambda By\| \geq \|A\|$ for $\lambda \leq 0$ combined give $\|A + \lambda B\| \geq \|A\|$ for each $\lambda \in \mathbb{R}$). Similar results for operators on an infinite-dimensional $\mathcal{X}$ as well as the description of smooth points in $\mathbb{B}(\mathcal{X})$ can also be found in this survey.

3.2 B-J Orthogonality in Commutative C-Algebras and Function Spaces*

The case of commutative C^*-algebras was discussed by Kečkić. Recall that for each unital commutative C^*-algebra $\mathscr{A}$ there is a compact Hausdorff space K such that $\mathscr{A}$ is $*$-isomorphic to the C^*-algebra $\mathcal{C}(K)$ of all continuous complex valued functions on K with the maximum norm $\|f\| = \max\{|f(t)|;\quad t \in K\}$, see [23, II.2.2.4 and II.1.1.3.(2)]. If $\mathscr{A}$ is a nonunital commutative C^*-algebra, then there is a noncompact locally compact Hausdorff space Ω such that $\mathscr{A}$ is isomorphic to $\mathcal{C}_0(\Omega)$, the C^*-algebra of all continuous complex functions on Ω vanishing at "infinity". If $K = \Omega \cup \{s_0\}$ is the one-point compactification of Ω, then we can identify $\mathcal{C}_0(\Omega)$ with the C^*-subalgebra $\{f \in \mathcal{C}(K);\quad f(s_0) = 0\}$ of $\mathcal{C}(K)$. In this way, since B-J orthogonality of two elements "happens" in the subspace spanned by these two elements, it is enough to obtain the characterization of B-J orthogonality in $\mathcal{C}(K)$.

Let us begin with some examples.

Example

(a) Let $f, g \in \mathcal{C}(K)$ be such that there is $t_0 \in K$ satisfying $|f(t_0)| = \|f\|$ and $g(t_0) = 0$. Then

$$\|f + \lambda g\| \geq |f(t_0) + \lambda g(t_0)| = \|f\|, \quad \forall \lambda \in \mathbb{C},$$

so $f \perp g$.

(b) Let $f, g \in \mathcal{C}(K)$. It follows from the first example that $f \perp (\|f\|^2 - |f|^2)g$. Observe the similarity with Corollary 3.2.

(c) Let $f, g \in C([0, 1])$ be defined as $f(t) = 1$ and $g(t) = e^{2\pi i t}$. It is easy to see that $f \perp g$. This example shows that the sufficient conditions stated in the first example are not necessary.

Theorem 3.7 (Cf. [55, Corollary 2.1]) *Let $\mathcal{C}(K)$ be the Banach space of all continuous complex valued functions on a compact Hausdorff space K, with the norm $\|f\| = \max\{|f(t)|; \quad t \in K\}$. For $f \in \mathcal{C}(K)$ define*

$$E_f = \{t \in K; \quad |f(t)| = \|f\|\}.$$

Then the following are equivalent for $f, g \in \mathcal{C}(K)$:

(i) $f \perp g$.

(ii) *The set $F = \{\overline{f(t)}g(t); \quad t \in E_f\}$ is not contained in an open half plane in $\mathbb{C}$ with a boundary that contains the origin, that is, the closed convex hull of F contains the origin.*

(iii) *There exists a probability measure μ concentrated at E_f, such that*

$$\int_K \overline{f(t)}g(t)\, d\mu(t) = 0.$$

Therefore, if $f \neq 0$ and $E_f = \{t_0\}$ is a singleton, then $f \perp g$ if and only if $g(t_0) = 0$.

Sketch of the Proof By [55, Theorem 2.1, Proposition 1.3] f is orthogonal to g if and only if

$$\inf_{\phi \in [0,2\pi)} \max_{t \in E_f} \mathrm{Re}(e^{i\phi} e^{-i \arg f(t)} g(t)) \geq 0,$$

that is, if and only if it holds that, under every rotation around the origin, the set $\{e^{-i \arg f(t)} g(t); \quad t \in E_f\}$ contains at least one value with nonnegative real part. Since $\overline{f(t)}g(t) = \|f\| e^{-i \arg f(t)} g(t)$ for each $t \in E_f$, this is equivalent to (ii).

Since the convex hull of the set F is the set of points of the form $\int_K \overline{f(t)}g(t)\, d\lambda(t)$ where λ is a probability measure supported on a finite subset of E_f, there is a sequence $(\lambda_n)_n$ such that

$$0 = \lim_{n\to\infty} \int_K \overline{f(t)} g(t)\, d\lambda_n(t).$$

By the Banach–Alaoglu theorem, there is a subsequence (λ_{n_k}) which w^*-converges to some probability measure μ. Obviously, the support of μ is contained in E_f and (iii) holds. Conversely, suppose that (iii) holds. Then for all $\lambda \in \mathbb{C}$ we have

$$\|f + \lambda g\|^2 \geq \int_{E_f} |f + \lambda g|^2(t)\, d\mu(t) = \|f\|^2 + |\lambda|^2 \int_{E_f} |g(t)|^2\, d\mu(t) \geq \|f\|^2,$$

which gives (i). □

In the same paper, Kečkić also considered a more general space $\mathcal{C}_b(X)$ of bounded, continuous complex valued functions on a locally compact, Hausdorff space X, equipped with the supremum norm. His result is as follows:

Proposition 3.8 (Cf. [55, Corollary 3.1]) *The following is equivalent for $f, g \in \mathcal{C}_b(X)$:*

(i) $f \perp g$.
(ii) *There exists a sequence of probability measures μ_n concentrated at $E_\delta := \{x \in X;\ \ |f(x)| \geq \|f\| - \delta\}$ such that*

$$\lim_{n\to\infty} \int_X \overline{f(x)} g(x)\, d\mu_n(x) = 0.$$

For $\mathbf{c}_0$, that is, the space of all complex-valued sequences which converge to zero, and equipped with supremum norm, the characterization of B-J orthogonality is, as expected, easier.

Proposition 3.9 (Cf. [54, Example 1.7]) *The following are equivalent for $x = (x_n)_n$, $y = (y_n)_n \in \mathbf{c}_0$:*

(i) $x \perp y$.
(ii) *There does not exist an acute open angle $D = \{z;\ \ \alpha < \arg z < \beta\}$ with $\beta - \alpha < \pi$, such that $\overline{x_n} y_n \in D$ for all those n for which $|x_n| = \|x\|$ holds.*

Some other classical spaces are ℓ_p and $L_p(\mu)$ with $p \in [1, \infty)$. Here $L_p(\mu)$ denotes the space of complex valued functions on a measurable space Ω with a positive measure μ whose p-th degree is summable. In these spaces B-J orthogonality was characterized by James [47] and Kečkić [54] by means of supporting functionals and Gateaux derivatives of the norm. We remark that James considered the real case and $\Omega = [0, 1]$ with Lebesgue measure μ only, but the proof can be transferred easily to the complex case and an arbitrary measurable space Ω.

Proposition 3.10 (Cf. [47, Example 8.1]) *Let $x = (x_j)_{j\in\mathbb{N}}$ and $y = (y_j)_{j\in\mathbb{N}}$ belong to ℓ_p with $p \geq 1$. Then $x \perp y$ if and only if one of the following conditions holds:*

(i) $p = 1$ *and*

$$\left|\sum_{x_j\neq 0} \frac{\overline{x_j}}{|x_j|} y_j\right| \leq \sum_{x_j=0} |y_j|;$$

(ii) $p > 1$ *and*

$$\sum_{j\in\mathbb{N}} |x_j|^{p-2}\overline{x_j} y_j = 0,$$

where any occurrence of $|0|^{p-2}0$ in the sum above is interpreted as zero.

It follows that if $p = 1$ then $x \neq 0$ is a smooth point of the norm if and only if $x_j \neq 0$ for any $j \in \mathbb{N}$, and if $p > 1$ then the norm on ℓ_p is smooth.

Proposition 3.11 (Cf. [47, Example 8.2] and [54, Example 1.6]) *Let μ be a positive measure on a measurable space Ω. Then for $f, g \in L_p(\mu)$, $p \geq 1$, it holds $f \perp g$ if and only if one of the following conditions is satisfied:*

(i) $p = 1$ *and*

$$\left|\int_{\Omega\setminus f^{-1}(0)} \frac{\overline{f(t)}}{|f(t)|} g(t)\, d\mu(t)\right| \leq \int_{f^{-1}(0)} |g(t)|\, d\mu(t);$$

(ii) $p > 1$ *and*

$$\int_{\Omega} |f(t)|^{p-2}\overline{f(t)} g(t)\, d\mu(t) = 0.$$

Similarly, if $p = 1$ then $f \neq 0$ is a smooth point of the norm if and only if $\mu(f^{-1}(0)) = 0$, and if $p > 1$ then the norm on $L_p(\mu)$ is smooth.

3.3 B-J Orthogonality in General C*-Algebras and Hilbert C*-Modules

A characterization of B-J orthogonality in general C^*-algebras (and, more generally, in Hilbert C^*-modules) was discussed in Arambašić–Rajić [10] and later, by using a different approach, in Bhattacharyya–Grover [21]. They obtained a characterization in terms of *states* on a C^*-algebra $\mathscr{A}$, that is, positive linear functionals on $\mathscr{A}$ of norm one.

For example, if $x \in \mathcal{H}$ is a normalized vector, then $\phi\colon \mathbb{B}(\mathcal{H}) \to \mathbb{C}$ given by $\phi(T) = \langle Tx, x\rangle$ is a state on $\mathbb{B}(\mathcal{H})$, while in the case of $C(K)$ examples of states are evaluations at a point. There are other states than these (for example, if $T \in \mathbb{B}(\mathcal{H})$ is a positive operator which does not achieve its norm, then by [68, Theorem 5.1.11] there exists a state ϕ such that $\phi(T^2) = \|T^2\|$, while $\langle T^2x, x\rangle = \|Tx\|^2 < 1$ for each normalized x).

Observe that Theorem 3.1 can be rewritten in terms of states. Namely, $0 \in W_A(B^*A)$ means that there exists a sequence of normalized vectors $(x_n)_n$ such that

$$\langle A^*Ax_n, x_n\rangle \xrightarrow{n\to\infty} \|A\|^2 \quad \text{and} \quad \langle B^*Ax_n, x_n\rangle \xrightarrow{n\to\infty} 0.$$

Then for each n, $\phi_n : T \mapsto \langle Tx_n, x_n\rangle$ is a state. Recall that the set of states is closed in w^*-topology, hence compact by the Banach–Alaoglu theorem.

Proposition 3.12 (Cf. [10, Theorem 2.7], [21, Proposition 4.1]) *The following is equivalent in a C^*-algebra $\mathscr{A}$:*

(i) $a \perp b$.
(ii) *There exists a state ϕ on $\mathscr{A}$ such that*

$$\phi(a^*a) = \|a\|^2 \quad \textit{and} \quad \phi(a^*b) = 0.$$

Proof Let ϕ be a state as in (ii). Then for all $\lambda \in \mathbb{C}$ we have

$$\|a\|^2 = |\phi(a^*(a + \lambda b))| \le \|a^*(a + \lambda b)\| \le \|a\|\|a + \lambda b\|,$$

which gives $a \perp b$.

Conversely, suppose that $a \perp b$. By Gelfand–Naimark theorem, we embed $\mathscr{A}$ into $\mathbb{B}(\mathcal{H})$ and then use a sequence of states $\phi_n : T \mapsto \langle Tx_n, x_n\rangle$ provided by Theorem 3.1. Due to w^*-compactness, this sequence has a subsequence which w^*-converges to a desired state ϕ. □

By using the linking algebra of a Hilbert C^*-module, this result can be extended from C^*-algebras to Hilbert C^*-modules. The concept of a Hilbert C^*-module has been introduced by Kaplansky [52] and Paschke [71] in an investigation of right modules over a C^*-algebra which possess a C^*-valued inner product respecting the module action. More precisely, a Hilbert C^*-module $\mathcal{X}$ over a C^*-algebra $\mathscr{A}$ is a right $\mathscr{A}$-module equipped with an $\mathscr{A}$-valued inner product $\langle\cdot\,,\,\cdot\rangle : \mathcal{X} \times \mathcal{X} \to \mathscr{A}$ which satisfies

(1) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$ for $x, y, z \in \mathcal{X}$, $\alpha, \beta \in \mathbb{C}$,
(2) $\langle x, ya\rangle = \langle x, y\rangle a$ for $x, y \in \mathcal{X}$, $a \in \mathcal{A}$,
(3) $\langle x, y\rangle^* = \langle y, x\rangle$ for $x, y \in \mathcal{X}$,
(4) $\langle x, x\rangle \ge 0$ for $x \in \mathcal{X}$; if $\langle x, x\rangle = 0$ then $x = 0$,

and which is a Banach space with respect to the norm defined as $\|x\| = \sqrt{\|\langle x, x\rangle\|}$. We say that a Hilbert C^*-module $\mathcal{X}$ over a C^*-algebra $\mathcal{A}$ is *full* if the inner products of elements from $\mathcal{X}$ span a dense subset in $\mathcal{A}$, in short, if $\langle \mathcal{X}, \mathcal{X}\rangle = \mathscr{A}$. A left Hilbert C^*-module can be defined in a similar way. Every right Hilbert $\mathscr{A}$-module is also a left Hilbert C^*-module over a C^*-algebra $\mathbb{K}(\mathcal{X})$ of 'compact' operators on $\mathcal{X}$, that is, the C^*-algebra spanned by the operators $\theta_{x,y}$, $x, y \in \mathcal{X}$, defined as $\theta_{x,y}(z) = x\langle y, z\rangle$. It is easy to show that $\|x\| = \sqrt{\|\theta_{x,x}\|}$.

Besides Hilbert spaces, one of the most important examples of Hilbert C^*-modules are C^*-algebras. If $\mathscr{A}$ is a C^*-algebra, then we can regard it as a Hilbert C^*-module over itself with the algebra multiplication as a (right) module action and an inner product defined as $\langle a, b\rangle = a^*b$.

The linking algebra $\mathcal{L}(\mathcal{X})$ of a Hilbert $\mathscr{A}$-module $\mathcal{X}$ is defined as the C^*-algebra of all 'compact' operators acting on the Hilbert $\mathscr{A}$-module $\mathscr{A} \oplus \mathcal{X}$. It can be written in the matrix form

$$\mathcal{L}(\mathcal{X}) = \left\{ \begin{bmatrix} T_a & l_y \\ r_x & T \end{bmatrix};\ \ a \in \mathscr{A},\ x, y \in \mathcal{X},\ T \in \mathbb{K}(\mathcal{X}) \right\},$$

where the maps $r_x : \mathscr{A} \to \mathcal{X}$, $l_y : \mathcal{X} \to \mathscr{A}$ and $T_a : \mathscr{A} \to \mathscr{A}$ are given by $r_x(a) = xa$, $l_y(z) = \langle y, z\rangle$ and $T_a(b) = ab$. For more details on Hilbert C^*-modules we refer to [23] and [64].

The following result was proved in [10] and [21].

Theorem 3.13 (Cf. [10, Theorem 2.7] and [21, Theorem 4.4]) *Let $\mathcal{X}$ be a Hilbert C^*-module over a C^*-algebra $\mathscr{A}$. Let $x, y \in \mathcal{X}$. Then $x \perp y$ if and only if there is a state ϕ on $\mathscr{A}$ such that $\phi(\langle x, x\rangle) = \|x\|^2$ and $\phi(\langle x, y\rangle) = 0$.*

Sketch of the Proof Let $x, y \in \mathcal{X}$ be such that $x \perp y$. Then for $X = \begin{bmatrix} 0 & 0 \\ r_x & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & 0 \\ r_y & 0 \end{bmatrix}$, elements of the C^*-algebra $\mathcal{L}(\mathcal{X})$, it holds $X \perp Y$. Applying Proposition 3.12 to $\mathcal{L}(\mathcal{X})$, we get a state Φ on $\mathcal{L}(X)$ such that $\Phi(X^*X) = \|X\|^2$ and $\Phi(X^*Y) = 0$. If $\phi : \mathscr{A} \to \mathbb{C}$ is defined by the formula $\phi(a) = \Phi\left(\begin{bmatrix} T_a & 0 \\ 0 & 0 \end{bmatrix}\right)$, we easily see that ϕ is a state on $\mathscr{A}$ satisfying $\phi(\langle x, x\rangle) = \|x\|^2$ and $\phi(\langle x, y\rangle) = 0$.

The converse is similar to the C^*-algebra case. □

If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, then the Banach space $\mathbb{B}(\mathcal{H}, \mathcal{K})$ of all bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$ is a Hilbert C^*-module over $\mathbb{B}(\mathcal{H})$. Therefore, as a corollary of the previous theorem we get a generalization of Theorem 3.1.

Corollary 3.14 *Let $A, B \in \mathbb{B}(\mathcal{H}, \mathcal{K})$. Then $A \perp B$ if and only if there is a sequence of normalized vectors $(x_n)_n$ in $\mathcal{H}$ such that* $\lim_{n\to\infty} \|Ax_n\| = \|A\|$ *and* $\lim_{n\to\infty}\langle B^*Ax_n, x_n\rangle = 0$.

We remark that Singla [85, Theorem 1.3] very recently extended Corollary 3.14 and classified when an operator $A \in \mathbb{B}(\mathcal{H}, \mathcal{K})$, which is not far away from compact operators, is B-J orthogonal to a subspace $\mathcal{B} \subseteq \mathbb{B}(\mathcal{H}, \mathcal{K})$. This was achieved as a consequence of his study of norm's directional derivatives D_+ at elements in a C^*-algebra (which are not far from some, possibly nonproper, closed ideal).

3.4 Strong B-J Orthogonality on Hilbert C^*-Modules

In addition to the types of orthogonality that we have already mentioned, in a Hilbert C^*-module $\mathcal{X}$ over a C^*-algebra $\mathscr{A}$ there are two types of orthogonality that rely on C^*-modular structure of $\mathcal{X}$. The first one is orthogonality which directly generalizes orthogonality in inner product spaces: two elements x and y of a Hilbert C^*-module $\mathcal{X}$ are *orthogonal with respect to the C^*-valued inner product* in $\mathcal{X}$ if $\langle x, y\rangle = 0$. This is a very strong type of orthogonality; for example, in a Hilbert C^*-module $\mathbb{B}(H, K)$ over $\mathbb{B}(H)$, where the inner product of A and B is defined as A^*B, this is exactly the range orthogonality of operators. The second one generalizes B-J orthogonality in a way that the role of scalars is taken by elements of the underlying C^*-algebra $\mathscr{A}$ (see [11]): for $x, y \in \mathcal{X}$ we say that x is *strongly B-J orthogonal* to y, denoted as $x \perp^s y$, if $\|x + ya\| \geq \|x\|$ for every $a \in \mathscr{A}$, that is, if the distance from x to the $\mathscr{A}$-submodule $\overline{y\mathcal{A}}$ of $\mathcal{X}$ generated by y is exactly $\|x\|$.

Obviously, $x \perp^s y$ holds if and only if $x \perp ya$ for every $a \in \mathscr{A}$. However, as we show in the following theorem, it turns out that instead of checking that $x \perp ya$ holds for every $a \in \mathscr{A}$, it is enough to check that $x \perp ya$ holds for one special element $a \in \mathscr{A}$. Also, this result gives a characterization of strong B-J orthogonality.

Theorem 3.15 (Cf. [11, Theorem 2.5]) *Let $\mathcal{X}$ be a Hilbert C^*-module over a C^*-algebra $\mathscr{A}$, and let $x, y \in \mathcal{X}$. Then the following statements are equivalent:*

(a) $x \perp^s y$.
(b) $x \perp y\langle y, x\rangle$.
(c) *There is a state ϕ on $\mathscr{A}$ such that $\phi(\langle x, x\rangle) = \|x\|^2$ and $\phi(\langle x, y\rangle\langle y, x\rangle) = 0$.*

Proof The implication (a)$\Longrightarrow$(b) is obvious, while the equivalence (b)$\Longleftrightarrow$(c) follows from Theorem 3.13. It remains to prove (c)$\Longrightarrow$(a).

Suppose that there is a state ϕ on $\mathscr{A}$ such that $\phi(\langle x, x\rangle) = \|x\|^2$ and $\phi(\langle x, y\rangle\langle y, x\rangle) = 0$. Let $a \in \mathscr{A}$ be arbitrary. By the Bunyakovsky-Cauchy-Schwarz inequality applied to $(a, b) \mapsto \phi(a^*b)$ we get

$$|\phi(\langle x, ya\rangle)|^2 = |\phi(\langle x, y\rangle a)|^2 \leq \phi(\langle x, y\rangle\langle y, x\rangle)\phi(a^*a) = 0,$$

so $\phi(\langle x, ya\rangle) = 0$, and therefore $x \perp ya$. This gives that $x \perp^s y$. □

In combination with Theorems 3.1 and 3.7 this result gives the following characterizations of strong B-J orthogonality in C^*-algebras $C(K)$ and $\mathbb{B}(H)$

(regarded as Hilbert C^*-modules over itself):

(1) If $f, g \in \mathcal{C}(K)$, then $f \perp^s g$ if and only if there is $t_0 \in X$ such that $|f(t_0)| = \|f\|$ and $g(t_0) = 0$ (cf. [13, Proposition 4.2]).
(2) If $A, B \in \mathbb{B}(H)$, then $A \perp^s B$ if and only if there is a sequence of normalized vectors (x_n) in H such that $\lim_{n\to\infty} \|Ax_n\| = \|A\|$ and $\lim_{n\to\infty} B^*Ax_n = 0$. In particular, if $\dim H < \infty$, then $A \perp^s B$ if and only if there is a normalized vector $x \in H$ such that $\|Ax\| = \|A\|$ and $B^*Ax = 0$ (cf. [11, Proposition 2.8]).

It follows from Theorem 3.15 and the definition of strong B-J orthogonality that in every Hilbert C^*-module $\mathcal{X}$ the following implications hold for all $x, y \in \mathcal{X}$:

$$\langle x, y\rangle = 0 \Longrightarrow x \perp^s y \Longrightarrow x \perp y.$$

Now when we have characterizations of B-J and strong B-J orthogonality in $\mathcal{X}$, we easily see that the converse implications do not hold in general. For example, let $\mathcal{X} = \mathcal{C}([0, 1])$. In order to see that the converse of the first implication does not hold, take $f, g \in C([0, 1])$ defined as $f(t) = 1$ and $g(t) = 2t - 1$ for $t \in [0, 1]$. Then $\langle f, g\rangle = \overline{f}g \neq 0$, while $f \perp^s g$ holds because $f(\frac{1}{2}) = \|f\|$ and $g(\frac{1}{2}) = 0$. As an example of functions which show that the converse of the second implication does not hold, we can use the same function f and the function $h(t) = e^{2\pi it}$. By Theorem 3.7, it follows that $f \perp h$, while $f \not\perp^s h$, since $h(t) \neq 0$ for all $t \in [0, 1]$.

It is evident that these three types of orthogonality coincide in a Hilbert space (regarded as a Hilbert C^*-module over the C^*-algebra of complex numbers). Now it is natural to ask if there is any other example of a Hilbert C^*-module in which two of these three orthogonalities coincide. These questions were discussed in [12] for the class of full Hilbert C^*-modules, and the following results were obtained.

Theorem 3.16 (Cf. [12, Theorem 3.5, Corollary 4.9]) *Let $\mathcal{X} \neq \{0\}$ be a full Hilbert C^*-module over a C^*-algebra $\mathscr{A}$.*

(1) *The equivalence $x \perp y \Longleftrightarrow x \perp^s y$ holds for every $x, y \in \mathcal{X}$ if and only if $\mathscr{A}$ is isomorphic to $\mathbb{C}$.*
(2) *If $\mathcal{X}$ is not singly generated, then the equivalence $\langle x, y\rangle = 0 \Longleftrightarrow x \perp^s y$ holds for every $x, y \in \mathcal{X}$ if and only if $\mathscr{A}$ is isomorphic to $\mathbb{C}$. If $\mathcal{X}$ is singly generated, then the equivalence $\langle x, y\rangle = 0 \Longleftrightarrow x \perp^s y$ holds for every $x, y \in \mathcal{X}$ if and only if $\mathbb{K}(\mathcal{X})$ is isomorphic to $\mathbb{C}$.*

Recall that a right Hilbert C^*-module $\mathcal{X}$ over a C^*-algebra $\mathcal{A}$ is also a full left Hilbert C^*-module over the C^*-algebra $\mathbb{K}(\mathcal{X})$. The norm on $\mathcal{X}$ is defined as $\|x\| = \sqrt{\|\langle x, x\rangle\|}$, so we can say that it is induced by the C^*-norm on $\mathscr{A}$ and the $\mathscr{A}$-valued inner product $(x, y) \mapsto \langle x, y\rangle$ on $\mathcal{X}$. It is easy to see that for every $x \in \mathcal{X}$ it holds $\|x\| = \sqrt{\|\theta_{x,x}\|}$, which can be interpreted as if the norm on $\mathcal{X}$ were induced by the C^*-norm on $\mathbb{K}(\mathcal{X})$ and the $\mathbb{K}(\mathcal{X})$-valued inner product $(x, y) \mapsto \theta_{x,y}$ on $\mathcal{X}$. Therefore, if strong B-J orthogonality coincides on $\mathcal{X}$ with either B-J orthogonality or orthogonality with respect to the $\mathscr{A}$-valued inner product, then $\mathcal{X}$ is an ordinary Hilbert space. In other words, whenever neither $\mathscr{A}$ nor $\mathbb{K}(\mathcal{X})$ are isomorphic to $\mathbb{C}$, strong B-J orthogonality is a new type of orthogonality on $\mathcal{X}$. Therefore, this is a very interesting topic and many papers discuss this subject in general or for some special classes of Hilbert C^*-modules. For example, in [67] the authors characterize strong B-J orthogonality for elements of a general C^*-algebra (regarded as a Hilbert C^*-module over itself) as well as for the special classes of elements of $\mathbb{B}(\mathcal{H})$. In the same paper some characterizations of standard B-J orthogonality for elements of a Hilbert $\mathbb{K}(\mathcal{H})$-module and of $\mathbb{B}(\mathcal{H})$ are obtained. Let us also mention the paper [93] where the author studies B-J orthogonality for elements and finite-dimensional subspaces of a pre-Hilbert C^*-module in terms of a convex hull of continuous linear functionals.

4 Applications of B-J Orthogonality

Of all the inequivalent orthogonalities listed in Sect. 1, B-J orthogonality has arguably found the most applications. Part of the reason is that several key properties of orthogonality in inner product spaces are inherited within B-J orthogonality. For example, given a vector x in an inner product space $\mathcal{X}$, its orthogonal projection to a subspace $\mathcal{Y} \subseteq \mathcal{X}$ is its best approximate (within $\mathcal{Y}$). That is,

$$\|x - y_0\| = \inf_{y \in \mathcal{Y}} \|x - y\| =: d(x, \mathcal{Y}) \tag{11}$$

if and only if $\langle x - y_0, y\rangle = 0$ for each $y \in \mathcal{Y}$. This holds also for general normed spaces:

Proposition 4.1 *Let $\mathcal{X}$ be a normed space. Among all the vectors from a subspace $\mathcal{Y} \subseteq \mathcal{X}$ a vector $y_0 \in \mathcal{Y}$ is the best approximate to x in a sense of* (11) *precisely when $x - y_0$ is B-J orthogonal to $\mathcal{Y}$.*

Proof From the definition of B-J orthogonality we have that $(x - y_0) \perp \mathcal{Y}$ if and only if $\|(x - y_0) + y\| \geq \|x - y_0\|$ for every $y \in \mathcal{Y}$. □

In infinite-dimensional Banach spaces the sum of two closed subspaces may fail to be closed. For a classical concrete example, consider the Hilbert space $\mathcal{H} := \ell_2$ with $(e_n)_{n\geq 0}$ as a standard orthonormal basis, let $\mathcal{M}$ be a closed subspace spanned by pairwise orthogonal vectors $\left(e_{2n} + \frac{1}{n+1}e_{2n+1}\right)_n$ and let $\mathcal{N}$ be a closed subspace spanned by orthonormal vectors $(e_{2n})_n$. Then $\mathcal{M} \cap \mathcal{N} = 0$ and $\mathcal{M} + \mathcal{N}$ is dense in ℓ_2, since it contains e_n for any $n \geq 0$. However, $\mathcal{M} + \mathcal{N}$ does not contain the vector $\sum_{n\geq 0} \frac{1}{n+1} e_{2n+1} \in \ell_2$, because otherwise the only way to write it would be $\sum_{n\geq 0}(e_{2n} + \frac{1}{n+1}e_{2n+1}) - \sum_{n\geq 0} e_{2n}$, but $\sum_{n\geq 0} e_{2n} \notin \ell_2$.

However, it is easily seen that, in Hilbert spaces, the sum of two orthogonal, closed subspaces is again closed. Again, this can be generalized easily to general Banach spaces (see Anderson [8, Remark 1.3]). To avoid misunderstanding, we say that a subset S_1 of a normed space $\mathcal{X}$ is B-J orthogonal to a subset $S_2 \subseteq \mathcal{X}$ (in symbols: $S_1 \perp S_2$) if $s_1 \perp s_2$ for every tuple of vectors $(s_1, s_2) \in S_1 \times S_2$.

Proposition 4.2 *Let $\mathcal{M}, \mathcal{N}$ be closed subspaces in a (real or complex) Banach space $\mathcal{X}$. If $\mathcal{M} \perp \mathcal{N}$, then $\mathcal{M} \cap \mathcal{N} = 0$ and $\mathcal{M} + \mathcal{N}$ is closed.*

Proof Firstly, if $m = n \in \mathcal{M} \cap \mathcal{N}$, then $\|m+n\| \geq \|m\|$ implies $m = n = 0$, so B-J orthogonal subspaces intersect trivially. Secondly, a projection P from $\mathcal{M} + \mathcal{N}$ to $\mathcal{N}$ is bounded with norm one because $\|P(m+n)\| = \|m\| \leq \|m+n\|$ (by definition of the fact that $m \perp n$). Therefore, if $(m_k + n_k)_k \in \mathcal{M} + \mathcal{N}$ is convergent, then $m_k := P(m_k + n_k) \in \mathcal{M} = \overline{\mathcal{M}}$ is a Cauchy sequence, so it converges to some $m \in \mathcal{M}$ ($\mathcal{M}$ is a closed, hence Banach, subspace of a Banach space $\mathcal{X}$). Then also $n_k = (n_k + m_k) - m_k$ converges to an element in $\overline{\mathcal{N}} = \mathcal{N}$, so $\lim_k(n_k + m_k) \in \mathcal{M} + \mathcal{N}$. □

Bhatia and Šemrl [20] used B-J orthogonality to prove that the diameter of the unitary orbit of a given complex matrix $A \in M_n(\mathbb{F})$ equals $2d(A, \mathbb{C}I)$. The unitary orbit of a complex square matrix $A \in M_n(\mathbb{F})$ is the set $\{UAU^*; \quad U \in M_n(\mathbb{F}) \text{ is unitary}\}$, so its diameter is given by

$$\begin{aligned} d_A &= \max\{\|VAV^* - UAU^*\|; \quad U, V \text{ are unitary}\} \\ &= \max\{\|A - UAU^*\|; \quad U \text{ is unitary}\}. \end{aligned}$$

Here the norm is assumed to be the spectral norm.

Theorem 4.3 (Cf. [20, Theorem 1.2]) *For each $A \in M_n(\mathbb{C})$ it holds $d_A = 2d(A, \mathbb{C}I)$.*

Proof Notice that $d_A = 0$ if and only if A is a scalar matrix, so the statement holds for scalar matrices.

Suppose that A is not a scalar matrix. The inequality $d_A \leq 2d(A, \mathbb{C}I)$ is easy to prove, since for every unitary matrix U and $\lambda \in \mathbb{C}$ we have

$$\|A - UAU^*\| = \|(A - \lambda I) - U(A - \lambda I)U^*\| \leq 2\|A - \lambda I\|.$$

For the converse inequality, let $\lambda_0 \in \mathbb{C}$ be such that $d(A, \mathbb{C}I) = \|A + \lambda_0 I\|$. Then $A_0 := A + \lambda_0 I \neq 0$, and for all $\lambda \in \mathbb{C}$ it holds

$$\|A_0 + \lambda I\| = \|A + (\lambda + \lambda_0)I\| \geq \|A + \lambda_0 I\| = \|A_0\|,$$

so $A_0 \perp I$. By Corollary 3.3, there is a normalized vector $x \in \mathbb{C}^n$ such that $\|A_0 x\| = \|A_0\|$ and $\langle A_0 x, x\rangle = 0$. Denote $y = \frac{1}{\|A_0\|} A_0 x$. Then x and y are normalized vectors such that $\langle x, y\rangle = 0$ and $\langle A_0 x, y\rangle = \|A_0\|$. It follows from the second equality that $\langle A_0 x, y\rangle = \|A_0 x\|\|y\|$, so we have the case of equality in the Bunyakovsky-Cauchy-Schwarz inequality. Therefore, it has to be $A_0 x = \|A_0\| y$. Now for a unitary U such that $Ux = x$ and $Uy = -y$ we have

$UA_0U^*x = -\|A_0\|y$, and then

$$d_A = d_{A_0} \geq \|A_0x - UA_0U^*x\| = 2\|A_0\| = 2d(A, \mathbb{C}I). \qquad \square$$

This result can be used for calculating the norm of inner derivations. Recall that, for a given $A \in \mathbb{B}(\mathcal{H})$, an inner derivation is the operator δ_A given by $\delta_A(X) = AX - XA$. In [86] Stampfli proved that the norm of δ_A is equal to $2d(A, \mathbb{C}I)$. Bhatia and Šemrl gave a simpler proof of this by using Theorem 4.3. Namely, each $X \in M_n(\mathbb{C})$ such that $\|X\| = 1$ can be written in the form $X = \frac{1}{2}(V + W)$, where V and W are unitary matrices (this follows from singular value decomposition of X by writing each singular value, which is a number between 0 and 1, as $\frac{1}{2}(e^{i\theta} + e^{-i\theta})$ for some $\theta \in \mathbb{R}$). Then we have

$$\begin{aligned}\|\delta_A\| &= \max\{\|AX - XA\|; \quad \|X\| = 1\}\\ &= \max\{\|AU - UA\|; \quad U \text{ is unitary}\} = d_A = 2d(A, \mathbb{C}I).\end{aligned}$$

This can be extended to the infinite-dimensional case by a limiting argument, see [20, Remark 3.2]. Namely, one can consider an increasing sequence $\{P_n\}$ of finite rank projections, set $A_n = P_nA$ and show that $\|\delta_A\| \geq 2d(A_n, \mathbb{C}I)$ for all $n \in \mathbb{N}$. It then follows that $\|\delta_A\| \geq 2d(A, \mathbb{C}I)$. For the converse inequality, it is sufficient to note that for any $X \in \mathbb{B}(\mathcal{H})$ with $\|X\| = 1$ and any $\lambda \in \mathbb{C}$ it holds $\|AX - XA\| = \|(A - \lambda I)X - (A - \lambda I)X\| \leq 2\|(A - \lambda I)\|$.

In this respect we mention also a classical result by Anderson [8] that, for a normal operator N, the range of δ_N is always B-J orthogonal to its kernel. In other words, if $NS = SN$, then for every $X \in \mathbb{B}(\mathcal{H})$ it holds

$$\|NX - XN + S\| \geq \|S\|.$$

This result has been greatly extended in various directions, see [25, 34, 56].

It was shown in [10] that B-J orthogonality provides a convenient criterion for two elements of a normed space $\mathcal{X}$ to satisfy the equality in the triangle inequality.

Proposition 4.4 (Cf. [10, Proposition 4.1]) *Let $\mathcal{X}$ be a real or complex normed space, $x, y \in \mathcal{X}$. Then the following conditions are equivalent:*

(i) $\|x + y\| = \|x\| + \|y\|$*;*
(ii) $x \perp (\|y\|x - \|x\|y)$*;*
(iii) $y \perp (\|y\|x - \|x\|y)$.

Proof It follows from the Hahn–Banach theorem that $\|x + y\| = \|x\| + \|y\|$ if and only if there exists a norm-one linear functional $f : \mathcal{X} \to \mathbb{F}$ such that $f(x) = \|x\|$ and $f(y) = \|y\|$, see [69, Theorem 2]. If $x \neq 0$, then the latter condition can be restated as $f(x) = \|x\|$ and $f(\|y\|x - \|x\|y) = 0$. Such a linear functional f exists if and only if $x \perp (\|y\|x - \|x\|y)$, so the equivalence (i) $\Leftrightarrow$ (ii) is proved. Similarly for (i) $\Leftrightarrow$ (iii). □

This proposition, together with the obtained characterizations of B-J orthogonality in different normed spaces, gives characterizations of the case of equality in the triangle inequality, which were directly obtained earlier in [9] and [69]. Let us formulate the version for C^*-algebras.

Corollary 4.5 (Cf. [69, Theorem 1], [9, Remark 2.2]) *Let $\mathcal{A}$ be a C^*-algebra and $a, b \in \mathcal{A}$ be nonzero. Then the equality $\|a + b\| = \|a\| + \|b\|$ holds if and only if there is a state ϕ on $\mathcal{A}$ such that $\phi(a^*b) = \|a\|\|b\|$.*

Proof Suppose that $\|a+b\| = \|a\|+\|b\|$ holds. Then, by the previous proposition, $a \perp (\|b\|a - \|a\|b)$ and, by Proposition 3.12, there is a state ϕ on $\mathcal{A}$ such that $\phi(a^*a) = \|a\|^2$ and $\phi(a^*(\|b\|a - \|a\|b)) = 0$. These two relations give $\phi(a^*b) = \|a\|\|b\|$.

Conversely, suppose there is a state ϕ on $\mathcal{A}$ such that $\phi(a^*b) = \|a\|\|b\|$. From the Bunyakovsky-Cauchy-Schwarz inequality applied to $(a, b) \mapsto \phi(a^*b)$ we get

$$\|a\|\|b\| = |\phi(a^*b)| \le \phi(a^*a)^{\frac{1}{2}}\phi(b^*b)^{\frac{1}{2}} \le \|a^*a\|^{\frac{1}{2}}\|b^*b\|^{\frac{1}{2}} = \|a\|\|b\|,$$

wherefrom $\phi(a^*a) = \|a\|^2$ and $\phi(b^*b) = \|b\|^2$. Thus, $\phi(a^*(\|b\|a - \|a\|b)) = \|b\|\phi(a^*a) - \|a\|\phi(a^*b) = 0$, so, by Proposition 3.12, $a \perp (\|b\|a - \|a\|b)$. □

Among other applications of B-J orthogonality we mention also Cheng–Mashreghi–Ross's [28] bounds for the zeros of an analytic function on a disk, where B-J orthogonality on ℓ_p with $p \in (1, \infty)$ is used, see Proposition 3.10 above.

5 Preservers of B-J Orthogonality

It is an easy exercise to prove that a linear map between inner product spaces which preserves orthogonality must be a scalar multiple of an isometry. This result can also be generalized to B-J orthogonality. Koldobsky [57] was the first to show that linear preservers of B-J orthogonality on real normed spaces are scalar multiples of isometries. Later, Blanco and Turnšek [24] extended his result to complex normed spaces. We present a simplified proof due to Wójcik [94, Theorem 2.1]. To counter nonsmooth norms, Wójcik relied on the following standard identities:

Proposition 5.1 (Cf. [33, Theorem 18]) *In the notations of Sect. 2.1, for any nonzero $x, y \in \mathcal{X}$ it holds that*

$$\lim_{t \nearrow 0} D_-(x + ty; y) = D_-(x; y) \;\; \textit{and} \;\; \lim_{t \searrow 0} D_+(x + ty; y) = D_+(x; y). \tag{12}$$

Sketch of the Proof For $t \in \mathbb{R}$ let $\phi_t \in J(x + ty)$ be a supporting functional such that $\operatorname{Re}\phi_t(y) = D_+(x + ty; y)$. Then $|\operatorname{Re}\phi_t(x)| \le \|\operatorname{Re}\phi_t\| \cdot \|x\| = \|x\|$, so

$\|x + ty\| = \operatorname{Re}\phi_t(x + ty) = \operatorname{Re}\phi_t(x) + t\operatorname{Re}\phi_t(y) \le \|x\| + t\operatorname{Re}\phi_t(y)$, and therefore, with $t > 0$,

$$\frac{\|x + ty\| - \|x\|}{t} \le \operatorname{Re}\phi_t(y) = D_+(x + ty; y). \tag{13}$$

Moreover, for each fixed $t > 0$ and each x we have $D_+(x; y) \le \frac{\|x+ty\|-\|x\|}{t}$. By replacing x with $x + ty$ we get, in combination with (13),

$$\frac{\|x + ty\| - \|x\|}{t} \le D_+(x + ty; y) \le \frac{\|x + 2ty\| - \|x + ty\|}{t}.$$

Both the left and the right sides converge to $D_+(x; y)$ as $t \searrow 0$, which gives the second estimate of (12); the first one is completely similar. □

Theorem 5.2 (Cf. [24, 57, 94]) *Let $T : \mathcal{X} \to \mathcal{Y}$ be a (conjugate) linear map between two normed spaces $\mathcal{X}, \mathcal{Y}$ over the field $\mathbb{F}$. Then the following are equivalent:*

(i) *T preserves B-J orthogonality, i.e., $x \perp y \Longrightarrow Tx \perp Ty$ for every $x, y \in \mathcal{X}$.*
(ii) *T is a scalar multiple of an isometry, i.e., there is $\gamma \ge 0$ such that $\|Tx\| = \gamma\|x\|$ for every $x \in \mathcal{X}$.*

Sketch of the Proof Injectivity is straightforward: Indeed, assume $Tc = 0$ for some nonzero c. Notice that $0 = \|c+(-1)c\| \ngeq \|c\|$, so there exists a small enough $\varepsilon > 0$ such that if $\|x\| < \varepsilon$, then $\|(c + x) + (-1)c\| \ngeq \|c + x\|$, i.e., $(c + x) \not\perp c$. By Proposition 1.4, we can find a scalar $\lambda = \lambda_x$ with $(c + x) \perp (\lambda(c + x) + c)$, giving $Tx = T(c + x) \perp T(\lambda(c + x) + c) = \lambda^\sigma Tx$, where $\sigma : \mathbb{F} \to \mathbb{F}$ is either identity or a complex conjugation. This is possible only if $\lambda = 0$ (contradicting $(c + x) \not\perp c$) or if $Tx = 0$. Hence, if T is not injective, then $T = 0$.

Choose linearly independent $x, y \in \mathcal{X}$ and fix $\phi \in J(x)$. By Proposition 1.4 with $\alpha := \frac{\phi(y)}{\|x\|}$, we have $x \perp (\alpha x - y)$, so also $Tx \perp (\alpha^\sigma Tx - Ty)$. By Proposition 1.4 again, there exists $f \in J(Tx)$ which annihilates $(\alpha^\sigma Tx - Ty)$. Therefore, $\phi(y) = \alpha\|x\| = \frac{\|x\|}{\|Tx\|} f(Ty)^\sigma$, so also

$$\begin{aligned}\operatorname{Re}\phi(y) = \frac{\|x\|}{\|Tx\|}\operatorname{Re} f(Ty) &\in \frac{\|x\|}{\|Tx\|}\left[\inf_{f\in J(Tx)} \operatorname{Re} f(Ty), \sup_{f\in J(Tx)} \operatorname{Re} f(Ty)\right]\\ &= \frac{\|x\|}{\|Tx\|}\Big[D_-(Tx; Ty)\,,\ D_+(Tx; Ty)\Big].\end{aligned}$$

By taking infimum and supremum, respectively, over all $\phi \in J(x)$, we get

$$\tfrac{\|x\|}{\|Tx\|} D_-(Tx; Ty) \le D_-(x; y) \le D_+(x; y) \le \tfrac{\|x\|}{\|Tx\|} D_+(Tx; Ty).$$

If the norm is smooth at Tx, i.e., if $D_-(Tx; Ty) = D_+(Tx; Ty)$, then the above inequality simplifies into $D_\pm(Tx; Ty) = \frac{\|Tx\|}{\|x\|} D_\pm(x; y)$. This holds also if Tx is a nonsmooth point, namely, one can consider a two-dimensional subspace of $\mathcal{Y}$ spanned by Tx, Ty; smooth points are dense there, and then (12) can be applied.

Let $b(x) := \frac{\|Tx\|}{\|x\|}$. Then

$$
\begin{aligned}
0 &= D_+(Tx; Ty) - \frac{\|Tx\|}{\|x\|} D_+(x; y) \\
&= \lim_{t \searrow 0} \left(\frac{\|T(x + ty)\| - \|Tx\|}{t} - \frac{\|Tx\|}{\|x\|} \frac{\|x + ty\| - \|x\|}{t} \right) \\
&= \lim_{t \searrow 0} \left(\frac{\|T(x + ty)\| \cdot \|x\| - \|Tx\| \cdot \|x + ty\|}{t \cdot \|x\|} \right) \\
&= \lim_{t \searrow 0} \left(\frac{b(x + ty) - b(x)}{t} \cdot \|x + ty\| \right) = \|x\| \cdot \lim_{t \searrow 0} \left(\frac{b(x + ty) - b(x)}{t} \right).
\end{aligned}
$$

Likewise one shows that $\lim_{t \nearrow 0} \left(\frac{b(x+ty)-b(x)}{t} \right) = 0$. Hence, the function b is constant. □

Blanco and Turnšek considered in [24] (possibly nonlinear) bi-preservers of B-J orthogonality on projective Banach space $\mathbb{P}\mathcal{X} := \{[x] = \mathbb{F}x; \quad x \in \mathcal{X} \setminus \{0\}\}$. Their result is the following:

Theorem 5.3 (Cf. [24, Corollary 3.4]) *Let $\mathcal{X}$ be an infinite-dimensional, reflexive, smooth Banach space. If $\Phi\colon \mathbb{P}\mathcal{X} \to \mathbb{P}\mathcal{X}$ is a bijective map such that*

$$[x] \perp [y] \iff \Phi([x]) \perp \Phi([y])$$

then there exists a linear or conjugate linear surjective isometry $U\colon \mathcal{X} \to \mathcal{X}$, so that

$$\Phi([x]) = [Ux].$$

The smoothness assumption is indispensable here, for example, in $\mathbb{P}\mathbf{c}_0$ there exists bijective bi-preservers which are not induced by (conjugate) linear map, let alone isometry (see [24, Example 3.5]). It turns out, however, that the assumption about infinite dimensionality is not required. We will show this in our last chapter.

We also remark that there do exist complex reflexive Banach spaces which are conjugate-linear isometric but are not even isomorphic. The examples were constructed by Bourgain [27] and Kalton [51].

6 Graph Induced by B-J Orthogonality

It is possible to study B-J orthogonality on a normed space $(\mathcal{X}, \|\cdot\|)$ over the field $\mathbb{F}$ also with the help of a (directed) graph, $\hat{\Gamma} = \hat{\Gamma}(\mathcal{X})$. Its vertex set consists of all lines, i.e., points in a projective space $\mathbb{P}\mathcal{X} = \{[x] = \mathbb{F}x;\quad x \in \mathcal{X} \setminus \{0\}\}$, and vertices $[x], [y]$ form a directed edge $([x], [y])$ if $x \perp y$. Notice that its edges are well defined because of the homogeneity of B-J orthogonality. To simplify things, we will frequently not distinguish between a nonzero vector $x \in \mathcal{X}$ and the line $[x] = \mathbb{F}x \subseteq \mathcal{X}$ passing through it. Thus, we will often denote vertices of di-orthograph $\hat{\Gamma}(\mathcal{X})$ simply by x instead of $[x]$ but we do call them lines. The results of this section are from our recent paper [14].

6.1 Property Recognition

The fundamental question which we address here is the following:

> Let $\mathcal{P}$ be a given property on a normed space. Can we decide, using B-J orthogonality alone, if a space has property $\mathcal{P}$ or not?

Below we collect, mostly without proofs, some partial answers; we refer to [14] for proofs. The first one is merely a restatement of Corollary 2.12.

Proposition 6.1 *A normed space $\mathcal{X}$ over $\mathbb{F}$ is infinite-dimensional if and only if $\hat{\Gamma}(\mathcal{X})$ contains an infinite clique.*

Di-orthograph alone can compute the dimension of the underlying space.

Proposition 6.2 (Cf. [14, Lemma 2.3]) *Let $\mathcal{X}$ be a normed space over $\mathbb{F}$ and $n \geq 2$. Then the following statements are equivalent.*

(i) *For any $(n-1)$-tuple of lines $(x_1, \ldots, x_{n-1})$, one can always find $x_n \in \hat{\Gamma}(\mathcal{X})$ with $x_i \perp x_n$, $1 \leq i \leq n-1$.*
(ii) $\dim \mathcal{X} \geq n$.

Corollary 6.3 *A normed space $\mathcal{X}$ has dimension $n < \infty$ if and only if its di-orthograph $\hat{\Gamma}(\mathcal{X})$ satisfies item (i) with $k = 1, \ldots, n-1$ and does not satisfy item (i) for larger k.*

Di-orthograph can detect the presence of nonsmooth points. Compare with Theorem 2.6(ii) where smoothness was related to right uniqueness and right additivity of B-J orthogonality.

Proposition 6.4 (Cf. [14, Lemma 2.5]) *Let $\mathcal{X}$ be a normed space over $\mathbb{F}$ with $2 \leq \dim \mathcal{X} = n < \infty$. Then the norm in $\mathcal{X}$ is nonsmooth if and only if there exist $n-1$ lines $x_1, \ldots, x_{n-1} \in \hat{\Gamma}(\mathcal{X})$ and two additional distinct lines y_n, y_n' such that $x_i \perp x_j$ for $1 \leq i < j \leq n-1$, and $x_i \perp y_n$, $x_i \perp y_n'$ for $1 \leq i \leq n-1$.*

Di-orthograph alone can also detect if the norm is strictly convex (compare again with Theorem 2.6 (i) or (iv)). Given a vertex z in a di-orthograph $\hat{\Gamma}$, let $N_z := \{v \in \hat{\Gamma};\quad (z, v) \in E(\hat{\Gamma})\}$ denote its neighborhood; here $E(\hat{\Gamma})$ is the set of all directed edges of $\hat{\Gamma}$. Note that $N_z \neq \varnothing$ if the underlying normed space is at least two-dimensional.

Proposition 6.5 (Cf. [14, Lemma 2.6]) *A normed space $\mathcal{X}$ with $\dim \mathcal{X} \geq 2$ over $\mathbb{F}$ is strictly convex if and only if the function $\hat{\Gamma}(\mathcal{X}) \to 2^{\hat{\Gamma}(\mathcal{X})}$ which maps a vertex z to its neighborhood N_z is injective.*

6.2 Isomorphism Problem

How much information on the norm is encoded in the di-orthograph $\hat{\Gamma}(\mathcal{X})$? Clearly, if $A\colon \mathcal{X} \to \mathcal{Y}$ is a linear bijective isometry between two Banach spaces, then A induces an isomorphism of di-orthographs $\hat{\Gamma}(\mathcal{X})$ and $\hat{\Gamma}(\mathcal{Y})$. We show next the converse of this fact. Note that this problem is closely related to characterization of preservers of B-J orthogonality which was discussed in Sect. 5. Namely, bijective bi-preservers of B-J orthogonality between projective Banach spaces $\mathbb{P}(\mathcal{X})$ and $\mathbb{P}(\mathcal{Y})$ are exactly isomorphisms between $\hat{\Gamma}(\mathcal{X})$ and $\hat{\Gamma}(\mathcal{Y})$.

In the lemma below, *a curve* is a subset in $\mathbb{C}$, which is the image of *a path*, i.e., the image of a continuous map $\mathbf{r}\colon [a, b] \to \mathbb{C}$ where $[a, b] \subseteq \mathbb{R}$ is an interval with at least two different points. We say that a function $f : \mathbb{C} \to \mathbb{C}$ is bounded from above on a subset $\Gamma \subseteq \mathbb{C}$ if $\sup_{z\in\Gamma} |f(z)| < \infty$.

Lemma 6.6 (Cf. [14, Lemma 3.1]) *Suppose that a nonzero ring homomorphism $\sigma\colon \mathbb{C} \to \mathbb{C}$ is bounded from above on a curve $\Gamma \subseteq \mathbb{C}$ with more than one point. Then σ is continuous, hence either identity or a complex conjugation.*

Sketch of the Proof The full proof is relatively long and can be found in [14]. We first manipulate Γ by a finite number of rotations/translations/taking unions to obtain a closed curve $\hat{\Gamma}$ which separates the complex plane. Observe that the ring homomorphism σ remains bounded from above on $\hat{\Gamma}$. Then we use the fact, inspired by a paper of Simon and Taylor [81], that $\hat{\Gamma} - \hat{\Gamma} := \{\gamma_1 - \gamma_2;\quad \gamma_1, \gamma_2 \in \hat{\Gamma}\}$ has a nonempty interior. Since σ is clearly bounded from above also on $\hat{\Gamma} - \hat{\Gamma}$, it is hence bounded on an open set, and hence it must be continuous (see, e.g., [1, Corollary 5, p. 15]). □

We remark that, with the help of dimension theory for separable metric spaces (see, e.g., a monograph by Hurewitz and Wallman [44]), an even more general result was obtained by Shchepin [80]: If $\Gamma_1, \ldots, \Gamma_n$ are $n \geq 1$ compact connected subsets in $\mathbb{R}^n$ such that (i) $0 \in \bigcap_{i=1}^n \Gamma_i$ and (ii) there exist n linearly independent points $a_i \in \Gamma_i$, then the sum $\sum_{i=1}^n \Gamma_i := \{\gamma_1 + \cdots + \gamma_n;\quad \gamma_i \in \Gamma_i\} \subseteq \mathbb{R}^n$ is of dimension at least n and therefore it contains an open ball (see [44, Theorem IV.3]).

It was shown by Rätz [74] (see also Sundaresan [87, Lemma 1]) that B-J orthogonality in real normed spaces is Thalesian, that is, if x, y are B-J orthogonal vectors in a real normed space, then for every $\lambda_0 > 0$ there exists a scalar α such that $(x + \alpha y) \perp (\lambda_0 x - \alpha y)$. This fact was used by Wójcik [94, Proof of Theorem 3.1] to give an alternative proof that linear maps which preserve B-J orthogonality between real normed spaces are scalar multiples of isometries. The main idea of the proof of the next lemma comes from Wójcik's paper (see [94, Proof of Theorem 3.1]) and uses a partial extension of Thalesian property for B-J orthogonality in *complex* normed spaces: we show that if normalized vectors x, y are mutually B-J orthogonal, then there exists a curve $\Gamma \subseteq \mathbb{C}$ with more than one point, such that for every $\lambda \in \Gamma$ we can find $\alpha \in \mathbb{C}$ with $(x + \alpha y) \perp (\lambda x - \alpha y)$.

Recall that an additive map $\Phi\colon \mathcal{X} \to \mathcal{Y}$ between $\mathbb{F}$-vector spaces $\mathcal{X}$ and $\mathcal{Y}$ is σ-quasilinear if $\Phi(\lambda x) = \sigma(\lambda)\Phi(x)$ holds where $\sigma : \mathbb{F} \to \mathbb{F}$ is a field homomorphism (if σ is surjective, such maps are semilinear).

Proposition 6.7 (Cf. [14, Lemma 3.4]) *Let $\mathcal{X}, \mathcal{Y}$ be smooth normed complex spaces of dimension at least two, and let $\sigma : \mathbb{C} \to \mathbb{C}$ be a field homomorphism. If a nonzero σ-quasilinear map $\Phi\colon \mathcal{X} \to \mathcal{Y}$ preserves B-J orthogonality, then σ is identity or a complex conjugation.*

Sketch of the Proof By Proposition 2.13(i), there exist two mutually B-J orthogonal normalized vectors $x, y \in \mathcal{X}$ such that $\Phi(x) \neq 0$. Identify the two-dimensional subspace $\mathrm{span}_{\mathbb{C}}\{x, y\}$ with $(\mathbb{C}^2, \|\cdot\|)$, so that $x = (1, 0)$ and $y = (0, 1)$. By (5), a $\mathbb{C}$-linear supporting functional at a point $(1, \alpha) \in \mathbb{C}^2$ equals

$$f_\alpha = \left(\tfrac{\partial\|(1,\alpha)\|}{\partial \overline{z_1}}, \tfrac{\partial\|(1,\alpha)\|}{\partial \overline{z_2}}\right)^*, \quad (z_k = x_k + i y_k).$$

and its kernel contains a row vector

$$\left(\tfrac{\partial\|(1,\alpha)\|}{\partial z_2}, -\tfrac{\partial\|(1,\alpha)\|}{\partial z_1}\right). \tag{14}$$

Therefore, $(1, \alpha) \perp \left(\tfrac{\partial\|(1,\alpha)\|}{\partial z_2}, -\tfrac{\partial\|(1,\alpha)\|}{\partial z_1}\right)$. Since x, y are mutually B-J orthogonal, their supporting functionals equal

$$\left(\tfrac{\partial\|(1,0)\|}{\partial \overline{z_1}}, \tfrac{\partial\|(1,0)\|}{\partial \overline{z_2}}\right)^* = \mu_x \cdot (1, 0)^* \;\text{ and }\; \left(\tfrac{\partial\|(0,1)\|}{\partial \overline{z_1}}, \tfrac{\partial\|(0,1)\|}{\partial \overline{z_2}}\right)^* = \mu_y \cdot (0, 1)^*,$$

for some nonzero $\mu_x, \mu_y \in \mathbb{C}$, respectively. Since partial derivatives of a smooth norm are continuous (see Rockafellar [77, Theorem 25.5]), the first supporting functional equals the limit of $\left(\tfrac{\partial\|(1,\alpha)\|}{\partial \overline{z_1}}, \tfrac{\partial\|(1,\alpha)\|}{\partial \overline{z_2}}\right)^*$ as $\mathbb{R} \ni \alpha \to 0$. Next, by positive homogeneity of the norm, $\frac{\partial\|(1,\alpha)\|}{\partial \overline{z_k}} = \frac{\partial\|(1/\alpha,1)\|}{\partial \overline{z_k}}$ $(\alpha > 0)$, so the second supporting functional equals the limit of $\left(\frac{\partial\|(1,\alpha)\|}{\partial\overline{z_1}}, \frac{\partial\|(1,\alpha)\|}{\partial\overline{z_2}}\right)^*$ as $\mathbb{R} \ni \alpha \to \infty$. Hence,

$$w_\alpha = \frac{\frac{\partial\|(1,\alpha)\|}{\partial z_2}}{\frac{\partial\|(1,\alpha)\|}{\partial z_1}}\alpha x - \alpha y$$

is a well-defined vector-valued function of α around $\alpha = 0$ and parallel to (14), so $(x + \alpha y) \perp w_\alpha$. Also, its first component, i.e.,

$$\lambda(\alpha) := \frac{\frac{\partial\|(1,\alpha)\|}{\partial z_2}}{\frac{\partial\|(1,\alpha)\|}{\partial z_1}}\alpha,$$

is a continuous function which cannot vanish identically. As such, with α restricted to a suitable closed interval $I = [0, \varepsilon]$, its range is a curve $\Gamma \subseteq \mathbb{C}$ with more than one point.

Thus, for every $\alpha \in [0, \varepsilon]$ we have $(x + \alpha y) \perp (\lambda(\alpha)x - \alpha y)$, and therefore $\Phi(x + \alpha y) \perp \Phi(\lambda(\alpha)x - \alpha y)$. From here, the very definition of B-J orthogonality of $\Phi(x)$ and $\Phi(y)$ gives

$$\begin{aligned}\|\Phi(x)\| &\le \|\Phi(x) + \sigma(\alpha)\Phi(y)\| = \|\Phi(x + \alpha y)\| \\ &\le \|\Phi(x + \alpha y) + \Phi(\lambda(\alpha)x - \alpha y)\| = |1 + \sigma(\lambda(\alpha))| \cdot \|\Phi(x)\|\end{aligned}$$

so $|\sigma|$ is bounded from below by 1 on the curve $1 + \Gamma \subseteq \mathbb{C}$. Being multiplicative, σ is then bounded from above on the reciprocal curve, $\frac{1}{1+\Gamma}$. The result then follows from Lemma 6.6. □

We can now prove a version of Theorem 5.3, which is valid also for finite-dimensional complex normed spaces. The proof follows almost verbatim Blanco and Turnšek [24]. The only major difference is that at a final step of the proof they relied on Molnár's result, [66, Corollary 1], to show that a σ-quasilinear bijection which preserves B-J orthogonality must be (conjugate) linear. Instead, we replace [66, Corollary 1] with Proposition 6.7 which is valid also in finite-dimensional complex spaces.

Theorem 6.8 (Cf. [24, Corollary 3.4]) *Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be smooth Banach spaces over $\mathbb{F}$. Suppose that $\mathcal{X}$ is reflexive and $3 \le \dim \mathcal{X} \le \infty$. If $T: \hat{\Gamma}(\mathcal{X}) \to \hat{\Gamma}(\mathcal{Y})$ is an isomorphism of di-orthographs, then there exists a linear or conjugate-linear surjective map $U: \mathcal{X} \to \mathcal{Y}$ such that*

$$T[x] = [Ux] \quad \text{and} \quad \|Ux\|_{\mathcal{Y}} = \|x\|_{\mathcal{X}}.$$

Proof By Proposition 6.1, $\dim \mathcal{X} < \infty$ if and only if $\dim \mathcal{Y} < \infty$. It then follows from Corollary 6.3 that $\dim \mathcal{X} = \dim \mathcal{Y}$.

Define a map $S\colon \mathbb{P}\mathcal{X}^* \to \mathbb{P}\mathcal{Y}^*$ between projectivizations of dual spaces of $\mathcal{X}$ and $\mathcal{Y}$ by $[f] \mapsto [f_{T[x]}]$. Here, x is (any) normalized vector where f achieves its norm (it exists because of reflexivity of $\mathcal{X}$) and $f_{T[x]}$ is a unique supporting functional at some nonzero vector in a line $T[x] \in \mathbb{P}\mathcal{Y}$. Note that S is well-defined because if f achieves its norm on normalized vectors $x, y \in \mathcal{X}$, then f is a unique supporting functional for $[x]$ and $[y]$, giving that $N_{[x]} = N_{[y]} = \ker f$ (notations copied from Proposition 6.5). Therefore, also $N_{T[x]} = N_{T[y]}$ and, since $\mathcal{Y}$ is smooth, the supporting functionals at $T[x], T[y] \in \mathbb{P}\mathcal{Y}$ coincide.

The last argument shows at once that S is not only well defined but also injective. Also, the surjectivity of T implies that the range of S consists of all lines spanned by norm-attaining functionals. It is easily seen that, given $f \in \mathcal{X}^*$ and $z \in \mathcal{X}$, we have $f(z) = 0$ if and only if, for every functional $g \in S[f]$ and vector $y \in T[z]$, we have $g(y) = 0$.

We claim that T is a morphism of projective spaces. Namely, assume from the contrary that there exist the lines $[x], [y], [z] \in \hat{\Gamma}$ which satisfy

$$[z] \subseteq [x] + [y] \tag{15}$$

but their images $T[x], T[y], T[z]$ span a three-dimensional space. Then we can choose a norm-attaining normalized functional $g\colon \mathcal{Y} \to \mathbb{F}$ which annihilates $T[x]$ and $T[y]$ but not $T[z]$. Clearly, the line spanned by such g belongs to the range of S, so $[g] = S[f]$, and f annihilates $[x]$ and $[y]$ but not the line $[z]$ contradicting (15).

Indeed, T is a bijective morphism of projective spaces. By the (nonsurjective version of) Fundamental Theorem of Projective Geometry (see Faure [35]), there exist a field isomorphism $\sigma : \mathbb{F} \to \mathbb{F}$ and a σ-quasilinear map $V : \mathcal{X} \to \mathcal{Y}$ such that $T[x] = [Vx]$ for $x \in \mathcal{X} \setminus \{0\}$. Clearly, V must be orthogonality preserving, so, by Proposition 6.7, σ is either identity or a complex conjugation. By Theorem 5.2, the (conjugate) linear orthogonality preserving map $V\colon (\mathcal{X}, \|\cdot\|_{\mathcal{X}}) \to (\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ is a scalar multiple of an isometry (i.e., there exists a scalar $\mu \in \mathbb{C}$ such that $U := \mu V$ satisfies $\|Ux\|_{\mathcal{Y}} = \|x\|_{\mathcal{X}}$). □

We remark that, if $\mathcal{X}$ is finite-dimensional and smooth, then in the theorem above the smoothness of $\mathcal{Y}$ need not be assumed in advance; it follows from Proposition 6.4 (after Corollary 6.3 establishes that $\dim \mathcal{X} = \dim \mathcal{Y}$). Note also that Theorem 6.8 is not valid for $\dim \mathcal{X} = \dim \mathcal{Y} = 2$ because of the Radon planes; see also [14, Example 4.1] or [88, Theorem 4.1]. Here is a restatement of Theorem 6.8 in terms of preservers of B-J orthogonality.

Corollary 6.9 *Let $\mathcal{X}, \mathcal{Y}$ be normed spaces over $\mathbb{F}$, with $3 \le \dim \mathcal{X} < \infty$. Assume also that $\mathcal{X}$ is smooth. If there exists a bijection $\Phi\colon \mathbb{P}\mathcal{X} \to \mathbb{P}\mathcal{Y}$ such that*

$$[x] \perp [y] \Longleftrightarrow \Phi([x]) \perp \Phi([y]),$$

then there exists a surjective (conjugate) linear isometry $U\colon \mathcal{X} \to \mathcal{Y}$ such that

$$\Phi([x]) = [Ux] \quad \textit{and} \quad \|Ux\|_{\mathcal{Y}} = \|x\|_{\mathcal{X}}.$$

We have similar results valid in Banach spaces rather than their projectivizations. Let us show first that, in smooth spaces, B-J orthogonality can check for linear independence among two vectors:

Lemma 6.10 *Let x, y be nonzero vectors in a smooth normed space $\mathcal{X}$ over $\mathbb{F}$. Then the following conditions are equivalent:*

(i) $\mathbb{F}x \neq \mathbb{F}y$.
(ii) *There exists a vector $z \in \mathcal{X}$ such that $z \perp x$ but $z \not\perp y$.*

Proof (i) $\Longrightarrow$ (ii). Choose a normalized linear functional f which attains its norm at some vector $z \in \mathcal{X}$ and which satisfies $f(x) = 0$ and $f(y) \neq 0$. Then f is the unique supporting functional at z, and (ii) follows from Proposition 1.4.

(ii) $\Longrightarrow$ (i). Follows from homogeneity of B-J orthogonality for any normed space $\mathcal{X}$. □

Example The above lemma does not hold in nonsmooth spaces. Say, in $(\mathbb{R}^3, \|\cdot\|_\infty)$ we have that $u = (1, 1/2, 0)$ and $v = (1, 1/3, 0)$ are independent, and yet they are B-J orthogonal to the same vectors: $N_u = N_v = \{0\} \times \mathbb{R}^2$, and the same vectors are B-J orthogonal to them: ${}_vN = {}_uN = \{(a, b, c);\quad (|c| \geq \max\{|a|, |b|\}) \vee (a + b = 0 \wedge |c| \leq |b|)\}$, and also they are mutually B-J orthogonal to the same set of vectors, i.e., to $\{(0, b, c);\ |b| \leq |c|\}$.

Corollary 6.11 *Let $\mathcal{X}, \mathcal{Y}$ be smooth normed spaces over $\mathbb{F}$, with $3 \leq \dim \mathcal{X} < \infty$. If there exists a bijection $\Phi\colon \mathcal{X} \to \mathcal{Y}$ such that*

$$x \perp y \Longleftrightarrow \Phi(x) \perp \Phi(y),$$

then there exists a surjective (conjugate) linear isometry $U\colon \mathcal{X} \to \mathcal{Y}$ and a scalar-valued function $\gamma\colon \mathcal{X} \to \mathbb{F}$ such that

$$\Phi(x) = \gamma(x)Ux \quad \text{and} \quad \|Ux\|_{\mathcal{Y}} = \|x\|_{\mathcal{X}}.$$

Proof Since $x \perp x$ if and only if $x = 0$, we see that $\Phi(0) = 0$, and hence $\Phi(\mathcal{X}\backslash\{0\}) = \mathcal{Y}\backslash\{0\}$. By Lemma 6.10, Φ preserves in both directions linear dependence among two vectors, so it induces an isomorphism of di-orthographs $T\colon \hat{\Gamma}(\mathcal{X}) \to \hat{\Gamma}(\mathcal{Y})$. The rest follows from Corollary 6.9. □

By applying Theorem 6.8 instead of Corollary 6.9 we could state a similar result for reflexive, smooth, infinite-dimensional Banach spaces $\mathcal{X}, \mathcal{Y}$. Moreover, it turns out that reflexivity and completeness assumptions can be omitted, see [14, Theorem 3.11] for more details.

Finally, we remark that Tanaka [88, 89] has also studied preservers of B-J orthogonality between Banach spaces without the linearity assumption. In [88, Theorem 2.5] he managed to prove a version of Theorem 6.8 for real smooth Banach spaces without the reflexivity assumption. Besides, in [88, Theorem 4.3] he gave an example of strongly B-J isomorphic (see [88, Definition 3.8]) but not isometric nonsmooth real Banach spaces of arbitrary dimension. This shows that smoothness assumption cannot be omitted. We note that his construction uses Radon planes and "extends" them in a suitable way.

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Approximate Birkhoff-James Orthogonality in Normed Linear Spaces and Related Topics

Jacek Chmieliński

Abstract The classical Birkhoff-James orthogonality (BJ-orthogonality) in a real normed linear space is one of many possible, but arguably the most adequate, generalizations of the usual orthogonality relation in an inner product space. In this work, however, we are dealing not so much with the *exact* BJ-orthogonality as with its *approximate* version. In the first section of this chapter we introduce basic definitions connected with the notion of *approximate BJ-orthogonality*. Then we present a package of equivalent statements, defining in various ways the introduced concept. Some of these characterizations are known but some other are new. The second part of the paper is a survey on selected results depicting the areas where the approximate BJ-orthogonality can be applied or where it stimulates further studies.

Keywords Birkhoff-James orthogonality · Approximate Birkhoff-James orthogonality · Normed spaces · Inner product spaces · Linear operators · Orthogonality preserving mappings · Orthogonality of operators

1 Introduction and Preliminaries

This work is concentrated around the notions of *orthogonality* and *approximate orthogonality*. Whereas in inner product spaces they both can be defined naturally by comparison to zero the value of the inner product, it becomes a challenge to transfer these concepts to normed spaces. Even though it is possible and in many ways, none of them is fully satisfactory.

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1.1 Orthogonality in Inner Product Spaces

In an inner product space $(\mathcal{H}, \langle\cdot|\cdot\rangle)$ with the standard *orthogonality* relation: $x \perp y \Leftrightarrow \langle x|y\rangle = 0$, we have an equally natural notion of the *approximate orthogonality* or, more specifically, the ε*-orthogonality* (with a given $\varepsilon \in [0, 1)$):

$$x \perp^{\varepsilon} y \quad \Longleftrightarrow \quad |\langle x|y\rangle| \leq \varepsilon \|x\| \|y\|, \qquad x, y \in \mathcal{H}. \tag{1}$$

Of course, 0-orthogonality is just orthogonality. Equivalently, one can write (1) in the form

$$x \perp^{\varepsilon} y \quad \Longleftrightarrow \quad \frac{|\langle x|y\rangle|}{\|x\| \|y\|} \leq \varepsilon, \qquad x, y \in \mathcal{H} \setminus \{0\},$$

where the quotient $\frac{\langle x|y\rangle}{\|x\| \|y\|}$ can be interpreted as a *cosine* of the angle between vectors x and y.

We notice easily that the approximate orthogonality means the same as the exact orthogonality to some nearby vector; more precisely:

$$x \perp^{\varepsilon} y \quad \Longleftrightarrow \quad \exists\, z \in \operatorname{Lin}\{x, y\} : \; x \perp z \text{ and } \|z - y\| \leq \varepsilon \|y\|. \tag{2}$$

Indeed, if $x \perp^{\varepsilon} y$, then the vector $z = -\frac{\langle x|y\rangle}{\|x\|^2} x + y$ (or $z = y$ if $x = 0$) does the job. Conversely, assuming $x \perp z$ and $\|z - y\| \leq \varepsilon \|y\|$, we have

$$|\langle x|y\rangle| = |\langle x|y - z\rangle| \leq \|x\| \|y - z\| \leq \varepsilon \|x\| \|y\|,$$

i.e., $x \perp^{\varepsilon} y$. Since the relations $\perp$ and $\perp^{\varepsilon}$ are here symmetric, the roles of x, y and z in the above statement can be interchanged.

1.2 Orthogonalities in Normed Spaces

Although inner product spaces are the most natural venue for orthogonality, analogous relations may be considered also in normed linear spaces (cf. e.g. the survey [1]). Among the most natural, with clear geometrical background, are the *isosceles orthogonality*:

$$x \perp_{\mathrm{I}} y \quad \Longleftrightarrow \quad \|x + y\| = \|x - y\|$$

or the *Pythagorean orthogonality*:

$$x \perp_{\mathrm{P}} y \quad \Longleftrightarrow \quad \|x - y\|^2 = \|x\|^2 + \|y\|^2.$$

Many other, not considered here, orthogonality relations are defined, including axiomatic definitions in linear spaces or other structures (cf. [24, 26, 44]).

Birkhoff-James Orthogonality One of the most important orthogonality relation in a normed space, is the Birkhoff-James orthogonality (sometimes called the Birkhoff orthogonality). This concept is well known, extensively studied (cf. [1, 2, 6, 29–31]) and crucial for the present paper.

Let $\mathcal{X}$ be a real normed linear space and $x, y \in \mathcal{X}$. The *Birkhoff-James ortohogonality* (BJ-orthogonality) of x and y (in the given order) is defined as follows:

$$x \perp_{B} y \iff \forall\, \lambda \in \mathbb{R}: \ \|x + \lambda y\| \geq \|x\|.$$

For a fixed $x \in \mathcal{X} \setminus \{0\}$ we consider the (always nonempty) set of its *supporting functionals*:

$$J(x) = \{\varphi \in \mathcal{X}^* : \ \|\varphi\| = 1, \ \varphi(x) = \|x\|\},$$

where $\mathcal{X}^*$ denotes the dual space. With the Hahn-Banach theorem behind, the following characterization can be given [30, Corollary 2.2]:

$$x \perp_{B} y \iff \exists\, \varphi \in J(x): \ \varphi(y) = 0. \tag{3}$$

Norm Derivatives and Semi-inner Product To introduce yet another type of orthogonality let us recall the notion of *norm derivatives* in a real normed linear space $\mathcal{X}$ (cf. [3] for the background and properties of these functionals).

$$\begin{aligned} \rho'_{\pm}(x, y) &= \lim_{t \to 0^{\pm}} \frac{\|x + ty\|^2 - \|x\|^2}{2t} \\ &= \|x\| \lim_{t \to 0^{\pm}} \frac{\|x + ty\| - \|x\|}{t}, \qquad x, y \in \mathcal{X}. \end{aligned}$$

We define the related orthogonality relations

$$x \perp_{\rho_{\pm}} y \iff \rho'_{\pm}(x, y) = 0$$

and then the corresponding approximate orthogonalities (cf. [17, 18])

$$x \perp^{\varepsilon}_{\rho_{\pm}} y \iff |\rho'_{\pm}(x, y)| \leq \varepsilon \|x\|\,\|y\|.$$

Since not every norm is generated by an inner product, the following definition is sometimes very much helpful. Due to [25, 35] (see also the monograph [23]) each norm in a linear space $\mathcal{X}$ (over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$) admits a *semi-inner product* generating this norm, that is a functional $[\cdot|\cdot] : \mathcal{X} \times \mathcal{X} \to \mathbb{K}$ satisfying the conditions:

- $[\lambda x + \mu y|z] = \lambda\,[x|z] + \mu\,[y|z]\,,\quad x, y, z \in \mathcal{X},\ \lambda, \mu \in \mathbb{K};$
- $[x|\lambda y] = \overline{\lambda}\,[x|y]\,,\quad x, y \in \mathcal{X},\ \lambda \in \mathbb{K};$
- $|\,[x|y]\,| \le \|x\|\,\|y\|,\quad x, y \in \mathcal{X}$

and

- $[x|x] = \|x\|^2,\quad x \in \mathcal{X}.$

Generally, unless the norm is smooth, there could be many different semi-inner products related to a given norm in $\mathcal{X}$. If the norm in $\mathcal{X}$ comes from an inner product, then the inner product itself is the unique semi-inner product. For a fixed semi-inner product $[\cdot|\cdot]$ and $x, y \in \mathcal{X}$, we define the *semi-orthogonality*

$$x \perp_{s} y \quad \iff \quad [y|x] = 0$$

and the ε-*semi-orthogonality* (*approximate semi-orthogonality*)

$$x \perp_{s}^{\varepsilon} y \quad \iff \quad |\,[y|x]\,| \le \varepsilon\|x\|\,\|y\|.$$

It is known (cf. [23]) that for a real normed space $\mathcal{X}$ the values of ρ'_{+} and ρ'_{-} can be obtained as the supremum or infimum, respectively, over the values taken by all semi-inner products in $\mathcal{X}$ on a given pair of vectors. Namely,

$$\rho'_{+}(x, y) = \sup\{[y|x] : [\cdot|\cdot] \text{ is a semi-inner product on } \mathcal{X}\} \tag{4}$$

and

$$\rho'_{-}(x, y) = \inf\{[y|x] : [\cdot|\cdot] \text{ is a semi-inner product on } \mathcal{X}\}. \tag{5}$$

There is an apparent connection of the BJ-orthogonality and semi-orthogonality (cf. [23]): for any $x, y \in \mathcal{X}$ with $x \perp_{B} y$ there exists a semi-inner product $[\cdot|\cdot]$ such that $x \perp_{s} y$ (i.e., $[y|x] = 0$).

2 Approximate Birkhoff-James Orthogonality in Normed Linear Spaces

The objective of this part of the paper is to define and develop the concept of an *approximate* Birkhoff-James orthogonality. Various attempts and several characterizations will be discussed. While many of the facts presented here have been already known, there is some novelty in this section, both in results and in the presentation and proofs. For the convenience of the reader, we tried to make this part as detailed and self-contained as possible; only in few places we refer to results from the outside. From now on, let $\mathcal{X}$ be a real normed space with $\dim \mathcal{X} \ge 2$.

There are at least two notions of ε-BJ-orthogonality (with given $\varepsilon \in [0, 1)$). The first one was given by Dragomir [22]

$$x \underset{\varepsilon}{\perp}_{\mathrm{B}} y \quad \Longleftrightarrow \quad \forall \lambda \in \mathbb{R}: \ \|x + \lambda y\| \geq (1 - \varepsilon)\|x\|$$

and the other one, by Chmieliński [10]

$$x \perp_{\mathrm{B}}^{\varepsilon} y \quad \Longleftrightarrow \quad \forall \lambda \in \mathbb{R}: \ \|x + \lambda y\|^2 \geq \|x\|^2 - 2\varepsilon\|x\| \, \|\lambda y\|. \tag{6}$$

One can check that if the norm comes from an inner product then (6) coincides exactly with (1), whereas $\underset{\varepsilon}{\perp}_{\mathrm{B}}$ gives $\perp^{\eta}$ with $\eta = \sqrt{1 - (1 - \varepsilon)^2}$. The latter causes that it is sometimes more convenient to use a modification of the Dragomir's definition, replacing $1 - \varepsilon$ by $\sqrt{1 - \varepsilon^2}$:

$$x \perp_{\mathrm{D}}^{\varepsilon} y \quad \Longleftrightarrow \quad \forall \lambda \in \mathbb{R}: \ \|x + \lambda y\| \geq \sqrt{1 - \varepsilon^2}\|x\|.$$

Some relationships between $\perp_{\mathrm{B}}^{\varepsilon}$ and $\underset{\varepsilon}{\perp}_{\mathrm{B}}$ were established in [10, 19, 38]. In particular, if $\mathcal{X}$ is a real normed space and $\varepsilon \in \left[0, \frac{1}{2}\right)$, then (cf. [19])

$$x \perp_{\mathrm{B}}^{\varepsilon} y \quad \Longrightarrow \quad x \underset{2\varepsilon}{\perp}_{\mathrm{B}} y.$$

If $\mathcal{X}$ is a real uniformly smooth normed space and $\delta_{\mathcal{X}^*}$ denotes the modulus of convexity for the dual space $\mathcal{X}^*$, then for $\varepsilon \in [0, 2\delta_{\mathcal{X}^*}(1))$ and for any $x, y \in \mathcal{X}$ we have (cf. [38]):

$$x \underset{\varepsilon}{\perp}_{\mathrm{B}} y \quad \Longrightarrow \quad x \perp_{\mathrm{B}}^{\eta} y$$

with $\eta = \delta_{\mathcal{X}^*}^{-1}\left(\frac{\varepsilon}{2}\right)$.

In the sequel we will consider mainly the approximate BJ-orthogonality $\perp_{\mathrm{B}}^{\varepsilon}$, defined by (6).

It is known [18, Theorem 3.2] that if $\mathcal{X}$ is a real normed space, then for any semi-inner product on $\mathcal{X}$ and $\varepsilon \in [0, 1)$, there is $\perp_{\mathrm{s}}^{\varepsilon} \subset \perp_{\mathrm{B}}^{\varepsilon}$ (in particular $\perp_{\mathrm{s}} \subset \perp_{\mathrm{B}}$) and in a smooth space both relations coincide, that is $\perp_{\mathrm{s}}^{\varepsilon} = \perp_{\mathrm{B}}^{\varepsilon}$ (cf. also [50]).

In our considerations we will also use the following characterization of the ε-BJ-orthogonality (actually, it is a special case of a more general result given in [18, Theorem 3.1]):

Lemma 2.1 *Let $\mathcal{X}$ be a real normed linear space and let $\varepsilon \in [0, 1)$. Then, for arbitrary $x, y \in \mathcal{X}$ we have*

$$x \perp_{B}^{\varepsilon} y \quad \Longleftrightarrow \quad \rho'_{-}(x, y) - \varepsilon\|x\| \, \|y\| \leq 0 \leq \rho'_{+}(x, y) + \varepsilon\|x\| \, \|y\|. \tag{7}$$

Proof Assume $x \perp_{\mathrm{B}}^{\varepsilon} y$. Then, from (6), we have

$$-\varepsilon\|x\|\,\|y\| \leq \frac{\|x+\lambda y\|^2 - \|x\|^2}{2\lambda} \quad \text{for } \lambda > 0$$

and

$$\frac{\|x+\lambda y\|^2 - \|x\|^2}{2\lambda} \leq \varepsilon\|x\|\,\|y\| \quad \text{for } \lambda < 0.$$

Letting $\lambda \to 0^{\pm}$, we get $-\varepsilon\|x\|\,\|y\| \leq \rho'_{+}(x, y)$ and $\rho'_{-}(x, y) \leq \varepsilon\|x\|\,\|y\|$, respectively, as required.

Now, let us prove the converse. Since it is obvious for $x = 0$ or $y = 0$, we assume that $x, y \in \mathcal{X} \setminus \{0\}$. The first inequality in (7) can be written as

$$\lim_{\lambda \to 0^-} \frac{\|x+\lambda y\|^2 - \|x\|^2}{\lambda} \leq 2\varepsilon\|x\|\|y\|$$

and with an arbitrarily choosen $\gamma \in (0, 1)$ we have

$$\lim_{\lambda \to 0^-} \frac{\|x+\lambda y\|^2 - \|x\|^2}{\lambda} < 2(\varepsilon+\gamma)\|x\|\|y\|.$$

It follows that there exists $\delta_1 < 0$ such that

$$\forall\,\lambda \in [\delta_1, 0): \quad \frac{\|x+\lambda y\|^2 - \|x\|^2}{\lambda} < 2(\varepsilon+\gamma)\|x\|\|y\|,$$

whence

$$\forall\,\lambda \in [\delta_1, 0): \quad \|x\|^2 < \|x+\lambda y\|^2 + 2(\varepsilon+\gamma)\|x\|\|\lambda y\|. \tag{8}$$

Analogously, from the second inequality in (7) we get (for the same γ as above and for some $\delta_2 > 0$)

$$\forall\,\lambda \in (0, \delta_2]: \quad \|x\|^2 < \|x+\lambda y\|^2 + 2(\varepsilon+\gamma)\|x\|\|\lambda y\|. \tag{9}$$

Define $\varphi\colon \mathbb{R} \to \mathbb{R}$ by $\varphi(\lambda) := \|x+\lambda y\|^2 + 2(\varepsilon+\gamma)\|x\|\|\lambda y\|$. It can be easily shown that this mapping is convex. Inequalities (8) and (9) yield $\varphi(0) = \min\{\varphi(\lambda)\colon \lambda \in [\delta_1, \delta_2]\}$ and convexity of φ gives $\varphi(0) = \min\{\varphi(\lambda)\colon \lambda \in \mathbb{R}\}$. Thus

$$\|x\|^2 < \|x+\lambda y\|^2 + 2(\varepsilon+\gamma)\|x\|\|\lambda y\|, \qquad \lambda \in \mathbb{R} \setminus \{0\}. \tag{10}$$

Since γ was arbitrarily chosen from the interval (0, 1), letting $\gamma \to 0^+$ in (10) we obtain

$$\|x\|^2 \leq \|x + \lambda y\|^2 + 2\varepsilon\|x\|\|\lambda y\|, \qquad \lambda \in \mathbb{R} \setminus \{0\}.$$

Obviously, the above inequality holds true also for $\lambda = 0$, thus finally we get $x \perp_B^\varepsilon y$. □

Now we formulate a useful auxiliary result (originally stated in [15]).

Lemma 2.2 *Let $\mathcal{X}$ be a real normed space and let $x, y \in \mathcal{X}$ be such that $x \perp_B^\varepsilon y$ (with some $\varepsilon \in [0, 1)$). Then, for each $n \in \mathbb{N}$ there exists a semi-inner product $[\cdot|\cdot]_n$ in $\mathcal{X}$ such that*

$$|\,[y|x]_n\,| \leq \left(\varepsilon + \frac{1}{n}\right)\|x\|\,\|y\|. \tag{11}$$

Proof Applying (7) and (4)–(5), for each integer n we may choose semi-inner products $[\cdot|\cdot]_n'$ and $[\cdot|\cdot]_n''$ such that

$$[y|x]_n' < \left(\varepsilon + \frac{1}{n}\right)\|x\|\,\|y\| \quad \text{and} \quad -\left(\varepsilon + \frac{1}{n}\right)\|x\|\,\|y\| < [y|x]_n''.$$

It follows that for some $\lambda_n \in [0, 1]$ we have

$$-\left(\varepsilon + \frac{1}{n}\right)\|x\|\,\|y\| \leq \lambda_n\,[y|x]_n' + (1 - \lambda_n)\,[y|x]_n'' \leq \left(\varepsilon + \frac{1}{n}\right)\|x\|\,\|y\|.$$

Now, we consider a semi-inner product $[\cdot|\cdot]_n := \lambda_n\,[\cdot|\cdot]_n' + (1 - \lambda_n)\,[\cdot|\cdot]_n''$ and from the above inequalities it follows

$$-\left(\varepsilon + \frac{1}{n}\right)\|x\|\,\|y\| \leq [y|x]_n \leq \left(\varepsilon + \frac{1}{n}\right)\|x\|\,\|y\|.$$

□

Characterizations of the Approximate Orthogonality The notion of approximate BJ-orthogonality $\perp_B^\varepsilon$, as defined by (6), can be characterized in various ways. Now, we collect the already known as well as new characterizations in one theorem.

Theorem 2.3 (Characterization of $\perp_B^\varepsilon$) *Let $\mathcal{X}$ be a real normed linear space, $x, y \in \mathcal{X}$ and let $\varepsilon \in [0, 1)$. The following conditions are equivalent (each of them can be treated as a definition of $x \perp_B^\varepsilon y$):*

(i) $\exists\, c > 0\ \forall\, \lambda \in [-c, c] : \ \|x + \lambda y\|^2 \geq \|x\|^2 - 2\varepsilon\|x\|\,\|\lambda y\|$,
(ii) $\forall\, \lambda \in \mathbb{R} : \ \|x + \lambda y\|^2 \geq \|x\|^2 - 2\varepsilon\|x\|\,\|\lambda y\|$,
(iii) $\exists\, z \in \operatorname{Lin}\{x, y\} : \ x \perp_B z, \ \|z - y\| \leq \varepsilon\|y\|$,
(iv) $\exists\, \varphi \in J(x) : \ |\varphi(y)| \leq \varepsilon\|y\|$,

(v) $\forall\,\lambda \in \mathbb{R}:\ \|x+\lambda y\| \geq \|x\| - \varepsilon\|\lambda y\|$,
(vi) $\exists\, c>0\ \forall\,\lambda \in [-c,c]:\ \|x+\lambda y\| \geq \|x\| - \varepsilon\|\lambda y\|$,

Proof (i)⇒(ii) Assume (i) and define

$$f(\lambda) := \|x+\lambda y\|^2 + 2\varepsilon\|x\|\,\|\lambda y\|, \qquad \lambda \in \mathbb{R}.$$

Clearly, the mapping $f\colon \mathbb{R} \to [0,\infty)$ is convex and $f(0) = \|x\|^2$. Moreover, it follows from (i) that $f(\lambda) \geq \|x\|^2$ whenever $|\lambda| \leq c$. Assume, contrary to our claim, that (ii) does not hold and $\|x+\lambda_0 y\| < \|x\|^2 - 2\varepsilon\|x\|\,\|\lambda_0 y\|$ for some $\lambda_0 \in \mathbb{R}$. That would mean $f(\lambda_0) < \|x\|^2$. Taking $n \in \mathbb{N}$ big enough so that $\left|\frac{\lambda_0}{n}\right| \leq c$ and using convexity of f we would have then:

$$\begin{aligned}
\|x\|^2 &\leq f\left(\frac{\lambda_0}{n}\right) = f\left(\frac{1}{n}\lambda_0 + \left(1-\frac{1}{n}\right)\cdot 0\right)\\
&\leq \frac{1}{n} f(\lambda_0) + \left(1-\frac{1}{n}\right) f(0)\\
&< \frac{1}{n}\|x\|^2 + \left(1-\frac{1}{n}\right)\|x\|^2 = \|x\|^2,
\end{aligned}$$

a contradiction. Thus (ii) holds true.

(ii)⇒(iii) Suppose that (ii) holds and that $x \neq 0$ (otherwise the result is trivial). It follows from Lemma 2.2 that for an arbitrary $n \in \mathbb{N}$ there exists a semi–inner product $[\cdot|\cdot]_n$ in $\mathcal{X}$ such that (11) holds. Let $\perp_{s,n}$ denote the corresponding semi-orthogonality relation. Defining

$$z_n := -\frac{[y|x]_n}{\|x\|^2}x + y \in \operatorname{Lin}\{x,y\},$$

it is easy to see that $x \perp_{s,n} z_n$ and since $\perp_{s,n} \subset \perp_{\mathrm{B}}$, it follows that $x \perp_{\mathrm{B}} z_n$. Applying (11) we estimate $\|z_n - y\|$ and we get

$$x \perp_{\mathrm{B}} z_n \qquad \text{and} \qquad \|z_n - y\| \leq \left(\varepsilon + \frac{1}{n}\right)\|y\|. \tag{12}$$

Notice that $\|z_n\| \leq 2\|y\|$ whence the elements of the sequence $(z_n)_{n=1,2,\ldots}$ belong to a closed ball in a two-dimensional space $\operatorname{Lin}\{x,y\}$. Thus there exists $z \in \operatorname{Lin}\{x,y\}$ and a subsequence (z_{n_k}) convergent to z. Finally, (12) and continuity of the norm yield $x \perp_{\mathrm{B}} z$ and $\|z-y\| \leq \varepsilon\|y\|$.

(iii)⇒(iv) Assuming (iii) and applying (3), there exists $\varphi \in J(x)$ such that $\varphi(z) = 0$. Therefore, $|\varphi(y)| = |\varphi(z) - \varphi(y)| \leq \|z-y\| \leq \varepsilon\|y\|$, as claimed.

(iv)⇒(v) For an arbitrary $\lambda \in \mathbb{R}$ we have from (iv)

$$\begin{aligned}\|x+\lambda y\| &\geq |\varphi(x+\lambda y)| \geq |\varphi(x)| - |\varphi(\lambda y)| \\ &= \|x\| - \varepsilon\|\lambda y\|,\end{aligned}$$

i.e., (v) is satisfied.

(v)⇒(vi) Trivial.

(vi)⇒(i) Assuming that (vi) holds true, let $d := \min\left\{\frac{\|x\|}{\varepsilon\|y\|}, c\right\}$ (we assume that $y \neq 0$, otherwise (i) follows trivially). Now, if $|\lambda| \leq d$, then $\|x\| - \varepsilon\|\lambda y\| \geq 0$ whence:

$$\begin{aligned}\|x+\lambda y\|^2 &\geq (\|x\| - \varepsilon\|\lambda y\|)^2 = \|x\|^2 - 2\varepsilon\|x\|\,\|\lambda y\| + \varepsilon^2\|\lambda y\|^2 \\ &\geq \|x\|^2 - 2\varepsilon\|x\|\,\|\lambda y\|\end{aligned}$$

and we are done. □

Remarks 2.4

1. The equivalence of (ii), (iii) and (iv) was proved already in [15].
2. Conditions (v) and (vi) are new characterizations of the $\perp_B^\varepsilon$ relation, perhaps more convenient to use.
3. Notice also that (i) and (vi) prove that it is sufficient to verify either of the inequalities $\|x+\lambda y\|^2 \geq \|x\|^2 - 2\varepsilon\|x\|\,\|\lambda y\|$ or $\|x+\lambda y\| \geq \|x\| - \varepsilon\|\lambda y\|$ for λ in some neighbourhood of zero only.
4. It is clear that (iii) generalizes (2). However, since the BJ-orthogonality is not symmetric, the roles of x and y cannot be interchanged now.

Another Definition of an Approximate Birkhoff-James Orthogonality

We conclude this section with an attempt to propose a new definition of an approximate BJ-orthogonality, joining the advantages of the definitions $\underset{\varepsilon}{\perp}_B$ and $\perp_B^\varepsilon$.

Definition 2.5 Let $\mathcal{X}$ be a real normed linear space. For $x, y \in \mathcal{X}$ and $\varepsilon \in [0, 1)$ we define:

$$x \overset{\varepsilon}{\perp}_B y \iff \forall\,\lambda \in \mathbb{R}: \ \|x+\lambda y\| \geq \|x\| - \varepsilon \cdot \min\{\|x\|, \|\lambda y\|\}. \tag{13}$$

Remark 2.6 It is quite straightforward that for $x, y \in \mathcal{X} \setminus \{0\}$

$$x \overset{\varepsilon}{\perp}_B y \iff \|x+\lambda y\| \geq \begin{cases} \|x\| - \varepsilon\|\lambda y\|, & |\lambda| \leq \frac{\|x\|}{\|y\|}, \\ (1-\varepsilon)\|x\|, & |\lambda| > \frac{\|x\|}{\|y\|}. \end{cases} \tag{14}$$

Notice that if $|\lambda| \geq (2-\varepsilon)\frac{\|x\|}{\|y\|}$, then we have always

$$\|x+\lambda y\| \geq \|\lambda y\| - \|x\| \geq (2-\varepsilon)\|x\| - \|x\| = (1-\varepsilon)\|x\|$$

so in (13) (or (14)) one can restrict to considering $|\lambda| < (2-\varepsilon)\frac{\|x\|}{\|y\|}$.

Proposition 2.7 *For an arbitrary real normed space $\mathcal{X}$, $x, y \in \mathcal{X}$ and $\varepsilon \in [0, 1)$ we have:*

$$x \overset{\varepsilon}{\perp}_B y \qquad \Longleftrightarrow \qquad x \underset{\varepsilon}{\perp}_B y \quad \textit{and} \quad x \perp_B^{\varepsilon} y.$$

Proof Assume that $x \overset{\varepsilon}{\perp}_{\mathrm{B}} y$. The assertion is trivial for $x = 0$ or $y = 0$ so we assume that $x, y \in \mathcal{X} \setminus \{0\}$. If $|\lambda| \leq \frac{\|x\|}{\|y\|}$, then $-\varepsilon\|\lambda y\| \geq -\varepsilon\|x\|$ and from (14) we have

$$\|x+\lambda y\| \geq \|x\| - \varepsilon\|\lambda y\| \geq (1-\varepsilon)\|x\|.$$

Thus $\|x+\lambda y\| \geq (1-\varepsilon)\|x\|$ holds true for any λ and we get $x \underset{\varepsilon}{\perp}_{\mathrm{B}} y$.

It follows from (14) that $\|x+\lambda y\| \geq \|x\| - \varepsilon\|\lambda y\|$ for $|\lambda| \leq c$ where $c := \frac{\|x\|}{\|y\|}$, whence the condition (vi) in Theorem 2.3 yields $x \perp_{\mathrm{B}}^{\varepsilon} y$.

The reverse implication follows easily from the definition of $\underset{\varepsilon}{\perp}_{\mathrm{B}}$ and the characterization (v) of $\perp_{\mathrm{B}}^{\varepsilon}$ in Theorem 2.3. □

3 Applications and Generalizations—Review of Selected Results

This part of the paper is a survey on known results, with the aim to give at least an impression of various places where the notions of BJ-orthogonality and its approximate counterpart can be considered or applied. Actually, we will concentrate ourselves here on approximate orthogonality. The role of the (exact) BJ-orthogonality in studies on the geometry of Banach spaces is well described, e.g., in [42].

This section is organized as follows. At first, we discuss the natural extension from general normed spaces to the space of linear bounded operators. Then we consider an issue belonging to *linear preserver problems*, namely preservation or approximate preservation (by linear operators) of the BJ-orthogonality relation. In the third subsection we are pointing out some recent studies on the approximate symmetry of the BJ-orthogonality, which is a topic very much connected to approximate orthogonality as well as to the geometry of the considered space.

3.1 Approximate Birkhoff-James Orthogonality in Operator Theory

There is a vast literature devoted to various aspects of BJ-orthogonality in the space of linear bounded operators. Let $\mathcal{H}$ be a Hilbert space. Denote by $\mathbb{B}(\mathcal{H})$ the space of all linear and bounded operators on $\mathcal{H}$ and by $\mathbb{K}(\mathcal{H})$ the subspace of $\mathbb{B}(\mathcal{H})$ consisting of all compact operators. For a given $T \in \mathbb{B}(\mathcal{H})$ we denote by M_T the set of unit vectors at which T attains its norm, i.e.,

$$M_T = \{x \in S_{\mathcal{H}} : \ \|Tx\| = \|T\|\},$$

where $S_{\mathcal{H}}$ stands for the unit sphere in $\mathcal{H}$. We begin with presenting a canonical result given by Bhatia and Šemrl [8] (and independently by Paul [41]).

Theorem 3.1 ([8, Theorem 1.1, Remark 3.1]) *Let $\mathcal{H}$ be a Hilbert space and let $T, A \in \mathbb{B}(\mathcal{H})$. Then, the following conditions are equivalent:*

(1) $T \perp_B A$;
(2) $\exists\, (x_n)_{n=1}^{\infty} \subset S_{\mathcal{H}} : \quad \lim_{n\to\infty} \|Tx_n\| = \|T\|, \quad \lim_{n\to\infty} \langle Tx_n | Ax_n \rangle = 0.$

Moreover, if $\dim \mathcal{H} < \infty$*, then each of the above conditions is equivalent to:*

(3) $\exists\, x_0 \in M_T : \quad Tx_0 \perp Ax_0.$

The above result was developed by various authors. Benítez, Fernández and Soriano [7] showed that the equivalence (1)⇔(3) is valid if and only if $\mathcal{H}$ is a Hilbert space (cannot be replaced by a Banach space). Generalizations of Theorem 3.1 have been obtained, e.g., by Arambašić and Rajić [4], Sain, Paul and Hait [45, 46], Grover [27] and Wójcik [51].

In [15] authors provided the first extension of the result of Bhatia and Šemrl to *approximate* orthogonality in $\mathbb{B}(\mathcal{H})$.

Theorem 3.2 ([15, Theorem 3.2]) *Let $\mathcal{H}$ be a real Hilbert space, let $T, A \in \mathbb{B}(\mathcal{H})$ and let $\varepsilon \in [0, 1)$. Then, the following conditions are equivalent:*

(1) $T \perp_B^{\varepsilon} A$;
(2) $\exists\, (x_n)_{n=1}^{\infty} \subset S_{\mathcal{H}}$:

$$\lim_{n\to\infty} \|Tx_n\| = \|T\|, \quad \lim_{n\to\infty} |\langle Tx_n | Ax_n \rangle| \leq \varepsilon \|T\|\,\|A\|.$$

Moreover, if $\dim \mathcal{H} < \infty$*, then each of the above conditions is equivalent to:*

(3) $\exists\, x_0 \in M_T : \quad |\langle Tx_0 | Ax_0 \rangle| \leq \varepsilon \|T\|\,\|A\|.$

If $\dim \mathcal{H} < \infty$ *and, additionally, $M_T \subset M_A$, then each of the above three conditions is equivalent also to:*

(4) $\exists\, x_0 \in M_T : \quad Tx_0 \perp^{\varepsilon} Ax_0.$

The condition $\dim \mathcal{H} < \infty$ can be replaced by compactness of T.

Theorem 3.3 ([15, Theorem 3.4]) *Let $\mathcal{H}$ be a real Hilbert space, let $T, A \in \mathbb{B}(\mathcal{H})$ and let $\varepsilon \in [0, 1)$. Assume that $M_T \subset M_A$ and $T \in \mathbb{K}(\mathcal{H})$. Then $T \perp_B^\varepsilon A$ if and only if*

$$\exists\, x_0 \in M_T : \quad Tx_0 \perp^\varepsilon Ax_0.$$

The following characterization of the approximate orthogonality in $\mathbb{B}(\mathcal{H})$ was given later by Paul et al. [43].

Theorem 3.4 ([43, Theorem 3.1])

(1) *Let $T \in \mathbb{B}(\mathcal{H})$. Then for any $A \in \mathbb{B}(\mathcal{H})$, $T \perp_B^\varepsilon A \Leftrightarrow |\langle Tx|Ax\rangle| \leq \varepsilon\|T\|\,\|A\|$ for some $x \in M_T$ if and only if $M_T = S_{\mathcal{H}_0}$ for some finite dimensional subspace $\mathcal{H}_0$ of $\mathcal{H}$ and $\|T\|_{\mathcal{H}_0^\perp} < \|T\|$.*

(2) *Moreover, if $M_T \subset M_A$ then $T \perp_B^\varepsilon A \;\Leftrightarrow\; Tx \perp^\varepsilon Ax$ for some $x \in M_T$ if and only if $M_T = S_{\mathcal{H}_0}$ for some finite dimensional subspace $\mathcal{H}_0$ of $\mathcal{H}$ and $\|T\|_{\mathcal{H}_0^\perp} < \|T\|$.*

Now, let $\mathcal{X}, \mathcal{Y}$ be normed linear spaces. By $\mathbb{B}(\mathcal{X}, \mathcal{Y})$ we denote the space of all bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$ and by $\mathbb{K}(\mathcal{X}, \mathcal{Y})$ its subspace consisting of compact operators.

In the above quoted paper [43], some characterizations were given for the approximate BJ-orthogonality for compact operators defined on a reflexive Banach space.

Theorem 3.5 ([43, Theorem 3.2]) *Let $\mathcal{X}$ be a reflexive Banach space and let $\mathcal{Y}$ be a normed space. Let $T, A \in \mathbb{K}(\mathcal{X}, \mathcal{Y})$ and $M_T = D \cup (-D)$, where D is a nonempty compact connected subset of $S_\mathcal{X}$. Then $T \perp_B^\varepsilon A$ if and only if there exists $x \in M_T$ such that $\|Tx + \lambda Ax\|^2 \geq \|T\|^2 - 2\varepsilon\|T\|\,\|\lambda A\|$. Moreover if $M_T \subset M_A$, then $T \perp_B^\varepsilon A$ if and only if $Tx \perp_B^\varepsilon Ax$.*

Theorem 3.6 ([43, Theorem 3.3]) *Let $\mathcal{X}$ be a reflexive Banach space and $\mathcal{Y}$ be a normed space. Let $T, A \in \mathbb{K}(\mathcal{X}, \mathcal{Y})$. Then $T \perp_B^\varepsilon A$ if and only if there exist $x, y \in M_T$ such that $\|Tx + \lambda Ax\|^2 \geq \|T\|^2 - 2\varepsilon\|T\|\,\|\lambda A\|$ for all $\lambda \geq 0$ and $\|Ty + \lambda Ay\|^2 \geq \|T\|^2 - 2\varepsilon\|T\|\,\|\lambda A\|$ for all $\lambda \leq 0$.*

For arbitrary normed spaces and linear bounded operators the following characterization was given.

Theorem 3.7 ([43, Theorem 3.4]) *Let $\mathcal{X}, \mathcal{Y}$ be two normed spaces. Suppose $T \in \mathbb{B}(\mathcal{X}, \mathcal{Y})$ be nonzero. Then for any $A \in \mathbb{B}(\mathcal{X}, \mathcal{Y})$, $T \perp_B^\varepsilon A$ if and only if either of the conditions in* (a) *or* (b) *holds.*

(a) *There exists a sequence (x_n) of unit vectors such that*

$$\|Tx_n\| \to \|T\| \text{ and } \lim_{n\to\infty} \|Ax_n\| \leq \varepsilon\|A\|.$$

(b) *There exist two sequences* (x_n), (y_n) *of unit vectors and two sequences of positive real numbers* (ε_n), (δ_n) *such that*

(i) $\varepsilon_n \to 0$, $\delta_n \to 0$, $\|Tx_n\| \to \|T\|$, $\|Ty_n\| \to \|T\|$ *as* $n \to \infty$*;*
(ii) $\|Tx_n+\lambda Ax_n\|^2 \geq (1-\varepsilon_n^2)\|Tx_n\|^2 - 2\varepsilon\sqrt{1-\varepsilon_n^2}\|Tx_n\|\,\|\lambda A\|$ *for all* $\lambda \geq 0$*;*
(iii) $\|Ty_n+\lambda Ay_n\|^2 \geq (1-\delta_n^2)\|Ty_n\|^2 - 2\varepsilon\sqrt{1-\delta_n^2}\|Ty_n\|\,\|\lambda A\|$ *for all* $\lambda \leq 0$.

Under some additional conditions on the norm attainment set, further characterizations of the approximate BJ-orthogonality of operators between normed linear spaces were obtained in particular by Mal et al. [36]. For bilinear operators the topic was studied recently by Khurana and Sain [33]. For a positive operator $A \in \mathbb{B}(\mathcal{H})$ acting on a Hilbert space $\mathcal{H}$, Sen et al. [48] defined A-orthogonality in $\mathcal{H}$ and A-BJ-orthogonality in $\mathbb{B}(\mathcal{H})$. Next, they introduced the notions of (ε, A)-orthogonality and (ε, A)-BJ-orthogonality in $\mathcal{H}$, and finally (ε, A)-BJ-orthogonality in $\mathbb{B}(\mathcal{H})$. In particular, some characterizations of those approximate orthogonalities were given. Investigations concerning BJ-orthogonality in semi-Hilbertian spaces were carried on also by Zamani [52].

Now, let us consider the space $\mathcal{C}(K)$ of all real continuous mappings defined on a locally compact topological space K endowed with the supremum norm. A subspace $\mathcal{C}_0(K)$ of $\mathcal{C}(K)$ defined by

$$\mathcal{C}_0(K) := \{f \in \mathcal{C}(K) : \ \forall\, \varepsilon > 0, \ \text{the set } \{t \in K : \ |f(t)| \geq \varepsilon\} \text{ is compact}\}$$

has the property that for $f \in \mathcal{C}_0(K)$ the set $M_f := \{t \in K : |f(t)| = \|f\|\}$ is always nonempty and compact. In [15] a characterization of approximate BJ-orthogonality on $\mathcal{C}_0(K)$ was given.

Theorem 3.8 ([15, Theorem 3.6]) *Let* $f, g \in \mathcal{C}_0(K)$, $f \neq 0 \neq g$. *Assume that* M_f *is connected. Then, the following conditions are equivalent:*

(a) $f \perp_B^{\varepsilon} g$*;*
(b) $\exists\, t_1 \in M_f : \ |g(t_1)| \leq \varepsilon\|g\|$.

3.2 Operators Approximately Preserving Orthogonality

A linear mapping T between two inner product spaces $\mathcal{H}$ and $\mathcal{K}$ which *preserves orthogonality*, i.e., such that

$$x \perp y \quad \Longrightarrow \quad Tx \perp Ty, \qquad x, y \in \mathcal{H},$$

is necessarily a linear similarity, i.e., an isometry multiplied by a constant (quite elementary proof can be found, e.g., in [11, Theorem 2.1]). Koldobsky [34] and then Blanco and Turnšek [9] extended this result to normed linear spaces with BJ-orthogonality.

A related study on linear mappings *approximately preserving orthogonality* for inner product spaces was started in [11, 12] and continued in [49]. A generalization to C^*-modules was given by Ilišević and Turnšek [28].

Another generalization, toward normed spaces, was tackled first for the isosceles-orthogonality in [16]. Later, Mojškerc and Turnšek [38] have shown that each linear mapping between normed spaces, approximately preserving BJ-orthogonality, is an approximate similarity.

Theorem 3.9 ([38, Theorem 3.5]) *Let $\mathcal{X}, \mathcal{Y}$ be normed spaces, $\varepsilon \in [0, \frac{1}{2})$ and $T\colon \mathcal{X} \to \mathcal{Y}$ a linear mapping satisfying*

$$x \perp y \quad \Longrightarrow \quad Tx \perp_B^{\varepsilon} Ty, \qquad x, y \in \mathcal{X}.$$

Then

$$(1 - 16\varepsilon)\|T\|\,\|x\| \le \|Tx\| \le \|T\|\,\|x\|, \qquad x \in \mathcal{X}.$$

The assumption $\varepsilon < \frac{1}{2}$ is needed to prove that T is bounded, i.e., $\|Tx\| \le \|T\|\,\|x\|$ holds true. The left-hand side inequality in the assertion is redundant unless $\varepsilon < \frac{1}{16}$. The constant 16ε can be diminished to 8ε for real spaces (cf. [38, Remark 3.1]).

A more extended review on the described above results and related topics can be found in [13].

3.3 Approximate Symmetry of the Birkhoff-James Orthogonality

Approximate orthogonality has been used to introduce the notion of approximate symmetry of the BJ-orthogonality. It is known that BJ-orthogonality generally is not symmetric; even more—its symmetry characterizes inner product spaces among normed spaces of the dimension greater than or equal to 3. Only for a 2-dimensional linear space it is possible to find a norm which does not come from an inner product but the corresponding $\perp_B$ orthogonality is symmetric (more on such norms, so called Radon norms, in [37]). In [19] the following definition was introduced.

Definition 3.10 The BJ-orthogonality relation in a normed linear space $\mathcal{X}$ is called *approximately symmetric*, or more precisely: *ε-symmetric* for some $\varepsilon \in [0, 1)$, if for any $x, y \in \mathcal{X}$

$$x \perp_B y \quad \Longrightarrow \quad y \perp_B^{\varepsilon} x.$$

In [19] there were given several conditions sufficient for approximate symmetry of $\perp_B$. On the other hand, there were given examples of (classes of) normed spaces for which the BJ-orthogonality is not approximately symmetric. Moreover, approximate symmetry of $\perp_B$ does not imply inner product structure (regardless of the dimension of the underlying space).

It is obvious that symmetry of the approximate orthogonality implies approximate symmetry of the orthogonality. But not vice versa; actually even the exact symmetry of the orthogonality does not imply symmetry of the approximate orthogonality (cf. [19, Example 2.5]).

In [19, Theorem 4.1] it was proved that in each real uniformly convex normed space, BJ-orthogonality is approximately symmetric. The same assertion is true (cf. [19, Theorem 4.2]) for real finite-dimensional and smooth normed spaces. Moreover, if $\mathcal{X}$ is a real uniformly convex and smooth Banach space, then (cf. [19, Theorem 4.6]) there exists $\varepsilon \in [0, 1)$ such that the BJ-orthogonality relations in $\mathcal{X}$, $\mathcal{X}^*$ and $\mathcal{X}^{**}$ are ε-symmetric.

Whereas the above described approximate symmetry of $\perp_B$ has a global setting, its local version was considered in [14]. Namely, the following notion was introduced.

Definition 3.11 Let $\mathcal{X}$ be a normed linear space and let $x \in \mathcal{X}$. We say that x is *approximately left-symmetric* if there exists $\varepsilon_x \in [0, 1)$ such that whenever $y \in \mathcal{X}$ and $x \perp_B y$, it follows that $y \perp_B^{\varepsilon_x} x$. Analogously, we define the *approximate right-symmetry* of x.

In particular, it was proved in [14, Theorem 3.10] that the approximate symmetry of the orthogonality and the local approximate left-symmetry at each point of the unit sphere are equivalent properties of any finite-dimensional polyhedral Banach space. Moreover a geometrical characterization of this property was given. An analogous definition was introduced also for Dragomir's definition of approximate orthogonality and it was proved that the BJ-orthogonality is approximately symmetric in the sense of Dragomir in all finite-dimensional Banach spaces.

Recently, approximate symmetry of the BJ-orthogonality has been studied by Set et al. [48] for semi-Hilbertian structures induced by positive operators acting on a Hilbert space. The notion of (ε, A)-approximate right (left) symmetry of the BJ-orthogonality of linear A-bounded operators on $\mathcal{H}$ was introduced.

3.4 Varia

Apart from mentioned above, there are other areas of research where the notion of approximate BJ-orthogonality is involved. Without giving too much details we only signal their presence.

Approximate Birkhoff-James Orthogonality in Hilbert Modules In an inner product module over a C^*-algebra one can define an orthogonality relation by using both the inner product as well as the corresponding norm. This gives rise to various types of orthogonality (see Arambašić and Rajić [4, 5]). Approximate

BJ-orthogonality in such realm has been also considered by Moslehian and Zamani [39].

Orthogonality Sets Two notions of *approximate Birkhoff-James orthogonality sets* have been introduced by Sain et al. [47]. Given $x \in \mathcal{X}$ and $\varepsilon \in [0, 1)$ one can consider

$$F(x, \varepsilon) := \{y \in \mathcal{X} : \; x \perp_{\mathrm{D}}^{\varepsilon} y\}; \quad G(x, \varepsilon) := \{y \in \mathcal{X} : \; x \perp_{\mathrm{B}}^{\varepsilon} y\}.$$

A geometrical description of these two sets in an arbitrary normed space was given: each of them is a union of two-dimensional normal cones.

A similar notion of Birkhoff–James ε-orthogonality sets for matrices and for matrix polynomials, based on the Dragomir's definition of the approximate BJ-orthogonality, were defined and studied in [20] (see also [21, 32, 40]).

The list of topics and results connected with the approximate BJ-orthogonality which we dealt with in this section is by no means complete. It was our purpose just to show that the considered concepts may be applied in a variety of further studies.

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Orthogonally Additive Operators on Vector Lattices

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Abstract An orthogonally additive operator $T: E \to F$ between vector lattices E, F is a map which satisfies $T(x + y) = T(x) + T(y)$ for all disjoint elements $x, y \in E$. We summarize some results and open problems on this class of operators. We focus mainly on the vector lattice structure of different partial subclasses of the vector space of all orthogonally additive operators, some versions of order continuity, certain domination problems, representation theorems and Banach lattice structure of orthogonally additive operators.

Keywords Orthogonally additive operator · Abstract Uryson operator · Nemytskii operator · C-compact operator · AM-compact operator · Narrow operator · Vector lattice · Lateral ideal · Lateral band

1 Introduction

The work of famous mathematicians Drewnowski et al. [15, 16, 32, 33] has led to the appearance and study of orthogonally additive operators (OAOs) on vector lattices. OAOs generalize linear ones (see for a definition below) and naturally appear in different areas of modern mathematics, e.g. partial differential equations, convex geometry, dynamical systems and stochastic processes [31, 50, 57]. It is worth mentioning that some classical operators of nonlinear analysis including Uryson, Hammerstein and Nemyskii operators are orthogonally additive in appropriate

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function spaces [35]. The theory of OAOs is being developed by different authors in many directions [1, 4, 5, 19–21, 47, 49].

The aim of this paper is to discuss some known results and state open problems which can be useful for further development of the theory. Let us describe the content of this article. In the next section, we briefly present necessary background on vector lattices and orthogonally additive operators. In Sect. 3, we discuss the lateral partial order on vector lattices, which is useful for the study of OAOs. Section 4 presents several extension theorems. In Sect. 5 we provide results on the order structure of different classes of OAOs, including elegant formulas of Riesz-Kantorovich type for the lattice operations over OAOs. In Sect. 6, we discuss on C-compact and AM-compact OAOs. In particular, we show that, under some mild conditions, the set of all C-compact OAOs is a projection band in $\mathcal{OA}_r(E, F)$ and present a solution to the domination problem for AM-compact abstract Uryson operators. Section 7 is devoted to the relationships between different partial order continuities, which are clear for linear operators, however become involved for OAOs. In Sect. 8 we present some theorems on narrow OAOs, including a deep result on the representation of regular operators as the sum of a pseudo-embedding and a diffuse (= narrow) operator, both for linear and orthogonally additive settings. In Sect. 9, we endow the vector lattice of order bounded OAOs with a norm, such that the set of all OAOs having finite norm becomes a Banach lattice in which the subspace of all linear bounded operators is contractive complemented by means of plenty projections called linear sections of OAOs. Final section contains some open problems. Several results are provided with proofs, which are not new.

The standard reference books on the theory of vector and Banach lattices are [8, 37]. All vector lattices we consider below are supposed to be Archimedean. We write $x = \bigsqcup_{i=1}^n x_i$ to express that $x = \sum_{i=1}^n x_i$ and $x_i \perp x_j$ for all $i \neq j$. In particular, for $n = 2$ we use the notation $x = x_1 \sqcup x_2$. We say that y is a *fragment* (a *component*) of $x \in E$ and use the notation $y \sqsubseteq x$, if $y \perp (x - y)$. The set of all fragments of $x \in E$ is denoted by $\mathfrak{F}_x$. We say that $x_1, x_2 \in \mathfrak{F}_x$ are *mutually complemented* if $x = x_1 \sqcup x_2$. It is a standard exercise to show that $\sqsubseteq$ is a partial order on E, called the *lateral order* (see [38] for a detailed study of this order).

2 Definition and Main Examples of OAOs

Definition 2.1 Let E be a vector lattice and X a real vector space. A function $T : E \to X$ is called an *orthogonally additive operator* (OAO in short) provided $T(x + y) = T(x) + T(y)$ for any disjoint elements $x, y \in E$.

It is not hard to check that $T(0) = 0$. The set of all OAOs from E to X is a real vector space with respect to the natural linear operations.

Definition 2.2 Let E, F be vector lattices. An OAO $T: E \to F$ is said to be:

- *positive* if $Tx \geq 0$ holds in F for all $x \in E$;
- *regular* if $T = S_1 - S_2$, where S_1, S_2 are positive OAOs from E to F;
- *order bounded*, or an *abstract Uryson operator*, if it maps order bounded sets in E to order bounded sets in F;
- *disjointness preserving*, if $Tx \perp Ty$ for every disjoint $x, y \in E$;
- *non-expanding*, if $E = F$ and $Tx \in \{x\}^{dd}$ for every $x \in E$;
- *C-bounded* or a *Popov operator*, if the set $T(\mathfrak{F}_x)$ is order bounded in F for every $x \in E$.

Observe that if $T: E \to F$ is a positive OAO and $x \in E$ is such that $T(x) \neq 0$ then $T(-x) \neq -T(x)$. So, the positivity of OAOs is completely different from that of linear operators, and the only linear operator which is positive in the sense of OAOs is zero. A positive OAO need not be order bounded. Indeed, every function $T: \mathbb{R} \to \mathbb{R}$ with $T(0) = 0$ is an OAO, and, obviously, not all such functions are order bounded.

The cone of all positive OAOs from E to F is denoted by $\mathcal{OA}_+(E, F)$. The vector spaces of all regular, abstract Uryson and C-bounded operators we denote by $\mathcal{OA}_r(E, F)$, $\mathcal{U}(E, F)$ and $\mathcal{P}(E, F)$ respectively. We note that the inclusion $\mathcal{U}(E, F) \subset \mathcal{P}(E, F)$ is strict even in the one-dimensional case.

Example Let $E = F = \mathbb{R}$ and map $T: \mathbb{R} \to \mathbb{R}$ be defined by the formula

$$T(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Since for every $x \in \mathbb{R}$ the set $\mathfrak{F}_x$ contains only two elements $\{0, x\}$, one has $T \in \mathcal{P}(E, F)$. On the other hand, for the order bounded subset $(-1, 1) \subset \mathbb{R}$, the subset $T(-1, 1)$ is order unbounded and hence $T \notin \mathcal{U}(E, F)$.

Proposition 2.3 *Let E, F be vector lattices. Then $\mathcal{OA}_+(E, F) \subseteq \mathcal{P}(E, F)$.*

Proof Given any $T \in \mathcal{OA}_+(E, F)$, $x \in E$ and $y \in \mathfrak{F}_x$, one has

$$T(x) = T\big(y \sqcup (x - y)\big) = T(y) + T(x - y) \geq Ty \geq 0,$$

hence $T(x) = \sup T(\mathfrak{F}_x)$, and $T \in \mathcal{P}(E, F)$ is proved. □

Consider some traditional examples of OAOs which motivate the development of the general theory of OAOs.

Definition 2.4 Let (A, Σ, μ) and (B, Ξ, ν) be finite measure spaces. By $(A \times B, \mu \otimes \nu)$ we denote the completion of their product measure space. A map

$K: A \times B \times \mathbb{R} \to \mathbb{R}$ is said to be a *Carathéodory function* if the following conditions hold:

(C_1) $K(\cdot, \cdot, r)$ is $\mu \otimes \nu$-measurable for all $r \in \mathbb{R}$;
(C_2) $K(s, t, \cdot)$ is continuous on $\mathbb{R}$ for $\mu \otimes \nu$-almost all $(s, t) \in A \times B$.

We say that a Carathéodory function K is *normalized* if $K(s, t, 0) = 0$ for $\mu \otimes \nu$-almost all $(s, t) \in A \times B$.

Example ([18, Proposition 3.2]) Let E be an order ideal of $L_0(\nu)$, $K: A \times B \times \mathbb{R} \to \mathbb{R}$ a normalized Carathéodory function and let the inequality

$$\int_B |K(s, t, f(t))| \, d\nu(t) < \infty$$

hold for every $f \in E$ and almost all $s \in A$. Then the formula

$$Tf(s) = \int_B K(s, t, f(t)) \, d\nu(t) \tag{1}$$

defines a regular OAO $T: E \to L_0(\mu)$. A special case is the *Hammerstein operator* defined by setting

$$(Tf)(s) := \int_B L(s, t) N(t, f(t)) \, d\nu(t),$$

where $L(\cdot, \cdot)$ is a $\mu \times \nu$-measurable function on $A \times B$ and $N: B \times \mathbb{R} \to \mathbb{R}$ a normalized Carathéodory function.

We note that the integral operator T defined by (1) is known in the literature as a *Uryson integral operator* and the normalized Carathéodory function K is called the *kernel* of T. The theory of such operators is widely represented in the literature [28, 35, 36, 55].

Example Let (A, Σ, μ) be a finite measure space. We say that $N: A \times \mathbb{R} \to \mathbb{R}$ is a *superpositionally measurable function*, or *sup-measurable* for shortness, if $N(\cdot, f(\cdot))$ is μ-measurable for every $f \in L_0(\mu)$. A sup-measurable function N is said to be *normalized* if $N(s, 0) = 0$ for μ-almost all $s \in A$. Every normalized sup-measurable function N generates an OAO $\mathcal{N}: L_0(\mu) \to L_0(\mu)$ defined by setting $\mathcal{N}(f)(s) = N(s, f(s)), \quad f \in L_0(\mu)$.

Remark that the operator $\mathcal{N}$ is known in the literature as a nonlinear superposition operator or Nemytskii operator. This operator arises in various problems of modern mathematics (see [9, 10, 31]).

3 The Lateral Order and Related Notions

3.1 Basic Properties

The given partial order $\leq$ on a vector lattice E induces another partial order $\sqsubseteq$ on E, which was formally introduced and studied in [38].

Definition 3.1 Let E be a vector lattice. The partial order $\sqsubseteq$ on E we call the *lateral order* on E. A subset $G \subseteq E$ is said to be *laterally bounded* in E if $G \subseteq \mathfrak{F}_x$ for some $x \in E$. We do not mention here "from above" because every subset is automatically laterally bounded from below by zero.

The lateral supremum and infimum with respect to the lateral order $\sqsubseteq$ on E are denoted using the bold symbols $\bigsqcup$, $\sqcup$ and $\bigsqcap$, $\sqcap$ respectively.

Proposition 3.2 ([38, 52]) *Let E be a vector lattice and $e \in E$. Then the following assertions hold.*

1. *The set $\mathfrak{F}_e$ of all fragments of e is a Boolean algebra with zero 0, unit e with respect to the operations $\sqcup$ and $\sqcap$. Moreover, $x \sqcup y = (x_+ \vee y_+) - (x_- \vee y_-)$ and $x \sqcap y = (x_+ \wedge y_+) - (x_- \wedge y_-)$ for all $x, y \in \mathfrak{F}_e$.*
2. *Assume $e \geq 0$. Then the following holds.*
   (a) *The lateral order $\sqsubseteq$ on $\mathfrak{F}_e$ coincides with the lattice order $\leq$.*
   (b) *Let a nonempty subset A of $\mathfrak{F}_e$ have a lateral supremum $a = \bigsqcup A$ (respectively, a lateral infimum $a = \bigsqcap A$).*
      (i) *If $y = \sup A$ (respectively, $y = \inf A$) exists in E then $y = a$.*
      (ii) *If, moreover, E has the principal projection property then* $\sup A$ *(respectively,* $\inf A$*) exists in E and by* (i) *equals a.*

Remark that there exist a vector lattice E, an element $e \in E_+$ and subsets A and B of $\mathfrak{F}_e$ such that $\bigsqcup A$ and $\bigsqcap B$ exist, while $\sup A$ and $\inf B$ do not exist in E [52, Example 1.2].

A vector lattice E is said to be

- *C-complete* if every nonempty laterally bounded subset G of E has a lateral supremum $\bigsqcup G \in E$;
- *laterally complete* if every disjoint family from E_+ has a supremum.

If a vector lattice E is either Dedekind complete or laterally complete then E is C-complete [38, Corollary 5.8]. The Banach lattice $C[0, 1]$ is a C-complete vector lattice which is neither Dedekind complete, nor laterally complete.

The following statement is easy to prove (cf. [8, Theorem 1.49], [18]).

Proposition 3.3 *Let E be a C-complete vector lattice. Then for every $x \in E$ the Boolean algebra $\mathfrak{F}_x$ is Dedekind complete.*

The lateral order is of great importance for the study of OAOs.

Proposition 3.4 *Let E, F be vector lattices, $e, f \in E$ with $e \sqsubseteq f$ and $T\colon E \to F$ be a positive OAO. Then $Te \le Tf$.*

Proof $f = (f - e) \sqcup e$ and $T(f - e) \ge 0$ imply $Tf = T(f - e) + Te \ge Te$. □

Definition 3.5 Let E be a vector lattice. We say that, a net $(e_\alpha)_{\alpha \in A}$ in E *horizontally* converges (or *laterally* converges in another terminology) to an element $e \in E$ (notation $e_\alpha \xrightarrow{h} e$) if the net $(e_\alpha)_{\alpha \in A}$ order converges to e and $e_\alpha \sqsubseteq e_\beta$ for all $\alpha, \beta \in A$ with $\alpha \le \beta$.

By Proposition 3.3, if $e_\alpha \xrightarrow{h} e$ then $e_\alpha \sqsubseteq e$ for all $\alpha \in A$.

Definition 3.6 Let E be a vector lattice and X either a normed space or a vector lattice, depending on each case. An OAO $T\colon E \to X$ is said to be:

1. *horizontally-to-norm continuous* if for every net $(x_\alpha)_{\alpha \in A}$ in E and every $x \in E$ the relation $x_\alpha \xrightarrow{h} x$ implies $\|T(x_\alpha) - T(x)\| \to 0$;
2. *horizontally continuous* if for every net $(x_\alpha)_{\alpha \in A}$ in E and every $x \in E$ the condition $x_\alpha \xrightarrow{h} x$ implies $T(x_\alpha) \xrightarrow{h} T(x)$;
3. *horizontally-to-order continuous* for every net $(x_\alpha)_{\alpha \in A}$ in E horizontally convergent to $x \in E$ the net $(T(x_\alpha))_{\alpha \in A}$ order converges to $T(x)$.

Example ([18, Proposition 4.7]) Let (A, Σ, μ) be a finite measure space and let $N : B \times \mathbb{R} \to \mathbb{R}$ be a sup-measurable function. Then the Nemytskii operator $\mathcal{N}\colon L_0(\mu) \to L_0(\mu)$ associated with N is horizontally-to-order continuous.

Example ([18, Proposition 4.8]) Let (A, Σ, μ), (B, Ξ, ν) be finite measure spaces, let E be an order ideal of $L_0(\nu)$ and let $T\colon E \to L_0(\mu)$ be an integral Uryson operator with a kernel K. Then T is a horizontally-to-order continuous OAO.

Some properties of horizontally-to-order continuous OAOs were studied in [22, 23, 47, 49, 51].

3.2 Lateral Ideal and Lateral Bands

Definition 3.7 Let E be a vector lattice. A subset $\mathcal{I}$ of E is said to be a *lateral ideal* if the following hold:

1. $x \sqcup y \in \mathcal{I}$ for every disjoint $x, y \in \mathcal{I}$;
2. if $x \in \mathcal{I}$ then $y \in \mathcal{I}$ for all $y \in \mathfrak{F}_x$.

Example Let E be a vector lattice and $\mathcal{I}$ be an order ideal of E. Then $\mathcal{I}$ is a lateral ideal of E.

Example Let E be a vector lattice, $x \in E$. Then $\mathfrak{F}_x$ is a lateral ideal of E.

Example Let E, F be vector lattices and $T : E \to F$ a positive, OAO. Then the kernel $\ker(T) = \{y \in E : T(y) = 0\}$ is a lateral ideal of E.

Theorem 3.8 ([39, Theorem 3.1]) *A subset I of a vector lattice E is a lateral ideal if and only if I is the kernel of some positive OAO $T : E \to F$ where F is a suitable Dedekind complete vector lattice.*

We recall that a net $(x_\lambda)_{\lambda\in\Lambda}$ in a vector lattice E is called *order fundamental* if the net $(x_\lambda - x_{\lambda'})_{(\lambda,\lambda')\in\Lambda\times\Lambda}$ order converges to zero.

Definition 3.9 An order fundamental net $(x_\lambda)_{\lambda\in\Lambda}$ in E is called *horizontally fundamental* if $x_\lambda \sqsubseteq x_{\lambda'}$ for all $\lambda, \lambda' \in \Lambda$ with $\lambda \leq \lambda'$. A subset D of the vector lattice E is called *horizontally closed*, if every horizontally fundamental net $(x_\lambda)_{\lambda\in\Lambda}$ in D order converges to some $x \in D$. Horizontally closed lateral ideal $\mathcal{B}$ is said to be a *lateral band* of E.

Example Every band $\mathcal{B}$ of a Dedekind complete vector lattice E is a lateral band of E.

Example Let E be a Dedekind complete vector lattice. Since for every $x \in E$ the lateral ideal $\mathfrak{F}_x$ is laterally closed (see Proposition 3.3), it follows that $\mathfrak{F}_x$ is a lateral band.

Obviously, the intersection of any nonempty family of lateral ideals (or lateral bands) is a lateral ideal (respectively, a lateral band). The *lateral ideal* (or *lateral band*) *generated by* a nonempty subset A of E is defined to be the intersection of all lateral ideals (respectively, lateral bands) of E including A. For every $e \in E$ the set $\mathfrak{F}_e$ is simultaneously the lateral ideal and lateral band generated by the singleton $\{e\}$, and is called the *principal lateral ideal* and *principal lateral band* of E.

Remark 3.10 Let $\mathcal{B}$ be a lateral band of a vector lattice E and $x \in E$. Then the set-theoretical intersection $\mathfrak{F}_x \cap \mathcal{B}$ contains zero and hence is nonempty.

Proposition 3.11 *Suppose E is a C-complete vector lattice, $x \in E$ and $\mathcal{B}$ is a lateral band of E. Then $\mathfrak{F}_x \cap \mathcal{B}$ has a $\sqsubseteq$-greatest element, which we denote by $x^{\mathcal{B}} := \bigsqcup(\mathfrak{F}_x \cap \mathcal{B})$. In particular, for $\mathcal{B} = \mathfrak{F}_y$, $y \in E$, one has $x^{\mathfrak{F}_y} = x \Cap y$.*

Proof The first part is a consequence of Proposition 3.3, and the second one follows from the remark before Theorem 3.14 below. □

Proposition 3.12 ([47, Lemma 3.5]) *Let E be a vector lattice, $x, y, z, v \in E$ and $z \sqcup v = x \sqcup y$. Then there exist elements $z_1, z_2, v_1, v_2 \in E$ such that*

(i) $z = z_1 \sqcup z_2$; $v = v_1 \sqcup v_2$;
(ii) $x = z_1 \sqcup v_1$; $y = z_2 \sqcup v_2$.

Proposition 3.13 *Let E be a vector lattice, $\mathcal{B}$ a lateral band of E, $x \in E$ and $x = y \sqcup z$. Then $x^{\mathcal{B}} = y^{\mathcal{B}} \sqcup z^{\mathcal{B}}$*

Proof Since $x^{\mathcal{B}} \sqsubseteq x$ by Proposition 3.12, there exists a decomposition $x^{\mathcal{B}} = u \sqcup v$ where $u \sqsubseteq y$ and $v \sqsubseteq z$. We claim that $u = y^{\mathcal{B}}$ and $u = z^{\mathcal{B}}$. Indeed, it is clear that

$u \in \mathcal{B} \cap \mathfrak{F}_y$ and $v \in \mathcal{B} \cap \mathfrak{F}_z$. Thus $u \sqsubseteq y^{\mathcal{B}}$ and $v \sqsubseteq z^{\mathcal{B}}$. Assume that either $u \neq y^{\mathcal{B}}$ or $v \neq z^{\mathcal{B}}$. Then $x^{\mathcal{B}} \sqsubseteq y^{\mathcal{B}} \sqcup z^{\mathcal{B}} \in \mathcal{B} \cap \mathfrak{F}_x$ and $x^{\mathcal{B}} \neq y^{\mathcal{B}} \sqcup z^{\mathcal{B}}$, a contradiction with the maximality of $x^{\mathcal{B}}$. □

3.3 The Intersection Property

By (1) of Proposition 3.2, every finite laterally bounded subset of E has a lateral supremum and lateral infimum. However, there is a vector lattice E and a two-point subset $\{x, y\}$ of E which (being laterally bounded from below by 0) has no lateral infimum [38, Example 3.11]. A vector lattice E is said to have the *intersection property* if every two-point subset $\{x, y\}$ of E has a lateral infimum $x \Cap y$. Remark that if $x \Cap y$ exists for some $x, y \in E$ then $x \Cap y = \bigcup(\mathfrak{F}_x \cap \mathfrak{F}_y)$ is the $\sqsubseteq$-maximal common fragment of x and y [52, Proposition 1.12]. The intersection property is a lateral analogue of the principal projection property, see Proposition 4.9.

The following result describes the relationships between the intersection property and some other known properties of vector lattices.

Theorem 3.14 *Let E be a vector lattice.*

1. *If E has the principal projection property then E possesses the intersection property. Moreover,* $(\forall x, y \in E)\ \mathfrak{F}_{x \Cap y} = \mathfrak{F}_x \cap \mathfrak{F}_y$.
2. *The C-completeness of E implies the intersection property of E.*
3. *The vector lattice $C[0, 1]$ is C-complete and does not have the principal projection property. As a consequence, the intersection property does not imply the principal projection property.*
4. *There exists a vector lattice with the principal projection property which is not C-complete. As a consequence, the intersection property does not imply the C-completeness.*

Item (1) follows from [38, Theorem 3.13]. (2) is a part of [52, Proposition 1.12]. (3) The C-completeness of C[0,1] is proved in [43, Proposition 4.2], and the fact that $C[0, 1]$ fails the principal projection property is well known and easily seen. (4) A corresponding example is provided in [39, Proposition 2.5].

4 Extension of Orthogonally Additive Maps

Definition 4.1 Let E be a vector lattice and $\mathcal{I}$ a lateral ideal of E. We say that a subset D of $\mathcal{I}$ is *absolutely order bounded in* $\mathcal{I}$, provided

$$(\exists y \in \mathcal{I})(\forall x \in D)\ |x| \leq |y|.$$

Definition 4.2 Let E, F be vector lattices and $\mathcal{I}$ a lateral ideal of E. A map $T : \mathcal{I} \to F$ is said to be

- *orthogonally additive* provided $T(x + y) = Tx + Ty$ for all disjoint elements x, $y \in \mathcal{I}$;
- *positive* provided $Tx \geq 0$ for every $x \in \mathcal{I}$;
- *order bounded* provided T maps absolutely order bounded in $\mathcal{I}$ subsets of $\mathcal{I}$ to order bounded subsets of F.

Theorem 4.3 ([45, Theorems 1,2]) *Let E, F be vector lattices with F Dedekind complete, $\mathcal{I} \subseteq E$ a lateral ideal and $T : \mathcal{I} \to F$ a positive order bounded orthogonally additive map. Then the map $\widetilde{T}_{\mathcal{I}} : E \to F$ defined by*

$$\widetilde{T}_{\mathcal{I}}x = \sup\{Ty : y \in \mathfrak{F}_x \cap \mathcal{I}\} \quad (x \in E),$$

is a positive OAO from E to F, that is, $\widetilde{T}_{\mathcal{I}} \in \mathcal{OA}_+(E, F)$. Moreover, $\widetilde{T}_{\mathcal{I}}x = Tx$ for all $x \in \mathcal{I}$.

Now we present a refinement of [24, Theorem 3]. Given a vector lattice E, an OAO $T : E \to E$ is said to be *laterally non-expanding*, if $T(x) \sqsubseteq x$ for all $x \in E$. Obviously, every laterally non-expanding OAO preserves disjointness. A laterally non-expanding projection (that is, $T^2 = T$) is called a *lateral retraction*. A subset A of E is called a *lateral retract* if A is the image of some lateral retraction $T : E \to E$, that is, $T(E) = A$. A lateral band A of E, which is a lateral retract, is called a *projection lateral band*, and the lateral retraction of E onto A is called the *lateral band projection* of E onto A.

Theorem 4.4 ([27, Theorem 2.6]) *Let E be a vector lattice.*

1. *For each lateral retract A in E there is a unique lateral retraction of E onto A.*
2. *Every lateral retraction is horizontally continuous.*
3. *Every lateral retract in E is a lateral band.*

The following theorem generalizes Theorem 3 of [24] and asserts, in particular, that every lateral band in a C-complete vector lattice is a lateral retract, and hence, a projection lateral band.

Theorem 4.5 *Every lateral band $\mathcal{B}$ of a C-complete vector lattice E is a lateral retract, and the function $\mathfrak{p}_{\mathcal{B}} : E \to E$ defined by setting for every $x \in E$*

$$\mathfrak{p}_{\mathcal{B}}(x) = \bigsqcup (\mathfrak{F}_x \cap \mathcal{B}) = x^{\mathcal{B}} \tag{2}$$

is the lateral band projection of E onto $\mathcal{B}$

Proof By Proposition 3.11, the map $\mathfrak{p}_{\mathcal{B}}$ is well defined. By Theorem 3 of [24], we have to prove the orthogonal additivity of $\mathfrak{p}_{\mathcal{B}}$ only. Fix disjoint $y, z \in E$ and let

$x = y \sqcup z$. Then by Proposition 3.13 we have that

$$\mathfrak{p}_{\mathcal{B}}(x) = x^{\mathcal{B}} = y^{\mathcal{B}} \sqcup z^{\mathcal{B}} = \mathfrak{p}_{\mathcal{B}}(y) \sqcup \mathfrak{p}_{\mathcal{B}}(z).$$

Hence, $\mathfrak{p}_{\mathcal{B}}$ is an OAO. □

The following theorem provides a partial case of formula (2) for principal lateral bands, however proved under a less restrictive assumption on E to have the intersection property.

Theorem 4.6 ([52, Theorem 1.6]) *Let E be a vector lattice with the intersection property. Then for every $e \in E$ the function $Q_e : E \to E$ defined by setting*

$$Q_e x = x \cap e \quad \textit{for all} \;\; x \in E$$

is a lateral retraction, the image of which is the principal lateral band $\mathfrak{F}_e$.

To provide an interesting consequence of Theorem 4.6, we need the following definition.

Definition 4.7 Let E be a vector lattice and $x, y \in E$. We say that x is *laterally disjoint* to y and write $x \dagger y$ if $\mathfrak{F}_x \cap \mathfrak{F}_y = \{0\}$. We say that two subsets H and D of E are *laterally disjoint* and use the notation $H \dagger D$ if $x \dagger y$ for every $x \in H$ and $y \in D$. The *laterally disjoint complement* to a subset A of E is defined as follows: $A^{\dagger} := \{x \in E : (\forall a \in A)\, x \dagger a\}$.

Observe that $x \perp y$ implies $x \dagger y$ for all $x, y \in E$, and the converse implication is false. However, one can show that, $x \dagger y$ implies $x \perp y$ for every laterally bounded pair $x, y \in E$.

Now we show that the intersection property is a lateral analogue of the principal projection property.

Definition 4.8 An element e of a vector lattice E is called a *laterally projection element* provided E is decomposed into a nonlinear direct sum $E = \mathfrak{F}_e \sqcup \mathfrak{F}_e^{\dagger}$, that is, every $x \in E$ has a unique representation $x = y \sqcup z$, where $y \in \mathfrak{F}_e$ and $z \in \mathfrak{F}_e^{\dagger}$.

Proposition 4.9 *A vector lattice E has the intersection property if and only if every element of E is laterally projective.*

Proof Let E have the intersection property and $e \in E$ and $x \in E$. Then $x = Q_e x \sqcup (x - Q_e x)$, which gives the desired decomposition by Theorem 4.6.

Assume that every element of E is laterally projective and fix any $x, y \in E$. Our goal is to show that there exists the lateral infimum $x \cap y$, which is the $\sqsubseteq$-maximal common fragment of x and y. Using the decomposition $E = \mathfrak{F}_y \sqcup \mathfrak{F}_y^{\dagger}$, write $x = y' \sqcup z$, where $y' \in \mathfrak{F}_y$ and $z \dagger y$. Then $y' \in \mathfrak{F}_x \cap \mathfrak{F}_y$, and we prove that y' is the maximal common fragment of x and y. Assume on the contrary this is false. Then there exists $t \in \mathfrak{F}_x \cap \mathfrak{F}_y$ such that $t \not\sqsubseteq y'$. Hence for $w := y' \cup t$ we obtain $w \in \mathfrak{F}_x \cap \mathfrak{F}_y$, $y' \sqsubseteq w \sqsubseteq x$ and $y' \neq w$. By the above, $x = y' \sqcup (x - y')$ and $x = w \sqcup (x - w) = y' \sqcup (w - y') \sqcup (x - w)$ and therefore, $x - y' = (w - y') \sqcup (x - w)$, which yields $w - y' \sqsubseteq x - y' = z$. On the other hand, $0 \neq w - y' \sqsubseteq y$, which contradicts $z \dagger y$. □

5 The Order Structure on the Vector Lattice of OAOs

5.1 Order Calculus and Riesz-Kantorovich Types Formulas

Two fundamental results in this direction were obtained by Mazón and Segura de León in 1990.

Theorem 5.1 ([35, Theorem 3.2]) *Let E and F be vector lattices with F Dedekind complete. Then $\mathcal{U}(E, F)$ is a Dedekind complete vector lattice. Moreover, for each $S, T \in \mathcal{U}(E, F)$ and $x \in E$ the following conditions hold:*

1. $(T \vee S)(x) = \sup\{T(y) + S(z) : x = y \sqcup z\}$.
2. $(T \wedge S)(x) = \inf\{T(y) + S(z) : x = y \sqcup z\}$.
3. $T_+(x) = \sup\{Ty : y \sqsubseteq x\}$.
4. $T_-(x) = -\inf\{Ty : y \sqsubseteq x\}$.
5. $|T(x)| \leq |T|(x)$.

The second one represents the lattice operations on $\mathcal{U}(E, F)$ in terms of directed systems.

Theorem 5.2 ([36, Lemma 3.2]) *Let E and F be vector lattices with F Dedekind complete. Then for all $T, S \in \mathcal{U}(E, F)$ and $x \in E$ one has*

1. $\left\{ \sum_{i=1}^n T(y_i) \wedge S(y_i) : \ x = \bigsqcup_{i=1}^n y_i;\ n \in \mathbb{N} \right\} \downarrow (S \wedge T)(x)$.
2. $\left\{ \sum_{i=1}^n T(y_i) \vee S(y_i) : \ x = \bigsqcup_{i=1}^n y_i;\ n \in \mathbb{N} \right\} \uparrow (S \vee T)(x)$.
3. $\left\{ \sum_{i=1}^n |T(y_i)| : \ x = \bigsqcup_{i=1}^n y_i;\ n \in \mathbb{N} \right\} \uparrow |T|(x)$.

Similar results, obtained by the first named author and Ramdane in 2018, concern a more wide class $\mathcal{P}(E, F)$ of C-bounded OAOs.

Theorem 5.3 ([47, Theorem 3.6]) *Let E and F be vector lattices with F Dedekind complete. Then $\mathcal{P}(E, F)$ is a Dedekind complete vector lattice. Moreover, $\mathcal{P}(E, F) = \mathcal{OA}_r(E, F)$ and for all $S, T \in \mathcal{P}(E, F)$ and $x \in E$ conditions 1–5 from Theorem 5.1 hold.*

The following proposition strengthens the inclusion $\mathcal{U}(E, F) \subset \mathcal{P}(E, F)$.

Proposition 5.4 ([47, Proposition 3.7]) *Let E, F be vector lattices with F Dedekind complete. Then $\mathcal{U}(E, F)$ is an order ideal of $\mathcal{P}(E, F)$.*

The next example shows that $\mathcal{U}(E, F)$ need not be a band in $\mathcal{P}(E, F)$.

Example Let $E = F = \mathbb{R}$ and the operator $T \in \mathcal{P}(\mathbb{R})$ be

$$T(x) = \begin{cases} \frac{1}{|x|}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

It is clear that T is an C-bounded OAO and $T \notin \mathcal{U}(\mathbb{R})$. Let

$$T_n(x) = \begin{cases} Tx, & \text{if } Tx \leq n \\ 0, & \text{if } Tx > n. \end{cases}$$

It is not difficult to check that $T_n \in \mathcal{U}(\mathbb{R})$ for every $n \in \mathbb{N}$ and $T_n \uparrow T$.

We remark that, without the assumption of Dedekind completeness of F, the vector spaces $\mathcal{U}(E, F)$ and $\mathcal{P}(E, F)$ are, in general, not vector lattices. Nevertheless, for any $S \in \mathcal{P}(E, F)$ and $T \in \mathcal{U}(E, F)$ the relation $0 \leq S \leq T$ implies the inclusion $S \in \mathcal{U}(E, F)$. Indeed, take a order bounded subset D of E and let $z \in F_+$ be a some upper boundary of the set $T(D)$. Then $Sx \leq z$ for any $x \in D$ and $S \in \mathcal{U}(E, F)$.

The set of all horizontally-to-order continuous (σ-continuous) C-bounded OAOs is denoted by $\mathcal{P}_c(E, F)$ ($\mathcal{P}_{\sigma c}(E, F)$).

Proposition 5.5 *Let E, F be vector lattices. Then every horizontally-to-order continuous OAO $T : E \to F$ is C-bounded.*

Proof By Proposition 3.2, for every $x \in E$ the set $\mathfrak{F}_x$ is directed with respect to the partial order $\sqsubseteq$. Consider $\Lambda = \mathfrak{F}_x \times \mathfrak{F}_x$ as a directed set with the lexicographical order. Define a net $\big(x_{(u,v)}\big)_{(u,v)\in\Lambda}$ by setting $x_{(u,v)} = u \cap v$ for all $(u, v) \in \Lambda$. Note that $x_{(u,v)} \xrightarrow{h} x$. Moreover, for every $u_0 \in \mathfrak{F}_x$ one has $x_{(u_0,v)} \xrightarrow{h}_v u_0$ and for every $v_0 \in \mathfrak{F}_x$, $\big(x_{(u,v_0)}\big)_{u\in\mathfrak{F}_x} \xrightarrow{h}_u v_0$. By the horizontal-to-order continuity of T, there exists a net $(e_{(u,v)})_{(u,v)\in\Lambda} \subset F_+$ with the same index set Λ such that $|Tx - Tx_{(u,v)}| \leq e_{(u,v)} \leq e_{(u_0,v_0)}$ for all $(u, v) \geq (u_0, v_0)$. Given any $v \in \mathfrak{F}_x$, for $u := u_0 \cup v$ one has $u_0 \sqsubseteq u$ and $v \sqsubseteq u$. Then $x_{(u,v)} = u \cap v = v$ and we may write $|Tx - Tv| = |Tx - Tx_{(u,v)}| \leq e_{(u_0,v_0)}$ and hence $|Tv| \leq e_{(u_0,v_0)} + |Tx|$. Thus, $T(\mathfrak{F}_x)$ is order bounded in F. □

Theorem 5.6 ([47, Theorem 3.13]) *Let E, F be vector lattices with F Dedekind complete. Then $\mathcal{P}_c(E, F)$ and $\mathcal{P}_{\sigma c}(E, F)$ are bands in the vector lattice $\mathcal{P}(E, F)$.*

5.2 The Boolean Algebra of Fragments of a Positive OAO

Let E, F be vector lattices with F Dedekind complete and $T \in \mathcal{U}_+(E, F)$. The purpose of this section is to describe the fragments of T. That is

$$\mathfrak{F}_T = \{S \in \mathcal{U}_+(E, F) : S \wedge (T - S) = 0\}.$$

First we consider elementary fragments. For a subset $\mathcal{A}$ of a vector lattice W we use the following notation: $\mathcal{A}^{\uparrow} = \{x \in W : \exists$ a net (x_α) in $\mathcal{A}$ with $x_\alpha \uparrow x\}$. The meaning of $\mathcal{A}^{\downarrow}$ is analogous. As usual, we also write $\mathcal{A}^{\downarrow\uparrow} = (\mathcal{A}^{\downarrow})^{\uparrow}$. It is clear that $\mathcal{A}^{\downarrow\downarrow} = \mathcal{A}^{\downarrow}$, $\mathcal{A}^{\uparrow\uparrow} = \mathcal{A}^{\uparrow}$. Since $\mathfrak{F}_T$ is a Boolean algebra, it is closed under finite suprema and infima. In particular, all "ups and downs" of $\mathfrak{F}_T$ are likewise closed under finite suprema and infima, and therefore are also directed upward and, respectively, downward.

Let $T \in \mathcal{U}_+(E, F)$ and $D \subseteq E$ be a lateral ideal. Then for every $x \in E$ the following formula defines a map $\pi^D T : E \to F_+$

$$\pi^D T(x) = \sup\{Ty : y \in \mathfrak{F}_x \cap D\}. \tag{3}$$

Proposition 5.7 ([11, Lemma 3.6]) *Let E, F be vector lattices with F Dedekind complete, $\rho \in \mathfrak{B}(F)$, $T \in \mathcal{U}_+(E, F)$ and D be a lateral ideal. Then $\pi^D T$ is a positive abstract Uryson operator and $\rho\pi^D T \in \mathfrak{F}_T$.*

If $D = \mathfrak{F}_x$ then the operator $\pi^D T$ is denoted by $\pi^x T$. Let F be a vector lattice. Any fragment of the form $\sum_{i=1}^n \rho_i \pi^{x_i} T$, $n \in \mathbb{N}$, where $\rho_1, \dots, \rho_n$ is a finite family of mutually disjoint order projections of F, is called an *elementary* fragment of T. The set of all elementary fragments of T we denote by $\mathcal{A}_T$. The following theorem describes the structure of $\mathfrak{F}_T$ for a positive abstract Uryson operator T.

Theorem 5.8 ([42, Theorem 3.12]) *Let E, F be vector lattices, F Dedekind complete and $T \in \mathcal{U}_+(E, F)$. Then $\mathfrak{F}_T = \mathcal{A}_T^{\uparrow\downarrow\uparrow}$.*

Remark that, for linear positive operators a similar theorem and its modifications were proved by de Pagter, Aliprantis and Burkinshaw, Kusraev and Strizhevski, see [7, 13, 29].

6 Compact Orthogonally Additive Operators

In this section we study C-compact and AM-compact OAOs taking values in Banach lattices.

6.1 The Projection Band of C-Compact Orthogonally Additive Operators

In this subsection, following [43] we show that the set of all C-compact regular OAOs from a vector lattice E to a Banach lattice F with an order continuous norm is a band in the vector lattice of all OAOs from E to F.

Definition 6.1 Let E be a vector lattice and Y a normed space. An OAO $T: E \to Y$ is said to be *C-compact*, if $T(\mathfrak{F}_x)$ is relatively compact in Y for all $x \in E$. For a Banach lattice F, by $\mathcal{COA}_r(E, F)$ we denote the set of all C-compact regular OAOs from E to F.

Example We note that $\mathcal{OA}_r(\mathbb{R}, \mathbb{R})$ is exactly the set of all real-valued functions such that $f(0) = 0$. Define an OAO $T: \mathbb{R} \to \mathbb{R}$ by

$$T(x) = \begin{cases} \frac{1}{x^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Since any element $0 \neq x \in \mathbb{R}$ is an atom, one has $\mathfrak{F}_x = \{0, x\}$ for any $x \in \mathbb{R}$. It follows that T is C-compact. On the other hand, $T([0, 1])$ is an unbounded set in $\mathbb{R}$ and therefore T is not AM-compact.

Let (A, Σ, μ) be a σ-finite complete measure space. A Banach space E which is a linear subspace of $L_0(\mu)$ is called a *Banach function space*, provided for every $x \in L_0(\mu)$ and $y \in E$ the condition $|x| \leq |y|$ implies that $x \in E$ and $\|x\| \leq \|y\|$. Obviously, a Banach function space is a Banach lattice.

Proposition 6.2 *Let E be a Banach function space on a σ-finite measure space (B, Σ, ν) and $T: E \to \mathbb{R}$ the Uryson integral functional defined by*

$$Tf = \int_B K(t, f(t))\, d\nu(t), \quad f \in E$$

with a kernel K. Then T is C-compact.

Proof Given any $f \in E$, one has $\mathfrak{F}_f = \{f 1_D : D \in \Sigma\}$. Then for every $D \in \Sigma$

$$T(f 1_D) = \int_B K(t, f 1_D(t))\, d\nu(t) = \int_D K(t, f(t))\, d\nu(t) \leq \int_B |K(t, f(t))|\, d\nu(t) = M.$$

Hence the set $T(\mathfrak{F}_f)$ is order bounded in $\mathbb{R}$ and therefore T is C-compact. □

We mention that a C-compact order bounded OAO $T: E \to F$ from a Banach lattice E to a σ-Dedekind complete Banach lattice F is AM-compact if, in addition, T is uniformly continuous on order bounded subsets of E [36, Theorem 3.4].

Recall that, a Banach lattice with an order continuous norm is Dedekind complete (see [8, Theorem 12.9]).

Next is the main result of the subsection, which we provide with a proof.

Theorem 6.3 ([43, Theorem 3.9]) *Let E be a vector lattice and F a Banach lattice with an order continuous norm. Then the set of all C-compact regular OAOs from E to F is a projection band in $\mathcal{OA}_r(E, F)$.*

In our proof we use the following lemma.

Lemma 6.4 ([43, Lemma 2.2]) *Let $T \in \mathcal{COA}_r(E, F)$ under the assumptions of Theorem 6.3. Then $\mathfrak{F}_T \subset \mathcal{COA}_r(E, F)$.*

Proof of Theorem 6.3 We prove some properties of $\mathcal{COA}_r(E, F)$:

(a) Clearly, $\mathcal{COA}_r(E, F)$ is a vector subspace of $\mathcal{OA}_r(E, F)$.

(b) We show that $\mathcal{COA}_r(E, F)$ is even a vector sublattice of $\mathcal{OA}_r(E, F)$. Take $S, T \in \mathcal{COA}_r(E, F)$. Then $T - S \in \mathcal{COA}_r(E, F)$. By Lemma 6.4 $\mathfrak{F}_T \subset \mathcal{COA}_r(E, F)$ and hence $T_+ \in \mathcal{COA}_r(E, F)$. Therefore due to the equalities

$$S + (T - S)_+ = S + (T - S) \vee 0 = T \vee S, \quad S \wedge T = -((-S) \vee (-T))$$

which are valid in $\mathcal{OA}_r(E, F)$ we obtain that $\mathcal{COA}_r(E, F)$ is a sublattice of $\mathcal{OA}_r(E, F)$.

(c) Now we show that, if $0 \leq T_\lambda \uparrow T$ in $\mathcal{OA}_r(E, F)$ and any $T_\lambda \in \mathcal{COA}_r(E, F)$ then $T \in \mathcal{COA}_r(E, F)$. Indeed, take $x \in E$ and $\varepsilon > 0$. Since the Banach lattice F is order continuous, it follows from $T_\lambda x \uparrow Tx$ that $\|Tx - T_{\lambda_0}x\| < \frac{\varepsilon}{4}$ for some λ_0. We claim that, moreover, $\|Ty - T_{\lambda_0}y\| < \frac{\varepsilon}{4}$ for any $y \in \mathfrak{F}_x$. Indeed, consider $x = y \sqcup z$ for some $z \in E$. Then

$$0 \leq Ty - T_{\lambda_0}y \leq Ty - T_{\lambda_0}y + Tz - T_{\lambda_0}z = Tx - T_{\lambda_0}x$$

implies $\|Ty - T_{\lambda_0}y\| \leq \|Tx - T_{\lambda_0}x\|$. Since $T_{\lambda_0} \in \mathcal{COA}_r(E, F)$, there exists a finite subset D of $\mathfrak{F}_x$ with the property that for any $y \in \mathfrak{F}_x$ there exists $u \in D$ satisfying

$$\|T_{\lambda_0}u - T_{\lambda_0}y\| < \frac{\varepsilon}{2}.$$

So we obtain $\|Tu - Ty\| \leq \|Tu - T_{\lambda_0}u\| + \|Ty - T_{\lambda_0}y\| + \|T_{\lambda_0}u - T_{\lambda_0}y\| < \varepsilon$, which establishes the relative compactness of $T(\mathfrak{F}_x)$ in F.

(d) Finally we prove that $\mathcal{COA}_r(E, F)$ is an order ideal in $\mathcal{OA}_r(E, F)$. Let $0 \leq R \leq T$, where $R \in \mathcal{OA}_r(E, F)$ and $T \in \mathcal{COA}_r(E, F)$. Then $R \in I_T$ and by the Freudenthal's spectral theorem [8, Theorem 2.8], there exists a sequence $(S_n)_{n \in \mathbb{N}}$ in $\mathcal{OA}_r(E, F)$ of T-step-functions with $0 \leq S_n \uparrow R$. Taking into account that[1] $S_n \in \mathcal{COA}_r(E, F)$ for all $n \in \mathbb{N}$ and, what has been established in c), we deduce that $R \in \mathcal{COA}_r(E, F)$. So, $\mathcal{COA}_r(E, F)$ is a band in $\mathcal{OA}_r(E, F)$.

(e) Due to the Dedekind completeness of $\mathcal{OA}_r(E, F)$, it is a projection band.

□

[1] This follows from the fact that together with T each fragment T_i of T belongs to $\mathcal{COA}_r(E, F)$.

6.2 Domination Problem for AM-Compact Abstract Uryson Operator

In this section, we consider the following domination problem for AM-compact abstract Uryson operators. Let E, F be vector lattices and $S, T: E \to F$ be OAOs with $0 \le S \le T$. Let $\mathcal{P}$ be some property of OAOs, so $\mathcal{P}(R)$ means that an OAO $R: E \to F$ possesses $\mathcal{P}$. Does $\mathcal{P}(T)$ imply $\mathcal{P}(S)$?

Definition 6.5 Let E be a vector lattice and Y a normed space. An OAO $T: E \to Y$ is said to be *AM-compact* provided T maps order bounded subsets of E to relatively compact subsets of Y.

Example ([36, Theorem 3.5]) Let E, F be Banach function spaces with F having an order continuous norm. Then every integral Uryson operator $T \in \mathcal{U}(E, F)$ is AM-compact.

The next theorem is the main result of the subsection.

Theorem 6.6 ([40, Theorem 3.19]) *Let E be a Dedekind complete vector lattice, F a Banach lattice with an order continuous norm, and $T \in \mathcal{U}_+(E, F)$ an AM-compact operator. Then every operator $S \in \mathcal{U}_+(E, F)$ with $0 \le S \le T$ is AM-compact.*

Remark that the same property for linear operators was proved earlier by Dodds and Fremlin in [14].

Taking into account that every integral Uryson operator $T: E \to F$ is AM-compact (see [36, Theorem 3.5]), we obtain the following consequence of Theorem 6.6.

Corollary 6.7 ([40, Corollary 3.21]) *Let E, F be Banach function spaces with F having an order continuous norm and $T \in \mathcal{U}_+(E, F)$ be an integral Uryson operator. Then every abstract Uryson operator $S \in \mathcal{U}(E, F)$ such that $0 \le S \le T$ is AM-compact.*

7 Partial Order Continuities of Orthogonally Additive Operators

Throughout this section, let E, F be vector lattices with F Dedekind complete. Here we discuss the relationships between the order continuity of an abstract Uryson operator $T: E \to F$ and its modulus $|T|$, as well as some partial order continuities, like horizontal-to-order and uniformly-to-order ones. Similar questions for linear operators are simpler, see the next proposition.

Proposition 7.1 ([35, Proposition 3.9]) *Let E be a vector lattice with the principal projection property, F a Dedekind complete vector lattice and $S\colon E \to F$ a regular linear operator. Then the following assertions hold:*

1. *if S is horizontally-to-order continuous then S is order continuous;*
2. *if S is horizontally-to-order σ-continuous then S is order σ-continuous.*

This is not longer true for OAOs due to the following example.

Example Let $0 \le p \le \infty$. There exists a horizontally-to-order continuous orthogonally additive functional $f\colon L_p \to \mathbb{R}$ which is not order continuous. Moreover, f is not uniformly-to-order continuous.

Define a function $\varphi\colon \mathbb{R} \to [-1, 1]$ by setting $\varphi(t) = t$ as $|t| \le 1$ and $\varphi(t) = 0$ for $|t| > 1$. Then define $f\colon L_p \to \mathbb{R}$ by setting

$$f(x) = \int_{[0,1]} \varphi\big(x(t)\big)\, d\mu(t).$$

Detailed proof that f is as desired the reader can find in [22, Example 2.1].

We say that a function $f\colon E \to F$ is *uniformly-to-order continuous* if for every net (x_α) in E and every $x \in E$ the condition $x_\alpha \rightrightarrows x$ implies $f(x_\alpha) \xrightarrow{o} f(x)$.

Assume $T \in \mathcal{U}(E, F)$. Then the function $\widehat{T}\colon E \to F$ defined by setting

$$\widehat{T}(x) = \sup_{|y| \le |x|} |T|(y), \quad x \in E \tag{4}$$

is a positive abstract Uryson operator (that is, $\widehat{T} \in \mathcal{U}(E, F)^+$) [35, Proposition 3.4]. Following [52], we say that $\widehat{T}$ is the *envelope* of T. Remark that the envelope has the following properties (see propositions 3.4 and 3.5 of [52] for details).

Proposition 7.2 *Let E, F be vector lattices with F Dedekind complete and $S, T \in \mathcal{U}(E, F)$. Then*

1. *$T(x) \le \widehat{T}(x)$ for all $x \in E$;*
2. *if $x \le y$ for $x, y \in E$ then $\widehat{T}(x) \le \widehat{T}(y)$;*
3. *if $0 \le S \le T$ then $\widehat{S}(x) \le \widehat{T}(x)$ for all $x \in E$;*
4. *$\widehat{S + T}(x) \le \widehat{S}(x) + \widehat{T}(x)$ for all $x \in E$;*
5. *if, moreover, E has the principal projection property then $\widehat{\widehat{T}} = \widehat{T}$.*

Theorem 7.3 ([22, Theorem 2.2]) *Let E be a vector lattice with the principal projection property, F a Dedekind complete vector lattice and $T \in \mathcal{U}(E, F)$. Consider the following statements.*

(i) T is order continuous.
(ii) $|T|$ is order continuous.
(iii) The envelope $\widehat{T}$ of T is horizontally-to-order continuous.

Then the following assertions hold.

(A) (i) *and* (ii) *imply* (iii).
(B) *Suppose in addition that* E *has the horizontal Egorov property and* (iii) *holds. Then the uniformly-to-order continuity of* T *implies* (i), *and the uniformly-to-order continuity of* $|T|$ *yields* (ii).

Remark that conditions (i) and (ii) are equivalent for linear operators [8, Theorem 1.56], however for OAOs this is not true, see the next example.

Example Set $E = F = \mathbb{R}$ and define an order bounded orthogonally additive functional $f: E \to F$ by setting

$$f(x) = \begin{cases} 0 & \text{for } -\infty < x \le 0, \\ x & \text{for } 0 < x \le 1, \\ -1 & \text{for } 1 < x < +\infty. \end{cases}$$

Then

$$|f|(x) = \begin{cases} 0 & \text{for } -\infty < x \le 0, \\ x & \text{for } 0 < x \le 1, \\ 1 & \text{for } 1 < x < +\infty. \end{cases}$$

Obviously, $|f|$ is order continuous, however f is not.

A limited space does not allow us presenting all the results of [22]. We just remark that not everything is now clear in this direction, see open problems in the final section.

8 Narrow OAOs and Representation of Regular Operators

8.1 Narrow Operators

Narrow linear operators were introduced and studied in 1990 by Plichko and the second named author in [41] as a generalization of compact operators defined on symmetric function spaces. But actually these operators were investigated by different mathematicians earlier. In 2009 narrow operators were naturally generalized to linear operators defined on vector lattices in [34] (see also [53] and references therein). After the monograph [53] was published, the notion was (not less naturally) generalized to OAOs in paper by the authors [44], and then developed in some other papers (see e.g., [21]). A recent paper by the authors [46] contains a representation theorem for regular operators (two versions for both linear and orthogonally additive settings) which generalizes a number of known results in this direction.

Definition 8.1 An OAO $T: E \to F$ between vector lattices E, F is said to be *order narrow* if for every $e \in E$ there is a net of decompositions $e = e'_\alpha \sqcup e''_\alpha$ such that the net $\big(T(e'_\alpha) - T(e''_\alpha)\big)_\alpha$ order converges to zero. An OAO $T: E \to G$ from a vector lattice E to a Banach space G is called *narrow* if for every $e \in E$ and every $\varepsilon > 0$ there is a decomposition $e = e' \sqcup e''$ such that $\|T(e') - T(e'')\| < \varepsilon$. An OAO $T: E \to V$ from a vector lattice E to a linear space V is called *strictly narrow* if for every $e \in E$ there is a decomposition $e = e' \sqcup e''$ such that $T(e') = T(e'')$.

Obviously, every strictly narrow operator is both narrow and order narrow for suitable range lattices. Let us briefly demonstrate of why is every "small" operator strictly narrow. Let E be a Banach function space on [0, 1]. Then **every horizontally-to-norm continuous OAO $f: E \to \mathbb{R}$ is strictly narrow.** Indeed, given any $e \in E$, the function $\varphi: [0,1] \to \mathbb{R}$ defined by $\varphi(t) = f(e \cdot 1_{[0,t]})$ is continuous, $\varphi(0) = 0$ and $\varphi(1) = f(e)$. Choosing $s \in [0,1]$ so that $\varphi(s) = f(e)/2$, we obtain $e = e \cdot 1_{[0,s]} \sqcup e \cdot e \cdot 1_{(s,1]}$ and

$$f\left(e \cdot 1_{[0,s]}\right) = \frac{f(e)}{2} = f\left(e \cdot 1_{(s,1]}\right).$$

Theorem 8.2 ([53, Proposition 2.1]) *If a Banach function space E on a finite measure space (Ω, Σ, μ) has an absolutely continuous norm on the unit (i.e., $\lim_{\mu(A)\to 0} \|1_A\| = 0$) and X is a Banach space then every compact and every AM-compact linear operator $T: E \to X$ is narrow.*

A strictly narrow operator need not have "small" range and can be "very non-compact": under the same assumptions on $\mathbb{T} = (\Omega, \Sigma, \mu)$, for every rearrangement invariant Banach space E on $\mathbb{T}$ there exists a strictly narrow linear projection of E onto a subspace E_0 which is isometrically isomorphic to E [53, Theorem 4.17]. The assumption on E to have an absolutely continuous norm on the unit is essential: there are nonnarrow continuous linear functionals on L_∞ [53, Example 10.12].

The following theorem extends Theorem 8.2 to OAOs.

Theorem 8.3 ([44, Theorem 3.2]) *Let E be an atomless Dedekind complete vector lattice and X a Banach space. Then every orthogonally additive horizontally-to-norm continuous C-compact operator $T: E \to X$ is narrow.*

8.2 Representation of Regular Operators

In this subsection, we present some recent (unpublished) authors' results [46] which generalize known representation theorems by different authors: Kwapień [30] and Kalton's representation theorem for continuous linear operators on $L_p(\mu)$ for $0 \le p \le 1$ [26], Rosenthal's version of the same theorem for operators on $L_1[0,1]$ [54], Weis' representation theorem for order continuous linear operators on function vector lattices [58], Huijsmans and de Pagter's theorem on representation of regular linear operators on vector lattices [25], O. Maslyuchenko, Mykhaylyuk and the second named author's representation theorem of order continuous linear operators on vector lattices [34] and the authors' representation of order bounded OAOs [44]. The main contribution of the result under presentation is ridding of the order continuity assumption on an operator and extending to OAOs. Earliest of the above mentioned results are formulated using the language of measures and integral operators on suitable function spaces, and the later ones used lattice terminology, which is shorter and allows obtaining more general results. The general idea of all representation theorems is to split the vector lattice $X(E, F)$ of all operators $T: E \to F$ from a certain class into a direct sum of orthogonal bands $X(E, F) = Y(E, F) \oplus Z(E, F)$, where operators from $Y(E, F)$ are in some sense atomic and operators from $Z(E, F)$ are kind of continuous, which then yields a unique desirable representation of every operator $T = T_a + T_c$.

Any representation theorem for linear operators can be stated as follows: let $\mathcal{H}(E, F)$ be the band generated by all lattice homomorphisms (in other words, by all disjointness preserving operators) from E to F, and $\mathcal{D}(E, F) = \mathcal{H}(E, F)^d$ be the disjoint complement of $\mathcal{H}(E, F)$ in $\mathcal{L}_b(E, F)$. Then, by the Dedekind completeness of F, we obtain the following decomposition of $\mathcal{L}_b(E, F)$ into orthogonal bands

$$\mathcal{L}_b(E, F) = \mathcal{H}(E, F) \oplus \mathcal{D}(E, F). \tag{5}$$

Now any representation theorem is reduced to characterization of operators which belong to the summands of (5). One convenient characterization of operators from the first summand [53, Theorem 1.33] asserts: for every $T \in \mathcal{L}_b(E, F)$ the relation $T \in \mathcal{H}(E, F)$ holds if and only if T is the sum of an absolutely order summable family $T = \sum_{j \in J} T_j$ of disjointness preserving operators $T_j \in \mathcal{L}_b(E, F)$.

The very effective characterizations of the summands of (5) concern the case $E = F = L_1 = L_1[0, 1]$ and were obtained by Kalton [26] and Rosenthal [54] due to the fact that the set $\mathcal{L}(L_1)$ of all continuous linear operators on L_1 coincides with $\mathcal{L}_b(L_1)$. Representation (5) for this particular case has the following form

$$\mathcal{L}(L_1) = \mathcal{H}(L_1) \oplus \mathcal{D}(L_1). \tag{6}$$

The next two combined theorems by Enflo, Kalton, Rosenthal and Starbird characterize summands of (6) (cf. theorems 7.38 and 7.39 in [53]).

Theorem 8.4 ([54, Theorem 3.2]) *For every operator $T \in \mathcal{L}(L_1)$ the following assertions are equivalent:*

1. $T \in \mathcal{H}(L_1)$;
2. *T equals a pointwise absolutely convergent*[2] *series $T = \sum_{n=1}^{\infty} T_n$ of disjointness preserving operators $T_n \in \mathcal{L}(L_1)$.*

[2] Strong ℓ_1-convergent, in Rosenthal's original terminology.

Moreover, every nonzero operator $T \in \mathcal{H}(L_1)$ *possesses the following property: for every* $\varepsilon > 0$ *there exists a measurable subset* A *of* $[0, 1]$ *such that the restriction* $T\big|_{L_1(A)}$ *is an into isomorphism with*

(i) $\big\|T\big|_{L_1(A)}\big\| \geq \|T\| - \varepsilon$;
(ii) $\big\|T\big|_{L_1(A)}\big\| \cdot \big\|T\big|^{-1}_{L_1(A)}\big\| < 1 + \varepsilon$.

Before stating of the second theorem, we provide with a definition of the Enflo-Starbird function for OAOs, which is the same as for linear operators.

Definition 8.5 Let E, F be vector lattices with F Dedekind complete and $T \in \mathcal{OA}_r(E, F)$. We define a function $\lambda_T : E_+ \to F_+$, called the *Enflo-Starbird function* of T, by setting for all $x \in E_+$

$$\lambda_T(x) = \inf\left\{ \sup_{1 \leq i \leq m} |T|x_i \, : \, x = \bigsqcup_{i=1}^{m} x_i, \; x_i \in E_+, \; m \in \mathbb{N} \right\}.$$

Theorem 8.6 ([26, 54]) *For every* $T \in \mathcal{L}(L_1)$ *the following assertions are equivalent:*

1. $T \in \mathcal{D}(L_1)$*;*
2. T *is narrow;*
3. *the Enflo-Starbird function* λ_T *of* T *equals zero;*
4. *for every measurable subset* A *of* $[0, 1]$ *the restriction* $T\big|_{L_1(A)}$ *is not an into isomorphism.*

Implication (4) ⇒ (3) and the technique of λ-function is due to Enflo and Starbird [17]; equivalences (1) ⇔ (3) ⇔ (4) can be deduced from Kalton's results [26], and the final equivalence (1) ⇔ (2), as well as noticing of the entire theorem the reader can find in Rosenthal's paper [54].

In view of Theorem 8.4, elements of $\mathcal{H}(E, F)$ we call *pseudo-embeddings*, and following terminology of Weis [58] and Huijsmans-de Pagter [25], elements of $\mathcal{D}(E, F)$ we call *diffuse operators*. Using this terminology, we provide below one of the main results of [34] generalizing Theorem 8.6 to vector lattices (see also [53, Theorem 10.40]).

Theorem 8.7 ([34]) *Let* E, F *be Dedekind complete vector lattices such that* E *is atomless and* F *is an ideal of some order continuous Banach lattice. Then for every regular order continuous operator* $T : E \to F$ *the following assertions are equivalent:*

1. $T \in \mathcal{D}(E, F)$*;*
2. T *is order narrow;*
3. *the Enflo-Starbird* λ*-function of* T *is zero:* $\lambda_T = 0$.

Hence, each regular order continuous operator $T : E \to F$ *is uniquely represented in the form* $T = T_a + T_c$ *where* T_a *is a sum of an order absolutely* *summable family of disjointness preserving order continuous operators and T_c is an order continuous order narrow operator.*

The main contribution of Theorem 8.7 was equivalence (1) $\Leftrightarrow$ (2), because equivalence (1) $\Leftrightarrow$ (3) easily follows (even without the assumption of order continuity on T) from the results of [25]. The main open questions remained unsolved in [34] (see also [53, Problem 10.42] and [53, Problem 10.43]) are: is Theorem 8.7 true for regular operators, which are not order continuous?[3] Is the set of all order narrow regular operators $T: L_\infty \to L_\infty$ a band in the vector lattice $L_r(L_\infty)$ of all regular linear operators on L_∞? Remark that, the set of all narrow regular operators $T: L_\infty \to L_\infty$ is not a band in $L_r(L_\infty)$ [34], [53, Theorem 10.5].

Results, which we present below, give affirmative answers to both questions not only for linear operators, but for OAOs. Moreover, in these results the Dedekind complete assumption on E is replaced with a less restrictive assumption of possessing the principal projection property, and the atomlessness assumption on E is removed (the later adjustment is not a big deal, because every narrow and order narrow operator must send atoms to zero, making the representation trivial on the atomic part of E). Being more general at first glance, the results for OAOs do not formally imply similar results for linear operators, because the set of linear operators is just a linear subspace of the vector lattice of OAOs, but not a sublattice, due to different orders.

Theorem 8.8 ([46, Theorem A]) *Let E be a vector lattice with the principal projection property and F a Dedekind complete vector lattice being an ideal of some order continuous Banach lattice G. Then for every regular OAO $T: E \to F$ the following assertions are equivalent*

1. *T is diffuse;*
2. *$T: E \to F$ is order narrow;*
3. *$T: E \to G$ is order narrow;*
4. *$T: E \to G$ is narrow;*
5. *the Enflo-Starbird λ-function of T is zero: $\lambda_T = 0$ (both in F and G).*

Hence, every regular OAO $T: E \to F$ is uniquely represented as follows $T = T_a + T_c$, where T_a is an absolutely order convergent sum of disjointness preserving regular OAOs and T_c is a regular order narrow OAO.

Similar result holds for linear operators.

Theorem 8.9 ([46, Theorem B]) *Let E be a vector lattice with the principal projection property and F a Dedekind complete vector lattice being an ideal of some order continuous Banach lattice G. Then for every $T \in \mathcal{L}_b(E, F)$ assertions 1-5 of Theorem 8.8 are equivalent.*

To get Theorem 8.9 as a consequence of Theorem 8.8, we define the *canonical embedding* $\varphi: \mathcal{L}_b(E, F) \to \mathcal{OA}_r(E, F)$ by setting $\varphi(T)\,x = T|x|$ for a given

[3] Weis' representation theorem [58] was proved under the assumption of order continuity.

$T \in \mathcal{L}_b^+(E, F)$ and all $x \in E$, and by $\varphi(T) = \varphi(T^+) - \varphi(T^-)$ for an arbitrary $T \in \mathcal{L}_b(E, F)$, that is, $\varphi(T)\,x = T^+|x| - T^-|x|$ for all $x \in E$. If E has the principal projection property then φ is a lattice monomorphism, that is, an injective lattice homomorphism.

9 Banach Lattices of OAOs

Throughout this section, we consider a pair of normed lattices E, F with F both norm and Dedekind complete. There is a naturally defined norm on the vector lattice $\mathcal{U}(E, F)$, called the absolute norm. However, the normed sublattice $\mathcal{AB}(E, F)$ of $\mathcal{U}(E, F)$, consisting of all bounded with respect to this norm OAOs, need not be norm complete. Another natural somewhat greater norm, called the uniform norm, makes the sublattice $\mathcal{UB}(E, F)$ of both $\mathcal{AB}(E, F)$ and $\mathcal{U}(E, F)$ of all bounded with respect to the latter norm OAOs to be a Banach lattice, which is ought to be investigated.

9.1 Absolute and Uniform Norms of an Abstract Uryson Operator

Definition 9.1 Let E be a normed lattice and F a Dedekind complete Banach lattice. An OAO $T \in \mathcal{U}(E, F)$ is said to be *absolutely norm bounded* if there is $M \in [0, +\infty)$ such that, for every $x \in E$ one has $\big\||T|(x)\big\| \le M\|x\|$.

The set of all such operators is denoted by $\mathcal{AB}(E, F)$ and endowed with the following nonnegative value

$$\|T\|_{abs} := \sup_{x \in E \setminus \{0\}} \frac{\big\||T|(x)\big\|}{\|x\|},$$

which we call the *absolute norm*. The following statement, in particular, asserts that it is a norm.

Theorem 9.2 ([52, Theorem 3.2]) *Let E be a normed lattice and F a Dedekind complete Banach lattice. Then $\mathcal{AB}(E, F)$ is a normed lattice with respect to the absolute norm, and is a sublattice of $\mathcal{U}(E, F)$.*

The following example (with non-obvious proof) shows that the normed lattice $\mathcal{AB}(E, F)$ need not be norm complete.

Example ([52, Example 3.3]) Let $E = F = L_1[0, 1]$. Then the normed space $\mathcal{AB}(E, F)$ is not norm complete.

To introduce a complete norm on $\mathcal{U}(E, F)$, we must restrict the sublattice $\mathcal{AB}(E, F)$ to a much more narrow class of operators.

Definition 9.3 Let E be a normed lattice and F a Dedekind complete Banach lattice. An abstract Uryson operator $T : E \to F$ is said to be *uniformly order bounded*, if there is $L \in [0, +\infty)$ such that for every $x \in E$ one has

$$\Big\| \sup_{|y| \le |x|} |T|(y) \Big\| \le L\|x\|.$$

In other words, T is uniformly order bounded provided its envelope[4] is absolutely norm bounded, that is, $\widehat{T} \in \mathcal{AB}(E, F)$. The set of all uniformly order bounded abstract Uryson operators we denote by $\mathcal{UB}(E, F)$ and endow with the following nonnegative value

$$\|T\|_u := \sup_{x \in E \setminus \{0\}} \frac{\big\|\sup_{|y| \le |x|} |T|(y)\big\|}{\|x\|} = \sup_{x \in E \setminus \{0\}} \frac{\big\|\widehat{T}(x)\big\|}{\|x\|}$$

which we call the *uniform norm*. Obviously, $\mathcal{UB}(E, F) \subseteq \mathcal{AB}(E, F)$ and $\|T\|_{abs} \le \|T\|_u = \|\widehat{T}\|_{abs}$ for every $T \in \mathcal{UB}(E, F)$.

There inclusion $\mathcal{UB}(E, F) \subset \mathcal{AB}(E, F)$ is strict for the class of AL-spaces [52, Example 3.8].

Theorem 9.4 ([52, Theorem 3.9]) *Let E be a normed lattice and F a Dedekind complete Banach lattice. Then $\mathcal{UB}(E, F)$ is a Dedekind complete Banach lattice with respect to the uniform norm, and is a sublattice of $\mathcal{U}(E, F)$.*

9.2 Consistent Sets and Levels in a Vector Lattice

A subset G of a vector lattice E is said to be *consistent* if every two-point subset $\{x, y\}$ of G is laterally bounded in E, that is, there exists $e \in E$ such that $x \sqsubseteq e$ and $y \sqsubseteq e$ (equivalently, every finite subset of G is laterally bounded [38, Proposition 5.2]). The lateral band in E generated by a consistent set G is consistent [38, Theorem 6.10].

Definition 9.5 A consistent lateral band in a vector lattice E is called a *level* of E. A level which is not included in another level is called a *maximal level*. A level $\mathbf{L}$ in E is called a *principal level* provided $\mathbf{L} = \mathfrak{F}_e$ for some $e \in E$.

[4] See Sect. 7 for the definition of the envelope.

Example Let (Ω, Σ, μ) be a finite atomless measure space, $0 \le p \le \infty$ and $E = L_p(\mu)$. Fix any $z \in L_0(\mu)$ and set $\mathbf{L}_z = \{x \in E : x \sqsubseteq z\}$. Then

1. $\mathbf{L}_z$ *is a level in* E;
2. $\mathbf{L}_z$ *is a maximal level in* E *if and only if* $\operatorname{supp} z = \Omega$;
3. $\mathbf{L}_z$ *is a principal level* $\mathbf{L}_z = \mathfrak{F}_z$ *if and only if* $z \in E$.

Proposition 9.6 ([52, Proposition 2.4]) *A vector lattice* E *is laterally complete if and only if every maximal level is a principal level.*

Obviously, if $\mathbf{L}'$ and $\mathbf{L}''$ are *orthogonal levels* in a vector lattice E, that is, $e' \perp e''$ for all $e' \in \mathbf{L}'$ and $e'' \in \mathbf{L}''$, then the *direct sum* defined by setting

$$\mathbf{L}' \oplus \mathbf{L}'' = \{x + y : x \in \mathbf{L}',\ y \in \mathbf{L}''\}$$

is a level as well.

A level $\mathbf{L}$ in a vector lattice E is said to be *positive* (respectively, *negative*) provided $\mathbf{L} \subset E_+$ (respectively, $x \le 0$ for each $x \in \mathbf{L}$). The relation $\mathbf{L} \ge 0$ (respectively, $\mathbf{L} \le 0$) means that the level $\mathbf{L}$ is positive (respectively, negative).

Proposition 9.7 ([52, Proposition 2.5]) *Every level* $\mathbf{L}$ *in a vector lattice* E *admits a unique decomposition into a direct sum of levels* $\mathbf{L} = \mathbf{L}_+ \oplus \mathbf{L}_-$*, where* $\mathbf{L}_+ \ge 0$ *and* $-\mathbf{L}_- \ge 0$*. In particular, for any principal level* $\mathbf{L} = \mathfrak{F}_e$ *one has* $(\mathfrak{F}_e)_+ = \mathfrak{F}_{(e_+)}$ *and* $(\mathfrak{F}_e)_- = \mathfrak{F}_{(-e_-)}$.

9.3 Linear Sections of Orthogonally Additive Operators

This subsection is devoted to construction of norm one projections of the Banach lattice $\mathcal{UB}(E, F)$ onto its subspace $\mathcal{L}(E, F)$ of all linear bounded operators. We present some results asserting the existence of plenty norm one projections of $\mathcal{UB}(E, F)$ onto $\mathcal{L}(E, F)$.

Given an OAO $T : E \to F$ between vector lattices and a level $\mathbf{L}$ in E, we construct a linear operator $S : E \to F$ having the same values on $\mathbf{L}$ and vanishing on the disjoint complement to $\mathbf{L}$. First, we define such an operator on the linear subspace $E_{\mathbf{L}} \oplus \mathbf{L}^d$ of E and then find a possibility to extend it to the entire space E preserving some important properties (by $E_{\mathbf{L}}$ we denote the minimal ideal of E including $\mathbf{L}$). Our final purpose is to find assumptions on E, F, T and $\mathbf{L}$ under which there is a unique linear operator with the desired properties. Then such a linear operator will be picked as a canonical projection of $\mathcal{UB}(E, F)$ onto $\mathcal{L}(E, F)$.

Definition 9.8 Let E be a vector lattice, $\mathbf{L}$ a level in E, F a linear space and $T : E \to F$ an OAO. A linear operator $S : E_{\mathbf{L}} \oplus \mathbf{L}^d \to F$ is called a *linear section of* T *by* $\mathbf{L}$ if $S|_{\mathbf{L}} = T|_{\mathbf{L}}$ and $S|_{\mathbf{L}^d} = 0$. A linear operator $S : E \to F$ is called an *extended linear section of* T *by* $\mathbf{L}$ if $S|_{\mathbf{L}} = T|_{\mathbf{L}}$ and $S|_{\mathbf{L}^d} = 0$.

The first theorem concerns the very general case and asserts the existence of an extended linear section without any additional properties.

Theorem 9.9 ([52, Theorem 4.2]) *Let E be a vector lattice, F a vector space and $T: E \to F$ an OAO. Then for every level $\mathbf{L}$ of E there exists an extended linear section $S: E \to F$ of T by $\mathbf{L}$.*

Definition 9.10 Let E, F be vector lattices and $D \subseteq E$. A function $f: D \to F$ is said to be *vertically order σ-continuous* on D if D is an ideal of E and for every $w \in D_+$, every $x \in E_w$ and every increasing sequence $(x_n)_{n=1}^{\infty}$ in E_w such that $0 \le x - x_n \le \frac{1}{n} w$ one has $f(x_n) \xrightarrow{\text{o}} f(x)$.

One can easily show that every regular linear operator $T: E \to F$ is vertically order σ-continuous on E, once F is Archimedean [52, Proposition 4.6].

Next is the main technical tool for the construction of linear sections.

Theorem 9.11 ([52, Theorem 4.7]) *Let E, F be vector lattices. Assume E has the principal projection property, F is Dedekind complete and $T \in \mathcal{U}(E, F)$. Then for every level $\mathbf{L}$ of E there is a unique regular linear section $S = \Psi_{\mathbf{L}}(T): E_{\mathbf{L}} \oplus \mathbf{L}^d \to F$ of T by $\mathbf{L}$. Moreover, if $\mathbf{L} \ge 0$ then $S^+ = (\Psi_{\mathbf{L}}(T))^+ = \Psi_{\mathbf{L}}(T^+)$. In particular, if T is positive as an OAO and $\mathbf{L} \ge 0$ then S is positive as a linear operator.*

Definition 9.12 Let E be a vector lattice with the principal projection property, F a Dedekind complete vector lattice, $T \in \mathcal{U}(E, F)$ and $\mathbf{L}$ a level of E. The regular linear section $S = \Psi_{\mathbf{L}}(T) : E_{\mathbf{L}} \oplus \mathbf{L}^d \to F$ of T by $\mathbf{L}$, the existence and uniqueness of which Theorem 9.11 asserts, is called the *canonical linear section* of T by $\mathbf{L}$.

Theorem 9.11 yields the following **properties of the canonical linear section $S = \Psi_{\mathbf{L}}(T)$ of T by L**:

1° *S is a regular linear operator.*
If, in addition, $\mathbf{L} \ge 0$ then
2° $S^+ = (\Psi_{\mathbf{L}}(T))^+ = \Psi_{\mathbf{L}}(T^+)$;
3° *if $T \ge 0$ as an OAO then $S \ge 0$ as a linear operator.*

The next property is expressed in the following theorem.

Theorem 9.13 ([52, Theorem 4.12]) *Let E be a vector lattice with the principal projection property, F a Dedekind complete vector lattice, $T \in \mathcal{U}(E, F)$ and $\mathbf{L}$ a positive level of E. If T is horizontally-to-order continuous (horizontally-to-order σ-continuous) on $\mathbf{L}$ then the canonical linear section $S = \Psi_{\mathbf{L}}(T)$ of T by $\mathbf{L}$ is order continuous (order σ-continuous) on its domain.*

The following theorem asserts that the canonical linear section is a linear operator from $\mathcal{UB}(E, F)$ to $\mathcal{L}_r(E_{\mathbf{L}} \oplus \mathbf{L}^d, F)$ with some useful properties.

Theorem 9.14 ([52, Theorem 4.15]) *Let E, F be vector lattices. Assume E has the principal projection property, F is Dedekind complete and $\mathbf{L}$ is a level in E. Then the corresponding canonical linear section as a mapping $\Psi_{\mathbf{L}}: \mathcal{UB}(E, F) \to \mathcal{L}_r(E_{\mathbf{L}} \oplus \mathbf{L}^d, F)$ is a disjointness preserving linear operator. If, moreover, $\mathbf{L} \ge 0$ then $\Psi_{\mathbf{L}}$*

is a lattice homomorphism, and if $\mathbf{L} \leq 0$ *then* $-\Psi_{\mathbf{L}}$ *is a lattice homomorphism. Consequently, in the general case* $\Psi_{\mathbf{L}}$ *is a difference of two lattice homomorphisms.*

The existence of a unique extended linear section, which is a linear projection of $\mathcal{UB}(E, F)$ onto $\mathcal{L}(E, F)$, can be obtained in some partial cases.

Theorem 9.15 ([52, Theorem 5.1]) *Let* E *be an* AL*-space,* F *a Dedekind complete Banach lattice and* $T \in \mathcal{UB}(E, F)$*. Then for every positive level* $\mathbf{L}$ *of* E *there is a unique extended linear bounded regular section* $S = \Phi_{\mathbf{L}}(T) : E \to F$ *of* T *by* $\mathbf{L}$ *with* $\|S\| \leq \|T\|_u$*. Moreover,* $S = \Phi_{\mathbf{L}}(T)$ *is linear with respect to* T *and if* $T \geq 0$ *then* $S \geq 0$*.*

In some natural cases, an extended linear section does not exist.

Example Let $1 \leq p < r < \infty$. Denote by $\mathbf{1}$ the characteristic function of $[0, 1]$, by $\mathbf{L}$ the principal level $\mathbf{L} = \mathfrak{F}_{\mathbf{1}}$ in L_p, and by $J : \mathbf{L} \to L_r$ denote the identity embedding. Define a function $T : L_p \to L_r$ by setting

$$T(x) = J\big(x \Cap \mathbf{1}\big) \quad \text{for all} \quad x \in L_p. \tag{7}$$

Then $T \in \mathcal{U}(L_p, L_r)$ and there is no extended linear section of T by $\mathbf{L}$.

Proof Since L_p is Dedekind complete, L_p has the intersection property and T is well defined by (7). By Theorem 4.6, T is an OAO, and the inequality $|Tx| \leq |x|$ implies that T is order bounded. Thus, $T \in \mathcal{U}(L_p, L_r)$. Observe that $\mathbf{L}^d = \{0\}$, $E_{\mathbf{L}} = L_\infty$ and the canonical linear section of T by $\mathbf{L}$ is the identity operator $S = \Psi_{\mathbf{L}}(T) : L_\infty \to L_r$, $Sx = x$ for all $x \in L_\infty \subset L_p$. So S, being unbounded on its domain, cannot be extended to the entire L_p. □

As a vector subspace of $\mathcal{UB}(E, F)$, the Banach lattice $\mathcal{L}_r(E, F)$ is not a sublattice of $\mathcal{UB}(E, F)$, because the lattice order on $\mathcal{L}_r(E, F)$ is completely different from that of $\mathcal{UB}(E, F)$. Moreover, $\mathcal{UB}(E, F)_+ \cap \mathcal{L}_r(E, F) = \{0\}$. However, for the next result, it is enough to consider the case where

$$\begin{aligned} &E \text{ is an } AL\text{-space and } F \text{ a Dedekind complete Banach lattice that satisfy} \\ &\mathcal{L}_r(E, F) = \mathcal{L}(E, F) \text{ and } \big(\forall T \in \mathcal{L}(E, F)\big)\ \big\||T|\big\| = \|T\| \end{aligned} \tag{8}$$

Observe that, under the above assumptions on E and F, the Banach space $\mathcal{L}(E, F)$ is a subspace of $\mathcal{UB}(E, F)$.

Recall that a Banach lattice F is called a KB*-space* if every increasing norm bounded sequence in F_+ is norm convergent (equivalently, if the canonical image of F in its second dual F'' is a band [8, Theorem 4.60]). By [8, Theorem 4.75], every KB-space F satisfies (8) for any AL-space E. Nevertheless, there is a strictly wider class of Banach lattices F than the KB-spaces (including e.g. infinite dimensional $L_\infty(\mu)$-spaces which are not KB-spaces), possessing (8) for any AL-space E.

A norm $\|\cdot\|$ on a Banach lattice F is said to be:

- a *Fatou norm* provided for every net (f_α) in F^+ and $f \in F^+$ the condition $f_\alpha \uparrow f$ implies $\|f\| = \lim_\alpha \|f_\alpha\|$;
- a *Levi norm* provided every net (f_α) in F with $0 \le f_\alpha \uparrow$ and $\|f_\alpha\| \le 1$ for all α has a supremum in F;
- a *Fatou-Levi norm* provided that its norm is both Fatou and Levi.

By [6, Theorem 4.1], a Banach lattice F satisfies (8) for every AL-space E if and only if F has a Fatou-Levi norm.

Next is the main result of the section.

Theorem 9.16 ([52, Theorem 5.4]) *Let E be an AL-space, F a Banach lattice with a Fatou-Levi norm. Then $\mathcal{L}(E, F)$ is a 1-complemented subspace of $\mathcal{UB}(E, F)$. Moreover, for every maximal positive level $\boldsymbol{L}$ in E, $\Phi_{\boldsymbol{L}}$ is a contractive projection of $\mathcal{UB}(E, F)$ onto $\mathcal{L}(E, F)$ such that*

(i) *$\Phi_{\boldsymbol{L}}(T)|_{\boldsymbol{L}} = T|_{\boldsymbol{L}}$ for all $T \in \mathcal{UB}(E, F)$;*
(ii) *if $\boldsymbol{L}'$ and $\boldsymbol{L}''$ are distinct maximal levels in E then $\Phi_{\boldsymbol{L}'} \neq \Phi_{\boldsymbol{L}''}$.*

10 Open Problems

10.1 An Analytic Representation of OAOs

The criterions of an Uryson and Hammerstein type integral representability of order bounded OAOs were obtained in [55, 56].

Problem 10.1 Obtain the criterions of integral representability of regular, (in general order unbounded) OAOs.

10.2 Disjointness Preserving OAOs

Different classes of disjointness preserving OAOs were investigated in [2, 4, 12]. It is well known that the sum of disjointness preserving OAOs need not be a disjointness preserving operator.

Problem 10.2 Obtain the criterion for an OAO T to be the sum of n disjointness preserving OAOs.

We recall that an OAO $T: E \to E$ is non-expanding if $Tx \in \{x\}^{dd}$. By $\mathcal{N}(E)$ we denote the set of all non-expanding OAOs on E. It is not hard to verify, that $T + S \in \mathcal{N}(E)$ and $T \circ S \in \mathcal{N}(E)$ for every $T, S \in \mathcal{N}(E)$. Actually, $\mathcal{N}(E)$ is an (noncommutative) algebra over $\mathbb{R}$.

Problem 10.3 Investigate algebraic properties of $\mathcal{N}(E)$. In particular describe automorphisms of $\mathcal{N}(E)$, left (right) ideals of $\mathcal{N}(E)$.

10.3 Compact OAOs

Suppose E is a vector lattice and F is a Banach lattice with an order continuous norm. By Theorem 6.3, $\mathcal{COA}_r(E, F)$ is a projection band of $\mathcal{OA}_r(E, F)$.

Problem 10.4 Does Theorem 6.3 continue to hold if F is a Dedekind complete Banach lattice?

Problem 10.5 Obtain a formula for the band projection from $\mathcal{OA}_r(E, F)$ onto $\mathcal{COA}_r(E, F)$.

10.4 Order Projections

Order projections in different spaces of OAOs were studied in [2, 3, 12, 48],

Problem 10.6 Obtain a formula for the order projection of $\mathcal{OA}_r(E, F)$ onto the band generated by an arbitrary positive OAO $T: E \to F$.

Problem 10.7 Obtain a formula for the order projection in $\mathcal{OA}_r(E, F)$ onto the band by all disjointness preserving OAOs from in E to F.

10.5 Partial Order Continuities of Orthogonally Additive Operators

Problem 10.8 Under what assumptions on vector lattices E, F with F Dedekind complete every abstract Uryson operator $T: E \to F$, which is both horizontally-to-order continuous and uniformly-to-order continuous, is order continuous?

Problem 10.9 Do there exist a vector lattice with the principal projection property E, a Dedekind complete vector lattice F and an order continuous abstract Uryson (or, at least, laterally-to-order bounded) operator $T: E \to F$ such that $|T|$ is not order continuous?

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Part III
Inequalities Related to Types of Operators

Normal Operators and their Generalizations

Pietro Aiena

Abstract In this chapter we study the spectral properties of some classes of operators which generalize normal operators on Hilbert spaces. In particular, we consider for these operators some aspects of local spectral theory and Fredholm theory.

Keywords Normal operators · Fredholm theory · Localized single valued extension property · Weyl type theorems

1 Introduction

It is well-known that a normal operator on a Hilbert space possesses a rich spectral theory. Many classes of operators that generalize normal operators, have been introduced and studied in the last years. These classes of operators are defined by means of some (order) inequalities that involve the operator T and its adjoint T^*. Precisely, most of them are defined by relaxing the condition of normality $TT^* = T^*T$.

In this chapter we shall consider the spectral properties of these classes of operators, and we show that such operators share with the normal operators many spectral properties, mostly of them concerning Fredholm theory and local spectral theory.

Our main interest regards the isolated points of the spectrum of these operators, as well as the isolated points of the approximate point spectrum. Many times these points are poles (or left poles) of the resolvent. In this context the concept of *polaroid operator* (or *a-polaroid operator*), jointly with the single-valued extension property (SVEP), provide a very useful tool for studyng the structure of the spectrum. We also introduce the quasi-$\mathcal{THN}$ operators (*quasi totally hereditarily*

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normaloid operators), in order to determine a general theoretical framework from which we can state that several versions of Weyl type theorems hold for all these operators in their classical form, as well as in their generalized form.

2 Notations and Preliminary Results on Spectral Theory

Since this chapter concerns the spectral theory of bounded linear operators, we always assume that the Banach spaces, or the Hilbert spaces, are complex infinite-dimensional. If X, Y are Banach spaces, by $L(X, Y)$ we denote the Banach space of all bounded linear operators from X into Y. Recall that if $T \in L(X, Y)$, the norm of T is defined by

$$\|T\| := \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}.$$

If $X = Y$ we write $L(X) := L(X, X)$. By $X^* := L(X, \mathbb{C})$ we denote the *dual* of X. If $T \in L(X, Y)$ by $T' \in L(Y^*, X^*)$ we denote the *dual operator* of T, defined by

$$(T'f)(x) := f(Tx) \quad \text{for all } x \in X,\ f \in Y^*.$$

The identity operator on X will be denoted by I_X, or simply I if no confusion can arise.

We reassume now some of the basic definitions of Hilbert space operators. We refer to the books Rudin [53], Heuser [39] for details and proof. Let H be a complex Hilbert space with an inner product $\langle \cdot, \cdot \rangle$. The inner product satisfies the *Schwarz inequality*, i.e.,

$$|\langle x, y \rangle| \leq \|x\| \|y\| \quad \text{for all } x, y \in H.$$

The dual of a Hilbert space is described by the following theorem.

Theorem 2.1 (Frechét-Riesz Representation Theorem) *For each fixed element $z \in H$ the map $f : x \in H \to \langle x, z \rangle$ defines a continuous linear form on H, Conversely, for every continuous linear form f on H there exists exactly a vector $z \in H$ such that $f(x) = \langle x, z \rangle$ for all $x \in H$, Furthermore, $\|f\| = \|z\|$.*

A consequence of this theorem is that every Hilbert space is isometrically isomorphic to its dual. If $T \in L(H)$, for a fixed $y \in H$ define

$$f(x) := \langle Tx, y \rangle.$$

According Theorem 2.1, there exists a unique element $z \in H$ such that

$$f(x) = \langle Tx, y \rangle = \langle x, z \rangle.$$

The *adjoint* operator $T^\star \in L(H)$ is then defined by

$$\langle Tx, y\rangle = \langle x, z\rangle = \langle x, T^* y\rangle \text{ for all } x \in H.$$

By the Frechét-Riesz representation theorem there exists a conjugated-linear isometry $U : H \to H'$, H' the dual of H, that associates to every $y \in H$ the linear form defined $f_y(x) :=< x,\ y >$. The dual T' and the adjoint T^* of T are related as follows:

$$(\bar{\lambda} I - T^*) = U^{-1}(\lambda I - T')U \quad \text{for every } \lambda \in \mathbb{C}.$$

Hence

$$U(\bar{\lambda} I - T^*) = (\lambda I - T')U \quad \text{and} \quad (\bar{\lambda} I - T^*)U^{-1} = U^{-1}(\lambda I - T'). \tag{1}$$

Given a bounded operator $T \in L(X, Y)$ between Banach spaces, the *kernel* of T is the set

$$\ker T := \{x \in X : Tx = 0\},$$

while the *range* of T is denoted by $T(X)$. In the sequel, for every bounded operator $T \in L(X, Y)$, we shall denote by $\alpha(T)$ the *nullity* of T, defined as $\alpha(T) := \dim \ker T$, while the *deficiency* $\beta(T)$ of T is the dimension of the *cokernel* of $T(X)$, i.e., $\beta(T) := \dim Y/T(X) = \operatorname{codim} T(X)$. The *spectrum* of $T \in L(X)$ defined as

$$\sigma(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bijective}\}.$$

It is well-known that the spectrum is a compact subset of $\mathbb{C}$ and $\sigma(T) = \sigma(T^*)$ for all $T \in L(X)$, while for the adjoint of a Hilbert space operator we have $\sigma(T) = \overline{\sigma(T^*)}$. If X is a complex Banach space then every $T \in L(X)$ has non-empty spectrum. The complement $\rho(T) := \mathbb{C} \setminus \sigma(T)$ is called the *resolvent* of T. We have $\sigma(T) = \sigma(T')$

Recall that an operator $T \in L(X)$ is said to be *bounded below* if T is injective and has closed range. An basic result shows that $T \in L(X, Y)$ is bounded below if and only if there exists $K > 0$ such that

$$\|Tx\| \geq K\|x\| \quad \text{for all } x \in X. \tag{2}$$

The classical *approximate point spectrum* $\sigma_{\rm ap}(T)$ is defined by

$$\sigma_{\rm ap}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\},$$

while the *surjectivity spectrum* $\sigma_{\rm s}(T)$ is defined by

$$\sigma_{\rm su}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not onto}\}.$$

These two spectra are nonempty and dual one to each other, i.e., $\sigma_a(T^*) = \sigma_s(T)$ and $\sigma_s(T^*) = \sigma_a(T)$. Both spectra $\sigma_{ap}(T)$ and $\sigma_{su}(T)$ contain the boundary $\partial\sigma(T)$ of the spectrum, see [4, Theorem 1.12]. Furthermore, it is easily seen that for Hilbert space operators for the adjoint T^* and the dual T' we have:

$$\sigma_{ap}(T') = \overline{\sigma_{ap}(T^*)} \quad \textit{and} \quad \sigma_{su}(T') = \overline{\sigma_{su}(T^*)}.$$

Given a linear operator T on a vector space X, T is said to have *finite ascent* if $\mathcal{N}^\infty(T) = \ker T^k$ for some positive integer k. Clearly, in such a case there is a smallest positive integer $p = p(T)$ such that $\ker T^p = \ker T^{p+1}$. The positive integer p is called the *ascent* of T. If there is no such integer we set $p(T) := \infty$. Analogously, T is said to have *finite descent* if $T^\infty(X) = T^k(X)$ for some k. The smallest integer $q = q(T)$ such that $T^{q+1}(X) = T^q(X)$ is called the *descent* of T. If there is no such integer we set $q(T) := \infty$. The proof of following basic results may be found in [4, Chapter 1].

Theorem 2.2 *Let T be a linear operator on a vector space X. If both $p(T)$ and $q(T)$ are finite then $p(T) = q(T)$.*

The *defects* $\alpha(T)$, $\beta(T)$, the ascent and the descent are related as follows:

Theorem 2.3 *If T is a linear operator on a vector space X then the following properties hold:*

(i) *If $p(T) < \infty$ then $\alpha(T) \le \beta(T)$;*
(ii) *If $q(T) < \infty$ then $\beta(T) \le \alpha(T)$;*
(iii) *If $p(T) = q(T) < \infty$ then $\alpha(T) = \beta(T)$ (possibly infinite);*
(iv) *If $\alpha(T) = \beta(T) < \infty$ and if either $p(T)$ or $q(T)$ is finite then $p(T) = q(T)$.*

The finiteness of the ascent and the descent of a linear operator T is related to a certain decomposition of X.

Theorem 2.4 *Suppose that T is a linear operator on a vector space X. If $p = p(T) = q(T) < \infty$ then we have the decomposition*

$$X = T^p(X) \oplus \ker T^p.$$

Conversely, if for a natural number m we have the decomposition $X = T^m(X) \oplus \ker T^m$ then $p(T) = q(T) \le m$. In this case $T|T^p(X)$ is bijective.

The following subspace has been introduced by Vrbová [59] and later studied by Mbekhta [47].

Definition 2.5 Let X be a Banach space and $T \in L(X)$. The *analytic core* of T is the set $K(T)$ of all $x \in X$ such that there exists a sequence $(u_n) \subset X$ and a constant $\delta > 0$ such that:

(1) $x = u_0$, and $Tu_{n+1} = u_n$ for every $n \in \mathbb{Z}_+$;
(2) $\|u_n\| \le \delta^n \|x\|$ for every $n \in \mathbb{Z}_+$.

It is easily seen that $K(T)$ is a linear subspace of X and $T(K(T)) = K(T)$. Another important invariant subspace for a bounded operator $T \in L(X)$, X a Banach space, is defined as follows:

Definition 2.6 Let $T \in L(X)$, X a Banach space. The *quasi-nilpotent part* of T is defined to be the set

$$H_0(T) := \{x \in X : \lim_{n\to\infty} \|T^n x\|^{1/n} = 0\}.$$

Clearly $H_0(T)$ is a linear subspace of X, generally not closed. Obviously $\ker(T^m) \subseteq H_0(T)$ for every $m \in \mathbb{N}$.

Theorem 2.7 *Let X be a Banach space. Then $T \in L(X)$ is quasi-nilpotent if and only if $H_0(T) = X$.*

Proof Suppose that T is quasi-nilpotent, i.e., $\lim_{n\to\infty} \|T^n\|^{1/n} = 0$, Then $\|T^n x\| \leq \|T^n\|\|x\|$ for every $x \in X$, so $\lim_{n\to\infty} \|T^n x\|^{1/n} = 0$. This shows that $H_0(T) = X$.

Conversely, assume that $H_0(T) = X$. By the n-th root test the series

$$\sum_{n=0}^{\infty} \frac{\|T^n x\|}{|\lambda|^{n+1}},$$

converges for each $x \in X$ and $\lambda \neq 0$. Define

$$y := \sum_{n=0}^{\infty} \frac{T^n x}{\lambda^{n+1}}.$$

It is easy to verify that $(\lambda I - T)y = x$, thus $(\lambda I - T)$ is surjective for all $\lambda \neq 0$. On the other hand, for every $\lambda \neq 0$ we have that

$$\{0\} = \ker(\lambda I - T) \cap H_0(T) = \ker(\lambda I - T) \cap X = \ker(\lambda I - T),$$

which shows that $\lambda I - T$ is invertible and therefore $\sigma(T) = \{0\}$, i.e. T is quasi-nilpotent. ■

The chain lengths of $(\lambda I - T)$ are intimately related to the poles of the resolvent $R(\lambda, T)$. If $f : \mathbb{D}(\lambda_0, \delta) \setminus \{\lambda_0\} \to X$, X a Banach space, is a analytic function defined in the open disc centered at λ_0 with values in X, then, by the *Laurent expansion*, f can be represented in the form

$$f(\lambda) = \sum_{k=0}^{\infty} a_k(\lambda - \lambda_0)^k + \sum_{k=1}^{\infty} \frac{b_k}{(\lambda - \lambda_0)^k},$$

with $a_k, b_k \in X$, and $\lambda \in \mathbb{D}(\lambda_0, \delta) \setminus \{\lambda_0\}$. The coefficients are given by the formulas

$$a_k = \frac{1}{2\pi i} \int_\Gamma \frac{f(\lambda)}{(\lambda - \lambda_0)^{k+1}} d\lambda, \quad \text{and} \quad b_k = \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda - \lambda_0)^{k-1} d\lambda,$$

where Γ is a positively oriented circle $|\lambda - \lambda_0| = r$, with $0 < r < \delta$, see Proposition 46.7 of [39] for details. We say that λ_0 is a *pole of order* p if $b_p \neq 0$ and $b_n = 0$ for all $n > p$.

Let λ_0 be an isolated point of $\sigma(T)$ and let us consider the particular case of the Laurent expansion of the analytic function $R_\lambda : \lambda \in \rho(T) \to (\lambda I - T)^{-1} \in L(X)$ in a neighborhood of λ_0. According the previous considerations, we have

$$R_\lambda = \sum_{k=0}^{\infty} Q_k (\lambda - \lambda_0)^k + \sum_{k=1}^{\infty} \frac{P_k}{(\lambda - \lambda_0)^k} \quad \text{with } P_k, Q_k \in L(X).$$

for all $0 < |\lambda - \lambda_0| < \delta$. The coefficients are calculated according the formulas

$$Q_k = \frac{1}{2\pi i} \int_\Gamma \frac{R_\lambda}{(\lambda - \lambda_0)^{k+1}} d\lambda \tag{3}$$

$$P_k = \frac{1}{2\pi i} \int_\Gamma R_\lambda (\lambda - \lambda_0)^{k-1} d\lambda, \tag{4}$$

where Γ is a sufficiently small, positively oriented circle around λ_0.

Let $\mathcal{H}(\sigma(T))$ be the set of all complex-valued functions which are locally analytic on an open set containing $\sigma(T)$. Suppose that $f \in \mathcal{H}(\sigma(T))$, $\Delta(f)$ be the domain of f, and let Γ denote a contour in $\Delta(f)$ that surrounds $\sigma(T)$. This means a positively oriented finite system $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ of closed rectifiable curves in $\Delta(f) \setminus \sigma(T)$ such that $\sigma(T)$ is contained in the inside of Γ and $\mathbb{C} \setminus \Delta(f)$ in the outside of Γ. Then, from the classical functional calculus,

$$f(T) := \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda I - T)^{-1} d\lambda,$$

is well-defined and does not depend on the particular choice of Γ. It should be noted that *mutatis mutandis* all the arguments and notions introduced above may be extended to Banach algebras with unit u: if $a \in \mathcal{H}(\sigma(a))$ is an analytic function defined on open set containing $\sigma(a)$ then

$$f(a) := \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda u - a)^{-1} d\lambda,$$

is defined in a similar way as we have done for the elements of the Banach algebra $L(X)$.

In the particular case that of functions which are equal to 1 on certain parts of $\sigma(T)$ and equal to 0 on others we get idempotent operators. To see this, suppose that σ is a spectral set (i.e. σ and $\sigma(T) \setminus \sigma$ are both closed) and $\Delta := \Delta_1 \cup \Delta_2$ is an open covering of $\sigma(T)$ such that $\Delta_1 \cap \Delta_2 = \emptyset$ and $\sigma \subseteq \Delta_1$, define $h(\lambda) := 1$ for λ on Δ_1 and $h(\lambda) := 0$ for λ on Δ_2. Consider the operator $P_\sigma := h(T)$. It is easy to check that $P_\sigma^2 = P_\sigma$, so P_σ is a projection called the *spectral projection associated with* σ, and obviously

$$P_\sigma = \frac{1}{2\pi i} \int_\Gamma (\lambda I - T)^{-1} d\lambda, \tag{5}$$

where Γ is a curve enclosing σ and which separates σ from the remaining part of the spectrum.

Let us consider again the case of an isolated point λ_0 of $\sigma(T)$. Then $\{\lambda_0\}$ is a spectral set, so we can consider the spectral projection P_0 associated with $\{\lambda_0\}$. It is easy to check that if P_k are defined according (4) then

$$P_1 = P_0, \quad P_k = (T - \lambda_0 I)^{n-1} P_0 \quad (k = 1, 2, \dots) \tag{6}$$

Equation (6) show that either $P_k \neq 0$, or that there exists a natural p such that $P_k \neq 0$ for $k = 1, \dots, p$ but $P_k = 0$ for $k > p$. In the second case the isolated point λ_0 is pole of order p of T.

The spectral sets produce the following decomposition see [39, §49].

Theorem 2.8 *If σ is a spectral set (possibly empty) of $T \in L(X)$ then the projection in (5) generates the decomposition $X = P_\sigma(X) \oplus \ker P_\sigma$. The subspaces $P_\sigma(X)$ and $\ker P_\sigma$ are invariant under every $f(T)$ with $f \in \mathcal{H}(\sigma(T))$; the spectrum of the restriction $T|P_\sigma(X)$ is σ and the spectrum of $T|\ker P_\sigma$ is $\sigma(T) \setminus \sigma$.*

The proof of the following basic result may be found in [39, Proposition 50.2].

Theorem 2.9 *Let $T \in L(X)$. Then $\lambda_0 \in \sigma(T)$ is a pole of $R(\lambda, T)$ if and only if $0 < p(\lambda_0 I - T) = q(\lambda_0 I - T) < \infty$. Moreover, if $p := p(\lambda_0 I - T) = q(\lambda_0 I - T)$ then p is the order of the pole, every pole $\lambda_0 \in \sigma(T)$ is an eigenvalue of T, and if P_0 is the spectral projection associated with $\{\lambda_0\}$ then*

$$P_0(X) = \ker (\lambda_0 I - T)^p, \quad \ker P_0 = (\lambda_0 I - T)^p(X).$$

In the following result, due to Schmoeger [54], we show that for an isolated point λ_0 of $\sigma(T)$ the quasi-nilpotent part $H_0(\lambda_0 I - T)$ and the analytical core $K(\lambda_0 I - T)$ may be precisely described as a range or a kernel of a projection.

Theorem 2.10 *Let $T \in L(X)$, where X is a Banach space, and suppose that λ_0 is an isolated point of $\sigma(T)$. If P_0 is the spectral projection associated with $\{\lambda_0\}$, then:*

(i) $P_0(X) = H_0(\lambda_0 I - T)$;

(ii) $\ker P_0 = K(\lambda_0 I - T)$. *Consequently,*

$$X = H_0(\lambda_0 I - T) \oplus K(\lambda_0 I - T).$$

In particular, if $\{\lambda_0\}$ is a pole of the resolvent, or equivalently $p := p(\lambda_0 I - T) = q(\lambda_0 I - T) < \infty$, then

$$H_0(\lambda_0 I - T) = \ker(\lambda_0 I - T)^p,$$

and

$$K(\lambda_0 I - T) = (\lambda_0 I - T)^p(X).$$

Recall that $T \in L(X)$ is said to be *upper semi-Fredholm*, $T \in \Phi_+(X)$, if $\alpha(T) < \infty$ and $T(X)$ is closed, while $T \in L(X)$ is said to be *lower semi-Fredholm*, $T \in \Phi_-(X)$ if $\beta(T) < \infty$. The class of *Fredholm* operators is defined by $\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$, while the class of *semi-Fredholm* operators is defined by $\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)$. If $T \in \Phi_\pm(X)$ then the index is defined by ind $(T) := \alpha(T) - \beta(T)$. The *semi-Fredholm spectrum* is defined as the set

$$\sigma_{\mathrm{sf}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not semi-Fredholm}\}.$$

A bounded operator $T \in L(X)$, X a Banach space, is said to be a *semi-regular* operator if T has closed range $T(X)$ and $\ker T \subseteq T^n(X)$ for every $n \in \mathbf{N}$. Elementary examples of semi-regular operators are a surjective operator or an operator that is bounded below.

Definition 2.11 An operator $T \in L(X)$, X a Banach space, is said to admit a *generalized Kato decomposition*, abbreviated as GKD, if there exists a pair of T-invariant closed subspaces (M, N) such that $X = M \oplus N$, the restriction $T|M$is semi-regular and $T|N$ is quasi-nilpotent. If $T|N$ is assumed to be nilpotent of order d then T is said to be of *Kato type of operator of order d*. An operator is said to be *essentially semi-regular* if it admits a GKD (M, N) such that N is finite-dimensional.

Note that if T is essentially semi-regular then $T|N$ is nilpotent, since every quasi-nilpotent operator on a finite dimensional space is nilpotent. A celebrate result of Kato shows that every semi-Fredholm operator is essentially semi-regular, in particular of Kato type, see Müller [51] for details.

Note that if T is of Kato type then also T' is of Kato type. More precisely, the pair $(N^\perp, M^\perp)$ is a GKD for T' with $T'|N^\perp$ semi-regular and $T'|M^\perp$ nilpotent, see Theorem 1.43 of [1].

3 Some Notions of Local Spectral Theory

For many reasons the most satisfactory generalization to the general Banach space setting of the normal operators on a Hilbert space is the concept of decomposable operator. In fact the class of these operators possesses a spectral theorem and a rich lattice structure for which it is possible to develop what it is called a *local spectral theory*, i.e. a local analysis of their spectra. Decomposability may be defined in several ways, for instance by means of the concept of *glocal spectral subspace*.

Definition 3.1 For an arbitrary bounded linear operator on a Banach space $T \in L(X)$ and a closed subset F of $\mathbb{C}$, the *glocal spectral subspace* $\mathcal{X}_T(F)$ is defined as the set of all $x \in X$ such that there is an analytic X-valued function $f : \mathbb{C} \setminus F \to X$ for which

$$(\lambda I - T) f(\lambda) = x$$

for all $\lambda \in \mathbb{C} \setminus F$.

The quasi-nilpotent part may be described as a glocal subspace, indeed we have

$$H_0(\lambda I - T) = \mathcal{X}_T(\{\lambda\}) \quad \text{for all } \lambda \in \mathbb{C},$$

see [4, Chapter 2].

Another important concept of local spectral theory is that of local spectrum of an operator $T \in L(X)$ at a point $x \in X$.

Definition 3.2 Given an arbitrary operator $T \in L(X)$, X a Banach space, let $\rho_T(x)$ denote the set of all $\lambda \in \mathbb{C}$ for which there exists an open neighborhood $\mathcal{U}_\lambda$ of λ in $\mathbb{C}$ and an analytic function $f : \mathcal{U}_\lambda \to X$ such that the equation

$$(\mu I - T) f(\mu) = x \quad \text{holds for all } \mu \in \mathcal{U}_\lambda. \tag{7}$$

If the function f is defined on the set $\rho_T(x)$ then f is called a *local resolvent function* of T at x. The set $\rho_T(x)$ is called the *local resolvent* of T at x. The *local spectrum* $\sigma_T(x)$ of T at the point $x \in X$ is defined to be the set

$$\sigma_T(x) := \mathbb{C} \setminus \rho_T(x).$$

Definition 3.3 For every subset F of $\mathbb{C}$ the *local spectral subspace* of an operator $T \in L(X)$ associated with $F \subseteq \mathbb{C}$ is the set

$$X_T(F) := \{x \in X : \sigma_T(x) \subseteq F\}.$$

It is easily seen that $X_T(F)$ is a linear subspace, and that

$$\mathcal{X}_T(F) \subseteq X_T(F) \quad \text{for every closed subset } F \subseteq \mathbb{C}.$$

Note that T has SVEP if and only if $\mathcal{X}_T(F) = X_T(F)$ for every closed subset $F \subseteq \mathbb{C}$, see [4, Theorem 2.23]. Obviously, if $F_1 \subseteq F_2 \subseteq \mathbb{C}$ then $X_T(F_1) \subseteq X_T(F_2)$ and

$$X_T(F) = X_T(F \cap \sigma(T).$$

Indeed, $X_T(F) \cap \sigma(T)) \subseteq X_T(F)$. Conversely, if $x \in X_T(F)$ then $\sigma_T(x) \subseteq F \cap \sigma(T)$, and hence $x \in X_T(F \cap \sigma(T))$. Moreover, from the basic properties of the local spectrum it is easily seen that $X_{\lambda I+T}(F) = X_T(F \setminus \{\lambda\})$.

The analytic core is a local spectral subspace, precisely:

$$K(T) = X_T(\mathbb{C} \setminus \{0\}) = \{x \in X : 0 \notin \sigma_T(x)\},$$

see [4, theorem 2.20].

We now introduce an important property, defined for bounded linear operators on complex Banach spaces, the so called *single-valued extension property* (SVEP). This property dates back to the early days of local spectral theory and has received a more systematic treatment in the classical texts by Dunford and Schwartz [25], as well as those by Colojoară and Foiaş [23], by Vasilescu [58] and, more recently, by Laursen and Neumann [43], and Aiena [1] . The single-valued extension property has a basic importance in local spectral theory since it is satisfied by a wide variety of linear bounded operators in the spectral decomposition problem.

Definition 3.4 $T \in L(X)$ is said to have *the single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open disc U of λ_0, the only analytic function $f : U \to X$ which satisfies the equation

$$(\lambda I - T)f(\lambda) = 0 \quad \text{for all } \lambda \in U$$

is the function $f \equiv 0$. An operator $T \in L(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$.

Evidently, both T and T' have SVEP at the points λ which belong to the boundary $\partial\sigma(T)$ of the spectrum. In particular both T and T' have SVEP at the isolated points of the spectrum. Furthermore, if $\sigma_{\mathrm{ap}}(T) \subseteq \partial\sigma(T)$ then T has SVEP, and dually the inclusion $\sigma_{su}(T) \subseteq \partial\sigma(T)$ entails that T' has SVEP. Furthermore, it is not difficult to see that

$$\sigma_{ap}(T) \text{ does not cluster at } \lambda \Rightarrow T \text{ has SVEP at } \lambda,$$

and dually,

$$\sigma_{su}(T) \text{ does not cluster at } \lambda \Rightarrow T' \text{ has SVEP at } \lambda.$$

From the equality (1) it is easily follows that for Hilbert space operators we have

$$T' \text{ has SVEP} \Leftrightarrow T^* \text{ has SVEP}.$$

The proof of the following theorem may be found in [1, Theorem 3.16, Theorem 3.17].

Theorem 3.5 *Let $T \in L(X)$, X a Banach space and suppose that $\lambda I - T$ is of Kato type . Then we have:*

(i) *T has SVEP at λ $\Leftrightarrow$ $p(\lambda I - T) < \infty$.*
(ii) *T^* has SVEP at $\lambda \Leftrightarrow q(\lambda I - T) < \infty$.*
In particular the equivalences (i) and (ii) hold for semi-Fredholm operators.

Recall that a bounded operator $K \in L(X)$ is said to be *algebraic* if there exists a non-constant polynomial h such that $h(K) = 0$. Trivially, every nilpotent operator is algebraic and it is well-known that if $K^n(X)$ has finite dimension for some $n \in \mathbb{N}$ then K is algebraic. An operator $T \in L(X)$ is said to be *Riesz* if $\lambda I - T$ is Fredholm for every $\lambda \neq 0$.

The SVEP is also stable under algebraic commuting or Riesz commuting perturbations, see [5, 6]:

Theorem 3.6 *Let $T \in L(X)$ and K be algebraic which commutes with T. If T has SVEP then $T + K$ has SVEP. An anologous result holfd for Riesz commuting perturbations.*

Definition 3.7 A bounded operator T is said to be *decomposable* if, for any open covering $\{\mathcal{U}_1, \mathcal{U}_2\}$ of the complex plane $\mathbb{C}$ there are two closed T-invariant subspaces Y_1 and Y_2 of X such that $Y_1 + Y_2 = X$ and $\sigma(T|Y_k) \subseteq \mathcal{U}_k$ for $k = 1, 2$.

A bounded operator $T \in L(X)$ on a Banach space X is said to have the *Dunford property* (C) if every glocal spectral subspace is closed. We have

$$T \text{ is decomposable} \Leftrightarrow T \text{ has both property } (C) \text{ and property } (\delta),$$

see [43], where *property* (δ) means that for every open covering (U, V) of $\mathbb{C}$ we have $X = \mathcal{X}_T(\overline{U}) + \mathcal{X}_T(\overline{V})$. Standard examples of decomposable operators are normal operators on Hilbert spaces and operators which have totally disconnected spectra, as for instance compact operators.

Two important properties in local spectral theory related to property (C) are the so-called *property* (β). This property has been introduced by Bishop [16] and is defined as follows. Let U be an open subset of $\mathbb{C}$ and denote by $\mathcal{H}(U, X)$ the Fréchet space of all analytic functions $f : U \to X$ with respect the pointwise vector space operations and the topology of locally uniform convergence. $T \in L(X)$ has Bishop's *property* (β) if for every open $U \subseteq \mathbb{C}$ and every sequence $(f_n) \subseteq \mathcal{H}(U, X)$ for which $(\lambda I - T) f_n(\lambda)$ converges to 0 uniformly on every compact subset of U, then also $f_n \to 0$ in $\mathcal{H}(U, X)$.

Examples of operators which have property (β), are provided by the weighted right shift on $\ell^2(\mathbb{N})$ for which the weight sequence is increasing, see [43]. Note that

$$\text{property } (\beta) \Rightarrow \text{property } (C) \Rightarrow \text{ SVEP},$$

and

$$T \text{ is decomposable} \Leftrightarrow T \text{ has both property } (\beta) \text{ and property } (\delta),$$

see [43].

Let T' denote the dual of T. Property (β) and property (δ) are dual each other, i.e., $T \in L(X)$ satisfies (β) (respectively (δ)) if and only if T' satisfies (δ) (respectively, (β)), see [43]. Consequently,

$$T \in L(X) \text{ is decomposable} \Leftrightarrow \text{both } T \text{ and } T' \text{ have property } (\beta).$$

Examples of operators satisfying property (β) but not decomposable may be found among multipliers of semi-simple commutative Banach algebras, see [43].

In the sequel, for every set $F \subseteq \mathbb{C}$ we set $\overline{F} := \{\overline{\lambda} : \lambda \in F\}$ and F^{cl} the closure of F. In the case of Hilbert space operators the dual T' may be replaced by the adjoint T^*. Let $x \in \mathcal{H}_{T^*}(F)$, for some closed $F \subseteq \mathbb{C}$. Then there exists an analytic function $f : \mathbb{C} \to H$ such that $(\lambda I - T^*) f(\lambda) = x$ for all $\lambda \in \mathbb{C} \setminus F$. From (1) we know that $U(\lambda I - T^*) = (\bar{\lambda} I - T')U$, so

$$Ux = U(\lambda I - T^*) f(\lambda) = (\bar{\lambda} I - T')U\, f(\lambda) \quad \text{for all } \lambda \in \mathbb{C} \setminus F.$$

The function $g(\bar{\lambda}) := U\, f(\lambda)$ for $\bar{\lambda} \in \mathbb{C} \setminus \bar{F}$ is analytic, so $Ux \in \mathcal{H}_{T'}(\bar{F})$. This shows that $U\mathcal{H}_{T^*}(F) \subseteq \mathcal{H}_{T'}(\bar{F})$. Analogously, it can be shown that $\mathcal{H}_{T'}(\overline{F}) \subseteq U\mathcal{H}_{T^*}(F)$ for every closed set $F \subseteq \mathbb{C}$, so $\mathcal{H}_{T'}(\overline{F}) = \mathcal{H}_{T'}(\bar{F})$.

Now, if T^* has property (δ) then $H = \mathcal{H}_{T^*}(V^{cl}) + \mathcal{H}_{T^*}(W^{cl})$ for every cover $\{V, W\}$ of $\mathbb{C}$, so

$$H = UH = U\mathcal{H}_{T^*}(V^{cl}) + U\mathcal{H}_{T^*}(W^{cl}) = \mathcal{H}_{T'}(\bar{V}^{cl}) + \mathcal{H}_{T'}(\bar{W}^{cl}),$$

and hence, T' has property (δ). An analogous argument shows that if T' has property (δ) then T^* has property (δ), so we have.

$$T' \text{ has property } (\delta) \Leftrightarrow T^* \text{ has property } (\delta), \tag{8}$$

By duality, T' has property $(\beta) \Leftrightarrow T = (T^*)^*$ has property (δ), and hence, by (8), if and only if $(T^*)'$ has property (δ), from which we conclude that

$$T' \text{ has property } (\beta) \Leftrightarrow T^* \text{ has property } (\beta).$$

Consequently,

$$T' \text{ is decomposable} \Leftrightarrow T^* \text{ is decomposable.}$$

Lemma 3.8 *Let $T \in L(X)$, X a Banach space, and $\lambda \in \rho(T)$. Then $\lambda(\lambda I - T)^{-1}x \to x$ for every $x \in X$ as $|\lambda| \to +\infty$.*

Proof Fix $x \in X$ and define $f(\lambda) := (\lambda I - T)^{-1}x : \rho(T) \to X$. It is known that $f(\lambda) \to 0$ when $|\lambda| \to +\infty$. We have, for every $\lambda \in \rho(T)$,

$$\lambda(\lambda I - T)^{-1}x - x = \lambda(\lambda I - T)^{-1}x - (\lambda I - T)(\lambda I - T)^{-1}x = T(\lambda I - T)^{-1}x$$

hence $\lambda(\lambda I - T)^{-1}x - x \to 0$, so $\lambda(\lambda I - T)^{-1}x \to x$ as $|\lambda| \to +\infty$. ∎

Theorem 3.9 *Let $T \in L(H)$, then $H_0(T^*) \subseteq K(T)^{\perp}$.*

Proof Let $x \in H_0(T^*) = \mathcal{H}_{T^*}(\{0\})$ and fix an arbitrary $y \in \mathcal{K}(T)$. We have to show that $\langle x, y\rangle = 0$. As already observed $K(T) = \{x : 0 \in \rho_T(x)\}$, so there exist two analytic functions $f : \mathbb{C} \setminus \{0\} \to H$ and $g : \mathbb{D}_0 \to H$, $\mathbb{D}_0$ an open disc centered at 0, such that

$$(\bar{\lambda} I - T^\star) f(\bar{\lambda}) = x,\ \lambda \in \mathbb{C} \setminus \{0\}, \quad \text{and} \quad (\lambda I - T)g(\lambda) = y,\ \lambda \in \mathbb{D}_0.$$

Both the functions $f(\lambda)$ and $g(\lambda)$ are defined in $\mathbb{D}_0 \setminus \{0\}$ and for $\mu \in \mathbb{D}_0 \setminus \{0\}$ we have

$$\langle f(\bar{\mu}), y\rangle = \langle f(\bar{\mu}), (\mu I - T)g(\mu)\rangle = \langle(\bar{\mu} I - T^*) f(\bar{\mu}), g(\mu)\rangle = \langle x, g(\mu)\rangle.$$

Define

$$h(\mu) := \begin{cases} \langle f(\bar{\mu}), y\rangle \text{ if } \mu \neq \mathbb{D}_0, \\ \langle x, g(\mu)\rangle \text{ if } \mu \in \mathbb{D}_0, \end{cases}$$

The function $h(\mu)$ is well-defined and is analytic on $\mathbb{C}$. Since $f(\bar{\mu}) = (\bar{\mu} I - T^*)^{-1}x$ for all $\bar{\mu} \in \rho(T^*)$, see [4, Remark 2.11] and $f(\bar{\mu}) \to 0$ for $|\bar{\mu}| \to +\infty$, then $h(\mu) \to 0$ as $|\mu| \to +\infty$, so, by the classical Liouville theorem, $h \equiv 0$ on $\mathbb{C}$. From

Lemma 3.8 we have also have $\bar{\mu}(\bar{\mu}I - T^*)^{-1}x = -x$, as $|\mu| \to +\infty$, $\bar{\mu} \in \rho(T^*)$, hence

$$\langle x, y\rangle = \lim_{|\mu|\to+\infty} \langle \bar{\mu}(\bar{\mu}I - T^*)^{-1}x, y\rangle = \lim_{|\mu|\to+\infty} \langle \bar{\mu} f(\bar{\mu}), y\rangle = \mu h(\mu) = 0,$$

so $x \in K(T)^\perp$, as desired. ∎

Next we want to show that if T is decomposable then $H_0(T^*) = K(T)^\perp$. To do this we need some preliminary results. Suppose that M is a closed T-invariant subspace of a Banach space X and denote by $T/M : X/M \to X/M$ the canonical quotien mapping defined on the quotient X/M by $(T/M)(x + M) := Tx + M$.

For an open disc $\mathbb{D}$ of $\mathbb{C}$ centered at 0, let $\mathbf{D}$ denote its closure.

Lemma 3.10 *Suppose that $T \in L(X)$, X a Banach space, is decomposable. If $M := X_T(\mathbb{C} \setminus \mathbb{D})$ then $\sigma(T/M)$ is contained in $\boldsymbol{D}$.*

Proof This follows as a particular case of Theorem 1.2.23, part (b), of [43], by taking $F = \mathbb{C} \setminus \mathbb{D}$. ∎

If Y is a closed T-invariant subspace by $T|Y$ we denote the restriction of T to Y.

Lemma 3.11 *Suppose that $T \in L(H)$ is decomposable. If $\mathbb{D}$ is an open disc centered at* 0 *then $H_T(\mathbb{C} \setminus \mathbb{D})^\perp \subseteq H_{T^*}(\boldsymbol{D})$.*

Proof Let $M := X_T(\mathbb{C} \setminus \mathbb{D})$. Recall that M is a closed invariant subspace of T, since a decomposable operator has property (C), while $M^\perp$ is a closed subspace invariant under T^*. We show first that

$$\sigma(T^*|M) \subseteq \mathbf{D} \tag{9}$$

If $S : M^\perp \to H/M$ is defined by $S(x) = x + M$ for every $x \in M^\perp$, then S is bijective and an isometry. It is easily seen that $S(T^*|M^\perp) = (T/M)^*S$, so $T^*|M^\perp$ and $(T/M)^*$ are similar. Therefore,

$$\sigma(T^*|M^\perp) = \sigma(T/M)^* = \overline{\sigma(T/M)}.$$

By Lemma 3.10 then

$$\sigma(T^*|M^\perp) = \sigma(T^*|X_T(\mathbb{C} \setminus \mathbb{D})^\perp) \subseteq \mathbf{D},$$

since $\mathbf{D}$ is the closure of $\overline{\mathbb{D}} = \mathbb{D}$. Thus, the inclusion (9) is proved. From part (e) of [43, Proposition 1.2.16] we then obtain $H_T(\mathbb{C} \setminus \mathbb{D})^\perp = M^\perp \subseteq H_{T^*}(\mathbf{D})$. ∎

Theorem 3.12 *Let $T \in L(H)$ be decomposable then $H_0(T^*) = K(T)^\perp$ and $H_0(T) = K(T^*)^\perp$.*

Proof To show the equality $H_0(T^*) = K(T)^\perp$, let $\{\mathbf{D}_\alpha\}_\alpha$ denote the set of all closed discs of $\mathbb{C}$ centered at 0. Since T has SVEP we have

$$H_0(T^*) = \mathcal{H}_{T^*}(\{0\}) = H_{T^*}(\{0\}) = \bigcap_\alpha H_{T^*}(\mathbf{D}_\alpha),$$

see [4, Theorem 2.13, part (iv)]. To show the equality $H_0(T^*) = K(T)^\perp$ we need to prove, by Theorem 3.9, the inclusion $K(T)^\perp \subseteq H_0(T^*)$, and for this it suffices to prove that $K(T)^\perp \subseteq H_{T^*}(\mathbf{D})$, where $\mathbf{D}$ is any closed disc centered at 0. Evidently,

$$H_T(\mathbb{C} \setminus \mathbb{D}) \subseteq H_T(\mathbb{C} \setminus \{0\}) = K(T),$$

so $K(T)^\perp \subseteq H_T(\mathbb{C} \setminus \mathbf{D})^\perp$ and $H_T(\mathbb{C} \setminus \mathbb{D})^\perp \subseteq H_{T^*}(\mathbf{D})$, by Lemma 3.10, so the proof of the first equality is complete. The second equality is clear, since T', and hence T^*, is decomposable, so we have

$$H_0(T) = H_0((T^*)^*) = K(T^*)^\perp.$$

∎

Remark 3.13 It should be noted that the identity $K(T) = H_0(T^*)^\perp$ in general does not hold even if T is decomposable. For instance, if $T \in L(H)$ is Riesz operator which has infinite spectrum then T is decomposable, but $K(T)$ is not closed, since in this case $\sigma(T)$ would be finite, see [50]. Hence $K(T) \neq H_0(T^*)^\perp$, since $H_0(T^*)^\perp$ is closed.

Corollary 3.14 *If $T \in L(H)$ is self-adjoint then $H_0(T) = K(T)^\perp$.*

Proof T is decomposable and $T = T^*$. ∎

4 Normal Type Operators

Perhaps the most important class of operators in Hilbert spaces is given by the normal operators defined on a Hilbert space. Recall that if H is a complex infinite-dimensional Hilbert space a bounded linear operator $T \in L(H)$ is said to be *normal* if

$$TT^* = T^*T \tag{10}$$

Normal operators have several important spectral properties that will be next recalled.

(A) Every isolated spectral point of a normal operator T is a simple pole of the resolvent. If every isolated spectral point of an operator T on a Banach space is a pole then T is said to be *polaroid*.

(B) $\|T\| = r(T)$, where $r(T)$ denotes the *spectral radius* of T defined as

$$r(T) := \sup_{\lambda\in\sigma(T)} |\lambda|.$$

An operator $T \in L(X)$, X a Banach space, for which $\|T\| = r(T)$ is said to be *normaloid*.

(C) An operator $T \in L(X)$ is said to be *Weyl*, ($T \in W(X)$), if $T \in \Phi(X)$ and $\operatorname{ind} T = 0$. The Weyl spectrum is denoted by $\sigma_{\mathrm{w}}(T)$. For a normal operator T we have

$$\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi_{00}(T), \tag{11}$$

where

$$\pi_{00}(T) := \{\lambda \in \operatorname{iso}\sigma(T) : 0 < \alpha(T) < \infty\},$$

i.e, the spectral points for which $\lambda I - T \in W(X)$ are exactly the eigenvalues which have finite multiplicity. An operator $T \in L(X)$, X a Banach space, for which the equality (11) holds is said to *satisfy Weyl theorem*.

(D) T is *decomposable*; in particular both T and T^* have SVEP.

Normal operators may be generalized in several ways:

Hyponormal Operators This class of operators on Hilbert spaces is defined whenever the condition of normality (10) is relaxed to the inequality

$$T^*T \geq TT^*. \tag{12}$$

An operator $T \in L(H)$ which satisfies (12) is said to be *hyponormal* . It is easily seen that T is hyponormal if and only if

$$\|T^*x\| \leq \|Tx\| \quad \text{for all } x \in H.$$

Indeed, $T^*T \geq TT^*$ means that

$$\langle T^*Tx, x\rangle \geq \langle TT^*x, x\rangle \quad \text{for all} x \in H,$$

or equivalently

$$\|T^*x\|^2 = \langle T^*x, T^*x\rangle = \langle TT^*x, x\rangle \leq \langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = \|Tx\|^2.$$

Thus, $\|T^*x\| \leq \|Tx\|$.

A routine computation shows that a weighted right shift on the Hilbert space $\ell^2(\mathbb{N})$ is hyponormal if and only if the corresponding weight sequence is increasing.

Other examples of hyponormal operators are the quasi-normal operators, see Conway [22] or Furuta [33], where $T \in L(H)$ is said to be *quasi-normal* if

$$T(T^*T) = (T^*T)T.$$

A very easy example of quasi-normal operator is given by the unilateral right shift R on the Hilbert space $\ell_2(\mathbb{N})$. Recall that a such operator is defined by

$$R(x_1, x_2, \dots) := (0, x_1, x_2, \dots) \quad \text{for all } (x_n) \in \ell_2(\mathbb{N}).$$

The adjoint of R is the left shift L, defined by

$$L(x_1, x_2, \dots) := (x_2, x_3, \dots) \quad \text{for all } (x_n) \in \ell_2(\mathbb{N}).$$

and obviously $R(R^*R) = (RR^*)R$. Note that R is not normal, since

$$RR^* = RL \neq R^*R = LR = I.$$

An operator $T \in L(H)$ is said to be *subnormal* if there exists an normal extension N, i.e. there exists a Hilbert space K such that $H \subseteq K$ and a normal operator $N \in L(K)$ such that $N|H = T$. We havept

$$T \text{ quasi-normal} \Rightarrow T \text{ subnormal} \Rightarrow T \text{ hyponormal},$$

for details see Furuta [33, p. 105]. In the sequel we show some relevant properties of hyponormal operators.

Theorem 4.1 *Let $T \in L(H)$ be hyponormal. Then we have:*

(i) *$\lambda I - T$ is hyponormal for every $\lambda \in \mathbb{C}$.*
(ii) *If M is a closed invariant subspace of H then $T|M$ is hyponormal.*

Proof

(i) We have

$$(\lambda I - T)^*(\lambda I - T) - (\lambda I - T)(\lambda I - T)^* =$$
$$(\overline{\lambda} I - T)(\lambda I - T) - (\lambda I - T)(\overline{\lambda} I - T) = T^*T - TT^* \leq 0,$$

thus, $\lambda I - T$ is hyponormal.

(ii) If P_M is the projection of T onto M, then $(T|M)^* = (P_M T^*)|M$. For every $x \in M$ then we have

$$\|(T|M)^*x\| = \|(P_M T^*)x\| \leq \|P_M\| \|T^*x\| =$$
$$\|T^*x\| \leq \|Tx\| = \|T|Mx\|,$$

thus $T|M$ is hyponormal. ∎

Lemma 4.2 *Let $T \in L(H)$ be a self-adjoint operator such that $\lambda I \leq T$ for some $\lambda \geq 0$. Then T is invertible. In particular, if $I \leq T$ then $0 \leq T^{-1} \leq I$.*

Proof To show the first assertion, observe that by the Schwarz inequality we have

$$\|Tx\|\|x\| \geq (Tx, x) \geq c\|x\|^2,$$

so $\|Tx\| \geq c\|x\|$, and hence T is bounded below. Let y be an orthogonal element to $T(H)$, that is

$$0 = (y, Tx) = (Ty, x) \text{ for all } x \in H.$$

Then $Ty = 0$ and since T is injective we then have $y = 0$. Therefore, $T(H)^\perp = \overline{T(H)}^\perp = \{0\}$, and hence T is surjective, thus T is invertible.

To show the second assertion, note that if $I \leq T$ then T is invertible and T^{-1} is also positive. Since the product of two commuting positive operators is also positive, it then follows that

$$T^{-1}(T - I) = I - T^{-1} \geq 0,$$

thus $T^{-1} \leq I$. ■

It is easily seen that if T is self-adjoint then STS^* is also self-adjoint for every $S \in L(H)$. Moreover, if T is positive then $STS^* \geq 0$ for all $S \in L(H)$.

Theorem 4.3 *If $T \in L(H)$ is an invertible hyponormal operator then its inverse T^{-1} is also hyponormal.*

Proof Suppose that T is hyponormal. Then $T^*T - TT^* \geq 0$ and hence, as noted above, the product

$$T^{-1}(T^*T - TT^*)(T^{-1})^*$$

is positive. From this we obtain

$$T^{-1}(T^*T)(T^{-1})^* - I \geq 0,$$

and hence

$$T^{-1}(T^*T)(T^{-1})^* \geq I,$$

thus, by Lemma 4.2, the product $T^{-1}(T^*T)(T^{-1})^*$ is invertible with

$$0 \leq [T^{-1}(T^*T)(T^{-1})^*]^{-1} \leq I.$$

From the last inequality we then obtain that

$$S := I - T^*(T^{-1}(T^*)^{-1})T$$

is positive, so

$$T^{-1}ST^{-1} \geq 0,$$

from which we easily obtain that

$$(T^{-1})^*T^{-1} - T^{-1}(T^{-1})^* \geq 0.$$

Hence T^{-1} is hyponormal. ∎

Paranormal Operators An operator $T \in L(X)$ on a Banach space X is said to be *paranormal* if

$$\|Tx\| \leq \|T^2x\| \quad \text{for all unit vectors } x \in X. \tag{13}$$

Evidently, the restriction $T|M$ of a paranormal operator $T \in L(X)$ to a closed subspace M is evidently paranormal. Moreover, any scalar multiple, and the inverse (if it exists) of a paranormal operator, is paranormal.

Theorem 4.4 *If $T \in L(X)$ is paranormal then we have:*

(i) *Every power T^n is paranormal.*
(ii) *T is normaloid.*

Proof

(i) Observe that from the definition (13) we have

$$\frac{\|T^{k+1}x\|}{\|T^kx\|} \leq \frac{\|T^{k+2}x\|}{\|T^{k+1}x\|}$$

from which we obtain

$$\begin{aligned}\frac{\|T^nx\|}{\|x\|} &= \frac{\|Tx\|}{\|x\|}\frac{\|T^2x\|}{\|Tx\|}\cdots\frac{\|T^nx\|}{\|T^{n-1}x\|} \\ &\leq \frac{\|T^{n+1}x\|}{\|T^nx\|}\frac{\|T^{n+2}x\|}{\|T^{n+1}x\|}\cdots\frac{\|T^{2n}x\|}{\|T^{2n-1}x\|} = \frac{\|T^{2n}x\|}{\|T^nx\|}.\end{aligned}$$

Therefore, $\|T^nx\|^2 \leq \|(T^n)^2x\|\|x\|$.

(ii) For every paranormal operator we have

$$\|Tx\|^2 \leq \|T^2x\|\|x\| \leq \|T^2\|\|x\|^2,$$

thus $\|T^2\| = \|T\|^2$. Since T^n is paranormal then $\|T^{2n} = \|T\|^{2n}$ for every $n \in \mathbb{N}$. Hence

$$r(T) = \lim_{n\to\infty} \|T^{2^n}\|^{\frac{1}{2^n}} = \|T\|.$$

■

Definition 4.5 Given a class of operators $\mathbf{L} \subseteq L(X)$, an operator T is said to be *algebraically* $\mathbf{L}$ if there exists a non-trivial polynomial h for which $h(T)$ belongs to $\mathbf{L}$. T is said to be *analytically* $\mathbf{L}$ if there exists an analytic function h such that $h(T)$ belongs to $\mathbf{L}$, where h is defined on an open neighborhood of $\sigma(T)$, non-costant on each of the components of its domain.

Since every paranormal operator $T \in L(X)$ is normaloid, then we have

$$T \quad \text{quasi-nilpotent paranormal} \Rightarrow T = 0. \tag{14}$$

Recall that an invertible operator $T \in L(X)$ is said to be *doubly power-bounded* if $\sup\{\|T^n\| : n \in \mathbb{Z}\} < \infty$. Algebraic paranormal operators have been studied in [2].

Theorem 4.6 *Suppose that $T \in L(X)$ is algebraically paranormal and quasi-nilpotent. Then T is nilpotent.*

Proof Suppose that h is a polynomial for which $h(T)$ is paranormal. From the spectral mapping theorem we have

$$\sigma(h(T)) = h(\sigma(T)) = \{h(0)\}.$$

We claim that $h(T) = h(0)I$. To see that let us consider the two possibilities: $h(0) = 0$ or $h(0) \neq 0$.

If $h(0) = 0$ then $h(T)$ is quasi-nilpotent, so from the implication (14), we deduce that $h(T) = 0$, hence the equality $h(T) = h(0)I$ trivially holds.

Suppose the other case $h(0) \neq 0$, and set $h_1(T) := \frac{1}{h(0)} h(T)$. Clearly, $h_1(T)$ has spectrum $\{1\}$ and $\|h_1(T)\| = 1$. Moreover, $h_1(T)$ is invertible and also its inverse $h_1(T)^{-1}$ has norm 1. The operator $h_1(T)$ is then doubly power-bounded and by a classical theorem due to Gelfand, see [43, Theorem 1.5.14] for a proof, it then follows that $h_1(T) = I$, and hence $h(T) = h(0)I$, as claimed.

Now, from the equality $h(0)I - h(T) = 0$, we see that there exist some natural $n \in \mathbb{N}$ and $\mu \in \mathbb{C}$ for which

$$0 = h(0)I - h(T) = \mu\, T^m \prod_{i=1}^{n} (\lambda_i I - T) \quad \text{with } \lambda_i \neq 0,$$

where all $\lambda_i I - T$ are invertible. This obviously implies that $T^m = 0$, so T is nilpotent. ■

Theorem 4.7 *Every algebraically paranormal operator $T \in L(X)$, X a Banach space has SVEP.*

Proof We show first the SVEP for paranormal operators. If $\lambda \neq 0$ and $\lambda \neq \mu$ then, by Theorem 2.6 of [20], we have

$$\|x + y\| \geq \|y\| \quad x \in \ker(\mu I - T),\ y \in \ker(\lambda I - T).$$

It then follows that if U is an open disc and $f : U \to X$ is an analityc function such that $0 \neq f(z) \in \ker(zI - T)$ for all $z \in U$, then f fails to be continuous at every $0 \neq \lambda \in U$. Finally, if T is algebraically paranormal then $h(T)$ is paranormal for some non-trivial polynomial h, and hence $h(T)$ has SVEP. This implies that T has SVEP, see [4, Corollary 2.89]. ■

Theorem 4.8 *If $T \in L(X)$ is algebraically paranormal then every isolated point of the spectrum $\sigma(T)$ is a pole of the resolvent; i.e. T is polaroid.*

Proof For every isolated point λ of $\sigma(T)$ we have $p(\lambda I - T) = q(\lambda I - T) < \infty$. Indeed, let λ be an isolated point of $\sigma(T)$. If $M := K(\lambda I - T)$ and $N := H_0(\lambda I - T)$, then $H = H_0(\lambda I - T) \oplus K(\lambda I - T)$, by Theorem 2.10. Furthermore, since $\sigma(T|N) = \{\lambda\}$, while $\sigma(T|M) = \sigma(T) \setminus \{\lambda\}$, so the restriction $\lambda I - T|N$ is quasi-nilpotent and $\lambda I - T|M$ is invertible. Since $\lambda I - T|N$ is algebraically paranormal then Lemma 5.2 implies that $\lambda I - T|N$is nilpotent. In other worlds, $\lambda I - T$ is an operator of Kato Type.

Now, both T and its dual T^* have SVEP at λ, since λ is isolated in $\sigma(T) = \sigma(T^*)$, and this implies, by Theorem 3.5, that both $p(\lambda I - T)$ and $q(\lambda I - T)$ are finite. Therefore, λ is a pole of the resolvent. ■

The property of being paranormal is not translation-invariant, see Chō and J. I. Lee [18]. An operator $T \in L(X)$ is called *totally paranormal* if $\lambda I - T$ is paranormal for all $\lambda \in \mathbb{C}$.

Theorem 4.9 *Every hyponormal operator $T \in L(H)$ is totally paranormal.*

Proof To show that T is totally paranormal it suffices to prove, by Lemma 4.1 to prove that every hyponormal operator is paranormal. Since T is hyponormal we have, for every $x \in H$,

$$\begin{aligned} \|Tx\|^2 = (Tx, Tx) = (T^*Tx, x) &\leq \|T^*(Tx)\|\|x\| \\ &\leq \|T(Tx)\|\|x\| = \|T^2x\|\|x\|, \end{aligned}$$

Taking $\|x\| = 1$ we then have $\|Tx\|^2 \leq \|T^2x\|$, so T is paranormal. ■

Remark 4.10 In [33, p. 113] it is shown that there exists a hyponormal operator T for which T^2 is not hyponormal. Since every hyponormal operator is paranormal, then T is paranormal and hence, by Theorem 4.4, T^2 is paranormal. Therefore T^2 provides an example of operator which is paranormal, but not hyponormal.

Theorem 4.11 *For every totally paranormal operator $T \in L(X)$ we have*

$$H_0(\lambda I - T) = \ker(\lambda I - T) \quad \textit{for every } \lambda \in \mathbb{C}. \tag{15}$$

Proof In fact, if $x \in H_0(\lambda I - T)$ then $\|(\lambda I - T)^n x\|^{1/n} \to 0$ and since T is totally paranormal then

$$(\lambda I - T)^n x\|^{1/n} \geq \|(\lambda I - T)x\|.$$

Therefore, $H_0(\lambda I - T) \subseteq \ker(\lambda I - T)$, and since the reverse inclusion holds for every operator, then we have $H_0(\lambda I - T) = \ker(\lambda I - T)$. ∎

Equation (15) has a remarkable conseguence. To see this, observe that if $\lambda \in \operatorname{iso}\sigma(T)$, where $\operatorname{iso} K$ denotes the isolated points of a set $K \subseteq \mathbb{C}$, then by Theorem 2.10, we have

$$H = H_0(\lambda I - T) \oplus K(\lambda I - T) = \ker(\lambda I - T) \oplus K(\lambda I - T),$$

and hence

$$(\lambda I - T)(H) = (\lambda I - T)(K(\lambda I - T)) = K(\lambda I - T).$$

Therefore,

$$H = \ker(\lambda I - T) \oplus (\lambda I - T)(H).$$

By Theorem 2.4 then $p(\lambda I - T) = q(\lambda I - T) = 1$, so λ is a simple pole of the resolvent.

Corollary 4.12 *Every isolated point of the spectrum of a totally paranormal operator is a simple pole of the resolvent.*

An operator $T \in L(X)$ is said to be *hereditarily polaroid* if every restriction $T|M$ on a closed invariant subspace is polaroid. From Lemma 4.1 we know that the restriction $T|M$ on a closed invariant subspace of a hyponormal operator is hyponormal too, so, by Theorem 4.9 and Corollary 4.12 we deduce:

Corollary 4.13 *Every hyponormal operator is hereditarily polaroid.*

If $T \in L(X)$, $S \in L(Y)$ a quasi-affinity is an operator $A \in L(X, Y)$ injective with dense range for which $SA = AT$. It is easily seen that the property of being hereditarily polaroid is similarity invariant, but is not preserved by a quasi-affinity. It is easily seen that the property of being hereditarily polaroid is similarity invariant, but is not preserved by a quasi-affinity. We now want to show that every hereditarily polaroid operator has SVEP. First we need to introduce two concepts of orthogonality on Banach spaces.

Definition 4.14 A closed subspace M of a Banach space X is said to be *orthogonal* to a closed subspace N of X in the sense of Birkoff and James, in symbol $M \perp N$ if $\|x\| \leq \|x + y\|$ for all $x \in M$ and $y \in N$.

A study of this concept of orthogonality may be found in [25]. Note that this concept of orthogonality is asymmetric and reduces to the usual definition of orthogonality in the case of Hilbert spaces. This concept of orthogonality may be weakened as follows:

Definition 4.15 A closed subspace M of a Banach space X is said to be *approximate orthogonal* to a closed subspace N of X, in symbol $M \perp_a N$, if there exists a scalar $\alpha \geq 1$ such that $\|x\| \leq \alpha\|x + y\|$ for all $x \in M$ and $y \in N$.

What $M \perp_a N$ means is that M meets N at an angle θ, $0 \leq \theta \leq \frac{\pi}{2}$, where by definition

$$\operatorname{sen} \theta = \inf\{\|x - y\|, \|y\| = 1\} \quad \text{for all } x \in M, y \in N.$$

If $\theta = \frac{\pi}{2}$, then M is orthogonal in the sense Birkoff-James sense. If M meets N at an angle $\theta > 0$ then N meets M at an angle $\phi > 0$, where in general $\theta \neq \phi$.

Theorem 4.16 *Every hereditarily polaroid operator $T \in L(X)$ has SVEP.*

Proof Let T be hereditarily polaroid. For distinct eigenvalues λ and μ of T, let M denote the subspace generated by $\ker(\lambda I - T)$ and $\ker(\mu I - T)$. Set $S := T|M$. Then S is polaroid and $\sigma(S) = \{\lambda, \mu\}$. Denote by P_μ the spectral projection corresponding to the spectral set $\{\mu\}$. Then

$$P_\mu(M) = \ker(\mu I - S) = \ker(\mu I - T),$$

while

$$\ker P_\mu = (I - P_\mu)(M) = \ker(\lambda I - S) = \ker(\lambda I - T).$$

Set $\alpha := \|P_\mu\|$. Then $\alpha \geq 1$, and

$$\|x\| = \|p_\mu x\| = \|P_\mu(x.y)\| \leq \alpha\|x - y\|$$

for all $x \in P_\mu(M) = \ker(\mu I - T)$ and $y \in (I - P_\mu)(M) = \ker(\lambda I - T)$.

Now, suppose that T does not have SVEP at a point $\delta_0 \in \mathbb{C}$. Then there exists an open disc $\mathbb{D}_0$ centered at δ_0 and a non-trivial analytic function $f : \mathbb{D}_0 \to X$ such that

$$f(\delta) \in \ker(\delta I - T) \text{ for all } \delta \in \mathbb{D}_0.$$

Let $\lambda \in \mathbb{D}_0$ and $\mu \in \mathbb{D}_0$ be two distinct complex numbers such that $f(\lambda)$ and $f(\mu)$ are non-zero. Since $\ker(\mu I - T) \perp_a \ker(\lambda I - T)$, then

$$0 < \|f(\mu)\| \le \alpha \|f(\mu) - f(\lambda)\|.$$

But then f is not continuous at μ, a contradiction. Hence T has SVEP. ∎

We have seen that T^* is polaroid if and only if T is polaroid. An immediate consequence of Theorem 4.16 is that this equivalence in general does not hold for hereditarily polaroid operators. Indeed, the right shift R is trivially hereditarily polaroid while its dual, the left shift L, cannot be hereditarily polaroid, since it does not have SVEP.

A class of hereditarily polaroid operator may be defined by extending the property (15) observed in Theorem 4.11. Consider a function $\mathbf{p} : \lambda \in \mathbb{C} \to \mathbf{p}(\lambda) \in \mathbb{N}$.

Definition 4.17 An operator $T \in L(X)$, X a Banach space, is said to be a $H(\mathbf{p})$-operator if

$$H_0(\lambda I - T) = \ker(\lambda I - T)^{\mathbf{p}(\lambda)} \quad \text{for every } \lambda \in \mathbb{C}.$$

Obviously, every hyponormal operator is $H(\mathbf{1})$, where $\mathbf{1}$ is the constant function $\mathbf{1}(\lambda) = 1$.

The property $H(p)$ is inherited by the restrictions on closed invariant subspaces:

Theorem 4.18 *Let $T \in L(X)$ be a bounded operator on a Banach space X. If T has the property $H(\mathbf{p})$ and Y is a closed T-invariant subspace of X then $T|Y$ has the property $H(\mathbf{p})$.*

Proof If $H_0(\lambda I - T) = \ker(\lambda I - T)^{\mathbf{p}(\lambda)}$ then

$$H_0((\lambda I - T)|Y) \subseteq \ker(\lambda I - T)^{\mathbf{p}(\lambda)} \cap Y = \ker((\lambda I - T)|Y)^{\mathbf{p}(\lambda)},$$

from which we obtain $H_0((\lambda I - T)|Y) = \ker((\lambda I - T)|Y)^{\mathbf{p}(\lambda}$. ∎

Theorem 4.19 *Every $H(\mathbf{p})$-operator T is hereditarily polaroid.*

Proof By Theorem 4.18 it suffices to prove that a $H(\mathbf{p})$-operator T is polaroid. If $\lambda \in \operatorname{iso} \sigma(T)$, then by Theorem 2.10, we have

$$H = H_0(\lambda I - T) \oplus K(\lambda I - T) = \ker(\lambda I - T)^{\mathbf{p}(\lambda)} \oplus K(\lambda I - T),$$

and hence

$$(\lambda I - T)^{\mathbf{p}(\lambda)}(H) = (\lambda I - T)^{\mathbf{p}(\lambda)}(K(\lambda I - T)) = K(\lambda I - T).$$

Therefore,

$$H = \ker(\lambda I - T)^{\mathbf{p}(\lambda)} \oplus (\lambda I - T)^{\mathbf{p}(\lambda)}(H).$$

By Theorem 2.4 then $p(\lambda I - T) = q(\lambda I - T) = p(\lambda)$, so λ is a pole of the resolvent. ∎

The class of $H(\mathbf{p})$-operators is very large. To see this, we first introduce a special class of operators which has an important role in local spectral theory. Let $\mathcal{C}^\infty(\mathbb{C})$ denote the Fréchet algebra of all infinitely differentiable complex-valued functions on $\mathbb{C}$.

Definition 4.20 An operator $T \in L(X)$, X a Banach space, is said to be *generalized scalar* if there exists a continuous algebra homomorphism $\Psi : \mathcal{C}^\infty(\mathbb{C}) \to L(X)$ such that

$$\Psi(1) = I \quad \text{and} \quad \Psi(Z) = T,$$

where Z denotes the identity function on $\mathbb{C}$.

The interested reader can be find a well organized treatment of generalized scalar operators in Laursen and Neumann [43, Section 1.5]). It should be noted that:

(a) every quasi-nilpotent generalized scalar operator is nilpotent, [43, Proposition 1.5.10].
(b) An operator similar to a restriction of a generalized scalar operator to one of its closed invariant subspaces is called *subscalar*.
(c) If T is generalized scalar then T is decomposable and hence has *Dunford property* (C), from which it follows that $H_0(\lambda I - T) = \mathcal{X}_T(\{\lambda\})$ is closed, see [43, Theorem 1.5.4 and Proposition 1.4.3], or [4, Chapter 4].

An important result due to Putinar [52] , every hyponormal operator is similar to a subscalar operator, see also [43, section 2.4], hence any hiponormal operator is decomposable.

Theorem 4.21 *Every generalized scalar, as well as every subscalar operator, $T \in L(X)$ is $H(\boldsymbol{p})$. Consequently, every generalized scalar and every subscalar operator is hereditarily polaroid.*

Proof By Lemma 4.18 and Theorem 4.22 we may assume that T is generalized scalar. Consider a continuous algebra homomorphism $\Psi : \mathcal{C}^\infty(\mathbb{C}) \to L(X)$ such that $\Psi(1) = I$ and $\Psi(Z) = T$. Let $\lambda \in \mathbb{C}$. Since every generalized scalar operator has the property (C), then $H_0(\lambda I - T)$ is closed. On the other hand, if $f \in \mathcal{C}^\infty(\mathbb{C})$ then

$$\Psi(f)(H_0(\lambda I - T)) \subseteq H_0(\lambda I - T),$$

because $T = \Psi(Z)$ commutes with $\Psi(f)$. Define

$$\tilde{\Psi} : \mathcal{C}^\infty(\mathbb{C}) \to L(H_0(\lambda I - T))$$

by

$$\tilde{\Psi}(f) = \Psi(f)|H_0(\lambda I - T) \quad \text{for every } f \in \mathcal{C}^\infty(\mathbb{C}).$$

Clearly, $T|H_0(\lambda I - T)$ is generalized scalar and quasi-nilpotent, so it is nilpotent. Thus there exists $p \geq 1$ for which $H_0(\lambda I - T) = \ker(\lambda I - T)^p$. ■

Two operators $T \in L(X)$, $S \in L(Y)$, X and Y Banach spaces, are said to be *intertwined* by $A \in L(X, Y)$ if $SA = AT$; and A is said to be a *quasi-affinity* if it has a trivial kernel and dense range. If T and S are intertwined by a quasi-affinity then T is called a *quasi-affine transform* of S, and we write $T \prec S$. If both $T \prec S$ and $S \prec T$ hold then T and S are said to be *quasi-similar*.

The next result shows that property $H(\mathbf{p})$ is preserved by quasi-affine transforms.

Theorem 4.22 *Suppose that $S \in L(Y)$ has property $H(\boldsymbol{p})$ and $T \prec S$. T, then T has property $H(\boldsymbol{p})$.*

Proof Suppose S has property $H(\mathbf{1})$, $SA = AT$, with A injective. If $\lambda \in \mathbb{C}$ and $x \in H_0(\lambda I - T)$ then

$$\|(\lambda I - S)^n Ax\|^{1/n} = \|A(\lambda I - T)^n x\|^{1/n} \leq \|A\|^{1/n}\|(\lambda I - T)^n x\|^{1/n},$$

from which it follows that $Ax \in H_0(\lambda I - S) = \ker\,(\lambda I - S)$. Hence $A(\lambda I - T)x = (\lambda I - S)Ax = 0$ and, since A is injective, this implies that $(\lambda I - T)x = 0$, i.e. $x \in \ker\,(\lambda I - T)$. Therefore $H_0(\lambda I - T) = \ker\,(\lambda I - T)$ for all $\lambda \in \mathbb{C}$.

The more general case of $H(\mathbf{p})$-operators is proved by using a similar argument.

■

Log-Hyponormal Operators In order to introduce a new class of operators we need first to recall some basic fact. For any positive operator $T \in L(H)$ there exists an unique operator S such that $S^2 = T$. The operator S is called the *square root* of T and denoted by $T^{\frac{1}{2}}$. Let M be a closed subspace of H. Then $H = M \oplus M^\perp$, where $M^\perp$ is the orthogonal complement of M, i.e.,

$$M^\perp := \{y \in H : (x, y) = 0 \text{ for all } x \in M.\}$$

The projection P_M of H onto M along $M^\perp$ is called the *orthogonal projection* from H onto M. Recall that a projection P is orthogonal if and only if P is self-adjoint. Every orthogonal projection P_M has norm equal to 1, moreover

$$0 \leq P_M \leq I,$$

see Furuta [33].

An operator $U \in L(H)$ is said to be a *partial isometry* if there exists a closed subspace M such that

$$\|Ux\| = \|x\| \text{ for any } x \in M, \text{ and} Ux = 0 \text{ for any } x \in M.$$

The subspace M is said to be the *initial space* of U, while the range $N := U(H)$ is said to be the *final space* of U. Evidently, U is an isometry if and only if U is a partial isometry and $M = H$, while U is unitary if and only if U is a partial isometry and $M = N = H$, see [33] .

Theorem 4.23 *Let $U \in L(H)$ be a partial isometry with initial space M and final space N.Then we have*

(i) $UP_M = U$ and $U'U = P_M$.
(ii) N is a closed subspace of H.
(iii) The adjoint U^* is a partial isometry with initial space N and final space M.

Note that an operator $U \in L(H)$ is a partial isometry if and only if U^* is a partial isometry, and in this case UU^* and U^*U are projection. Set $|T| := (T^*T)^{\frac{1}{2}}$. It is easily seen that $\ker T = \ker |T|$.

Theorem 4.24 (Polar Decomposition) *For every $T \in L(H)$ there exists a partial isometry U such that $T = U|T|$. The initial space of U is $M := \overline{|T|(H)} = \overline{T^*(H)}$, the final space is $N := \overline{T(H)}$. Moreover,*

$$\ker U = \ker |T| \quad \textit{and} \quad U^*U|T| = |T|.$$

If U is as in Theorem 4.24 the product $T = U|T|$ is called the *polar decomposition* of T. The partial isometry U is uniquely determined. If $T = U|T|$ is the polar decomposition of T then $T^* = U^*|T^*|$ is the polar decomposition of T^*. Some important properties are transmitted from T to U, for instance if T is normal then U is normal, if T is self-adjoint then U is self-adjoint, if T is positive then U is positive.

For $T \in L(H)$ let $T = W|T|$ be the polar decomposition of T. The operator defined by Aluthge in [11] as

$$R := |T|^{1/2}W|T|^{1/2}$$

is said to be the *Aluthge transform* of T.

Definition 4.25 An operator $T \in L(H)$ is said to be *log-hyponormal* if T is invertible and satisfies

$$\log (T^*T) \geq \log (TT^*).$$

If $R = V|R|$ is the polar decomposition of R, let define

$$\widetilde{T} := |R|^{1/2} V |R|^{1/2}.$$

If T is log-hyponormal then $\widetilde{T}$ is hyponormal and $T = K\widetilde{T}K^{-1}$, where $K := |R|^{1/2}|T|^{1/2}$, see Tanahashi [56], and M. Chō, Jeon and J. I. Lee [19]) . Hence T is similar to a hyponormal operator and therefore, by Theorem 4.22, has property $H(\mathbf{1})$.

***p*-Hyponormal Operators** An operator $T \in L(H)$ is said to be *p-hyponormal*, with $0 < p \le 1$, if

$$(T'T)^p \ge (TT')^p.$$

If $p = \frac{1}{2}$, T is said to be semi-hyponormal . The class of p-hyponormal operators have been studied by Aluthge [11] , while semi-hyponormal operators have been introduced by Xia [61]. Any p-hyponormal operator is q-hyponormal if $q < p$, but there are examples to show that the converse is not true, see [11]. Every invertible p-hyponormal is subscalar, see Ko [41], and is quasi-similar to a log-hyponormal operator. Consequently, by Theorem 4.22, every invertible p-hyponormal is operator has property $H(\mathbf{1})$. This is also true for p-hyponormal operators which are not invertible, see Duggal and Jeon [30]. Every p-hyponormal operator is paranormal, see [12] or [17].

***M*-Hyponormal Operators** Recall that $T \in L(H)$ is said to be *M-hyponormal* if there exists $M > 0$ such that

$$TT^* \le MT^*T.$$

Every M-hyponormal operator is subscalar ([43, Proposition 2.4.9]) and hence $H(\mathbf{p})$.

***w*-Hyponormal Operators** If $T \in L(H)$ and $T = U|T|$ is the polar decomposition, define

$$\hat{T} := |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}.$$

$T \in L(H)$ is said to be *w-hyponormal* if

$$|\hat{T}| \ge |T| \ge |\hat{T}^*|.$$

Examples of w-hyponormal operators are p-hyponormal operators and log-hyponormal operators. All w-hyponormal operators are subscalar, together with its Aluthge transformation, see M. Chō, H. Jeon, and J. I. Lee [46], and hence $H(p)$. In [38, Theorem 2.5] it is shown that for every isolated point λ of the spectrum of a w-hyponormal operator T we have $H_0(\lambda I - T) = \ker(\lambda I - T)$ and hence λ is a simple pole of the resolvent.

p-Quasihyponormal Operators A Hilbert space operator $T \in L(H)$ is said to be *p-quasihyponormal* for some $0 < p \leq 1$ if

$$T^*|T^*|^{2p}T \leq T * |T|^{2p}T.$$

Every p-quasi-hyponormal is paranormal [45].

Class A Operators An operator $T \in L(H)$ is said to be *a class A operator* if

$$|T^2| \geq |T|^2.$$

Every log-hyponormal operator is a class A operator [34], but the converse is no true, see [33, p. 176]. Every class A operator is paranormal (an example of a paranormal operator which is not a class A operator can be found in [33, p. 177]). Therefore every class A operator, as well as every algebraically class A operator, is polaroid.

Quasi-Class A Operators An operator $T \in L(H)$ is said to be *a quasi-class A operator* if $T^*|T^2|T \geq T^*|T|^2T$. The quasi-class A operators contains the class of al p-quasinormal operators and the class of all class A operators. In [31] it is given an example of a quasi-class A operator which is not paranormal. Every quasi-class A operator has SVEP, since $p(\lambda I - T) \leq 1$ for all $\lambda \in \mathbb{C}$, while every non-zero λ_0 isolated point of the spectrum is a simple pole of T and $H_0(\lambda_0 I - T) = \ker(\lambda_0 I - T)$, see [31]. It has been observed in [31, Example 0.2] that a quasi-class A operator need not to be normaloid.

$*$-Paranormal Operators A bounded operator $T \in L(H)$ is said to be *$*$-paranormal* if $\|T^*x\|^2 \leq \|T^2x\|$ for every unit vector $x \in H$. Paranormality is independent of $*$-paranormality and, evidently, hyponormal operators are both paranormal and $*$-paranormal. It is known [13] that

$$T \text{ is } * \text{-paranormal} \Leftrightarrow T^{*2}T^2 - 2\lambda TT^* + \lambda^2 \geq 0 \text{ for each } \lambda > 0.$$

Every $*$-paranormal operator T is normaloid. Moreover, $\ker(\lambda I - T) \subseteq \ker(\overline{\lambda} I - T^*)$ for all $\lambda \in \mathbb{C}$, from which it easily follows that $p(\lambda I - T) < \infty$ for all $\lambda \in \mathbb{C}$, thus T has SVEP.

Totally $*$-Paranormal $T \in L(H)$ is said to be *totally $*$-paranormal* if $\lambda I - T$ is $*$-paranormal for every $\lambda \in \mathbb{C}$. An example of a $*$-paranormal operator which is not totally $*$-paranormal may be found in [37]. It is not known to the author if every totally $*$-paranormal operator has property (C). Observe that every totally algebraically $*$-paranormal operator is $H(1)$ and hence hereditarily polaroid. Indeed, $\mu I - T$ is normaloid for all $\mu \in \mathbb{C}$, so

$$\|(\lambda I - T)x\| \leq \|(\lambda I - T)^n x\|^{\frac{1}{n}} \quad \text{for all } x \in X, \lambda \in \mathbb{C},$$

so that $H_0(\lambda I - T) \subseteq \ker(\lambda I - T)$ for all $\lambda \in \mathbb{C}$.

The class of p-quasihyponormal may be extended as follows:

(p,k)-Quasihyponormal Operators An operator $T \in L(H)$ is said to be *(p,k)-quasihyponormal* for some $0 < p \leq 1$ and $k \in \mathbb{N}$ if

$$T^{*k}|T^*|^{2p}T^k \leq (T^{*k}|T|^{2p}T^k.$$

Evidently,

1. a $(1, 1)$-quasihyponormal operator is quasihyponormal;
2. a $(p, 1)$-quasihyponormal operator is p-quasihyponormal;
3. a $(p, 0)$-quasihyponormal operator is p-hyponormal if $0 < p < 1$ and hyponormal if $p = 1$.

In [57, Theorem 6] it has been proved that every (p, k)-quasihyponormal operator $T \in L(H)$ is hereditarily polaroid. Moreover, every isolated point $\lambda \neq 0$ is a simple pole of the resolvent.

It should be noted that the class of totally $*$-paranormal operators, as well as the class of M-hyponormal operators, are independent of the classes (p, k)-quasi-hyponormal.

5 Totally Hereditarily Normaloid Operators

We have already observed that a normal operator is normaloid, i.e. $\|T\| = r(T)$. Since the restriction of a normal operator is still normal, these motivates the following definition.

Definition 5.1 An operator $T \in L(X)$ is said to be *hereditarily normaloid*, X a Banach space, $(T \in \mathcal{HN})$, if the restriction $T|M$ of T, to any closed T-invariant subspace M, is normaloid. Finally, $T \in L(X)$ is said to be *totally hereditarily normaloid*, $T \in \mathcal{THN}$, if $T \in \mathcal{HN}$ and every invertible restriction $T|M$ has a normaloid inverse.

Totally hereditarily operators were introduced by Duggal and S.V. Djordjevíc in [29].

Recall that an invertible operator $T \in L(X)$ is said to be *doubly power-bounded* if $\sup\{\|T^n\| : n \in \mathbb{Z}\} < \infty$.

Theorem 5.2 *Suppose that $T \in L(X)$ is quasi-nilpotent. If T is an analytically $\mathcal{THN}$ operator, then T is nilpotent.*

Proof Let $T \in L(X)$ and suppose that $f(T)$ is a $\mathcal{THN}$-operator for some $f \in \mathcal{H}_{nc}(\sigma(T))$. From the spectral mapping theorem we have

$$\sigma(f(T)) = f(\sigma(T)) = \{f(0)\}.$$

We claim that $f(T) = f(0)I$. To see this, let us consider the two possibilities: $f(0) = 0$ or $f(0) \neq 0$.

If $f(0) = 0$ then $f(T)$ is quasi-nilpotent and $f(T)$ is normaloid, and hence $f(T) = 0$. The equality $f(T) = f(0)I$ then trivially holds.

Suppose the other case $f(0) \neq 0$, and set $f_1(T) := \frac{1}{f(0)} f(T)$. Clearly, $\sigma(f_1(T)) = \{1\}$ and $\|f_1(T)\| = 1$. Further, $f_1(T)$ is invertible and is $\mathcal{THN}$. This easily implies that its inverse $f_1(T)^{-1}$ has norm 1. The operator $f_1(T)$ is then doubly power-bounded and, by a classical theorem due to Gelfand (see [43, Teorem 1.5.14] for an elegant proof), it then follows that $f_1(T) = I$, and consequently $f(T) = f(0)I$, as claimed.

Now, define $g(\lambda) := f(0) - f(\lambda)$. Clearly, $g(0) = 0$, and g may have only a finite number of zeros in $\sigma(T)$. Let $\{0, \lambda_1, \dots, \lambda_n\}$ be the set of all zeros of g, where $\lambda_i \neq \lambda_j$, for all $i \neq j$, and λ_i has multiplicity $n_i \in \mathbb{N}$. Write

$$g(\lambda) = \mu\lambda^m \prod_{i=1}^{n} (\lambda_i I - T)^{n_i} h(\lambda),$$

where $h(\lambda)$ has no zeros in $\sigma(T)$. The equality $g(T) = f(0)I - f(T) = 0$ then implies that

$$0 = g(T) = \mu\, T^m \prod_{i=1}^{n} (\lambda_i I - T)^{n_i} h(T) \quad \text{with } \lambda_i \neq 0,$$

where all the operators $\lambda_i I - T$ and $h(T)$ are invertible. This, obviously, implies that $T^m = 0$, i. e. T is nilpotent. ■

If $T \in L(X)$ the *numerical range* of T is defined as

$$W(T) := \{f(T) : f \in L(X)^*, ||f\| = f(I) = 1\},$$

while the *numerical radius* of T is defined by

$$w(T) := \sup\{|\lambda| : \lambda \in W(T)\}.$$

In the case of Hilbert space operator the numerical range may be described as the set

$$W(T) = \{(Tx, x) : \|x\| = 1\},$$

and the well-known *Toeplitz-Hausdorff theorem* establishes that $W(T)$ is a convex set in the complex plane (for a proof, see Furuta [33, p. 91]). Furthermore,

$$r(T) \leq w(T) \leq \|T\|.$$

The next nontrivial result has been proved in Sinclair [55], we omit the not simple proof.

Theorem 5.3 *Let $T \in L(X)$ and suppose that 0 is in the boundary of the numerical range of T. Then the kernel of T is orthogonal (in the sense of Birkoff-James) to the range of T.*

In the case of paranormal operators we have:

Theorem 5.4 *Suppose that $T \in L(X)$ is totally hereditarily normaloid, $\lambda, \mu \in \mathbb{C}$, with $\lambda \neq 0$ and $\lambda \neq \mu$. Then $\ker(\lambda I - T) \perp \ker(\mu I - T)$, i.e. $\ker(\lambda I - T)$ is orthogonal to $\ker(\mu I - T)$ in the Birkhoff and James sense.*

Proof Suppose first that $|\lambda| \geq |\mu|$ and let $x \in \ker(\lambda I - T)$, $y \in \ker(\mu I - T)$. Then $Tx = \lambda x$ and $Ty = \mu y$. Denote by M the subspace generated by x and y and set $T_M := T|M$. Clearly, $\sigma(T|M) = \{\lambda, \mu\}$ and being $T|M$ normaloid then

$$\|T|M\| = r(T|M) = |\lambda, |$$

so that $\mu(T|M) = |\lambda|$. Consequently, λ belongs to the boundary of the numerical range of $T|M$ and hence, by Theorem 5.3, $\ker(\lambda I - T|M) \perp (\lambda I - T|M)(M)$. Evidently, λ and μ are poles of the resolvent of $T|M$ having order 1. Denoting by P_λ and P_μ the spectral projections for $T|M$ associated with $\{\lambda\}$ and $\{\mu\}$, respectively, we then have

$$(\lambda I - T|M)(M) = (I - P_\lambda)(M) = P_\mu(M) = \ker(\beta I - T|M)$$

Now, $x \in \ker(\lambda I - T|M)$ and $y \in \ker(\mu I - T|M)$, so $\|x + y\| \geq \|x\|$.

Consider now the case where $|\lambda| < |\mu|$. Then $|\mu| > 0$, so $T|M$ is invertible and

$$\sigma(T|M)^{-1} = \{\frac{1}{\lambda}, \frac{1}{\mu}\},$$

with $|\frac{1}{\lambda}| > |\frac{1}{\mu}|$. Since $T|M$ is normaloid then the inverse $(T|M)^{-1}$ is also normaloid. As in the first case we then see that the kernels $\ker(\frac{1}{\lambda}I - (T|M)^{-1})$ and $\ker(\frac{1}{\mu}I - (T|M)^{-1})$ are orthogonal. Obviously, $x \in \ker(\frac{1}{\lambda}I - (T|M)^{-1})$ and $y \in \ker(\frac{1}{\mu}I - (T|M)^{-1})$, so the proof is complete. ∎

Theorem 5.5 *Every totally hereditarily normaloid operator T on a separable Banach space has SVEP.*

Proof To prove the first assertion, we observe first that the point spectrum $\sigma_p(T)$ is countable, hence its interior part is empty. If $\sigma_p(T)$ were not countable we would have an uncountable set of unit vectors such that

$$\|x_i - x_j\| \geq 1.$$

Since X is separable this is not possible. ∎

Remark 5.6 It is rather simple to see that if $T \in L(X)$ is $\mathcal{THN}$ and M is a T-invariant closed subspace of X then the restriction $T|M$ is also $\mathcal{THN}$. In the sequel by $\overline{Y}$ we denote the closure of $Y \subseteq X$.

Definition 5.7 An operator $T \in L(X)$, X a Banach space, is said to be *k-quasi totally hereditarily normaloid*, k a nonnegative integer, if the restriction $T|\overline{T^k(X)}$ is $\mathcal{THN}$.

Evidently, every $\mathcal{THN}$-operator is quasi-$\mathcal{THN}$, and if $T^k(X)$ is dense in X then a quasi-$\mathcal{THN}$ operator T is $\mathcal{THN}$.

Lemma 5.8 *If $T \in L(X)$ is quasi-$\mathcal{THN}$ and M is a closed T-invariant subspace of X, then $T|M$ is quasi-$\mathcal{THN}$.*

Proof Let k a nonnegative integer such that $T_k := T|\overline{T^k(X)}$ is $\mathcal{THN}$. Let T_M denote the restriction $T|M$. Clearly, $\overline{{T_M}^k(M)} \subseteq \overline{T^k(X)}$, so $\overline{{T_M}^k(M)}$ is T_k-invariant subspace of $\overline{T^k(X)}$. By Remark 5.6 it then follows that

$$T_M|\overline{{T_M}^k(M)} = T_k|\overline{{T_M}^k(M)}$$

is $\mathcal{THN}$. ∎

We recall now some elementary algebraic facts. Suppose that $T \in L(X)$ and $X = M \oplus N$, with M and N closed subspaces of X, M invariant under T. With respect to this decomposition of X it is known that T may be represented by a *upper triangular operator matrix* $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, where $A \in L(M)$, $C \in L(N)$ and $B \in L(N, M)$. It is easily seen that for every $x = \begin{pmatrix} x \\ 0 \end{pmatrix} \in M$ we have $Tx = Ax$, so $A = T|M$. Let us consider now the case of operators T acting on a Hilbert space H, and suppose that $T^k(H)$ is not dense in H. In this case we can consider the nontrivial orthogonal decomposition

$$H = \overline{T^k(H)} \oplus \overline{T^k(H)}^{\perp}, \tag{16}$$

where $\overline{T^k(H)}^{\perp} = \ker(T^*)^k$, T^* the adjoint of T. Note that the subspace $\overline{T^k(H)}$ is T-invariant, since

$$T(\overline{T^k(H)}) \subseteq \overline{T(T^k(H))} = \overline{T^{k+1}(H)} \subseteq \overline{T^k(H)}.$$

Thus we can represent, with respect the decomposition (16), T as an upper triangular operator matrix

$$\begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}, \tag{17}$$

where $T_1 = T|\overline{T^k(H)}$. Moreover, T_3 is nilpotent. Indeed, if $x \in \overline{T^k(X)}^{\perp}$, an easy computation yields

$$T^k x = T \begin{pmatrix} 0 \\ x \end{pmatrix} = {T_3}^k x.$$

Hence ${T_3}^k x = 0$, since $T^k x \in \overline{T^k(H)} \cup \overline{T^k(H)}^{\perp} = \{0\}$. Therefore we have:

Theorem 5.9 *Suppose that $T \in L(H)$ and $T^k(H)$ non dense in H. Then, according the decomposition (16), $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ is quasi-$\mathcal{THN}$ if and only if T_1 is $\mathcal{THN}$. Furthermore,*

$$\sigma(T) = \sigma(T_1) \cup \sigma(T_3) = \sigma(T_1) \cup \{0\}.$$

Proof The first assertion is clear, since $T_1 = T|\overline{T^k(H)}$. The second assertion follows from the following general result: if $T := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is an upper triangular operator matrix acting on some direct sum of Banach spaces and $\sigma(A) \cap \sigma(B)$ has no interior points, then $\sigma(T) = \sigma(A) \cup \sigma(B)$; see [44] for a proof. ■

In the sequel we give some examples of operators which are quasi totally hereditarily normaloid.

1. The class of quasi-paranormal operators may be extended as follows: $T \in L(H)$ is said to be (n, k)-*quasiparanormal* if

$$\|T^{k+1}x\| \leq \|T^{1+n}(T^k x)\|^{\frac{1}{1+n}} \|T^k x\|^{\frac{n}{1+n}} \quad \text{for all } x \in H.$$

   The class of $(1, k)$- quasiparanormal operators has been studied in [48]. If $T^k(H)$ is not dense then, in the triangulation $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$, $T_1 = T|\overline{T^k(H)}$ is n-quasiparanormal, and hence $\mathcal{THN}$, see [62].
2. An extension of class A operators is given by the class of all *k-quasiclass A operators*, where $T \in L(H)$, H a separable infinite dimensional Hilbert space, is said to be a k-quasiclass A operator if

$$T^{*k}(|T|^2 - |T|^2)T^k \geq 0.$$

   Every k-quasiclass A operator is quasi-$\mathcal{THN}$. Indeed, if T has dense range then T is a class A operator and hence paranormal. If T does not have dense range then T with respect the decomposition $H = \overline{T^k(H)} \oplus \ker T^{*k}$ may be represented as a matrix $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$, where $T_1 := T|\overline{T^k(H)}$ is a class A operator, and hence $\mathcal{THN}$, see [56].

We have observed a quasi-class A operator (i.e. $k = 1$), need not to be normaloid. This shows that, in general, a quasi-$\mathcal{THN}$ operator is not normaloid, so the class of quasi-$\mathcal{THN}$ operators properly contains the class of $\mathcal{THN}$ operators.

3. An operator $T \in L(H)$, H a separable infinite dimensional Hilbert space, is said to be *k-quasi *-paranormal*, $k \in \mathbb{N}$, if

$$\|T^*T^k x\|^2 \leq \|T^{k+2}x\|\|T^k x\| \quad \text{for all unit vectors } x \in H.$$

This class of operators contains the class of all quasi- $*$-paranormal operators (which corresponds to the value $k = 1$). Every k-quasi *-paranormal operator is quasi-$\mathcal{THN}$. Indeed, if T^k has dense range then T is $*$-paranormal and hence $\mathcal{THN}$. If T^k does not have dense range then T may be decomposed, according the decomposition $H = \overline{T^k(H)} \oplus \ker T^{*k}$, as $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$, where $T_1 = T|\overline{T^k(H)}$ is $*$-paranormal, hence $\mathcal{THN}$, see [49, Lemma 2.1].

Every (p, k)-quasihyponormal operator T with respect to the decomposition $H = \overline{T^k(H)} \oplus \ker T^{*k}$, may be represented as a matrix $T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}$, where $T_1 := T|\overline{T^k(H)}$ is k-hyponormal (hence paranormal) and consequently $\mathcal{THN}$, see [40]. The next result generalizes the result of Lemma 5.2.

Theorem 5.10 *Suppose that $T \in L(H)$, H a Hilbert space, is analytically quasi-$\mathcal{THN}$ and quasi-nilpotent. Then T is nilpotent.*

Proof Suppose first that T is quasi-nilpotent and k-quasi $\mathcal{THN}$. If $T^k(H)$ is dense then T is $\mathcal{THN}$, so T is nilpotent by Theorem 5.2. Suppose that $T^k(H)$ is not dense and write $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$, where T_1 is $\mathcal{THN}$, ${T_3}^k = 0$, and $\sigma(T) = \sigma(T_1) \cup \{0\}$. Since $\sigma(T) = \{0\}$ and $\sigma(T_1)$ is not empty, we then have $\sigma(T_1) = \{0\}$, thus T_1 is a quasi-nilpotent $\mathcal{THN}$ operator and hence $T_1 = 0$. Therefore $T = \begin{pmatrix} 0 & T_2 \\ 0 & T_3 \end{pmatrix}$. An easy computation yields that

$$T^{k+1} = \begin{pmatrix} 0 & T_2 \\ 0 & T_3 \end{pmatrix}^{k+1} = \begin{pmatrix} 0 & T_2 T_3^k \\ 0 & T_3^{k+1} \end{pmatrix}^{k+1} = 0,$$

so that T is nilpotent.

Finally, suppose that T is quasi-nilpotent and analytically k-quasi $\mathcal{THN}$. Let h be analytic on an open neighborhood of $\sigma(T)$, and non-constant on the components of its domain, be such that $h(T)$ is quasi-$\mathcal{THN}$. We claim that $h(T)$ is nilpotent. If $h(T)^k$ has dense range then $h(T)$ is $\mathcal{THN}$ and hence, by Lemma 5.2, $h(T)$ is nilpotent. Suppose that $h(T)^k$ has not dense range. Then with respect the decomposition $X = \overline{h(T)^k(H)} \oplus \overline{h(T)^k(H)}^{\perp}$, the operator $h(T)$ has a triangulation

$h(T) = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, such that $A = h(T)|\overline{h(T)^k(H)}$ is $\mathcal{THN}$ and

$$\sigma(h(T)) = \sigma(A) \cup \{0\}.$$

By the spectral mapping theorem we have

$$\sigma(h(T)) = h(\sigma(T)) = \{h(0)\}.$$

Consequently, $0 \in \{h(0)\}$, i. e. $h(0) = 0$, and therefore $h(T)$ is quasi-nilpotent. Since $h(T)$ is quasi-$\mathcal{THN}$, by the first part of proof it then follows that $h(T)$ is nilpotent. Now, $h(0) = 0$ so we can write

$$h(\lambda) = \mu\lambda^n \prod_{i=1}^{n} (\lambda_i I - T)^{n_i} g(\lambda),$$

where $g(\lambda)$ has no zeros in $\sigma(T)$ and $\lambda_i \neq 0$ are the other zeros of g with multiplicity n_i. Hence

$$h(T) = \mu\, T^n \prod_{i=1}^{n} (\lambda_i I - T)^{n_i} g(T),$$

where all $\lambda_i I - T$ and $g(T)$ are invertible. Since $h(T)$ is nilpotent then also T is nilpotent. ∎

Theorem 5.11 *If $T \in L(H)$ is an analytically quasi $\mathcal{THN}$ operator, then T is polaroid.*

Proof We show that for every isolated point λ of $\sigma(T)$ we have $p(\lambda I - T) = q(\lambda I - T) < \infty$. Let λ be an isolated point of $\sigma(T)$. Then $H = H_0(\lambda I - T) \oplus K(\lambda I - T)$, by Theorem 2.10. Furthermore, since $\sigma(T|H_0(\lambda I - T)) = \{\lambda\}$, while $\sigma(T|K(\lambda I - T)) = \sigma(T) \setminus \{\lambda\}$, so the restriction $\lambda I - T|H_0(\lambda I - T)$ is quasi-nilpotent and $\lambda I - T|K(\lambda I - T)$ is invertible. Since $\lambda I - T|H_0(\lambda I - T)$ is analytically quasi $\mathcal{THN}$, then Lemma 5.10 implies that $\lambda I - T|H_0(\lambda I - T)$ is nilpotent. In other worlds, $\lambda I - T$ is an operator of Kato Type.

Now, both T and the dual T^* have SVEP at λ, since λ is isolated in $\sigma(T) = \sigma(T^*)$, and this implies, by Theorem 3.5, that both $p(\lambda I - T)$ and $q(\lambda I - T)$ are finite. Therefore, λ is a pole of the resolvent. ∎

Lemma 5.12 *Suppose that $T \in L(X)$ admits, with respect to the decomposition $X = M \oplus N$, the representation $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$, where T_3 is nilpotent. Then T has SVEP if and only if T_1 has SVEP.*

Proof Suppose that T_1 has SVEP. Fix arbitrarily $\lambda_0 \in \mathbb{C}$ and let $f : U \to X$ be an analytic function defined on open disc U centered at λ_0 such that $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in U$. Set $f(\lambda) := f_1(\lambda) \oplus f_2(\lambda)$ on $X = M \oplus N$. Then we can write

$$0 = (\lambda I - T)f(\lambda) = \begin{pmatrix} \lambda I - T_1 & -T_2 \\ 0 & -\lambda I - T_3 \end{pmatrix} \begin{pmatrix} f_1(\lambda) \\ f_2(\lambda) \end{pmatrix}$$

$$= \begin{pmatrix} (\lambda I - T_1)f_1(\lambda) - T_2 f_2(\lambda) \\ (\lambda I - T_3)f_2(\lambda) \end{pmatrix}.$$

Then $(\lambda I - T_3)f_2(\lambda) = 0$ and $(\lambda I - T_1)f_1(\lambda) - T_2 f_2(\lambda) = 0$. Since a nilpotent operator has SVEP then $f_2(\lambda) = 0$, and consequently $(\lambda I - T_1)f_1(\lambda) = 0$. But T_1 has SVEP at λ_0 , so $f_1(\lambda) = 0$ and hence $f(\lambda) = 0$ on U. Thus, T has SVEP at λ_0. Since λ_0 is arbitrary then T has SVEP.

Conversely, suppose that T has SVEP. Since T_1 is the restriction of T to M and the SVEP from T is inherited by the restriction to closed invariant subspaces, then T_1 has SVEP. ■

Theorem 5.13 *If $T \in L(H)$ is analytically quasi $\mathcal{THN}$, then T is hereditarily polaroid and hence has SVEP.*

Proof Let $f \in \mathcal{H}_{nc}(\sigma(T))$ such that $f(T)$ is quasi $\mathcal{THN}$. If M is a closed T-invariant subspace of X, we know that $f(T)|M$ is quasi $\mathcal{THN}$, by Lemma 5.8, and $f(T)|M = f(T|M)$, so $f(T|M)$ is polaroid, by Theorem 5.11, and consequently, $T|M$ is polaroid, see [4, Theorem 4.19]. ■

6 Weyl Type Theorems for Analytically Quasi $\mathcal{THN}$ Operators

An operator $T \in L(X)$, X a Banach space, is said to be *a-polaroid* if every $\lambda \in \text{iso}\,\sigma_a(T)$ is a pole of the resolvent of T. It easily seen that every a-polaroid operator is polaroid. Indeed, if $\lambda \in \text{iso}\,\sigma(T)$ then λ belongs to the boundary of the spectrum, so $\lambda \in \text{iso}\,\sigma_a(T)$ and hence is a pole. Evidently, T is polaroid if and only if T' is polaroid (T^* in the case of Hilbert space operators). If T has SVEP then $\sigma(T') = \sigma_{ap}(T')$, so if T is polaroid then T' is a-polaroid, and analogously, if T' has SVEP then $\sigma(T) = \sigma_{ap}(T)$ and hence T is a-polaroid.

Recall that an operator $T \in L(X)$ is said to be *Weyl* ($T \in W(X)$), if T is Fredholm and the *index* $\operatorname{ind} T := \alpha(T) - \beta(T) = 0$. The *Weyl spectrum* of $T \in L(X)$ is defined by

$$\sigma_{\mathrm{w}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W(X)\}.$$

The class of *upper semi-Weyl operators* is defined by $W_+(X) := \{T \in \Phi_+(X) : \operatorname{ind} T \leq 0\}$, and class of *lower semi-Weyl operators* is defined by $W_-(X) := \{T \in \Phi_-(X) : \operatorname{ind} T \geq 0\}$. Clearly, $W(X) = W_+(X) \cap W_-(X)$. The classes of operators above defined generate the following spectra: the *upper semi-Weyl spectrum*

$$\sigma_{\mathrm{uw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W_+(X)\},$$

and the *lower semi-Weyl spectrum* defined as

$$\sigma_{\mathrm{lw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W_-(X)\}.$$

An operator $T \in L(X)$ is said to be *Browder* ($T \in B(X)$), if T is *Fredholm* and $p(T) = q(T) < \infty$. The *Browder spectrum* of $T \in L(X)$ is defined by

$$\sigma_{\mathrm{b}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin B(X)\}.$$

Following Coburn [21], we say that *Weyl's theorem holds* for $T \in L(X)$ (in symbol, (W)) if

$$\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi_{00}(T), \tag{18}$$

where

$$\pi_{00}(T) := \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(\lambda I - T) < \infty\}.$$

Note that T satisfies (W) if and only if T satisfies *Browder's theorem*, (i.e., $\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{w}}(T)$) and $\pi_{00}(T) = p_{00}(T)$, where $p_{00}(T) := \sigma(T) \setminus \sigma_{\mathrm{b}}(T)$, see for instance [4, Theorem 6.40].

The concept of Fredholm operators has been generalized in the following way [15]: for every $T \in L(X)$ and a nonnegative integer n let us denote by $T_{[n]}$ the restriction of T to $T^n(X)$ viewed as a map from the space $T^n(X)$ into itself (we set $T_{[0]} = T$). $T \in L(X)$ is said to be *B-Fredholm* if for some integer $n \geq 0$ the range $T^n(X)$ is closed and $T_{[n]}$ is a Fredholm operator. In this case $T_{[m]}$ is a Fredholm operator for all $m \geq n$ [15]. This enables one to define the index of a Fredholm as $\operatorname{ind} T = \operatorname{ind} T_{[n]}$. A bounded operator $T \in L(X)$ is said to be *B-Weyl* ($T \in BW(X)$) if for some integer $n \geq 0$ $T^n(X)$ is closed and $T_{[n]}$ is Weyl. The *B-Weyl spectrum* $\sigma_{\mathrm{bw}}(T)$ is defined

$$\sigma_{\mathrm{bw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin BW(X)\}.$$

Another version of Weyl's theorem has been introduced by Berkani and Koliha ([14] as follows: $T \in L(X)$ is said to verify *generalized Weyl's theorem*, (in symbol (gW)) if

$$\sigma(T) \setminus \sigma_{\mathrm{bw}}(T) = E(T), \tag{19}$$

where

$$E(T) := \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(\lambda I - T)\}.$$

An operator $T \in L(X)$ is said to be *Drazin invertible* if $p(T)$ and $q(T)$ are finite, and this happens if and only if T is invertible or 0 is a pole of the resolvent. The Drazin spectrum id denoted by $\sigma_{\mathrm{d}}(T)$. Note that $(gW))$ holds for T if and only if T satisfies *generalized Browder's theorem*, (i.e., $\sigma_{\mathrm{bw}} = \sigma_{\mathrm{d}}$, or equivalently, Browder's theorem, see [4, Theorem 5.15]) and $E(T) = \Pi(T)$, where $\Pi(T)$ is the set of all poles of the resolvent of T. Note that generalized Weyl's theorem entails Weyl's theorem, see [4, Theorem 6.60].

The following result shows that in presence of SVEP the polaroid condition entails Weyl type theorems.

Theorem 6.1 *Let $T \in L(X)$ be polaroid and suppose that either T or T' has SVEP. Then both T and T' satisfy generalized Weyl's theorem.*

Proof If T is polaroid also T' is polaroid, and Weyl's theorem and generalized Weyl's theorem for T, or T', are equivalent, see [10, Theorem 3.7]. The assertion then follows from [10, Theorem 3.3]. ∎

Remark 6.2 In the case of a Hilbert space operator $T \in L(H)$ it is more appropriated to consider the Hilbert adjoint T^* instead of the dual T'. Note that T' satisfies (gW) if and only if T^* does. This easily follows from the well known equalities, $\sigma_{\mathrm{w}}(T^*) = \overline{\sigma_{\mathrm{w}}(T')}$, where $\overline{E}$ is the conjugate of $E \subseteq \mathbb{C}$, $\sigma_{\mathrm{b}}(T^*) = \overline{\sigma_{\mathrm{b}}(T')}$, $E(T^*) = \overline{E}(T')$, and $\Pi(T^*) = \overline{\Pi(T')}$. Furthermore, T' satisfies SVEP if and only if T^* satisfies SVEP, so, in the statement of Theorem 6.1, T' may be replaced by the Hilbert adjoint T^*.

In [3] it is shown that if T is hereditarily polaroid and has SVEP, and K is an algebraic operator which commutes with T then $T + K$ is polaroid and $T' + K'$ is a-polaroid.

The following perturbation result has been proved in [3, Theorem 3.12].

Theorem 6.3 *Suppose that $T \in L(X)$ and $K \in L(X)$ an algebraic operator commuting with $T \in L(X)$. If $T \in L(X)$, or T', has SVEP and T, or T', is hereditarily polaroid, then $f(T + K)$ and $f(T' + K')$) satisfies (gW) for every $f \in \mathcal{H}_{nc}(\sigma(T + K))$.*

Observe that in the case of Hilbert space operators

$$T^* + K^* \text{ is } a\text{-polaroid} \Leftrightarrow T' + K' \text{ is } a\text{-polaroid},$$

see Theorem [10, Theorem 2.3].

Theorem 6.4 *Let $T \in L(H)$ be an analytically quasi $\mathcal{THN}$ operator on a Hilbert space H, and let $K \in L(H)$ be an algebraic operator commuting with T. Then both $f(T+K)$ and $f(T'+K')$ satisfies (gW) for every $f \in \mathcal{H}_{nc}(\sigma(T+K))$.*

Proof Suppose that $T \in L(H)$ is analytically quasi $\mathcal{THN}$, and let $f \in \mathcal{H}_{nc}(\sigma(T))$ be such that $f(T)$ is quasi $\mathcal{THN}$, Since T has SVEP then $f(T)$ has SVEP. Now, by Theorem 5.13 T is hereditarily polaroid, and hence the results of Theorem 6.3 apply. ∎

Theorem 6.4 gives to us a general framework and applies to all classes of the operators above considered in this paper (and much more!). Moreover, Theorem 6.4 considerably improves most the existing results in literature concerning Weyl type theorems for these classes of operators. Observe that, always in the situation of Theorem 6.4, the fact that $f(T+K)$ is polaroid entails that all Weyl type theorems (as property (gw) and a-Weyl's theorem) hold for $f(T'+K')$, see [10] for definitions and details, in particular Theorem 3.10.

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On Wold Type Decomposition for Closed Range Operators

H. Ezzahraoui, M. Mbekhta, and E. H. Zerouali

Abstract This survey aims to give a brief introduction to Wold-type decomposition for some closed range operators satisfying some operator inequalities. As a cornerstone in the theory of the Hardy space, Beurling theorem for unweighted shift is our starting point that we try to transfer to regular operators. Also, several results on left invertible operators close to isometries, as extensions of the Hardy shifts, are listed and extended to the case of regular operators. We define and study the Cauchy dual for such operators by using the Moore-Penrose inverse of closed range operators. The Cauchy dual plays the role of the left inverse in our approach for this general setting.

Keywords Wold-type decomposition · Beurling-type theorem · Regular operator · Moore-Penrose inverse · Cauchy dual

1 Introduction

Let us denote first by $\mathcal{H}$ a Hilbert space and by $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. For an operator $T \in \mathcal{L}(\mathcal{H})$, we denote by $R(T)$ and $N(T)$ the range and the kernel sub-spaces of T respectively. We also write $R^\infty(T) = \bigcap_{k=0}^{\infty} R(T^k)$ and $N^\infty(T) = \bigcup_{n\geq 0} N(T^n)$ for the generalized range and the generalized kernel of T respectively. We will say that T is a pure operator if $R^\infty(T) = \{0\}$.

A subspace $E \subset \mathcal{H}$ is said to be $T-$invariant if $T(E) \subseteq E$, the lattice of all closed $T-$invariant sub-spaces in $\mathcal{H}$ will be denoted by $Lat(T, \mathcal{H})$. For any given

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subspace E of $\mathcal{H}$, we set $[E]_T = \bigvee_{k=0}^{\infty} T^k(E)$ for the smallest closed T-invariant subspace of $\mathcal{H}$ containing E. A subspace E is called a *reducing subspace* for T if it is invariant for both T and T^* in which T^* is the adjoint operator of T.

Recall that the operator $T \in \mathcal{L}(\mathcal{H})$ is *bounded below* if there is $c > 0$ satisfying $\|Tx\| \geq c\|x\|$ for every $x \in \mathcal{H}$. It is not difficult to see that an operator T is bounded below if and only if it is one to one and has a closed range. This is also equivalent to T^*T invertible and hence to T is left invertible. A standard left inverse of T is given by $L = (T^*T)^{-1}T^*$. The *reduced minimum modulus* $\gamma(T)$ of T is defined by the formula:

$$\gamma(T) = \inf\{\|Tx\| : \|x\| = 1, \ x \in N(T)^{\perp}\}$$

if T is not the null operator and $\gamma(T) = \infty$ if $T = 0$. The reduced minimum modulus encodes several properties of T. It is not difficult to show that $\gamma(T) = \gamma(T^*)$ for every $T \in \mathcal{L}(\mathcal{H})$, for more information see [17]. Clearly, T has a closed range if and only if $\gamma(T) > 0$. The operator $T \in \mathcal{L}(\mathcal{H})$ is *contractive* if $\|Tx\| \leq \|x\|$ for every $x \in \mathcal{H}$ and is said to be *expansive* if $\|Tx\| \geq \|x\|$ for every $x \in \mathcal{H}$. It is not difficult to see that T is an expansive operator if and only if $\gamma(T) \geq 1$. In particular expansive operators are left invertible with contractive left inverses.

The classical Wold decomposition theorem states that if U is an isometry on a Hilbert space $\mathcal{H}$, then $\mathcal{H}$ is the direct sum of two reducing subspaces for U, $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_p$ such that $U_{|\mathcal{H}_u} \in \mathcal{L}(\mathcal{H}_u)$ is unitary and $U_{|\mathcal{H}_p} \in \mathcal{L}(\mathcal{H}_p)$ is unitarily equivalent to a unilateral shift. This decomposition is unique and the canonical subspaces are defined by

$$\mathcal{H}_u := \bigcap_{n=1}^{\infty} U^n\mathcal{H} \text{ and } \mathcal{H}_p := \bigoplus_{n=1}^{\infty} U^n E,$$

where $E := N(U^*) = \mathcal{H} \ominus U\mathcal{H}$. The subspace E is called a wandering subspace for U, it is characterized by the property $U^n E \perp U^m E$, for $n \neq m$.

In 1961 Halmos published a paper concerning shift operators on Hilbert spaces (see [9]). The paper introduced the notion of the wandering subspace and its connections with invariant subspaces of unilateral and bilateral shifts. This notion is closely related to the famous Wold decomposition of isometries.

A famous result is the theorem of A. Aleman, S. Richter and C. Sundberg. In the standard Bergman space $L_a^2(\mathbb{D})$ of square area integrable analytic functions on the unit disc, if T is the operator of multiplication by the complex coordinate acting in this space and $\mathcal{M}$ is an arbitrary z-invariant subspace, then

$$\mathcal{M} = [\mathcal{M} \ominus T\mathcal{M}]_T.$$

Later on, S.Shimorin introduced a more general property named the Wold-type decomposition which generalizes the classical Wold decomposition and proved in [24, Theorem 3.6]. S. Shimorin utilized a concept from the paper of Richter from

1988, who showed that an analytic 2-concave operator has the wandering subspace property (see [20]). Shimorin obtain a weak analog of the Wold decomposition theorem, representing operator close to isometry in some sense as a direct sum of a unitary operator and a shift operator acting in some reproducing kernel Hilbert space of vector-valued holomorphic functions defined on a disc. The construction of the Shimorin's model for a left-invertible analytic operator becomes as a powerful tool in the model theory of left-invertible operators.

1.1 Beurling Theorems for Shift Operators

Let $\omega = (\omega(n))_{n\geq 0}$ be a sequence of nonegative numbers and

$$H_\omega = \{x = \sum_{n\geq 0} x_n e_n : \sum_{n\geq 0} |x_n|^2 \omega(n)^2 < \infty\}.$$

be a Hilbert space endowed with some orthonormal basis $(e_n)_{n\geq 0}$. The weighted shift S_ω on H_ω is defined by $S_\omega e_n = \omega(n) e_{n+1}$. We devote this section to some classical results concerning Beurling theorems for weighted shift operators.

1.1.1 The Hardy Space

The hardy space $H^2(\mathbb{D})$ of analytic functions on the unit disc $\mathbb{D}$ is given as the Hilbert space

$$H^2(\mathbb{D}) := \left\{ f(z) = \sum_{n\geq 0} a_n z^n \ : \ \|f\|^2_{H^2(\mathbb{D})} := \sum_{n\geq 0} |a_n|^2 < \infty \right\}.$$

The family $e_n(z) = z^n;\ n \geq 0$ is hence an orthonormal basis of $H^2(\mathbb{D})$.

The unilateral shift operator of $H^2(\mathbb{D})$ is the linear operator defined by $M_z(e_n) = e_{n+1}$. It is clear that $M_z(f) = zf$ for every $f \in H^2(\mathbb{D})$.

The shift operator on a Hilbert space of analytic functions H is defined by this last remark provided that $zH \subset H$. Since its introduction, the shift operator becomes the principal tool in the study of the Hardy spaces (and even in all spaces of analytic functions). It is the perfect bridge between the theory of analytic functions and operator theory. As an example we cite the fact that every cyclic subnormal operator is unitarily equivalent to the shift operator on some Hilbert space of analytic functions.

Let T be a bounded operator on $\mathcal{H}$. Following P. Halmos [12], $\mathcal{E}$ is called *wandering* (for T) if

$$\mathcal{E} \perp T^k(\mathcal{E}),\ \text{for every } k \geq 1.$$

We clearly have $\mathcal{E} = N(T^*)$ is always a wandering subspace for T since $T^k\mathcal{E} \subset R(T) \perp N(T^*)$. Notice that in the case where T is an isometry, then $\mathcal{E}$ is wandering if and only if $T^i\mathcal{E} \perp T^j\mathcal{E}$ for every $j \neq i$.

Also, a vector e is said to be *wandering* for the operator T if the subspace $\mathbb{C}.e$ is a wandering subspace for T, the later is equivalent to $e \perp T^j e$ for every $j \geq 1$.

In [4], Arne Beurling described the lattice of invariant subspaces of M_z in the Hardy space $H^2(\mathbb{D})$.

Theorem 1.1 *(Beurling's Theorem) A closed subspace E is invariant for M_z in the Hardy space $H^2(\mathbb{D})$, if and only if there exists an inner function θ such that $E = \theta H^2(\mathbb{D})$. Moreover, $\mathcal{E} =: E \ominus \mathcal{M}_z E = \mathbb{C}.\theta$ is a one dimensional wandering subspace such that*

$$E = [E \ominus \mathcal{M}_z E]_{\mathcal{M}_z}.$$

1.1.2 The Bergman Space

The standard Bergman space $L_a^2(\mathbb{D})$ of square area integrable analytic functions on the unit disc is given as the Hilbert space

$$L_a^2(\mathbb{D}) = \{f(z) = \sum_{n\geq 0} a_n z^n : \|f\|^2 = \frac{1}{2\pi}\int_{\mathbb{D}} |f(z)|^2 dA(z) = \sum_{n\geq 0} \frac{1}{n+1}|a_n|^2 < \infty.\}$$

Here dA denotes the area Lebesgue measure on the complex plane $\mathbb{C}$.

The family $e_n(z) = (n+1)z^n;\ n \geq 0$ is hence an orthonormal basis of $L_a^2(\mathbb{D})$. For the unilateral shift operator $M_z(z^n) = z^{n+1}$ on the standard Bergman space $L_a^2(\mathbb{D})$, the situation is quite different. A. Aleman, S. Richter and C. Sundberg, showed the next Beurling type theorem

Theorem 1.2 ([2, Theorem 3.5]) *Let E be an invariant subspace of M_z in the Bergman space $L_a^2(\mathbb{D})$, then $E = [E \ominus \mathcal{M}_z E]_{\mathcal{M}_z}$.*

In contrast with the Hardy case, the dimension of wandering subspaces $E \ominus zE$ in Bergman shift M_z ranges from 1 to ∞.

1.1.3 The Direchlet Space

The Dirichlet space $\mathcal{D}(\mathbb{D})$ consists of analytic functions on the unit disc $\mathbb{D}$ is

$$\mathcal{D}(\mathbb{D}) = \{f(z) = \sum_{n\geq 0} a_n z^n : D(f) := \int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty\},$$

here $dA(z) = \frac{1}{\pi} r dr dt$ denotes normalized area measure on $\mathbb{D}$. A norm on $\mathcal{D}(\mathbb{D})$ is defined by

$$\|f\|_{\mathcal{D}}^2 = \|f\|_{H^2(\mathbb{D})}^2 + D(f) = \sum_{n=0}^{\infty} (n+1)|a_n|^2.$$

Endowed with this norm, $\mathcal{D}(\mathbb{D})$ is a Hilbert space in which

$$\{e_n(z) = \frac{1}{n+1} z^n \,:\, n \geq 0\}$$

is a canonical orthonormal basis. The main theorem of Beurling type in the case of Dirichlet space is given by Richter in [20]. In this space, every z-invariant subspace E of $\mathcal{D}(\mathbb{D})$ is generated by an extremal function. More precisely, $E = \phi D(m_\phi)$, where ϕ is a normalized extremal function, m_ϕ is a certain absolutely continuous measure on the unit circle $\mathbb{T}$, and $D(m_\phi)$ is a Dirichlet-type space associated with m_ϕ. Moreover, $\mathcal{E} := E \ominus zE = \mathbb{C}\phi$ is a one dimensional wandering subspace such that

$$E = [E \ominus \mathcal{M}_z E]_{\mathcal{M}_z}.$$

For more information, see for example [20].

1.1.4 More on Beurling's Theorem for Hilbert Spaces of Analytic Functions

In the Hardy space on the bi-disc $H^2(\mathbb{D}^2)$, Beurling theorem fails in general. Indeed, W. Rudin provided two examples showing that none of the equalities in Beurling theorem hold. Recall that a closed subspace $E \subset H^2(\mathbb{D}^2)$ is invariant under the bi-shift $M_{(z_1,z_2)} = (M_{z_1}, M_{z_2})$ if and only if $(z_1E + z_2E) \subset E$. Again $E \ominus (z_1E + z_2E)$ is a wandering subspace.

Example ([21]) The invariant subspace $[z_1 - z_2]_{M_{(z_1,z_2)}}$ is not of the form $\theta H^2(\mathbb{D}^2)$ for any two variable inner function $\theta \in H^2(\mathbb{D}^2)$.

Example ([21]) Let E be the set of all functions $f \in H^2(\mathbb{D}^2)$ which have a zero of order greater than or equal to n at $(1 - n^{-3}, 0)$ for $n = 1, 2, \cdots$. Then E is a not finitely generated invariant subspace of the bi-shift, i.e., there exists no finite set $f_1, f_2, \cdots, f_n \in H^2(\mathbb{D}^2)$ such that $E = [f_1, f_2, \cdots, f_n]_{M_{(z_1,z_2)}}$.

We also have the next result.

Theorem 1.3 ([13, Theorem 3.6]) *There exists a nontrivial function $f \in H^2(\mathbb{D}^2)$ such that $[f]_{M_{(z_1,z_2)}} \ominus (z_1[f]_{M_{(z_1,z_2)}} + z_2[f]_{M_{(z_1,z_2)}})$ does not generate $[f]$. The* *subspace* $\mathcal{E} =: E \ominus \mathcal{M}_z E = \mathbb{C}.\theta$ *is a one dimensional wandering subspace such that*

$$E = [E \ominus \mathcal{M}_z E]_{\mathcal{M}_z}.$$

1.2 Beurling's Type Theorem for Left Invertible Operators Close to Isometries

Motivated by the previous discussion, the next definition has been introduced in several papers.

Definition 1.4 We shall say that an operator $T \in \mathcal{L}(\mathcal{H})$ admits *Wold-type decomposition*, if $R^{\infty}(T)$ is closed and,

(i) $R^{\infty}(T)$ is reducing for T for which the restriction operator $T_{|R^{\infty}(T)}$ is unitary.
(ii) $\mathcal{H} = [\mathcal{H} \ominus T\mathcal{H}]_T \oplus R^{\infty}(T)$.

Definition 1.5 An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have the *Wandering subspace property* if $\mathcal{H} = [\mathcal{H} \ominus T\mathcal{H}]_T$ and we say the *Beurling-type theorem* holds for T if $M = [M \ominus TM]_T$ for every $M \in Lat(T, \mathcal{H})$.

It is clear that if Beurling-type theorem holds for T, it will follow that T admits the Wandering subspace property. Also for a pure operator, T has Wandering space property if and only if it admits Wold decomposition. From the preceding remarks the Hardy shift, the Bergman shift and the Dirichlet shift satisfy the Beurling-type theorem. We discuss below the contributions of several authors that have been interested in the class of operators satisfying the Beurling-type theorem. The problem of describing all weighted shifts that satisfy Beurling-type theorem remains open.

The case of left invertible operators has been widely studied in the two last decades. It is always assumed that T satisfies some operator inequalities close to isometries. A pioneer result goes to S. Richter (see [20]), that provides a sufficient condition on an operator $S \in \mathcal{L}(\mathcal{H})$ to admit the Wandering subspace property. More precisely,

Theorem 1.6 ([20, Theorem 1]) *Let* $S \in \mathcal{L}(\mathcal{H})$ *be pure such that*

$$\|S^2x\| + \|x\|^2 \leq 2\|Sx\|^2; \quad \textit{for every } x \in \mathcal{H}.$$

If $\mathcal{M} \in Lat(S, \mathcal{H})$*, then there exists a wandering subspace* $\mathcal{B}$ *for* S *such that*

$$\mathcal{M} = \bigvee_{n \geq 0} S^n \mathcal{B}.$$

In particular, Richter's result states that the Dirichlet shift satisfies the Wandering subspace property.

Later in [24] S. Shimorin gave a different approach, to prove the following theorem.

Theorem 1.7 ([24]) *Let $T \in \mathcal{L}(\mathcal{H})$ be pure such that;*

$$\|Tx + y\|^2 \leq 2(\|x\|^2 + \|Ty\|^2), \tag{1}$$

for every $x, y \in \mathcal{H}$. Then T satisfies the Wandering subspace property.

This result extends and provides a simpler proof of Aleman-Richter-Sundberg theorem to more general weighted spaces of analytic functions. On the other hand, A. Olofsson in [19] extended Richter's theorem in the following way,

Theorem 1.8 ([19, Theorem 2.1]) *Let $T \in \mathcal{L}(\mathcal{H})$ be pure such that;*

(i) T is expansive.
(ii) There exists some positive constants c_k, c with $\sum_{k\geq 2} \frac{1}{c_k} = \infty$ such that

$$\|T^k x\|^2 \leq c_k(\|Tx\|^2 - \|x\|^2) + c\|x\|^2. \tag{2}$$

for every $x \in \mathcal{H}$. Then T satisfies the Wandering subspace property.

Also, O. Olofsson gave a more precise relation between these conditions. His result can be stated as follows,

Proposition 1.9 ([19, Proposition 1.2]) *Let $T \in \mathcal{L}(\mathcal{H})$ be left invertible, $T' = T(T^*T)^{-1}$ and c be a nonnegative constant. Then the following two statements are equivalent:*

(i) $\|T'^2 x\|^2 - \|T'x\|^2 \leq c(\|T'x\|^2 - \|x\|^2)$ *for every $x \in \mathcal{H}$.*
(ii) $\|Q(Tx + y)\|^2 \leq (1 + \frac{1}{c})(\|x\|^2 + c\|Ty\|^2)$ *for every $x, y \in \mathcal{H}$*

where Q is the orthogonal projection of $\mathcal{H}$ onto $R(T)$.

Moreover, from [19, Corollary 2.1], if $T \in \mathcal{L}(\mathcal{H})$ is an expansive operator which satisfies Inequality (2) for $c = 1$, then $\mathcal{H} = [\mathcal{H} \ominus T\mathcal{H}]_T \oplus R^\infty(T)$ and the restriction $T_{|R^\infty(T)}$ is unitary, that is, T admits Wold-type decomposition.

Remark 1.10 We notice at this level that, in the previous theorem, it is necessary that $c \geq 1$. To see this, observe first that $\inf\limits_{\|x\|=1} \|Tx\| = 1$. Otherwise, there exists $r > 1$ such that $\|Tx\| \geq r\|x\|$ for every $x \neq 0$. It will follow by induction, that $r^{2k} - c \leq c_k(\|T\|^2 - 1)$ and then that $\sum_{k\geq 2} \frac{1}{c_k}$ is finite. Now taking the infinimum,

$$1 = \|x\|^2 \leq \|T^k x\|^2 \leq c_k(\|Tx\|^2 - 1) + c,$$

we will get $1 \leq c$.

In 2009, S. Sun and D. Zheng (see [25]) gave another proof of the Beurling-type theorem by proving some new identities in the Bergman space and later, K. J. Izuchi, K. H. Izuchi and Y. I. Izuchi used these ideas in [14] to prove the next theorem.

Theorem 1.11 ([14, Theorem 1.1]) *Let $T \in \mathcal{L}(\mathcal{H})$. If T satisfies the following conditions:*

(i) $\|Tx\|^2 + \|T^{*2}Tx\|^2 \leq 2\|T^*Tx\|^2$ *for all* $x \in \mathcal{H}$;
(ii) T *is bounded below;*
(iii) $\|T\| \leq 1$;
(iv) $\|T^{*k}x\| \longrightarrow 0$ *as* $k \longrightarrow \infty$ *for every* $x \in \mathcal{H}$.

Then $\mathcal{H} = [\mathcal{H} \ominus T\mathcal{H}]_T$

The main purpose of this survey is to present the abstract approach to the problem. We extend the previous results to the more general class of regular operators introduced by M. Mbekhta in [17] and developed in [8].

We devote Sect. 2 to some well known properties of regular operators and the basic tools of this class of operators. Section 3 is focused on the generalization of the previous results to the class of regular operators. More precisely, we give under the same conditions on orbits, an extension of Wold-type decomposition for regular operators. See Theorem 3.9).

Section 4 is devoted to the duality between a bi-regular operator T and its Cauchy dual $\omega(T)$. This duality is reflected in terms of extended Wold-type decomposition. Some applications and examples are widely given. In particular, we apply our results to regular bilateral weighted shifts.

2 Regular Operators

2.1 Moore-Penrose Generalized Inverse

An operator $S \in \mathcal{L}(\mathcal{H})$ is a *generalized inverse* of T if $TST = T$ and $STS = S$. It is not difficult to see that an operator admits a generalized inverse if and only if it has a closed range. We will focus in this survey on a particular generalized inverse for $T \in \mathcal{L}(\mathcal{H})$ with closed range. More precisely, a standard generalized inverse for T can be built as follows. We consider the operator $T_0 = T_{|N(T)^\perp} : N(T)^\perp \longrightarrow R(T)$ that is clearly bijective. Define $T^\dagger$ by

$$\begin{cases} T^\dagger x = T_0^{-1}x \text{ } if \text{ } x \in R(T) \\ T^\dagger x = 0 \qquad if \text{ } x \in R(T)^\perp \end{cases}.$$

We have $T^\dagger = T_0^{-1}P_{R(T)}$, where P_E denotes the orthogonal projection on a given subspace E. It is easy to see that, $TT^\dagger T = T$ and $T^\dagger TT^\dagger = T^\dagger$, and thus $T^\dagger$ is a generalized inverse of T satisfying $TT^\dagger$ and $T^\dagger T$ are orthogonal projections. The operator $T^\dagger$ is called the *Moore-Penrose inverse* of T, and has been widely studied in the literature. It is usually defined as the unique solution of the following four operator equations:

$$TT^\dagger T = T, \quad T^\dagger T T^\dagger = T^\dagger, \quad (TT^\dagger)^* = TT^\dagger, \quad (T^\dagger T)^* = T^\dagger T. \tag{3}$$

Among all generalized inverses, the Moore-Penrose inverse $T^\dagger$ received special attention by several authors. In the case of left invertible operators $T^\dagger$ coincides with the standard left inverse $(T^*T)^{-1}T^*$ and in the case where T is right invertible $T^\dagger = T^*(TT^*)^{-1}$. The next well known result links the minimum modulus and the Moore-Penrose inverse, it can be found in [18, Corollary 2.3]. For $T \in \mathcal{L}(\mathcal{H})$ with closed range, we have

$$\|T^\dagger\| = \frac{1}{\gamma(T)}.$$

We summarize in the proposition below some further properties of the Moore-Penrose inverse of T which will be used in the sequel.

Proposition 2.1 *Let $T \in \mathcal{L}(\mathcal{H})$ be with closed range. We have*

(a) $TT^\dagger = P_{R(T)}$, $T^\dagger T = P_{N(T)^\perp}$,
(b) $R(T^\dagger) = R(T^*) = N(T)^\perp$,
(c) $N(T^\dagger) = N(TT^\dagger) = N(T^*) = R(T)^\perp$,
(d) $R(T) = R(TT^\dagger) = R(T^{\dagger *})$,
(e) $N(T) = N(T^\dagger T) = N(T^{\dagger *})$,
(f) $(T^*)^\dagger = (T^\dagger)^*$,
(g) $(T^\dagger)^\dagger = T$,
(h) $T^*TT^\dagger = T^\dagger TT^* = T^*$.

2.2 Some Basic Properties of Regular Operators

Regular operators have been introduced as a natural family of operators close to semi invertible ones, where an operator is said to be *semi invertible* if it is left or right invertible. Since then, they have been widely studied, see [15] for example. We recall the definition of regular operators from [16],

Definition 2.2 An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *regular* if $R(T)$ is closed and if $N(T^k) \subset R(T)$, for every $k \geq 1$.

The regular resolvent, denoted $reg(T)$, is defined as the set of complex numbers λ for which, there is a neighborhood U_λ of λ and an analytic function $R_\lambda : U_\lambda \longrightarrow \mathcal{L}(\mathcal{H})$ such that $R_\lambda(\mu)$ is a generalized inverse of $T - \mu I$ for every $\mu \in U_\lambda$. If $0 \in reg(T)$, the operator T is said to be *Kato invertible*, see [16]. The *generalized spectrum* of T is defined as $\sigma_g(T) = \mathbb{C} \setminus reg(T)$.

It is clear that T is regular if and only if T^* is regular, that all injective operators with closed range and that all surjective operators are regular. We give next some classical known facts on regular operators. We refer to the corresponding books and papers for proofs and further information.

Following Saphar [22], the algebraic core $C(T)$ of T, is the greatest subspace M of $\mathcal{H}$ for which $T(M) = M$. In terms of sequences, we have

$$C(T) = \{x \in \mathcal{H} : \exists (u_n)_n \subset \mathcal{H} \text{ such that } x = u_0 \text{ and } Tu_{n+1} = u_n\}.$$

Proposition 2.3 ([7, Proposition 1]) *Let $T \in \mathcal{L}(\mathcal{H})$ be a regular operator. We have*

$$R^\infty(T) = T(R^\infty(T)) = C(T).$$

In particular, if T is regular, then $x \in R^\infty(T)$ if and only if $Tx \in R^\infty(T)$.

In the next proposition, we summarize some properties of the generalized range $R^\infty(T)$ in the case of regular operators.

Proposition 2.4 *Let $T \in \mathcal{L}(\mathcal{H})$ be a regular operator. e have*

(i) $R^\infty(T)$ is closed,

(ii) If $R^\infty(T) = \{0\}$, then T is left invertible,

For further information, We refer to [1], [3] and [15].

The next proposition is given in [3] and will be useful in the sequel,

Proposition 2.5 ([7, Proposition 3]) *Let $T \in \mathcal{L}(\mathcal{H})$ be regular. If S is such that $TST = T$, then*

$$T^n S^n T^n = T^n \ \textit{for every } n \geq 1.$$

In particular, if S is also regular, then S^n is a generalized inverse of T^n for every $n \geq 1$; that is

$$T^n S^n T^n = T^n \textit{ and } S^n T^n S^n = S^n \textit{ for every } n \geq 1.$$

Generalized inverses own various interesting properties. For example, if S is a generalized inverse of an operator T, then $TSx = x$ for every $x \in R(T)$. The next description of the generalized range for regular operators is useful for the proof of our main results.

Proposition 2.6 *Let $T \in \mathcal{L}(\mathcal{H})$ be a regular operator with a generalized inverse S. Then*

$$R^\infty(T) = \left\{x \in \mathcal{H} : T^n S^n x = x \textit{ for every } n \geq 0\right\}.$$

The next proposition from [3] summarizes additional properties of regular operators. See also [8].

Proposition 2.7 *Let $T \in \mathcal{L}(\mathcal{H})$ be regular and S be a generalized inverse of T. We have the following.*

(i) $T\overline{(N^\infty(T))} = \overline{N^\infty(T)}$;
(ii) $S(\overline{R^\infty(T)}) \subseteq \overline{R^\infty(T)}$;
(ii) $S(\overline{N^\infty(T)}) \subseteq \overline{N^\infty(T)}$;
(iv) $R^\infty(T)^\perp = \overline{N^\infty(T^*)}$.

We use generalized inverse to provide necessary and sufficient conditions for an operator to be regular.

Proposition 2.8 *Let $T \in \mathcal{L}(\mathcal{H})$ be with closed range and S be a generalized inverse of T. The following are equivalent*

(i) T *is regular;*
(ii) $S^k N(T) \subseteq R^\infty(T)$ *for every* $k \geq 0$;
(iii) $S^k N(T) \subseteq R(T)$ *for every* $k \geq 0$.

Proof $(i) \Rightarrow (ii)$. Since T is regular, the result follows immediately by induction from (ii) in the previous proposition.

$(ii) \Rightarrow (iii)$. Obvious.

$(iii) \Rightarrow (i)$. Since $R(T)$ is assumed to be closed, it remains to show that $N(T^n) \subseteq R(T)$ for every $n \geq 1$. So, suppose that $S^k N(T) \subseteq R(T)$ for every $k \geq 0$. For $n \geq 1$ and $x \in N(T^n)$, as in [8, Lemma 2], we have

$$\begin{aligned} x &= x - S^n T^n x \\ &= \textstyle\sum_{k=0}^{n-1} S^k P_{N(T)} T^k x. \end{aligned}$$

Since $P_{N(T)} T^k x \in N(T)$ and by our assumption $S^k N(T) \subseteq R(T)$ for every $k \geq 0$, we obtain $x \in R(T)$ and hence $N(T^n) \subseteq R(T)$. □

2.3 Restrictions of Regular Operators

The restriction of a regular operator to some invariant subspace do not need to be regular (one can take the kernel to be convinced). We consider in the following proposition, the restriction operator of T to $R^\infty(T)$. We have

Proposition 2.9 ([7, Proposition 6]) *Let $T \in \mathcal{L}(\mathcal{H})$ be a regular operator and $n \geq 1$. The restriction operator*

$$T_n = T_{|R^\infty(T) \cap R(T^{\dagger n})} : R^\infty(T) \cap R(T^{\dagger n}) \to R^\infty(T)$$

is bijective. In particular,

$$T_1 = T_{|R^\infty(T)\cap N(T)^\perp} : R^\infty(T) \cap N(T)^\perp \to R^\infty(T)$$

is bijective and hence

$$R^\infty(T) \cap R(T^{\dagger n}) = R^\infty(T) \cap N(T)^\perp \quad \textit{for every } n \geq 1.$$

Proof Since T is regular, it follows that $T(R^\infty(T)) = R^\infty(T)$ and the inclusion $T(R^\infty(T) \cap R(T^{\dagger n})) \subset T(R^\infty(T)) = R^\infty(T))$ derives immediately.

Let $x \in R^\infty(T)$. From Proposition 2.6, we have $x = T^n T^{\dagger n} x$. Since $R^\infty(T)$ is invariant for $T^\dagger$, we obtain $T^{\dagger n} x \in R^\infty(T) \cap R(T^{\dagger n})$. Finally, $x = T^n T^{\dagger n} x \in T(R^\infty(T) \cap R(T^{\dagger n}))$ and thus $T(R^\infty(T) \cap R(T^{\dagger n})) = R^\infty(T)$. Which leads to $T_{|R^\infty(T)\cap R(T^{\dagger n})}$ is onto. To show that $T_{|R^\infty(T)\cap R(T^{\dagger n})}$ is one to one, let $x \in R^\infty(T) \cap R(T^{\dagger n})$ be such that $T_{|R^\infty(T)\cap R(T^{\dagger n})} x = 0$, we have $x \in R^\infty(T) \cap R(T^{\dagger n}) \cap N(T)$. Since $R(T^{\dagger n}) \cap N(T) \subset R(T^\dagger) \cap N(T) = N(T)^\perp \cap N(T) = \{0\}$, we have $x = 0$ and so $T_{|R^\infty(T)\cap R(T^{\dagger n})}$ is one to one. Therefore, $T_{|R^\infty(T)\cap R(T^{\dagger n})}$ is bijective.

We derive the second affirmation for $n = 1$ because of $N(T)^\perp = R(T^\dagger)$ and the third one by taking the inverse of T_1. □

Theorem 2.10 ([7, Theorem 4]) *Let $T \in \mathcal{L}(\mathcal{H})$ be a regular operator. Then*

$$(T_{|R^\infty(T)})^\dagger = T^\dagger_{|R^\infty(T)}.$$

We also have the next characterization of regular operators.

Theorem 2.11 *[7, Theorem 5] Let $T \in \mathcal{L}(\mathcal{H})$ be with closed range. The following conditions are equivalent*

(i) *T is regular.*

(ii) *The map $\widehat{T} : R(T)^\perp \to R(T^n) \cap R(T^{n+1})^\perp$ defined by:*

$$\widehat{T}x = P_{R(T^n)\cap R(T^{n+1})^\perp} T^n x, \qquad \textit{for every } n \geq 0$$

is one to one.

(iii) $R(T)^\perp \underset{(bijection)}{\simeq} R(T^n) \cap R(T^{n+1})^\perp$ *for every $n \geq 0$.*

Corollary 2.12 *For $T \in \mathcal{L}(\mathcal{H})$ regular and for every integer $n \geq 1$, we have*

$$\dim R(T^n) \cap R(T^{n+1})^\perp = \dim R(T)^\perp.$$

We also have

Corollary 2.13 *Let* $T \in \mathcal{L}(\mathcal{H})$ *be regular such that* $\dim R(T)^{\perp} = 1$. *Then there is a sequence of orthogonal wandering vectors* $\{e_k\}_{k\geq 1}$ *such that*

$$\mathcal{H} = \bigoplus_{k\geq 1} \mathbb{C}.e_k \oplus R^{\infty}(T).$$

Proof By corollary 2.12, $\dim R(T^n) \cap R(T^{n+1})^{\perp} = 1$ for every $n \geq 0$. Let e_n be a nonzero vector in $R(T^{n-1}) \cap R(T^n)^{\perp}$ for $n \geq 1$. Since

$$\mathcal{H} = ((\mathcal{H} \cap R(T)^{\perp}) \oplus (R(T) \cap R(T^2)^{\perp}) \oplus (R(T^2) \cap R(T^3)^{\perp}) \oplus \cdots) \oplus R^{\infty}(T),$$

the relation $\mathcal{H} = \mathbb{C}.e_1 \oplus \mathbb{C}.e_2 \oplus \mathbb{C}.e_3 \cdots \oplus R^{\infty}(T)$ follows. But we know that $T^j e_n \in R(T^n)\ \forall j \geq 1$, thus, we have $e_n \perp T^j e_n\ \forall n, j \geq 1$, and so e_n is a wandering vector for T for all n. □

3 Wold Type Decomposition for Regular Operators

In the sequel, consider $T \in \mathcal{L}(\mathcal{H})$ such that $\gamma(T) \geq 1$. Fix the next notations

$$\mathcal{E} = \mathcal{H} \ominus T\mathcal{H} = N(T^*) \text{ and } \mathcal{E}^{\dagger} = \mathcal{H} \ominus T^{\dagger}\mathcal{H} = N(T^{\dagger *}) = N(T).$$

It follows from the identity $\|T^{\dagger}\| = \frac{1}{\gamma(T)}$ that $T^{\dagger}$ is contractive and then that $\|T^{\dagger}Tx\|^2 \leq \|Tx\|^2$ for every $x \in \mathcal{H}$. Since $T^{\dagger}T$ is an orthogonal projection, we conclude that $T^*T - T^{\dagger}T \geq 0$. We denote by D_T the operator given by

$$D_T = (T^*T - T^{\dagger}T)^{1/2}.$$

Clearly $\|D_T x\|^2 = \|Tx\|^2 - \|T^{\dagger}Tx\|^2$ for all $x \in \mathcal{H}$.

3.1 The Main Results

We start with the following proposition involving D_T.

Proposition 3.1 ([7, Proposition 7]) *Let* $T \in \mathcal{L}(\mathcal{H})$ *be an operator with* $\gamma(T) \geq 1$. *Let* $c > 0$ *and* $(c_k)_k$ $(k \geq 2)$ *be some positive sequence. The following are equivalent:*

(i)

$$\|T^k x\|^2 \leq c_k(\|D_T x\|^2) + c\|T^{\dagger}Tx\|^2; \quad \textit{for every } x \in \mathcal{H}, \tag{4}$$

(ii)

$$\|T^k x\|^2 \leq c_k(\|Tx\|^2 - \|x\|^2) + c\|x\|^2; \quad \textit{for every } x \in N(T)^{\perp}. \tag{5}$$

Proof We notice first that since $\gamma(T) > 0$, the operator $T^\dagger$ exists. Suppose that the inequality (4) holds and let $x \in N(T)^\perp$. Since $N(T)^\perp = R(T^\dagger)$, we get $T^\dagger T x = P_{R(T^\dagger)}x = x$ and so we have the result. Conversely, let $x \in \mathcal{H}$ $T^\dagger T x \in N(T)^\perp$. By substituting $T^\dagger T x \in N(T)^\perp$ for x in (5) and by using the identity $TT^\dagger T = T$ we obtain (4). □

In the case of expansive operators, these inequalities are equivalent to Inequality (2) introduced in [19].

We extend next some known results of left invertible operators to our setting.

Lemma 3.2 ([7, Lemma 1]) *Let $T \in \mathcal{L}(\mathcal{H})$ be such that $\gamma(T) \geq 1$ and let $n \geq 1$ be an integer. For every $x \in \mathcal{H}$, we have*

$$\|x\|^2 = \sum_{i=0}^{n-1} \|P_{\mathcal{E}}(T^\dagger)^i x\|^2 + \|(T^\dagger)^n x\|^2 + \sum_{i=1}^{n} \|D_T(T^\dagger)^i x\|^2 , \tag{6}$$

where $P_{\mathcal{E}} = I - TT^\dagger$.

Lemma 3.3 *Let $T \in \mathcal{L}(\mathcal{H})$ be with closed range and $n \geq 1$. We have,*

(i) $x - T^n(T^\dagger)^n x = \sum_{i=0}^{n-1} T^i P_{\mathcal{E}}(T^\dagger)^i x$ *for every* $x \in \mathcal{H}$;
(ii) $x - (T^\dagger)^n T^n x = \sum_{i=0}^{n-1} (T^\dagger)^i P_{\mathcal{E}^\dagger} T^i x$ *for every* $x \in \mathcal{H}$.

Proof We have:

$$\begin{aligned} I - T^n(T^\dagger)^n &= \sum_{i=0}^{n-1} T^i(T^\dagger)^i - T^{i+1}(T^\dagger)^{i+1} \\ &= \sum_{i=0}^{n-1} T^i(I - TT^\dagger)(T^\dagger)^i \\ &= \sum_{i=0}^{n-1} T^i P_{\mathcal{E}}(T^\dagger)^i. \end{aligned}$$

The second equality is obtained in a similar way. □

As a consequence we have the following useful result.

Proposition 3.4 ([7, Proposition 8]) *Let $T \in \mathcal{L}(\mathcal{H})$ have closed range and $n \geq 1$ be an integer. We have:*

(i) $N((T^\dagger)^n) \subset \bigvee_{i=0}^{n-1} \left\{T^i x_i \ , x_i \in \mathcal{E}\right\}.$

If moreover T is regular, we get

(ii) $N(T^n) = \bigvee_{i=0}^{n-1} \left\{(T^\dagger)^i x_i \ , x_i \in \mathcal{E}^\dagger\right\}$,
(ii) $N((T^*)^n) = \bigvee_{i=0}^{n-1} \left\{(T^{\dagger *})^i x_i \ , x_i \in \mathcal{E}\right\}$.

For an arbitrary operator, using the equalities:

$$\bigvee_{n\geq 0} N(T^n) = \bigvee_{n\geq 0} (R(T^*)^n)^\perp = \left\{\bigcap_{n\geq 0} R((T^*)^n)\right\}^\perp = R^\infty(T^*)^\perp,$$

we get the following duality formulas.

Corollary 3.5 *Let $T \in \mathcal{L}(\mathcal{H})$ be with closed range, then*

(i) $R^\infty(T^{\dagger *})^\perp \subset [\mathcal{E}]_T$.

If moreover T is regular, we have

(ii) $R^\infty(T^*)^\perp = [\mathcal{E}^\dagger]_{T^\dagger}$,
(iii) $R^\infty(T)^\perp = [\mathcal{E}]_{T^{\dagger *}}$.

To give further information for regular operators, we need the next lemma,

Lemma 3.6 *Let $T \in \mathcal{L}(\mathcal{H})$. If $R^\infty(T)$ reduces T, then*

$$[\mathcal{E}]_T \subset R^\infty(T)^\perp.$$

Proof We have $\mathcal{E} = R(T)^\perp \subset R^\infty(T)^\perp$. For $n \geq 1$, $x \in \mathcal{E}$ and $y \in R^\infty(T)$ we have $\langle T^n x, y\rangle = \langle x, T^{*n} y\rangle = 0$ because $T^{*n} y \in R^\infty(T)$. Finally $[\mathcal{E}]_T \subset R^\infty(T)^\perp$. □

We have

Theorem 3.7 ([7, Theorem 6]) *Let $T \in \mathcal{L}(\mathcal{H})$ be regular. If $R^\infty(T)$ reduces T, then $T^\dagger$ is regular.*

Theorem 3.8 ([7, Theorem 7]) *Let $T \in \mathcal{L}(\mathcal{H})$ be a regular operator with $\gamma(T) \geq 1$ and such that*

$$\|T^k x\|^2 \leq c_k(\|Tx\|^2 - \|T^\dagger T x\|^2) + \|T^\dagger T x\|^2 \text{ for every } x \in \mathcal{H} \tag{7}$$

with $\sum_{k\geq 2} \frac{1}{c_k} = \infty$. Then

$$\mathcal{H} = [\mathcal{E}]_T + R^\infty(T)$$

with $\mathcal{E} = \mathcal{H} \ominus T\mathcal{H}$.

We will show next that under the assumptions of Theorem 3.8, we have $\mathcal{H} = [\mathcal{E}]_T \oplus R^\infty(T)$. We collect next some additional results provided by the same assumptions.

Theorem 3.9 ([7, Theorem 8]) *Let $T \in \mathcal{L}(\mathcal{H})$ be a regular operator with $\gamma(T) \geq 1$. Under the assumptions of Theorem 3.8, the following assertions hold.*

(i) *The subspace $R^\infty(T)$ is reducing for T,*
(ii) $(T_{|R^\infty(T)})^\dagger = T^\dagger_{|R^\infty(T)} = T^*_{|R^\infty(T)} = (T_{|R^\infty(T)})^*$,
(iii) *$T^\dagger$ is regular,*
(iv) *the restriction*

$$T_{|R^\infty(T)\cap N(T)^\perp} : R^\infty(T) \cap N(T)^\perp \to R^\infty(T)$$

is a unitary operator,
(v) *$\mathcal{H}$ has an orthogonal decomposition*

$$\mathcal{H} = [\mathcal{E}]_T \oplus R^\infty(T).$$

The next duality formulas are immediate

Corollary 3.10 ([7, Corollary 7]) *Let $T \in \mathcal{L}(\mathcal{H})$. Under the assumptions of Theorem 3.8, we have*

(i) $[\mathcal{E}]_T = [\mathcal{E}]_{T^{\dagger *}}$ *and* $[\mathcal{E}^\dagger]_{T^\dagger} = [\mathcal{E}^\dagger]_{T^*}$*;*
(ii) $R^\infty(T^{\dagger *}) = R^\infty(T)$ *and* $R^\infty(T^*) = R^\infty(T^\dagger)$;
(iii) *$\mathcal{H}$ has an orthogonal decomposition*

$$\mathcal{H} = [\mathcal{E}^\dagger]_{T^\dagger} \oplus R^\infty(T^\dagger).$$

We derive that:

Corollary 3.11 ([7, Corollary 8]) *Let $T \in \mathcal{L}(\mathcal{H})$. Under the assumptions of Theorem 3.8, we have*

$$T^{\dagger n} = T^{n\dagger} \text{ on } R^\infty(T) \cap R^\infty(T^*).$$

The case of expansive operators is a particular case of $\gamma(T) \geq 1$, and in this case $T^\dagger$ is a left inverse of T, thus $T^\dagger T x = x, \forall x \in \mathcal{H}$. We retrieve Theorem 1.8 from [19, Theorem 2.1].

Moreover, we have $R^\infty(T) \cap N(T)^\perp = R^\infty(T)$, and using Theorem 3.9 we deduce the following corollary from [19, Corollary 2.1].

Corollary 3.12 ([7, Corollary 9]) *Let $T \in \mathcal{L}(\mathcal{H})$ be an expansive operator such that the inequality (4) holds for $x \in \mathcal{H}$ with $c = 1$ and $\sum_{k\geq 2} \frac{1}{c_k} = \infty$. Then, the space $\mathcal{H}$ has an orthogonal sum decomposition*

$$\mathcal{H} = [\mathcal{E}]_T \oplus R^\infty(T),$$

and the restriction

$$T_{|R^{\infty}(T)} : R^{\infty}(T) \to R^{\infty}(T)$$

is a unitary operator. That is T admits Wold-type decomposition.

3.2 Applications to Weighted Shifts

We apply our results to non necessarily left invertible weighted shift operators. To this purpose, we assume that $\mathcal{H}$ is a Hilbert space and $(e_n)_{n\in\mathbb{Z}}$ is an orthonormal basis of $\mathcal{H}$.

Let $(\omega_n)_{n\in\mathbb{Z}}$ be a bounded sequence and $S_\omega : \mathcal{H} \longrightarrow \mathcal{H}$ be the bilateral weighted bounded shift defined by $S_\omega e_n = \omega_n e_{n+1}$. It well known that S_ω is one to one if and only if $\omega_n \neq 0$ for every n. Indeed, let $x \in \mathcal{H}$ such that $x = \sum_{n\in\mathbb{Z}} x_n e_n$. We have

$$\|Tx\|^2 = \sum_{n\in\mathbb{Z}} |x_n|^2|\omega_n|^2 \geq \inf_{n\in\mathbb{Z}} |\omega_n|^2\|x\|^2.$$

Thus, S_ω is left invertible.

Regular bilateral weighted shifts operators S_ω that are not left invertible are easily characterized as those shifts such that there exists a unique $n_0 \in \mathbb{Z}$ such that $\omega_{n_0} = 0$. In this case, S_ω is the direct sum of a unilateral weighted shift (on the subspace spanned by $(e_n)_{n>0}$) and the adjoint of a unilateral weighted shift (on the orthogonal complement).

It is well known [23, Corollary 1] that S_ω is unitarily equivalent to the weighted shift operator with weight sequence $(|\omega_n|)_{n\in\mathbb{Z}}$. So, we can assume that $\omega_n \geq 0$ for every $n \in \mathbb{Z}$.

For more information and additional about weighted shifts operators see [23].

For simplicity we assume that $\omega_0 = 0$ and $\omega_n \neq 0$ for every $n \neq 0$.

Proposition 3.13 ([7, Proposition 9]) *Let $(\omega_k)_{k\in\mathbb{Z}}$ be a bounded sequence such that*

(i) $\omega_k = 1$ *for* $k < 0$, $\omega_0 = 0$ *and* $\omega_k \geq 1$ *for* $k \geq 1$*;*
(ii) $\omega_k^2.\omega_{k+1}^2 \cdots \omega_{k+n-1}^2 - 1 \leq c_n(\omega_k^2 - 1) \quad \forall k \geq 1; \quad n \geq 2$, *where* $(c_n)_n$ *is a positive sequence such that* $\sum_{n\geq 2} \frac{1}{c_n} = \infty$.

Then

(a) S_ω *satisfies the assumptions of Theorem 3.9. In particular, the inequality (4) holds for* S_ω *with* $c = 1$ *and for the sequence* $(c_n)_n$*,*
(b) $S_\omega^\dagger$ *is a regular contraction;*
(c) the subspace $\bigvee_{j\leq 0}\{e_j\}$ *is reducing for* S_ω *and the restriction of* S_ω *on* $\bigvee_{j<0}\{e_j\}$ *is unitary.*

Proof Notice first that since $\omega_k^2.\omega_{k+1}^2 \cdots \omega_{k+n-1}^2$ equals 1 or 0 for all $k < 0$ and $n \geq 2$, we get

$$\omega_k^2.\omega_{k+1}^2 \cdots \omega_{k+n-1}^2 - 1 \leq c_n(\omega_k^2 - 1) \quad \forall k \in \mathbb{Z}^*;\ n \geq 2. \tag{8}$$

Now, Let $x \in \mathcal{H}$ such that $x = \sum_{k\in\mathbb{Z}} a_k e_k$. Clearly, we have

$$\|S_\omega^n x\|^2 = \sum_{k\neq 0} |a_k|^2 \omega_k^2 \cdots \omega_{k+n-1}^2, \quad n \geq 1$$

and

$$\|S_\omega^\dagger S_\omega x\|^2 = \sum_{k\neq 0} |a_k|^2.$$

(a) Since $\|S_\omega^\dagger x\|^2 = \sum_{k\neq 1} \frac{1}{\omega_{k-1}^2} |a_k|^2$ and $\omega_k \geq 1$ for all integers $k \neq 0$, we conclude that $S_\omega^\dagger$ is a contraction (and so $\gamma(S_\omega) \geq 1$). Now, from Inequality (8), we get

$$\|S_\omega^n x\|^2 \leq c_n(\|S_\omega x\|^2 - \|S_\omega^\dagger S_\omega x\|^2) + c\|S_\omega^\dagger S_\omega x\|^2$$

where $c = 1$.

(b) From Theorem 3.9, we conclude that $S_\omega^\dagger$ is a regular operator.

(c) By Theorem 3.9, the subspace $R^\infty(S_\omega)$ is reducing for S_ω and the restriction of S_ω on $R^\infty(S_\omega) \cap N(S_\omega)^\perp$ is a unitary operator. On the other hand, clearly we have $R^\infty(S_\omega) = \bigvee_{j\leq 0} \{e_j\}$ and since $N(S_\omega)^\perp = \bigvee_{k\neq 0}\{e_k\}$, we get $R^\infty(S_\omega) \cap N(S_\omega)^\perp = \bigvee_{j<0} \{e_j\}$ and so $S_{\omega | \bigvee_{j<0}\{e_j\}}$ is unitary.

□

Remark 3.14 If we assume that $\omega_k = 1$ for all $k \neq 0$ and $\omega_0 = 0$, then we have $S_\omega^\dagger = S_\omega^*$, in this case; S_ω is called a partial isometry.

Example ([7]) Let $\mathcal{H}$ be a Hilbert space and $(e_n)_{n\in\mathbb{Z}}$ an orthonormal basis of $\mathcal{H}$. Let $S_\omega : \mathcal{H} \longrightarrow \mathcal{H}$ be a bilateral shift defined by $S_\omega(e_n) = \omega_n e_{n+1}$ where $(\omega_k)_{k\in\mathbb{Z}}$ is defined by:

$$\omega_k = \begin{cases} 1 & \text{for} k < 0, \\ 0 & \text{for } k = 0, \\ \sqrt{\frac{k+1}{k}} & \text{for } k \geq 1. \end{cases}$$

The sequence $(\omega_k)_{k\in\mathbb{Z}}$ is bounded. More precisely, we have $1 \leq \omega_k \leq \sqrt{2}$ for all $k \neq 0$. Take $c_n = n$. Clearly, we have

$$\sum_{n\geq 2} \frac{1}{c_n} = \sum_{n\geq 2} \frac{1}{n} = \infty.$$

For $k < 0$, we have $c_n(\omega_k^2 - 1) = 0$ and a simple computation shows that $\omega_k^2.\omega_{k+1}^2 \cdots \omega_{k+n-1}^2 - 1 \leq 0$. Thus

$$\omega_k^2.\omega_{k+1}^2 \cdots \omega_{k+n-1}^2 - 1 \leq c_n(\omega_k^2 - 1).$$

For $k \geq 1$, we have

$$\omega_k^2.\omega_{k+1}^2 \cdots \omega_{k+n-1}^2 - 1 = \frac{k+1}{k}.\frac{k+2}{k+1} \cdots \frac{k+n}{k+n-1} - 1 = \frac{n}{k}$$

and since $c_n(\omega_k^2 - 1) = n(\frac{k+1}{k} - 1) = \frac{n}{k}$, we conclude that

$$\omega_k^2.\omega_{k+1}^2 \cdots \omega_{k+n-1}^2 - 1 \leq c_n(\omega_k^2 - 1).$$

Thus, the inequality (8) is satisfied. Consequently, the operator S_ω satisfies all properties of Proposition 3.13.

4 Wold-Type Decomposition for Bi-Regular Operators

4.1 First Properties

It is not known whether if the Moore-Penrose inverse of a regular operator remains regular. This fact motivates the introduction of the class of bi-regular operators.

Definition 4.1 An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *bi-regular* if both T and $T^\dagger$ are regular.

We list below some examples of bi-regular operators.

Examples

(i) Left invertible operators and right invertible operators are bi-regular.

1. Regular partial isometries are bi-regular. In fact, a partial isometry satisfies $T^*TT^* = T^*$ and $TT^*T = T$, and it follows that $T^\dagger = T^*$ is regular.

(ii) Next, we provide an example of a bi-regular operator that is neither left invertible nor right invertible and that is not a partial isometry. Let $\mathcal{H}$ be a Hilbert space with an orthonormal basis $e_{i,j}$ where $(i, j) \in \mathbb{Z}^2$. Consider the operator $T_\alpha = \alpha(S^* \otimes S)P$, with S is the bilateral shift and P is the orthogonal projection onto $\bigvee\{e_i \otimes e_j : i, j \in \mathbb{Z}, i \neq j\}$, defined on $\mathcal{H}$ by

$$T_\alpha e_{i,j} = \alpha(1 - \delta_{i,j})e_{i-1,j+1},$$

where $\delta_{i,j}$ is the Kronecker symbol and α is a nonzero real number.

We first observe that $N(T_\alpha) = \bigvee_{i\in\mathbb{Z}}\{e_{i,i}\}$. Indeed, $T_\alpha e_{i,i} = 0$ for every $i \in \mathbb{Z}$ and hence $\bigvee_{i\in\mathbb{Z}}\{e_{i,i}\} \subset N(T_\alpha)$. Conversely, for $x = \sum_{i,j\in\mathbb{Z}} a_{i,j}e_{i,j} \in N(T_\alpha)$ for some $a_{i,j} \in \mathbb{C}$, we have

$$0 = T_\alpha x = \alpha \sum_{i,j\in\mathbb{Z}} (1 - \delta_{i,j})a_{i,j}e_{i-1,j+1}.$$

This implies that if $i \neq j$ we have $a_{i,j} = 0$. Thus, $x = \sum_{i\in\mathbb{Z}} a_{i,i}e_{i,i} \in \bigvee_{i\in\mathbb{Z}}\{e_{i,i}\}$.

Since $T_\alpha e_{i+1,i-1} = \alpha(1-\delta_{i+1,i-1})e_{i,i} = \alpha e_{i,i}$ and $T_\alpha e_{i+2,i-2} = \alpha e_{i+1,i-1}$, an induction argument shows that

$$e_{i,i} = \frac{1}{\alpha^n}T_\alpha^n e_{i+n,i-n} \text{ for every } n \geq 1.$$

Hence $N(T_\alpha) \subseteq R^\infty(T_\alpha)$. On the other hand, since $R(T_\alpha) = \bigvee_{j\neq i+2}\{e_{i,j}\}$, then $R(T_\alpha)$ is closed. Therefore, T_α is regular.

Now, simple computations shows that

$$T_\alpha^* e_{i,j} = \alpha(1 - \delta_{i+2,j})e_{i+1,j-1}, \tag{9}$$

and thus

$$N(T_\alpha^*) = \bigvee_{i\in\mathbb{Z}} \left\{e_{i,i+2}\right\}.$$

Let $(i, j) \in \mathbb{Z}^2$ be such that $j \neq i + 2$ and $(m, n) \in \mathbb{Z}^2$. If $m = n$ we get

$$e_{m,n} \in N(T_\alpha) = R(T_\alpha^\dagger)^\perp,$$

and hence $\langle T_\alpha^\dagger e_{i,j}, e_{m,n}\rangle = 0$. Suppose now that $m \neq n$. We have

$$\langle T_\alpha^\dagger e_{i,j}, e_{m,n}\rangle = \langle T_\alpha^\dagger e_{i,j}, \frac{1}{\alpha}T_\alpha^* e_{m-1,n+1}\rangle = \langle\frac{1}{\alpha}T_\alpha T_\alpha^\dagger e_{i,j}, e_{m-1,n+1}\rangle.$$

Since $j \neq i+2$, we have $e_{i,j} \in R(T_\alpha)$ and thus $T_\alpha T_\alpha^\dagger e_{i,j} = P_{R(T_\alpha)} e_{i,j} = e_{i,j}$. This implies $\langle T_\alpha^\dagger e_{i,j}, e_{m,n}\rangle = \langle \frac{1}{\alpha} e_{i,j}, e_{m-1,n+1}\rangle = \langle \frac{1}{\alpha} e_{i+1,j-1}, e_{m,n}\rangle$, and finally

$$T_\alpha^\dagger e_{i,j} = \frac{1}{\alpha}(1-\delta_{i+2,j}) e_{i+1,j-1} = \frac{1}{\alpha^2} T^* e_{i,j}. \tag{10}$$

It follows that $T_\alpha^\dagger$ is regular and hence that T_α is bi-regular.

Since T_α is a partial isometry if and only if $T_\alpha^* = T_\alpha^\dagger$, it follows from (10) that T_α is a partial isometry if and only if $\alpha = 1$.

We start with some elementary properties of bi-regular operators.

Proposition 4.2 *Let T be regular. We have*

(i) T *is bi-regular* $\iff$ $T^\dagger$ *is bi-regular* $\iff$ T^* *is bi-regular.*
(ii) If T is bi-regular, then T^n is bi-regular for every $n \geq 2$,
(iii) If T_1 and T_2 are bi-regular, then $T_1 \oplus T_2$ is bi-regular,
(iv) If T is bi-regular and E is a reducing subspace for T, then $T_{|E}$ is bi-regular.

We provide next a sufficient condition for an operator to be bi-regular.

Theorem 4.3 *Let $T \in \mathcal{L}(\mathcal{H})$ be regular. If $R^\infty(T^*)$ is T-invariant, then*

(i) $R^\infty(T^*) \subseteq C(T^\dagger)$. In particular, $R^\infty(T^*) \subseteq R^\infty(T^\dagger)$.
(ii) T is bi-regular.

Proof

(i) Since T is regular, T^* is also regular and by Proposition 2.7 we have $C(T^*) = R^\infty(T^*)$. For $x \in R^\infty(T^*)$, and $v_n := T^n x \in T^n R^\infty(T^*) \subseteq R^\infty(T^*) \subseteq R(T^*)$, we have $x = v_0$ and $T^\dagger v_{n+1} = T^\dagger T^{n+1} x = T^\dagger T v_n = v_n$. Thus $x \in C(T^\dagger)$ and hence $R^\infty(T^*) \subseteq C(T^\dagger)$.
(ii) From T^* is regular, we get $R(T^\dagger) = R(T^*)$ is closed and since moreover $N(T^\dagger) = N(T^*) \subseteq R^\infty(T^*) \subseteq C(T^\dagger) \subseteq R^\infty(T^\dagger)$, we derive that $T^\dagger$ is regular.

□

Remark 4.4 Under assumptions of theorem 4.3, since $T^\dagger$ is regular, we have $C(T^\dagger) = R^\infty(T^\dagger)$.

Theorem 4.3 implies the following corollary.

Corollary 4.5 *Let $T \in \mathcal{L}(\mathcal{H})$ be regular. If $R^\infty(T)$ or $R^\infty(T^*)$ is reducing for T, then T is bi-regular.*

4.2 The Cauchy Dual of a Closed Range Operator

The Cauchy dual of an operator $T \in \mathcal{L}(\mathcal{H})$ with closed range is introduced in [10] by the next formula

$$\omega(T) := T^{\dagger *}.$$

This definition extends the case where T is left invertible, in which $T^{\dagger} = (T^*T)^{-1}T^*$ and $\omega(T) = T^{\dagger *} = T(T^*T)^{-1}$. The Cauchy dual of a left invertible operator is introduced in [24] as a powerful tool in the model theory of left-invertible operators. The reader is referred for instance to [5, 19, 24] for more information.

In the following, we extend the result given in [24, Proposition 2.10].

Corollary 4.6 *Let $T \in \mathcal{L}(\mathcal{H})$ be a bi-regular operator. If $R^{\infty}(\omega(T))$ is $T^{\dagger}$-invariant, then*

$$R^{\infty}(\omega(T)) \subseteq R^{\infty}(T).$$

Proof Since $T^{\dagger}$ is regular and $R^{\infty}((T^{\dagger})^*) = R^{\infty}(\omega(T))$ is $T^{\dagger}$-invariant, by Theorem 4.3 we have $R^{\infty}(\omega(T)) \subseteq R^{\infty}(T)$. □

We also have the next extension of Olofsson's result that will allow to transfer Wold decomposition to Cauchy duals.

Proposition 4.7 *Let $T \in \mathcal{L}(\mathcal{H})$ be with closed range, $\omega(T)$ its Cauchy dual and let c be a nonnegative constant. Then the following statements are equivalent:*

(i)

$$\|\omega(T)^2x\|^2 - \|\omega(T)x\|^2 \leq c(\|\omega(T)x\|^2 - \|T^{\dagger}Tx\|^2), \quad \forall x \in \mathcal{H}. \tag{11}$$

(ii)

$$\|Q(Tx + T^{\dagger}Ty)\|^2 \leq (1 + \frac{1}{c})(\|x\|^2 + c\|Ty\|^2), \quad \forall x, y \in \mathcal{H}, \tag{12}$$

where Q is the orthogonal projection of $\mathcal{H}$ onto $R(T)$. In particular, if T satisfies (1), then T is bounded below and $\omega(T)$ is concave.

4.3 Extended Wold-Type Decomposition

Definition 4.8 We shall say that an operator $T \in \mathcal{L}(\mathcal{H})$ admits the *extended Wold-type decomposition* if

(i) $R^\infty(T)$ is closed and reduces T,
(ii) $T_{|R^\infty(T)\cap N(T)^\perp} : R^\infty(T) \cap N(T)^\perp \longrightarrow R^\infty(T)$ is unitary,
(iii) $\mathcal{H} = [\mathcal{E}]_T \oplus R^\infty(T)$.

Notice in passing that if T satisfies the extended Wold-type decomposition and if T is one to one or $T_{|R^\infty(T)}$ is unitary, then T admits the classical Wold type decomposition property investigated in [24].

For example, for a regular operator T and under the assumptions of Theorem 3.8, by Theorem 3.9, T admits the extended Wold-type decomposition.

Remark 4.9

(i) Corollary 3.5 implies that a regular operator T is pure if and only if $\omega(T)$ has the wandering subspace property.
(ii) If T is an operator with closed range, then $R(\omega(T))$ is closed. So, since $\omega((\omega(T)) = T$ and from (i) in Corollary 4.6 we have

$$R^\infty(T)^\perp \subseteq [\mathcal{E}]_{\omega(T)}.$$

The previous inclusion is strict in general as the following example shows.

Example Consider on $\mathbb{C}^3$, endowed with an orthonormal basis $\{e_1, e_2, e_3\}$, the operator defined by $Te_1 = Te_2 = 0$ and $Te_3 = e_1$. It is immediate that $TT^* = P_{\mathbb{C}e_1}$ and $T^*T = P_{\mathbb{C}e_3}$. It follows that $T^\dagger = T^*$ and hence that $T = \omega(T)$. Now $T^2 = 0$ implies $R^\infty(\omega(T)) = \{0\}$, and we obtain

$$R^\infty(\omega(T))^\perp = \mathbb{C}^3 \neq [\mathcal{E}]_T = \mathbb{C}\{e_2, e_3\}.$$

It is not clear whether the inclusion in Corollary 3.5, (i) can be replaced by the equality if T is an arbitrary regular operator. However, in the following example, we show that the equality holds if T is a regular weighted shift on $l^2(\mathbb{Z})$.

Example Let $\mathcal{H}$ be a Hilbert space endowed with an orthonormal basis $(e_n)_{n\in\mathbb{Z}}$ and let $\alpha = (\alpha)_{n\in\mathbb{Z}}$ be a bounded sequence. The weighted shift S_α on $\mathcal{H}$ associated with α is the bounded linear operator $S_\alpha : \mathcal{H} \longrightarrow \mathcal{H}$ defined by $S_\alpha e_n = \alpha_n e_{n+1}$. It is proved in [8] that if S_α is regular, then there exits at most $n_0 \in \mathbb{Z}$ such that $\alpha_{n_0} = 0$. Let S_α be a regular bilateral weighted shift such that $\alpha_0 = 0$. It is easy to see that

$$(S_\alpha^\dagger)^n e_k = \begin{cases} (\Pi_{i=1}^n \alpha_{k-i})^{-1} e_{k-n} & , \ if \ k \notin \{1, \cdots, n\}\,; \\ 0 & , \ if \ k \in \{1, \cdots, n\}\,. \end{cases}$$

We derive that $N((S_\alpha^\dagger)^n) = \bigvee_{1\le j\le n}\{e_j\}$ for every $n \ge 1$, and in particular, we have

$$\mathcal{E} = \mathcal{H} \ominus S_\alpha\mathcal{H} = N(S_\alpha^\dagger) = span\,\{e_1\}.$$

For $n \ge 1$ and $0 \le i \le n-1$, we have

$$(S_\alpha^\dagger)^n S_\alpha^i e_1 = \Pi_{j=1}^i \alpha_j (S_\alpha^\dagger)^n e_{i+1}.$$

Since $i+1 \in \{1, \cdots, n\}$, we get $(S_\alpha^\dagger)^n S_\alpha^i e_1 = 0$, and thus $S_\alpha^i e_1 \in N((S_\alpha^\dagger)^n)$ for every $0 \le i \le n-1$. *Hence* $\bigvee_{i=0}^{n-1}\{S_\alpha^i e_1\} \subset N((S_\alpha^\dagger)^n))$. *On the other hand, from [8, Proposition 8], we have* $N((S_\alpha^\dagger)^n) \subset \bigvee_{i=0}^{n-1}\{S_\alpha^i e_1\}$, *and thus*

$$N((S_\alpha^\dagger)^n) = \bigvee_{i=0}^{n-1}\left\{S_\alpha^i e_1\right\}.$$

Therefore

$$\bigvee_{n\ge 0} N((S_\alpha^\dagger)^n) = [e_1]_{S_\alpha}.$$

It is easy to check that $\bigvee_{j\le 0}\{e_j\} = R^\infty(S_\alpha) = R^\infty(\omega(S_\alpha))$ and that $\bigvee_{j\ge 1}\{e_j\} = [e_1]_{S_\alpha} = R^\infty(S_\alpha)^\perp$.

Proposition 4.10 *If $T \in \mathcal{L}(\mathcal{H})$ is a bi-regular operator, then*

$$\mathcal{H} = [\mathcal{E}]_T \oplus R^\infty(\omega(T)) = [\mathcal{E}]_{\omega(T)} \oplus R^\infty(T). \tag{13}$$

Proof Since $T^\dagger$ is regular, then $\omega(T) = T^{\dagger *}$ is also regular. Now, by substituting $\omega(T)$ for T in Corollary 3.5 and using the identities $\mathcal{E} = R(\omega(T))^\perp$ and $\omega(\omega(T)) = T$, we get the desired result. □

Proposition 4.10 extends the following duality result given by S. Shimorin in [24].

Corollary 4.11 ([24, Proposition 2.7]) *Let T be a left-invertible operator and let L be its left inverse defined by $L = (T^*T)^{-1}T^*$. Then*

$$\mathcal{H} = [\mathcal{E}]_T \oplus R^\infty(L^*) = [\mathcal{E}]_{L^*} \oplus R^\infty(T). \tag{14}$$

Corollary 4.5 and Proposition 4.10 imply the following results.

Corollary 4.12 *Let $T \in \mathcal{L}(\mathcal{H})$ be a regular operator such that $R^\infty(T)$ reduces T. Then*

$$\mathcal{H} = [\mathcal{E}]_T \oplus R^\infty(\omega(T)) = [\mathcal{E}]_{\omega(T)} \oplus R^\infty(T).$$

Corollary 4.13 *Let $T \in \mathcal{L}(\mathcal{H})$ be regular such that $R^\infty(T)$ reduces T and $R^\infty(\omega(T))$ reduces $\omega(T)$. Then*

$$[\mathcal{E}]_T = [\mathcal{E}]_{\omega(T)} \text{ and } R^\infty(T) = R^\infty(\omega(T)).$$

In particular,

$$\mathcal{H} = [\mathcal{E}]_T \oplus R^\infty(T).$$

Proof Since $R^\infty(T)$ reduces T, then T is bi-regular. The result follows from Corollaries 4.6 and 4.12. □

Remark 4.14

1. We notice that if T is a regular operator such that T^* and $T^\dagger$ are equal on $R^\infty(T)$, then by Proposition 2.7, $R^\infty(T)$ reduces T and then T will be bi-regular. We see easily that the equality $T^\dagger_{|R^\infty(T)} = T^*_{|R^\infty(T)}$ is equivalent to each one of the following.
   - (i) $TT^*P_{R^\infty(T)} = P_{R^\infty(T)}$, that is, $T^*_{|R^\infty(T)}$ is isometric;
   - (ii) $P_{R^\infty(T)}TT^* = P_{R^\infty(T)}TT^\dagger$;
   - (iii) $T^*P_{R^\infty(T)} = T^\dagger P_{R^\infty(T)}$;
   - (iv) $T^*TP_{R^\infty(T)} = T^\dagger TP_{R^\infty(T)}$;
   - (v) $T_{|R^\infty(T)\cap N(T)^\perp} : R^\infty(T) \cap N(T)^\perp \longrightarrow R^\infty(T)$ is unitary.
2. Under any of conditions (i)–(v), we have

$$(T_{|R^\infty(T)})^\dagger = T^\dagger_{|R^\infty(T)} = T^*_{|R^\infty(T)} = (T_{|R^\infty(T)})^*.$$

   In particular, $T_{|R^\infty(T)}$ is a surjective partial isometry.

Moreover, we have the following result.

Proposition 4.15 *Let $T \in \mathcal{L}(\mathcal{H})$ be regular. If one of the conditions (i)–(v) in Remark 4.14 is fulfilled, then*

(i) T is bi-regular;
(ii) $R^\infty(T^\dagger) = R^\infty(T^*)$.

Proof

(i) Follows from Remark 4.14.
(ii) From T is bi-regular, we obtain $R^\infty(T^\dagger)$ and $R^\infty(T^*)$ are closed. By Proposition 4.10 and since $\omega(T^*) = T^\dagger$, we have $R^\infty(T^\dagger)^\perp = [\mathcal{E}^\dagger]_{T^*}$ and $R^\infty(T^*)^\perp = [\mathcal{E}^\dagger]_{T^\dagger}$. On the other hand, because $\mathcal{E}^\dagger = N(T) \subseteq R^\infty(T)$, we get $T^{*n}x = T^{\dagger n}x$ for every $x \in \mathcal{E}^\dagger$ and for every $n \geq 0$. Then we have $[\mathcal{E}^\dagger]_{T^\dagger} = [\mathcal{E}^\dagger]_{T^*}$, which proves that $R^\infty(T^\dagger) = R^\infty(T^*)$.

□

Since T is regular if and only if its adjoint T^* is regular, then it is clear from Proposition 4.15 that if T is regular and one of the conditions (i)–(v) in Remark 4.14 is satisfied for T and for T^*, then $R^\infty(\omega(T)) = R^\infty(T)$. So, by Proposition 4.10, we get the following result.

Corollary 4.16 *Let $T \in \mathcal{L}(\mathcal{H})$ be regular. If one of the conditions* (i)–(v) *in Remark 4.14 is fulfilled for T and for T^*, then T and T^* admit the extended Wold-type decomposition.*

The duality between T and $\omega(T)$ is reflected in terms of extended Wold-type decomposition as follows.

Proposition 4.17 *Let $T \in \mathcal{L}(\mathcal{H})$ be bi-regular. Then T admits the extended Wold-type decomposition if and only if $\omega(T)$ admits it. In this case, we have*

$$R^\infty(T) = R^\infty(\omega(T)) \textit{ and } [\mathcal{E}]_T = [\mathcal{E}]_{\omega(T)}.$$

Proof Suppose that T admits the extended Wold-type decomposition. Then we have, $\mathcal{H} = [\mathcal{E}]_T \oplus R^\infty(T)$ and from Proposition 4.10 we get

$$R^\infty(T) = R^\infty(\omega(T)) \text{ and } [\mathcal{E}]_T = [\mathcal{E}]_{\omega(T)}.$$

On the other hand, since $\omega(T)$ is regular, then

$$\omega(T)(R^\infty(\omega(T)) = R^\infty(\omega(T)).$$

Let $x \in R^\infty(\omega(T))$. We have

$$\omega(T)^* x = T^\dagger x \in T^\dagger R^\infty(T) \subseteq R^\infty(T) = R^\infty(\omega(T)).$$

So, $R^\infty(\omega(T))$ reduces $\omega(T)$.

Its remains to show that

$$\omega(T)_{|R^\infty(T)\cap N(T)^\perp} : R^\infty(T) \cap N(T)^\perp \longrightarrow R^\infty(T)$$

is unitary. Let $x \in R^\infty(\omega(T)) \cap N(\omega(T))^\perp = R^\infty(T) \cap N(T)^\perp$. Since by assumption $T_{|R^\infty(T)\cap N(T)^\perp} : R^\infty(T) \cap N(T)^\perp \longrightarrow R^\infty(T)$ is unitary, we get $TT^*x = T^*Tx = x$ and thus,

$$T^*x = T^\dagger x.$$

Now, from identities $T^*T = \omega(T)^\dagger T$ and $\omega(T)\omega(T)^\dagger T = T$, it is clear that $T^*Tx = x$ implies

$$\omega(T)x = \omega(T)\omega(T)^\dagger Tx = Tx.$$

It follows that for every $x \in R^{\infty}(\omega(T)) \cap N(\omega(T))^{\perp}$ we have

$$\|\omega(T)x\| = \|Tx\| = \|x\|.$$

On the other hand, by Remark 4.14, we have $T^*x = T^{\dagger}x$ for every $x \in R^{\infty}(T)$. Thus,

$$\|\omega(T)^*x\| = \|T^{\dagger}x\| = \|T^*x\| = \|x\|, \ \forall x \in R^{\infty}(T).$$

So, $\omega(T)_{|R^{\infty}(T)\cap N(T)^{\perp}} : R^{\infty}(T) \cap N(T)^{\perp} \longrightarrow R^{\infty}(T)$ is unitary.

The converse follows by the duality $\omega(\omega(T)) = T$. □

Remark 4.15 In the case of left invertible operators, we retrieve Corollary 2.9 in [24], from the previous proposition.

We provide for bi-regular operators the next duality properties in the line of the ones given by D. Sutton for left invertible operators in [26].

Theorem 4.18 *Let $T \in \mathcal{L}(\mathcal{H})$ be a bi-regular operator. The following statements are equivalent:*

(i) $\mathcal{H} \neq [\mathcal{E}]_T$;
(ii) $R^{\infty}(\omega(T)) \neq \{0\}$;
(iii) there exists a closed subspace $\mathcal{M} \neq \{0\}$ *such that* $\mathcal{M} \subseteq \omega(T)\mathcal{M}$;
(iv) there exists a closed subspace $\mathcal{M} \subseteq R(T)$, $\mathcal{M} \neq \{0\}$ *such that* $T^*\mathcal{M} \subseteq \mathcal{M}$;
(v) there exists a closed subspace $\mathcal{M} \supseteq \mathcal{E}$, $\mathcal{M} \neq \mathcal{H}$ *such that* $T\mathcal{M} \subseteq \mathcal{M}$;
(vi) there exists a closed subspace $\mathcal{M} \supseteq \mathcal{E}$, $\mathcal{M} \neq \mathcal{H}$ *such that* $\mathcal{M} \subseteq T^{\dagger}\mathcal{M}$;
(vii) there exists closed subspaces $\mathcal{A}, \mathcal{B} \neq \{0\}$ *such that* $T\mathcal{A} \subseteq \mathcal{A}$, $T^{\dagger}\mathcal{A} \subseteq \mathcal{A} \oplus \mathcal{E}$, $P_{\mathcal{B}}T\mathcal{B} = \mathcal{B}$ *and* $R(T) = \mathcal{A} \oplus \mathcal{B}$, *where* $P_{\mathcal{B}}$ *is the orthogonal projection onto* $\mathcal{B}$.

Proof (i)$\Longleftrightarrow$(ii): This equivalence follows from Proposition 4.10.

(ii)$\Longrightarrow$(iii): Let $\mathcal{M} = R^{\infty}(\omega(T))$. By assumption we have $\mathcal{M} \neq \{0\}$. Since $T^{\dagger}$ is regular, $\omega(T) = T^{\dagger *}$ is also regular and then $R^{\infty}(\omega(T))$ is closed. Moreover, by Proposition 2.7, we have $\omega(T)R^{\infty}(\omega(T)) = R^{\infty}(\omega(T))$. Thus the condition $\mathcal{M} \subseteq \omega(T)\mathcal{M}$ is satisfied.

(ii)$\Longrightarrow$(iv): Let $\mathcal{M} = R^{\infty}(\omega(T)) \neq \{0\}$. Since $R^{\infty}(\omega(T)) \subseteq R(\omega(T))$ and $R(\omega(T)) = R(T)$, we have $\mathcal{M} \subseteq R(T)$. Now, since $\omega(T)$ is regular and T^* is a generalized inverse of $\omega(T)$, by Proposition 2.7, we get $T^*R^{\infty}(\omega(T)) \subseteq R^{\infty}(\omega(T))$. That is $T^*\mathcal{M} \subseteq \mathcal{M}$.

(i)$\Longrightarrow$(v): Let $\mathcal{M} = [\mathcal{E}]_T$, then $\mathcal{E} \subseteq \mathcal{M} \neq \mathcal{H}$. Moreover, from the definition of $[\mathcal{E}]_T$, it's clear that $T\mathcal{M} \subseteq \mathcal{M}$.

(i)$\Longrightarrow$(vi): Again, let $\mathcal{M} = [\mathcal{E}]_T$. Then $\mathcal{E} \subseteq \mathcal{M} \neq \mathcal{H}$. Since $\omega(T)$ is regular, we have $R^{\infty}(\omega(T))^{\perp} \subseteq N(\omega(T))^{\perp} = R(T^{\dagger})$. But by Proposition 4.10, we have $R^{\infty}(\omega(T))^{\perp} = [\mathcal{E}]_T$, thus $\mathcal{M} \subseteq R(T^{\dagger})$. Since $T\mathcal{M} \subseteq \mathcal{M}$ and $T^{\dagger}T = P_{R(T^{\dagger})}$ we conclude that $\mathcal{M} = T^{\dagger}T\mathcal{M} \subseteq T^{\dagger}\mathcal{M}$.

(i)$\Longrightarrow$(vii): Let $\mathcal{A} = \bigvee_{i=1}^{\infty} T^i\mathcal{E}$ and $\mathcal{B} = R^{\infty}(\omega(T))$. It follows from (i) and (ii) that $\mathcal{A}, \mathcal{B} \neq \{0\}$ and by Proposition 4.10 we have $\mathcal{A} \perp \mathcal{B}$. Since $R(T)$ is closed and $\mathcal{E} \perp T^i\mathcal{E}\ \forall i \geq 1$, by Proposition 4.10 again we obtain

$$R(T) = \mathcal{E}^{\perp} = ([\mathcal{E}]_T \oplus R^{\infty}(\omega(T))) \ominus \mathcal{E} = \bigvee_{i=1}^{\infty} T^i\mathcal{E} \oplus R^{\infty}(\omega(T)) = \mathcal{A} \oplus \mathcal{B}.$$

Since T is continuous, $T\mathcal{A} = T\bigvee_{i=1}^{\infty} T^i\mathcal{E} \subseteq \bigvee_{i=2}^{\infty} T^i\mathcal{E} \subseteq \bigvee_{i=1}^{\infty} T^i\mathcal{E} = \mathcal{A}$. On the other hand, $\mathcal{E} \perp T^i\mathcal{E}\ \forall i \geq 1$, thus $\mathcal{A} \oplus \mathcal{E} = \bigvee_{i=1}^{\infty} T^i\mathcal{E} \oplus \mathcal{E} = [\mathcal{E}]_T$. So, by Proposition 4.10 we have $(\mathcal{A} \oplus \mathcal{E})^{\perp} = R^{\infty}(\omega(T))$. But $\omega(T)$ is regular, so by Proposition 2.7 we obtain $T^{\dagger}R^{\infty}(\omega(T))^{\perp} \subseteq R^{\infty}(\omega(T))^{\perp}$. Equivalently, $T^{\dagger}(\mathcal{A} \oplus \mathcal{E}) \subseteq \mathcal{A} \oplus \mathcal{E}$, which implies that $T^{\dagger}\mathcal{A} \subseteq \mathcal{A} \oplus \mathcal{E}$. It remains to show the equality $P_{\mathcal{B}}T\mathcal{B} = \mathcal{B}$. Since $P_{\mathcal{B}}T\mathcal{B} \subseteq \mathcal{B}$, it suffices to show that $\mathcal{B} \subseteq P_{\mathcal{B}}T\mathcal{B}$. Let $x \in \mathcal{B} = R^{\infty}(\omega(T)) \subseteq R(T)$ such that $x = Ty$ for some $y \in \mathcal{H} = [\mathcal{E}]_T \oplus R^{\infty}(\omega(T))$ with $y = y_1 + y_2$, where $y_1 \in [\mathcal{E}]_T$ and $y_2 \in R^{\infty}(\omega(T)) = \mathcal{B}$. Since $Ty_1 \in [\mathcal{E}]_T \perp \mathcal{B}$, then $x = P_{\mathcal{B}}x = P_{\mathcal{B}}Ty_2 \in P_{\mathcal{B}}T\mathcal{B}$. Since $x \in \mathcal{B}$ is arbitrary, it follows that $P_{\mathcal{B}}T\mathcal{B} = \mathcal{B}$.

(iii)$\Longrightarrow$(ii): Applying $\omega(T)^j$ to the relation $\mathcal{M} \subseteq \omega(T)\mathcal{M}$, we obtain $\mathcal{M} \subseteq \omega(T)^j\mathcal{M}$ for every $j \geq 0$, and then that $\mathcal{M} \subseteq \bigcap_{j=0}^{\infty} \omega(T)^j\mathcal{M}$. Since $\bigcap_{j=0}^{\infty} \omega(T)^j\mathcal{M} \subseteq R^{\infty}(\omega(T))$, we get $\mathcal{M} \subseteq R^{\infty}(\omega(T))$. Now $\mathcal{M} \neq \{0\}$ implies that $R^{\infty}(\omega(T)) \neq \{0\}$.

(iv)$\Longrightarrow$(iii): We recall that $\omega(T)T^* = (TT^{\dagger})^* = TT^{\dagger}$ is an orthogonal projection onto $R(T)$. Thus, since $\mathcal{M} \subseteq R(T)$ we have $\omega(T)T^*\mathcal{M} = \mathcal{M}$. By applying $\omega(T)$ to the relation $T^*\mathcal{M} \subseteq \mathcal{M}$ we get $\omega(T)T^*\mathcal{M} = \mathcal{M} \subseteq \omega(T)\mathcal{M}$. By (iv) we have $\mathcal{M} \neq \{0\}$.

(v)$\Longrightarrow$(iv): Suppose that there exists a closed subset verifying the required properties of (v). Since $\mathcal{E} \subseteq \mathcal{M} \neq \mathcal{H}$, we have $\{0\} \neq \mathcal{M}^{\perp} \subseteq R(T)$, and by $T\mathcal{M} \subseteq \mathcal{M}$ we have $T^*\mathcal{M}^{\perp} \subseteq \mathcal{M}^{\perp}$. In particular, (iv) is satisfied with the closed subspace $\mathcal{M}^{\perp}$.

(vi)$\Longrightarrow$(iii) Since $\mathcal{E} \subseteq \mathcal{M} \neq \mathcal{H}$, we have $\{0\} \neq \mathcal{M}^{\perp} \subseteq R(T)$. We shall prove that $\mathcal{M}^{\perp} \subseteq \omega(T)\mathcal{M}^{\perp}$. To this aim, let $x \in \mathcal{M}^{\perp} \subseteq R(T) = R(\omega(T))$. There is $y \in \mathcal{H}$ such that $x = \omega(T)y$. Let $y = m_1 + m_2$ with $m_1 \in \mathcal{M}$ and $m_2 \in \mathcal{M}^{\perp}$, and suppose that $m_1 \neq 0$. Then for every $m \in \mathcal{M}$ we have $0 = \langle x, m\rangle = \langle \omega(T)(m_1 + m_2), m\rangle$, which implies that $\langle m_1 + m_2, T^{\dagger}m\rangle = 0$ for every $m \in \mathcal{M}$. Then $m_1 + m_2 \in (T^{\dagger}\mathcal{M})^{\perp} \subseteq \mathcal{M}^{\perp}$ because by assumption we have $\mathcal{M} \subseteq T^{\dagger}\mathcal{M}$. But $\langle m_1 + m_2, m_1\rangle = \langle m_1, m_1\rangle = \|m_1\|^2 \neq 0$, which contradicts the fact that $m_1 + m_2 \in \mathcal{M}^{\perp}$. We derive that $m_1 = 0$ and hence $x = \omega(T)m_2$. Since $x \in \mathcal{M}^{\perp}$ is arbitrary, we get that $\mathcal{M}^{\perp} \subseteq \omega(T)\mathcal{M}^{\perp}$. Therefore (iii) is satisfied with the closed subspace $\mathcal{M}^{\perp}$.

(vii)$\Longrightarrow$(v): Suppose that there exist closed subspaces $\mathcal{A}, \mathcal{B} \neq \{0\}$ such that $T\mathcal{A} \subseteq \mathcal{A}$, $T^{\dagger}\mathcal{A} \subseteq \mathcal{A} \oplus \mathcal{E}$, $P_{\mathcal{B}}T\mathcal{B} = \mathcal{B}$ and $R(T) = \mathcal{A} \oplus \mathcal{B}$. We shall prove first that $T\mathcal{E} \subseteq \mathcal{A}$. Let $x \in \mathcal{E}$, since $Tx \in R(T)$, then $Tx = a + b$ where $a \in \mathcal{A}$ and $b \in \mathcal{B} = P_{\mathcal{B}}T\mathcal{B}$. Thus $b = P_{\mathcal{B}}Tb_1$ for some $b_1 \in \mathcal{B}$. Now, from the orthogonal decomposition of $R(T)$, we see that $Tb_1 = a_2 + b$ for some $a_2 \in \mathcal{A}$. It follows that

$Tx = a + b = a - a_2 + Tb_1$, and then

$$Tx = a_1 + Tb_1, \tag{15}$$

where $a_1 = a - a_2$. Thus, $T^\dagger Tx = T^\dagger a_1 + T^\dagger Tb_1$. Now, since $T^\dagger$ is regular, we have $x \in \mathcal{E} = N(T^\dagger) \subseteq R(T^\dagger)$, and so $T^\dagger Tx = x = T^\dagger a_1 + T^\dagger Tb_1$. On the other hand, we have $T^\dagger a_1 \in T^\dagger \mathcal{A} \subseteq \mathcal{A} \oplus \mathcal{E} = \mathcal{B}^\perp$. Now, since $x \in \mathcal{E} \subseteq \mathcal{B}^\perp$, we get $T^\dagger Tb_1 \in \mathcal{B}^\perp = \mathcal{A} \oplus \mathcal{E}$. Thus $T^\dagger Tb_1 = a_3 + y$ where $a_3 \in \mathcal{A}$ and $y \in \mathcal{E}$. By applying T to this last equality and using the fact that $TT^\dagger T = T$, we get $Tb_1 = Ta_3 + Ty$ and thus we have that $a_3 - b_1 + y \in N(T)$. Now, since T is regular, we have $N(T) \subseteq R(T)$ and hence $a_3 - b_1 + y \in R(T)$. But $a_3 - b_1 \in \mathcal{A} \oplus \mathcal{B} = R(T)$, which yields $y \in R(T)$. The fact that $y \in \mathcal{E} = R(T)^\perp$ implies that $y = 0$ and hence that $Tb_1 = Ta_3$. By (15) we have $Tx = a_1 + Ta_3$. Since $a_3 \in \mathcal{A}$ and by assumption $Ta_3 \in T\mathcal{A} \subseteq \mathcal{A}$, we get $Tx \in \mathcal{A}$. But $x \in \mathcal{E}$ is arbitrary, so we deduce that $T\mathcal{E} \subseteq \mathcal{A}$. Now, because $T\mathcal{A} \subseteq \mathcal{A}$, we have $T(\mathcal{A} \oplus \mathcal{E}) \subseteq \mathcal{A} \subseteq \mathcal{A} \oplus \mathcal{E}$. So, for $\mathcal{M} := \mathcal{A} \oplus \mathcal{E}$, we have $T\mathcal{M} \subseteq \mathcal{M}$ and $\mathcal{E} \subseteq \mathcal{M}$. Since $\mathcal{M} \perp \mathcal{B}$ and $\mathcal{B} \neq \{0\}$, we have $\mathcal{M} \neq \mathcal{H}$. Therefore, (v) is satisfied with the closed subspace $\mathcal{M} = \mathcal{A} \oplus \mathcal{E}$. □

5 Some Open Questions

Several natural questions arise from this survey.

Problem 5.1 Given an operator T with closed range. Under which conditions, do we have $(T^\dagger)^n = (T^n)^\dagger$ for every $n \geq 1$?

An intermediate question is, suppose $(T^\dagger)^n = (T^n)^\dagger$ for some $n \geq 1$. Is it true that $(T^\dagger)^k = (T^k)^\dagger$ for some $k < n$?

A more general question is the so called reverse order law problem that investigates the identity $(AB)^\dagger = B^\dagger A^\dagger$ for A, B with closed range. See the survey [6] for example.

We show first that regular weighted shifts, answer this question positively. Let $\mathcal{H}$ be a Hilbert space, $(e_n)_{n\in\mathbb{Z}}$ be an orthonormal basis of $\mathcal{H}$, $(\omega)_{n\in\mathbb{Z}}$ be a bounded sequence and S_ω be the weighted shift associated with $(\omega)_n$.

It is not difficult to see that $R(S_\omega)$ is closed if and only if

$$\lim_{n\to+\infty} \left\{ \inf_{k>0} \omega_k \cdots \omega_{k+n-1} \right\}^{\frac{1}{n}} > 0$$

and

$$\lim_{n\to+\infty} \left\{ \inf_{k\geq 0} \omega_{-k-1} \cdots \omega_{-k-n} \right\}^{\frac{1}{n}} > 0.$$

If $\omega_0 = 0$ and $\omega_n \neq 0$ for every $n \neq 0$, then S_ω is regular. Moreover,

$$S_\omega^\dagger e_k = \begin{cases} (\omega_{k-1})^{-1} e_{k-1} & for\ k \neq 1; \\ 0 & for\ k = 1. \end{cases}$$

S_ω^n is the weighted shift of multiplicity n and it is easy to check that

$$(S_\omega^\dagger)^n = (S_\omega^n)^\dagger.$$

On the other hand, we provide provides an example disapproving the equality. *For* $T = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ *we have* $T^\dagger = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$ *and* $(T^\dagger)^2 T^2 = \frac{1}{2} T^\dagger T$ *which is not a projection. Thus* $(T^\dagger)^2 \neq (T^2)^\dagger$.

The equality $(T^n)^\dagger = (T^\dagger)^n$ may fail even for left invertible operators as shown by the examples below.

Recall that for T left invertible, we have $T^\dagger = (T^*T)^{-1}T^*$. Since T^n is also left invertible, we get $T^{n\dagger} = (T^{n*}T^n)^{-1}T^{*n}$ and $T^{\dagger n} = ((T^*T)^{-1}T^*)^n$.

Problem 5.2 When is the restriction of a bi-regular operator is bi-regular?

As for regular operators the restriction to the kernel is not bi-regular. It is interesting to see for which condition on the invariant subspace M of T, $T_{|M}$ is bi-regular. For example, if T is regular and E is an invariant subspace such that $N^\infty(T) \subset E$, then $T_{|E}$ is regular. Is this fact true for bi-regular operators?

Problem 5.3 Is every regular operator bi-regular?

In contrast with T is regular if and only if T^* is regular. We do not know if T is regular operator if and only if $T^\dagger$ is regular. A positive answer is given when T is regular such that $(T^\dagger)^n = (T^n)^\dagger$ every $n \geq 1$. In particular, regular weighted shifts are bi-regular.

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(Asymmetric) Dual Truncated Toeplitz Operators

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Abstract Multiplication operators on the space $L^2(\mathbb{T})$ on the unit circle $\mathbb{T}$ with Lebesgue measure are classical operators. So are Toeplitz operators on the Hardy space $H^2 \subset L^2(\mathbb{T})$. Sarason's paper (Oper Metrices 1:491–526, 2007) has started investigations of truncated Toeplitz operators (TTO), i.e., compressions of these multiplication operators to model spaces. If operators act between two different model spaces they are called asymmetric truncated Toeplitz operators (ATTO). Naturally the compressions of multiplication operators between orthogonal complements of model spaces can be investigated. They are called dual truncated Toeplitz operators (DTTO), or asymmetric dual truncated Toeplitz operators (ADTTO) if orthogonal complements to different model spaces are considered. In this chapter the properties of ADTTO are presented.

Keywords Model space · Multiplication operator · Dual truncated Toeplitz operator · Conjugation · Intertwining property · Commutativity of operators

1 Motivation and Basic Notations

TTO and ATTO are natural generalizations of Toeplitz matrices which appear in many contexts, such as in the study of finite–interval convolution equations, signal processing, control theory, probability and diffraction problems [10, 11, 19]. Model spaces, which provide the natural setting for TTO and ATTO, have generated enormous interest and they are relevant in connection with a variety of topics such as the Schrödinger operator, classical extremal problems in control theory,

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Hankel operators and Toeplitz matrices (see for instance [12] and [10]). Natural conjugations, which model spaces and the whole $L^2(\mathbb{T})$ possess (see [3]), make model spaces even more natural in the context of physics [11]. Their orthogonal complements in $L^2(\mathbb{T})$ also appear in numerous applications. In the equivalent setting of the real line [9, 14], using time and frequency as the natural variables, and taking the inner function $\theta = \theta_\lambda$ with $\theta_\lambda = \exp(i\lambda\xi)$ for $\xi \in \mathbb{R}$, they appear via the Fourier transform, for instance, as high frequency signals, which are of decisive importance in electronics, or as outputs of high–pass filters. DTTO and ADTTO, acting on these spaces have realizations, for example, in long distance communication links with several regenerators along the path that cancel low–frequency noise using high–pass filters, or in the description of wave propagation in the presence of finite–length obstacles. The chapter is mostly based on the papers [4–7, 18].

Let $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ be the unit disk and $\mathbb{T}$ the unit circle. Denote by $L^2(\mathbb{T}) := L^2(\partial\mathbb{D})$ the space of measurable and square integrable functions on $\mathbb{T}$ with respect to the normalized Lebesgue measure, by H^2 denote the classical Hardy space, and let $H^2_- = L^2(\mathbb{T}) \ominus H^2$. Let P^+ be the orthogonal projection from $L^2(\mathbb{T})$ onto H^2 and $P^- = I_{L^2(\mathbb{T})} - P^+$.

Let $\varphi \in L_\infty(\mathbb{T})$. Recall that the *multiplication operator* on $L^2(\mathbb{T})$ is defined as $M_\varphi f = \varphi f$ for $f \in L^2(\mathbb{T})$. The *Toeplitz operator* is defined by $T_\varphi f = P^+(\varphi f)$, and the operator $H_\varphi f = P^-(\varphi f)$, for $f \in H^2$, is called the *Hankel operator* (with symbol φ). The space of all Toeplitz operators is denoted by $\mathcal{T}(H^2)$ and space of all Hankel operators is denoted $\mathcal{H}(H^2, H^2_-)$.

Recall that the operator J, defined by $J \colon L^2(\mathbb{T}) \to L^2(\mathbb{T})$, $Jf(z) = \bar{z}\overline{f(z)}$, $z \in \mathbb{T}$, is an antilinear involution. Moreover, $J^{-1} = J = J^\sharp$ (by $\sharp$ we denote the antilinear adjoint) and $J(H^2) = H^2_-$, $J(H^2_-) = H^2$. More properties of antilinear operators can be found for example in [17]. Let θ be an inner function. Note that the multiplication operator M_θ maps H^2 bijectively onto θH^2 and $M_\theta^{-1} = M_{\bar{\theta}}$. Moreover, each of the operators J, M_θ and $M_{\bar{\theta}}$ preserves $L_\infty(\mathbb{T})$.

The following properties can be easily verified.

Proposition 1.1 *Let θ be an inner function. Then*

(a) $\langle f_1, f_2\rangle = \langle \theta f_1, \theta f_2\rangle = \langle Jf_2, Jf_1\rangle$ *for* $f_1, f_2 \in L^2(\mathbb{T})$;
(b) $P_{\theta H^2} = M_\theta P^+ M_{\bar{\theta}}$;
(c) $P^- = JP^+J$;
(d) $M_\theta(f_1 \otimes f_2)M_{\bar{\theta}} = \theta f_1 \otimes \theta f_2$ *for* $f_1, f_2 \in L^2(\mathbb{T})$;
(e) $J(f_1 \otimes f_2)J = Jf_1 \otimes Jf_2$ *for* $f_1, f_2 \in L^2(\mathbb{T})$;
(f) $M_\theta J M_\theta = J$;
(g) $JM_\varphi = M_{\bar{\varphi}}J$ *for* $\varphi \in L_\infty(\mathbb{T})$.

In particular $J1 = \bar{z}$ and $J(1 \otimes 1)J = \bar{z} \otimes \bar{z}$. Here, for $f_1, f_2 \in L^2(\mathbb{T})$, $f_1 \otimes f_2$ denotes the operator defined on $L^2(\mathbb{T})$ by $(f_1 \otimes f_2)(f) = \langle f, f_2\rangle f_1$,

For a nonconstant inner function θ denote by K_θ the *model space* defined as the orthogonal complement of θH^2 in H^2, i.e., $K_\theta = H^2 \ominus \theta H^2$, and let $P_\theta \colon L^2(\mathbb{T}) \to K_\theta$ be the orthogonal projection and let $P_\theta^\perp = I_{L^2(\mathbb{T})} - P_\theta$ be the orthogonal projection from $L^2(\mathbb{T})$ onto $(K_\theta)^\perp$. Hence we have following natural decompositions

$$H^2 = K_\theta \oplus \theta H^2 \quad \text{and} \quad L^2(\mathbb{T}) = K_\theta \oplus (K_\theta)^\perp = K_\theta \oplus \theta H^2 \oplus H^2_-.$$

There is a natural conjugation (an antiunitary involution) connected with a model space (see for instance [3, 11]). For an inner function θ define $C_\theta : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ by

$$C_\theta f(z) = \theta(z)\overline{zf(z)}, \quad |z| = 1. \tag{1}$$

Then C_θ is an antilinear isometric involution on $L^2(\mathbb{T})$, which implies that $\langle C_\theta f, C_\theta g\rangle = \langle g, f\rangle$ for $f, g \in L^2(\mathbb{T})$. One can easily verify that

$$C_\theta M_\varphi C_\theta = M_{\bar{\varphi}}. \tag{2}$$

It is well known [11] that C_θ preserves K_θ. Moreover, $C_\theta(\theta H^2) = H^2_-$ and $C_\theta(H^2_-) = \theta H^2$, so C_θ also preserves $(K_\theta)^\perp$. Hence,

$$C_\theta = \begin{bmatrix} C_{\theta|K_\theta} & 0 \\ 0 & C_{\theta|(K_\theta)^\perp} \end{bmatrix}.$$

Recall that $k_w = \frac{1}{1-\bar{w}z}$ is a reproducing kernel for all functions $f \in H^2$, i.e., $f(w) = \langle f, k_w\rangle$ for $w \in \mathbb{D}$. Let θ be a nonconstant inner function. Then $k_w^\theta = P_\theta k_w = (1 - \overline{\theta(w)}\theta)k_w$ is a reproducing kernel for all functions $f \in K_\theta$, i.e., $f(w) = \langle f, k_w^\theta\rangle$ for $w \in \mathbb{D}$. Denote $\tilde{k}_w^\theta = C_\theta k_w^\theta$, $\tilde{k}_w^\theta(z) = \frac{\theta(z)-\theta(w)}{z-w}$. Note that $C_\theta f(w) = \overline{\langle f, \tilde{k}_w^\theta\rangle}$ for $w \in \mathbb{D}$.

2 Restrictions of Multiplication Operators and Its Basic Properties

Let θ, α be nonconstant inner functions. Recall that $K_\theta^\infty := K_\theta \cap L_\infty(\mathbb{T})$ is a dense subset of K_θ, (see [10]). Since $\bar{z}\overline{H^\infty}$ is a dense subset of H^2_- and θH^∞ is a dense subset of θH^2, it follows that $K_\theta^\perp \cap L_\infty(\mathbb{T})$ is a dense subset of $K_\theta^\perp$. For $\varphi \in L^2(\mathbb{T})$ we can consider the densely defined multiplication operator

$$M_\varphi \colon (K_\theta \cap L_\infty(\mathbb{T})) \oplus (K_\theta^\perp \cap L_\infty(\mathbb{T})) \to K_\alpha \oplus K_\alpha^\perp. \tag{3}$$

Define also

$$A_\varphi^{\theta,\alpha} = P_\alpha M_{\varphi|K_\theta \cap L_\infty(\mathbb{T})}, \qquad \tilde{B}_\varphi^{\theta,\alpha} = P_\alpha M_{\varphi|K_\theta^\perp \cap L_\infty(\mathbb{T})} \quad \text{and}$$

$$B_\varphi^{\theta,\alpha} = P_\alpha^\perp M_{\varphi|K_\theta \cap L_\infty(\mathbb{T})}, \qquad D_\varphi^{\theta,\alpha} = P_\alpha^\perp M_{\varphi|K_\theta^\perp \cap L_\infty(\mathbb{T})}.$$

Remark 2.1 Note that for $g \in K_\theta^\perp \cap L_\infty(\mathbb{T})$, $h \in K_\alpha \cap L_\infty(\mathbb{T})$ we have

$$\langle \tilde{B}_\varphi^{\theta,\alpha} g, h\rangle = \langle P_\alpha M_{\varphi|K_\theta^\perp \cap L_\infty(\mathbb{T})} g, h\rangle = \langle \varphi g, h\rangle = \int \varphi g \bar{h}\, dm$$
$$= \langle g, \bar{\varphi} h\rangle \ = \langle g, P_\theta^\perp M_{\bar{\varphi}|K_\alpha \cap L_\infty(\mathbb{T})} h\rangle = \langle (B_{\bar{\varphi}}^{\alpha,\theta})^* g, h\rangle.$$

Hence, according to the decomposition (3), the action of the operator M_φ is given by the matrix

$$\begin{bmatrix} P_\alpha M_{\varphi|K_\theta \cap L_\infty(\mathbb{T})} & P_\alpha M_{\varphi|K_\theta^\perp \cap L_\infty(\mathbb{T})} \\ P_\alpha^\perp M_{\varphi|K_\theta \cap L_\infty(\mathbb{T})} & P_\alpha^\perp M_{\varphi|K_\theta^\perp \cap L_\infty(\mathbb{T})} \end{bmatrix} \begin{bmatrix} g \\ h \end{bmatrix} = \begin{bmatrix} A_\varphi^{\theta,\alpha} & (B_{\bar{\varphi}}^{\alpha,\theta})^* \\ B_\varphi^{\theta,\alpha} & D_\varphi^{\theta,\alpha} \end{bmatrix} \begin{bmatrix} g \\ h \end{bmatrix}. \tag{4}$$

If $A_\varphi^{\theta,\alpha}$ extends to the whole K_θ as a bounded operator, it is called an *asymmetric truncated Toeplitz operator* (ATTO). Similarly, if $B_\varphi^{\theta,\alpha}$ extends to a bounded operator from K_θ to $K_\alpha^\perp$, it is called an *asymmetric big truncated Hankel operator* (ATHO) (see [13, 15]), and if $D_\varphi^{\theta,\alpha}$ extends to the whole $K_\theta^\perp$ as a bounded operator, it is called an *asymmetric dual truncated Toeplitz operator* (ADTTO).

Let us fix the notation

$$\mathcal{T}(K_\theta, K_\alpha) = \{A_\varphi^{\theta,\alpha} :\ \varphi \in L^2(\mathbb{T}) \text{ and } A_\varphi^{\theta,\alpha} \text{ is bounded}\},$$

$$\mathcal{T}(K_\theta, K_\alpha^\perp) = \{B_\varphi^{\theta,\alpha} :\ \varphi \in L^2(\mathbb{T}) \text{ and } B_\varphi^{\theta,\alpha} \text{ is bounded}\},$$

$$\mathcal{T}(K_\theta^\perp, K_\alpha^\perp) = \{D_\varphi^{\theta,\alpha} :\ \varphi \in L^2(\mathbb{T}) \text{ and } D_\varphi^{\theta,\alpha} \text{ is bounded}\}.$$

In case $\theta = \alpha$ we will use the shorter notation A_φ^θ, B_φ^θ, D_φ^θ and $\mathcal{T}(K_\theta)$ and $\mathcal{T}(K_\theta, K_\theta^\perp)$, $\mathcal{T}(K_\theta^\perp)$, respectively.

The following basic properties of restrictions of multiplication operators hold.

Lemma 2.2 *Let* $A_\varphi^{\theta,\alpha} \in \mathcal{T}(K_\theta, K_\alpha)$, $B_\varphi^{\theta,\alpha} \in \mathcal{T}(K_\theta, K_\alpha^\perp)$, $D_\varphi^{\theta,\alpha} \in \mathcal{T}(K_\theta^\perp, K_\alpha^\perp)$.

(a) Then $(A_\varphi^{\theta,\alpha})^* = A_{\bar{\varphi}}^{\alpha,\theta}$, $(D_\varphi^{\theta,\alpha})^* = D_{\bar{\varphi}}^{\alpha,\theta}$.

(b) If $\psi \in L_\infty(\mathbb{T})$ *then* $A_{\psi\varphi}^{\theta,\alpha} \in \mathcal{T}(K_\theta, K_\alpha)$, $D_{\psi\varphi}^{\theta,\alpha} \in \mathcal{T}(K_\theta^\perp, K_\alpha^\perp)$.

The relations of restrictions of M_φ and conjugations will be now considered.

Proposition 2.3 *Let α, θ be inner functions and $\varphi \in L^2(\mathbb{T})$. Assume that $A_\varphi^{\theta,\alpha} \in \mathcal{T}(K_\theta, K_\alpha)$, $B_\varphi^{\theta,\alpha} \in \mathcal{T}(K_\theta, K_\alpha^\perp)$, $D_\varphi^{\theta,\alpha} \in \mathcal{T}(K_\theta^\perp, K_\alpha^\perp)$. Then*

(a) $C_\alpha A_\varphi^{\theta,\alpha} = A_{\alpha\bar{\varphi}\bar{\theta}}^{\theta,\alpha} C_\theta$;
(b) $C_\alpha D_\varphi^{\theta,\alpha} = D_{\alpha\bar{\varphi}\bar{\theta}}^{\theta,\alpha} C_\theta$;
(c) $C_\alpha B_\varphi^{\theta,\alpha} = B_{\alpha\bar{\varphi}\bar{\theta}}^{\theta,\alpha} C_\theta$.

Proof For $f \in L_\infty(\mathbb{T})$ we have

$$C_\alpha M_\varphi f = C_\alpha(\varphi f) = \alpha\bar{z}\bar{\varphi}\bar{f} = M_{\alpha\bar{\varphi}\bar{\theta}}\theta\bar{z}\bar{f} = M_{\alpha\bar{\varphi}\bar{\theta}} C_\theta f. \tag{5}$$

Now, since K_θ and $K_\theta^\perp$ are invariant for C_θ (the same holds for α), for $g \in K_\theta \cap L_\infty(\mathbb{T})$, $h \in K_\theta^\perp \cap L_\infty(\mathbb{T})$ using matrix representation (4) we get

$$\begin{bmatrix} C_{\alpha|K_\alpha} & 0 \\ 0 & C_{\alpha|K_\alpha^\perp} \end{bmatrix} \begin{bmatrix} A_\varphi^{\theta,\alpha} & (B_{\bar{\varphi}}^{\alpha,\theta})^* \\ B_\varphi^{\theta,\alpha} & D_\varphi^{\theta,\alpha} \end{bmatrix} \begin{bmatrix} g \\ h \end{bmatrix}$$
$$= \begin{bmatrix} A_{\alpha\bar{\varphi}\bar{\theta}}^{\theta,\alpha} & (B_{\bar{\alpha}\varphi\theta}^{\alpha,\theta})^* \\ B_{\alpha\bar{\varphi}\bar{\theta}}^{\theta,\alpha} & D_{\alpha\bar{\varphi}\bar{\theta}}^{\theta,\alpha} \end{bmatrix} \begin{bmatrix} C_{\theta|K_\theta} & 0 \\ 0 & C_{\theta|K_\theta^\perp} \end{bmatrix} \begin{bmatrix} g \\ h \end{bmatrix}.$$

Hence

(a) $C_\alpha A_\varphi^{\theta,\alpha} g = A_{\alpha\bar{\varphi}\bar{\theta}}^{\theta,\alpha} C_\theta g$ for $g \in K_\theta \cap L_\infty(\mathbb{T})$;
(b) $C_\alpha D_\varphi^{\theta,\alpha} h = D_{\alpha\bar{\varphi}\bar{\theta}}^{\theta,\alpha} C_\theta h$ for $h \in K_\theta^\perp \cap L_\infty(\mathbb{T})$;
(c) $C_\alpha (B_{\bar{\varphi}}^{\alpha,\theta})^* h = (B_{\bar{\alpha}\varphi\theta}^{\alpha,\theta})^* C_\theta h$ for $h \in K_\theta^\perp \cap L_\infty(\mathbb{T})$;
(d) $C_\alpha B_\varphi^{\theta,\alpha} g = B_{\alpha\bar{\varphi}\bar{\theta}}^{\theta,\alpha} C_\theta g$ for $g \in K_\theta \cap L_\infty(\mathbb{T})$.

□

Corollary 2.4 *Let θ be an inner function and $\varphi \in L^2(\mathbb{T})$. Assume that $A_\varphi^\theta \in \mathcal{T}(K_\theta)$, $D_\varphi^\theta \in \mathcal{T}(K_\theta^\perp)$. Then A_φ^θ and D_φ^θ are C_θ–symmetric, i.e., $C_\theta A_\varphi^\theta = A_{\bar{\varphi}}^\theta C_\theta$ and $C_\theta D_\varphi^\theta = D_{\bar{\varphi}}^\theta C_\theta$.*

The following properties of the operators B_z^θ and $B_{\bar{z}}^\theta$ can be verified.

Lemma 2.5 *Let θ be an inner function. Then*

(a) $B_z^\theta = \theta \otimes \widetilde{k}_0^\theta$;
(b) $(B_z^\theta)^* = \widetilde{k}_0^\theta \otimes \theta$;
(c) $B_{\bar{z}}^\theta = \bar{z} \otimes k_0^\theta$;
(d) $(B_{\bar{z}}^\theta)^* = k_0^\theta \otimes \bar{z}$.

Proof To show (a) let $f \in K_\theta$. Then

$$B_z^\theta f = P_\theta^\perp(zf) = \theta P^+\bar{\theta}(zf) = \theta P^+\overline{C_\theta f} = \overline{C_\theta f(0)}\,\theta = \overline{\langle C_\theta f, k_0^\theta\rangle}\,\theta = \overline{\langle C_\theta k_0^\theta, f\rangle}\,\theta = \langle f, \tilde{k}_0^\theta\rangle\,\theta = \big(\theta \otimes \tilde{k}_0^\theta\big) f.$$

(c) is a consequence of Proposition 2.3, since

$$B_{\bar{z}}^\theta = C_\theta B_z^\theta C_\theta = C_\theta\big(\theta \otimes \tilde{k}_0^\theta\big)C_\theta = \big(C_\theta\theta \otimes C_\theta\tilde{k}_0^\theta\big) = \bar{z} \otimes k_0^\theta.$$

(b) and (d) are straightforward. □

Proposition 2.6 *Let θ be a nonconstant inner function. Then*

(a) $D_z^\theta D_{\bar{z}}^\theta = I_{K_\theta^\perp} - (1 - |\theta(0)|^2)\theta \otimes \theta$;
(b) $D_{\bar{z}}^\theta D_z^\theta = I_{K_\theta^\perp} - (1 - |\theta(0)|^2)\bar{z} \otimes \bar{z}$;
(c) $B_z^\theta (B_{\bar{z}}^\theta)^* = \overline{\theta'(0)}\,\theta \otimes \bar{z}$;
(d) $(B_{\bar{z}}^\theta)^* B_z^\theta = 0.$

Proof Note that for $f, g \in K_\theta^\perp$ we have

$$\begin{aligned}\langle D_z^\theta D_{\bar{z}}^\theta f, g\rangle &= \langle z(\bar{z} f - P_\theta(\bar{z} f)), g\rangle \\ &= \langle f - zP_\theta(\bar{z} f), g\rangle = \langle f, g\rangle - \langle zP_\theta(\bar{z} f), g\rangle.\end{aligned}$$

Since

$$\begin{aligned}P_\theta(\bar{z} f) &= P^+\theta P^-\bar{\theta}(\bar{z}P^- f + \bar{z}P_{\theta H^2} f) \\ &= P^+\theta P^-\bar{\theta}\bar{z}P_{\theta H^2} f = \langle\bar{\theta} f, 1\rangle P^+\theta\bar{z} = \langle f, \theta\rangle\tilde{k}_0^\theta,\end{aligned}$$

we have

$$\langle D_z^\theta D_{\bar{z}}^\theta f, g\rangle = \langle f, g\rangle - \langle f, \theta\rangle\langle z\tilde{k}_0^\theta, g\rangle = \langle f, g\rangle - \langle f, \theta\rangle\langle P_\theta^\perp(z\tilde{k}_0^\theta), g\rangle.$$

Note that $P_\theta^\perp(z\tilde{k}_0^\theta) = (1 - |\theta(0)|^2)\theta$. Hence

$$\begin{aligned}\langle D_z^\theta D_{\bar{z}}^\theta f, g\rangle &= \langle f, g\rangle - \langle\langle f, \theta\rangle(1 - |\theta(0)|^2)\theta, g\rangle \\ &= \langle(I_{k_\theta^\perp} - (1 - |\theta(0)|^2)\theta \otimes \theta) f, g\rangle.\end{aligned}$$

To prove (b) note that, by Proposition 2.3,

$$\begin{aligned}D_{\bar{z}}^\theta D_z^\theta &= C_\theta D_z^\theta D_{\bar{z}}^\theta C_\theta = C_\theta(I_{k_\theta^\perp} - (1 - |\theta(0)|^2)\theta \otimes \theta)C_\theta \\ &= I_{k_\theta^\perp} - (1 - |\theta(0)|^2)C_\theta\theta \otimes C_\theta\theta = I_{K_\theta^\perp} - (1 - |\theta(0)|^2)\bar{z} \otimes \bar{z}.\end{aligned}$$

Calculating (c) we will use Lemma 2.5 and the formula for multiplication of rank-one operators (see [17])

$$B_z^\theta (B_{\bar{z}}^\theta)^* = (\theta \otimes \widetilde{k}_0^\theta)(k_0^\theta \otimes \bar{z}) = \langle k_0^\theta, \tilde{k}_0^\theta \rangle \, \theta \otimes \bar{z} = \overline{\theta'(0)}\, \theta \otimes \bar{z}.$$

The last formula can be obtained similarly. □

3 Basic Properties of ADTTO

It was shown that an asymmetric truncated Toeplitz operator can be bounded even if it has no bounded symbol. The same is true for an asymmetric big truncated Hankel operator. In the case of bounded dual asymmetric Toeplitz operators the symbol is always bounded and unique.

Proposition 3.1 *Let $\varphi \in L^2(\mathbb{T})$. Then $D_\varphi^{\theta,\alpha}$ is bounded if and only if $\varphi \in L_\infty(\mathbb{T})$. Moreover, in that case, $\|D_\varphi^{\theta,\alpha}\| = \|\varphi\|_\infty$.*

Proof Let $f \in H^\infty$. Then $\theta f \in \theta H^\infty \subset \theta H^2$ and

$$\begin{aligned}\|D_\varphi^{\theta,\alpha}(\theta f)\|^2 =& \|(P^- + \alpha P^+ \bar{\alpha})(\theta f)\|^2 = \|P^-(\theta f)\|^2 + \|\alpha P^+ \bar{\alpha}\theta f\|^2 \\ =& \|P^-(\theta f)\|^2 + \|\alpha P^+ \bar{\alpha}\theta f\|^2 \geqslant \|T_{\bar{\alpha}\varphi\theta} f\|^2.\end{aligned}$$

If the operator $D_\varphi^{\theta,\alpha}$ is bounded, then there is a constant $C > 0$ such that $\|D_\varphi^{\theta,\alpha}(\theta f)\| \leqslant C\|f\|$. Hence $\|T_{\bar{\alpha}\varphi\theta} f\| \leqslant C\|f\|$ for every $f \in H^\infty$, which implies that $T_{\bar{\alpha}\varphi\theta}$ is bounded and in consequence $\bar{\alpha}\varphi\theta \in L_\infty(\mathbb{T})$. Thus $\varphi \in L_\infty(\mathbb{T})$ and $\|\varphi\|_\infty = \|T_{\bar{\alpha}\varphi\theta}\| \leqslant \|D_\varphi^{\theta,\alpha}\|$.

If now $\varphi \in L_\infty(\mathbb{T})$, then for any $f \in K_\theta^\perp$ we have

$$\|D_\varphi^{\theta,\alpha} f\| = \|P_\alpha^\perp(\varphi f)\| \leqslant \|\varphi\|_\infty \|f\|.$$

Hence $D_\varphi^{\theta,\alpha}$ is bounded and, moreover, $\|D_\varphi^{\theta,\alpha}\| \leqslant \|\varphi\|_\infty$. □

The only compact Toeplitz operator is the zero operator. The same is true for asymmetric dual truncated Toeplitz operators.

Proposition 3.2 *Let $\varphi \in L_\infty(\mathbb{T})$. Then $D_\varphi^{\theta,\alpha}$ is compact if and only if $\varphi = 0$.*

Proof Let $f_n \in \theta H^2 \subset K_\theta^\perp$ be weakly convergent to 0 ($f_n \rightharpoonup 0$). Then $f_n = \theta \tilde{f}_n$ for $\tilde{f}_n \in H^2$. Note that $f_n \rightharpoonup 0$ if and only if $\tilde{f}_n \rightharpoonup 0$. If $D_\varphi^{\theta,\alpha}$ is compact, then $\|D_\varphi^{\theta,\alpha} f_n\| = \|D_\varphi^{\theta,\alpha}\theta \tilde{f}_n\| \to 0$. Since (as in the proof of Proposition 3.1) $\|T_{\bar{\alpha}\theta\varphi}\tilde{f}_n\| \leqslant \|D_\varphi^{\theta,\alpha}(\theta \tilde{f}_n)\| \to 0$, therefore compactness of $D_\varphi^{\theta,\alpha}$ implies compactness of the Toeplitz operator $T_{\bar{\alpha}\theta\varphi}$, which leads to the conclusion that $\varphi = 0$. □

An important consequence of Proposition 3.2 is that the only symbol for the zero asymmetric dual truncated Toeplitz operator is $\varphi = 0$. What follows is that each ADTTO has a unique symbol.

Proposition 3.3 *Let α, θ be inner functions and let $\varphi \in L_\infty(\mathbb{T})$. If $D_\varphi^{\theta,\alpha}$ is invertible, then φ is invertible in $L_\infty(\mathbb{T})$.*

Proof By Douglas [8, Corollary 4.24] we know that the operator M_φ is invertible in $L^2(\mathbb{T})$ if and only if φ is invertible in $L_\infty(\mathbb{T})$. Assume that $D_\varphi^{\theta,\alpha}$ is invertible, then there is a constant $c > 0$ such that

$$\|D_\varphi^{\theta,\alpha} f\| \geqslant c\|f\| \tag{6}$$

for all $f \in K_\theta^\perp$. Hence for all integers k and $g \in H^2$ we have

$$\|M_\varphi z^k g\| = \|\varphi z^k g\| = \|\varphi\theta g\| \geqslant \|P_\alpha^\perp \varphi\theta g\| = \|D_\varphi^{\theta,\alpha}\theta g\| \geqslant c\|\theta g\| = c\|z^k g\|.$$

Since the set $\{z^k g : k \in \mathbb{Z}, g \in H^2\}$ is dense in $L^2(\mathbb{T})$, we have that for $x \in L^2(\mathbb{T})$

$$\|M_\varphi x\| \geqslant c\|x\|.$$

Since $(D_\varphi^{\theta,\alpha})^* = D_{\bar\varphi}^{\alpha,\theta}$ is also invertible, thus $\|M_{\bar\varphi} x\| \geqslant c\|f\|$, we conclude that M_φ is invertible by Douglas [8, Corollary 4.9]. □

4 Intertwining Property for ADTTO

Since the unilateral shift S is unitarily equivalent to the Toeplitz operator T_z we are able to describe the commutant of the unilateral shift as

$$\{S\}' = \{T_z\}' = \{T_\varphi : \varphi \in H^\infty := L_\infty(\mathbb{T}) \cap H^2\}.$$

Now considering the compressions A_z^θ and D_z^θ, it is natural to try to describe the commutants $\{A_z^\theta\}'$ and $\{D_z^\theta\}'$. In the more general asymmetric setting we are searching for all operators intertwining A_z^θ and A_z^α in the case of two model spaces or searching for all operators intertwining D_z^θ and D_z^α in the case of orthogonal complements of two model spaces, i.e., we try to describe the following sets of operators

$$\mathcal{I}(K_\theta, K_\alpha) = \{B \in \mathbb{B}(K_\theta, K_\alpha)\colon A_z^\alpha B = B A_z^\theta\},$$

$$\mathcal{I}(K_\theta^\perp, K_\alpha^\perp) = \{B \in \mathbb{B}(K_\theta^\perp, K_\alpha^\perp)\colon D_z^\alpha B = B D_z^\theta\}.$$

The following result describes $\mathcal{T}(K_\theta^\perp, K_\alpha^\perp) \cap \mathcal{I}(K_\theta^\perp, K_\alpha^\perp)$.

Theorem 4.1 *Let α, θ be nonconstant inner functions and $\varphi \in L_\infty(\mathbb{T})$, $\varphi \neq 0$. Then*

(a) $D_\varphi^{\theta,\alpha} D_z^\theta - D_z^\alpha D_\varphi^{\theta,\alpha} = \alpha \otimes P_\theta^\perp(\bar{\varphi}\tilde{k}_0^\alpha) - P_\alpha^\perp(\varphi k_0^\theta) \otimes \bar{z}$;
(b) $D_\varphi^{\theta,\alpha} D_{\bar{z}}^\theta - D_{\bar{z}}^\alpha D_\varphi^{\theta,\alpha} = \bar{z} \otimes P_\theta^\perp(\bar{\varphi} k_0^\alpha) - P_\alpha^\perp(\varphi \tilde{k}_0^\theta) \otimes \theta$.

Proof Let $\varphi \in L_\infty(\mathbb{T}) \setminus \{0\}$. The commutation relation $M_\varphi M_z = M_z M_\varphi$ can be written as

$$\begin{bmatrix} A_\varphi^{\theta,\alpha} & (B_{\bar{\varphi}}^{\alpha,\theta})^* \\ B_\varphi^{\theta,\alpha} & D_\varphi^{\theta,\alpha} \end{bmatrix} \begin{bmatrix} A_z^\theta & (B_{\bar{z}}^\theta)^* \\ B_z^\theta & D_z^\theta \end{bmatrix} = \begin{bmatrix} A_z^\alpha & (B_{\bar{z}}^\alpha)^* \\ B_z^\alpha & D_z^\alpha \end{bmatrix} \begin{bmatrix} A_\varphi^{\theta,\alpha} & (B_{\bar{\varphi}}^{\alpha,\theta})^* \\ B_\varphi^{\theta,\alpha} & D_\varphi^{\theta,\alpha} \end{bmatrix}.$$

Thus we have

$$B_\varphi^{\theta,\alpha}(B_{\bar{z}}^\theta)^* + D_\varphi^{\theta,\alpha} D_z^\theta = B_z^\alpha (B_{\bar{\varphi}}^{\alpha,\theta})^* + D_z^\alpha D_\varphi^{\theta,\alpha}.$$

Clearly,

$$B_\varphi^{\theta,\alpha} k_0^\theta = P_\alpha^\perp(\varphi k_0^\theta) \quad \text{and} \quad B_{\bar{\varphi}}^{\alpha,\theta} \tilde{k}_0^\alpha = P_\theta^\perp(\bar{\varphi}\tilde{k}_0^\alpha).$$

and, by Proposition 2.3,

$$B_{\bar{\varphi}}^{\alpha,\theta} \tilde{k}_0^\alpha = B_{\bar{\varphi}}^{\alpha,\theta} C_\alpha k_0^\alpha = C_\theta B_{\theta\varphi\bar{\alpha}}^{\alpha,\theta} k_0^\alpha = C_\theta P_\theta^\perp(\theta\varphi\bar{\alpha}k_0^\alpha).$$

Therefore, by Lemma 2.5, we get

$$\begin{aligned} D_\varphi^{\theta,\alpha} D_z^\theta - D_z^\alpha D_\alpha^{\theta,\alpha} &= B_z^\alpha (B_\varphi^{\theta,\alpha})^* - B_\varphi^{\theta,\alpha}(B_{\bar{z}}^\theta)^* \\ &= (\alpha \otimes \tilde{k}_0^\alpha)(B_{\bar{\varphi}}^{\alpha,\theta})^* - B_\varphi^{\theta,\alpha}(k_0^\theta \otimes \bar{z}) \\ &= \alpha \otimes (B_{\bar{\varphi}}^{\alpha,\theta}\tilde{k}_0^\alpha) - (B_\varphi^{\theta,\alpha} k_0^\theta) \otimes \bar{z} \\ &= \alpha \otimes P_\theta^\perp(\bar{\varphi}\tilde{k}_0^\alpha) - P_\alpha^\perp(\varphi k_0^\theta) \otimes \bar{z} \end{aligned}$$

which proves (a).

To see (b) we start with the relation $M_\varphi M_{\bar{z}} = M_{\bar{z}} M_\varphi$, which can be written as

$$\begin{bmatrix} A_\varphi^{\theta,\alpha} & (B_{\bar{\varphi}}^{\alpha,\theta})^* \\ B_\varphi^{\theta,\alpha} & D_\varphi^{\theta,\alpha} \end{bmatrix} \begin{bmatrix} A_{\bar{z}}^\theta & (B_z^\theta)^* \\ B_{\bar{z}}^\theta & D_{\bar{z}}^\theta \end{bmatrix} = \begin{bmatrix} A_{\bar{z}}^\alpha & (B_z^\alpha)^* \\ B_{\bar{z}}^\alpha & D_{\bar{z}}^\alpha \end{bmatrix} \begin{bmatrix} A_\varphi^{\theta,\alpha} & (B_{\bar{\varphi}}^{\alpha,\theta})^* \\ B_\varphi^{\theta,\alpha} & D_\varphi^{\theta,\alpha} \end{bmatrix}.$$

Thus we have

$$B_\varphi^{\theta,\alpha}(B_z^\theta)^* + D_\varphi^{\theta,\alpha} D_{\bar{z}}^\theta = B_{\bar{z}}^\alpha (B_{\bar{\varphi}}^{\alpha,\theta})^* + D_{\bar{z}}^\alpha D_\varphi^{\theta,\alpha}.$$

Clearly,

$$B_{\varphi}^{\theta,\alpha} k_0^{\theta} = P_{\alpha}^{\perp}(\varphi k_0^{\theta}) \quad \text{and} \quad B_{\bar{\varphi}}^{\alpha,\theta} \tilde{k}_0^{\alpha} = P_{\theta}^{\perp}(\bar{\varphi}\tilde{k}_0^{\alpha}).$$

and, by Proposition 2.3,

$$B_{\bar{\varphi}}^{\alpha,\theta} \tilde{k}_0^{\alpha} = B_{\bar{\varphi}}^{\alpha,\theta} C_{\alpha} k_0^{\alpha} = C_{\theta} B_{\theta\varphi\bar{\alpha}}^{\alpha,\theta} k_0^{\alpha} = C_{\theta} P_{\theta}^{\perp}(\theta\varphi\bar{\alpha} k_0^{\alpha}).$$

Therefore, by Lemma 2.5, we get

$$\begin{aligned} D_{\varphi}^{\theta,\alpha} D_{\bar{z}}^{\theta} - D_{\bar{z}}^{\alpha} D_{\alpha}^{\theta,\alpha} &= B_{\bar{z}}^{\alpha} (B_{\varphi}^{\theta,\alpha})^* - B_{\varphi}^{\theta,\alpha} (B_{z}^{\theta})^* \\ &= (\bar{z} \otimes k_0^{\alpha})(B_{\bar{\varphi}}^{\alpha,\theta})^* - B_{\varphi}^{\theta,\alpha} (\tilde{k}_0^{\theta} \otimes \theta) \\ &= \bar{z} \otimes (B_{\bar{\varphi}}^{\alpha,\theta} k_0^{\alpha}) - (B_{\varphi}^{\theta,\alpha} \tilde{k}_0^{\theta}) \otimes \theta \\ &= \bar{z} \otimes P_{\theta}^{\perp}(\bar{\varphi} k_0^{\alpha}) - P_{\alpha}^{\perp}(\varphi \tilde{k}_0^{\theta}) \otimes \theta. \end{aligned}$$

□

Theorem 4.2 *Let α, θ be nonconstant inner functions and $\varphi \in L_{\infty}(\mathbb{T})$, $\varphi \neq 0$. Then $D_{\varphi}^{\theta,\alpha} D_{z}^{\theta} = D_{z}^{\alpha} D_{\varphi}^{\theta,\alpha}$ if and only if one of the following holds*

(a) $\alpha(0) = 0 = \theta(0)$ and $\varphi \in \frac{\alpha}{\gcd(\alpha,\theta)} K_{z\cdot\gcd(\alpha,\theta)}$, or

(b) $\alpha = \theta$ and $\varphi \in (k_0^{\theta})^{-1} K_{z\theta}$.

Proof Applying Theorem 4.1

$$D_{\varphi}^{\theta,\alpha} D_{z}^{\theta} = D_{z}^{\alpha} D_{\varphi}^{\theta,\alpha} \tag{7}$$

if and only if there is a constant $c \in \mathbb{C}$ such that

$$\begin{cases} P_{\alpha}^{\perp}(\varphi k_0^{\theta}) = c\,\alpha, \\ P_{\theta}^{\perp}(\bar{\varphi}\tilde{k}_0^{\alpha}) = \bar{c}\,\bar{z}. \end{cases} \tag{8}$$

Since

$$\begin{aligned} C_{\theta} P_{\theta}^{\perp}(\bar{\varphi}\tilde{k}_0^{\alpha}) = P_{\theta}^{\perp} C_{\theta}(\bar{\varphi}\tilde{k}_0^{\alpha}) = P_{\theta}^{\perp}(\theta\bar{\alpha}\varphi\alpha\bar{z}\overline{\tilde{k}_0^{\alpha}}) \\ = P_{\theta}^{\perp}(\theta\bar{\alpha}\varphi C_{\alpha}\tilde{k}_0^{\alpha}) = P_{\theta}^{\perp}(\theta\bar{\alpha}\varphi k_0^{\alpha}) \end{aligned}$$

the above is equivalent to

$$\begin{cases} P_{\alpha}^{\perp}(\varphi k_0^{\theta}) = c\,\alpha, \\ P_{\theta}^{\perp}(\theta\bar{\alpha}\varphi k_0^{\alpha}) = c\,\theta. \end{cases} \tag{9}$$

Hence, there are $g \in K_\alpha$ and $h \in K_\theta$ such that

$$\varphi k_0^\theta = c\alpha + g \qquad \text{and} \tag{10}$$

$$\theta\bar{\alpha}\varphi k_0^\alpha = c\theta + h. \tag{11}$$

Since the functions k_0^α, k_0^θ are bounded from below and analytic (in consequence we have $(k_0^\alpha)^{-1}, (k_0^\theta)^{-1} \in H^\infty$), we get by (10) that $\varphi \in H^2$. Let $\gamma = \gcd(\alpha, \theta)$. Then, by (11),

$$\varphi k_0^\alpha - c\alpha = \bar{\theta}\alpha h = \frac{\bar{\theta}}{\bar{\gamma}}\frac{\alpha}{\gamma}h \in H^2.$$

Hence h is divisible by $\frac{\theta}{\gamma}$ and since $h \in K_\theta = K_{\frac{\theta}{\gamma}} \oplus \frac{\theta}{\gamma}K_\gamma$, we have

$$h = \frac{\theta}{\gamma}h_1 \quad \text{with } h_1 \in K_\gamma.$$

Therefore, by (11),

$$\frac{\bar{\alpha}}{\bar{\gamma}}\varphi k_0^\alpha = c\gamma + h_1 \in H^2.$$

Since k_0^α is an outer function, it cannot be divisible by $\frac{\alpha}{\gamma}$, which implies that

$$\varphi = \frac{\alpha}{\gamma}\varphi_1 \quad \text{with } \varphi_1 \in H^2 \setminus \{0\}.$$

Hence by (10) we get $\frac{\alpha}{\gamma}\varphi_1 k_0^\theta = c\alpha + g$. Therefore $g = \frac{\alpha}{\gamma}g_1$, $g_1 \in K_\gamma$, and (10) and (11) are equivalent to

$$\varphi_1 k_0^\theta = c\gamma + g_1, \quad g_1 \in K_\gamma, \tag{12}$$

$$\varphi_1 k_0^\alpha = c\gamma + h_1, \quad h_1 \in K_\gamma. \tag{13}$$

Moreover,

$$P_\gamma(\varphi_1) = P_\gamma(\varphi_1 k_0^\theta + \overline{\theta(0)}\theta\varphi_1) = P_\gamma(c\gamma + g_1 + \overline{\theta(0)}\theta\varphi_1) = g_1.$$

Similarly, $P_\gamma(\varphi_1) = h_1$, so $g_1 = h_1$. Comparing (12) with (13) we thus get $\varphi_1(k_0^\alpha - k_0^\theta) = 0$. Since $\varphi_1, (k_0^\alpha - k_0^\theta) \in H^2$ and $\varphi_1 \neq 0$, we must have

$$k_0^\alpha - k_0^\theta = \overline{\theta(0)}\theta - \overline{\alpha(0)}\alpha = 0.$$

This is possible only in two cases:

1. if $\alpha(0) = 0 = \theta(0)$, then $k_0^\alpha = k_0^\theta = 1$ and by (12),

$$\varphi = \frac{\alpha}{\gamma}\varphi_1 \in \frac{\alpha}{\gamma}(K_\gamma \oplus \mathbb{C}\gamma) = \frac{\alpha}{\gamma}K_{z\cdot\gamma};$$

2. if $\alpha = \theta$, then (12) and (13) become the same condition, equivalent to $\varphi k_0^\theta \in K_\theta \oplus \mathbb{C}\theta = K_{z\theta}$, which leads to $\varphi \in (k_0^\theta)^{-1} K_{z\theta}$.

□

Corollary 4.3 *Let θ be a nonconstant inner function and let $\varphi \in L_\infty(\mathbb{T})$, $\varphi \neq 0$. Then $D_\varphi^\theta \in \{D_z^\theta\}'$ if and only if $\varphi \in (k_0^\theta)^{-1} K_{z\theta}$.*

Example Let $a \in \mathbb{D}$; we denote by B_a the Blaschke factor with zero at a, i.e., $B_a(z) = \frac{a-z}{1-\bar{a}z}$. Let $\theta = zB_b$ and $\alpha = zB_a$ with $a \neq b$, $a \neq 0$, $b \neq 0$, $a, b \in \mathbb{D}$. Then $\gcd(\theta, \alpha) = z$. By Theorem 4.2 for $\varphi \in L_\infty(\mathbb{T})$ we have $D_\varphi^{\theta,\alpha} \in \mathcal{I}(K_\theta^\perp, K_\alpha^\perp)$ if and only if

$$\varphi \in B_a K_{z^2} = \left\{ \frac{aa_0 + (aa_1 - a_0)z - a_1 z^2}{1-\bar{a}z} : a_0, a_1 \in \mathbb{D} \right\}.$$

Example Let $\theta(z) = \exp\{\frac{z+1}{z-1}\}$, $\alpha(z) = \exp\{a\,\frac{z+1}{z-1}\}$ with $0 < a < 1$. Then there are no dual truncated operators intertwining D_z^θ and D_z^α.

5 Other Relations with ADTTO

A classical result shows that an operator T on H^2 is Toeplitz if and only if $T - S^* T S = 0$. Sarason in [19] showed that if A is a TTO in the model space K_θ then $A - (A_z^\theta)^* A A_z^\theta$ is a specific rank two operator with easily seen symbol. First we try to calculate this expression for DTTO.

Proposition 5.1 *Let $D_\varphi^{\theta,\alpha} \in \mathcal{T}(K_\theta^\perp, K_\alpha^\perp)$. Then*

(a) $D_\varphi^{\theta,\alpha} - D_z^\alpha D_\varphi^{\theta,\alpha} D_{\bar{z}}^\theta = P_\alpha^\perp(\varphi\bar{z}\tilde{k}_0^\theta) \otimes \theta + \alpha \otimes P_\theta^\perp(\bar{\varphi}\bar{z}\tilde{k}_0^\alpha)$;
(b) $D_\varphi^{\theta,\alpha} - D_{\bar{z}}^\alpha D_\varphi^{\theta,\alpha} D_z^\theta = P_\alpha^\perp(\varphi\bar{z}k_0^\theta) \otimes \bar{z} + \bar{z} \otimes P_\theta^\perp(\bar{\varphi}\bar{z}k_0^\alpha)$.

Proof By Theorem 4.1 we have

$$D_\varphi^{\theta,\alpha} D_z^\theta D_{\bar{z}}^\theta - D_z^\alpha D_\varphi^{\theta,\alpha} D_{\bar{z}}^\theta = \alpha \otimes D_z^\theta P_\theta^\perp(\bar{\varphi}\tilde{k}_0^\alpha) - P_\alpha^\perp(\varphi k_0^\theta) \otimes D_z^\theta \bar{z}.$$

Since $D_z^\theta \bar{z} = P_\theta^\perp 1 = \overline{\theta(0)}\theta$, using Proposition 2.6 (a) we get

$$D_\varphi^{\theta,\alpha} - D_z^\theta D_\varphi^{\theta,\alpha} D_{\bar{z}}^\theta = \big((1 - |\theta(0)|^2) D_\varphi^{\theta,\alpha}\theta - \theta(0) P_\alpha^\perp(\varphi k_0^\theta)\big) \otimes \theta + \alpha \otimes D_z^\theta P_\theta^\perp(\bar{\varphi}\tilde{k}_0^\alpha). \tag{14}$$

Since $D_\varphi^{\theta,\alpha}(\theta) = P_\alpha^\perp(\theta\varphi)$, we obtain

$$\begin{aligned}(1-|\theta(0)|^2)D_\varphi^{\theta}\theta,\alpha-\theta(0)P_\alpha^\perp(\varphi k_0^\theta) &= P_\alpha^\perp(\theta\varphi - |\theta(0)|^2\theta\varphi - \theta(0)\varphi k_0^\theta)\\ &= P_\alpha^\perp(\theta\varphi - |\theta(0)|^2\theta\varphi - \theta(0)\varphi + |\theta(0)|^2\theta\varphi)\\ &= P_\alpha^\perp(\varphi(\theta-\theta(0))) = P_\alpha^\perp(\varphi\bar{z}\tilde{k}_0^\theta).\end{aligned}$$

The proof will be completed with

$$D_z^\theta P_\theta^\perp(\bar{\varphi}\tilde{k}_0^\alpha) = P_\theta^\perp(D_z^\theta(\bar{\varphi}\bar{z}(\alpha-\alpha(0))) = P_\theta^\perp(\bar{\varphi}(\alpha-\alpha(0))) = P_\theta^\perp(\bar{\varphi}\bar{z}\tilde{k}_0^\alpha).$$

To prove (b) write (a) for $\alpha\bar{\varphi}\bar{\theta} \in L_\infty(\mathbb{T})$ and apply C_α and C_θ, then

$$\begin{aligned}&C_\alpha D_{\alpha\bar{\varphi}\bar{\theta}}^{\theta,\alpha} C_\theta - (C_\alpha D_z^\alpha C_\alpha)(C_\alpha D_{\alpha\bar{\varphi}\bar{\theta}}^{\theta,\alpha} C_\theta)(C_\theta D_{\bar{z}}^\theta C_\theta)\\ &\qquad = C_\alpha\Big(P_\alpha^\perp(\alpha\bar{\varphi}\bar{\theta}(\theta-\theta(0)))\otimes\theta \;+\; \alpha\otimes P_\theta^\perp(\bar{\alpha}\varphi\theta(\alpha-\alpha(0)))\Big)C_\theta.\end{aligned}$$

Hence by Proposition 2.3 we have

$$\begin{aligned}&D_\varphi^{\theta,\alpha} - D_{\bar{z}}^\alpha D_\varphi^{\theta,\alpha} D_z^\theta\\ &\quad = \Big(C_\alpha P_\alpha^\perp(\alpha\bar{\varphi}\bar{\theta}(\theta-\theta(0)))\Big)\otimes C_\theta\theta + C_\alpha\alpha\otimes\Big(C_\theta P_\theta^\perp(\bar{\alpha}\varphi\theta(\alpha-\alpha(0)))\Big)\\ &\quad = \Big(P_\alpha^\perp C_\alpha(\alpha\bar{\varphi}(1-\theta(0)\bar{\theta}))\Big)\otimes\bar{z} \;+\; \bar{z}\otimes\Big(P_\theta^\perp C_\theta(\varphi\theta(1-\alpha(0)\bar{\alpha}))\Big)\\ &\quad = P_\alpha^\perp(\varphi\bar{z}k_0^\theta)\otimes\bar{z} \;+\; \bar{z}\otimes P_\theta^\perp(\bar{\varphi}\bar{z}k_0^\alpha).\end{aligned}$$

□

Corollary 5.2 *Let $D_\varphi^\theta \in \mathcal{T}(K_\theta^\perp)$. Then*

(a) $D_\varphi^\theta - D_z^\theta D_\varphi^\theta D_{\bar{z}}^\theta = P_\theta^\perp(\varphi\bar{z}\tilde{k}_0^\theta)\otimes\theta + \theta\otimes P_\theta^\perp(\bar{\varphi}\bar{z}\tilde{k}_0^\theta)$*;*
(b) $D_\varphi^\theta - D_{\bar{z}}^\theta D_\varphi^\theta D_z^\theta = P_\theta^\perp(\varphi\bar{z}k_0^\theta)\otimes\bar{z} + \bar{z}\otimes P_\theta^\perp(\bar{\varphi}\bar{z}k_0^\theta)$.

Example Let $\theta = B_a$ with $a \in \mathbb{D}$, then $k_0^\theta = \frac{1-|a|^2}{1-\bar{a}z}$ and $\tilde{k}_0^\theta = \frac{|a|^2-1}{1-\bar{a}z}$. Consider $\varphi \equiv 1$, then

$$P_\theta^\perp(\varphi\bar{z}\tilde{k}_0^\theta) = P_\theta^\perp(\bar{z}\tilde{k}_0^\theta) = P_\theta^\perp\Big((|a|^2-1)(\bar{z}+\bar{a}\tfrac{1}{1-\bar{a}z})\Big) = (|a|^2-1)\bar{z}.$$

Thus

$$D_1^\theta - D_z^\theta D_1^\theta D_{\bar{z}}^\theta = P_\theta^\perp(\bar{z}\tilde{k}_0^\theta)\otimes\theta + \theta\otimes P_\theta^\perp(\bar{z}k_0^\theta) = (|a|^2-1)(\theta\otimes\bar{z} + \bar{z}\otimes\theta).$$

Consider now $\varphi(z) = z(1 - \bar{a}z)$. Then

$$P_\theta^\perp(\varphi\bar{z}\tilde{k}_0^\theta) = P_\theta^\perp((1 - \bar{a}z)\tilde{k}_0^\theta) = P_\theta^\perp(|a|^2 - 1)$$
$$= (|a|^2 - 1)(1 - k_0^\theta) = (|a|^2 - 1)\bar{a}\theta$$

and

$$P_\theta^\perp(\bar{\varphi}\bar{z}\tilde{k}_0^\theta) = (|a|^2 - 1)P_\theta^\perp\big(\bar{z}^2(1 - a\bar{z})\,\tfrac{1}{1-\bar{a}z}\big) = (|a|^2 - 1)\,p(\bar{z}),$$

where $p(\bar{z}) = (\bar{a}(1 - |a|^2))\bar{z} + (1 - |a|^2)\bar{z}^2 - a\bar{z}^3$. Therefore

$$D^\theta_{z(1-\bar{a}z)} - D^\theta_z D^\theta_{z(1-\bar{a}z)} D^\theta_{\bar{z}} = (|a|^2 - 1)\big(\bar{a}\,\theta \otimes \theta + \theta \otimes p(\bar{z})\big).$$

We see the variety of the above expressions.

6 A Characterization of ADTTO

In this section our goal is to give a characterization of (asymmetric) dual truncated Toeplitz operators. The simplest approach is to require that the given operator has to fulfill some equation(s). Then we must have the possibility to find the symbol of the given operator. The classical Brown–Halmos result shows that a bounded linear operator $T \in \mathbb{B}(H^2)$ is a Toeplitz operator if and only if $T = (T_z)^* T T_z$. Similar characterizations (in terms of compressions of M_z) are known for Hankel operators and dual Toeplitz operators. In [19] D. Sarason characterized bounded truncated Toeplitz operators in terms of the compressions of M_z to K_θ. In particular, he proved that a bounded operator $A \in \mathbb{B}(K_\theta)$ is a truncated Toeplitz operator if and only if

$$A - A^\theta_z A A^\theta_{\bar{z}} = \psi \otimes k_0^\theta + k_0^\theta \otimes \chi \tag{15}$$

for some $\psi, \chi \in K_\theta$. In other words, the left hand side of (15) can be expressed as an operator of rank at most two. In this section our aim is to give similar expressions for operators from $\mathcal{T}(K_\theta^\perp, K_\alpha^\perp)$ using operators of rank at most two.

The characterization (15) proved in [19] for truncated Toeplitz operators immediately gives a symbol of the truncated Toeplitz operator $A = A^\theta_{\psi+\bar{\chi}}$. Moreover, the relation between ψ and χ is simple, see [19, Corollary after Theorem 3.1]. However, for any asymmetric dual truncated Toeplitz operator, the functions $\mu = P_\alpha^\perp(\varphi\bar{z}\tilde{k}_0^\theta)$, $\nu = P_\theta^\perp(\bar{\varphi}\bar{z}\tilde{k}_0^\alpha)$ in the formula (a) in Proposition 5.1, strongly and in a very complicated way depend on each other. Moreover, in case of dual truncated Toeplitz operators, having the rank-two operator on the right hand side of (5.1), $\mu\otimes\theta+\alpha\otimes\nu$ with $\mu, \nu \in K_\theta^\perp$, we are far from obtaining the symbol of D. For this reason, to answer a natural question when an operator $D \in \mathbb{B}(K_\theta^\perp, K_\alpha^\perp)$ is a ADTTO and to find its symbol, we will consider the matrix decomposition of D. This will be done in Theorem 6.7.

First the compressions of ADTTO's to certain subspaces of $K_\theta^\perp$ and $K_\alpha^\perp$ will be considered. Let θ, α be two inner functions. Using the decompositions $K_\theta^\perp = \theta H^2 \oplus H^2_-$ and $K_\alpha^\perp = \alpha H^2 \oplus H^2_-$ one can write each operator $D \in \mathbb{B}(K_\theta^\perp, K_\alpha^\perp)$ as a matrix

$$D = \begin{bmatrix} P_{\alpha H^2} D_{|\theta H^2} & P_{\alpha H^2} D_{|H^2_-} \\ P^- D_{|\theta H^2} & P^- D_{|H^2_-} \end{bmatrix}.$$

In particular, for $\varphi \in L_\infty(\mathbb{T})$, we obtain

$$D_\varphi^{\theta,\alpha} = \begin{bmatrix} \hat{T}_\varphi^{\theta,\alpha} & \check{\Gamma}_\varphi^\alpha \\ \hat{\Gamma}_\varphi^\theta & \check{T}_\varphi \end{bmatrix} = \begin{bmatrix} \hat{T}_\varphi^{\theta,\alpha} & (\hat{\Gamma}_{\bar{\varphi}}^\alpha)^* \\ \hat{\Gamma}_\varphi^\theta & \check{T}_\varphi \end{bmatrix}, \tag{16}$$

where

$$\hat{T}_\varphi^{\theta,\alpha} = P_{\alpha H^2} M_{\varphi|\theta H^2}, \quad \hat{\Gamma}_\varphi^\theta = P^- M_{\varphi|\theta H^2} \tag{17}$$

and

$$\check{\Gamma}_\varphi^\alpha = P_{\alpha H^2} M_{\varphi|H^2_-}, \quad \check{T}_\varphi = P^- M_{\varphi|H^2_-}. \tag{18}$$

Let us denote

$$\begin{aligned}
\mathcal{T}(\theta H^2, \alpha H^2) &= \{\hat{T} \in \mathbb{B}(\theta H^2, \alpha H^2) : \hat{T} = \hat{T}_\varphi^{\theta,\alpha} \text{ for some } \varphi \in L_\infty(\mathbb{T})\}, \\
\mathcal{T}(H^2_-) &= \{\check{T} \in \mathbb{B}(H^2_-) : \check{T} = \check{T}_\varphi \text{ for some } \varphi \in L_\infty(\mathbb{T})\}, \\
\mathcal{T}(\theta H^2, H^2_-) &= \{\hat{\Gamma} \in \mathbb{B}(\theta H^2, H^2_-) : \hat{\Gamma} = \hat{\Gamma}_\varphi^\theta = P^- M_{\varphi|\theta H^2} \text{ for } \varphi \in L_\infty(\mathbb{T})\}, \\
\mathcal{T}(H^2_-, \alpha H^2) &= \{\check{\Gamma} \in \mathbb{B}(H^2_-, \alpha H^2) : \check{\Gamma} = \check{\Gamma}_\varphi^\alpha = P_{\alpha H^2} M_{\varphi|H^2_-} \text{ for } \varphi \in L_\infty(\mathbb{T})\}.
\end{aligned}$$

As in [2] we will write $\mathcal{T}(\theta H^2)$ instead of $\mathcal{T}(\theta H^2, \theta H^2)$ and $\hat{T}_\varphi^\theta$ instead of $\hat{T}_\varphi^{\theta,\theta}$.

Each of the operators in (17) and (18) can be similarly defined for arbitrary $\varphi \in L^2(\mathbb{T})$. In that case $\hat{\Gamma}_\varphi^\alpha$ and $\hat{T}_\varphi$ are defined on a dense subset θH^∞ of θH^2, while $\check{\Gamma}_\varphi^\alpha$ and $\check{T}_\varphi$ are defined on $H^\infty_- = H^2_- \cap L_\infty(\mathbb{T}) = \overline{z H^\infty}$ which is a dense subset of H^2_-. However, in a moment we will justify the fact that, in a sense, one needs only to consider symbols from $L_\infty(\mathbb{T})$.

The following proposition is not difficult to verify using Proposition 1.1.

Proposition 6.1 *Let θ and α be two nonconstant inner functions and let $\varphi \in L^2(\mathbb{T})$. Then*

(a) $\hat{T}^{\theta,\alpha}_{\varphi|\theta H^\infty} = P_{\alpha H^2} M_{\varphi|\theta H^\infty} = M_\alpha T_{\bar{\alpha}\varphi\theta} M_{\bar{\theta}|\theta H^\infty} = M_\alpha T_{\bar{\alpha}\varphi|\theta H^\infty};$

(b) $\hat{\Gamma}^{\theta}_{\varphi|\theta H^\infty} = P^- M_{\varphi|\theta H^\infty} = H_{\varphi\theta} M_{\bar{\theta}|\theta H^\infty}$;

(c) $\check{T}_{\varphi|H^\infty_-} = P^- M_{\varphi|H^\infty_-} = J T_{\bar{\varphi}} J_{|H^\infty_-}$;

(d) $\check{\Gamma}^{\alpha}_{\varphi|H^\infty_-} = P_{\alpha H^2} M_{\varphi|H^\infty_-} = M_\alpha H^*_{\alpha\bar{\varphi}|H^\infty_-}$.

It now follows from Proposition 6.1(a) that $\hat{T} \in \mathcal{T}(\theta H^2, \alpha H^2)$ if and only if $M_{\bar{\alpha}} \hat{T} M_{\theta|H^2} \in \mathcal{T}(H^2)$ and so $\mathcal{T}(\theta H^2, \alpha H^2)$ is isomorphic to the space of classical Toeplitz operators $\mathcal{T}(H^2)$. Moreover, each $\hat{T}^{\theta,\alpha}_{\varphi}$ is uniquely determined by its symbol and extends to a bounded operator on θH^2 if and only if $\varphi \in L_\infty(\mathbb{T})$ (since classical Toeplitz operators have similar properties). Similarly, $\mathcal{T}(H^2_-)$ is isomorphic to $\mathcal{T}(H^2)$, each $\check{T}_\varphi$ is uniquely determined by its symbol and extends to a bounded operator on H^2_- if and only if $\varphi \in L_\infty(\mathbb{T})$.

On the other hand, both $\mathcal{T}(\theta H^2, H^2_-)$ and $\mathcal{T}(H^2_-, \alpha H^2)$ are isomorphic to the space of all Hankel operators $\mathcal{H}(H^2, H^2_-)$. It follows from properties of classical Hankel operators that $\hat{\Gamma}^{\theta}_{\varphi}$ and $\check{\Gamma}^{\alpha}_{\varphi}$ may be bounded even for $\varphi \notin L_\infty(\mathbb{T})$, and

$$\hat{\Gamma}^{\theta}_{\varphi} = \hat{\Gamma}^{\theta}_{\psi} \quad \text{if and only if} \quad (\varphi - \psi) \perp \overline{\theta z H^2} \tag{19}$$

and

$$\check{\Gamma}^{\alpha}_{\varphi} = \check{\Gamma}^{\alpha}_{\psi} \quad \text{if and only if} \quad (\varphi - \psi) \perp \alpha z H^2.$$

In particular, $\hat{\Gamma}^{\theta}_{\varphi} = 0$ if $\varphi \in \bar{\theta} H^\infty \supset H^\infty$ and $\check{\Gamma}^{\alpha}_{\varphi} = 0$ if $\varphi \in \alpha \overline{H^\infty} \supset \overline{H^\infty}$. However, since each bounded Hankel operator has a symbol from $L_\infty(\mathbb{T})$ [16, Theorem 1.3, Chapter 1], we see that the same is true for operators from $\mathcal{T}(\theta H^2, H^2_-)$ or $\mathcal{T}(H^2_-, \alpha H^2)$. Thus operators with bounded symbols form the spaces $\mathcal{T}(\theta H^2, H^2_-)$ and $\mathcal{T}(H^2_-, \alpha H^2)$.

Proposition 6.1 implies the following.

Corollary 6.2

(a) $\hat{T}^{\theta}_{z} = M_{z|\theta H^2}$, $\hat{T}^{\theta}_{\bar{z}} = M_{\bar{z}|\theta H^2} - (\theta\bar{z} \otimes \theta)_{|\theta H^2}$;

(b) $\check{T}_z = M_{z|H^2_-} - (1 \otimes \bar{z})_{|H^2_-}$, $\check{T}_{\bar{z}} = M_{\bar{z}|H^2_-}$;

(c) $\hat{\Gamma}^{\theta}_{z} = 0$, $\hat{\Gamma}^{\theta}_{\bar{z}} = (\bar{z} \otimes 1)_{|\theta H^2}$, $\hat{\Gamma}^{\theta}_{\bar{z}} = 0$ *if* $\theta(0) = 0$;

(d) $\check{\Gamma}^{\theta}_{z} = (\overline{\theta(0)}\,\theta \otimes \bar{z})_{|H^2_-}$, $\check{\Gamma}^{\theta}_{\bar{z}} = 0$, $\check{\Gamma}^{\theta}_{z} = 0$ *if* $\theta(0) = 0$.

Proof For (a) take $f \in \theta H^2$. Then

$$\begin{aligned}\hat{T}^{\theta}_{\bar{z}} f &= \theta P^+ \bar{\theta}\bar{z} f = \theta P^+ \bar{z}(\bar{\theta} f) = \theta\bar{z}(\bar{\theta} f - (\bar{\theta} f)(0)) \\ &= \bar{z} f - \theta\bar{z}\langle\bar{\theta} f, 1\rangle = \bar{z} f - \theta\bar{z}\langle f, \theta\rangle = \bar{z} f - (\theta\bar{z} \otimes \theta) f.\end{aligned}$$

To obtain (b) take $f \in H^2_-$,

$$\check{T}_z f = P^- zf = zf - P^+(zf) = zf - \langle f, \bar{z} \rangle = zf - (1 \otimes \bar{z})f.$$

Observe that (3) follows, since for $f \in H^2$, we have

$$\hat{\Gamma}^{\theta}_{\bar{z}}(\theta f) = P^- \bar{z}\theta f = \theta(0)f(0)\bar{z} = \langle \theta f, 1\rangle \bar{z} = (1 \otimes \bar{z})\theta f.$$

To note (d) take $f \in H^2_-$,

$$\begin{aligned}\check{\Gamma}^{\theta}_z f &= \theta P^+ \bar{\theta} z f = \theta P^+ z(\bar{\theta} f) = \theta \langle \bar{\theta} f, \bar{z} \rangle = \theta \langle f, \theta \bar{z} \rangle \\ &= \langle f, P^-(\theta \bar{z})\rangle \theta = \langle f, \theta(0)\bar{z}\rangle \theta = \overline{\theta(0)} \langle f, \bar{z} \rangle \theta = \overline{\theta(0)}(\theta \otimes \bar{z}) f.\end{aligned}$$

□

From Proposition 6.1 (a), (b) we obtain, in particular, that:

Corollary 6.3 *For $\varphi_1, \varphi_2 \in L_\infty(\mathbb{T})$,*

(a) $\hat{T}^{\theta}_{\varphi_1} \hat{T}^{\theta}_{\varphi_2} = M_\theta T_{\varphi_1} T_{\varphi_2} M_{\bar{\theta}|\theta H^2}$;
(b) $\check{T}_{\varphi_1} \check{T}_{\varphi_2} = J T_{\bar{\varphi}_1} T_{\bar{\varphi}_2} J$;
(c) $\hat{T}^{\theta}_{\bar{z}} \hat{T}^{\theta}_z = I_{\theta H^2}$;
(d) $\hat{T}^{\theta}_z \hat{T}^{\theta}_{\bar{z}} = M_\theta (I - 1 \otimes 1) M_{\bar{\theta}|\theta H^2} = I_{\theta H^2} - \theta \otimes \theta_{|\theta H^2}$;
(e) $\check{T}_{\bar{z}} \check{T}_z = J(I - 1 \otimes 1)J_{|H^2_-} = I_{H^2_-} - \bar{z} \otimes \bar{z}_{|H^2_-}$;
(f) $\check{T}_z \check{T}_{\bar{z}} = I_{H^2_-}$.

It is a part of common knowledge that the space of classical Toeplitz operators $\mathcal{T}(H^2)$ is isomorphic to the space $\mathcal{T}(\theta H^2)$ and also with $\mathcal{T}(\theta H^2, \alpha H^2)$ or also with $\mathcal{T}(H^2_-)$, but we present some lemmas for completeness and to fix the notations of these spacial isomorphisms. Recall also the notation $\mathcal{H}(H^2, H^2_-)$ for the space of all Hankel operators.

The following properties can be easily verified.

Proposition 6.4 *Let θ be a nonconstant inner function. Then*

(a) $\hat{T} \in \mathcal{T}(\theta H^2, \alpha H^2)$ *if and only of* $M_{\bar{\alpha}} \hat{T} M_{\theta|H^2} \in \mathcal{T}(H^2)$;
(b) $\check{T} \in \mathcal{T}(H^2_-)$ *if and only of* $J \hat{T} J_{|H^2} \in \mathcal{T}(H^2)$;
(c) $\hat{\Gamma} \in \mathcal{T}(\theta H^2, H^2_-)$ *if and only of* $\hat{\Gamma} M_{\theta|H^2} \in \mathcal{H}(H^2, H^2_-)$;
(d) $\check{\Gamma} \in \mathcal{T}(H^2_-, \theta H^2)$ *if and only of* $\left(M_{\bar{\theta}} \check{\Gamma}\right)^* \in \mathcal{H}(H^2, H^2_-)$.

Observe also that for $\varphi_1, \varphi_2 \in L_\infty(\mathbb{T})$,

$$\hat{T}^{\theta}_{\varphi_1} \hat{T}^{\theta}_{\varphi_2} = M_\theta T_{\varphi_1} T_{\varphi_2} M_{\bar{\theta}|\theta H^2} \quad \text{and} \quad \check{T}_{\varphi_1} \check{T}_{\varphi_2} = J T_{\bar{\varphi}_1} T_{\bar{\varphi}_2} J.$$

It thus follows from the properties of classical Toeplitz operators that if one of the functions φ_1, φ_2 belongs to H^∞, then

$$\hat{T}^\theta_{\bar{\varphi}_1}\hat{T}^\theta_{\varphi_2} = \hat{T}^\theta_{\bar{\varphi}_1\varphi_2} \quad \text{and} \quad \check{T}_{\varphi_1}\check{T}_{\bar{\varphi}_2} = \check{T}_{\varphi_1\bar{\varphi}_2}. \tag{20}$$

It is well known that classical Toeplitz and Hankel operators can be characterized in terms of compressions of M_z to H^2 and H^2_-.

Recall that

(A) if $T \in \mathbb{B}(H^2)$, then $T \in \mathcal{T}(H^2)$ if and only if $T = T_z^* T T_z$ and in that case $T = T_\varphi$ with $\varphi = T(1) + \overline{T^*(1) - \langle T^*1, 1\rangle}$, (it is Brown–Halmos result, see [10, Theorem 4.16]);

(B) if $H \in \mathbb{B}(H^2, H^2_-)$, then $H \in \mathcal{H}(H^2, H^2_-)$ if and only if $P^- zH = HT_z$ and in that case $P^-\varphi = H(1)$, see [16, Theorem 1.8, Chapter 1].

As a consequence of Proposition 6.1 we get the following.

Theorem 6.5 *Let θ and α be two nonconstant inner functions.*

(a) Let $\hat{T} \in \mathbb{B}(\theta H^2, \alpha H^2)$. Then $\hat{T} \in \mathcal{T}(\theta H^2, \alpha H^2)$ if and only if $\hat{T} = \hat{T}^\alpha_{\bar{z}}\hat{T}\hat{T}^\theta_z$. In that case $\hat{T} = \hat{T}^{\theta,\alpha}_\varphi$ with $\varphi = \bar{\theta}\hat{T}(\theta) + \alpha\overline{\hat{T}^*(\alpha)} - \alpha\bar{\theta}\langle\hat{T}\theta, \alpha\rangle \in L_\infty(\mathbb{T})$.

(b) Let $\check{T} \in \mathbb{B}(H^2_-)$. Then $\check{T} \in \mathcal{T}(H^2_-)$ if and only if $\check{T} = \check{T}_z\check{T}\check{T}_{\bar{z}}$ and in that case $\check{T} = \check{T}_\varphi$ with $\varphi = z\check{T}\bar{z} + \bar{z}\overline{\check{T}^*\bar{z}} - \langle\check{T}\bar{z}, \bar{z}\rangle \in L_\infty(\mathbb{T})$.

(c) Let $\hat{\Gamma} \in \mathbb{B}(\theta H^2, H^2_-)$. Then $\hat{\Gamma} \in \mathcal{T}(\theta H^2, H^2_-)$ if and only if $\check{T}_z\hat{\Gamma} = \hat{\Gamma}\hat{T}^\theta_z$ and in that case $\hat{\Gamma} = \hat{\Gamma}^\theta_\varphi$ with $P^-(\theta\varphi) = \hat{\Gamma}\theta$. Moreover, there exists such a φ belonging to $L_\infty(\mathbb{T})$.

(d) Let $\check{\Gamma} \in \mathbb{B}(H^2_-, \alpha H^2)$. Then $\check{\Gamma} \in \mathcal{T}(H^2_-, \alpha H^2)$ if and only if $\check{\Gamma}\check{T}_{\bar{z}} = \hat{T}^\alpha_{\bar{z}}\check{\Gamma}$ and in that case $\check{\Gamma} = \check{\Gamma}^\alpha_\varphi$ with $P^-(\alpha\bar{\varphi}) = \check{\Gamma}^*\alpha$. Moreover, there exists such a φ belonging to $L_\infty(\mathbb{T})$.

Proof By Proposition 6.4 (a) and (A), if $\hat{T} \in \mathcal{T}(\theta H^2, \alpha H^2)$, then $\mathcal{T}(H^2) \ni M_{\bar{\alpha}}\hat{T}M_{\theta|H^2} = T_z^* M_{\bar{\alpha}}\hat{T}M_\theta T_z$. Equivalently

$$\hat{T} = (M_\alpha T_{\bar{z}} M_{\bar{\alpha}})\hat{T}(M_\theta T_z M_{\bar{\theta}})_{|\theta H^2} = \hat{T}^\theta_{\bar{z}}\hat{T}\hat{T}^\theta_z.$$

The symbol φ of $\hat{T}$ can be obtained from the symbol ψ of $T = M_{\bar{\alpha}}\hat{T}M_\theta$, so

$$\begin{aligned}\varphi = \alpha\psi\bar{\theta} &= \alpha\Big(M_{\bar{\alpha}}\hat{T}M_\theta)(1) + \overline{(M_{\bar{\alpha}}\hat{T}M_\theta)^*(1) - \langle(M_{\bar{\alpha}}\hat{T}M_\theta)^*1, 1\rangle}\Big)\bar{\theta}\\ &= \alpha\Big(\bar{\alpha}\hat{T}\theta + \overline{\bar{\theta}\hat{T}^*\alpha - \langle\bar{\theta}T^*\alpha, 1\rangle}\Big)\bar{\theta} = \bar{\theta}\hat{T}\theta + \alpha\overline{\hat{T}^*\alpha} - \alpha\bar{\theta}\overline{\langle T^*\alpha, \theta\rangle}.\end{aligned}$$

To show (b) note that by Proposition 6.4 (b) and (A), if $\check{T} \in \mathcal{T}(H^2_-)$, then $\mathcal{T}(H^2) \ni J\check{T}J_{|H^2} = T_z^* J\check{T}JT_z$. Equivalently $\check{T} = (JT_{\bar{z}}J)\check{T}JT_zJ_{|H^2_-} = \check{T}_z\check{T}\check{T}_{\bar{z}}$. In that case its symbol is the conjugate of the symbol of $T = J\check{T}J \in \mathcal{T}(H^2)$, hence

$$\bar{\varphi} = J\check{T}J(1) + \overline{J\check{T}^*J(1)} - \overline{\langle J\check{T}^*J1, 1\rangle} = \bar{z}\check{T}(\bar{z}) + z\check{T}^*(\bar{z}) - \overline{\langle \check{T}^*\bar{z}, \bar{z}\rangle}.$$

To prove (c) we apply Proposition 6.4 (c) and (B). We have that $\hat{\Gamma} \in \mathcal{T}(\theta H^2, H^2_-)$ if and only if $P^- z\hat{\Gamma}M_{\theta|H^2} = \hat{\Gamma}M_\theta T_z$. Equivalently,

$$\check{T}_z\hat{\Gamma} = P^- zP^-\hat{\Gamma} = \hat{\Gamma}M_\theta T_z M_{\bar{\theta}|\theta H^2} = \hat{\Gamma}\hat{T}^\theta_z.$$

In that case $\hat{\Gamma} = \hat{\Gamma}^\theta_\varphi$ where $\theta\varphi$ is a symbol for the Hankel operator $\hat{\Gamma}M_{\theta|H^2}$ (by Proposition 6.1 (c)), thus $P^-(\theta\varphi) = \hat{\Gamma}M_\theta(1) = \hat{\Gamma}\theta$.

To obtain the last condition we apply Proposition 6.4 (d) and (B). Note that $\check{\Gamma} \in \mathcal{T}(H^2_-, \alpha H^2)$ if and only if $\check{\Gamma}^*M_{\alpha|H^2} \in \mathcal{H}(H^2, H^2_-)$. Equivalently, $P^- z\check{\Gamma}^*M_{\alpha|H^2} = \check{\Gamma}^*M_\alpha T_z$. Hence $\check{T}_z\check{\Gamma}^* = \check{\Gamma}^*M_\alpha T_z M_{\bar{\alpha}|\theta H^2}$. Finally, $\check{T}_z\check{\Gamma}^* = \check{\Gamma}^*\hat{T}^\alpha_z$, which is the same as $\check{\Gamma}\check{T}_{\bar{z}} = \hat{T}_{\bar{z}}\check{\Gamma}$. In that case $\check{\Gamma} = \check{\Gamma}^\alpha_\varphi$ where $\alpha\bar{\varphi}$ is a symbol of the Hankel operator $(M_{\bar{\alpha}}\check{\Gamma})^* = \check{\Gamma}M_{\alpha|H^2}$, so $P^-(\alpha\bar{\varphi}) = \check{\Gamma}^*M_\theta 1 = \check{\Gamma}^*\alpha$. □

We will now consider operators of the form

$$D = \begin{bmatrix} \hat{T}^{\theta,\alpha}_{\varphi_1} & \check{\Gamma}^\alpha_{\varphi_2} \\ \hat{\Gamma}^\theta_{\varphi_3} & \check{T}_{\varphi_4} \end{bmatrix}.$$

with $\varphi_i \in L^2(\mathbb{T})$ for $i = 1, 2, 3, 4$. Note that if D given above is bounded, then $\hat{T}^{\theta,\alpha}_{\varphi_1}$ and $\check{T}_{\varphi_4}$ are also bounded and so, as mentioned above, necessarily $\varphi_1, \varphi_4 \in L_\infty(\mathbb{T})$. On the other hand, even though for bounded D the compressions $\check{\Gamma}^\alpha_{\varphi_2}$ and $\hat{\Gamma}^\theta_{\varphi_3}$ are also bounded, the functions φ_2 and φ_3 may not belong to $L_\infty(\mathbb{T})$ (but there exist $\psi_2, \psi_3 \in L_\infty(\mathbb{T})$ such that $\check{\Gamma}^\alpha_{\varphi_2} = \check{\Gamma}^\alpha_{\psi_2}$ and $\hat{\Gamma}^\theta_{\varphi_3} = \hat{\Gamma}^\theta_{\psi_3}$).

We will now study relations of the operators (17) and (18) with respect to the conjugation C_θ (see (1)). Recall that C_θ can be expressed as $C_\theta = M_\theta J = JM_{\bar{\theta}}$, hence we obtain the following.

Proposition 6.6 *For $\varphi \in L_\infty(\mathbb{T})$,*

(a) $\hat{T}^{\theta,\alpha}_\varphi = C_\alpha\check{T}_{\alpha\bar{\varphi}\bar{\theta}}C_{\theta|\theta H^2} = C_\alpha\check{T}_\alpha\check{T}_{\bar{\varphi}}\check{T}_{\bar{\theta}}C_{\theta|\theta H^2}$;
(b) $\check{T}_\varphi = (P^-C_\alpha M_{\bar{\theta}})_{|\theta H^2}\hat{T}^{\alpha,\theta}_{\bar{\varphi}}(M_\theta C_\alpha)_{|H^2_-}$;
(c) $\hat{\Gamma}^\theta_\varphi = C_\theta\check{\Gamma}^\theta_{\bar{\varphi}}C_{\theta|\theta H^2}$.

Proof The proof of (c) can be found in [2]. A slight modification of the proof of [2, Proposition 22] (for $\alpha = \theta$) gives

$$\begin{aligned}\hat{T}^{\theta,\alpha}_\varphi &= M_\alpha T_{\bar{\alpha}\varphi\theta}M_{\bar{\theta}|\theta H^2} = M_\alpha(JP^-J)M_{\bar{\alpha}\varphi\theta}J(JM_{\bar{\theta}})_{|\theta H^2} \\ &= (M_\alpha J)P^-M_{\alpha\bar{\varphi}\bar{\theta}}P^-JM_{\bar{\theta}|\theta H^2} = C_\alpha\check{T}_{\alpha\bar{\varphi}\bar{\theta}}C_{\theta|\theta H^2} = C_\alpha\check{T}_\alpha\check{T}_{\bar{\varphi}}\check{T}_{\bar{\theta}}C_{\theta|\theta H^2},\end{aligned}$$

where the last equality follows from (20). Hence (a) holds.

To prove (b) recall that $JM_{\bar\theta} = M_\theta J$ and $JP^+ = P^- J$, which implies that

$$JT_{\bar\theta} = P^- M_\theta J_{|H^2} \quad \text{and} \quad T_\alpha J_{|H^2_-} = JP^- M_{\bar\alpha|H^2_-}.$$

Since $J = C_\alpha M_\alpha = M_{\bar\theta} C_\theta$, we get by Proposition 6.1

$$\begin{aligned}\check{T}_\varphi &= JT_{\bar\varphi} J_{|H^2_-} = JT_{\bar\theta} T_{\bar\alpha\bar\varphi\theta} T_\alpha J_{|H^2_-} = P^- M_\theta J T_{\bar\alpha\bar\varphi\theta} J P^- M_{\bar\alpha|H^2_-} \\ &= P^- M_\theta C_\alpha M_\alpha T_{\bar\alpha\bar\varphi\theta} M_{\bar\theta} C_\theta P^- M_{\bar\alpha|H^2_-} = P^- M_\theta C_\alpha \hat{T}^{\theta,\alpha}_{\bar\varphi} C_\theta M_{\bar\alpha|H^2_-}.\end{aligned}$$

The result follows since

$$M_\theta C_\alpha = M_\alpha C_\theta = C_\theta M_{\bar\alpha} = C_\alpha M_{\bar\theta}.$$

□

Note that for arbitrary $\varphi \in L^2(\mathbb{T})$ equalities (a)–(c) in Proposition 6.6 hold on the set of bounded functions.

Finally, we are ready to give the following characterization of ADTTO's.

Theorem 6.7 *Let θ and α be inner functions and let $D \in \mathbb{B}(K_\theta^\perp, K_\alpha^\perp)$. Then the operator D is an asymmetric dual truncated Toeplitz operator, $D \in \mathcal{T}(K_\theta^\perp, K_\alpha^\perp)$, if and only if the following conditions hold:*

(a) $P_{\alpha H^2} D_{|\theta H^2} = \hat{T}^\alpha_{\bar z} P_{\alpha H^2} D_{|\theta H^2} \hat{T}^\theta_z$;
(b) $P^- D_{|H^2_-} = (P^- C_\alpha M_{\bar\theta})_{|\theta H^2} \left(P_{\alpha H^2} D_{|\theta H^2}\right)^* (M_\theta C_\alpha)_{|H^2_-}$;
(c) $P^- D_{|\theta H^2} \hat{T}^\theta_z = \check{T}_z P^- D_{|\theta H^2}$ *and* $(P_{\alpha H^2} D_{|H^2_-})^* \hat{T}^\alpha_z = \check{T}_z (P_{\alpha H^2} D_{|H^2_-})^*$;
(d) $P^-(D(\theta)) = P^-(\theta\alpha\overline{D^*(\alpha)})$ *and* $P^-(D^*(\alpha)) = P^-(\theta\alpha\overline{D(\theta)})$.

In that case, $D = D^{\theta,\alpha}_\varphi$ with $\varphi \in L_\infty(\mathbb{T})$ given by

$$\varphi \;=\; \bar\theta\, P_{\alpha H^2} D(\theta) + \alpha\, \overline{P_{\theta H^2}(D^*(\alpha))} - \alpha\bar\theta\, \langle D(\theta), \alpha\rangle. \tag{21}$$

Proof Assume firstly that $D = D^{\theta,\alpha}_\varphi$ with $\varphi \in L_\infty(\mathbb{T})$. Then

$$D^{\theta,\alpha}_\varphi = \begin{bmatrix} \hat{T}^{\theta,\alpha}_\varphi & \check{\Gamma}^\alpha_\varphi \\ \hat{\Gamma}^\theta_\varphi & \check{T}_\varphi \end{bmatrix} = \begin{bmatrix} \hat{T}^{\theta,\alpha}_\varphi & (\hat{\Gamma}^\alpha_{\bar\varphi})^* \\ \hat{\Gamma}^\theta_\varphi & \check{T}_\varphi \end{bmatrix}.$$

Note that (a) follows from Theorem 6.5 (a). Moreover, (b) is satisfied by Proposition 6.6 (b) and (c) is satisfied by Theorem 6.5.

Moreover,

$$D(\theta) = P_\theta^\perp(\varphi\theta) = \theta P^+(\varphi) + P^-(\varphi\theta)$$

and

$$D^*(\theta) = P_\theta^\perp(\bar{\varphi}\theta) = \theta P^+(\bar{\varphi}) + P^-(\bar{\varphi}\theta).$$

It follows that $P^-(D(\theta)) = P^-(\varphi\theta)$ and

$$P^-(\theta^2\overline{D^*(\theta)}) = P^-(\theta\overline{P^+(\bar{\varphi})}) = P^-(\varphi\theta) = P^-(D(\theta)).$$

Similarly, $P^-(D^*(\theta)) = P^-(\bar{\varphi}\theta)$ and

$$P^-(\theta^2\overline{D(\theta)}) = P^-(\theta\overline{P^+(\varphi)}) = P^-(\bar{\varphi}\theta) = P^-(D^*(\theta)).$$

Thus (d) is also satisfied.

Assume now that $D = \begin{bmatrix} P_{\alpha H^2}D_{|\theta H^2} & P_{\alpha H^2}D_{|H^2_-} \\ P^-D_{|\theta H^2} & P^-D_{|H^2_-} \end{bmatrix} \in \mathbb{B}(K_\theta^\perp, K_\alpha^\perp)$ satisfies (a)–(d). It then follows from condition (a) and Theorem 6.5 (a) that $P_{\alpha H^2}D_{|\theta H^2} = \hat{T}_\varphi^{\theta,\alpha}$ for $\varphi \in L_\infty(\mathbb{T})$ given by

$$\varphi = \bar{\theta}\left(P_{\alpha H^2}D_{|\theta H^2}\right)(\theta) + \alpha\overline{\left(P_{\alpha H^2}D_{|\theta H^2}\right)^*(\alpha)} - \alpha\bar{\theta}\langle\left(P_{\alpha H^2}D_{|\theta H^2}\right)(\theta), \alpha\rangle. \tag{22}$$

By (b) and Proposition 6.6 (b),

$$\begin{aligned} P^-D_{|H^2_-} &= (P^-C_\alpha M_{\bar{\theta}})_{|\theta H^2}\left(P_{\alpha H^2}D_{|\theta H^2}\right)^*(M_\theta C_\alpha)_{|H^2_-} \\ &= (P^-C_\alpha M_{\bar{\theta}})_{|\theta H^2}\hat{T}_{\bar{\varphi}}^{\alpha,\theta}(M_\theta C_\alpha)_{|H^2_-} = \check{T}_\varphi. \end{aligned}$$

By (c) and Theorem 6.5 (c)–(d) there are functions $\psi, \chi \in L_\infty(\mathbb{T})$ such that $P^-D_{|\theta H^2} = \hat{\Gamma}_\psi^\theta$ with $P^-(\theta\psi) = P^-D_{|\theta H^2}(\theta)$ and $(P_{\theta H^2}D_{|H^2_-})^* = \hat{\Gamma}_\chi^\theta$ with $P^-(\alpha\chi) = (P_{\alpha H^2}D_{|H^2_-})^*(\alpha)$. We will now use (d) to show that

$$(\varphi - \psi) \perp \overline{\theta z H^2} \quad \text{and} \quad (\bar{\varphi} - \chi) \perp \overline{\alpha z H^2}. \tag{23}$$

Since φ is given by (22), using the first equality in (d) we get

$$\begin{aligned} P^-(\theta\varphi) &= P^-(\alpha\theta\overline{((P_{\theta H^2}D_{|\theta H^2})^*(\alpha))}) = P^-(\alpha\theta\overline{D^*(\alpha)}) \\ &= P^-(D(\theta)) = P^-D_{|\theta H^2}(\theta) = P^-(\theta\psi), \end{aligned}$$

and so $(\theta\varphi - \theta\psi) \perp H^2_-$. Similarly, using the second equality in (d) we get

$$\begin{aligned} P^-(\alpha\bar{\varphi}) &= P^-(\alpha\theta\overline{(P_{\theta H^2}D_{|\theta H^2}(\theta))}) = P^-(\alpha\theta\overline{D(\theta)}) \\ &= P^-(D^*(\alpha) = (P_{\alpha H^2}D_{|H^2_-})^*(\alpha) = P^-(\alpha\chi), \end{aligned}$$

hence $(\alpha\bar{\varphi} - \alpha\chi) \perp H^2_-$. Thus, by (19), we proved that $P^- D_{|\theta H^2} = \hat{\Gamma}^\theta_\varphi$ and $(P_{\alpha H^2} D_{|H^2_-})^* = \hat{\Gamma}^\theta_{\bar{\varphi}}$, that is, $P_{\alpha H^2} D_{|H^2_-} = (\hat{\Gamma}^\theta_{\bar{\varphi}})^* = \check{\Gamma}^\theta_\varphi$. Therefore,

$$D^\theta_\varphi = \begin{bmatrix} \hat{T}^{\theta,\alpha}_\varphi & \check{\Gamma}^\alpha_\varphi \\ \hat{\Gamma}^\theta_\varphi & \check{T}_\varphi \end{bmatrix}.$$

Moreover, by (22), we have

$$\begin{aligned} \varphi &= \bar{\theta} P_{\alpha H^2} D_{|\theta H^2}(\theta)) + \alpha\, \overline{(P_{\alpha H^2} D_{|\theta H^2})^*(\alpha)} - \alpha\bar{\theta}\langle P_{\alpha H^2} D_{|\theta H^2}(\theta), \alpha\rangle \\ &= \bar{\theta}\, P_{\alpha H^2} D(\theta) + \alpha\, \overline{P_{\theta H^2} D^*(\alpha)} - \alpha\bar{\theta}\, \langle D(\theta), \alpha\rangle. \end{aligned}$$

□

Remark 6.8 By (21), the symbol $\varphi \in L_\infty(\mathbb{T})$ of an asymmetric dual truncated Toeplitz operator D can be obtained by calculating $D(\theta)$ and $D^*(\alpha)$. The symbol φ can also be calculated using $D(\bar{z})$ and $D^*(\bar{z})$. To see this let $\varphi = \varphi^- + \varphi^+$, $\varphi^- \in H^2_-$, $\varphi^+ \in H^2$ and let $\hat{\varphi}(0)$ denote the 0–th Fourier coefficient of φ. Then, by the fact that $C_\alpha D^{\theta,\alpha}_\varphi C_{\theta|K^\perp_\theta} = D^{\theta,\alpha}_{\alpha\bar{\varphi}\bar{\theta}}$, we have

(a) $\hat{\varphi}(0) = \langle 1, \bar{\varphi}\rangle = \langle \alpha, D^{\theta,\alpha}_{\alpha\bar{\varphi}\bar{\theta}}\theta\rangle = \langle \alpha, C_\alpha D^{\theta,\alpha}_\varphi C_\theta\theta\rangle = \langle D\bar{z}, \bar{z}\rangle$,

(b) $\varphi^+ = P^+\bar{\theta} D^{\alpha,\theta}_{\bar{\alpha}\varphi\theta}\alpha = P^+\bar{\theta} C_\theta D^{\alpha,\theta}_{\bar{\varphi}} C_\alpha\alpha = J P^-(D^*\bar{z})$,

(c) $\varphi^- = \overline{P^+(\bar{\alpha} D^{\theta,\alpha}_{\alpha\bar{\varphi}\bar{\theta}}\theta)} - \hat{\varphi}(0) = \overline{P^+(\bar{\alpha} C_\alpha D^{\theta,\alpha}_\varphi C_\theta\theta)} - \hat{\varphi}(0)$
$= \overline{J P^-(D(\bar{z}))} - \hat{\varphi}(0) = P^-(z D(\bar{z}))$.

Hence,

$$\varphi = P^-(z D(\bar{z})) + J P^-(D^*\bar{z}) \in H^2_- + H^2. \tag{24}$$

Note that the decomposition (24) is orthogonal while (21) in general is not.

Remark 6.9 Let α and θ be nonconstant inner functions. If $D \in \mathbb{B}(K^\perp_\theta, K^\perp_\alpha)$, then D is an asymmetric dual truncated Toeplitz operator with an analytic symbol if and only if D satisfies conditions of Theorem 6.7 and moreover $P^-(z D(\bar{z})) = 0$. The last condition means that $D(\bar{z}) \perp \bar{z} H^2_-$.

7 A Brown–Halmos Type Theorem for DTTO

It is a classical result of Brown and Halmos [1] that the product of two Toeplitz operators is zero if and only if at least one of them has a zero symbol. The product of two Toeplitz operators $T_\varphi T_\psi$ ($\varphi, \psi \in L_\infty(\mathbb{T})$) is a Toeplitz operator if and only if $\bar{\varphi}$ or ψ is analytic. They also gave necessary and sufficient conditions for the commutativity of two Toeplitz operators. They proved that, for $\varphi, \psi \in L_\infty(\mathbb{T})$, $T_\varphi T_\psi = T_\psi T_\varphi$ if and only if either $\varphi, \psi \in H^2$ or $\varphi, \psi \in H^2_-$ or a nontrivial linear combination of φ and ψ is constant. In this section we will present similar results for dual truncated Toeplitz operators. Results are based on the paper [18]

Let $\varphi \in L_\infty(\mathbb{T})$. Then according to the decomposition $L^2(\mathbb{T}) = H^2 \oplus H^2_-$ the multiplication operator M_φ can be expressed as the operator matrix of the form

$$\begin{pmatrix} T_\varphi & H^*_{\bar{\varphi}} \\ H_\varphi & \check{T}_\varphi, \end{pmatrix} \tag{25}$$

where $H_\varphi \colon H^2 \to H^2_-$ is the Hankel operator, $H^*_{\bar{\varphi}} \colon H^2_- \to H^2$, $H^*_{\bar{\varphi}} h = P^+(\varphi h)$.

Assume now that $\varphi, \psi \in L_\infty(\mathbb{T})$. Since $M_\varphi M_\psi = M_{\varphi\psi}$, we have

$$T_{\varphi\psi} = T_\varphi T_\psi + H^*_{\bar{\varphi}} H_\psi; \tag{26}$$

$$H_{\varphi\psi} = H_\varphi T_\psi + \check{T}_\varphi H_\psi; \tag{27}$$

$$\check{T}_{\varphi\psi} = \check{T}_\varphi \check{T}_\psi + H_\varphi H^*_{\bar{\psi}}. \tag{28}$$

Note that

$$J H_\varphi J = H^*_\varphi \tag{29}$$

and

$$J \check{T}_\varphi J = T_\varphi. \tag{30}$$

Theorem 7.1 *Let θ be a nonconstant inner function and let $\varphi, \psi \in L_\infty(\mathbb{T})$. If $D^\theta_\varphi D^\theta_\psi = D^\theta_{\varphi\psi}$, then one of the following conditions holds:*

(a) $\varphi, \psi \in H^2$;
(b) $\bar{\varphi}, \bar{\psi} \in H^2$;
(c) either φ or ψ is constant.

Let $\lambda \in \mathbb{D}$ be fixed and let $K_\lambda = \frac{(1-|\lambda|^2)^{\frac{1}{2}}}{1-z\bar{\lambda}}$ be the normalized reproducing kernel. Denote, for each function $f \in L^2(\mathbb{T})$, by $f_+ = P^+ f$ and $f_- = P^- f$.

Lemma 7.2 *Let θ be a nonconstant inner function, and let $\varphi, \psi \in L_\infty(\mathbb{T})$. Then*

$$T_{\bar{\theta}\varphi} T_{\theta\psi} = T_\varphi T_\psi \tag{31}$$

if and only if $\bar{\varphi} \in H^2$ or $\psi \in H^2$.

Proof Assume $T_\varphi T_\psi = T_{\bar\theta\varphi} T_{\theta\psi}$. Since $T_{\bar\theta} T_\varphi = T_{\bar\theta\varphi}$ and $T_\psi T_\theta = T_{\theta\psi}$, then $T_\varphi T_\psi = T_{\bar\theta^n\varphi} T_{\theta^n\psi}$, for each positive integer n. Hence

$$\langle T_\varphi T_\psi K_\lambda, K_\lambda\rangle = \langle T_{\bar\theta^n\varphi} T_{\theta^n\psi} K_\lambda, K_\lambda\rangle.$$

Now we will use the properties of Berezin transform. Similarly as in [20, proof of Theorem 4.3], we have

$$\begin{aligned} &\varphi_+(\lambda)\psi_-(\lambda) - [\bar\theta^n\varphi]_+(\lambda)[\theta^n\psi]_-(\lambda) = \langle([\bar\theta^n\varphi]_-[\theta^n\psi]_+ - \varphi_-\psi_+)K_\lambda, K_\lambda\rangle \\ &+ [\bar\theta^n\varphi]_+(\lambda)[\theta^n\psi]_+(\lambda) + [\bar\theta^n\varphi]_-(\lambda)[\theta^n\psi]_-(\lambda) - \varphi_+(\lambda)\psi_+(\lambda) - \varphi_-(\lambda)\psi_-(\lambda). \end{aligned}$$

The right hand side of the above equation is harmonic on $\mathbb{D}$. Thus the left hand side is harmonic on $\mathbb{D}$ too. For any harmonic function $h(\lambda)$ we have $\langle hK_\lambda, K_\lambda\rangle = h(\lambda)$. Therefore

$$\langle(\varphi_+\psi_- - [\bar\theta^n\varphi]_+[\theta^n\psi]_-)K_\lambda, K_\lambda\rangle = \varphi_+(\lambda)\psi_-(\lambda)[\bar\theta^n\varphi]_+(\lambda)[\theta^n\psi]_-(\lambda).$$

Since T_θ is an isometry on H^2, by Wold decomposition, we have

$$H^2 = \bigcap_0^\infty \theta^n H^2 \oplus \left(\bigoplus_0^\infty \theta^n K_\theta\right).$$

An isometry T on H^2 is *pure* if $\bigcap\limits_{n=0}^{\infty} T^n H^2 = \{0\}$. A Toeplitz operator with an analytic symbol is a pure isometry if and only if its symbol is a nonconstant inner function. Thus

$$H^2 = \bigoplus_0^\infty \theta^n K_\theta$$

For any $\varphi \in L^2(\mathbb{T}) = H^2 \oplus \overline{zH^2}$, there are $\{x_j, y_j\} \subseteq K_\theta$ such that

$$\varphi = \sum_{j=0}^\infty \theta^j x_j + \bar z \sum_{l=0}^\infty \bar\theta^l \bar y^l$$

with $\|\varphi\|^2 = \sum\limits_{j=0}^\infty \|x_j\|^2 + \sum\limits_{l=0}^\infty \|y_l\|^2$. Hence

$$P^+\bar\theta^n\varphi = P^+\bar\theta^n \sum_{j=0}^\infty \theta^j x_j = P^+\left(\bar\theta^n \sum_{j\geqslant n} \theta^j x_j\right) + T_{\bar\theta^n}\left(\sum_{j=0}^{n-1} \theta^j x_j\right)$$

Note that $K_\theta = \ker T_{\bar{\theta}}$, thus

$$T_{\bar{\theta}^n}\left(\sum_{j=0}^{n-1}\theta^j x_j\right) = \sum_{j=0}^{n-1} T_{\bar{\theta}^{n-j-1}} T_{\bar{\theta}} x_j = 0.$$

Therefore

$$P^+\bar{\theta}^n\varphi = [\bar{\theta}^n\varphi]_+ = P^+\left(\bar{\theta}^n\sum_{j\geqslant n}\theta^j x_j\right)$$

and

$$\|P^+\bar{\theta}^n\varphi\| \leqslant \|\sum_{j\geqslant n}\theta^j x_j\| = (\sum_{j\geqslant n}\|x_j\|^2)^{\frac{1}{2}}.$$

In consequence we have that $\lim_{n\to\infty}\|P^+\bar{\theta}^n\varphi\| = 0$. For any fixed $\lambda \in \mathbb{D}$,

$$|[\bar{\theta}^n\varphi]_+(\lambda)| = |\langle P^+(\bar{\theta}^n\varphi), k_\lambda\rangle| \leqslant \|P^+(\bar{\theta}^n\varphi)\|\,\|k_\lambda\|$$

and

$$|[\theta^n\psi]_-(\lambda)| \leqslant \|[\theta^n\psi]_-\|\,\|k_\lambda\| \leqslant \|\psi\|\,\|k_\lambda\|,$$

where k_λ is the Hardy reproducing kernel. Hence,

$$\lim_{n\to\infty}[\bar{\theta}^n\varphi]_+(\lambda)[\theta^n\psi]_-(\lambda) = 0.$$

Moreover,

$$|\langle[\bar{\theta}^n\varphi]_+[\theta^n\psi]_-K_\lambda, K_\lambda\rangle| = \left|\int_{\partial D}[\bar{\theta}^n\varphi]_+[\theta^n\psi]_-|K_\lambda|^2\,dm\right| \leqslant \frac{1+|\lambda|}{1-|\lambda|}\|P^+\bar{\theta}^n\varphi\|\,\|\psi\|,$$

so

$$\lim_{n\to\infty}\langle[\bar{\theta}^n\varphi]_+[\theta^n\psi]_-K_\lambda, K_\lambda\rangle = 0.$$

Thus for every fixed $\lambda \in \mathbb{D}$, as $n \to \infty$, we obtain

$$\langle\varphi_+\psi_-K_\lambda, K_\lambda\rangle = \varphi_+(\lambda)\psi_-(\lambda).$$

Hence $\varphi_+(\lambda)\psi_-(\lambda)$ is harmonic on $\mathbb{D}$. It follows that $\bar{\varphi}$ or ψ is analytic on $\mathbb{D}$. □

Proof of Theorem 7.1 By assumption for any $f \in H^2$ we have

$$P^+ D^\theta_\varphi D^\theta_\psi(\theta f) = P^+ D^\theta_{\varphi\psi}(\theta f),$$

thus

$$P_{\theta H^2}\varphi P_\theta^\perp \psi\theta f = P_{\theta H^2}\varphi\psi\theta f.$$

Since

$$\begin{aligned} P_{\theta H^2}\varphi P_\theta^\perp \psi\theta f &= \theta P^+(\bar{\theta}\varphi(I - P^+ + \theta P^+\bar{\theta})(\theta\psi f)) \\ &= \theta(P^+(\varphi\psi f) - P^+(\bar{\theta}\varphi P^+\theta\psi f + P^+(\varphi P^+(\psi f))). \end{aligned}$$

Hence

$$\theta(P^+(\varphi\psi f) - P^+(\bar{\theta}\varphi P^+\theta\psi f + P^+(\varphi P^+(\psi f))) = \theta P^+(\bar{\theta}\varphi\psi\theta f),$$

which gives

$$P^+(\bar{\theta}\varphi P^+\theta\psi f) = P^+(\varphi P^+(\psi f)).$$

Thus $T_{\bar{\theta}\varphi}T_{\theta\psi} = T_\varphi T_\psi$, so by Lemma 7.2 $\bar{\varphi} \in H^2$ or $\psi \in H^2$.

Let now $g \in H^2_-$, then by assumption $P^- D_\varphi D_\psi g = P^- D_{\varphi\psi} g$. Hence $P^-(\varphi P_\theta^\perp(\psi g)) = P^-(\varphi\psi g)$. It follows that

$$\begin{aligned} 0 &= P^-(\varphi\theta P^+(\bar{\theta}\psi g)) + P^-(\varphi P^-(\psi g)) - P^-(\varphi\psi x) \\ &= P^-(\varphi\theta P^+(\bar{\theta}\psi g)) - P^-(\varphi P^+(\psi g))). \end{aligned}$$

Thus

$$H_{\theta\varphi}H^*_{\theta\bar{\psi}} = H_\varphi H^*_{\bar{\psi}}. \tag{32}$$

By (28) we obtain that

$$\check{T}_{\varphi\psi} - \check{T}_\varphi\check{T}_\psi = \check{T}_{\theta\varphi\bar{\theta}\psi} - \check{T}_{\theta\varphi}\check{T}_{\bar{\theta}\psi},$$

which implies that

$$\check{T}_\varphi\check{T}_\psi = \check{T}_{\theta\varphi}\check{T}_{\bar{\theta}\psi}.$$

Using the properties (30) of the conjugation J we get

$$T_{\bar{\varphi}}T_{\bar{\psi}} = J\check{T}_\varphi J J\check{T}_\psi J = J\check{T}_{\theta\varphi}JJ\check{T}_{\bar{\theta}\psi}J = T_{\bar{\theta}\bar{\varphi}}T_{\theta\bar{\psi}}.$$

So, using again Lemma 7.2 we have that $\varphi \in H^2$ or $\bar{\psi} \in H^2$. □

Lemma 7.3 *Let θ be a nonconstant inner function and let $\varphi, \psi \in H^\infty$. The following conditions are equivalent:*

(a) $D^\theta_{\bar\varphi} D^\theta_{\bar\psi} = D^\theta_{\overline{\varphi\psi}}$,
(b) $D^\theta_{\psi} D^\theta_{\varphi} = D^\theta_{\varphi\psi}$,
(c) $H_{\bar\varphi} H^*_{\bar\theta} H_{\bar\psi} = 0$.

Proof Note that (a) is equivalent to (b), since $(D^\theta_\varphi)^* = D^\theta_{\bar\varphi}$. Using (16) in that case we obtain

$$D^\theta_{\bar\varphi} = \begin{pmatrix} \hat T^\theta_{\bar\varphi} & 0 \\ \hat\Gamma^\theta_{\bar\varphi} & \check T_{\bar\varphi} \end{pmatrix}, \qquad D^\theta_{\bar\psi} = \begin{pmatrix} \hat T^\theta_{\bar\psi} & 0 \\ \hat\Gamma^\theta_{\bar\psi} & \check T_{\bar\psi} \end{pmatrix},$$

and

$$D^\theta_{\overline{\varphi\psi}} = \begin{pmatrix} \hat T^\theta_{\overline{\varphi\psi}} & 0 \\ \hat\Gamma^\theta_{\overline{\varphi\psi}} & \check T_{\overline{\varphi\psi}} \end{pmatrix}.$$

Since for $\varphi, \psi \in H^\infty$ we have $\hat T^\theta_{\bar\varphi} \hat T^\theta_{\bar\psi} = \hat T^\theta_{\overline{\varphi\psi}}$ and $\check T_{\bar\varphi} \check T_{\bar\psi} = \check T_{\overline{\varphi\psi}}$, thus $D^\theta_{\bar\varphi} D^\theta_{\bar\psi} = D^\theta_{\overline{\varphi\psi}}$ if and only if

$$\hat\Gamma^\theta_{\bar\varphi} \hat T^\theta_{\bar\psi} + \check T_{\bar\varphi} \hat\Gamma^\theta_{\bar\psi} = \hat\Gamma^\theta_{\overline{\varphi\psi}}. \tag{33}$$

Note that for any $f \in H^2$

$$\begin{aligned} \hat\Gamma^\theta_{\bar\varphi} \hat T^\theta_{\bar\psi}(\theta f) + \check T_{\bar\varphi} \hat\Gamma^\theta_{\bar\psi}(\theta f) &= P^-(\bar\varphi\theta P^+(\bar\theta\bar\psi\theta f)) + \check T_{\bar\varphi} P^-(\bar\psi\theta f) \\ &= (H_{\theta\bar\varphi} T_{\bar\psi} + \check T_{\bar\varphi} H_{\theta\bar\psi}) f \end{aligned}$$

and

$$\hat\Gamma^\theta_{\overline{\varphi\psi}}(\theta f) = P^-(\overline{\varphi\psi}\theta f) = H_{\theta\overline{\varphi\psi}} f.$$

Hence (33) is equivalent to

$$H_{\theta\bar\varphi} T_{\bar\psi} + \check T_{\bar\varphi} H_{\theta\bar\psi} = H_{\theta\overline{\varphi\psi}}. \tag{34}$$

Note that $H_{\theta\bar{\varphi}} = H_{\bar{\varphi}} T_\theta$ and by (26) $T_\theta T_{\bar{\psi}} = T_{\theta\bar{\psi}} - H^*_{\bar{\theta}} H_{\bar{\psi}}$. Therefore by (26)

$$\begin{aligned} H_{\theta\bar{\varphi}} T_{\bar{\psi}} + \check{T}_{\bar{\varphi}} H_{\theta\bar{\psi}} &= H_{\bar{\varphi}} T_\theta T_{\bar{\psi}} + \check{T}_{\bar{\varphi}} H_{\theta\bar{\psi}} \\ &= H_{\bar{\varphi}} (T_{\theta\bar{\psi}} - H^*_{\bar{\theta}} H_{\bar{\psi}}) + \check{T}_{\bar{\varphi}} H_{\theta\bar{\psi}} = H_{\bar{\varphi}} T_{\theta\bar{\psi}} + \check{T}_{\bar{\varphi}} H_{\theta\bar{\psi}} - H_{\bar{\varphi}} H^*_{\bar{\theta}} H_{\bar{\psi}} \\ &= H_{\theta\bar{\varphi}\bar{\psi}} - H_{\bar{\varphi}} H^*_{\bar{\theta}} H_{\bar{\psi}}. \end{aligned} \tag{35}$$

Hence (34) holds if and only if $H_{\bar{\varphi}} H^*_{\bar{\theta}} H_{\bar{\psi}} = 0$. □

Let $\lambda \in \mathbb{D}$. Denote by

$$\omega_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda} z}$$

the Möbius transform. The next Lemma we recall without proof

Lemma 7.4 *[21, Lemma 2.2] For $f_1, f_2, f_3 \in L_\infty(\mathbb{T})$ and $\lambda \in \mathbb{D}$ we have*

$$\begin{aligned} T^*_{\omega_\lambda} H^*_{f_1} H_{f_2} H^*_{f_3} \check{T}_{\omega_\lambda} - H^*_{f_1} H_{f_2} H^*_{f_3} = &- (H^*_{f_1} H_{f_2} H^*_{f_3} J K_\lambda) \otimes (J K_\lambda) \\ &- (J H_{f_1} K_\lambda) \otimes (J T_{\omega_\lambda} T^*_{\omega_\lambda} H^*_{f_3} H_{f_2} K_\lambda) \\ &+ (T^*_{\omega_\lambda} H^*_{f_1} H_{f_2} K_\lambda) \otimes (\check{T}^*_{\omega_\lambda} H_{f_3} K_\lambda). \end{aligned}$$

Lemma 7.5 *Let θ be a nonconstant inner function and let $\varphi, \psi \in H^\infty$. If neither φ nor ψ is constant, then $H_{\bar{\varphi}} H^*_{\bar{\theta}} H_{\bar{\psi}} = 0$ if and only if $\bar{\varphi}(\theta - \lambda)$, $\bar{\psi}(\theta - \lambda)$ and $\overline{\varphi\psi}(\theta - \lambda)$ are analytic for some $\lambda \in \mathbb{C}$.*

Proof Assume that $H_{\bar{\varphi}} H^*_{\bar{\theta}} H_{\bar{\psi}} = 0$. Using a property of the conjugation J we have that

$$0 = J H_{\bar{\varphi}} H^*_{\bar{\theta}} H_{\bar{\psi}} J = H^*_{\bar{\varphi}} H_{\bar{\theta}} H^*_{\bar{\psi}}.$$

Applying Lemma 7.4 with $\omega_0(z) = -z$ we have

$$\begin{aligned} T^*_z H^*_{\bar{\varphi}} H_{\bar{\theta}} H^*_{\bar{\psi}} \check{T}_z - H^*_{\bar{\varphi}} H_{\bar{\theta}} H^*_{\bar{\psi}} = &- (H^*_{\bar{\varphi}} H_{\bar{\theta}} H^*_{\bar{\psi}} J 1) \otimes (J 1) \\ &- (J H_{\bar{\varphi}} 1) \otimes (J T_z T^*_z H^*_{\bar{\psi}} H_{\bar{\theta}} 1) \\ &+ (T^*_z H^*_{\bar{\varphi}} H_{\bar{\theta}} 1) \otimes (\check{T}^*_z H_{\bar{\psi}} 1). \end{aligned}$$

Since $H^*_{\bar{\varphi}} H_{\bar{\theta}} H^*_{\bar{\psi}} = 0$, thus

$$(T^*_z H^*_{\bar{\varphi}} H_{\bar{\theta}} 1) \otimes (\check{T}^*_z H_{\bar{\psi}} 1) = (J H_{\bar{\varphi}} 1) \otimes (J T_z T^*_z H^*_{\bar{\psi}} H_{\bar{\theta}} 1). \tag{36}$$

Note that $\check{T}_z^* H_{\bar{\psi}} 1 = \bar{z}(\overline{\psi(z)} - \overline{\psi(0)})$ and $J H_{\bar{\varphi}} 1 = \bar{z}(\varphi(z) - \varphi(0))$ are not zero. Let $f \in K_\theta^\perp$ be such that $\langle f, \check{T}_z^* H_{\bar{\psi}} 1\rangle \neq 0$. Then

$$(T_z^* H_{\bar{\varphi}}^* H_{\bar{\theta}} 1) \otimes (\check{T}_z^* H_{\bar{\psi}} 1) f = \langle f, \check{T}_z^* H_{\bar{\psi}} 1\rangle T_z^* H_{\bar{\varphi}}^* H_{\bar{\theta}} 1$$

and

$$(J H_{\bar{\varphi}} 1) \otimes (J T_z T_z^* H_{\bar{\psi}}^* H_{\bar{\theta}} 1) f = \langle f, J T_z T_z^* H_{\bar{\psi}}^* H_{\bar{\theta}} 1\rangle J H_{\bar{\varphi}} 1,$$

so by (36)

$$T_z^* H_{\bar{\varphi}}^* H_{\bar{\theta}} 1 = \frac{\langle f, J T_z T_z^* H_{\bar{\psi}}^* H_{\bar{\theta}} 1\rangle}{\langle f, \check{T}_z^* H_{\bar{\psi}} 1\rangle} J H_{\bar{\varphi}} 1 = \lambda_1 J H_{\bar{\varphi}} 1. \tag{37}$$

Let $\lambda_1 = \frac{\langle f, J T_z T_z^* H_{\bar{\psi}}^* H_{\bar{\theta}} 1\rangle}{\langle f, \check{T}_z^* H_{\bar{\psi}} 1\rangle}$. Then (36) is equivalent to

$$(\lambda_1 J H_{\bar{\varphi}} 1) \otimes (\check{T}_z^* H_{\bar{\psi}} 1) = (J H_{\bar{\varphi}} 1) \otimes (J T_z T_z^* H_{\bar{\psi}}^* H_{\bar{\theta}} 1);$$
$$(J H_{\bar{\varphi}} 1) \otimes (\bar{\lambda}_1 \check{T}_z^* H_{\bar{\psi}} 1) = (J H_{\bar{\varphi}} 1) \otimes (J T_z T_z^* H_{\bar{\psi}}^* H_{\bar{\theta}} 1).$$

Hence

$$J T_z T_z^* H_{\bar{\psi}}^* H_{\bar{\theta}} 1 = \bar{\lambda}_1 \check{T}_z^* H_{\bar{\psi}} 1. \tag{38}$$

Since $J T_z J = \check{T}_z^*$ and the kernel of T_z is zero, the kernel of $\check{T}_z^*$ is also zero. Thus (38) is equivalent to

$$J T_z T_z^* H_{\bar{\psi}}^* H_{\bar{\theta}} 1 = \bar{\lambda}_1 J T_z J H_{\bar{\psi}} 1 = J T_z \lambda_1 J H_{\bar{\psi}} 1,$$

so

$$T_z^* H_{\bar{\psi}}^* H_{\bar{\theta}} 1 = \lambda_1 J H_{\bar{\psi}} 1.$$

On the other hand,

$$T_z^* H_{\bar{\psi}}^* H_{\bar{\theta}} 1 = P^+(\bar{z} P^+(\psi P^- \bar{\theta})) = P^+(\bar{z}\psi(\bar{\theta} - \overline{\theta(0)}))$$

and

$$\lambda_1 J H_{\bar{\psi}} 1 = \lambda_1 J P^- \bar{\psi} = \lambda_1 P^+(\bar{z}\psi).$$

Therefore $P^+(\bar{z}\psi(\bar{\theta}-\overline{\theta(0)}-\lambda_1))=0$ which implies that $\bar{z}\psi(\bar{\theta}-\bar{\lambda})\in\overline{zH^2}$, i.e., $\bar{\psi}(\theta-\lambda)\in H^2$, where $\lambda=\bar{\lambda}_1+\theta(0)$.

Similarly, using (37) one can show that $\bar{\varphi}(\theta-\lambda)\in H^2$. To prove the last condition note that

$$0=H_{\bar{\varphi}}H^*_{\bar{\theta}}H_{\bar{\psi}}1=H_{\bar{\varphi}}H^*_{\bar{\theta}-\bar{\lambda}}H_{\bar{\psi}}1=P^-(\bar{\varphi}P^+(\theta-\lambda)(\bar{\psi}-\overline{\psi(0)})).$$

Since $(\theta-\lambda)(\bar{\psi}-\overline{\psi(0)})\in H^2$, then

$$0=P^-\bar{\varphi}(\theta-\lambda)(\bar{\psi}-\overline{\psi(0)})=P^-(\bar{\varphi}\bar{\psi}(\theta-\lambda))-\overline{\psi(0)}P^-(\bar{\varphi}(\theta-\lambda)).$$

As we already have $\bar{\varphi}(\theta-\lambda)\in H^2$, it follows that $\bar{\varphi}\bar{\psi}(\theta-\lambda)\in H^2$.

For the converse implication assume that there is $\lambda\in\mathbb{C}$ such that $\bar{\varphi}(\theta-\lambda)$, $\bar{\psi}(\theta-\lambda)$ and $\overline{\varphi\psi}(\theta-\lambda)$ are analytic. For a reproducing kernel k_w in H^2, $w\in\mathbb{D}$ we have

$$\begin{aligned}H_{\bar{\varphi}}H^*_{\bar{\theta}}H_{\bar{\psi}}k_w=&H_{\bar{\varphi}}H^*_{\bar{\theta}-\bar{\lambda}}H_{\bar{\psi}}k_w=P^-(\bar{\varphi}P^+((\theta-\lambda)P^-\bar{\psi}k_w)\\ =&P^-(\bar{\varphi}P^+((\theta-\lambda)(\bar{\psi}k_w-\overline{\psi(w)}k_w))\\ =&P^-(\bar{\varphi}(\theta-\lambda)(\bar{\psi}-\overline{\psi(w)})k_w\\ =&P^-(\bar{\varphi}\bar{\psi}((\theta-\lambda)k_w)-\overline{\psi(w)}P^-(\bar{\varphi}(\theta-\lambda)k_w)=0.\end{aligned}$$

Since the set $\{k_w:\ w\in\mathbb{D}\}$ is linearly dense in H^2, we see that $H_{\bar{\varphi}}H^*_{\bar{\theta}}H_{\bar{\psi}}=0$. □

As a conclusion of Theorem 7.1, Lemma 7.3 and Lemma 7.5 we have the following.

Theorem 7.6 *Let θ be a nonconstant inner function and $\varphi,\psi\in L_\infty(\mathbb{T})$. Then $D^\theta_\varphi D^\theta_\psi=D^\theta_{\varphi\psi}$ if and only if one of the following conditions hold*

(a) $\varphi,\psi,\bar{\varphi}(\theta-\lambda),\bar{\psi}(\theta-\lambda),\overline{\varphi\psi}(\theta-\lambda)\in H^2$ for some constant λ,
(b) $\bar{\varphi},\bar{\psi},\varphi(\theta-\lambda),\psi(\theta-\lambda),\varphi\psi(\theta-\lambda)\in H^2$ for some constant λ,
(c) either φ or ψ is constant.

Example Let α,β be nonconstant inner functions and let $\theta=\alpha\beta$. It is easy to see that then the condition (1) of Theorem 7.6 is satisfied for the operators D^θ_α and D^θ_β. Hence

$$D^\theta_\alpha D^\theta_\beta=D^\theta_{\alpha\beta}.$$

Theorem 7.7 *Let θ be a nonconstant inner function and let $\varphi,\psi\in L_\infty(\mathbb{T})$. Then*

$$D^\theta_\varphi D^\theta_\psi=D^\theta_\psi D^\theta_\varphi \tag{39}$$

if and only if one of the following conditions hold

(a) $\varphi, \psi, \bar{\varphi}(\theta - \lambda), \bar{\psi}(\theta - \lambda) \in H^2$ *for some constant* λ*,*
(b) $\bar{\varphi}, \bar{\psi}, \varphi(\theta - \lambda), \psi(\theta - \lambda) \in H^2$ *for some constant* λ*,*
(c) a nontrivial linear combination of φ *and* ψ *is constant.*

The following easy example shows that the assumption that both symbols of dual truncated Toeplitz operators are analytic is not sufficient for their commutativity.

Example Let θ be an inner function such that $\theta(0) = 0$. Note that $D_z^\theta D_{\theta z}^\theta \bar{z} = \theta z$ and $D_{\theta z}^\theta D_z^\theta \bar{z} = 0$. Hence even if both symbols $z, \theta z$ are analytic, $D_z^\theta D_{\theta z}^\theta \neq D_{\theta z}^\theta D_z^\theta$.

Comparing Theorems 7.6 and 7.7 note that if $D_\varphi^\theta D_\psi^\theta = D_{\varphi\psi}^\theta$, then the operators D_φ^θ and D_ψ^θ commute. However, the converse is not true.

Example Let α, β be nonconstant inner functions and let $\theta = \alpha\beta$. Note that then by Theorem 7.7 the operators D_θ^θ and D_α^θ commute. On the other hand, $\bar{\alpha}\bar{\theta}\theta = \bar{\alpha}$ is not analytic, hence, by Theorem 7.6, $D_\theta^\theta D_\alpha^\theta \neq D_{\theta\alpha}^\theta$.

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Boundedness of Toeplitz Operators in Bergman-Type Spaces

Jari Taskinen and Jani A. Virtanen

Abstract The characterization of the bounded Toeplitz operators T_a in Bergman spaces is an open problem even in the simplest case of the unweighted Bergman-Hilbert space $A^2(\mathbb{D})$. We consider here recent partial results on the topic. These include sufficient conditions for the boundedness and compactness of T_a in terms of weak Carleson-types condition for the symbol a. The results were recently generalized to the case of spaces on the unit ball $\mathbb{B}_N$ of $\mathbb{C}^N$. The second approach is based on certain results on the structure of the Bergman-spaces, namely, representations of their weighted norms using finite-dimensional decompositions of the spaces. This approach provides a characterization of the boundedness and compactness in the case of operators in spaces with weighted sup-norms.

Keywords Bergman space · Weighted norm · Toeplitz operator · Little Hankel operator · Bounded operator · Compact operator

1 Introduction: The Spaces and Operators

The focus of this article is on recent results on the boundedness of Toeplitz operators on weighted Bergman spaces of holomorphic functions, mainly on the open unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$, although some of the results are also formulated on the unit ball $\mathbb{B}_N$ of $\mathbb{C}^N$, $N = 2, 3, \ldots$. The related question on the compactness is only considered when it can be dealt with parallel to boundedness, and certain more special recent results for compactness will remain out of this review.

We will concentrate on two circles of ideas. First, we deal with Toeplitz operators with oscillating symbols and weak Carleson-type sufficient conditions for

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boundedness. The starting point of this direction of research is the article [30]. The second approach applies to operators with radial symbols, and it is based on the results on the structure of weighted Bergman spaces which were pioneered in the works of W. Lusky [17–19] and adapted to the study of Toeplitz operators recently in the papers [4, 5]. This led to a characterization of the boundedness and compactness of Toeplitz operators in weighted H^∞-spaces.

Let us present the basic notation and definitions. The notation concerning the spaces on the unit ball $\mathbb{B}_N$ will only be needed and thus given at the end of Sect. 2. The normalized area measure on $\mathbb{D}$ is denoted by $dA = \pi^{-1} r dr d\theta$, where r and θ are the polar coordinates of $z = re^{i\theta} \in \mathbb{C}$. Given $1 \le p < \infty$ and the real parameter $\alpha > -1$ we define the weighted area measure by $dA_\alpha(z) = (1+\alpha)(1-r^2)^\alpha dA(z)$ and set

$$L^p_\alpha(\mathbb{D}) = \Big\{ g : \mathbb{D} \to \mathbb{C} \text{ measurable } : \|g\|^p_{p,\alpha} := \int_{\mathbb{D}} |g|^p dA_\alpha < \infty \Big\} \text{ and}$$

$$A^p_\alpha(\mathbb{D}) = \{ g \in L^p_v(\mathbb{D}) : g \text{ holomorphic } \};$$

in the case $\alpha = 0$ these spaces are denoted by $L^p(\mathbb{D})$ and $A^p(\mathbb{D})$, respectively. Here, $v(z) = (1-|z|^2)^\alpha$ are called standard weights.

We will also consider more general weighted Bergman spaces and their analogue, weighted Hardy space H^∞_v corresponding to $p = \infty$. In general, by a weight v we mean a continuous function $\mathbb{D} \to]0,\infty[$ which is radial, vanishing on the boundary and decreasing with the radius, i.e. there holds $v(z) = v(|z|)$ for all $z \in \mathbb{D}$, $\lim_{|z|\to 1} v(z) = 0$ and $v(r) \ge v(s)$ if $1 > s > r > 0$. We denote $v dA = dA_v$ and, for $1 \le p < \infty$,

$$L^p_v(\mathbb{D}) = \Big\{ g : \mathbb{D} \to \mathbb{C} \text{ measurable } : \|g\|^p_{p,v} := \int_{\mathbb{D}} |g|^p dA_v < \infty \Big\} \text{ and}$$

$$A^p_v(\mathbb{D}) = \{ g \in L^p_v(\mathbb{D}) : g \text{ holomorphic } \},$$

and

$$h^\infty_v(\mathbb{D}) = \{ g : \mathbb{D} \to \mathbb{C} : g \text{ harmonic}, \ \|g\|_v := \sup_{z\in\mathbb{D}} |g(z)| v(|z|) < \infty \}$$

and

$$H^\infty_v(\mathbb{D}) = \{ g \in h^\infty_v : g \text{ holomorphic } \};$$

we use the standard notation $H^\infty(\mathbb{D}) = (H^\infty(\mathbb{D}), \|\cdot\|_\infty)$ in the non-weighted case. In all of the above cases, the subspaces of holomorphic and harmonic functions are closed subspaces of the their superspaces.

We write $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Given α, the Bergman projection P_α is the orthogonal projection from the Hilbert space $L^2_\alpha(\mathbb{D})$ onto the closed subspace $A^2_\alpha(\mathbb{D})$. Given a function $a \in L^1(\mathbb{D})$, we also denote by M_a the pointwise multiplier $M_a : f \mapsto af$, where $f : \mathbb{D} \to \mathbb{C}$ is a measurable function (which is usually holomorphic or harmonic in the sequel). If $1 \le p < \infty$, then a Toeplitz operator T_a on $A^p_\alpha(\mathbb{D})$, with symbol a, is in principle defined as the composition

$$T_a f = P_\alpha M_a f, \tag{1}$$

but the assumptions made so far do not always suffice to guarantee that (1) makes sense, since M_a might map f outside $L^2_\alpha(\mathbb{D})$. In the case a is a bounded function, there is no problem with the definition, since P_α can be written with the help of the Bergman kernel as the integral operator

$$P_\alpha f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - z\overline{w})^{2+\alpha}} dA_\alpha(w)\,,$$

hence

$$P_\alpha M_a f = \int_{\mathbb{D}} \frac{a(w) f(w)}{(1 - z\overline{w})^{2+\alpha}} dA_\alpha(w), \tag{2}$$

and for every $z \in \mathbb{D}$, these integrals converge for all $f \in L^1_\alpha(\mathbb{D})$. Moreover, it is known that P_α is a bounded operator in the space $L^p_\alpha(\mathbb{D})$, when $1 < p < \infty$, which yields the boundedness of $T_a : A^p_\alpha(\mathbb{D}) \to A^p_\alpha(\mathbb{D})$ for bounded symbols.

It is not difficult to construct unbounded symbols a which still induce bounded Toeplitz operators, but the characterization of symbols $a \in L^1(\mathbb{D})$ such that $T_a : A^p_\alpha(\mathbb{D}) \to A^p_\alpha(\mathbb{D})$ is well-defined and bounded is a well-known open problem. Let us mention some partial results on it. The characterization of boundedness and compactness of Toeplitz operators with nonnegative symbols in terms of Carleson type measures first appeared in [24].

D. Luecking [15] proved that a Toeplitz operator T_a with a nonnegative symbol $a \in L^1(\mathbb{D})$ is bounded in $A^2(\mathbb{D})$, if and only if the average

$$|B(z,r)|^{-1} \int_{B(z,r)} a(w)\, dA(w)$$

is a bounded function of z. Here $B(z, r)$ denotes a disk in the Bergman metric, with center z and some fixed radius $r > 0$. Toeplitz operators with radial symbols in the space $A^2_\alpha(\mathbb{D})$ and analogues on higher dimensional domains were thoroughly considered in [10]: in this case the operator is unitarily equivalent with a sequence space multiplier, see also (44) below, and thus the boundedness properties can be determined. A partial generalization to the case $p \neq 2$ was established in [21]. The Berezin transform

$$B(f)(z) = (1 - |z|^2)^2 \int_{\mathbb{D}} \frac{f(w)}{|1 - z\bar{w}|^4} dA(w), \quad z \in \mathbb{D}, \tag{3}$$

is a useful tool for the theory of Toeplitz operators, although it will not be used in this article. N. Zorboska proved in [39] for symbols a of bounded mean oscillation that the Toeplitz operator $T_a : A^2(\mathbb{D}) \to A^2(\mathbb{D})$ is bounded if and only if $B(a)$ is bounded. The results of [15] and [39] generalize to other $A^p(\mathbb{D})$-spaces, $1 < p < \infty$, as well, see e.g. [30]. Here is a non-exhaustive list of other works dealing with the boundedness and compactness of Toeplitz operators in Bergman-type spaces: [8], [10], [9], [12], [11], [15], [16], [21], [22], [25], [27], [28], [29], [30], [34], [35], [36], [37], [39]. The monograph [38] is a standard reference for the topic, and we also mention the survey article [31].

In this article we will review in Sect. 2 the results of [30], [33], [12]. These consist of sufficient, weak Carleson-type conditions for the boundedness and compactness of Toeplitz operators in reflexive Bergman spaces with standard weights, both on the unit disk and the unit ball. Sections 3–6 are mainly based on the recent works [4, 5], which deal with operators on $H^\infty_v(\mathbb{D})$-spaces with quite general classes of weights. Theorem 5 of Sect. 4 states that there is a bounded harmonic symbol f for which T_f is unbounded in $H^\infty_v(\mathbb{D})$ for any radial weight v satisfying our general assumptions. The main result of Sect. 5, Theorem 7 contains a necessary and sufficient condition for the boundedness of T_f in $H^\infty_v(\mathbb{D})$, as well as the corresponding result for the compactness. These conditions are slightly abstract, and thus in Sect. 6 we derive some more concrete, easily formulated sufficient conditions based on the results of Sect. 5.

We conclude this section by a remark on the definition of Toeplitz operators as an improper integral. Here, we fix $\alpha > -1$ and assume the symbol a is radial. Formula (4) will be considered in detail in Sect. 2 even for more general, non-radial symbols. The proof of Proposition 1 is taken here from [14], although some versions of it have probably been known for specialists for a long time.

Proposition 1 *Let a be a radial symbol, i.e. $a(z) = a(|z|)$ for almost all $z \in \mathbb{D}$, belonging to $L^1_\alpha(\mathbb{D})$, $\alpha > -1$, and let $g(z) = \sum_{n=0}^\infty g_n z^n$ be a holomorphic function on $\mathbb{D}$. Then, the defining integral (2) of $T_a g$ exists in the improper sense as the limit*

$$T_a g(z) = \lim_{\rho \to 1} \int_{|w|<\rho} \frac{a(w)g(w)}{(1 - z\overline{w})^{2+\alpha}} dA_\alpha(w), \tag{4}$$

convergent for every $z \in \mathbb{D}$. Moreover,

$$T_a g = \sum_{n=0}^{\infty} \frac{\beta_{a,\alpha}(1,n) g_n}{(\alpha+1)B(n+1,\alpha+1)} z^n \tag{5}$$

and in particular the power series on the right converges for all $z \in \mathbb{D}$.

Here and in the next we denote by B and Γ Euler's beta- and gamma-functions,

$$B(n+1,c) = \frac{n!\Gamma(c)}{\Gamma(n+1+c)}, \quad c > 0,$$

and for $0 < \rho \leq 1$

$$\beta_{a,\alpha}(\rho,n) = (\alpha+1) \int_0^{\sqrt{\rho}} t^n (1-t)^\alpha a(\sqrt{t}) dt, \tag{6}$$

where the integral converges by the assumptions that a is radial and belongs to $L^1_\alpha(\mathbb{D})$.

Proof of Proposition 1 We start by the remark that for all $m \in \mathbb{N}_0$, the integral

$$\int_{\mathbb{D}} g(w)\overline{w}^m a(w) dA_\alpha(w)$$

exists in the improper sense for every holomorphic g on the disk $\mathbb{D}$. Namely, the rotational symmetry of a and the usual orthogonality relations of trigonometric polynomials yield for all $m \in \mathbb{N}_0$

$$\int_{|w|<\rho} g(w)\overline{w}^m a(w) dA_\alpha(w) = 2(\alpha+1) g_m \int_0^\rho r^{2m+1} a(r)(1-r^2)^\alpha dr. \tag{7}$$

Clearly, the limit exists, when $\rho \to 1$. For every $0 < \rho < 1$, $z \in \mathbb{D}$, we obtain by (7)

$$\begin{aligned}
&\int_{|w|<\rho} \frac{a(w)g(w)}{(1-z\overline{w})^{2+\alpha}} dA_\alpha(w) \\
&= \int_{|w|<\rho} g(w) \left(\sum_{n=0}^{\infty} \frac{(z\overline{w})^n}{(\alpha+1)B(n+1,\alpha+1)} \right) a(w) dA_\alpha(w) \\
&= \sum_{n=0}^{\infty} \frac{\beta_{a,\alpha}(\rho,n) g_n}{(\alpha+1)B(n+1,\alpha+1)} z^n.
\end{aligned} \tag{8}$$

Let $L \in \mathbb{N}$ be such that $L \geq |\alpha| + 1$. Then,

$$B(n+1, \alpha+1) \geq \frac{n!\Gamma(\alpha+1)}{(n+L)!} \geq C_L n^{-L} \tag{9}$$

for some constant $C_L > 0$. We also have

$$\beta_{a,\alpha}(\rho, n) \leq \beta_{a,\alpha}(1, n) = 2(\alpha+1) \int_0^1 t^{2n}(1-t^2)^\alpha a(t)dt \leq C_\alpha \tag{10}$$

for another constant $C_\alpha > 0$, for all ρ and n, since $a \in L^1_\alpha(\mathbb{D})$. Moreover, since g is a holomorphic function on $\mathbb{D}$, we have $\limsup_{n\to\infty} |g_n|^{\frac{1}{n}} \leq 1$, hence,

$$\limsup_{n\to\infty} \left| \frac{\beta_{a,\alpha}(1,n) g_n}{(\alpha+1)B(n+1,\alpha+1)} \right|^{\frac{1}{n}} \leq \limsup_{n\to\infty} \left(C_L C_\alpha n^L\right)^{\frac{1}{n}} \cdot \limsup_{n\to\infty} |g_n|^{\frac{1}{n}} \leq 1.$$

The same estimate holds, independently of ρ, when $\beta_{a,\alpha}(1,n)$ is replaced by $\beta_{1,\alpha}(\rho, n)$. Hence, by the elementary theory of power series, (5), (8) converge uniformly on compact subsets of the disk and define holomorphic functions. Moreover, we have $\beta_{a,\alpha}(\rho, n) \to \beta_{a,\alpha}(1, n)$ for every n as $\rho \to 1$, hence, considering truncated series (5), (8) shows that the limit on the right of (4) exists for every $z \in \mathbb{D}$ and coincides with (5).

2 Toeplitz Operators with Oscillating Symbols

If an unbounded, measurable function a is strongly oscillating, it may give rise to a Toeplitz operator via the improper integral (4), and the operator may even be bounded with respect to a Bergman norm. A sufficient condition for oscillating symbols to induce a bounded T_a was presented in the paper [30]. More precisely, in the reference it was shown that T_a is bounded under an averaging condition for the symbol itself rather than for its modulus. The result needs a generalized definition of Toeplitz operators, which, however, eventually coincides with the improper integral. The result also extends to little Hankel operators.

We will next review the mentioned approach. It is based on a decomposition of the disk into an infinite family of $(D_n)_{n=1}^\infty$ subdomains, which have essentially constant area with respect to the hyperbolic geometry. The geometry of the subdomains needs to be specified carefully, since an explicit integration by parts -argument is a crucial step in the argument. Here, the sets D_n are rectangles in the polar coordinates, but they could also be chosen differently, see the discussion below.

Let us consider a symbol $a : \mathbb{D} \to \mathbb{C}$, which is at least locally Lebesgue-integrable on $\mathbb{D}$. We also fix the parameter $\alpha > -1$.

Definition 1 Denote by $\mathcal{D}$ the family of the sets $D := D(r, \theta)$, where

$$D = \{\rho e^{i\phi} \mid r \le \rho \le 1 - \frac{1}{2}(1-r)\ ,\ \theta \le \phi \le \theta + \pi(1-r)\} \tag{11}$$

for all $0 < r < 1, \theta \in [0, 2\pi]$. Let $|D| := \int_D dA$ and, for $w = \rho e^{i\phi} \in D(r, \theta)$, let

$$\hat{a}_D(w) := \frac{1}{|D|} \int_r^\rho \int_\theta^\phi a(\varrho e^{i\varphi}) \varrho d\varphi d\varrho. \tag{12}$$

We will study symbols a for which there exists a constant $C > 0$ such that

$$|\hat{a}_D(w)| \le C \tag{13}$$

for all $D \in \mathcal{D}$ and all $w \in D$.

The sets $D\big(1 - 2^{-m+1}, 2\pi(k-1)2^{-m}\big) \in \mathcal{D}$, where $m \in \mathbb{N}, k = 1, \ldots, 2^{-m}$, form a decomposition of the disk $\mathbb{D}$. Let us re-index them somehow into a family $(D_n)_{n=1}^\infty$ with

$$D_n = \{\ z = re^{i\theta} \mid r_n < r \le r_n', \theta_n < \theta \le \theta_n'\} \tag{14}$$

where, for some m and k,

$$r_n = 1 - 2^{-m+1},\ r_n' = 1 - 2^{-m},\ \ \theta_n = \pi(k-1)2^{-m+1},\ \theta_n' = \pi k 2^{-m+1} \tag{15}$$

Given $f \in A_\alpha^p(\mathbb{D})$, we write for all $n = n(m, k)$

$$F_n f(z) = \int_{D_n} \frac{a(w) f(w)}{(1 - z\bar{w})^{2+\alpha}} dA_\alpha(w)\ ,\ \ z \in \mathbb{D}, \tag{16}$$

so that F_n can actually be considered as a conventional, bounded Toeplitz operator on $A_\alpha^p(\mathbb{D})$.

The following theorem, in the case of $\alpha = 0$, is the main result Theorem 2.3 of [30]. The weighted case was included in [12].

Theorem 1 *Let $1 < p < \infty$ and assume that the locally integrable function a satisfies the condition (13). Given $f \in A_\alpha^p(\mathbb{D})$, the series $\sum_{n=1}^\infty F_n f(z)$ converges pointwise, absolutely for almost all $z \in \mathbb{D}$, and the generalized Toeplitz operator*

$T_a : A^p \to A^p$, defined by

$$T_a f(z) = \sum_{n=1}^{\infty} F_n f(z) \tag{17}$$

is bounded for all $1 < p < \infty$, and there is a constant C_α, independent of a, such that

$$\|T_a\| \le C_\alpha \sup_{D \in \mathcal{D}, w \in D} |\hat{a}_D(w)|. \tag{18}$$

The main step of the proof consists of writing the integral (16) in polar coordinates and performing a double integration by parts (once with respect to both coordinates) such that there appear integrals of a and derivatives of $f(w)(1 - |w|^2)^\alpha (1 - z\bar{w})^{2+\alpha}$. The former can be estimated by using the assumption (13) and the latter by using bounds for the maximal Bergman projection and well known arguments and estimates related with hyperbolic geometry. One obtains a representation for the integral (2) as a pointwise convergent sum of the integrals (16) as in (17). We refer to [30] for the details. Improved versions of the proof appear in [33] and [12], and they yield our next theorem, although we do not repeat the proof here. We remark that every Toeplitz operator

$$T_{a_\rho} f(z) = \int_{|w|<\rho} \frac{a(w) f(w)}{(1 - z\bar{w})^{2+\alpha}} dA_\alpha(w) \tag{19}$$

is bounded $A^p_\alpha(\mathbb{D}) \to A^p_\alpha(\mathbb{D})$, since the support of the symbol is contained in a compact subset of $\mathbb{D}$.

Theorem 2 *Let $1 < p < \infty$ and $1/p + 1/q = 1$, and let the symbol a be as in Theorem 1. Then, the generalized Toeplitz operator $T_a : A^p_\alpha(\mathbb{D}) \to A^p_\alpha(\mathbb{D})$, defined in (17), can be written as*

$$T_a f = \lim_{\rho \to 1} T_{a_\rho} f \quad , \tag{20}$$

for all $f \in A^p_\alpha(\mathbb{D})$. The limit converges with respect to the strong operator topology. Moreover, the transposed operator $T_a^* : A^q_\alpha(\mathbb{D}) \to A^q_\alpha(\mathbb{D})$ (with respect to the standard complex dual pairing) satisfies

$$T_a^* f = \lim_{\rho \to 1} T_{\bar{a}_\rho} f \tag{21}$$

for $f \in A^q_\alpha(\mathbb{D})$ and for almost all $z \in \mathbb{D}$, and the limit also converges in the strong operator topology.

The limits in (20), (21) cannot in general converge in the operator norm, since the operators T_{a_ρ} are compact. We mention that, when $\alpha = 0$, the above results are formulated in [33] also for little Hankel operators

$$h_a f(z) = \int_{\mathbb{D}} \frac{a(w) f(w)}{(1-\bar{z}w)^2} dA(w) , \quad z \in \mathbb{D}. \tag{22}$$

Here, one also defines using the same decomposition of the unit disk as above

$$H_n f(z) = \int_{D_n} \frac{a(w) f(w)}{(1-\bar{z}w)^2} dA(w) , \quad z \in \mathbb{D}, \tag{23}$$

and defines the generalized little Hankel operator $h_a f(z)$ as $\sum_{n=1}^{\infty} H_n f(z)$. Then, if (13) holds for the symbol a, one obtains that $h_a : A^p(\mathbb{D}) \to L^p(\mathbb{D})$ is bounded for all $1 < p < \infty$, the operator norm of h_a has the same bound as in (18), and finally, the operator h_a and its transpose have representations as improper integrals similar to those in (20), (21).

The definition (17) of a generalized Toeplitz operator depends on the geometry of the special decomposition (14) of the unit disk, but Theorem 2 largely removes this unsatisfactory feature, since the improper integral in (20) is quite a natural one. We remark that in the literature there are versions of the result, which use different subdomains of the unit disk. In [36] the condition (13) is replaced by a similar one on Carleson squares

$$S_h^\alpha(e^{i\theta}) = \left\{ \rho e^{i\phi} \ : \ 1-h < \rho < 1, |\phi - \theta| < \pi \alpha h \right\}$$

where $0 < h < 1$, $0 \le \theta \le 2\pi$, $0 < \alpha \le 1$. The authors give a boundedness result for the Toeplitz operators and they also show that their sufficient condition is equivalent to that in Theorem 1. Finally, they also prove the important observation that the sufficient condition (13) is not necessary to the boundedness of $T_a : A_\alpha^p(\mathbb{D}) \to A_\alpha^p(\mathbb{D})$.

Another variant appears in [22, 23] where Toeplitz operators on Bergman spaces of simply connected planar domains are considered. In such domains any geometric symmetry is usually lost, and there does not exist a decomposition of the domain which is as natural as the one for the disk, see (14). However, the author uses a Whitney decomposition with Euclidean rectangles and obtains results which are analogous to Theorem 1. The Whitney decomposition can of course be applied also in the case of the disk, and it yields another sufficient condition for the boundedness of the Toeplitz operator. We do not know, if the condition is equivalent to (13).

In [32], we generalized Theorem 1 to the setting of $A^1(\mathbb{D})$, while bounded Toeplitz operators T_μ on $A_\alpha^1(\mathbb{B}_N)$ were characterized in terms of the reproducing kernels in [6] under additional conditions on the measure μ. We skip a detailed discussion on the boundedness problem in A^1-spaces and only note that the previous approach has not been worked out in the non-locally convex cases $0 < p < 1$.

Theorems 1 and 2, first proved in [30] and [33], have been generalized to the case of Toeplitz operators on the Bergman space of the unit ball of $\mathbb{C}^N$ in the recent work [12], but even presenting the results leads to non-trivial technical challenges. We do not directly need the Euclidean space $\mathbb{R}^3$ here, but since that dimension is still within the capabilities of the human imagination, we ask the reader to think about a radially symmetric decomposition of the unit ball of $\mathbb{R}^3$: that is indeed a challenge, since decomposing the ball surface into finitely many identical squares in spherical coordinates (corresponding to intervals $[\theta_n, \theta_n']$ in (14)–(15)) is impossible. For example, starting to fill the ball surface from the equator with spherical squares with one side parallel to the meridians, one runs into difficulties at latest when trying to fill the north and south caps.

The results of [12] are formulated for measures with standard weights and thus the proofs contain new information even in the case $N = 1$, since the earlier papers only contained the unweighted case. The basic idea of the proof is the same as in [30] and [33], but new non-trivial technical considerations are nevertheless needed. Let us review the approach of [12] superficially without going into all technical details. For $\alpha > -1$, we define the weighted Lebesgue measure dV_α on the unit ball $\mathbb{B}_N$, $N \in \mathbb{N}$, by $dV_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha dV(z)$, where dV is the unweighted N-dimensional (real) Lebesgue measure and c_α is a normalizing constant such that $\int_{\mathbb{B}_N} dV_\alpha = 1$. For $1 \le p < \infty$, we denote by $L^p_\alpha(\mathbb{B}_N)$ the L^p-space with respect to the measure dV_α and by $A^p_\alpha(\mathbb{B}_N)$ the weighted Bergman space of all holomorphic functions in $L^p_\alpha(\mathbb{B}_N)$. We also denote by P_α the orthogonal projection from $L^2_\alpha(\mathbb{B}_N)$ onto $A^2_\alpha(\mathbb{B}_N)$. It is known to be a bounded operator $L^p_\alpha(\mathbb{B}_N)$ onto $A^p_\alpha(\mathbb{B}_N)$ for all $1 < p < \infty$.

In the following it is useful to work with real variables by identifying $\mathbb{C}^N$ with $\mathbb{R}^n$, $n = 2N$, so that $\mathbb{B}_N$ equals $\mathbb{B}_n$ in real coordinates. Accordingly, any point $x \in \mathbb{B}_n$ with modulus $|x| = r$ can be written as

$$\begin{aligned} x &= (r\cos\theta_2, r\sin\theta_2\cos\theta_3, r\sin\theta_2\sin\theta_3\cos\theta_4, \cdots, \\ &\qquad r\sin\theta_2\cdots\sin\theta_{n-1}\cos\theta_n, r\sin\theta_2\cdots\sin\theta_{n-1}\sin\theta_n) \end{aligned}$$

in the spherical coordinates

$$\xi = (r, \theta_2, \cdots, \theta_n) \in [0, 1[\times \prod_{j=2}^{n-1} [0, \pi[\times [0, 2\pi[=: \mathbb{Q}_n,$$

and these determine the coordinate transform $\sigma : \mathbb{Q}_n \to \mathbb{B}_n$ by $x = \sigma(\xi)$. As in the case of the unit ball one needs to specify a suitable decomposition of the unit ball $\mathbb{B}_n$, but it turns out to be unexpectedly difficult in higher dimensions. We skip the detailed choice of the sets at this point, referring to Section 1 of [12] and only mention that it is possible to choose for every $m \in \mathbb{N}$ finitely many subsets

$B_{m,k}$, $k = 1, \ldots, K_m$, which are images under the mapping σ of certain rectangles $Q_{m,k} \subset \mathbb{Q}_n$ in polar coordinates, such that

- the volume of every $B_{m,k}$ is proportional to 2^{-nm},
- the union of all sets $B_{m,k}$ when $m \in \mathbb{N}$ and $k = 1, \ldots, K_m$, covers $\mathbb{B}_n$,
- there is a constant $N \in \mathbb{N}$ such that any point $x \in \mathbb{B}_n$ is contained in at most N of the sets $B_{m,k}$.

We enumerate the sets $Q_{m,k}$ and $B_{m,k}$ into sequences $(Q_j)_{j=1}^{\infty}$ and $(B_j)_{j=1}^{\infty}$. Then, we impose on $\mathbb{Q}_n$ the partial ordering

$$x \leqslant y \Longleftrightarrow x_1 \leq y_1,\ |\tfrac{\pi}{2} - x_2| \geq |\tfrac{\pi}{2} - y_2|,\ \ldots,\ |\tfrac{\pi}{2} - x_{n-1}| \geq |\tfrac{\pi}{2} - y_{n-1}|, \\ x_n \leq y_n. \tag{24}$$

On each Q_j we pick up the smallest and largest points $x^{(j)} = \big(x_1^{(j)}, \ldots, x_n^{(j)}\big)$ and $y^{(j)} = \big(y_1^{(j)}, \ldots, y_n^{(j)}\big)$ with respect to the given ordering, hence, there holds $Q_j = Q\big(x^{(j)}, y^{(j)}\big)$, where we denote, for $a, b \in \mathbb{Q}_n$ with $a \leqslant b$,

$$Q(a,b) = \big\{x \in \mathbb{R}^n \ :\ a \leqslant x \leqslant b\big\}, \quad B(a,b) = \sigma\big(Q(a,b)\big). \tag{25}$$

Note that for $x, y \in [0, 1) \times [0, \frac{\pi}{2}]^{n-2} \times [0, 2\pi]$ the order relation "$\leqslant$" coincides with the usual partial order of points in $\mathbb{R}^n$, which is then mirrored to all of $\mathbb{Q}_n$ to account for the construction of the sets Q_j and B_j. In particular, the $x^{(j)}$ and $y^{(j)}$ are two opposite corners of Q_j and we have $B_j = B(x^{(j)}, y^{(j)})$.

Let $a : \mathbb{B}_N \to \mathbb{C}$ be a locally integrable function and $1 < p < \infty$. The generalized Toeplitz operator is defined by

$$T_a f(z) := \sum_{j=1}^{\infty} T_a(\chi_j f)(z) = \sum_{j=1}^{\infty} P_\alpha(a\chi_j f)(z), \tag{26}$$

if the series converges for almost every $z \in \mathbb{B}_N$ and all $f \in A_\alpha^p(\mathbb{B}_N)$. Here χ_j denotes the characteristic function of the set B_j. The boundedness of the Bergman projection P_α in $L_\alpha^p(\mathbb{B}_N)$ implies that $T_a f = P_\alpha(af)$ whenever $af \in L_\alpha^p(\mathbb{B}_N)$. In particular, if a is bounded, then T_a is just the standard Toeplitz operator. As in the one-dimensional case, a "weak" Carleson-type condition (28) implies that T_a becomes a well-defined bounded linear operator and the definition coincides with the integral definition, when it is interpreted as an improper integral. Accordingly, given a locally integrable $a : \mathbb{B}_N \to \mathbb{C}$, we define for all $j \in \mathbb{N}$

$$\widehat{a}_j := \sup_{y \in B_j} \Bigg| \int_{B(x^{(j)}, y)} a \, dV_\alpha \Bigg| \tag{27}$$

and denote $|B|_\alpha = \int_B dV_\alpha$ for all measurable subsets $B \subset \mathbb{B}_N$.

Theorem 3 *Let $a : \mathbb{B}_N \to \mathbb{C}$ be locally integrable, $1 < p < \infty$ and the family $(B_j)_{j\in\mathbb{N}}$ be as above. If there exists a constant $C_a > 0$ such that*

$$\widehat{a}_j \le C_a |B_j|_\alpha \tag{28}$$

for all $j \in \mathbb{N}$, then the series (26) converges almost everywhere and in $L^p_\alpha(\mathbb{B}_N)$ and defines a bounded linear operator $A^p_\alpha(\mathbb{B}_N) \to A^p_\alpha(\mathbb{B}_N)$ with $\|T_a\| \le C_\alpha C_a$, for some constant $C_\alpha > 0$ independent of a.

Given the symbol a as above and $0 < \rho < 1$, we define $a_\rho(z) = a(z)$ for $|z| \le \rho$ and $a_\rho(z) = 0$ for $\rho < |z| < 1$; then every operator T_{a_ρ} is bounded on $A^p_\alpha(\mathbb{B}_N)$, since the supports of the symbols are compact subsets of the unit ball, or also by the previous theorem. As in the one-dimensional case, the assumption (28) allows the following representation of the Toeplitz operator, which does not depend on the decomposition $(B_j)_{j\in\mathbb{N}}$.

Theorem 4 *Let $1 < p < \infty$ and $1/p + 1/q = 1$, and suppose that $a \in L^1_{loc}$ satisfies (28). Then*

$$T_a f = \lim_{\rho\to 1} T_{a_\rho} f$$

for all $f \in A^p_\alpha(\mathbb{B}_N)$ and the transpose operator $T^*_a : A^q_\alpha(\mathbb{B}_N) \to A^q_\alpha(\mathbb{B}_N)$ can be expressed as

$$T^*_a f = \lim_{\rho\to 1} T_{\overline{a}_\rho} f$$

for $f \in A^p_\alpha(\mathbb{B}_N)$.

The transpose is defined respect to the standard duality of $A^p_\alpha(\mathbb{B}_N)$-spaces.

It would probably be possible and technically easier to formulate and prove a result analogous to Theorem 3 by using a rectangular Whitney decomposition of $\mathbb{B}_N$ instead of the one described here, but there would then be the disadvantage that the spherical symmetry would be lost and the condition for the boundedness would depend on the particular choice of the decomposition. In particular, it might be difficult or impossible to prove Theorem 4 with that approach.

3 Toeplitz Operators in H^∞_v-Spaces: Introduction

From now on we will deal with Toeplitz operators in spaces on $\mathbb{D}$ with quite general weights v satisfying the basic assumptions of Sect. 1. A typical, important example of weights considered in this section is the exponentially decreasing $v(r) = \exp(-1/(1-r))$. Because of such examples we need again to pay attention to the definition of Toeplitz operators in the spaces $A^p_v(\mathbb{D})$ and $H^\infty_v(\mathbb{D})$, namely, there is the problem that the Bergman projection may not be bounded. Actually we will show that this is always the case for $p = \infty$ for any weight, see Theorem 5, but even in the reflexive case there may be problems in this respect: in [7] it was shown that for the above mentioned exponential weight $v(z)$, the orthogonal projection $L_v^2(\mathbb{D}) \to A_v^2(\mathbb{D})$ is bounded in L_v^p if and only if $p = 2$. Moreover, in [19] W. Lusky proved that the mere existence of a bounded projection from $L_v^\infty(\mathbb{D})$ onto $H_v^\infty(\mathbb{D})$ is equivalent to v satisfying condition (B) of Definition 2, below. For example, the exponential weight v satisfies (B), but there also exist natural weights which do not, like $v(z) = (1 - \log(1 - |z|))^{-1}$ (see the statement after Theorem 1.2. of [19] and Example 2.4. of the same paper for other examples).

Yet, even in the spaces $H_v^\infty(\mathbb{D})$ and $A_v^p(\mathbb{D})$ with general weights, the definition of the Toeplitz operator involves the orthogonal projection $P_v : L_v^2(\mathbb{D}) \to A_v^2(\mathbb{D})$. It will be useful to consider the integral kernel of P_v, the so called Bergman kernel. In the next we follow well-known arguments, see e.g. [7]. We denote the inner product in the Hilbert spaces $L_v^2(\mathbb{D})$ and $A_v^2(\mathbb{D})$ by $\langle f, g\rangle = \int_{\mathbb{D}} f\overline{g}\, dA_v$. Then, the functions $e_k(z) = \Gamma_{2k}^{-1/2} z^k$, where $k \in \mathbb{N}_0$ and

$$\Gamma_k = 2\pi \int_0^1 r^{k+1} v(r) dr, \tag{29}$$

form an orthonormal basis of $A_v^2(\mathbb{D})$. We remark that the numbers Γ_k satisfy for all $0 < \varrho < 1$ and some constant $C_{v,\varrho} > 0$ the following lower bound

$$\Gamma_k \geq C_{v,\varrho} \varrho^k \tag{30}$$

for every $k \in \mathbb{N}_0$. This follows from (29) by considering the integral e.g. over the interval $[\varrho, 1 - (1 - \varrho)/2]$ only.

Convergence in the space $A_v^p(\mathbb{D})$, $1 < p < \infty$, with respect to the norm $\| \cdot \|_{p,v}$ implies pointwise convergence (hence $A_v^p(\mathbb{D})$ is a closed subspace of $L_v^p(\mathbb{D})$), and thus the point evaluation functionals at any point of $\mathbb{D}$ are bounded functionals on $A_v^p(\mathbb{D})$. Consequently, we find the Bergman kernel by using the Riesz representation theorem, which allows us to choose the family of functions $K_z \in A_v^2(\mathbb{D})$, $z \in \mathbb{D}$, such that

$$g(z) = \langle g, K_z \rangle = \int_{\mathbb{D}} g(w)\overline{K_z(w)}\, dA_v(w) \tag{31}$$

for all $g \in A_v^2(\mathbb{D})$. The integral operator defined by the right hand side can be extended to $L_v^2(\mathbb{D})$, and it actually defines the orthogonal projection from $L_v^2(\mathbb{D})$

onto $A_v^2(\mathbb{D})$, i.e. the Bergman projection P_v. Using the orthonormal basis $(e_k)_{k=0}^\infty$ we can write for all $z \in \mathbb{D}$

$$P_v g(z) = \sum_{k=0}^{\infty} \langle g, e_k \rangle e_k(z) = \int_{\mathbb{D}} \sum_{k=0}^{\infty} \frac{z^k \overline{w}^k}{\Gamma_{2k}} g(w) dA_v(w). \tag{32}$$

Here, the order of the summation and the integral can be changed, because (30) leads for any fixed $z \in \mathbb{D}$ to the estimate

$$\Big| \frac{z^k \overline{w}^k}{\Gamma_{2k}} \Big| \le c_{v,\varrho} \Big(\frac{|z|}{\varrho^2} \Big)^k, \tag{33}$$

and we can choose here $\varrho^2 > |z|$ so that the sum on the right-hand side of (32) converges well enough. Moreover, the estimate (33) implies that for every $z \in \mathbb{D}$ the Bergman kernel K_z is a bounded function:

$$|K_z(w)| \le C_z \ \text{ for all } w \in \mathbb{D}. \tag{34}$$

We obtain the following inference.

Lemma 1 *Let $f \in L^1(\mathbb{D})$. The integral defining the Toeplitz operator T_f with symbol f on H_v^∞,*

$$T_f g = \int_{\mathbb{D}} f(w) g(w) \overline{K_z(w)}\, dA_v(w), \tag{35}$$

converges for all $z \in \mathbb{D}$ and for all $g \in H_v^\infty(\mathbb{D})$,

Indeed, if $g \in H_v^\infty(\mathbb{D})$, then, by definition, $gv \in L^\infty(\mathbb{D})$. Hence, the result follows from (34).

We remark that the a priori assumption $f \in L^1(\mathbb{D})$ is usual also in the considerations on Toeplitz operators in the reflexive Bergman spaces, but in that case this assumption does not guarantee that the defining integral (35) converges for all $g \in A_v^p(\mathbb{D})$. From this point of view, the case $p = \infty$ is simpler. However, although $T_f g$ of (35) is a well-defined holomorphic function it might not be an element of $H_v^\infty(\mathbb{D})$ and T_f might not be a bounded operator $H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$. Actually it is an elementary consequence of the closed graph theorem that T_f is a bounded operator $H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ if and only if $T_f(H_v^\infty(\mathbb{D})) \subset H_v^\infty(\mathbb{D})$. We will soon turn to questions on the boundedness of the operator T_f.

If $g \in H_v^\infty(\mathbb{D})$ is such that $fg \in L_v^2(\mathbb{D})$, we also have

$$(T_f g)(z) = \sum_{n=0}^{\infty} \langle fg, e_n \rangle e_n(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_{2n}} \int_{\mathbb{D}} f(w) g(w) \overline{w}^n v(w) dA\ , \tag{36}$$

where the series converges in $L^2_v(\mathbb{D})$. However, the formula also holds for all $g \in H^\infty_v(\mathbb{D})$ (since we are assuming $f \in L^1(\mathbb{D})$) and the product fgv thus belongs to $L^1(\mathbb{D})$, and one can commute the summation and integration in (36), due to (33). In the latter case, the sum (36) converges uniformly for z in compact subsets of the disk.

4 Toeplitz Operators with Harmonic Symbols in $H^\infty_v(\mathbb{D})$-Spaces

In this section we will consider Toeplitz operators T_f with harmonic symbols $f : \mathbb{D} \to \mathbb{C}$ in weighted spaces $H^\infty_v(\mathbb{D})$. We assume that the weight v satisfies the basic requirements introduced in Sect. 1. In addition, the following notions will be needed here and in subsequent sections. For any function $g : \mathbb{D} \to \mathbb{C}$ and radius $0 \le r \le 1$ we will denote

$$M_\infty(g, r) = \sup_{|z|=r} |g(z)|. \tag{37}$$

Also, a weight v is called normal if

$$\sup_{n\in\mathbb{N}} \frac{v(1-2^{-n})}{v(1-2^{-n-1})} < \infty \quad \text{and} \quad \inf_{k\in\mathbb{N}} \limsup_{n\to\infty} \frac{v(1-2^{-n-k})}{v(1-2^{-n})} < 1. \tag{38}$$

For example, the standard weights $v(r) = (1-r^2)^\alpha$, $\alpha > 0$ are normal, whereas the weights of exponential type, $v(r) = \exp(-\alpha/(1-r)^\beta)$, $\alpha, \beta > 0$, are not. The Riesz projection P maps harmonic functions into holomorphic ones and it is defined by

$$P\Big(\sum_{k\in\mathbb{Z}} a_k r^{|k|} e^{ik\theta}\Big) = \sum_{k=0}^{\infty} a_k r^k e^{ik\theta}, \quad r \in [0, 1),\ \theta \in [0, 2\pi]. \tag{39}$$

For every $m > 0$ we denote by r_m be a point where the function $r \mapsto r^m v(r)$ attains its absolute maximum on $[0, 1]$. Due to the general assumptions on the weights it is easily seen that $r_n \ge r_m$ if $n \ge m$ and $\lim_{m\to\infty} r_m = 1$; see for example [17] for details.

We now turn to questions on the boundedness of Toeplitz operators T_f with harmonic symbols f. In the case f is even holomorphic, the operator T_f is just the multiplier M_f, and it is quite plain that T_f is bounded, if and only if $f \in H^\infty(\mathbb{D})$, i.e., f is a bounded function. Due to the generality of the weights, the details of this claim are exposed in [4, Section 2]. Allowing the symbol to be just a harmonic function changes the situation dramatically. The basic reason for this is the unboundedness of the Riesz and Bergman projections with respect to the sup-norm, but one can develop this idea as far as the following result. We repeat that in all of our results the weights v must satisfy the general assumptions made in Sect. 1.

Theorem 5 *There is a bounded harmonic function* $f : \mathbb{D} \to \mathbb{C}$ *such that* T_f *is not a bounded operator* $H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ *for any weight* v *on* $\mathbb{D}$.

This result implies the following conclusion.

Corollary 1 *For any weight* v*, the Bergman projection* P_v *is not a bounded mapping* $L_v^\infty(\mathbb{D}) \to L_v^\infty(\mathbb{D})$.

Namely, the pointwise multiplication with a bounded function f is always a bounded operator $H_v^\infty(\mathbb{D}) \to L_v^\infty(\mathbb{D})$. So, if P_v were bounded, this would imply $T_f : H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ is bounded for every $f \in L^\infty(\mathbb{D})$, which would contradict Theorem 5. We actually see that even the restriction of P_v onto $h_v^\infty(\mathbb{D})$ is unbounded.

In the sequel, the complex variable z will always be written in the polar coordinates as $z = re^{i\theta}$, unless otherwise indicated.

Proof of Theorem 5 Let us fix a weight v on $\mathbb{D}$ and define first the function $f_0 : \partial\mathbb{D} \to \mathbb{C}$ by

$$f_0(z) = \begin{cases} 1\,, & \text{if } -\pi/2 \le \theta \le \pi/2 \\ 0\,, & \text{if } -\pi \le \theta < -\pi/2 \text{ or } \pi/2 < \theta \le \pi . \end{cases}$$

The symbol f is defined as the harmonic extension of f_0 on $\mathbb{D}$ obtained from the Poisson integral, hence, we have $f \in h^\infty(\mathbb{D})$. Calculating the Fourier coefficients of f_0 we observe that

$$f(z) = \frac{1}{2} + \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \Big(z^{2k+1} + \bar{z}^{2k+1} \Big), \qquad z \in \mathbb{D}. \tag{40}$$

Indeed, let a_k, $k \in \mathbb{Z}$, be. Then we have

$$a_k = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{-ikt} dt = \frac{e^{ik\pi/2} - e^{-ik\pi/2}}{2k\pi i} = \frac{e^{i|k|\pi/2} - e^{-i|k|\pi/2}}{2|k|\pi i}$$

$$= \begin{cases} \frac{(-1)^j}{(2j+1)\pi}, & \text{if } |k| = 2j+1 \text{ for some } j \in \mathbb{N}_0, \\ 0 & \text{for other } k \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

Moreover, $a_0 = 1/2$. This implies (40).

Next we define the test functions, which will be used in showing the unboundedness of the Toeplitz operator: we set

$$g_m(z) = \frac{r^m e^{im\theta}}{r_m^m v(r_m)}, \quad z = re^{i\theta} \in \mathbb{D}$$

for all $m \in \mathbb{N}_0$, where the definition of the maximum point r_m was given in the beginning of the section so that we obviously have $\|g_m\|_v = 1$. We next show that for all $m \in \mathbb{N}_0$ there holds

$$T_f g_m(z) = \sum_{k=0}^{m} b_{k-m} \frac{\Gamma_{2m}}{\Gamma_{2k}} \frac{z^k}{r_m^m v(r_m)} + \sum_{k=m+1}^{\infty} b_{k-m} \frac{z^k}{r_m^m v(r_m)} \tag{41}$$

where $f(z) = \sum_{k=-\infty}^{\infty} b_k r^{|k|} e^{ik\theta}$ and Γ_k is as in (29). Indeed, this follows from

$$\begin{aligned} f(z)g(z) &= \sum_{j\in\mathbb{Z}} b_j \frac{r^{m+|j|} e^{i(j+m)\theta}}{r_m^m v(r_m)} \\ &= \sum_{k=m+1}^{\infty} b_{k-m} \frac{r^k e^{ik\theta}}{r_m^m v(r_m)} + \sum_{k=-\infty}^{m} b_{k-m} \frac{r^{2m-k} e^{ik\theta}}{r_m^m v(r_m)} \end{aligned}$$

and (36).

Let us now turn to the final proof showing that T_f is unbounded on $H_v^\infty(\mathbb{D})$. We define

$$f_1(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} \left(z^{2j+1} + \bar{z}^{2j+1} \right)$$

and note that it suffices to show that T_{f_1} is unbounded since $T_f = T_{1/2} + \pi^{-1} T_{f_1}$ and $T_{1/2}$ (multiplication by constant $1/2$) is bounded. Fix a positive integer m, say $m = 4m_0$ for $m_0 \in \mathbb{N}$. Then

$$k - m \text{ is } \begin{cases} \text{odd} & \text{if } k \text{ is odd} \\ \text{even} & \text{if } k \text{ is even} \end{cases} \quad \text{and} \quad j - 2m_0 \text{ is } \begin{cases} \text{odd} & \text{if } j \text{ is odd} \\ \text{even} & \text{if } j \text{ is even.} \end{cases}$$

We apply formula (41) with $b_k = 0$, if k is even, and with $b_k = (-1)^k/|2k+1|$ if k is odd, to obtain

$$T_{f_1} g_m(z) = \sum_{\substack{k=0,\\ k \text{ odd}}}^{m} b_{k-m} \frac{\Gamma_{2m}}{\Gamma_{2k}} \frac{z^k}{r_m^m v(r_m)} + \sum_{\substack{k=m+1,\\ k \text{ odd}}}^{\infty} b_{k-m} \frac{z^k}{r_m^m v(r_m)}. \tag{42}$$

Next, if S is the operator $Sf(z) = (f(z) - if(iz))/2$, we have

$$Sf(z) = \sum_{k=0}^{\infty} f_{4k+1} z^{4k+1} \quad \text{for} \quad f(z) = \sum_{k=0}^{\infty} f_{2k+1} z^{2k+1}, \tag{43}$$

since $1 - i \cdot i^{2k+1} = 1 + (-1)^k$. We obtain

$$ST_{f_1} g_m(z) = \sum_{0 \le 4j+1 \le m} b_{4j+1-m} \frac{\Gamma_{2m}}{\Gamma_{8j+2}} \frac{z^{4j+1}}{r_m^m v(r_m)} + \sum_{m+1 \le 4j+1 < \infty} b_{4j+1-m} \frac{z^{4j+1}}{r_m^m v(r_m)}.$$

Recall that $b_{4j+1-m} = 1/|4(j - m_0) + 1|$. So if we take $\theta = 0$ then all summands in the preceding sum are non-negative. Hence

$$\begin{aligned} \frac{r_m}{5} \log\left(\frac{1}{1 - r_m^4}\right) &= \frac{r_m}{5} \sum_{j=1}^{\infty} \frac{(r_m^4)^j}{j} \le \sum_{j=0}^{\infty} \frac{r_m^{4j+1}}{4j+1} \\ &= \sum_{m+1 \le 4j+1 < \infty} b_{4j+1-m} \frac{r_m^{4j+1} v(r_m)}{r_m^m v(r_m)} \le S(T_{f_1}(g_m))(r_m) v(r_m) \\ &\le \|S(T_{f_1}(g_m))\|_v \le \|T_{f_1}(g_m)\|_v. \end{aligned}$$

since trivially by the definition of the operator S we have $\sup_{|z|=r} |(Sf)(z)| \le \sup_{|z|=r} |f(z)|$. Since $\lim_{m \to \infty} r_m = 1$, the left-hand side of the preceding estimate grows to the infinity, when $m \to \infty$. Hence T_{f_1} and also T_f cannot be bounded.

5 General Result on Multipliers and Toeplitz Operators in $H_v^\infty(\mathbb{D})$ with Radial Symbols

We continue by considering a fixed radial weight v on $\mathbb{D}$ and Toeplitz operators $T_f : H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$, where $T_f = P_v M_f$. A function with radial symmetry on the disk can nearly never be harmonic, and the study of Toeplitz operators with radial symbols requires techniques different from those in Sect. 4. First we note that if $f \in L^1(\mathbb{D})$ is radial, i.e. $f(z) = f(|z|)$ for almost every $z \in \mathbb{D}$, then T_f is a coefficient multiplier. This is easily seen by expanding the kernel as in (32) and a calculation using the usual orthonormality relations of trigonometric polynomials,

$$\begin{aligned} T_f g(z) &= \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_{2n}} \int_0^1 \int_0^{2\pi} f(r) g(re^{i\theta}) r^{n+1} e^{-in\theta} v(r)\, d\theta dr \\ &= \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_{2n}} \int_0^1 f(r) r^{2n+1} v(r) g_n dr = \sum_{n=0}^{\infty} \gamma_n g_n z^n \end{aligned} \tag{44}$$

where $g = \sum_n g_n z^n$ and

$$\gamma_n = \frac{1}{\Gamma_{2n}} \int_0^1 r^{2n+1} v(r) f(r) dr. \tag{45}$$

We expose here the approach based mainly on the works [17], [19] and [20] dealing with the condition (B), below, which according to Theorem 1.1 of [19] characterizes those radial weights such that the space $H_v^\infty(\mathbb{D})$ is isomorphic to the Banach space ℓ^∞. Examples of weights satisfying (B) are all normal weights (38), in particular the standard weights, and the weights of exponential type $v(r) = \exp(-\gamma/(1-r)^\beta)$; see [19].

The very definition of condition (B) is somewhat technical and we cannot quite avoid other technical considerations in this survey either, however, one can follow our presentation without going into the depth of the arguments just by keeping in mind that condition (B) associates to the weight an increasing sequence of indices $(m_n)_{n=1}^\infty \subset (0,\infty)$ and radii $\left(r_{m_n}\right)_{n=1}^\infty \subset (0,1)$ such that $m_n \to \infty$ and $r_n \to 1$ as $n \to \infty$, and moreover, gives the very useful equivalent representation in Theorem 6 for the weighted sup-norm. We recall that the numbers $r_m \in]0,1[$ were defined in the beginning of Sect. 4.

Definition 2 The weight v satisfies the condition (B), if

$$\forall b_1 > 1\ \exists b_2 > 1\ \exists c > 0\ \forall m, n > 0$$
$$\left(\frac{r_m}{r_n}\right)^m \frac{v(r_m)}{v(r_n)} \le b_1 \text{ and } m, n, |m-n| \ge c \ \Rightarrow \left(\frac{r_n}{r_m}\right)^n \frac{v(r_n)}{v(r_m)} \le b_2.$$

Note that here m and n need not be integers. We now fix a number $b > 2$: it is shown in Lemma 5.1. of [19] that it is then possible to choose, by induction, an increasing, unbounded sequence $(m_n)_{n=1}^\infty \subset (0,\infty)$ such that

$$b = \min\left(\left(\frac{r_{m_n}}{r_{m_{n+1}}}\right)^{m_n} \frac{v(r_{m_n})}{v(r_{m_{n+1}})}, \left(\frac{r_{m_{n+1}}}{r_{m_n}}\right)^{m_{n+1}} \frac{v(r_{m_{n+1}})}{v(r_{m_n})}\right).$$

Next, for all $n \in \mathbb{N}$, for the given m_n, we define

$$w_{nk} = \begin{cases} \dfrac{|k| - [m_{n-1}]}{[m_n] - [m_{n-1}]}, & \text{if } m_{n-1} < |k| \le m_n, \text{ and} \\[2ex] \dfrac{[m_{n+1}] - |k|}{[m_{n+1}] - [m_n]} & \text{if } m_n < |k| \le m_{n+1}, \end{cases} \tag{46}$$

where $k \in \mathbb{Z}$ and $m_0 = 0$. Here $[r]$ is the largest integer not greater than r. With the help of these numbers we define the coefficient multipliers of de la Valleé Poisson type, acting on harmonic functions $f(z) = \sum_{k=-\infty}^{\infty} f_k r^{|k|} e^{ik\theta}$, by

$$W_n : \sum_{k=-\infty}^{\infty} f_k r^{|k|} e^{ik\theta} \mapsto \sum_{k=-\infty}^{\infty} w_{nk} f_k r^{|k|} e^{ik\theta}$$

We will need the following uniform boundedness property of the operators W_n, namely there exists a constant $C > 0$, depending on the weight only, such that

$$M_\infty(W_n g, r) \le C M_\infty(g, r) \tag{47}$$

for all $0 \le r \le 1$ and $g \in h_v^\infty(\mathbb{D})$. See (37) for the notation. The inequality (47) follows e.g. by combining an inequality in Theorem 1 of [20] with Lemma 3.3. of [19].

The operators W_n are important, since they decompose the space $H_v^\infty(\mathbb{D})$ into finite dimensional blocks with a useful representation for the norm. The result is from Theorem 1 of [20], see also Propositions 4.1. and 5.2. of [19].

Theorem 6 *Let v satisfy (B). Then there are constants $c_1, c_2 > 0$ such that, for all $g \in h_v^\infty(\mathbb{D})$,*

$$c_1 \sup_{n\in\mathbb{N}} M_\infty(W_n g, r_{m_n}) v(r_{m_n}) \le \|g\|_v \le c_2 \sup_{n\in\mathbb{N}} M_\infty(W_n g, r_{m_n}) v(r_{m_n}) \tag{48}$$

and

$$c_1 M_\infty(W_n g, r_{m_n}) v(r_{m_n}) \le \|W_n g\|_v \le c_2 M_\infty(W_n g, r_{m_n}) v(r_{m_n}) \tag{49}$$

for all $n \in \mathbb{N}$.

Moreover, it follows from Theorem 6 that if the numbers $f_k \in \mathbb{C}$, $k \in \mathbb{Z}$ satisfy

$$\sup_{n\in\mathbb{N}} \sup_{\theta\in[0,2\pi]} \left| \sum_{m_{n-1}<|k|\le m_{n+1}} w_{nk} f_k r_{m_n}^{|k|} e^{ik\theta} \right| v(r_{m_n}) < \infty, \tag{50}$$

then the series defining the harmonic function $f(re^{i\theta}) = \sum_{k=-\infty}^{\infty} f_k r^{|k|} e^{ik\theta}$ converges uniformly on compact subsets of $\mathbb{D}$ and f belongs to $h_v^\infty(\mathbb{D})$ and $\|g\|_v$ is bounded by a constant depending on the weight v. For this statement, see Remark 1, (iii) of [20].

Examples If v is normal then one can take $m_n = 2^{kn}$ for suitable fixed $k > 0$ (see [19, Example 2.4], and [17]). For $v(r) = \exp(-\alpha/(1-r)^\beta)$ one can take $m_n = \beta^2(\beta/\alpha)^{1/\beta} n^{2+2/\beta} - \beta^2 n^2$, see [2].

We now formulate one of the main results of this section, the characterization of boundedness and compactness for coefficient multipliers. The case of Toeplitz operators with radial symbols follows easily from this. The result was already proven for a more restricted class of weights in Theorem 4.1 of [18]. We will assume that a sequence $(\gamma_k)_{k=0}^{\infty}$ of complex numbers is given, and consider the formal series $f(\theta) = \sum_{k=0}^{\infty} \gamma_k e^{ik\theta}$, which may or may not converge. The formal series $W_n f$ is then naturally defined as

$$W_n f(\theta) = \sum_{k=0}^{\infty} w_{nk} \gamma_k e^{ik\theta}$$

where the numbers w_{nk} are as in (46). We denote by M_f the coefficient multiplier

$$M_f g(z) = \sum_{k=0}^{\infty} \gamma_k g_k r^k e^{ik\theta}, \quad z = re^{i\theta} \tag{51}$$

for harmonic functions $g(z) = \sum_{k=-\infty}^{\infty} g_k r^{|k|} e^{ik\theta}$. By definition, $M_f g$ is holomorphic, if the series (51) converges.

Theorem 7 *Let the weight v satisfy condition (B). Then M_f maps $h_v^{\infty}(\mathbb{D})$ into $H_v^{\infty}(\mathbb{D})$ and is bounded, if and only if*

$$\sup_{n \in \mathbb{N}} \int_0^{2\pi} |(W_n f)(\theta)| d\theta < \infty. \tag{52}$$

Moreover, assume (52) holds. Then $M_f : h_v^{\infty}(\mathbb{D}) \to H_v^{\infty}(\mathbb{D})$ is compact, if and only if

$$\int_0^{2\pi} |(W_n f)(\theta)| d\theta \to 0 \ \text{as } n \to \infty. \tag{53}$$

We present here the proof of the boundedness-statement, comment on the compact case only briefly and refer to [4] for the details. Let us first prove that (52) implies the boundedness of the operator. By (46), for every n there are only finitely many non-zero w_{nk}, hence, we can write $M_{W_n f}$, cf. (51), as a convolution

$$M_{W_n f} g(z) = \frac{1}{2\pi} \int_0^{2\pi} W_n f(\theta - \psi) g(re^{i\psi}) d\psi, \quad z = re^{i\theta} \in \mathbb{D}.$$

We obtain the estimate

$$\left| M_{W_n f} g(z) \right| v(r) \le \frac{1}{2\pi} \int_0^{2\pi} |(W_n f)(\theta)| d\theta \, \|g\|_v \tag{54}$$

for all $g \in h_v^\infty(\mathbb{D})$, Hence,

$$M_\infty(M_{W_n f} g, r)v(r) \le C\|g\|_v$$

for all n and r, where the constant $C > 0$ is the supremum on the left-hand side of (52). According to the remark concerning (50) the series on the right-hand side of (51) converges uniformly on compact subsets of $\mathbb{D}$, defines an element of $H_v^\infty(\mathbb{D})$ and is bounded by $\|g\|_v$. This means that M_f maps $h_v^\infty(\mathbb{D})$ continuously into $H_v^\infty(\mathbb{D})$.

As for the compactness of the operator M_f under the assumption (53), one takes a sequence $(g_j)_{j=1}^\infty$ contained in the closed unit ball of $h_v^\infty(\mathbb{D})$ and converging to 0 uniformly on compact subsets of $\mathbb{D}$. One needs to show that M_f maps such a sequence into a one converging to 0 with respect to the norm; see for example [26, Section 2.4]. Roughly speaking, one can improve the boundedness proof to get this, by using the assumption (53) together with the assumption on the convergence in the compact-open topology. One needs a more sophisticated use of Theorem 6.

As usual, the proof for the necessity of the condition (52) for the boundedness requires a careful enough choice of appropriate test functions. To this end we fix an arbitrary $0 < \varepsilon < 1$ as well as $n \in \mathbb{N}$ and $\varphi \in [0, 2\pi]$. Using the Fejer approximation theorem we find a trigonometric polynomial $g(z) = \sum_{k\in\mathbb{Z}} g_k r^{|k|} e^{ik\theta}$, depending on n, φ and ε, such that

$$\left| g(r_{m_n} e^{i\theta}) - \frac{\overline{W_n f(\varphi - \theta)}}{|W_n(\varphi - \theta)| v(r_{m_n})} \right| < \frac{\varepsilon}{v(r_{m_n})} \tag{55}$$

for all $\theta \in [0, 2\pi|$, in particular

$$M_\infty(g, r_{m_n}) v(r_{m_n}) \le 2. \tag{56}$$

As a consequence,

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} |(W_n f)(\theta)| d\theta &= \frac{1}{2\pi}\int_0^{2\pi} |(W_n f)(\varphi - \theta)| d\theta \\
\le \frac{1}{2\pi}&\left| \int_0^{2\pi} (W_n f)(\varphi - \theta) g(r_{m_n} e^{i\theta}) d\theta \right| v(r_{m_n}) + \varepsilon \\
= \frac{1}{2\pi}&\left| \int_0^{2\pi} f(\varphi - \theta)(W_n g)(r_{m_n} e^{i\theta}) d\theta \right| v(r_{m_n}) + \varepsilon \\
= |M_f W_n g(r_{m_n} e^{i\varphi})|\, &v(r_{m_n}) + \varepsilon \le \|M_f\| \cdot \|W_n g\|_v + \varepsilon.
\end{aligned} \tag{57}$$

Using Theorem 6 and (47), (56) we find a constant $C > 0$ such that

$$\|W_n g\|_v \le c_2 M_\infty(W_n g, r_{m_n}) v(r_{m_n}) \le c_2 d M_\infty(g, r_{m_n}) v(r_{m_n}) \le 2Cc_2.$$

Hence $\sup_n \int_0^{2\pi} |(W_n f)(\theta)| d\theta < \infty$.

The proof for the necessity of the condition (53) for the compactness of M_f needs a number of additional technical details. □

Since Riesz projection P, (39), is bounded by the assumptions of Theorem 7, it follows that the boundedness and compactness of $M_f : H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ are also equivalent to (52) and (53), respectively.

Let us turn back to Toeplitz operators. Let T_a be a Toeplitz operator on $H_v^\infty(\mathbb{D})$ with a given radial symbol $a \in L^1(\mathbb{D})$, i.e. $a(z) = a(|z|)$ for almost every z. Then, defining

$$\gamma_k = \frac{1}{\Gamma_{2k}} \int_0^1 r^{2k+1} v(r) a(r) dr, \; k \in \mathbb{N}_0 \quad \text{and} \quad f_a(\theta) = \sum_{k=0}^\infty \gamma_k e^{ik\theta}, \tag{58}$$

it was shown in (44)–(45) that T_a coincides with the Taylor multiplier with coefficients $(\gamma_k)_{k=0}^\infty$. The previous theorem thus yields the main result on the boundedness and compactness.

Theorem 8 *Let the weight satisfy* (B). *If* $a \in L^1$ *is radial then* T_a *is bounded as an operator* $H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ *if and only if*

$$\sup_n \int_0^{2\pi} |(W_n f_a)(\theta)| d\theta < \infty, \tag{59}$$

and T_a *is a compact operator* $H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$, *if and only if*

$$\lim_{n\to\infty} \int_0^{2\pi} |(W_n f_a)(\theta)| d\theta = 0. \tag{60}$$

We finally recall that Theorems 1.1 and 3.3 of the article [21] contain necessary and sufficient conditions for the boundedness and compactness of $T_a : A_v^p(\mathbb{D}) \to A_v^p(\mathbb{D})$ for $1 < p < \infty$, with minimal assumptions on the radial weights v. However, the characterization is in terms of the boundedness of coefficient multipliers in Hardy spaces, which is another open problem.

6 Supplementary Results on Toeplitz Operators with Radial Symbols

According to Theorem 5, the boundedness of the symbol does not suffice to imply the boundedness of the Toeplitz operator of $T_f : H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$. In this section we continue working with radial symbols and present results, where additional regularity or decay of the symbol at the boundary of the disk $\mathbb{D}$ implies the boundedness of T_a. The proofs are based on Theorem 8, although here we will only sketch some ideas of them.

In Theorem 8, the conditions for the boundedness and compactness of the Toeplitz operator may not be easy to verify for concrete weights and symbols, but the results of this section also serve the purpose of presenting some sufficient conditions that are quite easy to formulate and control. The setting for the spaces and symbols is the same as in the previous section, but in addition to condition (B) we also assume that, for some $\epsilon > 0$, v satisfies the following technical condition

$$\sup_{n\in\mathbb{N}} \frac{\int_0^1 r^{n-n^\epsilon} v(r)dr}{\int_0^1 r^n v(r)dr} < \infty. \tag{61}$$

It is not difficult to see that (61) holds for example for the important classes of standard, normal and exponential weights. For normal weights, condition (61) with $\epsilon = 1/2$ follows from Lemma 4.5. of [3]. In the case $v(r) = \exp(-1/(1-r))$ it is known that $\int_0^1 r^m v(r)dr$, $m > 1$, is proportional to the quantity $m^{-3/4}\exp(-Bm^{1/2})$ for some constant $B > 0$ independent of m (see e.g. Lemma 2.2. in [7] or Lemma 4.28 in [1]). Hence, assuming $\epsilon < 1/2$ we obtain

$$\int_0^1 r^{n-n^\epsilon} v(r)dr \le C(n-n^\epsilon)^{3/4}\exp(-B(n-n^\epsilon)^{1/2})$$

$$\le C'n^{3/4}\exp(-Bn^{1/2}+C'') \le C'''\int_0^1 r^n v(r)dr$$

for some positive constants C, C' etc., since

$$(n-n^\epsilon)^{1/2} = n^{1/2}(1-n^{\epsilon-1})^{1/2} = n^{1/2}\Big(1-\frac{1}{2}n^{\epsilon-1}+O(n^{2\epsilon-2})\Big)$$

$$= n^{1/2} - \frac{1}{2}n^{\epsilon-1/2} + O(n^{2\epsilon-3/2}) \ge n^{1/2} - C''$$

for all n. Thus, (61) holds. The same argument works for the more general weights $v(r) = \exp(-\alpha/(1-r)^\beta)$, $\alpha, \beta > 0$.

It was proven in [19] that normal and exponential weights satisfy (B).

Theorem 9 *Let v satisfy (B) and (61) and assume that the symbol $a \in L^1$ is real valued and radial. The operator T_a is a bounded operator $H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ in any of the following cases:*

(i) *The restriction $a|_{[\delta,1[}$ is differentiable (with respect to r) for some $\delta \in]0,1[$ and there holds*

$$\limsup_{r\to 1} a'(r) < \infty \qquad \text{or} \qquad \liminf_{r\to 1} a'(r) > -\infty, \tag{62}$$

and, in addition,

$$\limsup_{r\to 1} |a(r)\log(1-r)| < \infty \tag{63}$$

(ii) *The restriction $a|_{[\delta,1[}$ is differentiable for some $\delta \in]0,1[$, a' satisfies (62) and, for some constant $C > 0$, there holds the bound*

$$|a'(r)| \le \frac{C}{(1-r)\big(\log(1-r)\big)^2} \quad \text{for } r \in]\delta,1[. \tag{64}$$

(iii) *The symbol a is continuously differentiable on* $[0,1]$.

Theorem 9 holds also in the case of complex valued symbols a, namely, the assumptions need to be satisfied by both $\mathrm{Re}\, a$ and $\mathrm{Im}\, a$.

The symbol $a(r) = 1/(1-\log(1-r))$ satisfies the second condition (62) and, of course, (63) so that $T_a : H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ is bounded. The same is true for $a(r) = (1-r)^\delta$ with any $\delta > 0$. The latter symbol even induces a compact operator, as can be seen by the next result.

Theorem 10 *Let v satisfy (B) and (61) and assume that the symbol $a \in L^1$ is real valued and radial.*

(i) *If the restriction $a|_{[\delta,1[}$ is differentiable for some $\delta \in]0,1[$, satisfies (62) and, in addition,*

$$\limsup_{r\to 1} |a(r)\log(1-r)| = 0 \tag{65}$$

then the operator $T_a : H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ is compact.

(ii) *Assume that the restriction $a|_{[\delta,1[}$ is differentiable for some $\delta \in]0,1[$, satisfies (62), and there holds*

$$\lim_{r\to 1} |a'(r)|(1-r)\big(\log(1-r)\big)^2 = 0. \tag{66}$$

Then T_a is compact, if and only if $\lim_{r\to 1} a(r) = 0$.

Here, the case of complex valued symbols can be treated in the same way as in the previous theorem.

The item (i) in both Theorems 9 and 10 follows from Theorem 8. We do not present the proof but only refer to [5]. Recall that the coefficients of the series f_a in (59), (60) are given in (58), which involves integrals $\int_0^1 r^n a(r) v(r) dr$: the proofs of (i) of Theorems 9 and 10 are based on quite technical estimates and calculations with these expressions.

However, it is not so difficult to see that the sufficient condition (ii) essentially implies (i) in Theorem 9. Assume a is real-valued and that (64) holds. For all $r \in]\delta, 1[$ we get by the change of the integration variable $\log(1-s) =: x$ and $dx/ds = -1/(1-s)$ that

$$\int_r^1 |a'(s)| ds \leq C \int_r^1 \frac{1}{(1-s)\big(\log(1-s)\big)^2} ds = C \int_{-\infty}^{\log(1-r)} \frac{1}{x^2} dx = \frac{C}{|\log(1-r)|}. \tag{67}$$

This implies that we can extend a as a continuous function to $]\delta, 1]$ by setting

$$a(1) = \int_\delta^1 a'(s) ds + a(\delta) \ \big(= \lim_{r \to 1} a(r) \big).$$

Now, (67) yields for all $r \in]\delta, 1[$

$$|a(r) - a(1)| = \Big| \int_r^1 a'(s) ds \Big| \leq \frac{C}{|\log(1-r)|}, \tag{68}$$

which means that the function $a - a(1)$ satisfies (63). Note that the Toeplitz operator with the constant symbol $a(1)$ is bounded as it is just a constant multiplier.

It is plain that (iii) implies (ii) in Theorem 9.

Also, as regards to Theorem 10, the assumptions in (ii) imply those of (i). Namely, if (66) holds, then we can repeat the calculation (67)–(68) so that the constant C is replaced by a positive function $C(r)$ with $C(r) \to 0$ as $r \to 1$. Then, we see from the analogue of (68) that the function $a - a(1)$ even satisfies (65). If in addition $a(1) = 0$ then also a satisfies (65). Note that if $\lim_{r \to 1} a(r) = a(1) \neq 0$, then T_a is a compact perturbation of a non-zero multiple of the identity which is not compact, and thus it cannot be a compact operator.

In [5] it is shown that if v is a normal weight, the assumptions on a in the previous theorems can be relaxed, namely the boundedness of $T_a : H_v^\infty(\mathbb{D}) \to H_v^\infty(\mathbb{D})$ follows just from (63) and the compactness from (65) without any smoothness assumptions on the symbol. Also, in the case of exponential weights $v(r) = \exp\big(-\alpha/(1-r)^{\beta}\big)$, $\alpha, \beta > 0$, the smoothness requirements on a can be dropped, namely, if

$$\limsup_{r\to 1} |a(r)|(1-r)^{-1/2-\beta/4} < \infty, \tag{69}$$

then $T_a : H_v^{\infty}(\mathbb{D}) \to H_v^{\infty}(\mathbb{D})$ is bounded, and if

$$\limsup_{r\to 1} |a(r)|(1-r)^{-1/2-\beta/4} = 0, \tag{70}$$

then T_a is compact on $H_v^{\infty}(\mathbb{D})$.

Let us finally consider reflexive weighted Bergman spaces $A_v^p(\mathbb{D})$. For radial symbols, the boundedness of T_a as an operator from the Bergman-Hilbert space $A_v^2(\mathbb{D})$ into itself is characterized by the condition

$$\sup_{n\in\mathbb{N}} |\gamma_n| < \infty, \tag{71}$$

where the numbers γ_n are as in (45). The idea of trying to characterize the boundedness and compactness of $T_a : A_v^p(\mathbb{D}) \to A_v^p(\mathbb{D})$ for $2 < p < \infty$ (or $1 < p < 2$) by interpolating does not seem to work, but one can derive a sufficient condition similar to (52) for the boundedness of T_a in $A_v^p(\mathbb{D})$.

To formulate and sketch the proof of the result we need some modifications of the notions that were used in the case of weighted sup-norms. We again assume that the weight v satisfies condition (B). First, instead of the de la Valleé Poisson operators it is enough just to use the Dirichlet projections $Q_n g(z) = \sum_{k=0}^{n} g_k z^k$ for holomorphic $g(z) = \sum_{k=0}^{\infty} g_k z^k$. It is known that there are constants $c_p > 0$ with $M_p(Q_n g, r) \le c_p M_p(g, r)$ for all $0 < r < 1$, $1 < p < \infty$, where c_p does not depend on g, n or r and we write $M_p(g,r)^p = (2\pi)^{-1} \int_0^{2\pi} |g(re^{i\theta})|^p d\theta$.

Analogously with the case of weighted sup-norms one picks up suitable increasing numerical sequences $(\ell_n)_{n=1}^{\infty}$ with $\ell_1 = 0$ and $\lim_{n\to\infty} \ell_n = \infty$ and $(s_n)_{n=1}^{\infty} \subset (0,1)$ with $\lim_{n\to\infty} s_n = 1$ and then defines the operators

$$Z_n = Q_{[\ell_{n+1}]} - Q_{[\ell_n]}, \quad n \in \mathbb{N}.$$

These are used to derive an equivalent form of the weighted L^p-norm: for some constants $c_2 \ge c_1 > 0$, for every $f \in A_v^p(\mathbb{D})$, there holds

$$c_1 \|f\|_{p,v} \le \Big(\sum_{n=1}^{\infty} \omega_n^p M_p^p (Z_n f, s_n) \Big)^{1/p} \le c_2 \|f\|_{p,v}, \tag{72}$$

where the numbers ω_n are determined by the weight. The details of the definitions of the various parameters and proof of (72) can be found in [13] for $p = 1$ and in [20] for $1 < p < \infty$. Examples and calculations in concrete cases can be found in the paper [3]: there it is shown that one can obtain (72) for the exponential weights $v(r) = \exp\big(-\alpha/(1-r)^\beta\big)$, $\alpha, \beta > 0$ by using

$$\ell_n = \beta^{1+1/\beta}\alpha^{-1/\beta}n^{2+2/\beta} - \beta n^2, \quad s_n = 1 - \Big(\frac{\alpha}{\beta}\Big)^{1/\beta}\frac{1}{n^{2/\beta}}. \tag{73}$$

Proposition 2 *Let the weight satisfy* (B)*, let* $a \in L^1$ *be a radial function and let* $f_a(\theta) = \sum_{k=0}^{\infty}\gamma_k e^{ik\theta}$ *be as in* (45)*. Then the Toeplitz operator* $T_a : A_v^p(\mathbb{D}) \to A_v^p(\mathbb{D})$ *is a well-defined, bounded operator, if*

$$\sup_{n\in\mathbb{N}}\int_0^{2\pi}|(Z_n f_a)(\theta)|d\theta < \infty, \tag{74}$$

and $T_a : A_v^p(\mathbb{D}) \to A_v^p(\mathbb{D})$ *is compact, if*

$$\int_0^{2\pi}|(Z_n f_a)(\theta)|d\theta \to 0 \ \ \text{as } n \to \infty. \tag{75}$$

Proof Let M_f be the convolution operator, or the sequence space multiplier, corresponding to T_a, see (45). For all $g \in A_v^p(\mathbb{D})$ and $z = re^{i\theta} \in \mathbb{D}$ we get

$$(Z_n M_f g)(z) = (M_{Z_n f} g)(z) = \int_0^{2\pi} Z_n f(\theta - \psi) Z_n g(re^{id\psi})d\psi,$$

where we replaced g by $Z_n g$ by the usual orthogonality relations of trigonometric monomials. The Young inequality $\|a * b\|_{L^p(\partial\mathbb{D})} \le \|a\|_{L^1(\partial\mathbb{D})}\|b\|_{L^p(\partial\mathbb{D})}$ yields

$$M_p(Z_n M_f g, r) \le \int_0^{2\pi}|(Z_n f)(\theta)|d\theta\, M_p(Z_n g, r) \tag{76}$$

The inequality $\|M_f g\|_{p,v} \le C\|g\|_{p,v}$ thus follows by applying (74) and (72) to both $\|M_f g\|_{p,v}$ and $\|g\|_{p,v}$, and this implies the boundedness of T_a.

Assume next (75) holds, and let $(g_j)_{j=1}^{\infty}$ be a sequence which is contained in the unit ball of $A_v^p(\mathbb{D})$ and which converges to 0 uniformly on compact subsets of $\mathbb{D}$, and assume $\varepsilon > 0$ is given. We choose $N \in \mathbb{N}$ such that $\int_0^{2\pi}|(Z_n f)(\theta)|d\theta < \varepsilon$. The convergence of the sequence in the compact-open topology can be used to find a large enough $J \in \mathbb{N}$ such that

$$\sup_{|z|\le r_{m_n}}|Z_n M_f g_j(z)|v(z) < \frac{\varepsilon}{2\pi N\omega_n} \ \Rightarrow\ M_p(Z_n M_f g_j, r_{m_n}) < \frac{\varepsilon}{N\omega_n}$$

for all $n \le N$, all $j \ge J$. This, (76) and (72) imply

$$\|M_f g_j\|_{p,v}^p \le \sum_{n=1}^{N} \omega_n^p M_p(Z_n M_f g_j, r_{m_n})^p + \sum_{n=N+1}^{\infty} \omega_n^p M_p(Z_n M_f g_j, r_{m_n})^p$$

$$\le \varepsilon + \varepsilon \sum_{n=N+1}^{\infty} \omega_n^p M_p(Z_n g_j, r_{m_n})^p \le 2\varepsilon \|g_j\|_{p,v}^p \le 2\varepsilon.$$

We infer that the sequence $(g_j)_{j=1}^{\infty}$ converges to 0 in the norm of $A_v^p(\mathbb{D})$, which proves the compactness of the operator.

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Part IV
Inequalities in Various Banach Spaces

Disjointness Preservers and Banach-Stone Theorems

Denny H. Leung and Wee Kee Tang

Abstract Let Ω be a compact Hausdorff space. The space $C(\Omega)$ of continuous functions on Ω carries a number of structures. It is a Banach space (under the sup-norm), a vector lattice and a ring (under pointwise operations). The classical theorems of Banach-Stone, Kaplansky and Gelfand-Kolmogorov show that each of these structures on $C(\Omega)$ characterizes the space Ω up to homeomorphism. Within the last 30 years or so, a rich literature has been built up concerning mappings between function spaces that preserve the disjointness structure (biseparating maps or $\perp$-isomorphisms). These efforts have shown that in many cases, operators on function spaces that preserve various kinds of structures are $\perp$-isomorphisms. This lends a certain unity to various "preserver" results and highlights the utility of the concept of $\perp$-isomorphisms. In this chapter, we will describe a general theory of $\perp$-isomorphisms and survey a number of applications, including applications to order (lattice) isomorphisms, ring and multiplicative isomorphisms, isometries and nonvanishing preservers.

Keywords Banach-Stone theorems · Biseparating maps · Algebra isomorphisms · Order isomorphisms · Continuous functions · Uniformly continuous functions · Lipschitz functions · Differentiable functions

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1 Introduction

There is a long and fruitful tradition of studying a mathematical object by means of looking at the space of mappings from it into a simple object of the same sort. For example, the dual group is a fundamental object in abstract harmonic analysis; likewise, the dual space of a locally convex topological vector space is part and parcel of the theory of such spaces. If Ω is a compact Hausdorff space, the space $C(\Omega)$ of continuous real valued functions on Ω is a natural "dual space" of Ω. (We will generally take the scalar field to be $\mathbb{R}$, although most of what will be discussed in the paper applies equally well to complex scalars.) Moreover, $C(\Omega)$ carries with it a wealth of structures. It is a Banach space under the norm $\|f\| = \sup_{\omega\in\Omega} |f(\omega)|$, a ring (with unit) under pointwise addition and multiplication and a vector lattice under pointwise order. Each of these aspects of $C(\Omega)$ has been shown to determine the space Ω up to homeomorphism. These are the famous classical theorems of Banach-Stone [9, 38], Gelfand-Kolmogorov [24] and Kaplansky [28]. The aim of this paper is to give a survey of some developments that arise out of these classical results, which we will refer to as theorems of Banach-Stone type. Particularly, since the 1990s, mappings that preserve "disjointness structures"—biseparating maps in the linear case, $\perp$-isomorphisms more generally—have been studied by many researchers. An important point that we would like to make is to promote the use of $\perp$-isomorphisms as a unifying concept in the study of Banach-Stone type theorems. A recent example of such a point of view is given in the paper [15]. For a general survey of Banach-Stone theorems up to around the year 2000, see [22].

Let us briefly summarize the contents of the paper. In Sect. 2, we recall the statements of the three classical theorems mentioned above. The definition of a biseparating map is given and it is shown that if $T : C(\Omega) \to C(\Sigma)$ is either an isometry, an algebra (ring) isomorphism or a vector lattice isomorphism, then it is biseparating. A detailed proof is given of the fact that a biseparating map $T : C(\Omega) \to C(\Sigma)$ induces a homeomorphism $\varphi : \Sigma \to \Omega$, with respect to which T can be represented as a weighted composition operator. (See Theorem 2.6.) Consequently, the three classical theorems can be unified under Theorem 2.6. Section 3 develops the theory of $\perp$-isomorphisms. Minimal assumptions are made on the sets of functions and the mappings involved. Even so, it is found that a $\perp$-isomorphism induces an isomorphism between the Boolean algebras of regular open sets between the underlying domain spaces (Theorem 3.3). Under further conditions, it is shown that the Boolean isomorphism gives rise to a homeomorphism between the domain spaces. These can be viewed as "weak" Banach-Stone theorems. In Sect. 3.3, "strong" Banach-Stone theorems are given, that is, results where a $\perp$-isomorphism has a functional representation. Strong Banach-Stone theorems are seen to apply to a large class of function spaces. Finally, Sect. 4 contains applications of the results in Sect. 3 to a variety of settings. It is shown that in many cases lattice isomorphisms (Kaplansky's Theorem), ring isomorphisms (Gelfand-Kolmogorov Theorem), multiplicative isomorphisms (Milgram's Theorem), isometries (Banach-

Stone Theorem) and nonvanishing preservers are $\perp$-isomorphisms. Consequently, many results are consequences of, and can be extended by, characterization of $\perp$-isomorphisms.

2 Three Classical Theorems

Let Ω be a topological space. Denote by $C(\Omega)$ the vector space of all (real-valued) continuous functions on Ω. The space $C(\Omega)$ carries with it many structures. Indeed, it is an algebra under pointwise addition and multiplication. It is also a vector lattice under pointwise supremum and infimum. Finally, if Ω is compact Hausdorff, the space $C(\Omega)$ is a Banach space with the norm $\|f\| = \sup\{|f(\omega)| : \omega \in \Omega\}$. In the first half of the twentieth century, three remarkable theorems appeared that characterize the space Ω in terms of each of these structures of $C(\Omega)$.

Theorem 2.1 (Banach-Stone) *Let Ω, Σ be compact Hausdorff spaces and let $T : C(\Omega) \to C(\Sigma)$ be a linear isometry. There are a homeomorphism $\varphi : \Sigma \to \Omega$ and a function $h \in C(\Sigma)$ so that $|h(\sigma)| = 1$ for all $\sigma \in \Sigma$ and that*

$$Tf(\sigma) = h(\sigma) f(\varphi(\sigma)) \textit{ for all } f \in C(\Omega) \textit{ and all } \sigma \in \Sigma.$$

Theorem 2.1 was proved by Banach [9] for the case of compact metric spaces. The theorem was extended to compact Hausdorff spaces by Stone [38].

Theorem 2.2 (Gelfand-Kolmogorov [24]) *Let Ω, Σ be compact Hausdorff spaces and let $T : C(\Omega) \to C(\Sigma)$ be an algebra isomorphism. There is a homeomorphism $\varphi : \Sigma \to \Omega$ such that*

$$Tf(\sigma) = f(\varphi(\sigma)) \textit{ for all } f \in C(\Omega) \textit{ and all } \sigma \in \Sigma.$$

Theorem 2.3 (Kaplansky [28]) *Let Ω, Σ be compact Hausdorff spaces and let $T : C(\Omega) \to C(\Sigma)$ be a vector lattice isomorphism. There are a homeomorphism $\varphi : \Sigma \to \Omega$ and a function $h \in C(\Sigma)$ so that $h(\sigma) > 0$ for all $\sigma \in \Sigma$ and that*

$$Tf(\sigma) = h(\sigma) f(\varphi(\sigma)) \textit{ for all } f \in C(\Omega) \textit{ and all } \sigma \in \Sigma.$$

Theorem 2.3 is a special case of Kaplansky's result. For a discussion of the result in its full generality, see Sect. 4.1. In the intervening three quarters of a century, a large number of extensions and generalizations of these results have been obtained. A particularly fruitful concept that unifies the three classical theorems is that of disjointness preserving operators. Let Ω, Σ be topological spaces. Two functions $f, g \in C(\Omega)$, respectively, $C(\Sigma)$, are said to be *disjoint* if the pointwise product $fg = 0$. In terms of the lattice structure, f and g are disjoint if and only if $|f| \wedge |g| = 0$. Suppose that $A(\Omega)$ and $A(\Sigma)$ are vector subspaces of $C(\Omega)$ and $C(\Sigma)$ respectively. A linear operator $T : A(\Omega) \to A(\Sigma)$ is *disjointness preserving* if Tf, Tg are disjoint whenever f, g are disjoint functions in $A(\Omega)$. A *biseparating* operator is a linear bijection $T : A(\Omega) \to A(\Sigma)$ so that both T and T^{-1} are disjointness preserving. It is evident that if $A(\Omega)$ and $A(\Sigma)$ are algebras under pointwise operations, then every algebraic isomorphism $T : A(\Omega) \to A(\Sigma)$ is biseparating. A similar statement holds for lattice isomorphisms. Now we proceed to see that for compact Hausdorff spaces Ω and Σ, any linear isometry from $C(\Omega)$ onto $C(\Sigma)$ is biseparating. To do this, we make use of extreme points in the dual ball of $C(\Omega)$ and $C(\Sigma)$. Let C be a convex set in a vector space V. A point $x \in C$ is an *extreme point* of C if $x = \frac{1}{2}(y+z)$, $y, z \in C$, implies that $x = y = z$. Denote the set of extreme points of C by $\operatorname{ext} C$. It is easy to see that if V, W are vector spaces, $T : V \to W$ is a vector space isomorphism and x is an extreme point of $C \subseteq V$, then Tx is an extreme point of $T(C)$. The next result is due to Arens and Kelley, who used it in their proof of the Banach-Stone Theorem.

Proposition 2.4 ([7]) *Let Ω be a compact Hausdorff space and let $B_{C(\Omega)^*}$ be the closed ball of the dual space $C(\Omega)^*$. Then*

$$\operatorname{ext} B_{C(\Omega)^*} = \{\pm\delta_\omega : \omega \in \Omega\},$$

where δ_ω is the evaluation functional on $C(\Omega)$ given by $\delta_\omega(f) = f(\omega)$.

Proposition 2.5 *Let Ω and Σ be compact Hausdorff spaces. Every (onto) linear isometry $T : C(\Omega) \to C(\Sigma)$ is biseparating.*

Proof Let f, g be disjoint functions in $C(\Omega)$ and let $\sigma \in \Sigma$. By Proposition 2.4, $\delta_\sigma \in \operatorname{ext} B_{C(\Sigma)^*}$. Since T^* is a vector space isomorphism and $T^*(B_{C(\Sigma)^*}) = B_{C(\Omega)^*}$, $T^*\delta_\sigma \in \operatorname{ext} B_{C(\Omega)^*}$. By Proposition 2.4, there exists $\omega \in \Omega$ and $\varepsilon = \pm 1$ so that $T^*\delta_\sigma = \varepsilon\delta_\omega$. Thus

$$Tf(\sigma) \cdot Tg(\sigma) = (T^*\delta_\sigma)(f) \cdot (T^*\delta_\sigma)(g) = \varepsilon^2\delta_\omega(f) \cdot \delta_\omega(g) = f(\omega) \cdot g(\omega) = 0.$$

This proves that Tf and Tg are disjoint. Hence T is disjointness preserving. The same applies to T^{-1} by symmetry. □

The classical theorems of Banach-Stone, Gelfand-Kolmogorov and Kaplansky can now be unified and extended by the next result.

Theorem 2.6 ([27]) *Let Ω and Σ be compact Hausdorff spaces and let $T : C(\Omega) \to C(\Sigma)$ be a linear biseparating map. There are a homeomorphism $\varphi : \Sigma \to \Omega$ and a function $h \in C(\Sigma)$ so that $h(\sigma) \neq 0$ for all $\sigma \in \Sigma$ and that*

$$Tf(\sigma) = h(\sigma) f(\varphi(\sigma)) \text{ for all } f \in C(\Omega) \text{ and all } \sigma \in \Sigma.$$

In fact, Jarosz gave a description of general disjointness preserving linear maps $T : C(\Omega) \to C(\Sigma)$. As a result, he showed that every disjointness preserving linear bijection $T : C(\Omega) \to C(\Sigma)$ is biseparating and has the representation above. We will give a detailed proof of Theorem 2.6 that seems to us to be most amenable to generalization. For the remainder of the section, let Ω, Σ and T be as in Theorem 2.6. For any function $f \in C(\Omega)$, the *support* of f, supp f, is the closure of the set $\{\omega : f(\omega) \neq 0\}$. Similarly for functions in $C(\Sigma)$.

Proposition 2.7 ([5, Lemma 4]) *If* $f, g \in C(\Omega)$ *and* $\operatorname{supp} f \subseteq \operatorname{supp} g$*, then* $\operatorname{supp} Tf \subseteq \operatorname{supp} Tg$.

Proof Otherwise, there are $f, g \in C(\Omega)$ with $\operatorname{supp} f \subseteq \operatorname{supp} g$, yet $\operatorname{supp} Tf \not\subseteq \operatorname{supp} Tg$. Hence there exists $\sigma_0 \notin \operatorname{supp} Tg$ so that $Tf(\sigma_0) \neq 0$. Choose $h \in C(\Sigma)$ so that $h(\sigma_0) \neq 0$ and that h is disjoint from Tg. Since T^{-1} is disjointness preserving, $T^{-1}h$ and g are disjoint. Thus $T^{-1}h$ and f are disjoint. Therefore, h and Tf are disjoint, which contradicts the fact that h and Tf are both nonzero at σ_0. □

For any $\sigma \in \Sigma$, set

$$\mathcal{F}_\sigma = \{\operatorname{supp} f : f \in C(\Omega), (Tf)(\sigma) \neq 0\}.$$

Lemma 2.8 $\mathcal{F}_\sigma$ *has the finite intersection property.*

Proof Let $f_1, \ldots, f_m$ be functions in $C(\Omega)$ so that $Tf_i(\sigma) \neq 0$, $1 \leq i \leq m$. There exists a nonzero $g \in C(\Sigma)$ so that $\operatorname{supp} g \subseteq \operatorname{supp} Tf_i$, $1 \leq i \leq m$. Apply Proposition 2.7 to T^{-1} to see that $\operatorname{supp} T^{-1}g \subseteq \operatorname{supp} f_i$, $1 \leq i \leq m$. Since T is a bijection and $g \neq 0$, $T^{-1}g \neq 0$. Thus $\operatorname{supp} T^{-1}g$ is a nonempty set contained in $\bigcap_{i=1}^m \operatorname{supp} f_i$. □

Lemma 2.9 *For any* $\sigma \in \Sigma$, $\bigcap \mathcal{F}_\sigma$ *contains exactly one point in* Ω.

Proof Obviously, $\mathcal{F}_\sigma$ consists of closed sets in the compact Hausdorff space Ω. It follows from Lemma 2.8 that $\bigcap \mathcal{F}_\sigma$ is nonempty. Suppose, if possible, that ω_1, ω_2 are two distinct points in $\bigcap \mathcal{F}_\sigma$. Choose a pair of disjoint functions $h_1, h_2 \in C(\Omega)$ so that $h_i = 1$ on a neighborhood of ω_i, $i = 1, 2$. Let $f \in C(\Omega)$ be chosen so that $Tf(\sigma) \neq 0$. By definition, $\omega_i \in \operatorname{supp} f$, $i = 1, 2$. Since $h_1 f$ and $h_2 f$ are disjoint and T is disjointness preserving, $T(h_1 f)$ and $T(h_2 f)$ are disjoint. Without loss of generality, we may assume that $T(h_1 f)(\sigma) = 0$. Then $T((1 - h_1)f)(\sigma) = Tf(\sigma) \neq 0$. Since $\omega_1 \in \mathcal{F}_\sigma$, this would imply that $\omega_1 \in \operatorname{supp}(1 - h_1)f$, which is clearly false by choice of h_1. This completes the proof of the lemma. □

Define $\varphi : \Sigma \to \Omega$ by setting $\{\varphi(\sigma)\} = \bigcap \mathcal{F}_\sigma$. By symmetry, we may define

$$\mathcal{F}_\omega = \{\operatorname{supp} Tf : f \in C(\Omega), f(\omega) \neq 0\}$$

for any $\omega \in \Omega$. Then there is a well-defined function $\psi : \Omega \to \Sigma$ so that $\{\psi(\omega)\} = \bigcap \mathcal{F}_\omega$ for all $\omega \in \Omega$.

Lemma 2.10 $\varphi : \Sigma \to \Omega$ *is a homeomorphism with inverse* ψ.

Proof We will show that $\psi(\varphi(\sigma)) = \sigma$ for all $\sigma \in \Sigma$ and that φ is continuous. The lemma then follows by symmetry.

Suppose that $\sigma \in \Sigma$ and $\omega = \varphi(\sigma)$. Assume, if possible, that $\sigma' = \psi(\omega) \neq \sigma$. Let $f \in C(\Omega)$ be such that $Tf(\sigma) \neq 0$ and that $\sigma' \notin \operatorname{supp} Tf$. There exists $h \in C(\Sigma)$ disjoint from Tf so that $h = 1$ on a neighborhood of σ'. Choose $g \in C(\Omega)$ so that $g(\omega) \neq 0$. Since $h \cdot Tg$ and Tf are disjoint, so are $T^{-1}(h \cdot Tg)$ and f. As $Tf(\sigma) \neq 0$, $\omega \in \operatorname{supp} f$. Hence $T^{-1}(h \cdot Tg)(\omega) = 0$. Therefore,

$$0 \neq g(\omega) = T^{-1}(h \cdot Tg)(\omega) + T^{-1}((1-h) \cdot Tg)(\omega) = T^{-1}((1-h) \cdot Tg)(\omega).$$

It follows that $\sigma' \in \operatorname{supp}(1-h) \cdot Tg$, contrary to the choice of h. This completes the proof that $\psi(\varphi(\sigma)) = \sigma$.

If φ is not continuous, then making use of compactness of Ω, there is a net $(\sigma_\alpha)_\alpha$ in Σ converging to some σ_0 so that $(\varphi(\sigma_\alpha))_\alpha$ converges to $\omega' \neq \omega_0 := \varphi(\sigma_0)$. Let $f \in C(\Omega)$ be such that $Tf(\sigma_0) \neq 0$. There exists α_0 so that $Tf(\sigma_\alpha) \neq 0$ for all $\alpha \succeq \alpha_0$. By definition of φ, $\varphi(\sigma_\alpha) \in \operatorname{supp} f$ for all $\alpha \succeq \alpha_0$. Thus $\omega' \in \operatorname{supp} f$. But this shows that $\omega' \in \bigcap \mathcal{F}_{\sigma_0}$ and hence $\omega' = \omega$, contrary to the assumption. □

Proof of Theorem 2.6 We will show that if $f(\varphi(\sigma)) = 0$, then $Tf(\sigma) = 0$. Once this is shown, define $h = T1$. For any $f \in C(\Omega)$ and any $\sigma \in \Sigma$, $f - f(\varphi(\sigma))1$ vanishes at $\varphi(\sigma)$. Hence

$$0 = T[f - f(\varphi(\sigma))1](\sigma) = Tf(\sigma) - f(\varphi(\sigma))h(\sigma).$$

Thus $Tf(\sigma) = h(\sigma)f(\varphi(\sigma))$, as claimed. Furthermore, since T is a surjection, $h(\sigma) \neq 0$ for any $\sigma \in \Sigma$.

Suppose that, contrary to the claim above, there are $f \in C(\Omega)$ and $\sigma \in \Sigma$ so that $f(\varphi(\sigma)) = 0$ yet $Tf(\sigma) \neq 0$. By definition of φ, $\omega := \varphi(\sigma) \in \operatorname{supp} f$. Thus $\omega \in \overline{|f|^{-1}(0, r)}$ for any $r > 0$. Define

$$U_n = |f|^{-1}\left(\frac{1}{(3n+5)^2}, \frac{1}{(3n+1)^2}\right), \; n \in \mathbb{N}.$$

Then $\omega \in \overline{\bigcup_n U_n} = \overline{\bigcup_n U_{2n-1}} \cup \overline{\bigcup_n U_{2n}}$. Without loss of generality, assume that $\omega \in \overline{\bigcup_n U_{2n-1}}$. Set

$$V_n = |f|^{-1}\left(\frac{1}{(6n+3)^2}, \frac{1}{(6n-3)^2}\right), \; n \in \mathbb{N}.$$

Then (V_n) is a sequence of disjoint open sets so that $\overline{U_{2n-1}} \subseteq V_n$ for all n. For each n, choose a function $h_n \in C(\Omega)$ so that $0 \leq h_n \leq 1$, $h_n = 1$ on U_{2n-1} and $h_n = 0$ outside V_n. The sequence of functions $(nh_n f)_n$ is pairwise disjoint and $\|nh_n f\| \leq \frac{n}{(6n-3)^2} \to 0$. Hence the sum $g := \sum nh_n f$ converges in $C(\Omega)$. For each n, $g - nf = 0$ on the set U_{2n-1}. By definition of φ, this implies that $T(g - nf)(\sigma') = 0$ for all $\sigma' \in \varphi^{-1}(U_{2n-1})$. Choose a net (ω_α) in $\bigcup_n U_{2n-1}$ that converges to ω. Let $n_\alpha \in \mathbb{N}$ be such that $\omega_\alpha \in U_{2n_\alpha - 1}$. Set $\sigma_\alpha = \varphi^{-1}(\omega_\alpha)$. Since

$f(\omega) = 0$, $\omega \notin \overline{U_n}$ for any n. Thus $\lim_\alpha n_\alpha = \infty$. Note that (σ_α) converges to σ. Therefore,

$$Tf(\sigma) = \lim_\alpha Tf(\sigma_\alpha) = \lim_\alpha \frac{1}{n_\alpha} Tg(\sigma_\alpha) = 0,$$

contrary to the choices of f and σ. □

3 Isomorphism of Disjointness Structure

Results in Sect. 2 may serve to convince the reader that biseparating maps are worthy of study in their own right. Indeed, plenty of results concerning biseparating maps have been obtained in the past 30 years or so. Most of these are in the context of linear or at least additive maps. Since surjective additive maps between vector spaces are linear maps over the field of rational numbers, the results remain mainly "linear" in character. Very recently, several papers [15, 17, 18] appeared that took the study of isomorphisms of disjointness structure, or $\perp$-isomorphisms, to very general settings. It is shown that even for function spaces with minimal structure, analysis of $\perp$-isomorphisms can still bear fruitful results. The aim of this section is to describe this general approach to $\perp$-isomorphisms. Earlier results on biseparating maps, in spaces of (vector-valued) continuous functions, uniformly continuous functions, Lipschitz functions and differentiable functions, will be seen as consequences. Applications to theorems of Banach-Stone type will be considered in the next section.

3.1 $\perp$-Isomorphisms

Let Ω, X be Hausdorff topological spaces and let $A(\Omega, X)$ be a subset of $C(\Omega, X)$, the set of continuous functions $f : \Omega \to X$. For $f, g \in A(\Omega, X)$, let

$$[f \neq g] = \{\omega \in \Omega : f(\omega) \neq g(\omega)\},\ \operatorname{supp}_g f = \overline{[f \neq g]} \text{ and } \sigma_g(f) = \operatorname{int} \operatorname{supp}_g f.$$

Following [15], we define the following relations for $f, g, h \in A(\Omega, X)$.

1. $f \perp_h g$: $[f \neq h] \cap [g \neq h] = \emptyset$.
2. $f \subseteq_h g$: $\sigma_h(f) \subseteq \sigma_h(g)$.

The definitions of $\perp_h$ and $\subseteq_h$ may appear asymmetrical as one uses sets of the form $[f \neq h]$ while the other uses $\sigma_h(f)$. However, it is easy to see that $f \perp_h g$ if and only if $\sigma_h(f) \cap \sigma_h(g) = \emptyset$. Similarly, let $A(\Sigma, Y)$ be a subset of $C(\Sigma, Y)$, where Σ, Y are Hausdorff topological spaces. Assume that $T : A(\Omega, X) \to A(\Sigma, Y)$ is a bijection. Given $h \in A(\Omega, X)$, say that T is a $\perp_h$-*isomorphism* if

$$f \perp_h g \iff Tf \perp_{Th} Tg \text{ for all } f, g \in A(\Omega, X).$$

$\subseteq_h$-*isomorphism* is defined similarly. Clearly, a biseparating map in the sense of §2 is precisely a $\perp_0$ isomorphism, provided $T0 = 0$. $\perp_h$-isomorphism is a generalization of biseparating map to the nonlinear context. The set $A(\Omega, X)$ is said to be h-*weakly regular* for some $h \in A(\Omega, X)$ if

$$\Sigma_h = \{\sigma_h(f) : f \in A(\Omega, X)\} \text{ is a basis for the topology on } X.$$

Weak regularity is a basic assumption to ensure that there are sufficient functions in $A(\Omega, X)$ and $A(\Sigma, Y)$ to yield a nontrivial theory. The following simple yet important result is noted and used in [15]. Its ancestry can be traced back to at least [5, Lemma 4].

Proposition 3.1 *Let $T : A(\Omega, X) \to A(\Sigma, Y)$ be a bijection, where $A(\Omega, X)$ and $A(\Sigma, Y)$ are h- and Th-weakly regular respectively. Then T is a $\perp_h$-isomorphism if and only if it is a $\subseteq_h$-isomorphism.*

A set U in Ω is a *regular open set* if $U = \operatorname{int} \overline{U}$. All sets of the form $\sigma_h(f)$ are regular open sets. Denote the collection of all regular open sets in Ω by $\mathrm{RO}(\Omega)$. $\mathrm{RO}(\Omega)$ is a Boolean algebra with $0 = \emptyset$, $1 = \Omega$, lattice operations $U \wedge V = U \cap V$, $U \vee V = \operatorname{int} \overline{U \cup V}$ and negation $\neg U = \operatorname{int}(\Omega \backslash U)$. See [38]. If Ω is a regular topological space, then $\mathrm{RO}(\Omega)$ is a basis for the topology on Ω.

Let $T : A(\Omega, X) \to A(\Sigma, Y)$ be a $\subseteq_h$-isomorphism, where $A(\Omega, X)$ and $A(\Sigma, Y)$ are h- and Th-weakly regular respectively. Define a map $\theta_h : \mathrm{RO}(\Omega) \to \mathrm{RO}(\Sigma)$ by

$$\theta_h(U) = \operatorname{int} \overline{\bigcup \{\sigma_{Th}(Tf) : f \in A(\Omega, X), \sigma_h(f) \subseteq U\}}. \tag{1}$$

It can be shown that θ_h is a Boolean isomorphism from $\mathrm{RO}(\Omega)$ onto $\mathrm{RO}(\Sigma)$. In fact, its inverse is $\theta_{Th} : \mathrm{RO}(\Sigma) \to \mathrm{RO}(\Omega)$. Furthermore, if $f \in A(\Omega, X)$ and $U \in \mathrm{RO}(\Omega)$, then $f = h$ on U if and only if $Tf = Th$ on $\theta_h(U)$. In fact, we obtain a fundamental characterization of $\perp_h$-isomorphisms.

Theorem 3.2 *Let $T : A(\Omega, X) \to A(\Sigma, Y)$ be a bijection, where $A(\Omega, X)$ and $A(\Sigma, Y)$ are h- and Th-weakly regular respectively. Then T is a $\perp_h$-isomorphism if and only if there is a Boolean isomorphism $\theta_h : \mathrm{RO}(\Omega) \to \mathrm{RO}(\Sigma)$ so that for any $f \in A(\Omega, X)$ and $U \in \mathrm{RO}(X)$, $f = h$ on U if and only if $Tf = Th$ on $\theta_h(U)$.*

In Theorem 3.2, we say that θ_h is *associated with* (T, h). In general, if T is a $\perp_h$ isomorphism for different h's, the associated Boolean isomorphisms θ_h may well depend on h. Some way of "linking" different functions in $A(\Omega, X)$ and $A(\Sigma, Y)$ is needed in order to "uniformize" the θ_h's.

Call a set $A(\Omega, X) \subseteq C(\Omega, X)$ *weakly regular* if $A(\Omega, X)$ is h-weakly regular for all $h \in A(\Omega, X)$. Suppose that $T : A(\Omega, X) \to A(\Sigma, Y)$ is a bijection between weakly regular sets of functions that is a $\perp$*-isomorphism*, i.e., T is a $\perp_h$-isomorphism for all $h \in A(\Omega, X)$. Consider the following "linking" condition.

(L) If $h_1, h_2 \in A(\Omega, X)$, $U \in \mathrm{RO}(\Omega)$ and $\omega \notin \overline{U}$, then there exist $f \in A(\Omega, X)$ and $V \in \mathrm{RO}(\Omega)$ containing ω so that

$$f = \begin{cases} h_1 & \text{on } U \\ h_2 & \text{on } V. \end{cases}$$

A set of functions $A(\Omega, X)$ is *nowhere trivial* if for any $\omega \in \Omega$, there are $h_1, h_2 \in A(\Omega, X)$ so that $h_1(\omega) \neq h_2(\omega)$. If Ω is a regular topological space and $A(\Omega, X)$ is nowhere trivial and satisfies condition (L), then $A(\Omega, X)$ is weakly regular.

Theorem 3.3 *Let Ω, Σ be regular topological spaces. Assume that $A(\Omega, X)$ and $A(\Sigma, Y)$ are nowhere trivial and satisfy condition (L). A bijection $T : A(\Omega, X) \to A(\Sigma, Y)$ is a $\perp$-isomorphism if and only if there is Boolean isomorphism $\theta : \mathrm{RO}(\Omega) \to \mathrm{RO}(\Sigma)$ so that for all $f, g \in A(\Omega, X)$ and all $U \in \mathrm{RO}(\Omega)$, $f = g$ on U if and only if $Tf = Tg$ on $\theta(U)$.*

In order to prove Theorem 3.3, we first require a lemma.

Lemma 3.4 *Assume that $h_1, h_2 \in A(\Omega, X)$, $U \in \mathrm{RO}(\Omega)$ so that $h_1 = h_2$ on U. Then $\theta_{h_1}(U) = \theta_{h_2}(U)$.*

Proof Assume that $\theta_{h_2}(U) \not\subseteq \overline{\theta_{h_1}(U)}$. Since T is a bijection and $A(\Sigma, Y)$ is weakly regular, there exists $f \in A(\Omega, X)$ such that

$$\emptyset \neq \sigma_{Th_2}(Tf) \subseteq \theta_{h_2}(U) \backslash \overline{\theta_{h_1}(U)}.$$

By (1) and the fact that T is a $\subseteq$-isomorphism, $\theta_{h_2}(\sigma_{h_2}(f)) = \sigma_{Th_2}(Tf)$. Then $\emptyset \neq \sigma_{h_2}(f) = \theta_{h_2}^{-1}(\sigma_{Th_2}(Tf)) \subseteq U$. Hence $\sigma_{h_2}(f)$ is a nonempty set disjoint from $\overline{\neg U}$. By condition (L), there exist $g \in A(\Omega, X)$ and a nonempty set $V \in \mathrm{RO}(\Omega)$, $V \subseteq \sigma_{h_2}(f)$, so that

$$g = \begin{cases} h_1 & \text{on } \neg U, \\ f & \text{on } V. \end{cases}$$

Now

1. $\theta_f(V) \subseteq \theta_f(\sigma_{h_2}(f)) = \theta_f(\sigma_f(h_2)) = \sigma_{Tf}(Th_2) = \sigma_{Th_2}(Tf)$.
2. $Tg = Th_1$ on $\theta_{h_1}(\neg U) = \neg\theta_{h_1}(U) = \mathrm{int}[\theta_{h_1}(U)^c] \supseteq \sigma_{Th_2}(Tf)$.
3. $Tg = Tf$ on $\theta_f(V)$.
4. $Th_1 = Th_2$ on $\theta_{h_2}(U) \supseteq \sigma_{Th_2}(Tf)$.

Here, we have applied Theorem 3.2 for items 2–4. It follows that $Tf = Th_2$ on $\theta_f(V)$. However, $\theta_f(V)$ is a nonempty open subset of $\sigma_{Th_2}(Tf) = \mathrm{int}\,\overline{[Tf \neq Th_2]}$. So we have reached a contradiction. Therefore, $\theta_{h_2}(U) \subseteq \overline{\theta_{h_1}(U)}$. Since $\theta_{h_1}(U)$ is a regular open set, $\theta_{h_2}(U) \subseteq \theta_{h_1}(U)$. The lemma follows by symmetry. □

Proof of Theorem 3.3 Taking into account Theorem 3.2, it suffices to show that $\theta_{h_1} = \theta_{h_2}$ for any $h_1, h_2 \in A(\Omega, X)$. Suppose that there exists $U \in \mathrm{RO}(\Omega)$ so that $\theta_{h_2}(U) \nsubseteq \theta_{h_1}(U)$, so that in fact $\theta_{h_2}(U) \nsubseteq \overline{\theta_{h_1}(U)}$. By condition (L) for $A(\Sigma, Y)$, there exists $g \in A(\Sigma, Y)$ and a nonempty set $V \in \mathrm{RO}(\Sigma)$, $V \subseteq \theta_{h_2}(U)\backslash\overline{\theta_{h_1}(U)}$, so that

$$g = \begin{cases} Th_1 & \text{on } \theta_{h_1}(U), \\ Th_2 & \text{on } V. \end{cases}$$

Apply Lemma 3.4 on $A(\Sigma, Y)$. We find that

$$\theta_g(\theta_{h_1}(U)) = \theta_{Th_1}(\theta_{h_1}(U)) = U \quad \text{and} \quad \theta_g(V) = \theta_{Th_2}(V) = \theta_{h_2}^{-1}(V).$$

Since $\theta_{h_1}(U) \cap V = \emptyset$ and θ_g is a Boolean isomorphism,

$$U \cap \theta_{h_2}^{-1}(V) = \theta_g(\theta_{h_1}(V)) \cap \theta_g(V) = \emptyset.$$

However, $\theta_{h_2}^{-1}(V)$ is a nonempty subset of $\theta_{h_2}^{-1}(\theta_{h_2}(U)) = U$. The contradiction shows that $\theta_{h_2}(U) \subseteq \theta_{h_1}(U)$. The reverse inclusion follows by symmetry. □

In Theorem 3.3, say that θ is *associated with* T. We list a few examples of sets of functions satisfying condition (L). Another example is given in Lemma 4.2 below.

Example (a) Let Ω be a completely regular Hausdorff space and let X be a convex set in a Hausdorff topological vector space. The space $C(\Omega, X)$ consists of all continuous functions from Ω into X.

(b) Let Ω be a metric space and let X be a convex set in a normed space. Denote by $U(\Omega, X)$, $U_*(\Omega, X)$, $\mathrm{Lip}(\Omega, X)$, $\mathrm{Lip}_*(\Omega, X)$, respectively, the set of uniformly continuous functions, the set of bounded uniformly continuous functions, the set of Lipschitz functions and the set of bounded Lipschitz functions from Ω to X.

(c) Let Ω be an open set in a Banach space $\mathcal{Z}$ and let $\mathcal{X}$ be a Banach space. For $p \in \mathbb{N} \cup \{\infty\}$, denote by $C^p(\Omega, \mathcal{X})$ the space of all p-times continuously (Fréchet) differentiable $\mathcal{X}$-valued functions on Ω. To ensure that there are "sufficiently many" functions in $C^p(\Omega, \mathcal{X})$, we assume that there exists a bump function in $C^p(\mathcal{Z})$, i.e., a function $\xi \in C^p(\mathcal{Z})$ that has nonempty bounded support in $\mathcal{Z}$.

To see that all of the spaces $A(\Omega, X)$ above satisfy condition (L), first observe that if $\omega_0 \in \Omega$, $U \in \mathrm{RO}(\Omega)$ and $\omega_0 \notin \overline{U}$, then there exist $\xi : \Omega \to [0, 1]$, $V \in \mathrm{RO}(X)$ containing ω_0 so that $\xi = 0$ on V and $\xi = 1$ on U. Moreover, for the situation in (b), we can choose ξ to be Lipschitz, and for case (c), we can choose

$\xi \in C^p(\Omega)$. Given $h_1, h_2 \in A(\Omega, X)$, it is easy to verify that $f(\omega) = \xi(\omega)h_1(\omega) + (1-\xi(\omega))h_2(\omega)$ defines a function in $A(\Omega, X)$ that is equal to h_1 on U and h_2 on V.

Remark 3.5 Condition (L) is a condition on $A(\Omega, X)$, respectively, $A(\Sigma, Y)$, that guarantees that the Boolean isomorphisms θ_h are independent of h. Alternatively, we may impose conditions on T to warrant the same outcome. For example, if X, Y are Hausdorff topological groups, then $C(\Omega, X)$ is a topological group under pointwise group operations. Suppose that $A(\Omega, X), A(\Sigma, Y)$ are subgroups of $C(\Omega, X), C(\Sigma, Y)$ respectively and $T : A(\Omega, X) \to A(\Sigma, Y)$ is a group isomorphism as well as a $\perp_h$ isomorphism for some $h \in A(\Omega, X)$. Then routine verification shows that T is a $\perp$-isomorphism and for any $k \in A(\Omega, X)$, $\theta_h(U) = \theta_k(U)$ for all $U \in \mathrm{RO}(X)$. In particular, the situation occurs if X and Y are Hausdorff topological vector spaces, $A(\Omega, X), A(\Sigma, Y)$ are respective subspaces of $C(\Omega, X), C(\Sigma, Y)$, and $T : A(\Omega, X) \to A(\Sigma, Y)$ is an additive $\perp_0$-isomorphism.

3.2 Homeomorphism Associated with a ⊥-Isomorphism

Theorem 3.3 allows us to associate a Boolean isomorphism with a $\perp$-isomorphism. Unfortunately, in general, a Boolean isomorphism $\theta : \mathrm{RO}(\Omega) \to \mathrm{RO}(\Sigma)$ need not induce a homeomorphism $\varphi : \Omega \to \Sigma$.

Example ([15]) Let Ω be a topological space and let Σ be a dense open set in Ω. Then the map $\theta_\Sigma : \mathrm{RO}(\Omega) \to \mathrm{RO}(\Sigma)$ given by $\theta_\Sigma(U) = U \cap \Sigma$ is a Boolean isomorphism. In particular, let S^1 be the unit circle in the complex plane. The sets $(0, 1)$ and $S^1\backslash\{1\}$ are homeomorphic and open and dense in $[0, 1]$ and S^1 respectively. Hence we have a chain of Boolean isomorphisms

$$\mathrm{RO}([0, 1]) \leftrightarrow \mathrm{RO}((0, 1)) \leftrightarrow \mathrm{RO}(S^1\backslash\{1\}) \leftrightarrow \mathrm{RO}(S^1).$$

But of course $[0, 1]$ and S^1 are not homeomorphic.

The next result characterizes the Boolean isomorphisms that induce homeomorphisms. If $\omega \in \Omega$, where Ω is a topological space, let $\mathcal{N}_\omega$ be the family of open neighborhoods of ω.

Proposition 3.6 *Let* Ω, Σ *be Hausdorff topological spaces and let* $\theta : \mathrm{RO}(\Omega) \to \mathrm{RO}(\Sigma)$ *be a Boolean isomorphism. Assume that*

1. *For any* $\omega \in \Omega$*, there exists* $\sigma \in \Sigma$ *such that for any* $V \in \mathcal{N}_\sigma$*, there exists* $U \in \mathrm{RO}(\Omega)$ *containing* ω *such that* $\theta(U) \subseteq V$.
2. *For any* $\sigma \in \Sigma$*, there exists* $\omega \in \Omega$ *such that for any* $U \in \mathcal{N}_\omega$*, there exists* $V \in \mathrm{RO}(\Sigma)$ *containing* σ *such that* $\theta^{-1}(V) \subseteq U$.

Then there exists a homeomorphism $\psi : \Omega \to \Sigma$ *such that* $\psi(U) = \theta(U)$ *for any* $U \in \mathrm{RO}(X)$*. Conversely, if* Ω, Σ *are regular topological spaces and there is a* *homeomorphism* ψ *such that* $\psi(U) = \theta(U)$ *for any* $U \in \mathrm{RO}(X)$*, then conditions 1 and 2 hold.*

Given conditions 1 and 2, define $\psi(\omega) = \sigma$ when ω and σ are related by condition 1. Similarly, define $\varphi(\sigma) = \omega$ when σ and ω are related by condition 2. One can check that $\psi : \Omega \to \Sigma$ and $\varphi : \Sigma \to \Omega$ are continuous functions that are mutual inverses. Proposition 3.6 can be applied to obtain general versions of Theorem 2.6. A homeomorphism $\psi : \Omega \to \Sigma$ is *associated with* T if for any $U \in \mathrm{RO}(\Omega)$ and any $f, g \in A(\Omega, X)$, $f = g$ on U if and only if $Tf = Tg$ on $\psi(U)$.

Theorem 3.7 *Suppose that* $A(\Omega, X), A(\Sigma, Y)$ *are nowhere trivial subsets of* $C(\Omega, X)$ *and* $C(\Sigma, Y)$ *respectively that satisfy condition (L), where* X, Y *are Hausdorff spaces and* Ω, Σ *are compact Hausdorff. If* $T : A(\Omega, X) \to A(\Sigma, Y)$ *is a* $\perp$*-isomorphism, then there is a homeomorphism* $\psi : \Omega \to \Sigma$ *associated with* T.

Proof By Theorem 3.3, there is a Boolean isomorphism $\theta : \mathrm{RO}(\Omega) \to \mathrm{RO}(\Sigma)$ associated with T. Let us verify condition 1 in Proposition 3.6. Condition 2 follows by symmetry. Fix $\omega \in \Omega$. By assumption, there are functions $h_1, h_2 \in A(\Omega, X)$ so that $h_1(\omega) \neq h_2(\omega)$. The family $\{\overline{\theta(U)} : \omega \in U \in \mathrm{RO}(\Omega)\}$ has the finite intersection property and hence has nonempty intersection. Suppose that there are two distinct points $\sigma_1, \sigma_2 \in \bigcap\{\overline{\theta(U)} : \omega \in U \in \mathrm{RO}(\Omega)\}$. By condition (L), there are $V_1, V_2 \in \mathrm{RO}(\Sigma)$ and $f \in A(\Sigma, Y)$ so that $\sigma_i \in V_i$ and $f = Th_i$ on V_i, $i = 1, 2$. Thus $T^{-1}f = h_i$ on $\theta^{-1}(V_i)$. However, if $\omega \in U \in \mathrm{RO}(\Omega)$, then $\theta(U) \cap V_i \neq \emptyset$ and hence $U \cap \theta^{-1}(V_i) \neq \emptyset$. It follows that $\omega \in \overline{\theta^{-1}(V_1)} \cap \overline{\theta^{-1}(V_2)}$. By continuity of $T^{-1}f$, this would mean that $h_1(\omega) = T^{-1}f(\omega) = h_2(\omega)$, which is a contradiction. Therefore, the intersection $\bigcap\{\overline{\theta(U)} : \omega \in U \in \mathrm{RO}(\Omega)\}$ contains a unique point σ.

If condition 1 of Proposition 3.6 fails, there exists $V \in \mathcal{N}_\sigma$ such that $\overline{\theta(U)} \cap V^c \neq \emptyset$ for all $U \in \mathrm{RO}(\Omega) \cap \mathcal{N}_\omega$. Using compactness again, there exists σ' such that $\sigma' \in \overline{\theta(U)} \cap V^c$ for all $U \in \mathrm{RO}(\Omega) \cap \mathcal{N}_\omega$. Clearly, $\sigma' \neq \sigma$ and both belong to the intersection of the family $\{\overline{\theta(U)} : \omega \in U \in \mathrm{RO}(\Omega)\}$, contrary to the previous paragraph. □

Theorem 3.7 extends to the case of complete metric domains, provided the sets of functions satisfy an additional linking condition. Let Ω be a complete metric space, X be a Hausdorff topological space and let $A(\Omega, X)$ be a subset of $C(\Omega, X)$. A sequence (ω_n) in Ω is *separated* if $\inf_{m \neq n} d(\omega_m, \omega_n) > 0$.

(L_s) Let $h_1, h_2 \in A(\Omega, X)$ and let (ω_n) be a separated sequence in Ω. Then there exists $f \in A(\Omega, X)$ and $U_1, U_2 \in \mathrm{RO}(\Omega)$ so that each U_i contains infinitely many ω_n and that $f = h_i$ on U_i, $i = 1, 2$.

Theorem 3.8 *Suppose that* $A(\Omega, X), A(\Sigma, Y)$ *are nowhere trivial subsets of* $C(\Omega, X)$ *and* $C(\Sigma, Y)$ *respectively that satisfy conditions (L) and* (L_s)*, where* X, Y *are Hausdorff spaces and* Ω, Σ *are complete metric spaces. If* $T : A(\Omega, X) \to A(\Sigma, Y)$ *is a* $\perp$*-isomorphism, then there is a homeomorphism* $\psi : \Omega \to \Sigma$ *associated with* T.

Sketch of Proof By Theorem 3.3, there is a Boolean isomorphism $\theta : \mathrm{RO}(\Omega) \to \mathrm{RO}(\Sigma)$ associated with T. A bit of reflection shows that in order to verify condition 1 of Proposition 3.6, it is suffices to prove that if $\omega \in U_n \in \mathrm{RO}(\Omega)$, diam $U_n \to 0$ and $\sigma_n \in \theta(U_n)$ for all n, then (σ_n) converges in Σ. Fix functions $h_1, h_2 \in A(\Omega, X)$ so that $h_1(\omega) \neq h_2(\omega)$. If (σ_n) fails to be convergent, then either the sequence has no accumulation point, or at least two accumulation points. In either case, from condition (L) or (L_s), there are $f \in A(\Sigma, Y)$ and $V_1, V_2 \in \mathrm{RO}(\Sigma)$ so that V_i contain infinitely many σ_n and $f = Th_i$ on V_i, $i = 1, 2$. As in the proof of Theorem 3.7, $\omega \in \overline{\theta^{-1}(V_i)}$, $i = 1, 2$ and $T^{-1}f = h_i$ on $\overline{\theta^{-1}(V_i)}$, which leads to a contradiction. □

3.3 Representation

In many cases, it is possible to improve Theorems 3.7 and 3.8 by giving a functional representation of the $\perp$-isomorphism T.

Proposition 3.9 *Let Ω, Σ, X, Y be Hausdorff spaces. Suppose that $T : A(\Omega, X) \to A(\Sigma, Y)$ is a bijection, where $A(\Omega, X), A(\Sigma, Y)$ are subsets of $C(\Omega, X)$ and $C(\Sigma, Y)$ respectively. Assume that there is a homeomorphism $\psi : \Omega \to \Sigma$ that is associated with T. If $f, g \in A(\Omega, X)$, $\omega_0 \in \Omega$ and there exists $h \in A(\Omega, X)$ so that $\omega_0 \in \overline{\mathrm{int}\,[h = f]} \cap \overline{\mathrm{int}\,[h = g]}$, then $Tf(\psi(\omega_0)) = Tg(\psi(\omega_0))$.*

Proof Let $U = \mathrm{int}[h = f]$ and $V = \mathrm{int}[h = g]$. By assumption, $Th = Tf$ on $\psi(U)$ and $Th = Tf$ on $\psi(V)$. Since ψ is a homeomorphism and $\omega_0 \in \overline{U} \cap \overline{V}$, $\psi(\omega_0) \in \overline{\psi(U)} \cap \overline{\psi(V)}$. By continuity of Tf, Tg and Th, $Tf(\psi(\omega_0)) = Th(\psi(\omega_0)) = Tg(\psi(\omega_0))$. □

From the example following Theorem 3.3, the spaces listed there all satisfy condition (L).

Theorem 3.10 *Let Ω, Σ be a first countable compact Hausdorff topological space and let X, Y be convex sets in Hausdorff topological vector spaces, with X, Y containing more than one point. If $T : C(\Omega, X) \to C(\Sigma, Y)$ is a $\perp$-isomorphism, then there are a homeomorphism $\psi : \Omega \to \Sigma$ and a function $\Phi : \Omega \times X \to Y$ so that*

$$Tf(\psi(\omega)) = \Phi(\omega, f(\omega)) \text{ for all } f \in C(\Omega, X) \text{ and all } \omega \in \Omega.$$

Sketch of Proof As mentioned above, $C(\Omega, X)$ and $C(\Omega, Y)$ both satisfy condition (L). Hence there is a homeomorphism $\psi : \Omega \to \Sigma$ associated with T by Theorem 3.7. For any $x \in X$, let $g_x \in C(\Omega, X)$ be the constant function with value x. Define $\Phi : \Omega \times X \to Y$ by $\Phi(\omega, x) = (Tg_x)(\psi(\omega))$. Let $\omega_0 \in \Omega$ and

$f \in C(\Omega, X)$. Set $x = f(\omega_0)$. Using the first countability of Ω, one can easily construct $h \in C(\Omega, X)$ so that $\omega_0 \in \overline{\text{int}\,[h = f]} \cap \overline{\text{int}\,[h = g_x]}$. By Proposition 3.9,

$$Tf(\psi(\omega_0)) = (Tg_x)(\psi(\omega_0)) = \Phi(\omega_0, x) = \Phi(\omega_0, f(\omega_0)).$$

This completes the proof of the theorem. □

It is not hard to see that in this case Φ is a continuous function on $\Omega \times X$. In [18], it is shown that if $\mathcal{X}$ is a Banach space, then the spaces in the example (taking $X = \mathcal{X}$ where appropriate) satisfy condition (L_s). Hence we obtain the next result similarly.

Theorem 3.11 *Let $A(\Omega, \mathcal{X})$ be one of the spaces $U(\Omega, \mathcal{X})$, $U_*(\Omega, \mathcal{X})$, $\text{Lip}(\Omega, \mathcal{X})$ or $\text{Lip}_*(\Omega, \mathcal{X})$, where Ω is a complete metric space and $\mathcal{X}$ is a Banach space. Similarly for $A(\Sigma, \mathcal{Y})$. If $T : A(\Omega, \mathcal{X}) \to A(\Sigma, \mathcal{Y})$ is a $\perp$-isomorphism, then there are a homeomorphism $\psi : \Omega \to \Sigma$ and a function $\Phi : \Omega \times \mathcal{X} \to \mathcal{Y}$ so that*

$$Tf(\psi(\omega)) = \Phi(\omega, f(\omega)) \textit{ for all } f \in A(\Omega, \mathcal{X}) \textit{ and all } \omega \in \Omega.$$

In some instances, additional information concerning the functions ψ and Φ are known. For example, if $T : U(\Omega, \mathcal{X}) \to U(\Sigma, \mathcal{Y})$, then it can be shown that ψ is a uniform homeomorphism and Φ can be characterized. For details on this and for $\perp$-isomorphisms $T : \text{Lip}(\Omega, \mathcal{X}) \to \text{Lip}(\Sigma, \mathcal{Y})$, refer to [18].

Consider the space $C^p(\Omega, \mathcal{X})$, where $p \in \mathbb{N}$, Ω is an open set in a Banach space $\mathcal{Z}$ on which there is a C^p-bump function. It can be shown that if $f, g \in C^p(\Omega, \mathcal{X})$ satisfy $D^k f(\omega_0) = D^k g(\omega_0)$, $0 \le k \le p$, for some $\omega_0 \in \Omega$, then there exists $h \in C^p(\Omega, \mathcal{X})$ so that $\omega_0 \in \overline{\text{int}\,[h = f]} \cap \overline{\text{int}\,[h = g]}$. Therefore, we obtain the following counterpart of the preceding theorems for these spaces. For $k \in \mathbb{N}$, let $\mathcal{S}^k(\mathcal{Z}, \mathcal{X})$ be the space of all bounded symmetric k-linear operators from $\mathcal{Z}$ to $\mathcal{X}$.

Theorem 3.12 *Let $p, q \in \mathbb{N}$, Ω, Σ be open sets in a Banach spaces on which there are C^p, respectively, C^q-bump functions. Suppose that $T : C^p(\Omega, \mathcal{X}) \to C^q(\Sigma, \mathcal{Y})$ is a $\perp$-isomorphism. Denote by $\mathcal{Z}$ the Banach space containing Ω. Then there exist a homeomorphism $\psi : \Omega \to \Sigma$ and a function $\Phi : \Omega \times \mathcal{X} \times \mathcal{S}^1(\mathcal{Z}, \mathcal{X}) \times \cdots \times \mathcal{S}^p(\mathcal{Z}, \mathcal{X}) \to \mathcal{Y}$ so that*

$$Tf(\psi(\omega)) = \Phi(\omega, f(\omega), Df(\omega), \cdots, D^p f(\omega)), f \in C^p(\Omega, \mathcal{X}), \omega \in \Omega.$$

See [3] for a complete description of *additive* $\perp$-isomorphisms $T : C^p(\Omega, \mathcal{X}) \to C^q(\Sigma, \mathcal{Y})$.

4 Applications

We present several applications of the results in Sect. 3.

4.1 Order Isomorphism

In this subsection, let Ω, Σ be regular topological spaces and let X, Y be totally ordered sets endowed with the order topology, unless otherwise stated. Given subsets $A(\Omega, X)$, $A(\Sigma, Y)$ of $C(\Omega, X)$ and $C(\Sigma, Y)$ respectively, an *order isomorphism* is a bijection $T : A(\Omega, X) \to A(\Sigma, Y)$ that preserves the pointwise order: for all $f, g \in A(\Omega, X)$,

$$f(\omega) \le g(\omega) \text{ for all } \omega \in \Omega \iff Tf(\sigma) \le Tg(\sigma) \text{ for all } \sigma \in \Sigma.$$

If $A(\Omega, X)$ and $A(\Sigma, Y)$ are lattices (in the pointwise order), then an order isomorphism is a lattice isomorphism. Following [28], we say that $A(\Omega, X)$ is *X-normal* if for any disjoint closed sets F_1, F_2 in Ω and any $x_1, x_2 \in X$, there exists $f \in A(\Omega, X)$ so that $f = x_i$ on F_i, $i = 1, 2$. The following statement is Kaplansky's Theorem in its full generality.

Theorem 4.1 (Kaplansky [28]) *Let Ω, Σ be compact Hausdorff spaces and let X, Y be totally ordered sets with the order topology. If $C(\Omega, X)$ and $C(\Sigma, Y)$ are X- and Y-normal respectively and there exists an order isomorphism $T : C(\Omega, X) \to C(\Sigma, Y)$, then Ω and Σ are homeomorphic.*

We will see that Kaplansky's Theorem as well as similar results on other function spaces can be derived from considerations in Sect. 3. A function $f \in C(\Omega, X)$ is *bounded* if there are $x_1, x_2 \in X$ so that $x_1 \le f(\omega) \le x_2$ for all $\omega \in \Omega$. Clearly, if Ω is compact Hausdorff, or if X has both largest and smallest elements, then all functions in $C(\Omega, X)$ are bounded.

Lemma 4.2 *Let Ω be a regular topological space and let X be a totally ordered set with the order topology. Suppose that $A(\Omega, X)$ is a X-normal sublattice of $C(\Omega, X)$ that consists of bounded functions. Then $A(X, E)$ satisfies condition (L).*

Proof Let $h_1, h_2 \in A(\Omega, X)$, $U \in \mathrm{RO}(\Omega)$ and $\omega \notin \overline{U}$. Since Ω is regular, there exists $V \in \mathrm{RO}(\Omega)$ containing ω so that $\overline{V} \cap \overline{U} = \emptyset$. There are $x_1, x_2 \in X$ so that $x_1 \le h_1(\omega), h_2(\omega) \le x_2$ for all $\omega \in \Omega$. By X-normality, there are $k_1, k_2 \in A(\Omega, X)$ so that

$$k_1 = \begin{cases} x_2 & \text{on } \overline{V} \\ x_1 & \text{on } \overline{U} \end{cases} \quad \text{and} \quad k_2 = \begin{cases} x_1 & \text{on } \overline{V} \\ x_2 & \text{on } \overline{U} \end{cases}.$$

Set $k = (k_2 \vee h_1) \wedge (k_1 \vee h_2)$. Then $k \in A(\Omega, X)$. It is easy to see that $k = h_1$ on V and $k = h_2$ on U. This completes the verification of condition (L). □

Proposition 4.3 *Let Ω, Σ be regular topological spaces and let X, Y be totally ordered sets with the order topology. Suppose that $A(\Omega, X)$ is a sublattice of $C(\Omega, X)$ that satisfies condition (L). Similarly for $A(\Sigma, Y)$. If $T : A(\Omega, X) \to A(\Sigma, Y)$ is an order isomorphism, then T is a $\perp$-isomorphism.*

Proof Let $f, g, h \in A(\Omega, X)$ and suppose that $f \perp_h g$ and that $f, g \geq h$. Then $f \wedge g = h$ and hence $Tf \wedge Tg = Th$; whence $Tf \perp_{Th} Tg$. Similarly, $Tf \perp_{Th} Tg$ if $f \perp_h g$ and $f, g \leq h$.

Claim If $f, g, h \in A(\Omega, X)$, $f \perp_h g$ and $f \geq h \geq g$, then $Tf \perp_{Th} Tg$.

Otherwise, there exists $\sigma \in \Sigma$ so that $Tf(\sigma) > Th(\sigma) > Tg(\sigma)$. Let $U = \sigma_{Tg}(Th)$. By condition (L), there exists $k \in A(\Sigma, Y)$ so that $k(\sigma) = Tf(\sigma)$ and $k = Th = Tg$ on $\operatorname{int} U^c$. Replace k by $(k \vee Th) \wedge Tf$ if necessary to assume additionally that $Th \leq k \leq Tf$.

If $\sigma_g(T^{-1}k) \not\subseteq \overline{\sigma_g(h)}$, there exist a nonempty $W \in \mathrm{RO}(\Omega)$ and $l \in A(\Omega, X)$ so that $W \subseteq \sigma_g(T^{-1}k)$, $l = g$ on $\overline{\sigma_g(h)}$ and $l = T^{-1}k$ on W. Replace l by $l \vee g$ if necessary so that $l \geq g$. (Note that $g \leq h \leq T^{-1}k \leq f$.) Then $l, h \geq g$ and $l \perp_g h$. Hence $Tl \perp_{Tg} Th$. Since $k = Tg$ on $\operatorname{int} U^c$, $\sigma_{Tg}(k) \subseteq U$. Thus

$$\sigma_{Tg}(Tl) \cap \sigma_{Tg}(k) \subseteq \sigma_{Tg}(Tl) \cap U = \sigma_{Tg}(Tl) \cap \sigma_{Tg}(Th) = \emptyset.$$

So $Tl \perp_{Tg} k$. Since $l, T^{-1}k \geq g$ as well, $l \perp_g T^{-1}k$. But $l = T^{-1}k$ on W. Hence $T^{-1}k = g$ on W, which is absurd since W is a nonempty subset of $\sigma_g(T^{-1}k)$. This shows that $\sigma_g(T^{-1}k) \subseteq \overline{\sigma_g(h)}$ and thus $\sigma_g(T^{-1}k) \subseteq \sigma_g(h)$.

By assumption, $\sigma_h(f) \cap \sigma_g(h) = \sigma_h(f) \cap \sigma_h(g) = \emptyset$. Therefore, $\sigma_g(T^{-1}k) \cap \sigma_h(f) = \emptyset$. If $\omega \in \sigma_g(T^{-1}k)$, then $\omega \notin \sigma_h(f)$ and hence $f(\omega) = h(\omega)$. So $T^{-1}k(\omega) = h(\omega)$ since $f \geq T^{-1}k \geq h$. On the other hand, if $\omega \notin \sigma_g(T^{-1}k)$, then $g(\omega) = T^{-1}k(\omega)$ and hence $T^{-1}k(\omega) = h(\omega)$ since $T^{-1}k \geq h \geq g$. Combining the two cases, we see that $T^{-1}k = h$ and hence $k = Th$. This is impossible since they differ at σ. This completes the proof of the claim.

Finally, let $f, g, h \in A(\Omega, X)$ with $f \perp_h g$. Then $f \diamond h \perp_h g \diamond h$, where each $\diamond$ stands for one of the symbols (not necessarily the same) $\vee$ or $\wedge$. By the first paragraph and the Claim, $T(f \diamond h) \perp_{Th} T(g \diamond h)$. Since

$$\begin{aligned}[][Tf \neq Th] &= [(Tf \vee Th) \neq Th] \cup [(Tf \wedge Th) \neq Th] \\ &= [T(f \vee h) \neq Th] \cup [T(f \wedge h) \neq Th] \quad \text{and} \\ [Tg \neq Th] &= [T(g \vee h) \neq Th] \cup [T(g \wedge h) \neq Th],\end{aligned}$$

we see that $[Tf \neq Th] \cap [Tg \neq Th] = \emptyset$, i.e., $Tf \perp_{Th} Tg$. By symmetry, $Tf \perp_{Th} Tg$ implies $f \perp_h g$. This completes the proof of the proposition. □

The next result generalizes Kaplansky's Theorem and follows immediately from Theorem 3.7, Lemma 4.2 and Proposition 4.3.

Theorem 4.4 (See also [15]) *Let Ω, Σ be compact Hausdorff spaces and let X, Y be totally ordered sets with the order topology. If $A(\Omega, X)$ is a X-normal sublattice of $C(\Omega, X)$, $A(\Sigma, Y)$ is a Y-normal sublattice of $C(\Sigma, Y)$ and there is an order isomorphism $T : A(\Omega, X) \to A(\Sigma, Y)$, then Ω and Σ are homeomorphic.*

Proposition 4.3 and Theorem 3.11 also yield the following.

Theorem 4.5 *Let $A(\Omega)$ be one of the spaces of real valued functions $U(\Omega)$, $U_*(\Omega)$,* Lip(Ω) *or* Lip$_*(\Omega)$, *where Ω is a complete metric space. Similarly for $A(\Sigma)$. If $T : A(\Omega) \to A(\Sigma)$ is an order isomorphism, then there are a homeomorphism $\psi : \Omega \to \Sigma$ and a function $\Phi : \Omega \times \mathbb{R} \to \mathbb{R}$ so that*

$$Tf(\psi(\omega)) = \Phi(\omega, f(\omega)) \text{ for all } f \in A(\Omega) \text{ and all } \omega \in \Omega.$$

Linear and nonlinear lattice and order isomorphisms have been well studied in a variety of function spaces. Garrido and Jaramillo [20] showed that the unital vector lattices $U(\Omega)$ and $U_*(\Omega)$ determine Ω up to uniform homeomorphism. In [21], the same authors showed that as a unital vector lattice, Lip(Ω) determines Ω up to Lipschitz homeomorphism. For Lipschitz spaces defined on Banach spaces, F. and J. Cabello Sánchez showed that Lip($\mathbb{R}$) and Lip$_*(\mathbb{R})$ are isomorphic as vector lattices. However, if $\mathcal{X}$ is a Banach space of dimension > 1 and $\mathcal{Y}$ is a Banach space, then Lip$_*(\mathcal{X})$ is not isomorphic as a vector lattice to Lip($\mathcal{Y}$) [14].

As a lattice alone (i.e., disregarding linearity), Shirota [37] proved that if $U_*(\Omega)$ and $U_*(\Sigma)$ are lattice isomorphic, with Ω, Σ complete metric spaces, then Ω is uniformly homeomorphic to Σ. In the same paper, the claim was also made for lattice isomorphisms $T : U(\Omega) \to U(\Sigma)$; but the proof contains a gap. The gap was repaired by F. Cabello Sánchez [11] and F. and J. Cabello Sánchez [13]. The same authors also showed that if $T : C^p(\Omega) \to C^p(\Sigma)$ is an order isomorphism, where $p \in \mathbb{N} \cup \{\infty\}$ and Ω, Σ are manifolds modeled on Banach spaces that support C^p-bump functions, then Ω and Σ are homeomorphic [12]. A unified treatment of order isomorphisms between functions spaces can be found in [32].

4.2 Realcompact Spaces

One can also consider the situation for Theorem 4.4 away from the confines of compact Hausdorff spaces. A completely regular Hausdorff space Ω has a "largest" compactification, the *Stone-Čech compactification* $\beta\Omega$, characterized by the fact that every continuous function f from Ω into a compact Hausdorff space X has a unique continuous extension $\widehat{f} : \beta\Omega \to X$. A good source of information concerning the Stone-Čech compactification is [39]. For the purpose of extending the Gelfand-Kolmogorov Theorem, Hewitt [26] introduced the class of realcompact spaces. Let $\mathbb{R}_\infty$ be the one point compactification of $\mathbb{R}$. The (Hewitt) *realcompactification* $\upsilon\Omega$ consists of all $\omega_0 \in \beta\Omega$ such that for any continuous real-valued function f on Ω, its continuous extension $\widehat{f} : \beta\Omega \to \mathbb{R}_\infty$ satisfies $\widehat{f}(\omega_0) \in \mathbb{R}$. Ω is *realcompact* if $\Omega = \upsilon\Omega$. It is known that a space is realcompact if and only if it is homeomorphic to a closed subspace of $\mathbb{R}^\Gamma$ for some index set Γ; see, e.g., [22]. Hewitt showed that for realcompact spaces, $C(\Omega)$ as a ring determines Ω uniquely up to homeomorphism. The result was generalized by Araujo et al. [5], and subsequently by Araujo to vector valued functions [1, 2]. If $\mathcal{X}$ and $\mathcal{Y}$ are vector spaces, denote the set of all linear bijections from $\mathcal{X}$ onto $\mathcal{Y}$ by $I(\mathcal{X}, \mathcal{Y})$.

Theorem 4.6 ([2]) *Let Ω and Σ be realcompact spaces and let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces. If $T : C(\Omega, \mathcal{X}) \to C(\Sigma, \mathcal{Y})$ is a linear biseparating map (i.e., linear $\perp$-isomorphism), then there are a homeomorphism $\varphi : \Sigma \to \Omega$ and a function $J : \Sigma \to I(\mathcal{X}, \mathcal{Y})$ so that*

$$Tf(\sigma) = (J\sigma) f(\varphi(\sigma)) \text{ for all } f \in C(\Omega, \mathcal{X}) \text{ and all } \sigma \in \Sigma.$$

Without the assumption of linearity in Theorem 4.6, it is still possible to conclude that Ω and Σ are homeomorphic. But the representation of T may not hold.

Theorem 4.7 ([18]) *Let Ω, Σ be realcompact spaces and $\mathcal{X}, \mathcal{Y}$ be Hausdorff topological vector spaces. If $T : C(\Omega, \mathcal{X}) \to C(\Sigma, \mathcal{Y})$ is a $\perp$-isomorphism, then Ω and Σ are homeomorphic.*

In particular, using the arguments of Lemma 4.2 and Proposition 4.3, the result can be applied to lattice isomorphisms. We emphasize that the lattice isomorphism below need not be linear.

Theorem 4.8 ([32]) *Let Ω and Σ be realcompact spaces. If there is a lattice isomorphism $T : C(\Omega) \to C(\Sigma)$, then Ω and Σ are homeomorphic.*

4.3 Ring Isomorphism and Multiplicative Isomorphism

Let Ω be a Hausdorff topological space. The space $C(\Omega)$ of all real valued continuous functions on Ω is a (unital) ring under pointwise operations. The subring $C_*(\Omega)$ consists of the bounded functions. Clearly, the (pointwise) order on these rings is determined by the ring structure since $f \geq 0$ if and only if f is a square in the ring. It follows immediately that results from Sect. 4.1 give rise to corresponding results concerning ring isomorphisms. In particular, we cite Hewitt's generalization of the theorem of Gelfand-Kolmogorov as a consequence of Theorem 4.8.

Theorem 4.9 ([26]) *Let Ω and Σ be realcompact spaces. Suppose that $C(\Omega)$ and $C(\Sigma)$ are isomorphic as rings, then Ω and Σ are homeomorphic.*

If Ω is a complete metric space, then $\mathrm{Lip}_*(\Omega)$ is a ring under pointwise operations. Ring isomorphisms between such rings were described in [21]. Let Ω, Σ be metric spaces. A function $\psi : \Omega \to \Sigma$ is *Lipschitz in the small* if there exist $r, K > 0$ so that $d(\psi(\omega_1), \psi(\omega_2)) \leq K d(\omega_1, \omega_2)$ whenever $\omega_1, \omega_2 \in \Omega$ and $d(\omega_1, \omega_2) < r$. ψ is a *LS-homeomorphism* if it is a homeomorphism so that both ψ and ψ^{-1} are Lipschitz in the small.

Theorem 4.10 (Garrido and Jaramillo) *Let Ω, Σ be complete metric spaces. The following are equivalent.*

1. $\mathrm{Lip}_*(\Omega)$ *and* $\mathrm{Lip}_*(\Sigma)$ *are isomorphic as unital rings.*
2. $\mathrm{Lip}_*(\Omega)$ *and* $\mathrm{Lip}_*(\Sigma)$ *are isomorphic as unital vector lattices.*
3. Ω *and* Σ *are LS-homeomorphic.*

Garrido et al. showed that the ring of smooth functions $C^\infty(M)$ determines the manifold M up to smooth diffeomorphism. For notions and notation regarding global analysis on infinite dimensional manifolds, refer to [29].

Theorem 4.11 (Garrido et al. [23]) *Let M and N be paracompact Banach manifolds modeled on C^∞-smooth Banach spaces. The rings $C^\infty(M)$ and $C^\infty(N)$ are isomorphic if and only if M and N are C^∞-diffeomorphic.*

Instead of ring isomorphisms, one can disregard linearity and consider maps that preserve multiplication alone.

Proposition 4.12 *Let Ω, Σ be Hausdorff spaces and let $A(\Omega), A(\Sigma)$ be unital subrings of $C(\Omega)$ and $C(\Sigma)$ respectively. Assume that either*

1. Ω *and* Σ *are compact and* $A(\Omega), A(\Sigma)$ *satisfy condition (L); or*
2. Ω *and* Σ *are complete metric spaces and* $A(\Omega), A(\Sigma)$ *satisfy conditions (L) and* (L_s).

If $T : A(\Omega) \to A(\Sigma)$ is a multiplicative isomorphism, i.e., T is a bijection so that $T(fg) = Tf \cdot Tg$ for all $f, g \in A(\Omega)$, then there is a homeomorphism $\psi : \Omega \to \Sigma$ that is associated with T in the sense defined before Theorem 3.7. In particular, T is a $\perp$-isomorphism.

Proof By assumption, the constant functions belong to $A(\Omega)$ and $A(\Sigma)$. Let Let 0, 2 denote the constant functions with values 0, 2 respectively. Then

$$2 \cdot T0 = T(T^{-1}2 \cdot 0) = T0.$$

Hence $T0 = 0$. For any $f, g \in C(\Omega)$,

$$f \perp_0 g \iff fg = 0 \iff Tf \cdot Tg = T0 = 0 \iff Tf \perp_{T0} Tg.$$

Hence T is a $\perp_0$-isomorphism. Note that Ω is a regular topological space and that $A(\Omega)$ is nowhere trivial and satisfies condition (L). Hence $A(\Omega)$ is weakly regular. Similarly for $A(\Sigma)$. By Theorem 3.2, there is a Boolean isomorphism $\theta_0 : \mathrm{RO}(\Omega) \to \mathrm{RO}(\Sigma)$ associated with $(T, 0)$. The same argument from the proof of Theorem 3.7 or Theorem 3.8 shows that Proposition 3.6 applies to θ_0. Thus there is a homeomorphism $\psi : \Omega \to \Sigma$ so that for any $f \in C(\Omega)$ and any $U \in \mathrm{RO}(\Omega)$, $f = 0$ on U if and only if $Tf = 0$ on $\psi(U)$.

In fact, ψ is associated with T. Let $U \in \mathrm{RO}(\Omega)$ and let $f, g \in A(\Omega)$ be such that $f = g$ on U. For $\sigma \in \psi(U)$, it follows from condition (L) that there exists $h \in C(\Sigma)$ so that $h(\sigma) = 1$ and $h = 0$ on $\operatorname{int} \psi(U)^c = \psi(\operatorname{int} U^c)$. Hence $T^{-1}h =$

$T^{-1}0 = 0$ on int U^c. Thus $T^{-1}h \cdot f = T^{-1}h \cdot g$. Therefore, $h \cdot Tf = h \cdot Tg$. In particular, $Tf(\sigma) = Tg(\sigma)$. This proves that $Tf = Tg$ on $\psi(U)$ if $f = g$ on U. The reverse implication follows by symmetry.

Let $\theta : \mathrm{RO}(\Omega) \to \mathrm{RO}(\Sigma)$ be the Boolean isomorphism given by $\theta(U) = \psi(U)$. Then θ is associated with T. Hence T is a $\perp$-isomorphism by Theorem 3.3. □

Proposition 4.12 and Theorem 3.11 give.

Theorem 4.13 *Let Ω, Σ be complete metric spaces and let $A(\Omega)$ be one of the spaces $U_*(\Omega)$ or* $\mathrm{Lip}_*(\Omega)$. *Similarly for $A(\Sigma)$. Let $T : A(\Omega) \to A(\Sigma)$ be a multiplicative isomorphism. Then there are a homeomorphism $\psi : \Omega \to \Sigma$ and a function $\Phi : \Omega \times \mathbb{R} \to \mathbb{R}$ so that*

$$Tf(\psi(\omega)) = \Phi(\omega, f(\omega)) \text{ for all } f \in A(\Omega) \text{ and all } \omega \in \Omega.$$

Milgram [35] characterized all multiplicative isomorphisms $T : C(\Omega) \to C(\Sigma)$. A combination of Proposition 4.12 and Theorems 3.7, 3.10 gives a partial result in this regard. See [15] for a proof of Milgram's Theorem via $\perp$-isomorphisms. When $p \in \mathbb{N}$ and Ω is a C^p-manifold, Mrčun and Šemrl [36] showed that all multiplicative automorphisms T on $C^p(\Omega)$ are of the form $Tf = f \circ \psi$ for some C^p diffeomorphisms ψ. The result was extended to the case $p = \infty$ by Artstein-Avidan et al. [8]. See [30] for a survey on the multiplication operator and other operator functional equations.

4.4 Isometry

The study of isometries is probably the most well developed part among theorems of Banach-Stone type. Here we restrict ourselves to a much abridged survey. Further information can be found in the two-volume monograph [19].

Behrends [10] introduced the use of centralizers into Banach-Stone considerations. Let $\mathcal{X}$ be a Banach space and denote the set of extreme points of the ball in $\mathcal{X}^*$ by ext $\mathcal{X}^*$. A bounded linear operator $S : \mathcal{X} \to \mathcal{X}$ is a *multiplier* if every $x^* \in \operatorname{ext} \mathcal{X}^*$ is an eigenvector of S^*, i.e., $S^* x^* = a_S(x^*)x^*$ for some scalar $a_S(x^*)$. If R, S are multipliers, say that R is an *adjoint* of S if $a_R(x*) = \overline{a_S(x^*)}$ for all $x^* \in \mathcal{X}^*$. The *centralizer* $Z(\mathcal{X})$ of $\mathcal{X}$ consists of all multipliers S for which an adjoint exists. Note that for real Banach spaces, the centralizer is the same as the set of all multipliers. Multiples of the identity operator are always present in the centralizer. Say that $\mathcal{X}$ has *trivial centralizer* if there are no other operators in $Z(\mathcal{X})$. Many classes of Banach spaces have trivial centralizers; refer to [10, 19].

Theorem 4.14 (Behrends) *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces which have trivial centralizers. Suppose further that Ω and Σ are locally compact Hausdorff spaces and that there exists a surjective linear isometry $T : C_0(\Omega, \mathcal{X}) \to C_0(\Sigma, \mathcal{Y})$. Then* *there is a homeomorphism $\varphi : \Sigma \to \Omega$ and a continuous function V from Σ into the space of isometries from $\mathcal{X}$ onto $\mathcal{Y}$ (given the strong operator topology) such that*

$$Tf(\sigma) = V(\sigma)f(\varphi(\sigma)) \text{ for all } f \in C_0(\Omega, \mathcal{X}) \text{ and all } \sigma \in \Sigma.$$

Araujo [4] extended this result by way of finding a connection to $\perp$-isomorphisms (biseparating maps).

Theorem 4.15 (Araujo) *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces which have trivial centralizers. Assume one of the following situations.*

1. *Ω, Σ are realcompact spaces and $\mathcal{X}$, $\mathcal{Y}$ are infinite dimensional. $A(\Omega, \mathcal{X}) = C_*(\Omega, \mathcal{X})$, the space of bounded $\mathcal{X}$-valued continuous functions on Ω, with the sup-norm. $A(\Sigma, \mathcal{Y}) = C_*(\Sigma, \mathcal{Y})$.*
2. *Ω, Σ are complete metric spaces, $A(\Omega, \mathcal{X}) = U_*(\Omega, \mathcal{X})$, the space of bounded $\mathcal{X}$-valued uniformly continuous functions on Ω. $A(\Sigma, \mathcal{Y}) = U_*(\Sigma, \mathcal{Y})$.*

If $T : A(\Omega, \mathcal{X}) \to A(\Sigma, \mathcal{Y})$ is a surjective linear isometry, then it is a $\perp$-isomorphism. Consequently, there is a homeomorphism $\varphi : \Sigma \to \Omega$ and a continuous function V from Σ into the space of isometries from $\mathcal{X}$ onto $\mathcal{Y}$ (given the strong operator topology) such that

$$Tf(\sigma) = V(\sigma)f(\varphi(\sigma)) \text{ for all } f \in C_0(\Omega, \mathcal{X}) \text{ and all } \sigma \in \Sigma.$$

In case (2), φ is a uniform homeomorphism.

Let Ω be a complete metric space and let $\mathcal{X}$ be a Banach space. The space of $\mathcal{X}$-valued Lipschitz functions on Ω, $\mathrm{Lip}(\Omega, \mathcal{X})$ is a Banach space under the norm

$$\|f\| = \max\{\|f\|_\infty, L(f)\},$$

where

$$\|f\|_\infty = \sup_{\omega \in \Omega} \|f(\omega)\| \text{ and } L(f) = \sup_{\omega_1 \neq \omega_2} \frac{\|f(\omega_1) - f(\omega_2)\|}{d(\omega_1, \omega_2)}.$$

Araujo and Dubarbie [6] gave a complete description of isometries between vector-valued spaces of Lipschitz functions. We state a special case of their result here. Define an equivalence relation on Ω by $x \sim y$ if there are $x = x_1, \ldots, x_n = y$ in Ω so that $d(x_i, x_{i+1}) < 2$, $1 \le i < n$. The equivalence classes are called the 2-*components* of Ω.

Theorem 4.16 *Let Ω, Σ be complete metric spaces and let $\mathcal{X}, \mathcal{Y}$ be Banach spaces. Assume that $T : \mathrm{Lip}(\Omega, \mathcal{X}) \to \mathrm{Lip}(\Sigma, \mathcal{Y})$ is a surjective linear isometry so that for all $\sigma \in \Sigma$, there is a constant function $f \in \mathrm{Lip}(\Omega, \mathcal{X})$ so that $Tf(\sigma) \neq 0$. Then* *there is a homeomorphism $\varphi : \Sigma \to \Omega$ and a continuous function V from Σ into the space of isometries from $\mathcal{X}$ onto $\mathcal{Y}$ (given the strong operator topology) such that*

$$Tf(\sigma) = V(\sigma)f(\varphi(\sigma)) \text{ for all } f \in \mathrm{Lip}(\Omega, \mathcal{X}) \text{ and all } \sigma \in \Sigma.$$

Moreover, V is constant on each 2-component of Σ and $d_{\mathcal{Y}}(\varphi(\omega_1), \varphi(\omega_2)) = d_{\mathcal{X}}(\omega_1, \omega_2)$ if either of these quantities is < 2.

It is worth mentioning that in the course of the proof of Theorem 4.16, it is first shown that T is a $\perp$-isomorphism (biseparating). Characterization of linear isometries on certain spaces of scalar-valued Lipschitz functions was obtained earlier by Weaver [40].

4.5 Nonvanishing Preservers

In this part, assume that Ω, Σ are regular topological spaces and X, Y are Hausdorff spaces. Let $A(\Omega, X)$ and $A(\Sigma, Y)$ be subsets of $C(\Omega, X)$ and $C(\Sigma, Y)$ respectively. Given $n \in \mathbb{N}$ and $h \in A(\Omega, X)$, a bijection $T : A(\Omega, X) \to A(\Sigma, Y)$ is a $\cap_h^n$-*isomorphism* if for any $f_1, \ldots, f_n \in A(\Omega, X)$,

$$\bigcap_{i=1}^{n} [f_i = h] = \emptyset \iff \bigcap_{i=1}^{n} [Tf_i = Th] = \emptyset.$$

T is a $\cap^n$-*isomorphism* if it is a $\cap_h^n$-isomorphism for all $h \in A(\Omega, X)$. It is clear that every $\cap_h^n$-isomorphism is a $\cap_h^m$-isomorphism if $n > m$. Hence every $\cap^n$-isomorphism is a $\cap^m$-isomorphism if $n > m$. $\cap^n$-isomorphisms were introduced by Hernández and Ródenas [25]. Further results were given in [16, 31].

Proposition 4.17 *Let Ω, Σ be regular topological spaces. Suppose that $A(\Omega, X)$ and $A(\Sigma, Y)$ satisfy condition (L) and that there exists $k \in A(\Omega, X)$ so that $[k = h] = \emptyset$. If $T : A(\Omega, X) \to A(\Sigma, Y)$ is a $\cap_h^2$-isomorphism, then it is a $\perp_h$-isomorphism.*

Proof First of all, since T is a $\cap_h^2$-isomorphism, it is a $\cap_h^1$-isomorphism. Thus $[k = h] = \emptyset$ implies $[Tk = Th] = \emptyset$. Suppose that there are $f, g \in A(X, E)$ so that $f \perp_h g$ but $Tf \not\perp_{Th} Tg$. There exists $\sigma_0 \in \Sigma$ where $Tf(\sigma_0), Tg(\sigma_0) \neq Th(\sigma_0)$. Since Σ is a regular topological space, there exists $V \in \mathrm{RO}(\Sigma)$ containing σ_0 so that $V \subseteq [Tf \neq Th] \cap [Tg \neq Th]$. Then $\sigma_0 \notin \overline{\mathrm{int}\, V^c}$ and $\mathrm{int}\, V^c \in \mathrm{RO}(\Sigma)$. As $A(\Sigma, Y)$ satisfies condition (L), there exist $l \in A(\Sigma, Y)$ and $W \in \mathrm{RO}(Y)$ so that $\sigma_0 \in W$, $l = Tk$ on $\mathrm{int}\, V^c$ and $l = Th$ on W. Now

$$[Tf \neq Th] \cup [l \neq Th] \supseteq V \cup \overline{\mathrm{int}\, V^c} = \Sigma.$$

Thus $[Tf = Th] \cap [l = Th] = \emptyset$ and hence $[f = h] \cap [T^{-1}l = h] = \emptyset$. Similarly, $[g = h] \cap [T^{-1}l = h] = \emptyset$. But since $f \perp_h g$, $[f = h] \cup [g = h] = \Omega$. Therefore, $[T^{-1}l = h] = \emptyset$, whence $[l = Th] = \emptyset$, contradicting the fact that $l = Th$ on $W \neq \emptyset$. This completes the proof of the proposition. □

The next two results follow easily from Proposition 4.17, Theorem 3.7 and Theorem 3.11.

Theorem 4.18 *Let Ω, Σ be compact Hausdorff spaces and let $\mathcal{X}, \mathcal{Y}$ be normed spaces. If $T : C(\Omega, \mathcal{X}) \to C(\Sigma, \mathcal{Y})$ is a $\cap^2$-isomorphism, then there is a homeomorphism $\psi : \Omega \to \Sigma$ associated with T.*

Theorem 4.19 *Let Ω, Σ be complete metric spaces and let $\mathcal{X}, \mathcal{Y}$ be normed spaces. Suppose that $A(\Omega, \mathcal{X})$ is one of the spaces $U(\Omega, \mathcal{X}), U_*(\Omega, \mathcal{X})$, $\mathrm{Lip}(\Omega, \mathcal{X})$, $\mathrm{Lip}_*(\Omega, \mathcal{X})$. Similarly for $A(\Sigma, \mathcal{Y})$. If $T : A(\Omega, \mathcal{X}) \to A(\Sigma, \mathcal{Y})$ is a $\cap^2$-isomorphism, then there are a homeomorphism $\psi : \Omega \to \Sigma$ and a function $\Phi : \Omega \times \mathcal{X} \to \mathcal{Y}$ so that*

$$Tf(\psi(\omega)) = \Phi(\omega, f(\omega)) \textit{ for all } f \in A(\Omega, \mathcal{X}) \textit{ and all } \omega \in \Omega.$$

In general, a $\cap^1$-isomorphism need not be a $\cap^2$-isomorphism, as the following example shows.

Example Let $I = [0, 1]$. Define $T : C(I, I) \to C(I, I)$ by

$$Tf = \begin{cases} 1 - f & \text{if range } f = [0, 1] \\ f & \text{otherwise.} \end{cases}$$

Then T is a $\cap^1$-isomorphism but not a $\cap^2$-isomorphism, nor is T is a $\perp$-isomorphism.

Indeed, it is easy to check that if $f, g \in C(I, I)$, then $[f = g] \neq \emptyset$ if and only if $[Tf = Tg] = \emptyset$. However, let f be the constant function with value $\frac{1}{4}$ and let $g \in C(I, I)$ be such that $g(\frac{1}{4}) \neq \frac{1}{4} = g(\frac{3}{4})$ and range $g \neq [0, 1]$. Let h be the identity function $h(t) = t$ for all $t \in I$. Then $[f = h] \cap [g = h] = \emptyset$ but

$$\frac{3}{4} \in [f = 1 - h] \cap [g = 1 - h] = [Tf = Th] \cap [Tg = Th].$$

Hence T is not a $\cap^2$-isomorphism.

To see that T is not a $\perp$-isomorphism, consider the same h but take $f = h \vee \frac{1}{2}$, $g = h \wedge \frac{1}{2}$. It is clear that $f \perp_h g$ but that $Tf \not\perp_{Th} Tg$.

However, Li and Wong [33, 34] obtained a number of results regarding *linear* $\cap^1$-isomorphisms. The theorem below gives some special cases of their results.

Theorem 4.20 (Li and Wong) *Let Ω, Σ be Hausdorff completely regular topological spaces and let $\mathcal{X}, \mathcal{Y}$ be normed spaces. Assume that $A(\Omega, \mathcal{X})$ is the space*

$C(\Omega, \mathcal{X})$, or, where Ω is complete metric, $U(\Omega, \mathcal{X})$ or $\text{Lip}(\Omega, \mathcal{X})$. *Similarly for $A(\Sigma, \mathcal{Y})$. If $T : A(\Omega, \mathcal{X}) \to A(\Sigma, \mathcal{Y})$ is a linear $\cap^1$-isomorphism, then it is a $\perp$-isomorphism.*

We close with a positive result concerning nonlinear $\cap^1$-isomorphisms. Let Ω, Σ be Hausdorff spaces. We call a bijection $T : C(\Omega) \to C(\Sigma)$ an *anti-order isomorphism* if $f \geq g \iff Tg \geq Tf$ for all $f, g \in C(\Omega)$. Evidently, T is an anti-isomorphism if and only if the operator $-T$ is an order isomorphism, where $(-T)f := -Tf$.

Theorem 4.21 *Let Ω, Σ be connected compact Hausdorff spaces. If $T : C(\Omega) \to C(\Sigma)$ is a $\cap^1$-isomorphism, then T is an order isomorphism or an anti-order isomorphism. In particular, T is a $\perp$-isomorphism and hence Ω and Σ are homeomorphic.*

Proof Since Ω is connected, given any two functions $h, k \in C(\Omega)$ with $[h = k] = \emptyset$, either $h > k$ (i.e., $h(\omega) > k(\omega)$ for all ω) or $k > h$. Similarly for $C(\Sigma)$. We break the proof of the theorem into a series of steps.

Claim 1 If $f, g \in C(\Omega)$ and $f \leq g$, then either $Tf \leq Tg$ or $Tg \leq Tf$.
Otherwise, there are $\sigma_1, \sigma_2 \in \Sigma$ such that $Tf(\sigma_1) > Tg(\sigma_1)$ and $Tg(\sigma_2) > Tf(\sigma_2)$. Let $k_i \in C(\Sigma)$ be functions such that $k_2 > Tf > k_1$ and $k_i(\sigma_i) = Tg(\sigma_i)$, $i = 1, 2$. For $i = 1, 2$, $[T^{-1}k_i = f] = \emptyset$ and $[T^{-1}k_i = g] \neq \emptyset$. By the statement before Claim 1, $T^{-1}k_i > f$. Thus there exists $h \in C(\Omega)$ so that $h > f$ and $[h = T^{-1}k_i] \neq \emptyset$, $i = 1, 2$. But then $[Th = Tf] = \emptyset$ and $[Th = k_i] \neq \emptyset$, $i = 1, 2$; hence $Tf \not< Th$ and $Th \not< Tf$, contrary to the statement before Claim 1.

Claim 2 If $f, g, h \in C(\Omega)$ and $f \leq g, h$, then either $Tf \leq Tg, Th$ or $Tg, Th \leq Tf$.
If either of g, h equals f, then Claim 2 follows from Claim 1. Otherwise, we may choose $\omega_1, \omega_2 \in \Omega$ so that $g(\omega_1) > f(\omega_1)$ and $h(\omega_2) > f(\omega_2)$. Let $k \in C(\Omega)$ be such that $k > f$ and $k(\omega_1) = g(\omega_1)$, $k(\omega_2) = h(\omega_2)$. By the first statement of the proof, either $Tk > Tf$ or $Tk < Tf$. Assume the former. By Claim 1, either $Tg \geq Tf$ or $Tg \leq Tf$. Since $[Tg = Tk] \neq \emptyset$, we must have $Tg \geq Tf$. Similarly $Th \geq Tf$. If $Tk < Tf$, then we can show analogously that $Tg, Th \leq Tf$.

The following variant of Claim 2 can be established in the same way: if $g, h \leq f$, then either $Tg, Th \leq Tf$ or $Tf \leq Tg, Th$.

Claim 3 Let $f \in C(\Omega)$. Then either

$$g \leq f \leq h \implies Tg \leq Tf \leq Th \text{ or}$$

$$g \leq f \leq h \implies Th \leq Tf \leq Tg$$

In the first case, we say that T is order preserving with respect to f and in the second case, T is anti-order preserving with respect to f.

Otherwise, there are $g, h \neq f$, $g \leq f \leq h$ so that either $Tg, Th \geq Tf$ or $Tg, Th \leq Tf$. Apply Claim 2 or its variant to T^{-1} to see that either $g, h \geq f$ or $g, h \leq f$, contrary to the choices of g and h.

We now show that either T is an order isomorphism or an anti-order isomorphism. Otherwise, taking symmetry into account, by Claim 3, we may assume that there are h_1, h_2 so that T is order preserving with respect to h_1 and anti-order preserving with respect to h_2. Since T is a bijection, $h_1 \neq h_2$. Now $h_1 \wedge h_2 \leq h_1, h_2$. Thus $Th_2 \leq T(h_1 \wedge h_2) \leq Th_1$. On the other hand, by Claim 2, either $T(h_1 \wedge h_2) \leq Th_1, Th_2$ or $T(h_1 \wedge h_2) \geq Th_1, Th_2$. Assume the former case; the proof is similar in the latter case. We have $Th_2 \leq T(h_1 \wedge h_2) \leq Th_2$. Hence $h_1 \wedge h_2 = h_2$, i.e., $h_2 \leq h_1$. But since T is order preserving with respect to h_1 and anti-order preserving with respect to h_2, $Th_2 \leq Th_1$ and $Th_1 \leq h_2$. Thus $h_1 = h_2$, contrary to their choices. This concludes the proof that T is either an order isomorphism or anti-order isomorphism. Applying Proposition 4.3 to either T or $-T$, we see that T is a $\perp$-isomorphism. By Theorem 4.4, Ω and Σ are homeomorphic. □

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The Bishop–Phelps–Bollobás Theorem: An Overview

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Abstract In this survey, we provide an overview from 2008 to 2021 about the Bishop–Phelps–Bollobás theorem.

Keywords Norm attaining operators · Bishop–Phelps theorem · Bishop–Phelps–Bollobás property

1 Motivation and Historical Background

Before starting, we take a brief moment to introduce the necessary notation. Throughout the whole paper, we will be working with Banach spaces $\mathcal{X}$ over the field $\mathbb{K}$, which can be the set of real numbers, $\mathbb{R}$, or the set of complex numbers, $\mathbb{C}$. All the results are valid for both cases unless otherwise explicitly stated. We denote by $S_{\mathcal{X}}$ and $B_{\mathcal{X}}$ the unit sphere and the closed unit ball of the Banach space $\mathcal{X}$. We denote by $\mathcal{X}'$ the dual of the Banach space $\mathcal{X}$. The symbol $\mathbb{B}(\mathcal{X}, \mathcal{Y})$ stands for the Banach space of all bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$. When $\mathcal{Y} = \mathbb{K}$, and $\mathcal{X} = \mathcal{Y}$, we simply write $\mathbb{B}(\mathcal{X}, \mathbb{K})$ as $\mathcal{X}'$ and $\mathbb{B}(\mathcal{X}, \mathcal{X})$ as $\mathbb{B}(\mathcal{X})$. More in general, we denote by $\mathbb{B}(\mathcal{X}_1, \ldots, \mathcal{X}_N; \mathcal{Y})$ the Banach space of all N-linear mappings from $\mathcal{X}_1 \times \ldots \times \mathcal{X}_N$ into $\mathcal{Y}$ endowed with the supremum norm. We say that T attains its norm, or it is norm-attaining, if there exists $x_0 \in S_{\mathcal{X}}$ such that $\|T(x_0)\| = \|T\| = \sup_{x \in S_{\mathcal{X}}} \|T(x)\|$. We denote by $\mathrm{NA}(\mathcal{X}, \mathcal{Y})$ the set of all norm-attaining operators from $\mathcal{X}$ into $\mathcal{Y}$. The set of all norm-attaining functionals on $\mathcal{X}$ will be denoted by $\mathrm{NA}(\mathcal{X})$. Let L be a locally compact Hausdorff topological

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space. The space $C_0(L)$ is the space of all real or complex continuous functions defined on L with limit zero at infinity. We denote by $H^\infty(\mathbb{D})$ the algebra of all bounded analytic functions on the open unit disc $\mathbb{D}$ in the complex place $\mathbb{C}$. For a complex Banach space $\mathcal{X}$, we denote by $\mathcal{A}_u(B_\mathcal{X}; \mathcal{Y})$ (respectively, $\mathcal{A}_b(B_\mathcal{X}; \mathcal{Y})$) the set of all $\mathcal{Y}$-valued uniformly (respectively, bounded) continuous functions on $B_\mathcal{X}$ that are holomorphic on the interior of $B_\mathcal{X}$. The symbol $\mathcal{A}_{w^*u}(B_{\mathcal{X}'})$ stands for the unital algebra of all w^*-uniformly continuous functions from $B_{\mathcal{X}'}$ into $\mathbb{C}$ which are holomorphic on the interior of $B_{\mathcal{X}'}$ endowed with the supremum norm $\| \cdot \|_\infty$. Throughout Sect. 2.6, M will denote a pointed metric space, that is, a metric space with a distinguished point $0 \in M$ and such that $M \backslash \{0\} \neq \emptyset$, and d will denote the metric of M. If $\mathcal{Y}$ is a real Banach space, then $\mathrm{Lip}_0(M, \mathcal{Y})$ stands for the space of all Lipschitz mappings $f : M \to \mathcal{Y}$ with $f(0) = 0$ equipped with the Lipschitz norm. If $\mathcal{Y} = \mathbb{R}$, we may omit $\mathcal{Y}$ in the notation and simply write $\mathrm{Lip}_0(M)$. Finally, $\mathcal{F}(M)$ denotes the Lipschitz-free space associated to M (we refer the reader to the survey [111] and references therein for a solid background in Lipschitz-free spaces). We denote by $\mathcal{P}(^N\mathcal{X}; \mathcal{Y})$ the Banach space of all N-homogeneous polynomials from X into Y.

This survey is mainly motivated by two results due to three mathematicians: the Americans Errett Albert Bishop (1928–1983) and Robert Ralph Phelps (1926–2013), and the Hungarian Béla Bollobás (1943-). In fact, the whole story initiates with Robert Clarke James (1918–2003) who provided one of the most famous characterizations for reflexive spaces in Banach space theory known nowadays as the James theorem (see [117, 118]). In fact, in any first course in Functional Analysis, one is able to construct easily a bounded linear functional which *never* attains its norm and this opens the gate for a very natural question: when does a linear functional attain its norm? James, with outstanding techniques, proved that a Banach space $\mathcal{X}$ is reflexive if and only if *every* bounded linear functional attains its norm. It is not difficult to prove by using the Hahn-Banach theorem that when $\mathcal{X}$ is reflexive, every functional attains its norm. The real deal with the James theorem is of course the converse.

Since James characterized reflexive Banach spaces as those where every functional attained its norm, the concept of subreflexivity arose to denote the normed spaces for which the set of norm-attaining functionals is dense in the dual. Although there exist incomplete normed spaces which are not subreflexive (see [117]), in 1961, Bishop and Phelps showed that every Banach space is subreflexive; in other words, they proved that in any Banach space $\mathcal{X}$, given a functional $x^* \in \mathcal{X}'$ and an arbitrary positive number $\varepsilon > 0$, it is always possible to find a new functional $y^* \in \mathcal{X}'$ which attains its norm and satisfies $\|y^* - x^*\| < \varepsilon$ (see [41]). This means that, for every Banach space $\mathcal{X}$, the set $\mathrm{NA}(\mathcal{X})$ is norm dense in the dual space $\mathcal{X}'$.

Now, in order to be fair with the title of this survey, it remains to fit Bollobás' name somehow. This is so because Bollobás proved a strengthening of the Bishop–Phelps' result known nowadays as the Bishop–Phelps–Bollobás theorem [42]. The original Bollobás' result states the following.

Theorem 1.1 ([42, Theorem 1]) *Let $\mathcal{X}$ be a Banach space. Suppose that $x \in S_{\mathcal{X}}$ and $x^* \in S_{\mathcal{X}'}$ satisfy*

$$|x^*(x) - 1| \leq \frac{\varepsilon^2}{2},$$

where $0 < \varepsilon < \frac{1}{2}$. Then, there exist $y \in S_{\mathcal{X}}$ and $y^* \in S_{\mathcal{X}'}$ such that

$$y^*(y) = 1, \quad \|y - x\| < \varepsilon + \varepsilon^2, \quad \textit{and} \quad \|y^* - x^*\| \leq \varepsilon.$$

First of all, let us notice that it is clear that the Bishop–Phelps theorem is a particular case of the Bollobás' theorem. Indeed, Theorem 1.1 above contains much more information than the denseness of the functionals which attains their norms: it gives a simultaneous control of the involved points and functionals in a quantitative way and that is the great difference between these two theorems. At this point, it is worth mentioning the recent paper [63] where the authors sought the best possible constants in the Bollobás theorem (see, in particular, [63, Theorem 2.1]). We will be referring to the next result (extracted from [63, Corollary 2.4]) as the Bishop–Phelps–Bollobás Theorem, which is the sharpest version of Theorem 1.1 (see Sect. 3 for more information).

Theorem 1.2 (Bishop–Phelps–Bollobás Theorem) *Let $\mathcal{X}$ be a Banach space. Let $\varepsilon \in (0, 2)$ and suppose that $x \in B_{\mathcal{X}}$ and $x^* \in B_{\mathcal{X}'}$ satisfy*

$$\operatorname{Re} x^*(x) > 1 - \frac{\varepsilon^2}{2}.$$

Then, there exist $y \in S_{\mathcal{X}}$ and $y^* \in S_{\mathcal{Y}'}$ such that

$$y^*(y) = 1, \quad \|y - x\| < \varepsilon, \quad \textit{and} \quad \|y^* - x^*\| < \varepsilon.$$

Due to the generality of the Bishop–Phelps–Bollobás theorem (note that $\mathcal{X}$ is an arbitrary Banach space in Theorems 1.1 and 1.2), it seems reasonable to wonder whether an analogous result holds also for bounded linear operators. In fact, this question was asked by Bishop and Phelps at the end of their paper [41]: is it true that the set $\text{NA}(\mathcal{X}, \mathcal{Y})$ of all bounded linear operators which attain their norms is norm dense in $\mathbb{B}(\mathcal{X}, \mathcal{Y})$?

Joram Lindenstrauss (1936–2012) was the first one who gave a negative answer for this question in his seminal paper [137]. He exhibited an example of a Banach space $\mathcal{X}$ such that the set $\text{NA}(\mathcal{X}, \mathcal{X})$ is *not* dense in $\mathbb{B}(\mathcal{X})$ (see [137, Proposition 5]), answering in the negative Bishop–Phelps' question. This means, therefore, that there is no general version of the Bishop–Phelps theorem (and, consequently, no general version of the Bishop–Phelps–Bollobás theorem) for bounded linear operators. Nevertheless, Lindenstrauss did not stop there; he proved that, under some natural conditions, one can still have the denseness of the operators which attain their norms. For instance, this happens when $\mathcal{X}$ is reflexive (see [137, Theorem 1]) or when $\mathcal{Y}$ has the property β of Lindenstrauss (see [137, Proposition 3]). In other words, putting some extra conditions in the involved spaces, we can get versions of the Bishop–Phelps theorem for operators.

As a careful and curious reader may imagine, after Lindenstrauss' paper, a vast research on the topic has been done during the past sixty years in several directions. We would like to name just a few of them which provided a great impact on extensions of the Bishop–Phelps theorem: J. Bourgain, R. E. Huff, J. Johnson, W. Schachermayer, J. J. Uhl, J. Wolfe, and V. Zizler continued the study on the set of operators which attain their norms ([43, 116, 120, 144, 148, 151]); M. D. Acosta, R. M. Aron, F. J. Aguirre, Y. S. Choi, and R. Payá ([8, 35, 70]) considered some problems in the same line involving bilinear mappings; the second and the third authors of the present survey considered it for homogeneous polynomials (see [21, 36]); and more recently several problems on norm-attainment of Lipschitz mappings were also tackled (see for instance [54, 64, 111, 121]). We suggest the interested reader to check the excellent survey [4] from María Dolores Acosta to know more about the norm-attaining theory (up to 2006).

In this survey, we will be interested in (generalizations of) the Bishop–Phelps–Bollobás theorem. Let us notice that so far we have mentioned only possible versions of the Bishop–Phelps theorem for operators and nothing related to the Bollobás theorem for this class of functions was considered yet. The first time this question was addressed and systematically studied was in 2008, when the authors from [9] considered Theorem 1.2 for operators and provided several conditions under which this theorem holds in this more general manner.

The next definition is the main one of the present work and it is exactly what we will be discussing throughout the next sections.

Definition 1.3 (Bishop–Phelps–Bollobás Property) Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. We say that the pair $(\mathcal{X}, \mathcal{Y})$ has the Bishop–Phelps–Bollobás property for operators (BPBp, for short) if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $T \in \mathbb{B}(\mathcal{X}, \mathcal{Y})$ with $\|T\| = 1$ and $x \in S_{\mathcal{X}}$ satisfy

$$\|T(x)\| > 1 - \eta(\varepsilon), \tag{1}$$

there exist $S \in \mathbb{B}(\mathcal{X}, \mathcal{Y})$ with $\|S\| = 1$ and $x_0 \in S_{\mathcal{X}}$ such that

$$\|S(x_0)\| = 1, \quad \|x_0 - x\| < \varepsilon, \quad \text{and} \quad \|S - T\| < \varepsilon. \tag{2}$$

Finally, it is important to mention that María Dolores Acosta has an excellent survey on the recent progress on the BPBp (see [6]). Naturally, some of the fundamental results of the present survey will overlap with some of the contents of Acosta's. Nevertheless, we will focus on different research lines with the hope that the reader can get the whole picture of the theory by putting together both surveys.

2 The Bishop–Phelps–Bollobás Property

2.1 For Operators

In this section, we will be interested in providing some results on the Bishop–Phelps–Bollobás property for bounded linear operators. Chronologically, we should (and we do) start with the first paper [9] on the property. For this, we invite the reader to go back to Definition 1.3 once again and have in mind what we mean by BPBp for operators.

Before starting, let us make an obvious but essential observation on the BPBp: to get positive (or negative) results on this property, we strongly depend on the geometry of the unit ball of the involved Banach spaces. The reader should understand this literally as it is: every proof depends on the specific spaces that we are working with.

For finite-dimensional spaces, we have a positive result. The proof of the following theorem relies on the compactness of *both* unit balls $B_{\mathcal{X}}$ and $B_{\mathcal{Y}}$.

Theorem 2.1 ([9, Proposition 2.4]) *Let $\mathcal{X}, \mathcal{Y}$ be finite-dimensional Banach spaces. Then, the pair $(\mathcal{X}, \mathcal{Y})$ has the BPBp for operators.*

Note that Theorem 2.1 asks for both Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ to be finite-dimensional. If only $\mathcal{Y}$ is assumed to be finite-dimensional, it remains an open question to this day if even Bishop–Phelps Theorem is satisfied. About the analogous result where only the first space is considered to be finite-dimensional, Theorem 2.1 does not hold in general. Indeed, there exists a sequence of 2-dimensional polyhedral spaces such that, if we denote by $\mathcal{Z}$ their c_0-sum, then $(\ell_1^2, \mathcal{Z})$ fails to have the BPBp, where ℓ_1^2 denotes the 2-dimensional space $(\mathbb{R}^2, \|\cdot\|_1)$). This remarkable example can be found in [34, Example 4.1] (see also [34, Lemma 3.2]). Let us highlight this result.

Example (From [34, Example 4.1]) There exists a Banach space $\mathcal{Z}$ such that $(\ell_1^2, \mathcal{Z})$ fails the BPBp.

Note, however, that $\text{NA}(\ell_1^2, \mathcal{Y}) = \mathbb{B}(\ell_1^2, \mathcal{Y})$ for *every* Banach space $\mathcal{Y}$ and the set $\text{NA}(\mathcal{X}, \mathcal{Z})$ is dense in $\mathbb{B}(\mathcal{X}, \mathcal{Z})$ for *every* Banach space $\mathcal{X}$ (see the proof of [34, Theorem 4.2] which uses [21, Proposition 3]). This shows that the study of the Bishop–Phelps–Bollobás property is not merely a trivial extension of the study of the density of norm-attaining operators as one might think at a first glance.

Let us recall some definitions which we will need in what follows.

Definition 2.2 A Banach space $\mathcal{Y}$ satisfies property β of Lindenstrauss if there exist $\{y_\gamma : \gamma \in \Gamma\} \subseteq S_{\mathcal{Y}}$, $\{y_\gamma^* : \gamma \in \Gamma\} \subseteq S_{\mathcal{Y}'}$, and $0 \leq \rho < 1$ such that the following hold:

(a) $y_\gamma^*(y_\gamma) = 1$ for every $\gamma \in \Gamma$,
(b) $|y_\gamma^*(y_\beta)| \leq \rho < 1$ for every $\gamma \neq \beta$,
(c) $\|y\| = \sup_\gamma |y_\gamma^*(y)|$ for every $y \in \mathcal{Y}$.

We notice that the Banach spaces $c_0(\Gamma)$ and $\ell_\infty(\Gamma)$ satisfy property β of Lindenstrauss with $\rho = 0$ by using their biorthogonal systems. W. Schachermayer introduced a dual version of the previous property that is satisfied by spaces like ℓ_1.

Definition 2.3 ([145]) A Banach space $\mathcal{Y}$ satisfies property α of Schachermayer if there exist $A = \{y_\gamma : \gamma \in \Gamma\} \subseteq S_\mathcal{Y}$, $A^* = \{y_\gamma^* : \gamma \in \Gamma\} \subseteq S_{\mathcal{Y}'}$, and $0 \leq \rho < 1$ such that the following hold:

(a) $y_\gamma^*(y_\gamma) = 1$ for every $\gamma \in \Gamma$,
(b) $|y_\gamma^*(y_\beta)| \leq \rho < 1$ for every $\gamma \neq \beta$,
(c) $B_\mathcal{Y}$ is the closed absolutely convex hull of A.

J. Lindenstrauss also introduced in [137] two properties to denote spaces for which the density of norm-attaining operators was granted. Namely, a Banach space $\mathcal{X}$ has property A of Lindenstrauss if $\text{NA}(\mathcal{X}, \mathcal{Y})$ is always dense in $\mathbb{B}(\mathcal{X}, \mathcal{Y})$ for all Banach spaces $\mathcal{Y}$, and a Banach space $\mathcal{Y}$ has property B of Lindenstrauss if $\text{NA}(\mathcal{X}, \mathcal{Y})$ is always dense in $\mathbb{B}(\mathcal{X}, \mathcal{Y})$ for all Banach spaces $\mathcal{X}$. It is worth noting that property β of Lindenstrauss implies property B and property α of Schachermayer implies property A (see [137, 145]). The concepts of universal domain and range arose to represent those Banach spaces satisfying properties A and B, respectively. This concept was extended to the BPBp as follows.

Definition 2.4 ([34, Definition 1.2]) Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces. We say that

(a) $\mathcal{X}$ is a universal BPB domain space if, for every Banach space $\mathcal{Z}$, the pair $(\mathcal{X}, \mathcal{Z})$ has the BPBp.
(b) $\mathcal{Y}$ is a universal BPB range space if, for every Banach space $\mathcal{Z}$, the pair $(\mathcal{Z}, \mathcal{Y})$ has the BPBp.

Now, we have the two following results.

Theorem 2.5 ([9, Theorem 2.2]) *Suppose that $\mathcal{Y}$ has property β of Lindenstrauss. Then, the pair $(\mathcal{X}, \mathcal{Y})$ has the BPBp for operators for every Banach space $\mathcal{X}$. In other words, $\mathcal{Y}$ is a universal BPB range space.*

The next result is due to Sun Kwang Kim and Han Ju Lee (see also Theorem 2.39 below).

Theorem 2.6 ([125, Theorem 3.1]) *Suppose that $\mathcal{X}$ is uniformly convex. Then, $(\mathcal{X}, \mathcal{Y})$ has the BPBp for operators for every Banach space $\mathcal{Y}$. In other words, $\mathcal{X}$ is a universal BPB domain space.*

Remark 2.7 It is worth mentioning that Theorems 2.1, 2.5, and 2.6 were all generalized by considering bounded closed convex sets instead of the unit ball $B_\mathcal{X}$ (see [67, Theorems 3.1, 3.2 and Corollary 3.4], respectively). We strongly recommend the reader to take a look at that interesting paper due to Dong Hoon Cho and Yun Sung Choi, which contains a more general approach.

Let us notice that the authors in [9] were not interested in calculating the optimal constants in Theorem 2.5 (see the proof of [9, Theorem 2.2]). Nevertheless, by using some ideas from [54] (and also from [61, 62]), Vladimir Kadets and Mariia Soloviova studied in [123] estimates for these constants when the range space satisfies property β of Lindenstrauss (see Theorem 3.4). As far as we know no further research was done in this direction when $\mathcal{X}$ is taken to be uniformly convex. Regarding sharpness of the constants in Bishop–Phelps–Bollobás like theorems, we send the interested reader to Sect. 3 below.

We have already seen that reflexivity plays an important role when it comes to operators which attain their norms. Indeed, by using the James theorem, it is not difficult to construct a linear operator which never attains its norm when the domain space is non-reflexive. On the other hand, Lindenstrauss proved that reflexive spaces satisfy property A, that is, if $\mathcal{X}$ is reflexive, then $\text{NA}(\mathcal{X}, \mathcal{Y})$ is dense in $\mathbb{B}(\mathcal{X}, \mathcal{Y})$ for every Banach space $\mathcal{Y}$. Therefore, it is natural to wonder whether the same happens with the BPBp. However, this is *not* the case, as shown in the example after Theorem 2.1. Actually, there exists a reflexive space $\mathcal{X}$ such that the pair $(\mathcal{X}, \mathcal{X})$ does not have the BPBp. For this, take any $\mathcal{Y}$ which is reflexive and strictly convex but not uniformly convex and consider the reflexive space $\mathcal{X} = \ell_1^2 \oplus_1 \mathcal{Y}$. This space does the job (we send the interested reader to carefully check [34, Theorem 2.1, Corollary 3.3, and Example 3.4]).

Now, going back to the non-reflexive setting, the authors in [9] gave a complete characterization for all Banach spaces $\mathcal{Y}$ such that the pair $(\ell_1, \mathcal{Y})$ has the Bishop–Phelps–Bollobás property. They defined the approximate hyperplane series property (AHSP, for short) and proved that $(\ell_1, \mathcal{Y})$ has the BPBp for operators if and only if $\mathcal{Y}$ has the AHSP (see [9, Theorem 4.1]). This property is very technical and we will not treat it here. We send the interested reader in the recent progress on the AHSP (and its variants) to the already mentioned recent survey [6] (and the references therein), where M. D. Acosta exposes interesting facts about it in Section 3 (the interested reader may also check for instance the papers [10, 22, 60, 78, 108, 114] and their references for more information). For the sake of completeness, in the next theorem we exhibit some classical Banach spaces $\mathcal{Y}$ for which the pairs $(\ell_1, \mathcal{Y})$ satisfy the BPBp.

Theorem 2.8 *The pair $(\ell_1, \mathcal{Y})$ has the BPBp for operators when*

(a) $\mathcal{Y}$ *is finite-dimensional.*
(b) $\mathcal{Y}$ *is uniformly convex.*
(c) $\mathcal{Y} = C_0(L)$.
(d) $\mathcal{Y} = L_1(\mu)$, *where* μ *is any measure.*
(e) $\mathcal{Y} = \mathcal{A}(\mathbb{D})$.
(f) $\mathcal{Y} = H^\infty(\mathbb{D})$.
(g) $\mathcal{Y}$ *has the property* β *of Lindenstrauss.*
(h) $\mathcal{Y} = L_1(\mu, \mathcal{X})$, *where* μ *is a* σ*-finite measure and* $\mathcal{X}$ *is as (a)-(g).*
(i) $\mathcal{Y} = C_0(L, \mathcal{X})$, *where* L *is a locally compact Hausdorff space and* $\mathcal{X}$ *is as (a)-(g).*
(j) $\mathcal{Y} = \mathcal{A}_u(B_{\mathcal{X}})$, *the algebra of all uniformly continuous and holomorphic mappings on the open unit ball of a complex Banach space* $\mathcal{X}$.

Notice that Theorem 2.8.(g) is a particular case of Theorem 2.5 (see also the more general result [80, Proposition 2.10]). For the proofs of Theorem 2.8 we send the reader to [80, Section 2]. Item (i) from the previous theorem can be found in [114, Corollary 2.10], and item (j) can be found in [75, Corollary 2.17].

Note that the previous result shows once more a difference between the classical norm-attaining theory and the BPBp, since we know that if $\mathcal{X}$ has the Radon-Nikodým property, then for all Banach spaces $\mathcal{Y}$, $\text{NA}(\mathcal{X}, \mathcal{Y})$ is dense in $\mathbb{B}(\mathcal{X}, \mathcal{Y})$ (see [43]), but there are Banach spaces $\mathcal{Y}$ such that $(\ell_1, \mathcal{Y})$ fails the BPBp, and ℓ_1 satisfies the Radon-Nikodým property (see also [34, Section 3]). In fact, the authors of this survey are not aware of works studying when $(\mathcal{X}, \mathcal{Y})$ has the BPBp if $\mathcal{X}$ has the Radon-Nikodým property besides some particular cases. We do not know for example for what spaces $\mathcal{Y}$ we have that $(J, \mathcal{Y})$ satisfies the BPBp, where J is the James' space (see Question 6, and see [103, 138] for background on James' space).

Still on the AHSP, we have the following result which gives several examples on when the pair $(L_1(\mu), \mathcal{Y})$ satisfies the BPBp for operators: $(L_1(\mu), \mathcal{Y})$ has the BPBp if $\mathcal{Y}$ has the Radon-Nikodým property and the AHSP. We again send the reader to [6, pgs. 16-19] for a more complete discussion on the recent progress on the AHSP.

Theorem 2.9 ([77, Theorem 2.2]) *Suppose that $\mathcal{Y}$ has the Radon-Nikodým property. Let μ be a σ-finite measure. Then, the pair $(L_1(\mu), \mathcal{Y})$ has the BPBp for operators if and only if $\mathcal{Y}$ has the AHSP.*

On the other hand, Richard Aron, Yun Sung Choi, and the second and third authors of this survey proved that $(L_1(\mu), L_\infty[0, 1])$ has the BPBp for operators (see [33, Theorems 2.3 and 2.4]). This result was extended by Yun Sung Choi, Sun Kwang Kim, Han Ju Lee, and Miguel Martín (see [79, Theorem 4.1]).

Theorem 2.10 ([79, Theorem 4.1] (see also [33, Theorems 2.3 and 2.4])) *Let μ be an arbitrary measure and let ν be a localizable measure. Then, the pair $(L_1(\mu), L_\infty(\nu))$ has the BPBp for operators. In particular, $(L_1(\mu), L_\infty[0, 1])$ has the BPBp for operators.*

There is one more result that Theorem 2.9 above does not cover. Indeed, when the range space is an L_1-space, we have the following general result.

Theorem 2.11 ([79, Theorem 3.1]) *Let μ, ν be arbitrary measures. Then, the pair $(L_1(\mu), L_1(\nu))$ has the BPBp for operators.*

Regarding L_p-spaces, we borrow [79, Corollary 1.3] to summarize all the results on the pairs (L_p, L_q) that satisfy the BPBp for operators including Theorems 2.10 and 2.11. Let us notice that (b) below is an immediate consequence of Theorem 2.6. Item (c) follows from [11, Theorem 2.5] (see Theorem 2.13).

Theorem 2.12 ([79, Corollary 1.3]) *Let μ, ν be any measures. Then, the pair $(L_p(\mu), L_q(\nu))$ has the BPBp for operators when*

(a) $p = 1$ *and* $1 \leq q < \infty$.
(b) $1 < p < \infty$ *and* $1 \leq q \leq \infty$.

(c) $p = \infty$ *and* $q = \infty$ *(in the real case).*
(d) $p = 1$ *and* $q = \infty$ *whenever* ν *is a localizable measure.*

Moving towards another function space, we have the following positive result for operators from $C(K)$ into $C(S)$ in the real case. As far as we know the complex case of this result is still an open question (see Question 4). In fact, the analogous question for the density of norm-attaining operators from $C(K)$ into $C(S)$ is still open. Indeed, it seems to be a difficult task to adapt a proof from the real to the complex case when one is working with operators defined on $C(K)$-spaces.

Theorem 2.13 ([11, Theorem 2.5]) *Let* K, S *be compact Hausdorff topological spaces. Then, the pair* $(C(K), C(S))$ *has the BPBp for operators (in the real case).*

In the same direction, we have the following result due to Kim and Lee.

Theorem 2.14 ([126, Corollary 3.8]) *Let* S *be a locally compact metrizable space and* L *a locally compact Hausdorff space. Then, the pair* $(C_0(S), C_0(L))$ *has the BPB for operators (in the real case).*

A lot of progress has been made in pairs where the second space is uniformly convex. Let us highlight some of these results. It was shown in the original paper [9, Theorem 5.2] that when $\mathcal{X} = \ell_\infty^n$, we get a positive result.

Theorem 2.15 ([9, Theorem 5.2]) $(\ell_\infty^n, \mathcal{Y})$ *has the BPBp for operators for all* n *when* $\mathcal{Y}$ *is uniformly convex.*

Also, when one considers c_0 as the domain space and $\mathcal{Y}$ a uniformly convex Banach space in the range space, we always get a positive result. The next result is due to Sun Kwang Kim.

Theorem 2.16 ([124, Corollary 2.6]) *Let* $\mathcal{Y}$ *be a uniformly convex Banach space. Then, the pair* $(c_0, \mathcal{Y})$ *has the BPBp for operators.*

For a more general result, which covers Theorem 2.5 (when $\mathcal{X} = c_0$) and Theorem 2.16, we suggest the reader to go to [20], where the authors exhibit a new class of Banach spaces $\mathcal{Y}$ such that the pair $(c_0, \mathcal{Y})$ satisfies the BPBp for operators, a class of which covers all the Banach spaces with property β of Lindenstrauss and the uniformly convex Banach spaces (see also [6, page 23] for more details).

Kim, Lee and Lin showed in [129] the following when the domain is $L_\infty(\mu)$ (μ positive) or $c_0(\Gamma)$ (Γ index set) and the range is uniformly convex or $\mathbb{C}$-uniformly convex.

Theorem 2.17 ([129]) *Let* $\mathcal{X} = L_\infty(\mu)$ *with* μ *a positive measure or* $c_0(\Gamma)$ *where* Γ *is an index set. Then:*

(a) If $\mathcal{Y}$ *is uniformly convex, the pair* $(\mathcal{X}, \mathcal{Y})$ *has the BPBp for operators.*
(b) If $\mathcal{Y}$ *is* $\mathbb{C}$*-uniformly convex, the pair* $(\mathcal{X}, \mathcal{Y})$ *has the BPBp for operators in the complex case.*

In the real case, Kim and Lee proved that the pair $(C(K), \mathcal{Y})$ satisfies the BPBp for operators for any compact Hausdorff K whenever $\mathcal{Y}$ is uniformly convex [127].

Theorem 2.18 ([127, Theorem 2.2]) *If $\mathcal{Y}$ is uniformly convex, then the pair $(C(K), \mathcal{Y})$ has the BPBp for operators in the real case.*

In the complex case we should highlight the following nice result due to M. D. Acosta from 2016 (see [5]), which generalizes Theorem 2.17.(b).

Theorem 2.19 ([5, Theorem 2.4]) *The pair $(C_0(L), \mathcal{Y})$ has the BPBp for operators in the complex case whenever L is a locally compact Hausdorff and $\mathcal{Y}$ is any $\mathbb{C}$-uniformly convex complex space. In particular, the pairs*

(a) $(C_0(L), L_p(\nu))$,
(b) $(L_\infty(\mu), L_p(\nu))$

satisfy the BPBp for operators in the complex case for any positive measure μ and $1 \leq p < \infty$ and for every measure ν.

Let us note that Theorem 2.19.(b) was not covered by Theorem 2.12 above. It is worth mentioning that it seems to be an open problem whether or not the pair (c_0, ℓ_1) satisfies the Bishop–Phelps–Bollobás property for operators in the *real* case (see Question 5).

Whenever the pair (c_0, ℓ_1) has the BPBp for operators, then (ℓ_∞^n, ℓ_1) satisfies it uniformly for every $n \in \mathbb{N}$ (see [34, Theorem 2.1]), and the converse also holds. This fact motivated M. D. Acosta and José L. Dávila to characterize the Banach spaces $\mathcal{Y}$ such that the pairs of the form $(\ell_\infty^n, \mathcal{Y})$ satisfy the BPBp for operators (see [17, 18]; see also [14]) for a fixed $n \in \mathbb{N}$. In order to do so, they considered a geometric property: the approximate hyperplane sum property for ℓ_∞^n. A complete treatment of this property is done in Acosta's survey [6] in pages 22 and 23, and we strongly suggest the reader to check this and the references therein. Here, we highlight the consequences of such a property and exhibit the specific Banach spaces $\mathcal{Y}$ such that the pair $(\ell_\infty^n, \mathcal{Y})$ satisfies the BPBp for operators. We send the reader to check [6, Proposition 4.9] and the paragraph just after that. Let us notice that some of the items in Theorem 2.20 follow immediately from already mentioned results in this survey, nevertheless we include them for the sake of completeness.

Theorem 2.20 ([17, Theorem 3.3]) *Let $n \in \mathbb{N}$ be fixed with $n \geq 2$. The pair $(\ell_\infty^n, \mathcal{Y})$ has the BPBp for operators when*

(a) *$\mathcal{Y}$ is finite-dimensional.*
(b) *$\mathcal{Y}$ is uniformly convex.*
(c) *$\mathcal{Y}$ has property β of Lindenstrauss.*
(d) *$\mathcal{Y} \subseteq C(K)$ is a uniform algebra.*
(e) *$\mathcal{Y} = L_1(\mu)$ for every positive measure μ.*
(f) *$\mathcal{Y} = C_0(L, \mathcal{Z})$, where $\mathcal{Z}$ is one of the spaces in (a)-(e).*

Note that from the previous result, the pairs (ℓ_∞^n, ℓ_1) satisfy the BPBp for operators for all n, as desired. However, it is not known if they satisfy it uniformly, since the mappings η obtained in the proof depend on n, so the question of whether or not (c_0, ℓ_1) has the BPBp for operators remains open despite that.

We have been discussing some results and questions where the domain space was c_0 or ℓ_∞^n. Let us remark that, actually, some results are known when the domain space is Asplund. Richard M. Aron, Bernardo Cascales and Olena Kozhushkina showed in [32] the following result.

Theorem 2.21 ([32, Corollaries 2.6 and 2.7]) *$(\mathcal{X}, \mathcal{Y})$ has the BPBp for operators in the following cases:*

(a) $\mathcal{X}$ is Asplund and $\mathcal{Y} = C_0(L)$ for any locally compact Hausdorff L.
(b) $\mathcal{X}$ is any Banach space and $\mathcal{Y} = C_0(L)$, where L is a scattered locally compact Hausdorff space.

Soon after, Bernardo Cascales, Antonio J. Guirao and Vladimir Kadets extended Theorem 2.21.(a) to the case where the range space is a uniform algebra (see [55]).

Theorem 2.22 ([55, Theorem 3.6]) *$(\mathcal{X}, \mathcal{Y})$ has the BPBp for operators if $\mathcal{X}$ is Asplund and $\mathcal{Y} \subset C(K)$ is a uniform algebra.*

For positive results on the BPBp for operators when the range is an operator space, such as $\mathbb{K}(\mathcal{X}, C(K))$ or $\mathbb{W}(\mathcal{X}, C(K))$, we send the reader to [13, Theorem 3.1 and Corollary 3.2].

2.2 *For Some Classes of Operators*

In this section we consider the BPBp when restricted to some particular classes of operators. We may define in a natural way when a pair of Banach spaces $(\mathcal{X}, \mathcal{Y})$ satisfies the BPBp for some class. For instance, we say that $(\mathcal{X}, \mathcal{Y})$ has the BPBp for compact operators when one starts with a compact operator T in Definition 1.3 satisfying (1) and end up with another compact operator S satisfying conditions (2).

Theorem 2.23 ([79, Corollary 5.3]) *Let K be a compact Hausdorff space K and μ be a finite measure. Consider the real Banach space $L_1(\mu)$ and $C(K)$.*

(a) The pair $(L_1(\mu), C(K))$ has the BPBp for Bochner representable operators.
(b) The pair $(L_1(\mu), C(K))$ has the BPBp for weakly compact operators.

In fact, we have a more complete scenario when it comes to finite-rank, compact, weakly compact, and Radon-Nikodým operators when the domain is an L_1-space due to María Dolores Acosta, Julio Becerra Guerrero, Domingo García, Sun Kwang Kim, and Manuel Maestre.

Theorem 2.24 ([13, Proposition 2.2, Theorem 2.3, and Corollary 2.4]) *Let μ be a finite measure such that $L_1(\mu)$ is infinite-dimensional. The pair $(L_1(\mu), \mathcal{Y})$ has the BPBp for*

(1) finite-rank operators,
(2) compact operators,

(3) weakly compact operators,
(4) for Radon-Nikodým operators

when any of the following hold

(a) $\mathcal{Y}$ is finite-dimensional.
(b) $\mathcal{Y}$ is uniformly convex.
(c) $\mathcal{Y} = C(K)$, where K is a compact Hausdorff topological space.
(d) $\mathcal{Y} = L_1(\mu)$, where μ is a positive measure.
(e) $\mathcal{Y}$ has the property β of Lindenstrauss.
(f) $\mathcal{Y} = L_1(\mu, \mathcal{X})$, where μ is a σ-finite measure and $\mathcal{X}$

(f.1) is finite-dimensional.
(f.2) is uniformly convex.
(f.3) is lush and separable.
(f.4) is an almost-CL-space.
(f.5) has property β of Lindenstrauss.

Actually, item (f) is obtained in [80, Theorem 2.11 and Corollary 2.12].

As it was somehow mentioned before, in 2011, Richard Aron, Bernardo Cascales, and Olena Kozhushkina proved that the pair $(\mathcal{X}, C_0(L))$ satisfies the BPBp for Asplund operators for every Banach space $\mathcal{X}$ and every compact Hausdorff space K (see [32, Theorem 2.4]; a sharp version of this result can be found in [59, Theorem 5.5]). This was extended two years later by Antonio José Guirao, Vladimir Kadets and again Cascales to uniform algebras (see [55, Theorem 3.6]; we send the reader also to [13, Section 3] where the authors extend this last result to some $C(K, \mathcal{Y})$). Nevertheless, Bernardo's team contributions did not stop there. In 2018, Bernardo himself together with Guirao, Kadets, and Mariia Soloviova introduced a new Banach space property, called ACK$_\rho$-structure (see [56, Definition 3.1]), which gives Bishop–Phelps–Bollobás for a wider class of Banach spaces and also a wider class of operators. From our point of view such a paper contains many striking results which, as a consequence, provide a long collection of pairs of Banach spaces $(\mathcal{X}, \mathcal{Y})$ satisfying the BPBp for Asplund operators through a class of operators called Γ-flat operators (see [56, Definition 2.8]). We highlight only some of them although we strongly recommend [56]. We also send the reader to [55, Theorem 3.6 and Remarks R1, R2, R3 on page 380].

Theorem 2.25 ([56, Theorem 3.4, Corollary 4.6, Theorem 4.9]) *Let $\mathcal{X}$ be an arbitrary Banach space. The pair $(\mathcal{X}, \mathcal{Y})$ has the BPBp for*

(1) Asplund operators,
(2) finite rank operators,
(3) compact operators,
(4) p-summing operators, and
(5) weakly compact operators

whenever one of the following holds

(a) $\mathcal{Y} \subseteq C(K)$ *is a uniform algebra,*
(b) $\mathcal{Y}$ *has property* β *of Lindenstrauss.*

A different approach on how to prove that $(\mathcal{X}, \mathcal{Y})$ has the BPBp for compact operators when $\mathcal{Y}$ is a uniform algebra, can be found in [126, Theorem 3.9], where the authors use retractions as a tool for their proof.

Before going on, let us give some attention to an important tool for complex spaces used in the proofs of previous results: a Urysohn type lemma for uniform algebras proved by Cascales, Guirao, and Kadets (see [55]).

Lemma 2.26 ([55, Lemma 2.7]) *Let* $\mathcal{A} \subseteq C(K)$ *be a unital uniform algebra and* Γ_0 *its Choquet boundary. Then, for any open subset* U *of* K *with* $U \cap \Gamma_0 \neq \emptyset$ *and for* $0 < \varepsilon < 1$, *there exist* $f \in \mathcal{A}$ *and* $t_0 \in U \cap \Gamma_0$ *satisfying*

(a) $f(t_0) = \|f\|_\infty = 1.$
(b) $|f(t)| < \varepsilon$ *for every* $t \in K \setminus U.$
(c) $|f(t)| + (1-\varepsilon)|1 - f(t)| \leq 1$ *for every* $t \in K.$

We send the interested reader to [128] and [38] for related (and inspired by [55, Lemma 2.7]) Urysohn type lemmas for holomorphic functions. This lemma is also used in [75] to obtain results concerning the numerical index, the Daugavet equation, lushness and the AHSP on uniform algebras and also in [86] to give a version of Bishop–Phelps–Bollobás theorem for the unital uniform algebra $\mathcal{A}_{w^*u}(B_{\mathcal{X}'})$ for some complex Banach space $\mathcal{X}$.

Focusing on compact operators, the first three authors of this survey, together with Miguel Martín, studied the BPBp for this class of operators in a systematic way [88]. Although this survey is focused on the Bollobás theorem, it is worth mentioning that Martín gave a negative answer to an old open question on whether every compact operator can be approximated by norm-attaining operators [139] opening the gate for further research on this topic (see also [140] and the references therein).

Bearing in mind our Sect. 2.1, we already have a long list of pairs of Banach spaces $(\mathcal{X}, \mathcal{Y})$ that satisfy the BPBp for compact operators. Indeed, when analyzing the proofs of such results, when one starts with a compact operator, the new operator that we construct there which satisfy the Bollobás' conditions (2) is trivially compact. We borrow [88, Examples 1.5] and the results we have mentioned already, and list them in the following theorem.

Theorem 2.27 ([88, Examples 1.5]) *The pair* $(\mathcal{X}, \mathcal{Y})$ *has the BPBp for compact operators when*

(a) $\mathcal{X}$ *is arbitrary and* $\mathcal{Y}$ *has property* β *of Lindenstrauss.*
(b) $\mathcal{X}$ *is uniformly convex and* $\mathcal{Y}$ *is arbitrary.*
(c) $\mathcal{X}$ *is arbitrary and* $\mathcal{Y} \subseteq C(K)$ *is a uniform algebra.*

(d) $\mathcal{X} = L_1(\mu)$ *and* $\mathcal{Y} = L_1(\nu)$ *for* μ, ν *arbitrary.*
(e) $\mathcal{X} = L_1(\mu)$ *and* $\mathcal{Y} = L_\infty(\nu)$ *for any measure* μ *and any localizable measure* ν.
(f) $\mathcal{X}$ *arbitrary and* $\mathcal{Y}$ *an isometric predual of an* $L_1(\mu)$*-space.*
(g) $\mathcal{X} = L_1(\mu)$ *and* $\mathcal{Y}$ *as in Theorem 2.8 for every measure* μ.

Item (f) of Theorem 2.27 above was explicitly proven in [11, Theorem 4.2]. We also send the reader to check the proof of [11, Theorem 3.3] from the same paper where the authors prove that the pair $(C_0(L), \mathcal{Y})$ has the BPBp for compact operators whenever $\mathcal{Y}$ is uniformly convex.

At this point, and taking a look at Theorem 2.27 above, a natural question is to know what is the relation between the BPBp and the BPBp for compact operators. It turns out that the BPBp for compact operators *does not* imply the BPBp. Indeed, $(L_1[0,1], C[0,1])$ satisfies the BPBp for compact operators (check item (c) of Theorem 2.27 for instance) but the set $\text{NA}(L_1[0,1], C[0,1])$ is *not* dense in $\mathbb{B}(L_1[0,1], C[0,1])$ as proved by Schachermayer in 1983 (see [144]). On the other hand, the other implication seems to be still an open question (see Question 7).

By using technical tools (based on some results due to J. Johnson and J. Wolfe) the authors in [88] provide several results that allow passing the BPBp for compact operators from sequence spaces to function spaces (see [88, Lemma 2.1, Proposition 2.2, Corollaries 2.3 and 2.4, and Proposition 2.5]) as well as passing the BPBp to the BPBp for compact operators. These yield more examples of pairs of Banach spaces $(\mathcal{X}, \mathcal{Y})$ satisfying the BPBp for compact operators.

Theorem 2.28 ([88, Corollary 3.3]) *Let* $\mathcal{Y}$ *be a Banach space. If* $(c_0, \mathcal{Y})$ *has the BPBp, then* $(c_0, \mathcal{Y})$ *has the BPBp for compact operators. In particular,* $(c_0, \mathcal{Y})$ *has the BPBp for compact operators when*

(a) $\mathcal{Y}$ *has property* β *of Lindenstrauss.*
(b) $\mathcal{Y}$ *is uniformly convex.*

In fact, we have the following result.

Theorem 2.29 ([88, Corollary 3.5]) *The pair* $(C_0(L), \mathcal{Y})$ *has the BPBp for compact operators when*

(a) $\mathcal{Y}$ *has property* β *of Lindenstrauss.*
(b) $\mathcal{Y}$ *is uniformly convex.*

Although we have given only two specific examples for Theorems 2.28 and 2.29, there are Banach spaces $\mathcal{Y}$ (even 2-dimensional) which are neither uniformly convex nor satisfy property β of Lindenstrauss such that $(C_0(L), \mathcal{Y})$ satisfy the BPBp for compact operators (see [20] for a more general approach).

In the same direction, we have the following result.

Theorem 2.30 ([88, Corollaries 3.7 and 3.8]) *Let* $\mathcal{X}$ *be a Banach space such that its dual is isometrically isomorphic to* ℓ_1*. If* $\mathcal{Y}$ *is uniformly convex, then* $(\mathcal{X}, \mathcal{Y})$ *has the BPBp for compact operators.*

In fact, when the domain is ℓ_1, we provided the following characterization and we have an affirmative answer for Question 7 in this case.

Theorem 2.31 ([88, Corollary 3.11]) *Let $\mathcal{Y}$ be a Banach space. The following statements are equivalent.*

(a) $(\ell_1, \mathcal{Y})$ *has the BPBp for compact operators.*
(b) $(\ell_1, \mathcal{Y})$ *has the BPBp.*
(c) $(L_1(\mu), \mathcal{Y})$ *has the BPBp for compact operators for every positive measure* μ*.*

When it comes to strongly measurable function spaces, we have the following.

Theorem 2.32 ([88, Corollary 3.13]) *Let μ be a positive measure and let $\mathcal{X}, \mathcal{Y}$ be Banach spaces. The pair $(L_1(\mu, \mathcal{X}), \mathcal{Y})$ has the BPBp for compact operators when*

(a) $\mathcal{X}, \mathcal{Y}$ *are finite-dimensional.*
(b) $\mathcal{X}'$ *has the Radon-Nikodým property and $\mathcal{Y}$ is Hilbert such that the pair $(\mathcal{X}, \mathcal{Y})$ has the BPBp for compact operators.*

Item (b) of Theorem 2.32 above is a consequence of the proof of [131, Proposition 9]. When coming to range spaces, we have the following positive results.

Theorem 2.33 ([88, Theorem 3.15]) *Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces. If $(\mathcal{X}, \mathcal{Y})$ has the BPBp for compact operators, then so do*

(a) $(\mathcal{X}, L_\infty(\mu, \mathcal{Y}))$ *for every σ-finite positive measure μ.*
(b) $(\mathcal{X}, C(K, \mathcal{Y}))$.

Notice that we have plenty of examples of pairs $(\mathcal{X}, \mathcal{Y})$ satisfying the BPBp for compact operators from Theorem 2.32 and Theorem 2.33 by applying Theorem 2.27.

Recently, M. D. Acosta and M. Soleimani-Mourchehkhorti initiated the study of the Bishop–Phelps–Bollobás property for positive operators between Banach lattices (see [26, 28–30]). The reader may also check the necessary definitions and background of some of the concepts in the aforementioned papers as well. We summarize next some of their results.

Theorem 2.34 ([26, 28–30]) *The following pairs $(\mathcal{X}, \mathcal{Y})$ have the BPBp for positive operators.*

- $\mathcal{X} = c_0$ *or* $\mathcal{X} = L_\infty(\mu)$ *and* $\mathcal{Y} = L_1(\nu)$*, with μ and ν positive measures ([26, Theorems 1.6 and 1.7]).*
- $\mathcal{X} = c_0$ *or* $\mathcal{X} = L_\infty(\mu)$ *and $\mathcal{Y}$ is a uniformly monotone Banach lattice ([28, Theorems 2.5 and 3.2]; see also [28, Corollary 4.4]).*
- $(C_0(L), \mathcal{Y})$ *if $\mathcal{Y}$ is a uniformly monotone Banach function space, if $\mathcal{Y}$ is a uniformly monotone Banach lattice with a weak unit, or if $\mathcal{X} = C_0(L)$ is separable and $\mathcal{Y}$ is a uniformly monotone Banach lattice ([30, Theorem 2.8 and*

Corollaries 2.11 and 2.12]; see also [30, Proposition 2.13 and Corollary 2.14] for a partial converse).
- *If $\mathcal{X}$ and $\mathcal{Y}$ are finite dimensional Banach lattices ([29, Corollary 2.12]).*

Theorem 2.35 ([29]) *$\mathcal{X}$ has the BPBp for positive functionals when:*

- *$\mathcal{X}$ is uniformly monotone for orthogonal elements (this actually characterizes having a stronger property than the BPBp for positive functionals) ([29, Theorem 2.9]; see also [29, Remark 2.8]).*
- *$\mathcal{X}$ is a finite-dimensional Banach lattice ([29, Corollary 2.13]).*
- *$\mathcal{X}$ is strongly monotone and has the hereditary norm-attaining property ([29, Theorem 2.16]). In particular, $\mathcal{X} = C(K)$ (K compact Hausdorff), $\mathcal{X} = \mathcal{M}(K)$ (K compact Hausdorff) and $\mathcal{X} = L_p(\mu)$ ($1 \leq p < \infty$, μ positive measure) have this property ([29, Corollary 3.2]).*

They also provide some examples of Banach lattices that do not satisfy the BPBp for positive functionals (see [29, Section 3]).

In Hilbert spaces $\mathcal{H}$, Bishop–Phelps–Bollobás type properties have been studied for several classes of operators. In [104, Theirem 4.1], it was achieved a BPBp result on $(\mathcal{H}, \mathcal{H})$ for the Schatten-von Neumann class. In [71], a systematic study of BPBp-like properties was done in complex Hilbert spaces: the BPBpp (see Sect. 4 for the treatment of this property) and its natural adaptation to the numerical radius, the BPBpp-ν (see Sect. 2.7 for a numerical radius version of the BPBp). They proved in particular that complex Hilbert spaces have these two properties for many classes of operators (see [71, Theorems 3.1 and 4.1 and Propositions 3.2, 4.2 and 4.3]). We highlight these results as follows.

Theorem 2.36 ([71]) *Let $\mathcal{H}$ be a complex Hilbert space. Then $\mathcal{H}$ has the BPBpp and the BPBpp-ν for the following classes of operators: operators, self-adjoint operators, compact self-adjoint operators, anti-symmetric operators, unitary operators, normal operators, compact normal operators, compact operators, Schatten-von Neumann operators, positive operators, positive Schatten-von Neumann operators, self-adjoint Schatten-von Neumann operators, normal Schatten-von Neumann operators and compact positive operators.*

We send the reader to [44, Theorem 4.2.12 and Corollary 4.2.13] for new results related to the theorem above.

2.3 For Multilinear Mappings

In this section we will see some important results on the line of the Bishop–Phelps–Bollobás property for multilinear mappings which were obtained in the past few years focused on what has come after [4]. Throughout this section, we write $\mathcal{X}_1, \ldots, \mathcal{X}_N, \mathcal{Y}$ for arbitrary Banach spaces.

To start with, we need to adapt Definition 1.3 for this new context.

Definition 2.37 We say that $(\mathcal{X}_1, \ldots, \mathcal{X}_N; \mathcal{Y})$ has the Bishop–Phelps–Bollobás property for multilinear mappings (BPBp for multilinear mappings, for short) if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $A \in \mathbb{B}(\mathcal{X}_1, \ldots, \mathcal{X}_N; \mathcal{Y})$[1] with $\|A\| = 1$ and $(x_1, \ldots, x_N) \in S_{\mathcal{X}_1} \times \ldots \times S_{\mathcal{X}_N}$ satisfy

$$\|A\,(x_1, \ldots, x_N)\| > 1 - \eta(\varepsilon),$$

there are $B \in \mathbb{B}(\mathcal{X}_1, \ldots, \mathcal{X}_N; \mathcal{Y})$ with $\|B\| = 1$ and $\left(x_1^0, \ldots, x_N^0\right) \in S_{\mathcal{X}_1} \times \ldots \times S_{\mathcal{X}_N}$ such that

$$\left\|B\left(x_1^0, \ldots, x_N^0\right)\right\| = 1, \quad \max_{1 \le j \le N} \|x_j^0 - x_j\| < \varepsilon \quad \text{and} \quad \|B - A\| < \varepsilon. \tag{3}$$

First of all, we need to justify why the study of such a property for multilinear mappings is relevant. In 2009, Yun Sung Choi and Hyun Gwi Song proved that the Bishop–Phelps–Bollobás theorem does not hold for bilinear forms on $\ell_1 \times \ell_1$. Actually, they showed that given the bilinear form $T(e_i, e_j) := 1 - \delta_{ij}$, there is no norm-attaining bilinear form S on $\ell_1 \times \ell_1$ satisfying $\|S - T\| < 1$ (see [81, Theorem 2]). We highlight it below.

Theorem 2.38 ([81, Theorem 2]) *The triple $(\ell_1, \ell_1; \mathbb{K})$ fails the BPBp for bilinear forms.*

Nevertheless, Theorem 2.39 below gives a positive result. In particular, if $\mathcal{X}$ is uniformly convex, then $(\mathcal{X}, \mathcal{Y})$ has the BPBp for operators for every Banach space $\mathcal{Y}$ as in Theorem 2.6.

Theorem 2.39 ([15, Theorem 2.2]) *Let $\mathcal{X}_1, \ldots, \mathcal{X}_N$ be uniformly convex Banach spaces. Then, $(\mathcal{X}_1, \ldots, \mathcal{X}_N; \mathcal{Y})$ has the BPBp for multilinear mappings for every Banach space $\mathcal{Y}$.*

The authors of [15] also gave a complete characterization for the triple $(\ell_1 \times \mathcal{Y}; \mathbb{K})$ to have the Bishop–Phelps–Bollobás property for bilinear forms by using the AHSP for a pair $(\mathcal{X}, \mathcal{X}')$ which we will not treat here. We send the interested reader to [6, Definition 5.7] and the references therein. We give the most relevant (and specific) consequences of it in the next theorem.

Theorem 2.40 ([15, Theorem 3.6]) *The triple $(\ell_1, \mathcal{Y}; \mathbb{K})$ has the BPBp for bilinear forms when*

(a) $\mathcal{Y}$ is uniformly smooth.
(b) $\mathcal{Y}$ is finite-dimensional.
(c) $\mathcal{Y} = C_0(L)$ and, in particular, when $\mathcal{Y} = C(K)$ or $\mathcal{Y} = c_0$.
(d) $\mathcal{Y} = \mathbb{K}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space.

[1] We send the reader back to the first paragraph of the Introduction to check our notation.

The reader can find the proofs of Theorem 2.40.(a)–(d) in [15, Propositions 4.1, 4.2, 4.4 and 4.7], respectively.

Following the line of Theorem 2.38 above, we have the following negative result.

Theorem 2.41 ([15, Proposition 4.8]) *The triple* $(\ell_1, L_1(\mu); \mathbb{K})$ *fails the BPBp for bilinear forms whenever* $L_1(\mu)$ *is infinite-dimensional.*

It is worth mentioning that Yun Sung Choi showed in 1997 that the set of all norm-attaining bilinear forms on $L_1[0,1] \times L_1[0,1]$ is *not* dense in $\mathbb{B}(L_1[0,1], L_1[0,1]; \mathbb{K})$ (see [70, Theorem 3]). This means, in particular, that this triple cannot satisfy the BPBp for bilinear forms.

Theorem 2.42 *The triple* $(L_1[0,1], L_1[0,1]; \mathbb{K})$ *fails the BPBp for bilinear forms.*

On the other hand, the same authors of [15] together with Choi, Kim, and Lee provided a characterization for the triple $(L_1(\mu), \mathcal{Y}; \mathbb{K})$ to have the BPBp for bilinear forms in [12]. Let us notice that we have an additional assumption in [12, Theorem 2.6]: the Banach space $\mathcal{Y}$ is assumed to be Asplund. The tools and techniques used to prove the following result are also based on the AHSP.

Theorem 2.43 ([12, Theorem 2.6 and Corolary 2.7]) *Let* μ *be a* σ*-finite measure such that* $L_1(\mu)$ *is infinite-dimensional. Then,* $(L_1(\mu), \mathcal{Y}; \mathbb{K})$ *has the BPBp for bilinear forms when*

(a) $\mathcal{Y}$ *is uniformly smooth.*
(b) $\mathcal{Y}$ *is finite-dimensional.*
(c) $\mathcal{Y} = c_0$.

Remark 2.44 In the vein of Theorems 2.40 and 2.43, it is worth mentioning that there are positive results on Banach function spaces over a measure space. Indeed, Lucía Agud, José M. Calabuig, Sebastián Lajara, and Enrique A. Sánchez Pérez gave some applications to the BPBp (for operators and bilinear forms) from their results on Gâteaux and Fréchet smoothness, and the uniform smoothness of $L_p(m)$, where $m : \Sigma \longrightarrow \mathcal{X}$ is a vector measure (we invite the reader to visit Section 4 of [31] and check the necessary background in Sections 1, 2 and 3 of that paper).

There is another positive result when it comes to the BPBp for bilinear forms on $C_0(L_1) \times C_0(L_2)$ in the complex case due to Kim, Lee, and Martín, where L_1, L_2 are locally Hausdorff topological spaces (see [132]). Indeed, we have the following result.

Theorem 2.45 ([132, Theorem 2 and Corollary 3]) $(C_0(L_1), C_2(L_2); \mathbb{K})$ *has the BPBp for bilinear forms in the complex case. In particular, so does* $(c_0, c_0; \mathbb{K})$ *(in the complex case).*

We send the reader to [69] for some generalizations on the previous result.

We also have the following positive result when one of the factors is c_0 and the other one is an ℓ_p-space.

Theorem 2.46 ([124, Corollary 2.9]) *The triple* $(c_0, \ell_p; \mathbb{K})$ *has the BPBp for bilinear forms whenever* $1 < p < \infty$.

At this point, one might wonder what happens with Theorem 2.46 when c_0 is replaced by ℓ_1. The answer to this is that we have an analogous result, and it follows immediately from Theorem 2.8.(b) and [87, Proposition 2.6]. This was also observed explicitly in [82, Corollary 1.1]. We highlight this in the following result.

Theorem 2.47 ([82, Corollary 1.1]) *The triple* $(\ell_1, \ell_p; \mathbb{K})$ *has the BPBp for bilinear forms whenever* $1 < p < \infty$.

We have seen in Theorem 2.27 that when $\mathcal{Y}$ is an isometric L_1-predual, we have that the pair $(\mathcal{X}, \mathcal{Y})$ has the BPBp for compact operators. In turns out that, by using the metric approximation property on the L_1-predual, we have the following technical result for (compact) multilinear mappings which provides some positive results for the BPBp for this class of functions.

Theorem 2.48 ([87, Theorem 2.9]) *Suppose that* $\mathcal{Y}$ *is an* L_1*-predual and that the* N*-tuple* $(\mathcal{X}_1, \ldots, \mathcal{X}_N; \mathbb{K})$ *has the BPBp for multilinear forms. Then, the* $(N+1)$*-tuple* $(\mathcal{X}_1, \ldots, \mathcal{X}_N; \mathcal{Y})$ *has the BPBp for compact multilinear mappings.*

In particular, we have the following corollary.

Corollary 2.49 *Suppose that* $\mathcal{Y}$ *is an* L_1*-predual. The triple* $(\mathcal{X}, \mathcal{Z}; \mathcal{Y})$ *has the BPBp for compact bilinear mappings when*

(a) $\mathcal{X} = C_0(L_1)$ *and* $\mathcal{Z} = C_0(L_2)$ *in the complex case.*
(b) both $\mathcal{X}, \mathcal{Z}$ *are uniformly convex Banach spaces.*
(c) $\mathcal{X} = \ell_1$ *and*

- *(c.1)* $\mathcal{Z}$ *is uniformly smooth.*
- *(c.2)* $\mathcal{Z}$ *is finite-dimensional.*
- *(c.3)* $\mathcal{Z} = C_0(L)$ *and, in particular,* $\mathcal{Z} = C(K)$ *or* $\mathcal{Z} = c_0$.
- *(c.4)* $\mathcal{Z} = \mathbb{K}(\mathcal{H})$*, where* $\mathcal{H}$ *is a Hilbert space.*

(d) $\mathcal{X} = L_1(\mu)$*, where* μ *is a* σ*-finite measure such that* $L_1(\mu)$ *is infinite-dimensional and*

- *(d.1)* $\mathcal{Z}$ *is uniformly smooth.*
- *(d.2)* $\mathcal{Z}$ *is finite-dimensional.*
- *(d.3)* $\mathcal{Z} = c_0$.

(e) $\mathcal{X} = c_0$ *and* $\mathcal{Z} = \ell_p$ *for* $1 < p < \infty$.

Let us observe that items (c) and (d) of Corollary 2.49 are immediate consequences of Theorem 2.40 and Theorem 2.43, respectively. Item (b) follows from Theorem 2.39 and item (a) from Theorem 2.45. Item (e) follows from Theorem 2.46.

Remark 2.50 In [131], Kim, Lee, and Martín defined a more general geometric property than the ASHP in order to characterize the pairs $(\mathcal{X}, \mathcal{Y})$ such that $(\ell_1(\mathcal{X}), \mathcal{Y})$ satisfies the BPBp for operators. In the same directions, in [87] we defined the analogous property for bilinear forms with the idea to give a characterization for the triples of the form $(\ell_1(\mathcal{X}), \mathcal{Y}; \mathbb{K})$ to have the BPBp for bilinear forms.

For symmetric bilinear forms (Hermitian forms), the only positive known result is the following one:

Theorem 2.51 ([104]) *Let $\mathcal{H}$ be a Hilbert space.*

- *$(\mathcal{H}, \mathcal{H})$ has the BPBp for continuous symmetric bilinear forms ([104, Theorems 3.2 and 3.4]).*
- *If $\mathcal{H}$ is complex, then $(\mathcal{H}, \mathcal{H})$ has the BPBp for continuous Hermitian forms ([104, Corollary 2.2]).*

For negative results on the norm-attaining multilinear mappings (and, in particular, for the Bollobás version for this class of functions) we send the interested reader to [8, 47, 119].

2.4 For Homogeneous Polynomials

In this section, we present and discuss the progress on the Bishop–Phelps–Bollobás property for homogeneous polynomials. Again, the definition is easily adapted and we highlight it as follows. We send the reader back to the first paragraph of the Introduction to check our notation.

Definition 2.52 We say that the pair $(\mathcal{X}; \mathcal{Y})$ has the Bishop–Phelps–Bollobás property for N-homogeneous polynomials if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $P \in \mathcal{P}(^N\mathcal{X}; \mathcal{Y})$ with $\|P\| = 1$ and $x_0 \in S_{\mathcal{X}}$ satisfy

$$\|P(x_0)\| > 1 - \eta(\varepsilon),$$

there are $Q \in \mathcal{P}(^N\mathcal{X}; \mathcal{Y})$ with $\|Q\| = 1$ and $x_1 \in S_{\mathcal{X}}$ such that

$$\|Q(x_1)\| = 1, \quad \|x_1 - x_0\| < \varepsilon \quad \text{and} \quad \|Q - P\| < \varepsilon.$$

As in the case of operators (see Theorem 2.6) and multilinear mappings (see Theorem 2.39), we have the following universal result for polynomials.

Theorem 2.53 ([12, Theorem 3.1]) *Let $\mathcal{X}$ be a uniformly convex Banach space. Then, $(\mathcal{X}; \mathcal{Y})$ has the BPBp for N-homogeneous polynomials for every Banach space $\mathcal{Y}$.*

The technique used to prove Theorem 2.53 is based on the original Lindenstrauss argument. It is not known whether the analogous result for symmetric N-linear forms holds (see Question 8).

Concerning property β of Lindenstrauss, we have the following theorem. Item (a) of Theorem 2.54 below is an immediate consequence of [87, Proposition 2.3.(iii)].

Theorem 2.54 ([12, Proposition 3.3]) *Let $\mathcal{X}$ be a Banach space and $\mathcal{Y}$ be a Banach space with property β of Lindenstrauss. If $(\mathcal{X};\mathbb{K})$ has the BPBp for N-homogeneous polynomials, so does $(\mathcal{X};\mathcal{Y})$. In particular, $(\mathcal{X};\mathcal{Y})$ has the BPBp for N-homogeneous polynomials when*

(a) $\mathcal{X}$ is finite-dimensional.
(b) $\mathcal{X}$ is uniformly convex.

There is no Bishop–Phelps theorem either in scalar and vector-valued polynomials setting. For negative results on the norm-attaining N-homogeneous polynomials (and, in particular, for the Bollobás version for this class of functions) we send the interested reader to [47, 119].

2.5 For Holomorphic Functions

In this section, we consider the Bollobás theorem for holomorphic functions. Let us start by saying that it seems that not much has been done in this direction. Nevertheless, from our point of view, holomorphic functions deserve special attention and the reason is quite simple: it requires a complete different approach and interesting techniques.

To start with, we highlight a non-linear version of the Bishop–Phelps–Bollobás theorem. This was done in [86]. It was proven that a Bishop–Phelps–Bollobás type theorem holds on $\mathcal{A}_{w^*u}(B_{\mathcal{X}'})$ whenever $\mathcal{X}'$ is either a uniformly convex or a locally c-uniformly convex, order-continuous sequence space. For necessary notation, we send the reader to the first paragraph of Sect. 1.

Theorem 2.55 ([86, Theorem 1]) *Let $\mathcal{X}$ be a complex Banach space. Suppose that Γ_s is norm dense in $S_{\mathcal{X}'}$. Then, given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $f \in \mathcal{A}_{w^*u}(B_{\mathcal{X}'})$ with $\|f\|_\infty = 1$ and $x_0^* \in S_{\mathcal{X}'}$ satisfy*

$$|f(x_0^*)| > 1 - \eta(\varepsilon),$$

there are $g \in \mathcal{A}_{w^*u}(B_{\mathcal{X}'})$ with $\|g\|_\infty = 1$ and $x_1^* \in S_{\mathcal{X}'}$ such that

$$|g(x_1^*)| = 1, \quad \|g - f\|_\infty < \varepsilon, \quad \textit{and} \quad \|x_1^* - x_0^*\| < \varepsilon.$$

In Theorem 2.55, Γ_s represents the set of all strong peak points for $\mathcal{A}_{w^*u}(B_{\mathcal{X}'})$. Besides that, notice that in Theorem 2.55 we are requiring that the set Γ_s is dense in $S_{\mathcal{X}'}$; for results when this happens, we refer to [86, Proposition 5 and 6]. For necessary background we send the reader to [86, pages 8 and 9].

In the same direction, Kim and Lee proved similar results to Theorem 2.55 for the spaces $\mathcal{A}_u(B_{\mathcal{X}})$ and $\mathcal{A}_b(B_{\mathcal{X}})$ under some additional conditions on the complex Banach space $\mathcal{X}$.

Theorem 2.56 ([128, Corollary 8]) *Suppose that $\mathcal{X}$ is either a locally uniformly convex space or a locally c-uniformly convex, order-continuous sequence space. Let $\mathcal{Y}$ be a Banach space. If $\mathcal{A}$ is one of $\mathcal{A}_u(B_{\mathcal{X}})$ or $\mathcal{A}_b(B_{\mathcal{X}})$, then for $\varepsilon \in (0, 1)$, whenever a norm-1 element f in $\mathcal{A}(B_{\mathcal{X}}; \mathcal{Y})$ and an element $x_0 \in B_{\mathcal{X}}$ satisfy*

$$\|f(x_0)\| > 1 - \frac{\varepsilon}{6},$$

there are $x_1 \in S_{\mathcal{X}}$ and a strongly norm-attaining function $g \in \mathcal{A}(B_{\mathcal{X}}; \mathcal{Y})$ such that

$$\|g\| = \|g(x_1)\| = 1, \quad \|x_1 - x_0\| < \varepsilon, \quad \text{and} \quad \|f - g\|_\infty < \varepsilon.$$

Observe that if $\mathcal{X}$ is finite-dimensional (in particular, if $\mathcal{X} = \mathbb{D}$), then $\mathcal{A}_{w^*u}(B_{\mathcal{X}}) = \mathcal{A}_u(B_{\mathcal{X}}) = \mathcal{A}(B_{\mathcal{X}})$, and in particular, the two previous theorems hold for the classical $\mathcal{A}(\mathbb{D})$.

Recently it was proven that the pair $(H^\infty(\mathbb{D}), H^\infty(\mathbb{D}))$ has the BPBp for operators (see [38]). This result is due to Neeru Bala, Kousik Dhara, Jaydeb Sarkar, and Aryaman Sensarma.

Theorem 2.57 ([38, Theorem 3.1]) *The pair $(H^\infty(\mathbb{D}), H^\infty(\mathbb{D}))$ has the BPBp for operators.*

For negative results on the norm-attaining holomorphic functions (and, in particular, for the Bollobás version for this class of functions) we send the reader to very interesting paper [48] due to Daniel Carando and Martin Mazzitelli. More specifically, there is no Bishop–Phelps theorem for $\mathcal{A}_u(B_{\mathcal{X}})$.

2.6 For Lipschitz Mappings

In this section, we present and discuss the progress on the Bishop–Phelps–Bollobás property for Lipschitz mappings. Throughout this section, all the metric spaces will be considered to be complete and all Banach spaces will be considered to be *real*.

A quick glance at the definition of $\|\cdot\|_L$ suggests that the most natural way of defining a norm-attaining Lipschitz mapping is the following. We say that $f \in \mathrm{Lip}_0(M, \mathcal{Y})$ strongly attains its norm if there exist $x, y \in M$, $x \neq y$, such that

$$\|f\|_L = \frac{\|f(x) - f(y)\|}{d(x, y)}.$$

The subset of all Lipschitz mappings in $\mathrm{Lip}_0(M, \mathcal{Y})$ which attain their norms strongly is denoted by $\mathrm{SNA}(M, \mathcal{Y})$ (although the notations $\mathrm{SA}(M, \mathcal{Y})$ and $\mathrm{LipSNA}(M, \mathcal{Y})$ have also been used in the literature). Since there is a natural concept of norm attainment, it makes sense to study density problems or even Bishop–Phelps–Bollobás type properties in this scenario. Nevertheless, in [121, Theorem 2.3] it was shown that if $\mathcal{X}$ is a geodesic pointed metric space (in particular, if $\mathcal{X}$ is any Banach space), then $\mathrm{SNA}(\mathcal{X}, \mathbb{R})$ is *never* dense in $\mathrm{Lip}_0(\mathcal{X}, \mathbb{R})$ and later this was extended to metric length spaces [54].

Theorem 2.58 ([54, Theorem 2.2]) *Let M be a complete length pointed metric space. Then, the set $SNA(M, \mathbb{R})$ is not dense in $Lip_0(M, \mathbb{R})$.*

In fact, this result combined with Proposition 2.69 show that if M is a metric length space and $\mathcal{Y}$ is any other Banach space, a BPBp-like property is not possible (actually, in this case we do not even get density, see [65, Proposition 4.2]). However, this inconvenience has not stopped researchers from developing a rich theory on denseness of norm-attaining Lipschitz mappings by considering domains that are not Banach spaces or using weaker norm attainment concepts (see, for instance, [68, Section 1] for a clean exposition on several and the relations between them). As a matter of fact, as we are going to show throughout this section, many authors came up with several definitions in order to get positive results in this setting.

As far as the authors of this survey know, the study of the density of norm-attaining Lipschitz mappings was initiated independently in [112] and [121], and since then many authors have contributed to the topic. We refer the interested reader to [54, 64–66, 68, 83, 106, 107, 112, 121] and also the nice survey [111, Section 5] for a solid background on the topic. Four of those works also include Bishop–Phelps–Bollobás type properties for Lipschitz mappings. The rest of the section will focus on those results.

We start with a work by Vladimir Kadets, Miguel Martín, and Mariia Soloviova, who focused their study on the case where M is a Banach space, $\mathcal{Y} = \mathbb{R}$, and dealt with some weaker forms of norm attainment (see [121]). First, for the set of continuous seminorms on $\mathcal{X}$, $\mathrm{Sem}(\mathcal{X})$, they got a variation of a BPBp-like result.

Proposition 2.59 ([121, Proposition 3.4]) *Let $\mathcal{X}$ be a Banach space. Then for every $\varepsilon > 0$, there is $\delta > 0$ such that for every $p_0 \in Sem(\mathcal{X})$ with $\|p_0\| = 1$ and every $x_0 \in S_{\mathcal{X}}$ with $p_0(x_0) > 1 - \delta$, there exist $p \in Sem(\mathcal{X})$ with $\|p\| = 1$ and $x \in S_{\mathcal{X}}$ such that*

$$p(x) = 1 = \|p\|, \quad \|x - x_0\| < \varepsilon, \quad \|p - P_0\|_\infty = \sup_{y \in S_{\mathcal{X}}} |p(y) - p_0(y)| < \varepsilon.$$

Note that, in particular, this implies the uniform density of the set of norm attaining seminorms.

We now need the definition of a natural (but weaker) form of norm attainment in Lipschitz functionals in the setting of Banach spaces.

Definition 2.60 ([121, Definition 1.3]) Let $\mathcal{X}$ be a real Banach space. A Lipschitz functional $g \in \text{Lip}_0(\mathcal{X})$ attains its norm at the direction $u \in S_{\mathcal{X}}$ if there is a sequence of pairs $\{(x_n, y_n)\}$ in $\mathcal{X} \times \mathcal{X}$, with $x_n \neq y_n$, such that

$$\lim_{n\to\infty} \frac{x_n - y_n}{\|x_n - y_n\|} = u \quad \text{and} \quad \lim_{n\to\infty} \frac{g(x_n) - g(y_n)}{\|x_n - y_n\|} = \|g\|.$$

In this case, we say that g attains its norm directionally. The set of all those $f \in \text{Lip}_0(\mathcal{X})$ that attain their norm directionally is denoted by $\text{DA}(\mathcal{X})$.

This kind of norm attainment is a natural approach since it coincides with the usual norm attainment if g is linear. Also, if $\mathcal{X}$ is finite-dimensional, then $\text{DA}(\mathcal{X}) = \text{Lip}_0(\mathcal{X})$. In [121], the authors defined a BPBp-like property for Lipschitz mappings involving this kind of norm attainment.

Definition 2.61 ([121, Definition 1.4]) A Banach space $\mathcal{X}$ has the directional Bishop–Phelps–Bollobás property for Lipschitz functionals ($\mathcal{X} \in$ LipBPB, for short), if for every $\varepsilon > 0$, there is $\delta > 0$ such that for every $f \in \text{Lip}_0(\mathcal{X})$ with $\|f\| = 1$ and for every $x, y \in \mathcal{X}$ with $x \neq y$ satisfying

$$\frac{f(x) - f(y)}{\|x - y\|} > 1 - \delta,$$

there is $g \in \text{Lip}_0(\mathcal{X})$ with $\|g\| = 1$ and there is $u \in S_{\mathcal{X}}$ such that g attains its norm at the direction u,

$$\|g - f\| < \varepsilon, \quad \text{and} \quad \left\| \frac{x - y}{\|x - y\|} - u \right\| < \varepsilon.$$

We also need to define another kind of norm attainment and its associated BPBp-like property.

Definition 2.62 ([121, Definition 1.3]) Let $\mathcal{X}$ be a real Banach space. A Lipschitz functional $g \in \text{Lip}_0(\mathcal{X})$ attains its norm at a point $v \in \mathcal{X}$ at the direction $u \in S_{\mathcal{X}}$ if there is a sequence of pairs $\{(x_n, y_n)\}$ in $\mathcal{X} \times \mathcal{X}$, with $x_n \neq y_n$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = v$, such that

$$\lim_{n\to\infty} \frac{x_n - y_n}{\|x_n - y_n\|} = u \quad \text{and} \quad \lim_{n\to\infty} \frac{g(x_n) - g(y_n)}{\|x_n - y_n\|} = \|g\|.$$

In this case, we say that g attains its norm locally-directionally. The set of all those $f \in \text{Lip}_0(\mathcal{X})$ that attain their norm locally-directionally is denoted by $\text{LDA}(\mathcal{X})$.

Definition 2.63 ([121, Definition 1.4]) A Banach space $\mathcal{X}$ has the local directional Bishop–Phelps–Bollobás property for Lipschitz functionals ($\mathcal{X} \in$ LLipBPB for short), if for every $\varepsilon > 0$, there is $\delta > 0$ such that for every $f \in \text{Lip}_0(\mathcal{X})$ with

$\|f\| = 1$ and for every $x, y \in \mathcal{X}$ with $x \neq y$ satisfying

$$\frac{f(x) - f(y)}{\|x - y\|} > 1 - \delta,$$

there is $g \in \mathrm{Lip}_0(\mathcal{X})$ with $\|g\| = 1$ and there are $v \in \mathcal{X}$ and $u \in S_{\mathcal{X}}$ such that g attains its norm at the point v at the direction u,

$$\|g - f\| < \varepsilon, \quad \left\| \frac{x - y}{\|x - y\|} - u \right\| < \varepsilon, \quad \text{and} \quad \mathrm{dist}(v, [x, y]) < \varepsilon.$$

With the help of some lemmas (that might be of interest in themselves) involving the LipBPB and the LLipBPB (see [121, Lemmas 4.1 and 4.4]), the authors showed the following.

Theorem 2.64 ([121, Theorem 5.3]) *Every uniformly convex Banach space $\mathcal{X}$ has the local directional Bishop–Phelps–Bollobás property for Lipschitz functionals.*

Some time after, Rafael Chiclana and Miguel Martín did a systematic study of a vector-valued BPBp property for Lipschitz mappings for the strong norm attainment with the domain being a metric space. We begin with the following definition.

Definition 2.65 ([65, Definition 1.1]) Let M be a pointed metric space and let $\mathcal{Y}$ be a Banach space. We say that the pair $(M, \mathcal{Y})$ has the *Lipschitz Bishop–Phelps–Bollobás property* (Lip-BPB property for short) if given $\varepsilon > 0$, there is $\eta(\varepsilon) > 0$ such that for every norm-one $F \in \mathrm{Lip}_0(M, \mathcal{Y})$ and every $p, q \in M$, $p \neq q$ such that

$$\|F(p) - F(q)\| > (1 - \eta(\varepsilon))d(p, q),$$

there exist $G \in \mathrm{Lip}_0(M, \mathcal{Y})$ and $r, s \in M$, $r \neq s$, such that

$$\frac{\|G(r) - G(s)\|}{d(r, s)} = \|G\|_L = 1, \quad \|G - F\|_L < \varepsilon, \quad \frac{d(p, r) + d(q, s)}{d(p, q)} < \varepsilon.$$

If this holds for a class of linear operators from $\mathcal{F}(M)$ to $\mathcal{Y}$, we will say that the pair $(M, \mathcal{Y})$ has the Lip-BPB property for that class.

The authors also give a reformulation of that definition in [65, Remark 1.2.(a)] in terms of linear operators associated to the Lipschitz mappings. The first result for this property is the following.

Theorem 2.66 ([65, Theorem 2.1]) *Let M be a finite pointed metric space and let $\mathcal{Y}$ be a Banach space. If $(\mathcal{F}(M), \mathcal{Y})$ has the BPBp, then $(M, \mathcal{Y})$ has the Lip-BPB property.*

This happens for example if $\mathcal{Y}$ is finite-dimensional (see [65, Corollary 2.3]). The authors noted that we can not remove the condition of $(\mathcal{F}(M), \mathcal{Y})$ having the BPBp (see [65, Example 2.5]) or the finitude of M (see [65, Example 2.6]).

The next results will use a series of concepts related to pointed metric spaces. We refer the reader to [65, Section 3] and the references cited there for the definitions and necessary background.

Theorem 2.67 ([65, Theorem 3.3]) *Let M be a uniformly Gromov concave pointed metric space. Then, $(M, \mathcal{Y})$ has the Lip-BPB property for every Banach space $\mathcal{Y}$.*

This is the case for example when M is concave and $\mathcal{F}(M)$ has property α (see Definition 2.3), if M is concave and finite, if M is a pointed ultrametric space, and if M is a Hölder pointed metric space ([65, Corollaries 3.4, 3.5, 3.6, 3.7]).

In [65, Example 2.6] they showed that the BPBp of $(\mathcal{F}(M), \mathcal{Y})$ does not imply the Lip-BPB property of $(M, \mathcal{Y})$ in general. The other implication does not hold in general either.

Proposition 2.68 ([65, Proposition 3.9]) *Let M be a finite pointed metric space with more than two points. Then, there exists a Banach space $\mathcal{Y}$ such that $(\mathcal{F}(M), \mathcal{Y})$ fails the BPBp.*

Contrary to a conjecture they had, they showed that there is a Gromov concave pointed metric space such that $\mathcal{F}(M)$ has the RNP but $(M, \mathbb{R})$ fails the Lip-BPB property (see [65, Example 3.11]). They also studied relations between the scalar-valued and the vector-valued versions of the property. We will summarize some of the main results.

Proposition 2.69 ([65, Proposition 4.1]) *Let M be a pointed metric space. Suppose that there exists a Banach space $\mathcal{Y} \neq 0$ such that $(M, \mathcal{Y})$ has the Lip-BPB property. Then $(M, \mathbb{R})$ has the Lip-BPB property.*

Proposition 2.70 ([65, Proposition 4.4]) *Let M be a pointed metric space such that $(M, \mathbb{R})$ has the Lip-BPB property, and let $\mathcal{Y}$ be a Banach space satisfying property β of Lindenstrauss. Then $(M, \mathcal{Y})$ has the Lip-BPB property.*

The previous result does not hold for property quasi-β even though we have density of SNA in that case ([65, Proposition 4.7, Example 4.9]).

Finally, they managed to adapt some modifications of results from [65, Sections 3 and 4] and [88] to compact Lipschitz operators ([65, Propositions 4.10, 4.13, 4.16, 4.17]).

In a recent work by Geunsu Choi, Yun Sung Choi, and Miguel Martín ([68]), the authors studied a vector-valued variation of the LLipBPB (see Definition 2.63) for a slightly modified type of norm attainment as well as a version of it for compact operators. For a Banach space $\mathcal{X}$, let us denote $\tilde{\mathcal{X}}$ the set $\{(x, y) \in \mathcal{X}^2 : x \neq y\}$.

Definition 2.71 ([68, Definition 1.5]) We say that $f \in \text{Lip}_0(\mathcal{X}, \mathcal{Y})$ attains its norm locally directionally at the point $\overline{x} \in \mathcal{X}$ in the direction $u \in S_{\mathcal{X}}$ toward $z \in \mathcal{Y}$ if there exists $\{(x_n, y_n)_{n=1}^{\infty}\} \subseteq \tilde{\mathcal{X}}$ such that

$$\frac{f(x_n) - f(y_n)}{\|x_n - y_n\|} \to z \text{ with } \|z\| = \|f\|, \qquad \frac{x_n - y_n}{\|x_n - y_n\|} \to u \text{ and } x_n, y_n \to \overline{x}.$$

We denote by LDirA$(\mathcal{X}, \mathcal{Y})$ the set of every $f \in \mathrm{Lip}_0(\mathcal{X}, \mathcal{Y})$ which attains its norm locally directionally at some point $\overline{x} \in \mathcal{X}$ in some direction $u \in S_{\mathcal{X}}$ toward some point $z \in \mathcal{Y}$.

Definition 2.72 ([121, Definition 1.4]) A pair of Banach spaces $(\mathcal{X}, \mathcal{Y})$ is said to have the local directional Bishop–Phelps–Bollobás property for Lipschitz mappings (in short, LDirA-BPBp) if for every $\varepsilon > 0$, there is $\eta > 0$ such that whenever $f \in S_{\mathrm{Lip}_0(\mathcal{X},\mathcal{Y})}$ and $x, y \in \mathcal{X} \times \mathcal{X}$ with $x \neq y$ satisfy

$$\frac{\|f(x) - f(y)\|}{\|x - y\|} > 1 - \eta,$$

there exist $g \in S_{\mathrm{Lip}_0(\mathcal{X},\mathcal{Y})}$, $z \in S_{\mathcal{Y}}$, $u \in S_{\mathcal{X}}$ and $\overline{x} \in \mathcal{X}$ such that g attains its norm locally directionally at the point $\overline{x}$ in the direction u toward z,

$$\|g - f\| < \varepsilon, \quad \left\| \frac{x - y}{\|x - y\|} - u \right\| < \varepsilon, \quad \text{and} \quad \mathrm{dist}(\overline{x}, [x, y]) < \varepsilon.$$

Their main result extends [121, Theorem 5.3] in the case of compact operators.

Theorem 2.73 ([68, Theorem 4.1]) *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces such that $\mathcal{X}$ is uniformly convex and $(\mathcal{F}(\mathcal{X}), \mathcal{Y})$ has the BPBp for compact operators. Then, the pair $(\mathcal{X}, \mathcal{Y})$ has the LDirA-BPBp for Lipschitz compact mappings. In fact, we have something more: for every $\varepsilon > 0$, there exists $\eta > 0$ such that for any positive function $\rho : \tilde{\mathcal{X}} \to \mathbb{R}$ and whenever $f \in S_{Lip_{0\mathcal{K}}(\mathcal{X},\mathcal{Y})}$ and $(x, y) \in \tilde{\mathcal{X}}$ satisfy*

$$\frac{\|f(x) - f(y)\|}{\|x - y\|} > 1 - \eta,$$

there exist $g \in S_{Lip_{0\mathcal{K}}(\mathcal{X},\mathcal{Y})}$, $z \in S_{\mathcal{Y}}$, $u \in S_{\mathcal{X}}$ and $\overline{x} \in \mathcal{X}$ such that g attains its norm locally directionally at the point $\overline{x}$ in the direction u toward z, $\|g - f\| < \varepsilon$, $\left\| u - \frac{x-y}{\|x-y\|} \right\| < \varepsilon$ and $\mathrm{dist}(\overline{x}, [x, y]) < \varepsilon \rho(x, y)$.

In particular, if $\mathcal{X}$ is uniformly convex and $\mathcal{Y}$ has property β of Lindenstrauss or satisfies that $\mathcal{Y}' = L_1(\mu)$ for some μ, then $(\mathcal{X}, \mathcal{Y})$ has the LDirA-BPBp for Lipschitz compact mappings (see [68, Corollary 4.4]), and the same is also true if $\mathcal{Y}$ is a finite-dimensional polyhedral Banach space (see [68, Corollary 4.11]). The condition in the previous theorem that the pair $(\mathcal{F}(\mathcal{X}), \mathcal{Y})$ has the BPBp for compact operators is sufficient, but not necessary (see [68, Proposition 4.8] and the discussion right after it).

There is also a slightly different version of Theorem 2.73 where the domain space is a Hilbert space.

Theorem 2.74 ([68, Theorem 4.5]) *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{Y}$ be a Banach space. Suppose that $(\mathcal{F}(M), \mathcal{Y})$ has the BPBp for compact operators. Then for every $\varepsilon > 0$, there exists $\eta > 0$ such that whenever $f \in S_{Lip_{0\mathcal{K}}(\mathcal{H},\mathcal{Y})}$ and $(x, y) \in \tilde{\mathcal{H}}$ satisfy*

$$\frac{\|f(x) - f(y)\|}{\|x - y\|} > 1 - \eta,$$

there exist $g \in S_{Lip_{0\mathcal{K}}(\mathcal{H},\mathcal{Y})}$, $z \in S_{\mathcal{Y}}$ and $\overline{x} \in \mathcal{H}$ such that g attains its norm locally directionally at the point $\overline{x}$ in the direction $\frac{x-y}{\|x-y\|}$ toward z, $\|g - f\| < \varepsilon$, and $dist(\overline{x}, [x, y]) < \varepsilon \max\{\|x\|, \|y\|\}$.

In particular, the same holds if $\mathcal{Y}$ has property β of Lindenstrauss or if $\mathcal{Y}'$ is an $L_1(\mu)$ space (see [68, Corollary 4.6]), and the same is also true if $\mathcal{Y}$ is a finite-dimensional polyhedral Banach space (see [68, Corollary 4.12]).

Finally, they got some stronger versions of Theorems 2.73 and 2.74 for the case when $\mathcal{X} = \mathbb{R}$ (see [68, Propositions 4.7 and 4.8, Corollaries 4.9 and 4.10]).

To conclude this section, let us note that a very recent work by Rafael Chiclana and Miguel Martín ([66]) studied stability properties for the Bishop–Phelps–Bollobás property for Lipschitz mappings (see Definition 2.65). Let us start by defining the sum of a family of pointed metric spaces.

Definition 2.75 ([66, Definition 2.1], from [149, Definition 1.13]) Given a family of pointed metric spaces $\{(M_i, d_i)\}_{i\in I}$, the (metric) sum of the family is the disjoint union of all M_i's, identifying the base points, endowed with the following metric d: $d(x, y) = d_i(x, y)$ if both $x, y \in M_i$, and $d(x, y) = d_i(x, 0) + d_j(0, y)$ if $x \in M_i$, $y \in M_j$ and $i \neq j$. We will write $\bigsqcup_{i\in I} M_i$ to denote the sum of the family of metric spaces.

Proposition 2.76 ([66, Proposition 2.2]) *Let $M = M_1 \bigsqcup M_2$ be the sum of two pointed metric spaces and let $\mathcal{Y}$ be a Banach space. If the pair $(M, \mathcal{Y})$ has the Lip-BPB property, then so do $(M_1, \mathcal{Y})$ and $(M_2, \mathcal{Y})$.*

The version of this proposition for compact Lipschitz mappings remains true (see [66, Proposition 2.6]). However, none of the converses hold (see [66, Example 2.4]).

They also provided an extension of [65, Proposition 4.4]. We will include the result for the sake of completeness, but we refer the reader to [56] for the necessary definitions and background (see also [66, Section 3]) .

Theorem 2.77 ([66, Theorem 3.5]) *Let M be a pointed metric space such that $(M, \mathbb{R})$ has the Lip-BPB property. Let $\mathcal{Y}$ be a Banach space in ACK_ρ with associated 1-norming set $\Gamma \subseteq B_{\mathcal{Y}'}$, and let $\varepsilon > 0$. Then, there exists $\eta(\varepsilon, \rho) > 0$ such that if we take $\hat{T} \in L(\mathcal{F}(M), \mathcal{Y})$ a Γ-flat operator with $\|T\|_L = 1$ and $m \in Mol(M)$ satisfying $\|\hat{T}(m)\| > 1 - \eta(\varepsilon, \rho)$, then there exist an operator $\hat{S} \in L(\mathcal{F}(M), \mathcal{Y})$*

and a molecule $u \in Mol(M)$ *such that*

$$\|\hat{S}(u)\| = \|S\|_L = 1, \quad \|m - u\| < \varepsilon, \quad \|T - S\|_L < \varepsilon.$$

As a consequence, if M is a pointed metric space such that $(M, \mathbb{R})$ has the Lip-BPB property, then we get several pairs of spaces satisfying the Lip-BPB property for Γ-flat operators (see [66, Corollary 3.6] for the details). The inconvenience of having to work with Γ-flat operators disappears if we restrict the results to the compact operator case.

Proposition 2.78 ([66, Proposition 3.12]) *Let* M *be a pointed metric space such that* $(M, \mathbb{R})$ *has the Lip-BPB property and let* $\mathcal{Y}$ *be an* ACK_ρ *Banach space. Then, the pair* $(M, \mathcal{Y})$ *has the Lip-BPB for Lipschitz compact mappings.*

And again, a series of consequences can be derived from this (see [66, Corollary 3.13] for the details).

Using some results from [66] and [65, Proposition 4.16], we get the following result.

Corollary 2.79 ([66, Corollary 3.17]) *Let* M *be a pointed metric space and let* $\mathcal{Y}$ *be a Banach space such that the pair* $(M, \mathcal{Y})$ *has the Lip-BPB property for Lipschitz compact mappings.*

(1) For every compact Hausdorff topological space K*, the pair* $(M, C(K, \mathcal{Y}))$ *has the Lip-BPB property for Lipschitz compact mappings.*
(2) For $1 \leq p < \infty$*, if the pair* $(M, \ell_p(\mathcal{Y}))$ *has the Lip-BPB property for Lipschitz compact mappings, then so does* $(M, L_p(\mu, \mathcal{Y}))$ *for every positive measure* μ*.*
(3) For every σ*-finite positive measure* μ*, the pair* $(M, L_\infty(\mu, \mathcal{Y}))$ *has the Lip-BPB property for Lipschitz compact mappings.*

Finally, they study stability results for absolute sums of codomains.

Proposition 2.80 ([66, Proposition 4.3]) *Let* M *be a pointed metric space,* $\mathcal{Y}$ *be a Banach space and* $\mathcal{Y}_1$ *be an absolute summand of* $\mathcal{Y}$*. If the pair* $(M, \mathcal{Y})$ *has the Lip-BPB property with a function* $\varepsilon \mapsto \eta(\varepsilon)$*, then so does* $(M, \mathcal{Y}_1)$ *with the same function.*

As a consequence, we get an universal mapping η for universal Lip-BPB domain spaces.

Corollary 2.81 ([66, Corollary 4.4]) *Let* M *be a pointed metric space such that* $(M, \mathcal{Y})$ *has the Lip-BPB property for all Banach spaces* $\mathcal{Y}$*. Then, there exists a function* $\eta_M(\varepsilon)$*, which only depends on* M*, such that the pair* $(M, \mathcal{Y})$ *has the Lip-BPB property witnessed by the function* $\eta_M(\varepsilon)$ *for evey Banach space* $\mathcal{Y}$*.*

A result in the same direction is the following.

Proposition 2.82 ([66, Proposition 4.6]) *Let* M *be a pointed metric space,* $\mathcal{Y}$ *a Banach space, and* K *a compact Hausdorff topological space. If* $(M, C(K, \mathcal{Y}))$ *has* *the Lip-BPB property witnessed by a function $\eta(\varepsilon)$, then $(M, \mathcal{Y})$ has the Lip-BPB property witnessed by the same function.*

It is worth noting that the last 3 results are still valid in the case of compact operators (see [66, Proposition 4.8] and the discussion before it for the details).

Let $\mathcal{Y} = [\bigoplus_{i\in I} \mathcal{Y}_i]_{c_0}$ or $\mathcal{Y} = [\bigoplus_{i\in I} \mathcal{Y}_i]_{\ell_\infty}$ for some family of Banach spaces $\{\mathcal{Y}_i\}_{i\in I}$ and let M be a pointed metric space. By Proposition 2.80, if $(M, \mathcal{Y})$ has the Lip-BPB property, then all pairs $(M, \mathcal{Y}_i)$ have it with the same function η. We have the following result regarding the reverse implication:

Proposition 2.83 ([66, Proposition 4.9]) *Let M be a pointed metric space, let $\{\mathcal{Y}_i\}_{i\in I}$ be a family of Banach spaces and let $\mathcal{Y} = [\bigoplus_{i\in I} \mathcal{Y}_i]_{c_0}$ or $\mathcal{Y} = [\bigoplus_{i\in I} \mathcal{Y}_i]_{\ell_\infty}$. Assume that $(M, \mathcal{Y}_i)$ has the Lip-BPB property with the function $\eta_i(\varepsilon)$ for each $i \in I$. If $\inf\{\eta_i(\varepsilon) : i \in I\} > 0$ for every $\varepsilon > 0$, then $(M, \mathcal{Y})$ has the Lip-BPB property.*

Finally, let us notice that the compact operators version of the previous result is also true (see [66, Proposition 4.11]).

2.7 For Numerical Radius

In this section we will discuss an adaptation of the Bishop–Phelps–Bollobás property for the numerical radius.

Brailey Sims, in his 1972 Ph.D. dissertation (see [146]), raised a question that is, in nature, related to the one that Lindenstrauss tackled in 1963 [137]: the norm-denseness of the set of numerical radius attaining operators on a Banach space $\mathcal{X}$ (we will define this concept shortly). Ever since, many authors have made contributions regarding this question, such as Ira David Berg, Brailey Sims (see [40]), Carmen Silvia Cardassi (see [50–52]), María Dolores Acosta, Francisco José Aguirre, Rafael Payá, and Manuel Ruiz Galán (see for instance [2, 3, 7, 23–25, 141]). It is also worth noting for the interested reader that M. D. Acosta initiated a systematic study of this question in her nice Ph.D. dissertation [1].

Similar to what happened with the study of norm-attaining operators, it is a natural question whether or not we can have Bishop–Phelps–Bollobás type theorems for the numerical radius on some Banach spaces. The study of this question was first addressed in 2013 by Antonio José Guirao and Olena Kozhushkina in [115], where, paralleling the work [9], they introduced and studied the Bishop–Phelps–Bollobás property for numerical radius. We need some background before proceeding.

Given a Banach space $\mathcal{X}$, the set of states of $\mathcal{X}$ is $\Pi(\mathcal{X}) := \{(x, x^*) \in S_{\mathcal{X}} \times S_{\mathcal{X}'} : x^*(x) = 1\}$. Given an operator $T \in \mathbb{B}(\mathcal{X})$, its numerical radius is defined as $\nu(T) = \sup\{|x^*(T(x))| : (x, x^*) \in \Pi(\mathcal{X})\}$. It is immediate to check that ν is a seminorm, and that for all $T \in \mathbb{B}(\mathcal{X})$, we have $0 \leq \nu(T) \leq \|T\|$. The numerical index of a Banach space $\mathcal{X}$ is defined as the following number $n(\mathcal{X}) = \inf\{\nu(T) :$

$T \in \mathbb{B}(\mathcal{X})\} = \max\{k \geq 0 : k\|T\| \leq \nu(T)\}$, which somehow measures how "close" the norm and the numerical radius are for the given space $\mathcal{X}$. In particular, if $n(\mathcal{X}) = 1$, they coincide, and if $n(\mathcal{X}) > 0$, they are equivalent norms. In [134, Subsection 1.1], the authors provide an overview of the main known results about $n(\mathcal{X})$ up to 2016, and in that work, they also introduce the second numerical index of $\mathcal{X}$ as $n'(\mathcal{X}) := \inf\{\nu(T) : T \in \mathbb{B}(\mathcal{X}),\ \|T + Z(\mathcal{X})\| = 1\}$, where $Z(\mathcal{X}) := \{S \in \mathbb{B}(\mathcal{X}) : \nu(S) = 0\}$, and $\|T + Z(\mathcal{X})\|$ is the natural quotient norm in $\mathbb{B}(\mathcal{X})/Z(\mathcal{X})$.

We are now in a position to be able to define the main property of this section.

Definition 2.84 (Combining [115, Definition 1.2] and [130, Definition 5]) A Banach space $\mathcal{X}$ has the weak Bishop–Phelps–Bollobás property for the numerical radius (weak-BPBp-nu, for short) if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that, whenever $T \in \mathbb{B}(\mathcal{X})$ with $\nu(T) = 1$ and $(x, x^*) \in \Pi(\mathcal{X})$ satisfy $|x^*(T(x))| > 1 - \eta(\varepsilon)$, there exist $S \in \mathbb{B}(\mathcal{X})$ and $(y, y^*) \in \Pi(\mathcal{X})$ such that

$$\nu(S) = |y^*(S(y))|, \quad \|x - y\| < \varepsilon, \quad \|x^* - y^*\| < \varepsilon, \quad \|T - S\| < \varepsilon.$$

If, moreover, S can be chosen so that $\nu(S) = 1$, we say that $\mathcal{X}$ has the Bishop–Phelps–Bollobás property for the numerical radius (abbreviated BPBp-nu, although some authors use the notation BPBp-ν as well).

If the conditions from the previous definition hold within a subclass of $\mathbb{B}(\mathcal{X})$, we say that $\mathcal{X}$ has the (weak)-BPBp-nu for that class of operators (see [19]).

Acosta exhibits a nice overview of the main known results about the BPBp-nu in her survey [6, Section 6]. For the sake of completeness, we will briefly list some of the results about it without further detail. The following results are true both in the real and complex settings unless specified otherwise.

Theorem 2.85 *Let $\mathcal{X}, \mathcal{Y}$ be a Banach spaces, K be a compact Hausdorff topological space and μ be any measure.*

1. *If Γ is any index set, then the spaces $c_0(\Gamma)$ and $\ell_1(\Gamma)$ have the BPBp-nu ([115]).*
2. *$L_1(\mathbb{R})$ has the BPBp-nu ([101, Theorem 9]).*
3. *If $\mathcal{X}$ is finite-dimensional, then it has the BPBp-nu ([130, Proposition 2]).*
4. *$L_1(\mu)$ has the BPBp-nu ([130, Theorem 9]).*
5. *If $\mathcal{X}$ is both uniformly convex and uniformly smooth, then it has the weak-BPBp-nu ([130, Proposition 4]).*
6. *If $n(\mathcal{X}) > 0$, or if $n'(\mathcal{X}) > 0$, then $\mathcal{X}$ has the BPBp-nu if and only if it has the weak-BPBp-nu ([130, Proposition 6] and [134, Theorem 3.2]). In particular, all the $L_p(\mu)$ spaces have the BPBp-nu if $1 < p < \infty$ ([130, Example 8] and [134, Theorem 2.3]).*
7. *If K admits local compensation (see [37, Definition 2.1]), then the real space $C(K)$ has the BPBp-nu ([37, Theorem 2.2]). In particular, if K is metric, the real space $C(K)$ has the BPBp-nu (see [37, Section 3]).*
8. *If $\mathcal{X}$ is strongly lush (see [133, Definition 1.2]; strongly lush spaces include $C(K)$ spaces, $L_1(\mu)$ spaces and finite-codimensional subspaces of $C[0, 1]$) and*

$\mathcal{X} \oplus_1 \mathcal{Y}$ has the weak-BPBp-nu, then $(\mathcal{X}, \mathcal{Y})$ has the BPBp ([133, Theorem 2.1]).

9. *If $\mathcal{Y}$ is strongly lush and $\mathcal{X} \oplus_\infty \mathcal{Y}$ has the weak-BPBp-nu, then $(\mathcal{X}, \mathcal{Y})$ has the BPBp ([133, Theorem 2.3]).*
10. *If μ is finite, if $\mathcal{M}$ is any class of operators between the finite-rank operators and the Riesz-representable operators, then $L_1(\mu)$ has the BPBp-nu for the class $\mathcal{M}$ ([19, Theorem 2.1]). In particular, this includes weakly compact operators and compact operators ([19, Corollary 2.1]).*
11. *If $\mathcal{X}$ has the BPBp-nu and W is an absolute summand of $\mathcal{X}$ of type 1 or ∞ (see [72, Definition 1.2]), then W has the BPBp-nu, and the result is also true for the weak-BPBp-nu and for the compact operators case ([72, Section 4]).*

Regarding item (7), it is worth noting that the complex case and the general case remain open to this day, as far as the authors of this survey know (see Question 9).

The second, third and fourth authors of this survey and Miguel Martín studied recently the BPBp-nu for compact operators, that is, the BPBp-nu but where all the involved operators are compact (see [105]), paralleling the work done in [88], where the BPBp for compact operators was introduced and studied (see also [46], where the authors studied numerical radius attaining compact operators). We will continue this section by summarizing the main results from [105]. First of all, similar to how $n(\mathcal{X})$, $Z(\mathcal{X})$ and $n'(\mathcal{X})$ were defined, one may consider compact operator versions of those concepts, $n_K(\mathcal{X})$, $Z_K(\mathcal{X})$ and $n'(\mathcal{X})$, by restricting the original definitions to the setting of compact operators. Taking this into consideration, one can adapt the original proofs for the BPBp-nu to show that many Banach spaces have the BPBp-nu for compact operators.

Example ([105]) The following spaces have the BPBp-nu for compact operators:

1. Finite-dimensional spaces [130, Proposition 2].
2. c_0 and ℓ_1 (adapting the proofs given in [115, Corollaries 3.3 and 4.2]).
3. $L_1(\mu)$ for every measure μ (using [19, Corollary 2.1] for finite measures and adapting [130, Theorem 9] to compact operators for the general case).
4. $L_p(\mu)$ for every measure μ and all $1 < p < \infty$ (adapting the proofs from [130, Propositions 4 and 6] and [134, Theorem 2.3, Lemma 2.4 and Theorem 3.2]).

Regarding item (4), actually more is known: if a Banach space $\mathcal{X}$ is both uniformly convex and uniformly smooth, then it has the weak-BPBp-nu for compact operators, and if $n_K(\mathcal{X}) > 0$ or $n'_K(\mathcal{X}) > 0$, then having the weak-BPBp-nu for compact operators is equivalent to having the BPBp-nu for compact operators, since [130, Propositions 4 and 6] and [134, Theorem 3.2] remain true when properly adapted to compact operators.

As we mentioned earlier, in [72, Proposition 4.3] it is shown that if a Banach space $\mathcal{X}$ has the BPBp-nu for compact operators, then its absolute summands of type 1 and ∞ also have this property, and actually this is true with the same mapping η. This allows us to carry the property from some spaces to some projections of those spaces. It is natural to wonder whether something can be said in the opposite direction, perhaps by adding suitable extra conditions. In [88, Lemma 2.1] it was presented a tool (based in [120, Lemma 3.1]) that in particular allows to carry the BPBp for compact operators from some projections of a space to the space itself (check Sect. 2.2). In order to get a somewhat similar result for the numerical radius, one needs to control things both in the space and in its dual. The most general result obtained in this direction is the following lemma.

Lemma 2.86 ([105, Lemma 2.1]) *Let $\mathcal{X}$ be a Banach space satisfying that $n_K(\mathcal{X}) > 0$. Suppose that there is a mapping $\eta\colon (0,1) \longrightarrow (0,1)$ such that given $\delta > 0$, $x_1^*, \ldots, x_n^* \in B_{\mathcal{X}'}$ and $x_1, \ldots, x_\ell \in B_{\mathcal{X}}$, we can find norm one operators $\widetilde{P}\colon \mathcal{X} \longrightarrow \widetilde{P}(\mathcal{X})$, $i\colon \widetilde{P}(\mathcal{X}) \longrightarrow \mathcal{X}$ such that for $P := i \circ \widetilde{P}\colon \mathcal{X} \longrightarrow \mathcal{X}$, the following conditions are satisfied:*

(i) $\|P^*(x_j^*) - x_j^*\| < \delta$, *for* $j = 1, \ldots, n$.
(ii) $\|P(x_j) - x_j\| < \delta$, *for* $j = 1, \ldots, \ell$.
(iii) $\widetilde{P} \circ i = \mathrm{Id}_{\widetilde{P}(\mathcal{X})}$.
(iv) $\widetilde{P}(\mathcal{X})$ *satisfies the Bishop–Phelps–Bollobás property for numerical radius for compact operators with the mapping* η.
(v) Either P *is an absolute projection and* i *is the natural inclusion, or* $n_K(\widetilde{P}(\mathcal{X})) = n_K(\mathcal{X}) = 1$.

Then, $\mathcal{X}$ satisfies the BPBp-nu for compact operators.

Throughout [105, Section 2], Lemma 2.86 is used to show that if a Banach space $\mathcal{X}$ with positive compact index can be suitably projected into some net of spaces that have the BPBp-nu for compact operators with a common mapping η, then sometimes it is possible to show that $\mathcal{X}$ also has that property (see [105, Proposition 2.2]). This is used for instance to show that if $n_K(\mathcal{X}) > 0$, then if the spaces $\ell_\infty^n(\mathcal{X})$, $n \in \mathbb{N}$, all have the BPBp-nu for compact operators with the same η, then $c_0(\mathcal{X})$ also has the property [105, Corollary 2.3], and this is actually an equivalence, since the converse implication was already known (see [88, Proposition 4.3]). Another not so direct consequence of Lemma 2.86 is that if a Banach space $\mathcal{X}$ satisfies that $\mathcal{X}'$ is isometrically isomorphic to ℓ_1, then $\mathcal{X}$ has the BPBp-nu for compact operators (see [105, Corollary 2.6]). Let us highlight these two results.

Corollary 2.87 ([105, Corollary 2.3]) *Let $\mathcal{X}$ be a Banach space with $n_K(\mathcal{X}) > 0$. Then, the following statements are equivalent:*

(i) The space $c_0(\mathcal{X})$ *has the BPBp-nu for compact operators.*
(ii) There is a function $\eta : (0,1) \longrightarrow (0,1)$ *such that all the spaces* $\ell_\infty^n(\mathcal{X})$, *with* $n \in \mathbb{N}$, *have the BPBp-nu for compact operators with the function* η.

Moreover, if $\mathcal{X}$ is finite-dimensional, these properties hold whenever $c_0(\mathcal{X})$ or $\ell_\infty(\mathcal{X})$ have the BPBp-nu.

Corollary 2.88 ([105, Corollary 2.6]) *Let $\mathcal{X}$ be a Banach space such that $\mathcal{X}'$ is isometrically isomorphic to ℓ_1. Then $\mathcal{X}$ has the BPBp-nu for compact operators.*

Finally, in [105, Section 3], a series of tools involving some topological procedures and finding some suitable projections, was developed to study the BPBp-nu for compact operators in $C_0(L)$ spaces, where L is a locally compact Hausdorff topological space. By adequately splitting L, one can always find a projection from $C_0(L)$ to some finite-dimensional space ℓ_∞^p ($p \in \mathbb{N}$) that allows to use Lemma 2.86.

Theorem 2.89 ([105, Theorem 3.4]) *Let L be a locally compact space. Given points $\{f_1, \ldots, f_\ell\} \subset C_0(L)$ such that $\|f_j\| \leq 1$ for $j = 1, \ldots, \ell$, and given functionals $\{\mu_1, \ldots, \mu_n\} \subset C_0(L)'$ with $\|\mu_j\| \leq 1$ for $j = 1, \ldots, n$, for each $\varepsilon > 0$ there exists a norm one projection $P\colon C_0(L) \longrightarrow C_0(L)$ satisfying:*

(1) $\|P^*(\mu_j) - \mu_j\| < \varepsilon$, for $j = 1, \ldots, n$,
(2) $\|P(f_j) - f_j\| < \varepsilon$, for $j = 1, \ldots, \ell$,
(3) $P(C_0(L))$ is isometrically isomorphic to ℓ_∞^p for some $p \in \mathbb{N}$.

By using Theorem 2.89, Lemma 2.86 and its consequences, the main result from that paper is achieved.

Theorem 2.90 ([105, Theorem 1.6]) *If L is a locally compact Hausdorff space, then $C_0(L)$ has the BPBp-nu for compact operators.*

This fully answers the question for $C_0(L)$ spaces in the compact operators setting, so in particular, all the $C(K)$ spaces (K compact Hausdorff) and all the $L_\infty(\mu)$ spaces (μ measure) have the BPBp-nu for compact operators. Note that for the case of general operators, only some particular real $C(K)$ spaces (with K compact) are known to have the BPBp-nu (see [37]), and the general case, as well as the complex case, remain open, as we mentioned earlier (see Question 9).

Let us finish this section speaking about multilinear versions of the numerical radius (for multilinear versions on the calculation of numerical radius we refer the reader to [74, 76, 87]). In these papers, the authors show that the equality $v(A) = \|A\|$ holds for multilinear mappings A defined on c_0, ℓ_1, $\mathcal{A}(\mathbb{D})$, and $L_1(\mu)$ for any measure μ. It is natural, then, to study a multilinear version of the BPBp-nu. In [87, Section 5], the authors show that a bilinear version of the BPBp-nu on $L_1(\mu)$ is not possible. We highlight this result as follows.

Theorem 2.91 ([87, Theorem 5.3]) *The infinite-dimensional Banach space $L_1(\mu)$ fails to have the BPBp-nu for bilinear mappings for any measure μ.*

In the same paper, the authors show, however, that the BPBp-nu for multilinear mappings holds for finite-dimensional Banach spaces ([87, Proposition 5.2]). As far as we know there is no further research in this direction and we wonder whether or not there are more spaces $\mathcal{X}$ such that $\mathcal{X}$ satisfies the BPBp-nu for multilinear mappings.

3 Sharpness: The Bishop–Phelps–Bollobás Moduli

The vast majority of results we have seen so far were focused on finding pairs of Banach spaces for which a Bishop–Phelps–Bollobás theorem is valid. However, (almost) none of the proofs investigate the sharpness of the constants associated to those theorems. This interesting question initiated by Bollobás himself (see [42, Remark after Theorem 1]) has also been studied in the recent years. In this section we will briefly discuss some of the results obtained in this direction.

In order to do so, let us define the moduli of Bishop–Phelps–Bollobás in its general form.

Definition 3.1 ([123, Definition 4]) The Bishop–Phelps–Bollobás modulus of a pair of Banach spaces $(\mathcal{X}, \mathcal{Y})$ is the function $\Phi(\mathcal{X}, \mathcal{Y}, \cdot) : (0, 1) \to \mathbb{R}^+$ whose value in $\eta \in (0, 1)$ is defined as the infimum of those $\varepsilon > 0$ such that for every $(x, T) \in B_{\mathcal{X}} \times B_{\mathbb{B}(\mathcal{X},\mathcal{Y})}$ with

$$\|T(x)\| > 1 - \eta,$$

there is $(y, S) \in S_{\mathcal{X}} \times S_{\mathbb{B}(\mathcal{X},\mathcal{Y})}$ with

$$\|S(y)\| = 1, \quad \|x - z\| < \varepsilon, \quad \text{and} \quad \|T - S\| < \varepsilon.$$

We define the spherical Bishop-Phelps-Bollobás modulus as the analogous function $\Phi^S(\mathcal{X}, \mathcal{Y}, \cdot) : (0, 1) \to \mathbb{R}^+$ considering $(x, T) \in S_{\mathcal{X}} \times S_{\mathbb{B}(\mathcal{X},\mathcal{Y})}$ instead of $(x, T) \in B_{\mathcal{X}} \times B_{\mathbb{B}(\mathcal{X},\mathcal{Y})}$.

Roughly speaking, given $\eta > 0$, the symbol $\Phi^{(S)}(\mathcal{X}, \mathcal{Y}, \eta)$ represents the best possible ε for which the BPBp holds for the pair $(\mathcal{X}, \mathcal{Y})$. This concept somehow generalizes the BPB moduli, $\Phi^{(S)}_{\mathcal{X}}(\cdot)$, introduced in [63, Definition 1.2], where $\mathcal{Y}$ was considered to be the field $\mathbb{K}$, as they were working with functionals (the only difference between $\Phi^S(\mathcal{X}, \mathbb{K}, \eta)$ and $\Phi^S_{\mathcal{X}}(\eta)$ is that in the original definition of $\Phi^{(S)}_{\mathcal{X}}(\eta)$ they asked to have $y^*(y) = 1$ instead of $|y^*(y)| = 1$ for norm-attainment).

Remark 3.2 Note that in his original paper [42], Bollobás provided nice independent estimations for $\|x^* - y^*\|$ and $\|x - y\|$. The BPB moduli for functionals introduced in [63] serves as a common bound for those two values at the same time. Note also that although the lower and upper bound of the modulus from Bollobás' original theorem do not coincide, their difference is inessential when $\varepsilon \to 0$.

The systematic study of the BPB moduli was initiated in a 2014 paper by Mario Chica, Vladimir Kadets, Miguel Martín, Soledad Moreno-Pulido, and Fernando Rambla-Barreno (see [63]), where they wondered what is the *best* Bishop–Phelps–Bollobás theorem one can get in a given Banach space $\mathcal{X}$ while having a common bound for $\|x^* - y^*\|$ and $\|x - y\|$. As we mentioned implicitly in the introduction (see Theorem 1.2), they showed that

$$\Phi^S_{\mathcal{X}}(\eta) \leq \Phi_{\mathcal{X}}(\eta) \leq \sqrt{2\eta}$$

(see [63, Theorem 2.1 and Corollary 2.4]), that is, given an η, the best ε one can get that works for all Banach spaces at once is at most $\sqrt{2\eta}$. However, the authors did not stop there: not only they gave this upper bound, but they also showed that it is actually sharp, that is, there are Banach spaces where it can not be improved. We show just some of these examples.

Example ([63, Example 2.5 and Section 4]) Let $\mathcal{Y}$ be a Banach space. The Banach space $\mathcal{X}$ satisfies $\Phi^S_{\mathcal{X}}(\eta) = \Phi_{\mathcal{X}}(\eta) = \sqrt{2\eta}$ when $\mathcal{X}$ is

(a) The real space $\mathcal{X} = \ell^2_\infty$.
(b) $\mathcal{X} = L_1(\mu, \mathcal{Y})$, whenever $L_1(\mu)$ has dimension greater than one.
(c) $\mathcal{X} = L_\infty(\mu, \mathcal{Y})$, whenever $L_\infty(\mu)$ has dimension greater than one.
(d) $\mathcal{X} = c_0(\Gamma, \mathcal{Y})$, whenever Γ is an index set with more than one point.
(e) $\mathcal{X} = C_0(L, \mathcal{Y})$, whenever L is a locally compact Hausdorff topological space with at least two points.

They also found, however, spaces for which the bound of the modulus can be improved. Actually, they proved the following interesting result.

Theorem 3.3 ([63, Theorems 5.8 and 5.9]) *If $\mathcal{X}$ is an infinite-dimensional Banach space or a real finite-dimensional Banach space and there exists some $\eta \in (0, 1/2)$ such that $\Phi_{\mathcal{X}}(\eta) = \sqrt{2\eta}$, then $\mathcal{X}$ contains almost isometric copies of ℓ^2_∞.*

The converse of this result is not true in general (see [63, Section 6]). The previous theorem implies in particular that not many 2-dimensional real Banach spaces have the best possible BPB modulus. However, in [122], V. Kadets and M. Soloviova wondered what would be the modulus if in the Bishop–Phelps–Bollobás theorem the second functional was not asked to have norm 1 (the modulus happens to be $\sqrt{\eta}$ in this case), and they found out that, surprisingly, in this context many 2-dimensional real Banach spaces attain the maximum possible modulus (see [122, Subsection 2]).

Further refinements and properties of the BPB moduli for functionals were studied in [63] as well as in the more recent papers [61, 62] by Chica, Kadets, Martín, Merí, and Soloviova. For instance, it is known that $\Phi^{(S)}_{\mathcal{X}}(\eta)$ is continuous with respect to η (see [63, Proposition 3.1]) and with respect to $\mathcal{X}$ when using the Banach-Mazur distance (see [61, Section 3]).

In [123], Kadets and Soloviova studied the more general version of the modulus for operators in the case where the range space has property β of Lindenstrauss (note that in this case, $(\mathcal{X}, \mathcal{Y})$ is always granted to have the BPBp for operators, see Theorem 2.5). If $\mathcal{Y}$ satisfies property β of Lindenstrauss for some $\rho \geq 0$ (see Definition 2.2), we will denote it by $\beta(\mathcal{Y}) \leq \rho$. The authors found the following upper bound.

Theorem 3.4 ([123, Theorem 1]) *If $\mathcal{X}, \mathcal{Y}$ are Banach spaces and $\beta(\mathcal{Y}) \leq \rho$ for some $\rho \geq 0$, then for every $\varepsilon \in (0, 1)$,*

$$\Phi^S(\mathcal{X}, \mathcal{Y}, \eta) \leq \Phi(\mathcal{X}, \mathcal{Y}, \eta) \leq \min\left\{\sqrt{2\eta}\sqrt{\frac{1+\rho}{1-\rho}}, 2\right\}.$$

Note that they also found spaces for which that upper bound could be improved such that when $\mathcal{X}$ is uniformly non-square (see [123, Theorem 2]), or when $\mathcal{X} = \ell_1^2$ (see [123, Theorem 3]). Note that ℓ_1^2 attains the maximum possible modulus in the case of functionals, but this is not the case for operators. On the other hand, the authors do not provide any example of a pair of Banach spaces for which the estimation from the previous theorem is sharp. They did, however, show the existence of pairs of Banach spaces for which a lower bound of the modulus was *reasonably close* to the upper bound.

Theorem 3.5 ([123, Theorem 4]) *For every Banach space $\mathcal{Y}$,*

$$\Phi^S(\ell_1^2, \mathcal{Y}, \eta) \geq \min\{\sqrt{2\eta}, 1\},$$

and the equality is attained if $\beta(\mathcal{Y}) = 0$.

They also studied bounds for some range spaces $\mathcal{Y}$ depending on their associated ρ in the case where $\rho \in [1/2, 1)$ (see [123, Theorem 5]). An interesting remark is that the modulus for operators is not continuous with respect to the range space (see [123, Theorem 6]). Finally, the authors studied the growth of the modulus compared to $\sqrt{2\eta}$ (see [123, Section 3.4]) and the modulus of a version of the BPBp in which the operator S is not asked to have norm 1 (see [123, Section 4]). They left an open question that, as far as we know, remains open to this day (see Question 15).

For further research in this direction, we invite the reader to check [34, 59, 109, 110].

4 The Point and Operator Properties

In this section, we treat stronger properties than the BPBp, namely the Bishop–Phelps–Bollobás *operator* property and the Bishop–Phelps–Bollobás *point* property. Let us take a brief moment here to explain where the motivation to study such properties comes from.

In [125], Sun Kwang Kim and Han Ju Lee proved the following characterization for uniform convexity.

Theorem 4.1 ([125, Theorem 2.1]) *A Banach space $\mathcal{X}$ is uniformly convex if and only for every $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that for every $x^* \in S_{\mathcal{X}'}$ and $x \in B_{\mathcal{X}}$ such that*

$$|x^*(x)| > 1 - \eta(\varepsilon),$$

there is $x_0 \in S_{\mathcal{X}}$ such that

$$|x^*(x_0)| = 1 \quad \textit{and} \quad \|x - x_0\| < \varepsilon.$$

Let us notice that Theorem 4.1 is a Bollobás type theorem where the functional x^* does not change; in other words, the same functional that almost attains the norm at a point, actually attains its norm at a nearby point. At a first glance, the analogous property for bounded linear operators looks really restrictive in the sense that not many spaces would satisfy it. To make sure we are speaking the same language as the reader, let us give a name to such a (possible) property.

Definition 4.2 ([84, Definition 2.8]) We say that the pair $(\mathcal{X}, \mathcal{Y})$ has the Bishop–Phelps–Bollobás operator property (BPBop, for short) for operators if given $\varepsilon > 0$, there is $\eta(\varepsilon) > 0$ such that whenever $T \in \mathbb{B}(\mathcal{X}, \mathcal{Y})$ with $\|T\| = 1$ and $x \in S_{\mathcal{X}}$ satisfy

$$\|T(x)\| > 1 - \eta(\varepsilon),$$

there exists $x_0 \in S_{\mathcal{X}}$ such that

$$\|T(x_0)\| = 1 \quad \text{and} \quad \|x_0 - x\| < \varepsilon.$$

Notice now that Theorem 4.1 says simply that $\mathcal{X}$ is uniformly convex if and only $(\mathcal{X}, \mathbb{K})$ has the BPBop for linear functionals. It turns out that, for spaces $\mathcal{X}, \mathcal{Y}$ with dimension bigger or equal to 2, the BPBop for operators is *never* possible. Indeed, after several negative results presented in [84], the first author together with Kadets, Kim, Lee, and Martín proved the following result.

Theorem 4.3 ([94, Theorem 2.1]) *Let $\mathcal{X}, \mathcal{Y}$ be real Banach spaces of dimension greater or equal to 2. There exist $(T_n)_{n=1}^{\infty} \subseteq \mathrm{NA}(\mathcal{X}, \mathcal{Y}) \cap S_{\mathbb{B}(\mathcal{X},\mathcal{Y})}$ and $x_0 \in S_{\mathcal{X}}$ such that $\|T_n(x_0)\| \longrightarrow 1$ as $n \to \infty$ and*

$$\inf_{n\in\mathbb{N}} \Big\{ \mathrm{dist}\,(x_0, \{x \in S_{\mathcal{X}} : \|T_n(x)\| = 1\}) \Big\} > 0.$$

In particular, $(\mathcal{X}, \mathcal{Y})$ fails the BPBop for operators.

The proof of Theorem 4.3 is quite involved and requires finding an operator T defined on 2-dimensional spaces satisfying a very specific geometric condition so that we can construct the sequence satisfying the desired conditions (see [94, Propositions 2.4 and 2.5]).

Now that we have put away the operator property, we might think of a dual property of it remembering the duality between uniformly convex and uniformly smooth spaces. More specifically, one may consider the following property.

Definition 4.4 We say that $(\mathcal{X}, \mathcal{Y})$ has the Bishop–Phelps–Bollobás point property (BPBpp, for short) for operators if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that whenever $T \in \mathbb{B}(\mathcal{X}, \mathcal{Y})$ with $\|T\| = 1$ and $x \in S_{\mathcal{X}}$ satisfy

$$\|T(x)\| > 1 - \eta(\varepsilon),$$

there exists $S \in \mathbb{B}(\mathcal{X}, \mathcal{Y})$ with $\|S\| = 1$ such that

$$\|S(x)\| = 1 \quad \text{and} \quad \|S - T\| < \varepsilon.$$

Let us notice that the BPBpp implies immediately the BPBp. Moreover, as expected, it turns out that, for linear functionals, the Banach space $\mathcal{X}$ is uniformly smooth if and only if the pair $(\mathcal{X}, \mathbb{K})$ has the BPBpp for linear functionals (see [95, Theorem 2.1]). This implies (see [95, Proposition 2.3]) that whenever a pair $(\mathcal{X}, \mathcal{Y})$ satisfies the BPBpp for operators, the domain space $\mathcal{X}$ must be uniformly smooth. For that reason, we can see the great connection between the BPBpp and uniform smoothness as well as why the pairs of the form $(\ell_1, \mathcal{Y})$ and $(c_0, \mathcal{Y})$ always fail such a property. In particular, the BPBp and the BPBpp are not equivalent properties.

Next we will give the first positive results about the point property for operators. To start with, we present the following result, which says that Hilbert spaces are universal domain spaces. This is a consequence of the fact that Hilbert spaces have transitive norms, that is, if $x, y \in S_{\mathcal{H}}$ satisfy $\|x - y\| < \varepsilon$, then there exists a linear isometry $R : \mathcal{H} \longrightarrow \mathcal{H}$ such that $R(x) = y$ and $\|R - \mathrm{Id}_{\mathcal{H}}\| < \varepsilon$, where $\mathrm{Id}_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$.

Theorem 4.5 ([95, Theorem 2.5]) *Let $\mathcal{H}$ be a Hilbert space. The pair $(\mathcal{H}, \mathcal{Y})$ has the BPBpp for operators for every Banach space $\mathcal{Y}$.*

It is worth mentioning that there exists a more general result than Theorem 4.5 due to Cabello-Sánchez et al. [45]. They use the BPBpp as a tool to deal with Banach space $\mathcal{X}$ whose group of isometries acts micro-transitively on $S_{\mathcal{X}}$. In fact, they introduce a weakening of the micro-transitive, the uniform micro-semitransitive norms (see [45, Definition 2.2]). As a consequence of some results related to the BPBpp, they were able to prove that every Banach space whose norm is uniformly micro-semitransitive (in particular, if it is micro-transitive) is both uniformly convex and uniformly smooth (see [45, Corollary 2.13]). In fact, the *only* known spaces so far about micro-transitivity are Hilbert spaces. It is also worth mentioning that we do not know if micro-transitivity is a different property than uniform micro-semitransitivity.

Coming back to the point property and bearing in mind Theorem 4.5, it is natural to wonder whether the analogous result holds for $L_p(\mu)$-spaces. In other words, is it true that the pair $(L_p(\mu), \mathcal{Y})$ satisfies the BPBpp for every Banach space $\mathcal{Y}$? This question was addressed in [93] (where the authors called the BPBpp as pointwise BPB property). See also [95, Remark 2.6]. The answer to this question is not positive in general, as the following theorem shows.

Theorem 4.6 ([93, Corollary 3.6]) *Let $2 < p < \infty$ and let μ be a positive measure. If $\dim(L_p(\mu)) \geq 2$, then there exists a Banach space $\mathcal{Y}$ such that $(L_p(\mu), \mathcal{Y})$ fails to have the BPBpp for operators.*

Concerning Theorem 4.6, as far as we know, it is still open what happens when $1 < p < 2$ (see Question 10). Nevertheless, we have the following list of pairs satisfying the BPBpp for operators.

Theorem 4.7 ([94, Proposition 4.2 and Corollary 4.3]) *Let $\mathcal{X}$ be an arbitrary uniformly smooth Banach space. The pair $(\mathcal{X}, \mathcal{Y})$ has the BPBpp when*

(a) $\mathcal{Y}$ is a uniform algebra (in particular, when $\mathcal{Y} = C(K)$ and $\mathcal{Y} = C_0(L)$).
(b) $\mathcal{Y}$ has property β of Lindenstrauss.

Now another natural question arises. Do the pairs of the form $(\mathcal{X}, L_p(\mu))$ or $(\mathcal{X}, \ell_p)$ for $1 < p < \infty$ satisfy the BPBpp for operators for every uniformly smooth Banach space? The answer is again negative and can be checked in the following result. Let us put some emphasis on item (b) in Theorem 4.8 below: the pair $(\mathcal{X}_p, \ell_p^2)$ must have the BPBp for operators by Theorem 2.6 although this is no longer the case for the BPBpp.

Theorem 4.8 ([94, Theorem 4.4, Corollary 4.8, and Theorem 4.9])

(a) Let $1 < p < \infty$. Then, there exists a uniformly smooth Banach space $\mathcal{X}$ such that $(\mathcal{X}, L_p(\mu))$ fails to have the BPBpp for operators. In particular, there are uniformly smooth Banach spaces $\mathcal{X}$, $\mathcal{Z}$ such that $(\mathcal{X}, \ell_p)$ and $(\mathcal{Z}, \ell_p^n)$ for $n \geq 2$ fail the BPBpp.
(b) For each $2 \leq p < \infty$, there is a uniformly convex and uniformly smooth Banach space $\mathcal{X}_p$ such that $(\mathcal{X}_p, \ell_p^2)$ fails the BPBpp for operators.

It is worth mentioning that, similarly to how the BPBp has been studied for compact operators, the authors of [94] considered the BPBpp for compact operators. We send the reader to Section 5 of that paper. At this point, however, it is worth noting that it seems to be unknown whether the BPBpp and the BPBpp for compact operators are equivalent properties (see Question 11).

5 The Local Properties

Up to this point, the properties considered were uniform in nature. In this section we are going to tackle properties in which this uniform character is somehow lost. We will treat weakenings of the Bishop–Phelps–Bollobás (point and operator) properties. Besides their own interest, these properties were recently used successively as a tool in two different (in principle not connected) occasions. Indeed, on the one hand, they were used to defined exactly when the projective norm on $\mathcal{X}\widehat{\otimes}_\pi\mathcal{Y}$, the symmetric projective norm on $\widehat{\otimes}_{\pi,s,N}\mathcal{X}$, and the supremum norms on $\mathcal{P}(^N\mathcal{X};\mathcal{Y})$ and $\mathbb{B}(\mathcal{X}_1, \ldots, \mathcal{X}_N; \mathcal{Y})$ are (uniformly) strongly subdifferentiable (see [90, 96–99]). On the other hand, one of these properties was used as a tool to study norm attainment on $\mathcal{X}\widehat{\otimes}_\pi\mathcal{Y}$ (which is naturally connected to an important problem on norm-attaining theory (see Question 2)) and $\widehat{\otimes}_{\pi,s,N}\mathcal{X}$ (see [89, 92]).

5.1 Local Properties for Operators

As we have seen in Theorem 4.3, there is no version of the Bishop–Phelps–Bollobás operator property when one considers spaces with dimension greater or equal to 2. Therefore, the *only* hope to get positive results in this direction is considering a weakening of the mentioned property in the sense that, instead of requiring that the η from Definition 4.2 depends just on a positive real number $\varepsilon > 0$, it also depends on a previously fixed norm-one operator T. Indeed, this was done in [84] followed by [58, 143, 147] and we highlight as follows this new property for operators.

Definition 5.1 ([84, Definition 2.2.]) [2] We say that $(\mathcal{X}, \mathcal{Y})$ has the $\mathbf{L}_{o,o}$ for operators if given $\varepsilon > 0$ and $T \in \mathbb{B}(\mathcal{X}, \mathcal{Y})$ with $\|T\| = 1$, there exists $\eta(\varepsilon, T) > 0$ such that whenever $x \in S_{\mathcal{X}}$ satisfies

$$\|T(x)\| > 1 - \eta(\varepsilon, T),$$

there is $x_0 \in S_{\mathcal{X}}$ such that

$$\|T(x_0)\| = 1 \quad \text{and} \quad \|x_0 - x\| < \varepsilon.$$

Surprisingly, as the reader will see in a moment, there are several positive results on property $\mathbf{L}_{o,o}$. Beforehand, let us justify why the study of property $\mathbf{L}_{o,o}$ is not merely an attempt of forcing a property without much sense.

As we have mentioned already, Miguel Martín [139] proved that there are compact operators which cannot be approximated by norm-attaining operators. Perhaps the most important question at this very moment on norm-attaining theory is to known whether every finite-rank operator can be approximated by norm-attaining ones (see Question 2). Since the space of nuclear operators $\mathbb{N}(\mathcal{X}, \mathcal{Y})$ from $\mathcal{X}$ into $\mathcal{Y}$ satisfies that $\mathbb{F}(\mathcal{X}, \mathcal{Y}) \subseteq \mathbb{N}(\mathcal{X}, \mathcal{Y}) \subseteq \mathbb{K}(\mathcal{X}, \mathcal{Y})$, it seems to be completely reasonable to consider the class $\mathbb{N}(\mathcal{X}, \mathcal{Y})$ and address the analogous problem in this setting. This was done by the first and last author of this manuscript together with Luis Carlos García Lirola, Mingu Jung, and Abraham Rueda Zoca in the recent papers [89, 92]. Indeed, the authors used the $\mathbf{L}_{o,o}$ as a tool to provide positive results on the denseness of nuclear operators as well as tensors in the (symmetric) projective tensor product between Banach spaces.

Let us recall that, in the scalar-valued case, the BPBop and the BPBpp are dual properties in the sense that $(\mathcal{X}, \mathbb{K})$ has the BPBpp if and only if $(\mathcal{X}', \mathbb{K})$ has the BPBop (see [96, Proposition 2.2]). It seems to be natural also to consider the "local" version of the point property and this was done for the first time by the first author in a joint work with Sun Kwang Kim, Han Ju Lee, and Martin Mazzitelli.

[2] The symbol $\mathbf{L}_{o,o}$ comes from the fact that it is a local property and the double "o" means the *operator property* together with the requirement that η depends on an *operator*. In the paper [84], property $\mathbf{L}_{o,o}$ was called property 1 and in [143] strong BPB or sBPBp.

Definition 5.2 ([96, Definition 2.1]) [3] We say that $(\mathcal{X}, \mathcal{Y})$ has the $\mathbf{L}_{p,p}$ if given $\varepsilon > 0$ and $x \in S_{\mathcal{X}}$, there exists $\eta(\varepsilon, x) > 0$ such that whenever $T \in \mathbb{B}(\mathcal{X}, \mathcal{Y})$ with $\|T\| = 1$ satisfies

$$\|T(x)\| > 1 - \eta(\varepsilon, x),$$

there exists $S \in \mathbb{B}(\mathcal{X}, \mathcal{Y})$ with $\|S\| = 1$ such that

$$\|S(x)\| = 1 \quad \text{and} \quad \|S - T\| < \varepsilon.$$

Therefore, properties $\mathbf{L}_{o,o}$ and $\mathbf{L}_{p,p}$ must walk parallel with each other. It turns out that they are closely related to the strong subdifferentiability (usually denoted by SSD) of the norm of the Banach space as was observed by Gilles Godefroy, Vicente Montesinos, and Václav Zizler (see [113]).

Theorem 5.3 ([96, Theorem 2.3]) *Let $\mathcal{X}$ be a Banach space.*

(a) $(\mathcal{X}, \mathbb{K})$ *has the* $\mathbf{L}_{p,p}$ *if and only if* $\mathcal{X}$ *is SSD.*
(b) $(\mathcal{X}, \mathbb{K})$ *has the* $\mathbf{L}_{o,o}$ *if and only if* $\mathcal{X}$ *is reflexive and* $\mathcal{X}'$ *is SSD.*

Theorem 5.3 (together with [96, Proposition 2.6] (see also [143])) yields big differences between properties $\mathbf{L}_{p,p}$ and the BPBpp as well as property $\mathbf{L}_{o,o}$ and the BPBop (recall that the BPBpp and the BPBop for linear functionals give characterizations for uniformly smooth and uniformly convex Banach spaces, respectively). We suggest the interested reader to go to [97, page 47] or to the discussion in [97, pages 305 and 306] to find all necessary results and references about SSD.

Example The pair $(\mathcal{X}, \mathbb{K})$ has the

(a) $\mathbf{L}_{p,p}$ for linear functionals (but not the BPBpp) when

- (a1) $\mathcal{X} = c_0$.
- (a2) $\mathcal{X}$ is the predual of the Lorentz space $d_*(w, 1)$.
- (a3) $\mathcal{X}$ is the space of functions of vanishing mean oscillation (VMO), the predual of the Hardy space H^1.
- (a4) $\mathcal{X}$ is ℓ_1^n or ℓ_∞^n when $n \geq 2$.

(b) $\mathbf{L}_{o,o}$ for linear functionals (but not the BPBop) when

- (b1) $\mathcal{X}$ is ℓ_1^n or ℓ_∞^n.
- (b2) $\mathcal{X}$ is the space $(\oplus_{k=1}^{\infty} \ell_\infty^k)_{\ell_2}$.

Before moving forward to the results about operators, let us highlight one result on the $\mathbf{L}_{o,o}$ for linear functionals. The first author of this survey together with

[3] Analogous to the $\mathbf{L}_{o,o}$, we can justify the notation for property $\mathbf{L}_{p,p}$.

Abraham Rueda Zoca showed that there is a strong connection between this property and the compact operators (see also [143, Theorem 2.12] and [147, Theorem 2.3]).

Theorem 5.4 ([99, Theorem A]) *Let $\mathcal{X}$ be a reflexive space. The following are equivalent.*

(a) $(\mathcal{X}, \mathbb{K})$ *has the* $\mathbf{L}_{o,o}$ *for linear functionals.*
(b) $(\mathcal{X}, \mathcal{Y})$ *has the* $\mathbf{L}_{o,o}$ *for compact operators for every* $\mathcal{Y}$*.*
(c) $\mathcal{X}'$ *is SSD.*

Contrary to the BPBp (where one has to assume finite-dimensionality on *both* domain and range spaces, see Theorem 2.1), property $\mathbf{L}_{o,o}$ does not require such strong assumptions as we can see in the next result.

Theorem 5.5 ([84, Theorem 2.4]) *Let $\mathcal{X}$ be finite-dimensional. Then, the pair* $(\mathcal{X}, \mathcal{Y})$ *has the* $\mathbf{L}_{o,o}$ *for every Banach space* $\mathcal{Y}$*.*

We will come back to the $\mathbf{L}_{o,o}$ in a moment. Now, we sum up all the known results for bounded linear operators when it comes to the $\mathbf{L}_{p,p}$. We suggest the interested reader to check [96, Propositions 2.8, 2.9, 2.10, Theorem 2.12] and [97, Theorem 3.6].

Theorem 5.6 ([96, Section 2]) *The pair* $(\mathcal{X}, \mathcal{Y})$ *has the* $\mathbf{L}_{p,p}$ *when*

(a) $\mathcal{X}$ *and* $\mathcal{Y}$ *are finite-dimensional.*
(b) $(\mathcal{X}, \mathbb{K})$ *satisfies property* $\mathbf{L}_{p,p}$ *and* $\mathcal{Y}$ *has property* β *of Lindenstrauss.*
(c) $\mathcal{X} = \ell_1^n$ *and* $\mathcal{Y}$ *is*

- *(c1)* $\mathcal{Y}$ *is finite-dimensional.*
- *(c2)* $\mathcal{Y}$ *is uniformly convex.*
- *(c3)* $\mathcal{Y} = C_0(L)$.
- *(c4)* $\mathcal{Y} = L_1(\mu)$*, where* μ *is a positive measure.*
- *(c5)* $\mathcal{Y} = \mathcal{A}(\mathbb{D})$.
- *(c6)* $\mathcal{Y} = H^\infty(\mathbb{D})$.
- *(c7)* $\mathcal{Y}$ *has the property* β *of Lindenstrauss.*
- *(c8)* $\mathcal{Y} = L_1(\mu, \mathcal{X})$*, where* μ *is a* σ*-finite measure and* $\mathcal{X}$ *is as (c1)-(c7).*

(d) $\mathcal{X} = c_0$ *and* $\mathcal{Y} = L_p(\mu)$ *whenever* μ *is a positive measure and* $1 \leq p < \infty$*.*

In the same direction as properties $\mathbf{L}_{o,o}$ and $\mathbf{L}_{p,p}$, the authors in [96] considered also a local Bishop–Phelps–Bollobás property, where the η that appears in Definition 1.3 depends on a norm-one point or operator. Following the same notation, we set $\mathbf{L}_o$ to mean that we have a local BPBp when η depends on an operator and we set $\mathbf{L}_p$ when we have a local BPBp when η depends on a point. In that paper, the authors were interested in differentiating all of these properties and, as far as we know, there is not much done in the direction of properties $\mathbf{L}_o$ and $\mathbf{L}_p$ (see [96, Proposition 3.4 and Proposition 4.5]).

In Fig. 1, we sum up all the implications that hold and next justify why all these properties are different from each other. We do not know whether property $\mathbf{L}_p$ implies the denseness of the norm-attaning operators (see Question 13).

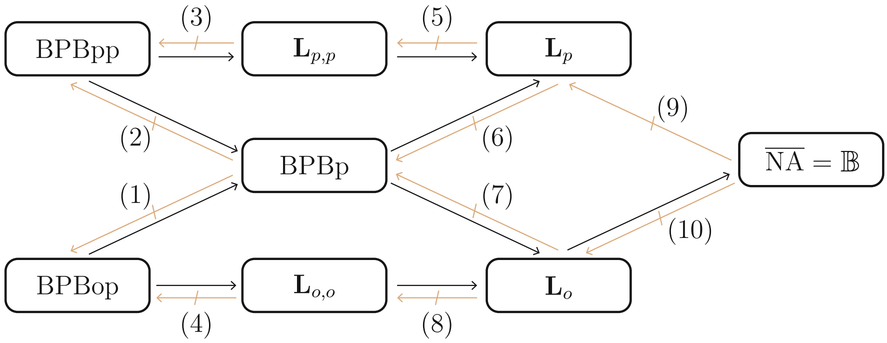

Fig. 1 Relations between the uniform and local BPBp

(1) BPBp $\not\Rightarrow$ BPBop: this follows immediately from Theorem 4.3 and any positive result for operators from Sect. 2.1. In the functional case the same happens: recall that the BPBop for linear functionals characterizes the uniformly convex Banach spaces and, on the other hand, the Bishop–Phelps–Bollobás theorem holds for *every* Banach space.

(2) BPBp $\not\Rightarrow$ BPBpp: recall that the BPBpp for functional characterizes uniformly smooth Banach spaces (see [95, Theorem 2.1]) and, in the operator case, if $(\mathcal{X}, \mathcal{Y})$ has the BPBpp, then $\mathcal{X}$ must be uniformly smooth (see [95, Proposition 2.3]). Therefore, we can take any positive result on the BPBp from Sect. 2.1 where the domain is not uniformly smooth and we are done.

(3) $\mathbf{L}_{p,p} \not\Rightarrow$ BPBpp: in the functional case, the BPBpp characterizes uniformly smooth Banach spaces (see [95, Theorem 2.1]) and the $\mathbf{L}_{p,p}$ characterizes the strong subdifferentiability of the norm (see [96, Theorem 2.3]). Therefore, the pair $(c_0, \mathbb{K})$ has the $\mathbf{L}_{p,p}$ but it cannot satisfy the BPBpp.

(4) $\mathbf{L}_{o,o} \not\Rightarrow$ BPBop: By Theorem 5.5, the pair $(\mathcal{X}, \mathcal{Y})$ always satisfies property $\mathbf{L}_{o,o}$ whenever $\mathcal{X}$ is finite-dimensional and $\mathcal{Y}$ is arbitrary. On the other hand, in the functional case, the BPBop characterizes uniformly convex spaces ([125, Theorem 2.1]) and in the operator case it does make any sense (Theorem 4.3).

(5) $\mathbf{L}_p \not\Rightarrow \mathbf{L}_{p,p}$: Since the BPBp implies property $\mathbf{L}_p$, we have $(\ell_1, \mathcal{Y})$ satisfies $\mathbf{L}_p$ in many cases (see, for instance, Theorem 2.8). Nevertheless, these pairs cannot have $\mathbf{L}_{p,p}$ since if $(\mathcal{X}, \mathcal{Y})$ has $\mathbf{L}_{p,p}$, then $\mathcal{X}$ must be SSD (see [97, Corollary 2.4]).

(6) $\mathbf{L}_p \not\Rightarrow$ BPBp: It is known that the set $\mathrm{NA}((\bigoplus_{k=2}^{\infty} \ell_k^2)_{\ell_2}, \mathcal{Y})$ is dense in $\mathbb{B}((\bigoplus_{k=2}^{\infty} \ell_k^2)_{\ell_2}, \mathcal{Y})$ [137] for every Banach space $\mathcal{Y}$. By [96, Proposition 3.4], the pair $((\bigoplus_{k=2}^{\infty} \ell_k^2)_{\ell_2}, \mathcal{Y})$ satisfies property $\mathbf{L}_p$ for every Banach space $\mathcal{Y}$. On the other hand, if $((\bigoplus_{k=2}^{\infty} \ell_k^2)_{\ell_2}, \mathcal{Y})$ had the BPBp for every Banach space $\mathcal{Y}$, then for every $\varepsilon \in (0, 1)$, there would exist a ε-dense uniformly strongly exposed family of the unit sphere of the space $(\bigoplus_{k=2}^{\infty} \ell_k^2)_{\ell_2}$ (see [34, Corollary 3.6]) by using the fact that $(\bigoplus_{k=2}^{\infty} \ell_k^2)_{\ell_2}$ is superreflexive. Nevertheless, by [96, Lemma 5.1 and Proposition 5.2], there exists $\varepsilon_0 \in (0, 1)$ such that

there is *no* ε_0-dense uniformly strongly exposed family on the unit sphere for $(\bigoplus_{k=2}^{\infty} \ell_k^2)_{\ell_2}$. Therefore, there exists a Banach space $\mathcal{Y}_0$ such that the pair $((\bigoplus_{k=2}^{\infty} \ell_k^2)_{\ell_2}, \mathcal{Y}_0)$ fails the BPBp.

(7) $\mathbf{L}_o \not\Rightarrow$ BPBp: Theorem 5.5 says that $(\mathcal{X}, \mathcal{Y})$ satisfies property $\mathbf{L}_{o,o}$ whenever $\mathcal{X}$ is finite-dimensional and $\mathcal{Y}$ is arbitrary. On the other hand, there exists a Banach space $\mathcal{Z}$ such that $(\ell_1^2, \mathcal{Z})$ fails the BPBp (see [34, Example 4.1]).

(8) $\mathbf{L}_o \not\Rightarrow \mathbf{L}_{o,o}$: The pairs $(\ell_1, \mathcal{Y})$ have the BPBp (and therefore property $\mathbf{L}_o$) in many occasions (see, for instance, Theorem 2.8). Nevertheless, such pairs cannot satisfy $\mathbf{L}_{o,o}$ since ℓ_1 is not reflexive.

(9) $\overline{\mathrm{NA}} = \mathbb{B} \not\Rightarrow \mathbf{L}_p$: Let $\mathcal{Y}$ be a strictly convex but not uniformly convex Banach space. Then, $(\ell_1^2, \mathcal{Y})$ fails to have $\mathbf{L}_p$ by [96, Proposition 3.2.(a)]. Nevertheless, $\mathrm{NA}(\ell_1^2, \mathcal{Y}) = \mathbb{B}(\ell_1^2, \mathcal{Y})$ for every Banach space $\mathcal{Y}$ by compactness.

(10) $\overline{\mathrm{NA}} = \mathbb{B} \not\Rightarrow \mathbf{L}_o$: Let $\mathcal{Y}$ be a strictly convex but not uniformly convex Banach space. Then, $(\ell_1, \mathcal{Y})$ fails property $\mathbf{L}_o$ by [96, Proposition 3.2.(b)]. Nevertheless, $\mathrm{NA}(\ell_1, \mathcal{Y})$ is dense in $\mathbb{B}(\ell_1, \mathcal{Y})$ for every Banach space $\mathcal{Y}$.

5.2 Local Properties for Bilinear Mappings

We can adapt Definitions 5.1 and 5.2 in a natural way to the context of bilinear mappings. This was done for the first time in [96] (and more recently in [99]) with the aim of classifying when the projective norm in the projective tensor product between Banach spaces is strongly subdifferentiable. We invite the reader to go to [96, Definition 2.1] for the formal definitions.

Concerning the $\mathbf{L}_{o,o}$ for bilinear mappings, we have the following general characterization which yields several particular interesting cases. It deals with the reflexivity of the projective tensor product $\mathcal{X}\widehat{\otimes}_\pi\mathcal{Y}$ and allows us to relate property $\mathbf{L}_{o,o}$ in different classes of functions (for linear functionals, operators, and bilinear forms).

Theorem 5.7 ([99, Theorem B]) *Let $\mathcal{X}$ be a strictly convex Banach space of a Banach space satisfying the Kadec-Klee property. Let $\mathcal{Y}$ be an arbitrary Banach space and assume that either $\mathcal{X}$ or $\mathcal{Y}$ satisfies the approximation property. The following statements are equivalent.*

(a) $(\mathcal{X}\widehat{\otimes}_\pi\mathcal{Y}, \mathbb{K})$ *has the* $\mathbf{L}_{o,o}$ *for linear functionals.*
(b) $\mathcal{X}\widehat{\otimes}_\pi\mathcal{Y}$ *is reflexive and both* $(\mathcal{X}, \mathbb{K})$ *and* $(\mathcal{Y}, \mathbb{K})$ *have the* $\mathbf{L}_{o,o}$ *for linear functionals.*
(c) $(\mathcal{X}, \mathcal{Y}; \mathbb{K})$ *has the* $\mathbf{L}_{o,o}$ *for bilinear forms.*

As a consequence, we have the following list of examples. The symbol q' stands for the conjugate index of q.

Example ([97, Proposition 2.2.(a), Lemma 2.6, and Theorem 2.7.(b)]) The triple $(\mathcal{X}, \mathcal{Y}; \mathbb{K})$ has the $\mathbf{L}_{o,o}$ for bilinear mappings whenever

(a) $\mathcal{X}, \mathcal{Y}$ are finite-dimensional.
(b) $\mathcal{X}$ is finite-dimensional and $\mathcal{Y}$ is uniformly convex.
(c) $\mathcal{X} = \ell_p$ and $\mathcal{Y} = \ell_q$ if and only if $p > q'$.

Another consequence of Theorem 5.7 is the following. We can classify *exactly* when the projective tensor norm on $\ell_p \widehat{\otimes}_\pi \ell_q$ is strongly subdifferentiable. As far as we know such a classification was done for the first time in [97]. Let us notice that the positive result obtained in Corollary 5.8 below will not happen so often since if the projective tensor product $\mathcal{X} \widehat{\otimes}_\pi \mathcal{Y}$ is SSD, then it is an Asplund space (see, for instance, [113, Theorem 2]). The following extends [97, Corollary 2.8] and it is a combination of [100, Exercise 16.5], [142, Corollary 4.24], and Theorem 5.7 above.

Corollary 5.8 *The norm of $\ell_p \widehat{\otimes}_\pi \ell_q$ is SSD if and only if $p^{-1} + q^{-1} < 1$.*

We conclude this section by inviting the reader to check Section 7, which contains a brief discussion of the possibility of using the point property for N-homogeneous polynomials as a tool to get results on the strong subdifferentiability of the norm of the symmetric projective tensor product $\widehat{\otimes}_{\pi,s,N} \mathcal{X}$.

6 Open Questions

In this section, we provide open questions on the different topics that we have treated in this survey.

1. *Bounded linear operators*

It is known that when both $\mathcal{X}, \mathcal{Y}$ are finite-dimensional Banach spaces, the pair $(\mathcal{X}, \mathcal{Y})$ satisfies the BPBp for operators (see Theorem 2.1) and the proof of such a result is done by contradiction using the compactness of the unit balls of both spaces. The following is due to Richard Aron.

Question 1 (Richard Aron) To provide a *direct* (constructive) proof for the fact that $(\mathcal{X}, \mathcal{Y})$ has the BPBp for operators whenever $\mathcal{X}$ and $\mathcal{Y}$ are finite-dimensional spaces.

Still in the finite-dimensional vein, we have the following question. It is worth noting that the following question is not known even when the range space is $\mathbb{R}^2$ endowed with the Euclidean norm.

Question 2 Is it true that all finite-rank operators can be approximated by norm-attaining ones?

A characterization is known for the Banach spaces $\mathcal{Y}$ such that the pairs of the form $(\ell_1, \mathcal{Y})$ satisfy the Bishop–Phelps–Bollobás property (see Theorem 2.8) through the AHSP. In the same line, we have the following question.

Question 3 ([79, page 240]) Characterize the topological Hausdorff compact spaces S such that the pair $(\mathcal{X}, C(S))$ satisfies the BPBp for operators for every Banach space $\mathcal{X}$.

Still with $\mathcal{X} = C(K)$-spaces, it is not known if the Bishop–Phelps–Bollobás theorem holds on these spaces in the complex case.

Question 4 ([11, page 326]) Let K, S be compact Hausdorff topological spaces. Is it true that the pair $(C(K), C(S))$ satisfies the BPBp for operators in the complex case? This question is unknown even for just the Bishop–Phelps property.

Surprisingly, the techniques used on $C(K)$-spaces do not seem to work for the following pair, and the following appears to be an open question.

Question 5 ([5, page 318]) Is it true that the pair (c_0, ℓ_1) has the BPBp for operators in the real case?

In Sect. 2.1, we commented that, as far as we are concerned, it is not known when $(\mathcal{X}, \mathcal{Y})$ has the BPBp assuming that $\mathcal{X}$ has the Radon-Nikodým property as well as what happens with the particular case of James' space J. This question was proposed by Abraham Rueda Zoca.

Question 6 (Abraham Rueda Zoca) Assume that $\mathcal{X}$ has the Radon-Nikodým property. For what Banach spaces $\mathcal{Y}$ is it true that $(\mathcal{X}, \mathcal{Y})$ has the BPBp? Assume for instance that $\mathcal{X} = J$ is the James' space.

In Sect. 2.2, we have shown the results on the BPBp for compact operators. Most of the results known for the BPBp also work for the BPBp for compact operators. The following is still unknown.

Question 7 ([88, page 57]) Is it true that the BPBp implies the BPBp for compact operators?

2. *Multilinear mappings*

Symmetric multilinear mappings seem to be (almost) always a headache when it comes to the BPBp. In view of Theorem 2.39, the following is a natural question.

Question 8 ([12, 4.6.(2)]) Let $\mathcal{X}$ be a uniformly convex Banach space. Is it true that $(\mathcal{X}, \ldots, \mathcal{X}; \mathbb{K})$ has the BPBp for symmetric N-linear forms?

The question has a positive answer for symmetric bilinear forms on Hilbert spaces ([104, Theorems 3.2 and 3.4]). It is worth mentioning that Hilbert spaces do not satisfy the Bishop–Phelps–Bollobás point property for symmetric multilinear mappings [90]. We also send the reader to the very interesting paper [49], where the authors characterize the sets of vectors $(x_1, \ldots, x_N)$ in $\mathcal{H} \times \ldots \times \mathcal{H}$ such that there exists an N-linear symmetric form attaining its norm at $(x_1, \ldots, x_N)$.

3. *Numerical Radius*

Thinking about Theorem 2.85 for $C(K)$-spaces, we do not know the following.

Question 9 Let K be an arbitrary compact Hausdorff topological space. Is it true that $C(K)$ has the BPBp-nu?

4. *The Bishop–Phelps–Bollobás point property*

By Theorem 4.6, we know that if $\dim(L_p(\mu)) \geq 2$, where $2 < p < \infty$ and μ is a positive measure, there exists a Banach space $\mathcal{Y}$ such that the pair $(L_p(\mu), \mathcal{Y})$ fails the Bishop–Phelps–Bollobás point property. Since L_1 and L_∞ are not uniformly smooth spaces, it does not make any sense to talk about the point property for these spaces. Nevertheless, we do not know what happens with L_p-spaces for $1 < p < 2$.

Question 10 ([93, Problem 6.3]) Is it true that the pair $(L_p(\mu), \mathcal{Y})$ has the BPBpp for every Banach $\mathcal{Y}$ whenever $1 < p < 2$?

In the same line of Sect. 2.2, we do not know the relation between the BPBpp and the BPBpp for compact operators.

Question 11 ([93, Problem 6.5]) Is it true that the BPBpp for operators and the BPBpp for compact operators are equivalent properties?

It is known that, for $2 < p, q < \infty$ the triple $(\ell_p, \ell_q; \mathbb{K})$ has the $\mathbf{L}_{p,p}$ for bilinear forms (see [97, Theorem 2.7.(a)]). Nevertheless, we do not know what happens for the stronger BPBpp for bilinear forms.

Question 12 ([97, Remark 2.9.(a)]) Is it true that the triple $(\ell_p, \ell_q; \mathbb{K})$ has the BPBpp for bilinear forms when $1 < p, q < 2$ or when $2 < p, q < \infty$?

5. *Local properties*

In view of the Diagram 1, the following implication seems to be open.

Question 13 ([96, page 322]) Is it true that if the pair $(\mathcal{X}, \mathcal{Y})$ has property $\mathbf{L}_p$, then the set $\text{NA}(\mathcal{X}, \mathcal{Y})$ is dense in $\mathbb{B}(\mathcal{X}, \mathcal{Y})$?

None of the current techniques seem to work when one tries to tackle the following question.

Question 14 ([96, page 322]) Is it true that the pair (ℓ_p^2, ℓ_q) has property $\mathbf{L}_{p,p}$ whenever $1 < p, q < \infty$?

6. *Modulus*

In [63] it was shown that for all Banach spaces $\mathcal{X}$, it holds that $\Phi^S_{\mathcal{X}}(\eta) \leq \sqrt{2\eta}$. In [123], a BPB modulus for operators was considered for a pair of Banach spaces $(\mathcal{X}, \mathcal{Y})$ which somehow generalizes the previously defined modulus for functionals. However, in the real case, $\Phi^S(\mathcal{X}, \mathbb{R}, \eta)$ and $\Phi^S_{\mathcal{X}}(\eta)$ have a slight difference in the definition concerning norm-attainment: in the first one, it is asked to satisfy $|y^*(y)| = 1$, while in the second one it is asked to satisfy $y^*(y) = 1$. The following natural question follows and was left open in [123]. The authors are thankful to Vladimir Kadets for reminding us about it.

Question 15 ([123, Problem 1]) Is it true that $\Phi^S(\mathcal{X}, \mathbb{R}, \eta) \leq \min\{\sqrt{2\eta}, 1\}$ for all real Banach spaces $\mathcal{X}$?

7 Further Research and New (or Recent) Possible Lines

In this section, we present some further research that has been done in the past few years. Moreover, we present possible new lines that the interested reader could follow. We divide the present section into subsections depending on the specific direction that we are considering.

Stability Results There are plenty of results on stability of the Bishop–Phelps–Bollobás property when it comes to (absolute) direct sums. This provides more examples of pairs of Banach spaces satisfying the BPBp (for operators, in particular) as well as counterexamples. It is worth mentioning that stability results appear quite commonly when one starts working on the BPBp; for this, we suggest, for instance, references [27, 69, 72, 131] and also [34, Section 2].

The Bollobás Theorem for Operators on $\mathcal{X}\widehat{\otimes}_\pi\mathcal{Y}$ After the papers [89] and [92], it seems to be natural to ask when a Bollobás theorem for operators defined on tensor products holds. In particular, the question of when it is possible to get the BPBp for pairs of the forms $(\mathcal{X}\widehat{\otimes}_\pi\mathcal{Y}, \mathcal{Z}\widehat{\otimes}_\pi W)$ seems to have its own interest.

The BPBp-nu for Multilinear Mappings As far as we known, Theorem 2.91, which says that $L_1(\mu)$ fails the Bishop–Phelps–Bollobás property for numerical radius for multilinear mappings, is the only result in this line. Perhaps, more research in this direction would provide interesting results. Other classes of mappings for which denseness of numerical radius attaining mappings have been studied include N-homogeneous polynomials (see for instance [16, 73]) and holomorphic mappings (see for instance [136, 150]), but as far as we know, no BPBp-nu property has been studied for those classes of mappings.

Minimum Norm-Attaining Operators A relatively new line of research studies the operators that attain their minimum norms. For $T \in \mathbb{B}(\mathcal{X}, \mathcal{Y})$, its minimum norm is defined as the number $m(T) := \inf_{x\in S_{\mathcal{X}}} \|T(x)\|$. Bollobás type theorems in this line could have their own relevance. We suggest the reader references [39, 53, 57, 58, 135].

Group Invariant Version of Bollobás Theorem Two very recent papers (see [85, 102]) consider versions of the Bishop–Phelps theorem for group invariant functionals and operators. In [102], Javier Falcó proved that a Bollobás theorem in this context does not hold in general (see [102, Example, page 1611]) but he also proved a possible extension for it (see [102, Theorem 5]). These properties for operators have their own interest.

The Set of Operators that Satisfy a BPB Theorem Very recently, the first and fourth authors of this survey, in a joint work with Mingu Jung, studied Bollobás type theorems from a different perspective (see [91]): instead of looking for spaces satisfying the Bollobás theorem, they studied the set of operators for which some Bollobás type theorem are valid ([91, Definition 1.1]). As far as we know there is no further research in this direction.

Strongly Subdifferentiability of $\widehat{\otimes}_{\pi,s,N}\mathcal{X}$ and $\mathcal{P}(^N\mathcal{X})$ In a upcoming paper, the first author together with Mingu Jung, Martin Mazzitelli, and Jorge Tomás Rodríguez, study the strongly subdifferentiability of the symmetric projective tensor product [90]. To do so, they study the Bishop–Phelps–Bollobás point property for N-homogeneous polynomials as a tool to provide when the symmetric projective tensor product $\widehat{\otimes}_{\pi,s,N}\mathcal{X}$ and $\mathcal{P}(^N\mathcal{X})$ have strongly differentiable norms.

Related Local Properties The reader can easily notice that one can consider the BPBop when η depends on a norm-one point x as well as the BPBpp when η depends on a norm-one operator T (see Definitions 5.1 and 5.2). This yields properties $\mathbf{L}_{o,p}$ and $\mathbf{L}_{p,o}$, respectively. There are not many results in this line and we invite the reader to check the recent paper [98], where the main aim of the authors is to distinguish all of them from each other. On the other hand, not much is known about the differences between the BPBp and its local versions, $\mathbf{L}_p$ and $\mathbf{L}_o$ (see [96, Section 3]).

8 Tables for Classical Banach Spaces: A Summary

The following tables gather and summarize known results about the Bishop–Phelps–Bollobás property for pairs of classical Banach spaces. In the pdf version of this document, each cell is hyperlinked to the corresponding result within this survey. The first column will represent domain spaces and the first row will represent range spaces.

Remark 8.1 As an important note, the table will be mostly focused on positive results, that is, when a pair of spaces has the BPBp. Negative results will be reserved exclusively for universal domains and ranges, and may not be exhaustive. For instance, in any pair where $\mathrm{NA}(\mathcal{X},\mathcal{Y})$ is not dense in $\mathbb{B}(\mathcal{X},\mathcal{Y})$, the BPBp automatically fails, but even in pairs where every operator attains its norm, we can find counterexamples (see Example after Theorem 2.1 for a 2-dimensional space failing to be a universal BPB domain). Actually, many classical Banach spaces fail to be universal BPB domains, such as c_0, ℓ_1, ℓ_∞ $L_1(\mu)$, $C(K)$ or any ℓ_1^n or ℓ_∞^n with $n > 1$ (except for maybe particular cases), and it is known also that ℓ_1, ℓ_p, $L_1(\mu)$, $L_p(\mu)$, and $C(K)$ spaces are not universal BPB range spaces in general (except for some particular cases). We encourage the interested reader to check the nice paper [34], where an exposition of this problem is shown, and see also the excellent survey [4] about norm-attaining operators.

Table 1 Classical Banach spaces with the BPBp (I)

BPBp	$\forall \mathcal{Y}$	$\mathbb{K}$	ℓ_1^m	ℓ_q^m	ℓ_∞^m	F.D.	c_0	ℓ_1	ℓ_q	ℓ_∞
$\forall \mathcal{X}$	✗	✓	✓		✓		✓	✗	✗	✓
$\mathbb{K}$	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
ℓ_1^2	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓
ℓ_1^n	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓
ℓ_p^n	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
ℓ_∞^n	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓
F.D.	✗	✓	✓	✓	✓	✓	✓			✓
c_0	✗	✓	✓	✓	✓		✓	$\checkmark_{\mathbb{C}}$	✓	✓
ℓ_1	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓
ℓ_p	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
ℓ_∞	✗	✓	✓	✓	✓		✓	$\checkmark_{\mathbb{C}}$	✓	✓
$L_1(\mu)$	✗	✓	✓	✓	✓	$\checkmark_{\sigma}$	✓	✓	✓	✓
$L_p(\mu)$	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
$L_\infty(\mu)$	✗	✓	✓	$\checkmark_{\mathbb{R} \mid \mathbb{C}}$	✓		✓	$\checkmark_{\mathbb{C}}$	$\checkmark_{\mathbb{R} \mid \mathbb{C}}$	✓
$C(K_1)$	✗	✓	✓	$\checkmark_{\mathbb{R} \mid \mathbb{C}}$	✓		✓	$\checkmark_{\mathbb{C}}$	$\checkmark_{\mathbb{R} \mid \mathbb{C}}$	✓
$C_0(L_1)$	✗	✓	✓	$\checkmark_{\mathbb{C}}$	✓		✓	$\checkmark_{\mathbb{C}}$	$\checkmark_{\mathbb{C}}$	✓

Table 2 Classical Banach spaces with the BPBp (II)

BPBp	$L_1(\nu)$	$L_q(\nu)$	$L_\infty(\nu)$	$C(K_2)$	$C_0(L_2)$
$\forall \mathcal{X}$	✗	✗		✗	✗
$\mathbb{K}$	✓	✓	✓	✓	✓
ℓ_1^2	✓	✓	✓	✓	✓
ℓ_1^n	✓	✓	✓	✓	✓
ℓ_p^n	✓	✓	✓	✓	✓
ℓ_∞^n	$\checkmark_{\mathbb{C} \mid \mathbb{R}, 2_+}$	✓	✓	✓	✓
F.D.			✓	✓	✓
c_0	$\checkmark_{\mathbb{C}}$	✓	✓	✓	✓
ℓ_1	✓	✓	✓	✓	✓
ℓ_p	✓	✓	✓	✓	✓
ℓ_∞	$\checkmark_{\mathbb{C}}$	✓	$\checkmark_{\mathbb{R}}$	$\checkmark_{\mathbb{R}}$	
$L_1(\mu)$	✓	✓	$\checkmark_{2_\circ}$		
$L_p(\mu)$	✓	✓	✓	✓	✓
$L_\infty(\mu)$	$\checkmark_{\mathbb{C}}$	$\checkmark_{\mathbb{R} \mid \mathbb{C}}$	$\checkmark_{\mathbb{R}}$	$\checkmark_{\mathbb{R}}$	
$C(K_1)$	$\checkmark_{\mathbb{C}}$	$\checkmark_{\mathbb{R} \mid \mathbb{C}}$	$\checkmark_{\mathbb{R}}$	$\checkmark_{\mathbb{R}}$	$\checkmark_{\mathbb{R}, 1_m}$
$C_0(L_1)$	$\checkmark_{\mathbb{C}}$	$\checkmark_{\mathbb{C}}$	$\checkmark_{\mathbb{R}, 1_m}$	$\checkmark_{\mathbb{R}, 1_m}$	$\checkmark_{\mathbb{R}, 1_m}$

Tables 1 and 2 use the following notation. Unless otherwise mentioned, $1 < p, q < \infty$, $m, n > 1$, μ, ν are any measures, K_1, K_2 are any compact Hausdorff spaces, L_1, L_2 are any locally compact Hausdorff spaces, and F.D. will denote finite-dimensional Banach spaces. Besides, we have the following additional notation.

- Symbol ✓ means that the pair has the BPBp in general, possibly under some extra conditions specified in the subindexes.

- Symbol ✗ means that there is at least 1 known counterexample.
- Subindex $\mathbb{R}$ means in the real case. Likewise, subindex $\mathbb{C}$ means in the complex case.
- When there is confusion, 1_{subindex} means that whatever comes next applies to the domain space, and 2_{subindex} means that whatever comes next applies to the range space.
- σ stands for a σ-finite measure.
- $+$ stands for a positive measure.
- $\circ$ stands for a localizable measure.
- m means that the corresponding (locally) compact Hausdorff space is metrizable.

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A New Proof of the Power Weighted Birman–Hardy–Rellich Inequalities

Fritz Gesztesy, Isaac Michael, and Michael M. H. Pang

Abstract In this chapter, we introduce a new method for proving the power-weighted Birman–Hardy–Rellich integral inequalities,

$$\int_0^\infty dx\, x^\alpha \big|f^{(m)}(x)\big|^2 \geqslant A(\ell,\alpha) \int_0^\infty dx\, x^{\alpha-2\ell}\big|f^{(m-\ell)}(x)\big|^2,$$
$$m \in \mathbb{N},\ \ell \in \{1,\dots,m\},\ \alpha \in \mathbb{R},\ f \in C_0^\infty((0,\infty)),$$

where $A(\ell,\alpha)$ is given by

$$A(\ell,\alpha) = 4^{-\ell}\prod_{j=1}^{\ell}(2j-1-\alpha)^2.$$

The new method of proof simultaneously establishes both the existence of such inequalities and their optimality (i.e., sharpness of the constant $A(\ell,\alpha)$ on the space $C_0^\infty((0,\infty))$ of infinitely differentiable functions of compact support in $(0,\infty)$). We also note that these inequalities are strict, that is, equality holds if and only if $f \equiv 0$.

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Extensions to homogeneous Sobolev spaces, that is, $f \in \overset{\bullet}{H}{}_0^m\big((0,\rho); x^\alpha dx\big)$, as well as the vector-valued case, where $f \in \overset{\bullet}{H}{}_0^m\big((0,\rho); x^\alpha dx; \mathcal{H}\big)$, $\rho \in (0,\infty)\cup\{\infty\}$, with $\mathcal{H}$ a complex, separable Hilbert space, are also discussed.

Keywords Birman–Hardy–Rellich inequalities · Logarithmic refinements

1 Introduction

The primary aim in this paper is to provide a new proof of the optimal version of the power-weighted sequence of Birman–Hardy–Rellich inequalities of the form,

$$\int_0^\infty dx\, x^\alpha \big|f^{(m)}(x)\big|^2 \geqslant A(\ell,\alpha) \int_0^\infty dx\, x^{\alpha-2\ell} \big|f^{(m-\ell)}(x)\big|^2, \\ m \in \mathbb{N},\ \ell \in \{1,\dots,m\},\ \alpha \in \mathbb{R},\ f \in C_0^\infty((0,\infty)), \tag{1}$$

where

$$A(\ell,\alpha) = 4^{-\ell} \prod_{j=1}^{\ell} (2j-1-\alpha)^2, \quad \ell \in \mathbb{N},\ \alpha \in \mathbb{R}. \tag{2}$$

The novelty of our proof lies in the fact that both the existence and the optimality of the constants $A(\ell,\alpha)$ in (1) are established simultaneously.

Moreover, we also prove these inequalities in the context of homogeneous Sobolev spaces, that is, for $f \in \overset{\bullet}{H}{}_0^m\big((0,\rho); x^\alpha dx\big)$, as well as in the vector-valued case for $f \in \overset{\bullet}{H}{}_0^m\big((0,\rho); x^\alpha dx; \mathcal{H}\big)$, $\rho \in (0,\infty) \cup \{\infty\}$, and the Sobolev space $H_0^m((\rho,\infty); x^\alpha dx; \mathcal{H})$, $\rho \in (0,\infty)$, where $\mathcal{H}$ is a complex, separable Hilbert space.

We recall that the special case $\alpha = 0$ appeared in work of Birman in 1961 (English translation in 1966) [3] (see also [9, pp. 83–84]). The case $m = 1$ in (1) represents Hardy's celebrated inequality [11], [12, Sect. 9.8] (see also [16, Chs. 1, 3, App.]), the case $m = 2$ is due to Rellich [24, Sect. II.7] (actually, in the multi-dimensional context). The inequalities (1) are known to be strict, that is, equality holds in (1) if and only if $f = 0$ on $(0,\infty)$. Moreover, they are known to be optimal, that is, the constants $A(\ell,\alpha)$ in (1) are sharp, although, this must be qualified as different authors frequently prove sharpness for different function spaces. In the present one-dimensional context at hand, sharpness of (1) is often proved in an integral form (rather than the currently presented differential form) where $f^{(m)}$ on the left-hand side is replaced by F and f on the right-hand side by m repeated integrals over F. For pertinent one-dimensional sources, we refer, for instance, to [2, p. 3–5], [4], [6, p. 104–105], [8, 10, 11], [12, p. 240–243], [16, Ch. 3], [17, p. 5–11], [19, 20, 23]. We also note that higher-order Hardy inequalities, including various weight functions, are discussed in [5], [15, Sect. 5], [16, Chs. 2–5], [17, Chs. 1–4], [18], and [22, Sect. 10] (however, with the exception of [5], Birman's sequence of inequalities, i.e., (1) for $\alpha = 0$, is not mentioned in these sources). There exists a wealth of multi-dimensional investigations of Hardy, Rellich, etc., inequalities on domains $\Omega \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geqslant 2$, which, when specialized to a ball in $\mathbb{R}^n$ and spherically symmetric functions f, yields one dimensional inequalities of the Birman–Hardy–Rellich-type with various weight functions. Since we included a very detailed bibliography in [7], including such multi-dimensional sources, we refrained from repeating it here and just focused on the one dimensional literature.

Briefly turning to the contents of each section, we introduce our new proof, a variant of a combination of transformations studied by Hartman [13], [14, p. 324–325] and Müller-Pfeiffer [21, p. 200–207], in Sect. 2. Generalizations to homogeneous Sobolev spaces and to the vector-valued case (replacing complex-valued $f(\cdot)$ by $f(\cdot) \in \mathcal{H}$, with $\mathcal{H}$ a complex, separable Hilbert space) appear in Sect. 3. For background on the vector-valued case we refer to Appendix A.

Throughout this paper, $\mathcal{H}$ represents a complex, separable Hilbert space with corresponding scalar product $(\cdot\,,\,\cdot)_{\mathcal{H}}$ (linear in the second factor) and associated norm $\|\cdot\|_{\mathcal{H}}$.

2 Power-Weighted Birman–Hardy–Rellich Inequalities

In this section we present our new proof of the power-weighted Birman–Hardy–Rellich inequalities for functions in $C_0^\infty((0,\infty))$.

We start by establishing the sequence of power-weighted Birman–Hardy–Rellich inequalities for the case $\ell = m$ in (1) using a slight modification of our proof in [7] that allows one to prove both the inequality and the optimality of the constant $A(m, \alpha)$ simultaneously.

Lemma 2.1 *Set*

$$A(m,\alpha) = 4^{-m} \prod_{j=1}^{m} (2j-1-\alpha)^2, \quad m \in \mathbb{N},\ \alpha \in \mathbb{R}. \tag{3}$$

Then,

$$\int_0^\infty dx\, x^\alpha \big|f^{(m)}(x)\big|^2 \geqslant A(m,\alpha) \int_0^\infty dx\, x^{\alpha-2m} \big|f(x)\big|^2, \\ m \in \mathbb{N},\ \alpha \in \mathbb{R},\ f \in C_0^\infty((0,\infty)). \tag{4}$$

Moreover, the constant $A(m, \alpha)$ in (4) is optimal and the inequality is strict, that is, equality holds in (4) if and only if $f \equiv 0$.

Proof Let $C \in (0, \infty)$ and define Q as the operator in $L^2((0,\infty); dx)$ given by

$$Q = \left[(-1)^m \frac{d^m}{dx^m}\left(x^\alpha \frac{d^m}{dx^m}\right) - C x^{\alpha-2m}\right]\Big|_{C_0^\infty((0,\infty))}. \tag{5}$$

Utilizing

$$\int_a^b dx\, x^\alpha \big|f^{(m)}(x)\big|^2 = (-1)^m \int_a^b dx\, \big(x^\alpha f^{(m)}(x)\big)^{(m)} \overline{f(x)}, \tag{6}$$
$$m \in \mathbb{N},\ \alpha \in \mathbb{R},\ f \in C_0^\infty((a,b)),\ 0 \leqslant a < b \leqslant \infty,$$

one concludes that

$$(f, Qf)_{L^2((0,\infty);dx)} = \int_0^\infty dx\, \Big\{x^\alpha \big|f^{(m)}(x)\big|^2 - C x^{\alpha-2m} \big|f(x)\big|^2\Big\},\ f \in C_0^\infty((0,\infty)). \tag{7}$$

Thus, to establish (4) and, simultaneously, optimality of the constant $A(m, \alpha)$, we will show that

$$Q \geqslant 0 \text{ if and only if } C \leqslant A(m, \alpha). \tag{8}$$

To this end, one introduces the following elementary variable transformation, an extension, and combination, of transformations considered by Hartman [13] (see also [14, p. 324–325]) and Müller-Pfeiffer [21, p. 200–207]: Assume temporarily that

$$\alpha \in \mathbb{R}\backslash\{j \mid 1 \leqslant j \leqslant 2m-1\}. \tag{9}$$

Given $f \in C_0^\infty((0,\infty))$, the transformation

$$x = e^t,\ x \in (0,\infty), \quad dx = e^t dt,\ t \in \mathbb{R}, \tag{10}$$

$$f(x) \equiv f(e^t) = e^{[(2m-1-\alpha)/2]t} w(t), \quad w \in C_0^\infty(\mathbb{R}), \tag{11}$$

yields

$$\big(x^\alpha f^{(m)}(x)\big)^{(m)} = e^{-[(2m+1-\alpha)/2]t} \sum_{\ell=0}^{2m} c_\ell(m,\alpha) w^{(\ell)}(t), \tag{12}$$

for appropriate constants $c_\ell(m, \alpha)$, $\ell = 0, 1, \ldots, 2m$ to be determined next.

The solutions of the differential equation

$$\big(x^\alpha f^{(m)}(x)\big)^{(m)} = 0, \tag{13}$$

are linear combinations of the following powers of x:

$$\begin{cases} x^j, & j = 0, 1, \dots, m-1, \\ x^{k-\alpha}, & k = m, \dots, 2m-1. \end{cases} \tag{14}$$

One notes that the solutions (14) are linearly independent due to (9).

Thus, recalling (10)–(12), it follows that the solutions of

$$\sum_{\ell=0}^{2m} c_\ell(m,\alpha) w^{(\ell)}(t) = 0, \quad t \in \mathbb{R}, \tag{15}$$

are the functions

$$e^{[(1+\alpha-2m)/2]t} x^j = e^{[(2j+1+\alpha-2m)/2])t}, \quad j = 0, 1, \dots, m-1, \tag{16}$$

and

$$e^{[(1+\alpha-2m)/2]t} x^{k-\alpha} = e^{(2k+1-\alpha-2m)/2]t} \quad k = m, \dots, 2m-1. \tag{17}$$

One observes that for $j = 0$ and $k = 2m-1$,

$$e^{[(2j+1+\alpha-2m)/2]t} = e^{[(1+\alpha-2m)/2]t}, \quad e^{[(2k+1-\alpha-2m)/2]t} = e^{-[(1+\alpha-2m)/2]t}, \tag{18}$$

and for $j = 1$ and $k = 2m-2$,

$$e^{[(2j+1+\alpha-2m)/2]t} = e^{[(3+\alpha-2m)/2]t}, \quad e^{[(2k+1-\alpha-2m)/2]t} = e^{-[(3+\alpha-2m)/2]t}. \tag{19}$$

Continuing iteratively, one concludes that the linearly independent solutions of (15) are of the form

$$e^{\pm[(2j+1-2m+\alpha)/2]t}, \quad j = 0, 1, \dots, m-1. \tag{20}$$

By a simple relabeling, given $\alpha \in \mathbb{R}\backslash\{j \mid 1 \leqslant j \leqslant 2m-1\}$, this is equivalent to

$$e^{\pm[(2j-1-\alpha)/2]t}, \quad j = 1, \dots, m, \ t \in \mathbb{R}, \tag{21}$$

are linearly independent solutions of (15). The zeros of the characteristic polynomial of (15) are thus the constant factors in the exponents of (21). Hence, the characteristic polynomial is given by

$$\begin{aligned}P_{m,\alpha}(\lambda) &= \sum_{\ell=0}^{2m} c_\ell(m,\alpha)\lambda^\ell \\ &= \left(\lambda^2 - \frac{(1-\alpha)^2}{4}\right)\left(\lambda^2 - \frac{(3-\alpha)^2}{4}\right)\cdots\left(\lambda^2 - \frac{(2m-1-\alpha)^2}{4}\right) \\ &= \prod_{j=1}^{m}\left(\lambda^2 - \frac{(2j-1-\alpha)^2}{4}\right). \end{aligned} \tag{22}$$

Thus, the coefficients $c_\ell(m,\alpha)$, $\ell = 0, 1, \dots, 2m$, satisfy the following properties:

$$\begin{aligned} &(i)\ c_{2j-1}(m,\alpha) = 0, \quad j = 1, \dots, m; \\ &(ii)\ c_{2j}(m,\alpha) = (-1)^{m-j}|c_{2j}(m,\alpha)|, \quad j = 0, 1, \dots, m; \\ &(iii)\ |c_0(m,\alpha)| = A(m,\alpha); \\ &(iv)\ c_{2m}(m,\alpha) = 1. \end{aligned} \tag{23}$$

Applying this transformation to (7) yields,

$$\begin{aligned}(f, Qf)_{L^2((0,\infty);dx)} &= \int_0^\infty dx\, \overline{f(x)}\Big\{\Big[(-1)^m\big(x^\alpha f^{(m)}(x)\big)^{(m)} - Cx^{\alpha-2m}f(x)\Big]\Big\} \\ &= \int_{-\infty}^\infty dt\, \bigg\{e^t\bigg[(-1)^m e^{-[(2m+1-\alpha)/2]t}\sum_{j=0}^m (-1)^{m-j}|c_{2j}(m,\alpha)|w^{(2j)}(t) \\ &\qquad - Ce^{(\alpha-2m)t+[(2m-1-\alpha)/2]t}w(t)\bigg]e^{[(2m-1-\alpha)/2]t}\overline{w(t)}\bigg\} \\ &= \int_{-\infty}^\infty dt\, \bigg\{\overline{w(t)}\bigg[\sum_{j=0}^m (-1)^j|c_{2j}(m,\alpha)|w^{(2j)}(t) - Cw(t)\bigg]\bigg\}. \end{aligned} \tag{24}$$

Hence, $Q \geqslant 0$ in $L^2((0,\infty); dx)$ if and only if the constant coefficient operator S in $L^2(\mathbb{R}; dt)$, defined by

$$S = \bigg[\sum_{j=0}^m (-1)^j|c_{2j}(m,\alpha)|\frac{d^{2j}}{dt^{2j}} - C\bigg]\bigg|_{C_0^\infty(\mathbb{R})}, \tag{25}$$

satisfies $S \geqslant 0$ in $L^2(\mathbb{R}; dt)$. Invoking the Fourier transform, the closure $\overline{S}$ of S in $L^2(\mathbb{R}; dt)$ is unitarily equivalent to the maximally defined operator of multiplication

T in $L^2(\mathbb{R}; d\xi)$ by the polynomial $\sum_{j=1}^m |c_{2j}(m,\alpha)|\xi^{2j} + \big(|c_0(m,\alpha)| - C\big)$, that is,

$$\begin{aligned} &(Tv)(\xi) = \sum_{j=1}^m |c_{2j}(m,\alpha)|\xi^{2j} v(\xi) + \big(|c_0(m,\alpha)| - C\big)v(\xi), \\ &v \in \operatorname{dom}(T) = \bigg\{u \in L^2(\mathbb{R}; d\xi) \,\bigg|\, \int_{-\infty}^{\infty} d\xi\, \xi^{4m} |u(\xi)|^2 < \infty\bigg\}. \end{aligned} \tag{26}$$

Recalling (23), part (iii), T (and hence Q) is nonnegative if and only if $C \leqslant A(m,\alpha)$. Moreover, if $C = A(m,\alpha)$, then $T \geqslant 0$ with trivial nullspace, $\ker(T) = \{0\}$. Thus, (4) is strict unless $f \equiv 0$.

The case $\alpha \in \{j \,|\, 1 \leqslant j \leqslant 2m-1\}$ then follows by taking the limits $\alpha \to k \in \{j \,|\, 1 \leqslant j \leqslant 2m-1\}$, noting that $A(m,\alpha)$ and $c_{2j}(m,\alpha)$ are continuous as polynomials in $\alpha \in \mathbb{R}$. □

Next, we extend Lemma 2.1 to include all intermediate cases $\ell \in \{1,\dots,m\}$ in (1). For this purpose we recall the following elementary identity: Given $A(m,\alpha)$, $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$, in (3), one has,

$$A(m,\alpha) = A(\ell,\alpha)A(m-\ell,\alpha-2\ell), \quad m \in \mathbb{N},\ \ell \in \{1,\dots,m\},\ \alpha \in \mathbb{R}. \tag{27}$$

Indeed,

$$\begin{aligned} A(\ell,\alpha)A(m-\ell,\alpha-2\ell) &= 4^{-\ell}\prod_{j=1}^{\ell}(2j-1-\alpha)^2\, 4^{-(m-\ell)}\prod_{k=1}^{m-\ell}[2k-1-(\alpha-2\ell)]^2 \\ &= 4^{-m}\prod_{j=1}^{\ell}(2j-1-\alpha)^2\prod_{k=1}^{m-\ell}[2(k+\ell)-1-\alpha]^2 \\ &= 4^{-m}\prod_{j=1}^{\ell}(2j-1-\alpha)^2\prod_{k=\ell+1}^{m}(2k-1-\alpha)^2 \\ &= 4^{-m}\prod_{j=1}^{m}(2j-1-\alpha)^2 \\ &= A(m,\alpha). \end{aligned} \tag{28}$$

Theorem 2.2 *One has*

$$\begin{aligned} &\int_0^\infty dx\, x^\alpha \big|f^{(m)}(x)\big|^2 \geqslant A(\ell,\alpha)\int_0^\infty dx\, x^{\alpha-2\ell}\big|f^{(m-\ell)}(x)\big|^2, \\ &\qquad m \in \mathbb{N},\ \ell \in \{1,\dots,m\},\ \alpha \in \mathbb{R},\ f \in C_0^\infty((0,\infty)). \end{aligned} \tag{29}$$

Moreover, the constants $A(\ell, \alpha)$*, for* $1 \leqslant \ell \leqslant m$*,* $\alpha \in \mathbb{R}\backslash\{2j-1\}_{\ell+1\leqslant j\leqslant m}$*, in* (29) *are optimal and the inequality is strict, that is, equality holds in* (29) *if and only if* $f \equiv 0$*.*

Proof Replacing f by $f^{(m-\ell)}$ in (4) yields (29).

Thus, it remains to prove the optimality of $A(\ell, \alpha)$ in (29). Arguing by contradiction, we suppose there exists $C > A(\ell, \alpha)$ such that

$$\int_0^\infty dx\, x^\alpha \big|f^{(m)}(x)\big|^2 \geqslant C \int_0^\infty dx\, x^{\alpha-2\ell}\big|f^{(m-\ell)}(x)\big|^2, \quad f \in C_0^\infty((0,\infty)). \tag{30}$$

Applying (4) again, one concludes with the help of (27),

$$\begin{aligned}\int_0^\infty dx\, x^\alpha \big|f^{(m)}(x)\big|^2 &\geqslant CA(m-\ell, \alpha-2\ell)\int_0^\infty dx\, x^{\alpha-2\ell-2(m-\ell)}\big|f(x)\big|^2\\ &> A(\ell,\alpha)A(m-\ell,\alpha-2\ell)\int_0^\infty dx\, x^{\alpha-2m}\big|f(x)\big|^2\\ &= A(m,\alpha)\int_0^\infty dx\, x^{\alpha-2m}\big|f(x)\big|^2, \quad f \in C_0^\infty((0,\infty)),\end{aligned} \tag{31}$$

contradicting the optimality of $A(m, \alpha)$ in (4). The condition $\alpha \in \mathbb{R}\backslash\{2j-1\}_{\ell+1\leqslant j\leqslant m}$ is used to guarantee that $A(m-\ell, \alpha-2\ell) \neq 0$. □

3 Some Generalizations

In this section we turn to generalizations regarding the use of homogeneous Sobolev spaces; we also treat the vector-valued case where $f(\,\cdot\,) \in \mathcal{H}$, with $\mathcal{H}$ a complex, separable Hilbert space. (For the necessary background in the vector-valued context we refer to Appendix A.)

We start with the following elementary result.

Lemma 3.1 *One has*

$$\begin{gathered}\int_0^\infty dx\, x^\alpha \big\|f^{(m)}(x)\big\|_{\mathcal{H}}^2 \geqslant A(\ell,\alpha)\int_0^\infty dx\, x^{\alpha-2\ell}\big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2,\\ m\in\mathbb{N},\ \ell\in\{1,\dots,m\},\ \alpha\in\mathbb{R},\ f\in C_0^\infty((0,\infty);\mathcal{H}).\end{gathered} \tag{32}$$

Proof Using the notations in the proof of Lemma A.3, Theorem 2.2 implies,

$$\begin{aligned}
\int_0^\infty dx\, x^\alpha \|f^{(m)}(x)\|_{\mathcal{H}}^2 &= \int_0^\infty dx\, x^\alpha \sum_{k=1}^\infty \big|\big(f^{(m)}(x), \varphi_k\big)_{\mathcal{H}}\big|^2 \\
&= \sum_{k=1}^\infty \int_0^\infty dx\, x^\alpha \big|\big(f_k\big)^{(m)}(x)\big|^2 \\
&\geqslant \sum_{k=1}^\infty A(\ell,\alpha) \int_0^\infty dx\, x^{\alpha-2\ell} \big|\big(f_k\big)^{(m-\ell)}(x)\big|^2 \\
&= A(\ell,\alpha) \sum_{k=1}^\infty \int_0^\infty dx\, x^{\alpha-2\ell} \big|\big(f^{(m-\ell)}\big)_k(x)\big|^2 \\
&= A(\ell,\alpha) \int_0^\infty dx\, x^{\alpha-2\ell} \sum_{k=1}^\infty \big|\big(f^{(m-\ell)}(x), \varphi_k\big)_{\mathcal{H}}\big|^2 \\
&= A(\ell,\alpha) \int_0^\infty dx\, x^{\alpha-2\ell} \|f^{(m-\ell)}(x)\|_{\mathcal{H}}^2. \qquad (33)
\end{aligned}$$

□

Definition 3.2 Let $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$.

(i) Define

$$\dot{H}^m\big((0,\infty); x^\alpha dx; \mathcal{H}\big) = \bigg\{ f : (0,\infty) \to \mathcal{H} \,\bigg|\, f^{(j)} \in AC_{loc}((0,\infty);\mathcal{H}); \; j = 0, 1, \dots, m-1; \int_0^\infty dx\, x^\alpha \|f^{(m)}(x)\|_{\mathcal{H}}^2 < \infty \bigg\}. \qquad (34)$$

We also introduce,

$$|||f|||_{m,\alpha}^2 = \int_0^\infty dx\, x^\alpha \|f^{(m)}(x)\|_{\mathcal{H}}^2, \quad f \in \dot{H}^m\big((0,\infty); x^\alpha dx; \mathcal{H}\big). \qquad (35)$$

(ii) Assume $A(m,\alpha) > 0$ and define

$$\dot{H}_0^m\big((0,\infty); x^\alpha dx; \mathcal{H}\big) = \bigg\{ f \in \dot{H}^m\big((0,\infty); x^\alpha dx; \mathcal{H}\big) \,\bigg|\, \text{there exists a Cauchy sequence } \{f_n\}_{n=1}^\infty \text{ in } \big(C_0^\infty((0,\infty);\mathcal{H}), |||\cdot|||_{m,\alpha}\big) \text{ such that, for } 0 \leqslant \ell \leqslant m, \text{ we have } \lim_{n\to\infty} \int_0^\infty dx\, x^{\alpha-2\ell} \|f_n^{(m-\ell)}(x) - f^{(m-\ell)}(x)\|_{\mathcal{H}}^2 = 0 \bigg\} \qquad (36)$$

$$= \Big\{ f \in \dot{H}^m\big((0,\infty); x^\alpha dx; \mathcal{H}\big) \,\Big|\, \text{there exists a Cauchy sequence } \{f_n\}_{n=1}^\infty \text{ in } \big(C_0^\infty((0,\infty);\mathcal{H}), |||\cdot|||_{m,\alpha}\big) \text{ such that, for all } 0<a<b<\infty, \lim_{n\to\infty}\int_a^b dx\, \|f_n(x)-f(x)\|_{\mathcal{H}}^2 = 0 \Big\}. \tag{37}$$

Remark 3.3 To see that (36) and (37) describe the same space, we first note that if $\{f_n\}_{n=1}^\infty \subseteq \big(C_0^\infty((0,\infty);\mathcal{H}), |||\cdot|||_{m,\alpha}\big)$ and $f \in \dot{H}^m\big((0,\infty); x^\alpha dx; \mathcal{H}\big)$ satisfy (36), then, by putting $\ell = m$ in (36), (37) is satisfied. Next, suppose $f \in \dot{H}^m\big((0,\infty); x^\alpha dx; \mathcal{H}\big)$ satisfies (37). By Lemma A.4 there exists a unique $g \in \dot{H}^m\big((0,\infty); x^\alpha dx; \mathcal{H}\big)$ such that for $\ell = 0, 1, \dots, m$,

$$\lim_{n\to\infty}\int_0^\infty dx\, x^{\alpha-2\ell}\big\|f_n^{(m-\ell)}(x) - g^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2 = 0. \tag{38}$$

Putting $\ell = m$ in (38) we have,

$$\lim_{n\to\infty}\int_a^b dx\, x^{\alpha-2m}\|f_n(x)-g(x)\|_{\mathcal{H}}^2 = 0 \tag{39}$$

for all $0<a<b<\infty$, hence $f=g$ since (37) is true by assumption. Thus, (38) implies (36). ◇

Clearly, $C_0^\infty((0,\infty)); \mathcal{H}) \subseteq \dot{H}^m\big((0,\infty); x^\alpha dx; \mathcal{H}\big)$. Moreover, by Lemma A.4, $\dot{H}_0^m\big((0,\infty); x^\alpha dx; \mathcal{H}\big)$ can be identified with the completion of $C_0^\infty((0,\infty);\mathcal{H})$ with respect to the norm $|||\cdot|||_{m,\alpha}$, that is,

$$\dot{H}_0^m\big((0,\infty); x^\alpha dx; \mathcal{H}\big) = \overline{C_0^\infty((0,\infty);\mathcal{H})}^{|||\cdot|||_{m,\alpha}}. \tag{40}$$

Theorem 3.4 *Let $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$, and assume $A(m,\alpha) > 0$. Then,*

$$\int_0^\infty dx\, x^\alpha \|f^{(m)}(x)\|_{\mathcal{H}}^2 \geqslant A(\ell,\alpha)\int_0^\infty dx\, x^{\alpha-2\ell}\|f^{(m-\ell)}(x)\|_{\mathcal{H}}^2, \quad \ell \in \{1,\dots,m\},\ f \in \dot{H}_0^m\big((0,\infty); x^\alpha dx; \mathcal{H}\big). \tag{41}$$

Proof Let $f \in \dot{H}_0^m\big((0,\infty); x^\alpha dx; \mathcal{H}\big)$ and let $\{f_n\}_{n=1}^\infty$ be a Cauchy sequence in $\big(C_0^\infty((0,\infty);\mathcal{H}), |||\,\cdot\,|||_{m,\alpha}\big)$ satisfying (36). Then, for all $\ell \in \{1,\dots,m\}$,

$$\begin{aligned}\int_0^\infty dx\, x^\alpha \big\|f^{(m)}(x)\big\|_{\mathcal{H}}^2 &= \lim_{n\to\infty}\int_0^\infty dx\, x^\alpha \big\|f_n^{(m)}(x)\big\|_{\mathcal{H}}^2 \\ &\geqslant A(\ell,\alpha)\lim_{n\to\infty}\int_0^\infty dx\, x^{\alpha-2\ell}\big\|f_n^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2 \\ &= A(\ell,\alpha)\int_0^\infty dx\, x^{\alpha-2\ell}\big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2, \end{aligned} \tag{42}$$

where we have used (36) and Lemma 3.1. □

Next, we turn to the case $A(m,\alpha)=0$.

Definition 3.5 Let $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$.

(i) Define

$$H^m((0,\infty); x^\alpha dx; \mathcal{H}) = \bigg\{ f : (0,\infty)\to\mathcal{H} \,\bigg|\, f^{(j)} \in AC_{loc}((0,\infty);\mathcal{H}), \\ 0\leqslant j\leqslant m-1;\ \|f\|_{m,\alpha}^2 = \sum_{j=0}^m \int_0^\infty dx\, x^\alpha \big\|f^{(j)}(x)\big\|_{\mathcal{H}}^2 < \infty \bigg\}. \tag{43}$$

(ii) Define

$$H_0^m((0,\infty); x^\alpha dx; \mathcal{H}) = \bigg\{ f \in H^m((0,\infty); x^\alpha dx; \mathcal{H}) \,\bigg|\, \text{there exists a Cauchy sequence } \{f_n\}_{n=1}^\infty \text{ in } \big(C_0^\infty((0,\infty);\mathcal{H}), \|\cdot\|_{m,\alpha}\big) \text{ such that } \lim_{n\to\infty}\|f_n - f\|_{m,\alpha} = 0 \bigg\} \tag{44}$$

$$= \bigg\{ f \in H^m((0,\infty); x^\alpha dx; \mathcal{H}) \,\bigg|\, \text{there exists a Cauchy sequence } \{f_n\}_{n=1}^\infty \text{ in } \big(C_0^\infty((0,\infty);\mathcal{H}), \|\cdot\|_{m,\alpha}\big) \text{ such that for any } 0<a<b<\infty \text{ we have } \lim_{n\to\infty}\int_a^b dx\,\|f_n(x)-f(x)\|_{\mathcal{H}}^2 = 0 \bigg\}. \tag{45}$$

Remark 3.6 To see that (44) and (45) describe the same space, we first note that if $\{f_n\}_{n=1}^\infty \subseteq \big(C_0^\infty((0,\infty);\mathcal{H}), \|\cdot\|_{m,\alpha}\big)$ and $f \in H^m\big((0,\infty); x^\alpha dx; \mathcal{H}\big)$ satisfy (44), then clearly (45) is immediately satisfied. Next, suppose $f \in$

$H^m\big((0,\infty);\ x^\alpha dx;\mathcal{H}\big)$ satisfies (45). By Lemma A.6 there is a unique $g \in H^m\big((0,\infty);x^\alpha dx;\mathcal{H}\big)$ such that

$$\lim_{n\to\infty}\int_0^\infty dx\, x^\alpha \big\| f_n(x)-g(x)\big\|_{\mathcal{H}}^2 \leqslant \lim_{n\to\infty}\|f_n-g\|_{m,\alpha}=0, \tag{46}$$

and hence, for all $0<a<b<\infty$,

$$\lim_{n\to\infty}\int_a^b dx\,\|f_n(x)-g(x)\|_{\mathcal{H}}^2=0. \tag{47}$$

By the assumption that (45) is true, this gives $f=g$. Thus (46) implies (44). ◇

Clearly $C_0^\infty((0,\infty));\mathcal{H})\subseteq H^m\big((0,\infty);x^\alpha dx;\mathcal{H}\big)$ and $\|\cdot\|_{m,\alpha}$ is a norm on $H^m\big((0,\infty);x^\alpha dx;\mathcal{H}\big)$. By Lemma A.6, $H_0^m((0,\infty);x^\alpha dx;\mathcal{H})$ can be identified with the completion of $C_0^\infty((0,\infty);\mathcal{H})$ with respect to $\|\cdot\|_{m,\alpha}$, that is,

$$H_0^m((0,\infty);x^\alpha dx;\mathcal{H})=\overline{C_0^\infty((0,\infty);\mathcal{H})}^{\|\cdot\|_{m,\alpha}}. \tag{48}$$

Theorem 3.7 *Let $m\in\mathbb{N}$, $\alpha\in\mathbb{R}$, and assume that $A(m,\alpha)=0$. Then,*

$$\begin{gathered}\int_0^\infty dx\,x^\alpha\big\|f^{(m)}(x)\|_{\mathcal{H}}^2\geqslant A(\ell,\alpha)\int_0^\infty dx\,x^{\alpha-2\ell}\big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2,\\ \ell\in\{1,\dots,m\},\ \ f\in H_0^m((0,\infty);x^\alpha dx;\mathcal{H}).\end{gathered} \tag{49}$$

Proof Let $f\in H_0^m((0,\infty);x^\alpha dx;\mathcal{H})$ and let $\{f_n\}_{n=1}^\infty$ be a Cauchy sequence in $\big(C_0^\infty((0,\infty);\mathcal{H}),\|\cdot\|_{m,\alpha}\big)$ satisfying (44). Let $\ell\in\{1,\dots,m\}$. Since

$$\lim_{n\to\infty}\int_0^\infty dx\,x^\alpha\big\|f_n^{(m-\ell)}(x)-f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2=0, \tag{50}$$

there exists a subsequence $\{f_{n_k}\}_{k=1}^\infty$ of $\{f_n\}_{n=1}^\infty$ such that

$$\lim_{k\to\infty}f_{n_k}^{(m-\ell)}(x)=f^{(m-\ell)}(x)\ \text{ for a.e. } x\in(0,\infty). \tag{51}$$

Hence, by Fatou's lemma,

$$\begin{aligned}&A(\ell,\alpha)\int_0^\infty dx\,x^{\alpha-2\ell}\big\|f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2\\ &\quad\leqslant\lim_{k\to\infty}\inf A(\ell,\alpha)\int_0^\infty dx\,x^{\alpha-2\ell}\big\|f_{n_k}^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2\\ &\quad\leqslant\lim_{k\to\infty}\inf\int_0^\infty dx\,x^\alpha\big\|f_{n_k}^{(m)}(x)\big\|_{\mathcal{H}}^2\qquad\text{(by Lemma 3.1)}\end{aligned}$$

$$= \lim_{k\to\infty} \int_0^\infty dx\, x^\alpha \big\| f_{n_k}^{(m)}(x) \big\|_{\mathcal{H}}^2$$

$$= \int_0^\infty dx\, x^\alpha \big\| f^{(m)}(x) \big\|_{\mathcal{H}}^2 \qquad \text{(by (44))}. \tag{52}$$

□

Next, we consider bounded intervals $(0, \rho)$, $0 < \rho < \infty$, and recall a simplified version of [7, Theorem 3.1 (iii)].

Lemma 3.8 *Let $\gamma \in (0, \infty)$, $\rho \in (0, \gamma)$, and set*

$$B(m, \alpha) = 4^{-m} \sum_{k=1}^{m} \prod_{j=1,\, j\neq k}^{m} (2j - 1 - \alpha)^2 > 0, \quad m \in \mathbb{N},\ \alpha \in \mathbb{R}. \tag{53}$$

Then,

$$\begin{gathered} \int_0^\rho dx\, x^\alpha \big| f^{(m)}(x) \big|^2 \geqslant B(\ell, \alpha) \int_0^\rho dx\, x^{\alpha - 2\ell} \big| f^{(m-\ell)}(x) \big|^2 [\ln(\gamma/\mathrm{x})]^{-2}, \\ m \in \mathbb{N},\ \ell \in \{1, \dots, m\},\ \alpha \in \mathbb{R},\ f \in C_0^\infty((0, \rho)). \end{gathered} \tag{54}$$

Lemma 3.9 *Let $\gamma \in (0, \infty)$, $\rho \in (0, \gamma)$. Then,*

$$\begin{gathered} \int_0^\rho dx\, x^\alpha \big\| f^{(m)}(x) \big\|_{\mathcal{H}}^2 \geqslant B(\ell, \alpha) \int_0^\rho dx\, x^{\alpha - 2\ell} \big\| f^{(m-\ell)}(x) \big\|_{\mathcal{H}}^2 [\ln(\gamma/\mathrm{x})]^{-2}, \\ m \in \mathbb{N},\ \ell \in \{1, \dots, m\},\ \alpha \in \mathbb{R},\ f \in C_0^\infty((0, \rho); \mathcal{H}). \end{gathered} \tag{55}$$

Proof Let $\{\varphi_k\}_{k=1}^\infty$ be an orthonormal basis of $\mathcal{H}$. For $k \in \mathbb{N}$, let $f_k \in C_0^\infty((0, \rho))$ be defined as in (A.4) so that (A.5) holds. Then

$$\int_0^\infty dx\, x^\alpha \big\| f^{(m)}(x) \big\|_{\mathcal{H}}^2 = \int_0^\rho dx\, x^\alpha \sum_{k=1}^\infty \big| \big(f^{(m)}(x), \varphi_k \big)_{\mathcal{H}} \big|^2$$

$$= \sum_{k=1}^\infty \int_0^\infty dx\, x^\alpha \big| \big(f_k \big)^{(m)}(x) \big|^2$$

$$\geqslant \sum_{k=1}^\infty B(\ell, \alpha) \int_0^\rho dx\, x^{\alpha - 2\ell} \big| \big(f_k \big)^{(m-\ell)}(x) \big|^2 [\ln(\gamma/x)]^{-2}$$

$$= B(\ell, \alpha) \sum_{k=1}^\infty \int_0^\rho dx\, x^{\alpha - 2\ell} \big| \big(f^{(m-\ell)} \big)_k(x) \big|^2 [\ln(\gamma/x)]^{-2}$$

$$= B(\ell, \alpha) \int_0^\rho dx\, x^{\alpha-2\ell} \sum_{k=1}^{\infty} \big|\big(f^{(m-\ell)}(x), \varphi_k\big)_{\mathcal{H}}\big|^2 [\ln(\gamma/x)]^{-2}$$

$$= B(\ell, \alpha) \int_0^\rho dx\, x^{\alpha-2\ell} \|f^{(m-\ell)}(x)\|_{\mathcal{H}}^2 [\ln(\gamma/x)]^{-2}. \tag{56}$$

□

Definition 3.10 Let $m \in \mathbb{N}$, $\rho \in (0, \infty)$, $\alpha \in \mathbb{R}$.

(*i*) Define

$$\begin{aligned} H_0^m((0, \rho); x^\alpha dx; \mathcal{H}) = \Big\{ f : (0, \rho) \to \mathcal{H} \,\Big|\, & f^{(j)} \in AC_{loc}((0, \rho)),\ 0 \leqslant j \leqslant m-1; \\ & \text{there exists a Cauchy sequence } \{f_n\}_{n=1}^\infty \text{ in } \big(C_0^\infty((0, \rho); \mathcal{H}), \|\cdot\|_{m,\alpha}\big) \\ & \text{such that } \lim_{n\to\infty} \int_0^\rho dx\, x^\alpha \|f_n^{(k)}(x) - f^{(k)}(x)\|_{\mathcal{H}}^2 = 0,\ 0 \leqslant k \leqslant m \Big\} \end{aligned} \tag{57}$$

$$\begin{aligned} = \Big\{ f : (0, \rho) \to \mathcal{H} \,\Big|\, & f^{(j)} \in AC_{loc}((0, \rho)),\ 0 \leqslant j \leqslant m-1; \text{ there exists a} \\ & \text{Cauchy sequence } \{f_n\}_{n=1}^\infty \text{ in } \big(C_0^\infty((0, \rho); \mathcal{H}), \|\cdot\|_{m,\alpha}\big) \text{ such that for any} \\ & 0 < a < b < \rho, \text{ we have } \lim_{n\to\infty} \int_a^b dx\, \|f_n(x) - f(x)\|_{\mathcal{H}}^2 = 0 \Big\}. \end{aligned} \tag{58}$$

(*ii*) Define

$$\begin{aligned} \overset{\bullet}{H}{}_0^m\big((0, \rho); x^\alpha dx; \mathcal{H}\big) = \Big\{ f : (0, \rho) \to \mathcal{H} \,\Big|\, & f^{(j)} \in AC_{loc}((0, \rho)),\ 0 \leqslant j \leqslant m-1; \\ & \text{there exists a Cauchy sequence } \{f_n\}_{n=1}^\infty \text{ in } \big(C_0^\infty((0, \rho); \mathcal{H}), |||\cdot|||_{m,\alpha}\big) \\ & \text{such that } \lim_{n\to\infty} \int_0^\rho dx\, x^\alpha \|f_n^{(m)}(x) - f^{(m)}(x)\|_{\mathcal{H}}^2 = 0 \Big\} \end{aligned} \tag{59}$$

$$\begin{aligned} = \Big\{ f : (0, \rho) \to \mathcal{H} \,\Big|\, & f^{(j)} \in AC_{loc}((0, \rho)),\ 0 \leqslant j \leqslant m-1; \text{ there exists a} \\ & \text{Cauchy sequence } \{f_n\}_{n=1}^\infty \text{ in } \big(C_0^\infty((0, \rho); \mathcal{H}), |||\cdot|||_{m,\alpha}\big) \text{ such that for} \\ & \text{any } 0 < a < b < \rho, \text{ we have } \lim_{n\to\infty} \int_a^b dx\, \|f_n(x) - f(x)\|_{\mathcal{H}}^2 = 0 \Big\}. \end{aligned} \tag{60}$$

Remark 3.11

(*i*) An argument similar to that from Remark 3.6 shows that the conditions (57) and (58) describe the same space.

(ii) Lemma A.7 together with (i) show that the conditions (59) and (60) describe the same space. $\diamond$

By Lemma A.7 one has

$$H_0^m((0,\rho); x^\alpha dx; \mathcal{H}) = \overset{\bullet}{H}{}_0^m\big((0,\rho); x^\alpha dx; \mathcal{H}\big), \quad m \in \mathbb{N},\ \alpha \in \mathbb{R},\ \rho \in (0,\infty). \tag{61}$$

Theorem 3.12 *Let $\rho \in (0,\infty)$. Then,*

$$\int_0^\rho dx\, x^\alpha \big\| f^{(m)}(x) \big\|_{\mathcal{H}}^2 \geqslant A(\ell,\alpha) \int_0^\rho dx\, x^{\alpha-2\ell} \big\| f^{(m-\ell)}(x) \big\|_{\mathcal{H}}^2, \\ m \in \mathbb{N},\ \ell \in \{1,\dots,m\},\ \alpha \in \mathbb{R},\ f \in H_0^m((0,\rho); x^\alpha dx; \mathcal{H}). \tag{62}$$

Proof Let $f \in H_0^m((0,\rho); x^\alpha dx; \mathcal{H})$ and let $\{f_n\}_{n=1}^\infty$ be a Cauchy sequence in $\big(C_0^\infty((0,\rho); \mathcal{H}), \|\cdot\|_{m,\alpha}\big)$ satisfying (57). Let $\ell \in \{1,\dots,m\}$. Since

$$\lim_{n\to\infty} \int_0^\rho dx\, x^\alpha \big\| f_n^{(m-\ell)}(x) - f^{(m-\ell)}(x) \big\|_{\mathcal{H}}^2 = 0, \tag{63}$$

there exists a subsequence $\{f_{n_k}\}_{k=1}^\infty$ of $\{f_n\}_{n=1}^\infty$ such that

$$\lim_{k\to\infty} f_{n_k}^{(m-\ell)}(x) = f^{(m-\ell)}(x) \text{ for a.e. } x \in (0,\rho). \tag{64}$$

Hence, by Fatou's lemma,

$$\begin{aligned} &A(\ell,\alpha) \int_0^\rho dx\, x^{\alpha-2\ell} \big\| f^{(m-\ell)}(x) \big\|_{\mathcal{H}}^2 \\ &\leqslant \liminf_{k\to\infty} A(\ell,\alpha) \int_0^\rho dx\, x^{\alpha-2\ell} \big\| f_{n_k}^{(m-\ell)}(x) \big\|_{\mathcal{H}}^2 \\ &\leqslant \liminf_{k\to\infty} \int_0^\rho dx\, x^\alpha \big\| f_{n_k}^{(m)}(x) \big\|_{\mathcal{H}}^2 \qquad \text{(by Lemma 3.1)} \\ &= \lim_{k\to\infty} \int_0^\rho dx\, x^\alpha \big\| f_{n_k}^{(m)}(x) \big\|_{\mathcal{H}}^2 \\ &= \int_0^\rho dx\, x^\alpha \big\| f^{(m)}(x) \big\|_{\mathcal{H}}^2 \qquad \text{(by (57))}. \end{aligned} \tag{65}$$

$\square$

We now establish analogous results on the exterior domain (ρ,∞), $\rho \in (0,\infty)$.

Lemma 3.13 *For $m \in \mathbb{N}$, $\rho \in (0, \infty)$, $\alpha \in \mathbb{R}$, let*

$$\dot{H}^m((\rho,\infty); x^\alpha dx; \mathcal{H}) = \Big\{ f : (\rho,\infty) \to \mathcal{H} \mid f^{(j)} \in AC_{loc}((\rho,\infty); \mathcal{H}) \text{ and } \lim_{x \downarrow \rho} f^{(j)}(x) = 0,\ j = 0, 1, \dots, m-1;\ f^{(m)} \in L^2((\rho,\infty); x^\alpha dx; \mathcal{H}) \Big\}. \tag{66}$$

For $f, g \in \dot{H}^m((\rho,\infty); x^\alpha dx; \mathcal{H})$ let

$$\langle f, g \rangle_{m,\alpha} = \int_\rho^\infty dx\, x^\alpha (f^{(m)}(x), g^{(m)}(x))_{\mathcal{H}}. \tag{67}$$

Then $\langle \cdot\,, \cdot \rangle_{m,\alpha}$ is an inner product on $\dot{H}^m((\rho,\infty); x^\alpha dx; \mathcal{H})$. In fact, $\big(\dot{H}^m((\rho,\infty); x^\alpha dx; \mathcal{H}), \langle \cdot\,, \cdot \rangle_{m,\alpha}\big)$ is a Hilbert space.

Proof The proof is analogous to the argument from [7, Proposition B.1]. □

Definition 3.14 For $m \in \mathbb{N}$, $\rho \in (0, \infty)$, $\alpha \in \mathbb{R}$ let $\dot{H}_0^m((\rho,\infty); x^\alpha dx; \mathcal{H})$ denote the closure of $C_0^\infty((\rho,\infty); \mathcal{H})$ in the space $\big(\dot{H}^m((\rho,\infty); x^\alpha dx; \mathcal{H}), \langle \cdot\,, \cdot \rangle_{m,\alpha}\big)$.

Lemma 3.15 *Let $f \in \dot{H}_0^m((\rho,\infty); x^\alpha dx; \mathcal{H})$. Then there is a sequence $\{f_n\}_{n=1}^\infty \subset C_0^\infty((\rho,\infty); \mathcal{H})$ such that*

$$\lim_{n\to\infty} \int_\rho^\infty dx\, x^\alpha \big\| f_n^{(m)}(x) - f^{(m)}(x) \big\|_{\mathcal{H}}^2 = 0 \tag{68}$$

and, for $k = 0, 1, \dots, m$, one has

$$\lim_{n\to\infty} f_n^{(k)}(x) = f^{(k)}(x) \text{ for a.e. } x \in (\rho, \infty). \tag{69}$$

Proof The proof is analogous to the argument from [7, Corollary B.2]. □

Theorem 3.16 *Let $\rho \in (0, \infty)$. Then,*

$$\begin{gathered} \int_\rho^\infty dx\, \big\| f^{(m)}(x) \big\|_{\mathcal{H}}^2 \geqslant A(\ell,\alpha) \int_\rho^\infty dx\, x^{\alpha-2\ell} \big\| f^{(m-\ell)}(x) \big\|_{\mathcal{H}}^2, \\ m \in \mathbb{N},\ \ell \in \{1, \dots, m\},\ \alpha \in \mathbb{R},\ f \in \dot{H}_0^m((\rho,\infty); x^\alpha dx; \mathcal{H}). \end{gathered} \tag{70}$$

for all $\ell = 1, \dots, m$, $\alpha \in \mathbb{R}$, $m \in \mathbb{N}$, and $f \in \dot{H}_0^m((\rho,\infty); x^\alpha dx; \mathcal{H})$.

Proof Let $\{f_n\}_{n=1}^{\infty} \subseteq C_0^{\infty}((\rho,\infty);\mathcal{H})$ be the sequence which satisfies (68) and (69). Then, by Fatou's lemma,

$$\begin{aligned}
&A(\ell,\alpha)\int_{\rho}^{\infty} dx\, x^{\alpha-2\ell}\,\|f^{(m-\ell)}(x)\|_{\mathcal{H}}^2\\
&\leqslant \liminf_{n\to\infty} A(\ell,\alpha)\int_{\rho}^{\infty} dx\, x^{\alpha-2\ell}\,\|f_n^{(m-\ell)}(x)\|_{\mathcal{H}}^2\\
&\leqslant \liminf_{n\to\infty} \int_{\rho}^{\infty} dx\, x^{\alpha}\,\|f_n^{(m)}(x)\|_{\mathcal{H}}^2 \qquad \text{(by Lemma 3.1)}\\
&= \lim_{n\to\infty} \int_{\rho}^{\infty} dx\, x^{\alpha}\,\|f_n^{(m)}(x)\|_{\mathcal{H}}^2\\
&= \int_{\rho}^{\infty} dx\, x^{\alpha}\,\|f^{(m)}(x)\|_{\mathcal{H}}^2. \qquad (71)
\end{aligned}$$

□

Optimality of $A(\ell,\alpha)$, and strictness of the inequalities in this section follow from Theorem 2.2.

Appendix A: Background for the Vector-Valued Case

For the remainder of this appendix, $\mathcal{H}$ denotes a separable complex Hilbert space.

Definition A.1

(i) Let $a, b \in \mathbb{R}$, $a < b$. A function $f : [a,b] \to \mathcal{H}$ is said to be absolutely continuous, denoted by $f \in AC([a,b];\mathcal{H})$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every finite collection $\{(a_j, b_j)\}_{j=1}^N$ of disjoint subintervals in $[a,b]$ with $\sum_{j=1}^N (b_j - a_j) < \delta$, one has

$$\sum_{j=1}^{N} \|f(b_j) - f(a_j)\|_{\mathcal{H}} < \varepsilon. \qquad (A.1)$$

(ii) A function $f : (c,d) \to \mathcal{H}$, $(c,d) \subseteq \mathbb{R}$, is said to be locally absolutely continuous on (c,d), denoted by $f \in AC_{loc}((c,d);\mathcal{H})$, if it is absolutely continuous on every compact subinterval $[a,b] \subset (c,d)$.

Lemma A.2 ([1, Propositions 1.2.2–1.2.4, Theorem 1.2.6]) *A map $f : (c,d) \to \mathcal{H}$, $(c,d) \subseteq \mathbb{R}$, is locally absolutely continuous, that is, $f \in AC_{loc}((c,d);\mathcal{H})$, if and only if there exists a locally Bochner integrable $g : (c,d) \to \mathcal{H}$ and $x_0 \in (c,d)$*

such that

$$f(x) = f(x_0) + \int_{x_0}^{x} dt\, g(t), \quad x \in (c, d). \tag{A.2}$$

If (A.2) *is satisfied, then*

$$f'(x) = g(x) \text{ for a.e. } x \in (c, d). \tag{A.3}$$

Lemma A.3 *Let $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$, and assume that $A(m, \alpha) > 0$. Then $|||\cdot|||_{m,\alpha}$, defined in* (35)*, is a norm on $C_0^\infty((0, \infty); \mathcal{H})$.*

Proof We only need to show that if $f \in C_0^\infty((0, \infty); \mathcal{H})$ and $|||f|||_{m,\alpha} = 0$, then $f = 0$. Let $\{\varphi_k\}_{k=1}^\infty$ be an orthonormal basis of $\mathcal{H}$. For $f \in C_0^\infty((0, \infty); \mathcal{H})$ and $k \in \mathbb{N}$ we write

$$f_k(x) = (f(x), \varphi_k)_{\mathcal{H}}, \quad x \in (0, \infty). \tag{A.4}$$

Then, for all $k, j \in \mathbb{N}$, we have

$$\big(f^{(j)}\big)_k(x) = \big(f_k\big)^{(j)}(x), \quad x \in (0, \infty). \tag{A.5}$$

Suppose $f \in C_0^\infty((0, \infty); \mathcal{H})$ and $|||f|||_{m,\alpha} = 0$. Then, for all $k \in \mathbb{N}$, $f_k \in C_0^\infty((0, \infty))$ and, applying Birman's inequalities to f_k, we have

$$\begin{aligned} 0 = |||f|||_{m,\alpha}^2 &= \int_0^\infty dx\, x^\alpha \big\| f^{(m)}(x)\big\|_{\mathcal{H}}^2 \geqslant \int_0^\infty dx\, x^\alpha \big|\big(f^{(m)}\big)_k(x)\big|^2 \\ &= \int_0^\infty dx\, x^\alpha \big|\big(f_k\big)^{(m)}(x)\big|^2 \geqslant A(m, \alpha) \int_0^\infty dx\, x^{\alpha - 2m} \big|f_k(x)\big|^2. \end{aligned} \tag{A.6}$$

Thus, since $A(m, \alpha) > 0$,

$$f_k(x) = 0, \quad x \in (0, \infty),\ k \in \mathbb{N}. \tag{A.7}$$

Hence, $f = 0$. □

Lemma A.4 *Let $m \in \mathbb{N}$ and $\alpha \in \mathbb{R}$. Suppose that $A(m, \alpha) > 0$ and let $\{f_n\}_{n=1}^\infty$ be a Cauchy sequence in $(C_0^\infty((0, \infty); \mathcal{H}), |||\cdot|||_{m,\alpha})$. Then there exists a unique $f \in \dot{H}^m\big((0, \infty); x^\alpha dx; \mathcal{H}\big)$, defined in Definition 3.2, such that, for all $0 \leqslant \ell \leqslant m$,*

$$\lim_{n \to \infty} \int_0^\infty dx\, x^{\alpha - 2\ell} \big\| f_n^{(m-\ell)}(x) - f^{(m-\ell)}(x)\big\|_{\mathcal{H}}^2 = 0. \tag{A.8}$$

Proof Since the sequence $\{f_n\}_{n=1}^\infty$ is Cauchy in $(C_0^\infty((0, \infty); \mathcal{H}), |||\cdot|||_{m,\alpha})$, the sequence $\{f_n^{(m)}\}_{n=1}^\infty$ is Cauchy in $L^2((0, \infty); x^\alpha dx; \mathcal{H})$, hence there exists $g_m \in$

$L^2((0,\infty); x^\alpha dx; \mathcal{H})$ such that

$$\lim_{n\to\infty} \int_0^\infty dx\, x^\alpha \big\| f_n^{(m)}(x) - g_m(x) \big\|_{\mathcal{H}}^2 = 0. \tag{A.9}$$

Since $A(m,\alpha) > 0$, $A(\ell,\alpha) > 0$ for all $1 \leqslant \ell \leqslant m-1$. Therefore, for $0 \leqslant \ell \leqslant m$ and $n_1, n_2 \in \mathbb{N}$, Lemma 3.1 implies

$$\begin{aligned} &A(\ell,\alpha) \int_0^\infty dx\, x^{\alpha-2\ell} \big\| f_{n_1}^{(m-\ell)}(x) - f_{n_2}^{(m-\ell)}(x) \big\|_{\mathcal{H}}^2 \\ &\quad \leqslant \int_0^\infty dx\, x^\alpha \big\| f_{n_1}^{(m)}(x) - f_{n_2}^{(m)}(x) \big\|_{\mathcal{H}}^2, \end{aligned} \tag{A.10}$$

hence $\{f_n^{(m-\ell)}\}_{n=1}^\infty$ is a Cauchy sequence in $L^2((0,\infty); x^{\alpha-2\ell}dx; \mathcal{H})$. Thus there exists $g_{m-\ell} \in L^2((0,\infty); x^{\alpha-2\ell}dx; \mathcal{H})$ such that, for $0 \leqslant \ell \leqslant m$,

$$\lim_{n\to\infty} \int_0^\infty dx\, x^{\alpha-2\ell} \big\| f_n^{(m-\ell)}(x) - g_{m-\ell}(x) \big\|_{\mathcal{H}}^2 = 0. \tag{A.11}$$

To complete the proof it remains to show

$$g_0^{(j)}(x) = g_j(x) \ \text{ for a.e. } x > 0,\ 1 \leqslant j \leqslant m. \tag{A.12}$$

By (A.11) there exist $K \subset (0,\infty)$ and a subsequence $\{f_{n_k}\}_{k=1}^\infty$ of $\{f_n\}_{n=1}^\infty$ such that $(0,\infty) \setminus K$ has zero Lebesgue measure and, for $0 \leqslant j \leqslant m$,

$$\lim_{k\to\infty} f_{n_k}^{(j)}(x) = g_j(x), \quad x \in K. \tag{A.13}$$

Fix $a \in K$. Then, for all $x \in K$,

$$\begin{aligned} g_0(x) &= \lim_{k\to\infty} f_{n_k}(x) \\ &= \lim_{k\to\infty} \bigg[f_{n_k}(a) + \int_a^x dt\, \{f_{n_k}'(t) - g_1(t)\} + \int_a^x dt\, g_1(t) \bigg], \end{aligned} \tag{A.14}$$

and, by (A.11) again,

$$\begin{aligned} &\bigg\| \int_a^x dt\, \{f_{n_k}'(t) - g_1(t)\} \bigg\|_{\mathcal{H}} \leqslant \int_a^x dt\, \big\| f_{n_k}'(t) - g_1(t) \big\|_{\mathcal{H}} \\ &\quad \leqslant \bigg(\int_a^x dt\, \big\| f_{n_k}'(t) - g_1(t) \big\|_{\mathcal{H}}^2 \bigg)^{1/2} |x-a|^{1/2} \\ &\quad = \bigg(\int_a^x dt\, t^{-[\alpha-2(m-1)]} t^{\alpha-2(m-1)} \big\| f_{n_k}'(t) - g_1(t) \big\|_{\mathcal{H}}^2 \bigg)^{1/2} |x-a|^{1/2} \end{aligned}$$

$$\leqslant \Big(\max\big\{a^{-[\alpha-2(m-1)]}, x^{-[\alpha-2(m-1)]}\big\}\Big)^{1/2} |x-a|^{1/2}$$
$$\times \bigg(\int_0^\infty dt\, t^{\alpha-2(m-1)} \big\| f_{n_k}'(t) - g_1(t) \big\|_{\mathcal{H}}^2 \bigg)^{1/2}$$
$$\underset{k\to\infty}{\longrightarrow} 0\,. \tag{A.15}$$

Thus, by (A.14) and (A.15),

$$g_0(x) = g_0(a) + \int_a^x dt\, g_1(t), \tag{A.16}$$

therefore g_0 is locally absolutely continuous on $(0,\infty)$ and

$$g_0'(x) = g_1(x) \text{ for a.e. } x \in (0,\infty). \tag{A.17}$$

Similarly, for $0 \leqslant j \leqslant m-1$,

$$\begin{aligned} g_j(x) &= \lim_{k\to\infty} f_{n_k}^{(j)}(x) \quad (x \in K) \\ &= \lim_{k\to\infty} \bigg[f_{n_k}^{(j)}(a) + \int_a^x dt\, \{f_{n_k}^{(j+1)}(t) - g_{j+1}(t)\} + \int_a^x dt\, g_{j+1}(t) \bigg], \end{aligned} \tag{A.18}$$

and, by (A.11),

$$\begin{aligned} &\bigg\| \int_a^x dt\, \big\{ f_{n_k}^{(j+1)}(t) - g_{j+1}(t) \big\} \bigg\|_{\mathcal{H}} \leqslant \int_a^x dt\, \big\| f_{n_k}^{(j+1)}(t) - g_{j+1}(t) \big\|_{\mathcal{H}} \\ &\quad \leqslant \bigg(\int_a^x dt\, \| f_{n_k}^{(j+1)}(t) - g_{j+1}(t) \|_{\mathcal{H}}^2 \bigg)^{1/2} |x-a|^{1/2} \\ &\quad = \bigg(\int_a^x dt\, t^{-[\alpha-2(m-j-1)]} t^{\alpha-2(m-j-1)} \big\| f_{n_k}^{(j+1)}(t) - g_{j+1}(t) \big\|_{\mathcal{H}}^2 \bigg)^{1/2} |x-a|^{1/2} \\ &\quad \leqslant \Big(\max\big\{a^{-[\alpha-2(m-j-1)]}, x^{-[\alpha-2(m-j-1)]}\big\}\Big)^{1/2} |x-a|^{1/2} \\ &\qquad \times \bigg(\int_0^\infty dt\, t^{\alpha-2(m-j-1)} \big\| f_{n_k}^{(j+1)}(t) - g_{j+1}(t) \big\|_{\mathcal{H}}^2 \bigg)^{1/2} \\ &\quad \underset{k\to\infty}{\longrightarrow} 0\,. \end{aligned} \tag{A.19}$$

Hence, by (A.18) and (A.19), for $0 \leqslant j \leqslant m-1$,

$$g_j(x) = g_j(a) + \int_a^x dt\, g_{j+1}(t), \quad x \in K, \tag{A.20}$$

thus g_j is locally absolutely continuous on $(0,\infty)$ and

$$g_j'(x) = g_{j+1}(x) \text{ for a.e. } x \in (0,\infty). \tag{A.21}$$

Putting $f = g_0$, we have $f \in \dot{H}^m\big((0,\infty); x^\alpha dx; \mathcal{H}\big)$, by (A.20) and (A.21), (A.8) follows from (A.11).

The uniqueness of f follows from (A.8) with $\ell = m$. □

Remark A.5 The condition (36) or (37) in Definition 3.2 (ii) in the context of $\dot{H}_0^m\big((0,\infty); x^\alpha dx; \mathcal{H}\big)$ is necessary to ensure that representation of an element in the completion of $\big(C_0^\infty((0,\infty);\mathcal{H}), |||\cdot|||_{m,\alpha}\big)$ by a function in $\dot{H}^m\big((0,\infty); x^\alpha dx; \mathcal{H}\big)$ is unique.

To illustrate this point, consider the following example with $m = 1$, $\alpha = 0$ and $\mathcal{H} = \mathbb{C}$: Let $g \in C_0^\infty((0,\infty))$ and put

$$\begin{aligned} f_n(x) &= g(x), \quad x \in (0,\infty),\ n \in \mathbb{N}, \\ F_j(x) &= j + g(x), \quad x \in (0,\infty),\ j \in \mathbb{N}. \end{aligned} \tag{A.22}$$

Then $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence in $\big(C_0^\infty((0,\infty)), |||\cdot|||_{m,\alpha}\big)$ and, for any $j \in \mathbb{N}$,

$$|||f_n - F_j|||_{1,0}^2 = \int_0^\infty dx\, \big|g'(x) - g'(x)\big|^2 = 0, \tag{A.23}$$

but $F_j \in \dot{H}^1\big((0,\infty); dx; \mathbb{C}\big)$ for all $j \in \mathbb{N}$ and $F_j \neq F_k$ for $j \neq k$.

This kind of "non-uniqueness" phenomenon is due to $|||\cdot|||_{m,\alpha}$ not being a norm on $\dot{H}^m\big((0,\infty); x^\alpha dx; \mathcal{H}\big)$. ◇

Next, we turn to the case $A(m,\alpha) = 0$.

Lemma A.6 *Let $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$. With the notation established in Definition 3.5 (i), let $\{f_n\}_{n=1}^\infty$ be a Cauchy sequence in $\big(C_0^\infty((0,\infty);\mathcal{H}), \|\cdot\|_{m,\alpha}\big)$. Then there exists a unique $f \in H^m\big((0,\infty); x^\alpha dx; \mathcal{H}\big)$ such that*

$$\lim_{n\to\infty} \|f_n - f\|_{m,\alpha} = 0. \tag{A.24}$$

Proof Since $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence in $\big(C_0^\infty((0,\infty);\mathcal{H}), \|\cdot\|_{m,\alpha}\big)$, $\{f_n^{(j)}\}_{n=1}^\infty$ is a Cauchy sequence in $L^2((0,\infty); x^\alpha dx; \mathcal{H})$ for $0 \leqslant j \leqslant m$, therefore there exists

$g_j \in L^2((0,\infty); x^\alpha dx; \mathcal{H})$ such that

$$\lim_{n\to\infty} \int_0^\infty dx\, x^\alpha \big\| f_n^{(j)}(x) - g_j(x) \big\|_{\mathcal{H}}^2 = 0. \tag{A.25}$$

We now prove that

$$g_0^{(j)}(x) = g_j(x) \ \text{ for a.e. } x \in (0,\infty), \ \ 1 \leqslant j \leqslant m. \tag{A.26}$$

By (A.25) there exists $K \subseteq (0,\infty)$ and a subsequence $\{f_{n_k}\}_{k=1}^\infty$ of $\{f_n\}_{n=1}^\infty$ such that $(0,\infty) \setminus K$ has zero Lebesgue measure and that, for $0 \leqslant j \leqslant m$,

$$\lim_{k\to\infty} f_{n_k}^{(j)}(x) = g_j(x), \quad x \in K. \tag{A.27}$$

Fix $a \in K$. Then, for $0 \leqslant j \leqslant m-1$,

$$\begin{aligned} g_j(x) &= \lim_{k\to\infty} f_{n_k}^{(j)}(x) \quad (x \in K) \\ &= \lim_{k\to\infty} \bigg[f_{n_k}^{(j)}(a) + \int_a^x dt\, \{ f_{n_k}^{(j+1)}(t) - g_{j+1}(t) \} + \int_a^x dt\, g_{j+1}(t) \bigg], \end{aligned} \tag{A.28}$$

and, by (A.25),

$$\begin{aligned} &\bigg\| \int_a^x dt\, \big\{ f_{n_k}^{(j+1)}(t) - g_{j+1}(t) \big\} \bigg\|_{\mathcal{H}} \leqslant \int_a^x dt\, \big\| f_{n_k}^{(j+1)}(t) - g_{j+1}(t) \big\|_{\mathcal{H}} \\ &\quad \leqslant \bigg(\int_a^x dt\, \big\| f_{n_k}^{(j+1)}(t) - g_{j+1}(t) \big\|_{\mathcal{H}}^2 \bigg)^{1/2} |x-a|^{1/2} \\ &\quad = \bigg(\int_a^x dt\, t^{-\alpha} t^\alpha \big\| f_{n_k}^{(j+1)}(t) - g_{j+1}(t) \big\|_{\mathcal{H}}^2 \bigg)^{1/2} |x-a|^{1/2} \\ &\quad \leqslant \Big(\max \big\{ a^{-\alpha}, x^{-\alpha} \big\} \Big)^{1/2} \bigg(\int_a^\infty dt\, t^\alpha \big\| f_{n_k}^{(j+1)}(t) - g_{j+1}(t) \big\|_{\mathcal{H}}^2 \bigg)^{1/2} |x-a|^{1/2} \\ &\quad \underset{k\to\infty}{\longrightarrow} 0\,. \end{aligned} \tag{A.29}$$

Hence, by (A.28) and (A.29), for $0 \leqslant j \leqslant m-1$,

$$g_j(x) = g_j(a) + \int_a^x dt\, g_{j+1}(t), \quad x \in K, \tag{A.30}$$

so g_j is locally absolutely continuous on $(0, \infty)$ and

$$g_j' = g_{j+1}(x) \text{ for a.e. } x \in (0, \infty). \tag{A.31}$$

Putting $f = g_0$, we have $f \in H^m((0, \infty); x^\alpha dx; \mathcal{H})$, by (A.30) and (A.31), (A.24) follows from (A.25).

Finally, uniqueness of f is a consequence of $\| \cdot \|_{m,\alpha}$ being a norm on the space $H^m((0, \infty); x^\alpha dx; \mathcal{H})$. □

Lemma A.7 *Let $m \in \mathbb{N}$, $\alpha \in \mathbb{R}$, $\rho \in (0, \infty)$. Then*

$$|||f|||_{m,\alpha} = \left(\int_0^\rho dx\, x^\alpha \| f^{(m)}(x) \|_{\mathcal{H}}^2 \right)^{1/2} \tag{A.32}$$

and

$$\| f \|_{m,\alpha} = \left(\sum_{k=0}^{m} \int_0^\rho dx\, x^\alpha \| f^{(k)}(x) \|_{\mathcal{H}}^2 \right)^{1/2} \tag{A.33}$$

are equivalent norms on $C_0^\infty((0, \rho); \mathcal{H})$.

Proof It suffices to show that, for $0 \leqslant k \leqslant m-1$, there exists a $C_k = C_k(m, \alpha, \rho) > 0$ such that

$$\int_0^\rho dx\, x^\alpha \| f^{(k)}(x) \|_{\mathcal{H}}^2 \leqslant C_k \int_0^\rho dx\, x^\alpha \| f^{(m)}(x) \|_{\mathcal{H}}^2, \quad f \in C_0^\infty((0, \rho); \mathcal{H}). \tag{A.34}$$

Let $\gamma = \rho + 1$. Choose $\eta \in (0, \rho)$ such that $x \mapsto x^{-2(m-k)}[\ln(\gamma/x)]^{-2}$ is strictly decreasing on $(0, \eta)$. Then, by Lemma 3.9,

$$\begin{aligned}
&\int_0^\rho dx\, x^\alpha \| f^{(m)}(x) \|_{\mathcal{H}}^2 \\
&\quad \geqslant B(m-k, \alpha) \int_0^\rho dx\, x^{\alpha - 2(m-k)} \| f^{(k)}(x) \|_{\mathcal{H}}^2 [\ln(\gamma/x)]^{-2} \\
&\quad = B(m-k, \alpha) \left(\int_0^\eta dx\, + \int_\eta^\rho dx \right) x^{\alpha - 2(m-k)} \| f^{(k)}(x) \|_{\mathcal{H}}^2 [\ln(\gamma/x)]^{-2} \\
&\quad \geqslant C_k(m, \alpha, \rho) \int_0^\rho dx\, x^\alpha \| f^{(k)}(x) \|_{\mathcal{H}}^2.
\end{aligned} \tag{A.35}$$

□

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An Excursion to Multiplications and Convolutions on Modulation Spaces

Nenad Teofanov and Joachim Toft

Abstract We give a self-contained introduction to (quasi-)Banach modulation spaces of ultradistributions, and review results on boundedness for multiplications and convolutions for elements in such spaces. Furthermore, we use these results to study the Gabor product. As an example, we show how it appears in a phase-space formulation of the nonlinear cubic Schrödinger equation.

Keywords Time–frequency analysis · Modulation spaces · Convolutions · Multiplications

1 Introduction

Modulation spaces were introduced in Feichtinger's seminal technical report [17], and prove themselves as useful family of Banach spaces of tempered distributions in time-frequency analysis, [4, 10, 28]. The main purpose of this survey article is to enlighten some properties of modulation spaces in a rather self-contained manner. In contrast to the most common situation, our analysis includes both quasi-Banach and Banach modulation spaces within the framework of ultradifferentiable functions and ultradistributions of Gelfand–Shilov type. For that reason we collect necessary background material in a rather detailed preliminary section.

Motivated by recent applications of modulation spaces in the context of nonlinear harmonic analysis and its applications, cf. [4–6, 14, 22, 38, 39, 47, 54] we focus our attention to boundedness for multiplications and convolutions for elements in such spaces. The basic results in that direction go back to the original contribution [17],

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and were thereafter reconsidered by many authors in different contexts. Let us give a brief, and unavoidably incomplete account on the related results.

In Sect. 3 we formulate in Theorems 3.5 and 3.7 bilinear versions of more general multiplication and convolution results in [54, Section 3]. The contents of Theorems 3.5 and 3.7 in the unweighted case for modulation spaces $M^{p,q}$ can be summarized as follows.

Proposition 1.1 *Let* $p_j, q_j \in (0, \infty]$, $j = 0, 1, 2$,

$$\theta_1 = \max\left(1, \frac{1}{p_0}, \frac{1}{q_1}, \frac{1}{q_2}\right) \quad \textit{and} \quad \theta_2 = \max\left(1, \frac{1}{p_1}, \frac{1}{p_2}\right).$$

Then

$$M^{p_1,q_1} \cdot M^{p_2,q_2} \subseteq M^{p_0,q_0}, \qquad \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_0}, \qquad \frac{1}{q_1} + \frac{1}{q_2} = \theta_1 + \frac{1}{q_0},$$

$$M^{p_1,q_1} * M^{p_2,q_2} \subseteq M^{p_0,q_0}, \qquad \frac{1}{p_1} + \frac{1}{p_2} = \theta_2 + \frac{1}{p_0}, \quad \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_0}.$$

The general multiplication and convolution properties in Sect. 3 also overlap with results by Bastianoni, Cordero and Nicola in [2], by Bastianoni and Teofanov in [1], and by Guo et al. in [32].

The multiplication relation in Proposition 1.1 for $p_j, q_j \geq 1$ was obtained already in [17] by Feichtinger. It is also obvious that the convolution relation was well-known since then (though a first formal proof of this relation seems to be given first in [48]). In general, these convolution and multiplication properties follow the rules

$$\ell^{p_1} * \ell^{p_2} \subseteq \ell^{p_0}, \quad \ell^{q_1} \cdot \ell^{q_2} \subseteq \ell^{q_0} \quad \Rightarrow \quad M^{p_1,q_1} * M^{p_2,q_2} \subseteq M^{p_0,q_0}$$

and

$$\ell^{p_1} \cdot \ell^{p_2} \subseteq \ell^{p_0}, \quad \ell^{q_1} * \ell^{q_2} \subseteq \ell^{q_0} \quad \Rightarrow \quad M^{p_1,q_1} \cdot M^{p_2,q_2} \subseteq M^{p_0,q_0},$$

which goes back to [17] in the Banach space case and to [25] in the quasi-Banach case. See also [19] and [42] for extensions of these relations to more general Banach function spaces and quasi-Banach function spaces, respectively.

In Sect. 3 we basically review some results from [54]. To make this survey self-contained we give the proof of Theorem 3.7 in unweighted case. In contrast to [32], we do not deduce any sharpness for our results.

To show Proposition 1.1 in the quasi-Banach setting, apart from the usual use of Hölder's and Young's inequalities, additional arguments are needed. In our situation we discretize the situations in similar ways as in [2] by using Gabor analysis for modulation spaces, and then apply some further arguments, valid in non-convex analysis. This approach is slightly different compared to what is used in [32] which follows the discretization technique introduced in [55], and which has some traces of Gabor analysis.

We refer to [54] for a detailed discussion on the uniqueness of multiplications and convolutions in Proposition 1.1.

In Sect. 4 we apply the results from previous parts in the framework of the so called Gabor product. It is introduced in [14] in order to derive a phase space analogue to the usual convolution identity for the Fourier transform. The main motivation is to use such kind of products in a phase-space formulation of certain nonlinear equations. As noticed in [14], among other interesting characteristics of phase-space representations, the initial value problem in phase-space may be well-posed for more general initial distributions. This means that the phase-space formulation could contain solutions other than the standard ones. We refer to [11–13], where the phase-space extensions are explored in different contexts. Here we illustrate this approach by considering the nonlinear cubic Schrödinger equation, which appear for example in Bose-Einstein condensate theory [35]. We also refer to [4, Chapter 7] for an overview of results related to well-posedness of the nonlinear Schrödinger equations in the framework of modulation spaces, see also [3, 38, 39].

2 Preliminaries

In this section we give an exposition of background material related to the definition and basic properties of modulation spaces. Thus we recall some facts on the short-time Fourier transform and related projections, the (Fourier invariant) Gelfand-Shilov spaces, weight functions, and mixed-norm spaces of Lebesgue type. We also recall convolution and multiplication in weighted Lebesgue sequence spaces.

2.1 The Short-Time Fourier Transform

In what follows we let $\mathscr{F}$ be the Fourier transform which takes the form

$$(\mathscr{F} f)(\xi) = \widehat{f}(\xi) \equiv (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(x) e^{-i\langle x,\xi\rangle}\, dx$$

when $f \in L^1(\mathbb{R}^d)$. Here $\langle \cdot\,, \cdot \rangle$ denotes the usual scalar product on $\mathbb{R}^d$. The same notation is used for the usual dual form between test functions and corresponding (ultra-)distributions. We recall that map $\mathscr{F}$ extends uniquely to a homeomorphism on the space of tempered distributions $\mathscr{S}'(\mathbb{R}^d)$, to a unitary operator on $L^2(\mathbb{R}^d)$ and restricts to a homeomorphism on the Schwartz space of smooth rapidly decreasing functions $\mathscr{S}(\mathbb{R}^d)$, cf. (29). We also observe with our choice of the Fourier transform, the usual convolution identity for the Fourier transform takes the forms

$$\mathscr{F}(f \cdot g) = (2\pi)^{-\frac{d}{2}} \widehat{f} * \widehat{g} \quad \text{and} \quad \mathscr{F}(f * g) = (2\pi)^{\frac{d}{2}} \widehat{f} \cdot \widehat{g} \tag{1}$$

when $f, g \in \mathscr{S}(\mathbb{R}^d)$.

In several situations it is convenient to use a localized version of the Fourier transform, called the short-time Fourier transform, STFT for short. The short-time Fourier transform of $f \in \mathscr{S}'(\mathbb{R}^d)$ with respect to the fixed *window function* $\phi \in \mathscr{S}(\mathbb{R}^d)$ is defined by

$$(V_\phi f)(x,\xi) \equiv (2\pi)^{-\frac{d}{2}} (f, \phi(\,\cdot\, - x) e^{i\langle\,\cdot\,,\xi\rangle})_{L^2}. \tag{2}$$

Here $(\,\cdot\,,\,\cdot\,)_{L^2}$ denotes the unique continuous extension of the inner product on $L^2(\mathbb{R}^d)$ restricted to $\mathscr{S}(\mathbb{R}^d)$ into a continuous map from $\mathscr{S}'(\mathbb{R}^d) \times \mathscr{S}(\mathbb{R}^d)$ to $\mathbb{C}$.

We observe that using certain properties for tensor products of distributions,

$$(V_\phi f)(x,\xi) = \mathscr{F}(f \cdot \overline{\phi(\,\cdot\, - x)})(\xi). \tag{2$'$}$$

(cf. [33, 52]). If in addition $f \in L^p(\mathbb{R}^d)$ for some $p \in [1, \infty]$, then

$$(V_\phi f)(x,\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(y)\overline{\phi(y-x)} e^{-i\langle y,\xi\rangle}\, dy. \tag{2$''$}$$

We observe that the domain of V_ϕ is $\mathscr{S}'(\mathbb{R}^d)$. The images are contained in $C^\infty(\mathbb{R}^{2d})$, the set of smooth functions defined on the phase space $\mathbb{R}^d \times \mathbb{R}^d \simeq \mathbb{R}^{2d}$.

The short-time Fourier transform appears in different contexts and under different names. In quantum mechanics it is rather common to call it the *coherent state transform* (see e.g. [37]). It is also closely related to the so-called Wigner distribution or radar ambiguity function (see e.g. [36]). In time-frequency analysis, it is also sometimes called the *Voice transform*.

The main idea with the design of short-time Fourier transform is to get the Fourier content, or the frequency resolution of localized functions and distributions. Roughly speaking, short-time Fourier transforms give a simultaneous information both on functions or distributions themselves as well as their Fourier transforms in the sense that the map

$$x \mapsto V_\phi f(x,\xi)$$

resembles on $f(x)$, while the map

$$\xi \mapsto V_\phi f(x,\xi)$$

resembles on $\widehat{f}(\xi)$.

As for the ordinary Fourier transform, there are several mapping properties which hold true for the short-time Fourier transform. As an elegant way to approach such properties in the framework of distributions, we may follow ideas given in [24] by Folland.

In fact, let T be the semi-conjugated tensor map

$$T(f,\phi) = f \otimes \overline{\phi}, \tag{3}$$

U be the linear pullback

$$(UF)(x,y) = U(y, y-x) \tag{4}$$

and $\mathscr{F}_2$ be the partial Fourier transform given by

$$(\mathscr{F}_2 F)(x,\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} F(x,y) e^{-i\langle y,\xi\rangle}\, dy. \tag{5}$$

Then

$$V_\phi f = (\mathscr{F}_2 \circ U \circ T)(f,\phi), \tag{6}$$

when $f, \phi \in \mathscr{S}(\mathbb{R}^d)$.

We observe that the mappings

$$T : \mathscr{S}(\mathbb{R}^d) \times \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^{2d}), \qquad U, \mathscr{F}_2 : \mathscr{S}(\mathbb{R}^{2d}) \to \mathscr{S}(\mathbb{R}^{2d}) \tag{7}$$

are continuous and uniquely extendable to continuous mappings

$$T : \mathscr{S}'(\mathbb{R}^d) \times \mathscr{S}'(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^{2d}), \quad U, \mathscr{F}_2 : \mathscr{S}'(\mathbb{R}^{2d}) \to \mathscr{S}'(\mathbb{R}^{2d}), \tag{8}$$

which in turn restricts to isometric mappings

$$T : L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d}), \qquad U, \mathscr{F}_2 : L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^{2d}). \tag{9}$$

Here that T is isometric means that

$$\|T(f,\phi)\|_{L^2(\mathbb{R}^{2d})} = \|f\|_{L^2(\mathbb{R}^d)} \|\phi\|_{L^2(\mathbb{R}^d)}.$$

It is now natural to define $V_\phi f$ as the right-hand side of (6) when $f, \phi \in \mathscr{S}'(\mathbb{R}^d)$, in which $V_\phi f$ is well-defined as an element in $\mathscr{S}'(\mathbb{R}^{2d})$.

Proposition 2.1 *The map*

$$(f,\phi) \mapsto V_\phi f : \mathscr{S}(\mathbb{R}^d) \times \mathscr{S}(\mathbb{R}^d) \ \to \mathscr{S}(\mathbb{R}^{2d}) \tag{10}$$

is continuous, which extends uniquely to a continuous map

$$(f,\phi) \mapsto V_\phi f : \mathscr{S}'(\mathbb{R}^d) \times \mathscr{S}'(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^{2d}), \tag{11}$$

which in turn restricts to an isometric map

$$(f,\phi) \mapsto V_\phi f : L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \ \to L^2(\mathbb{R}^{2d}). \tag{12}$$

If $\phi \in \mathscr{S}(\mathbb{R}^d)$ and $f \in \mathscr{S}'(\mathbb{R}^d)$, then (11) shows that $V_\phi f \in \mathscr{S}'(\mathbb{R}^{2d})$. On the other hand, it is easy to see that the right-hand side of (2) defines a smooth function. Consequently beside (11) and (10), we also have the continuous map

$$(f,\phi) \mapsto V_\phi f \,:\, \mathscr{S}'(\mathbb{R}^d) \times \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^{2d}) \cap C^\infty(\mathbb{R}^{2d}). \tag{13}$$

For short-time Fourier transform, the Parseval identity is replaced by the so-called Moyal identity, also known as the *orthogonality relation* given by

$$(V_\phi f, V_\psi g)_{L^2(\mathbb{R}^{2d})} = (\psi,\phi)_{L^2(\mathbb{R}^d)}(f,g)_{L^2(\mathbb{R}^d)}, \tag{14}$$

when $f, g, \phi, \psi \in \mathscr{S}(\mathbb{R}^d)$. The identity (14) is obtained by rewriting the short-time Fourier transforms by $(2)'$ and then applying the Parseval identity in suitable ways. We observe that the right-hand side makes sense also when f, g, ϕ and ψ belong to other spaces than $\mathscr{S}(\mathbb{R}^d)$. For example we may let

$$\begin{aligned} &(f,g,\phi,\psi) \in \mathscr{S}'(\mathbb{R}^d) \times \mathscr{S}(\mathbb{R}^d) \times \mathscr{S}(\mathbb{R}^d) \times \mathscr{S}'(\mathbb{R}^d), \\ &(f,g,\phi,\psi) \in \mathscr{S}(\mathbb{R}^d) \times \mathscr{S}'(\mathbb{R}^d) \times \mathscr{S}'(\mathbb{R}^d) \times \mathscr{S}(\mathbb{R}^d), \\ &(f,g,\phi,\psi) \in \mathscr{S}'(\mathbb{R}^d) \times \mathscr{S}(\mathbb{R}^d) \times L^q(\mathbb{R}^d) \times L^{q'}(\mathbb{R}^d) \\ \text{or} \quad &(f,g,\phi,\psi) \in L^p(\mathbb{R}^d) \times L^{p'}(\mathbb{R}^d) \times L^q(\mathbb{R}^d) \times L^{q'}(\mathbb{R}^d), \end{aligned} \tag{15}$$

when $p, p', q, q' \in [1,\infty]$ satisfy

$$\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1.$$

By Moyal's identity (14) it follows that if $\phi \in \mathscr{S}(\mathbb{R}^d) \setminus \{0\}$, then the identity operator on $\mathscr{S}'(\mathbb{R}^d)$ is given by

$$\mathrm{Id} = \left(\|\phi\|_{L^2}^{-2}\right) \cdot V_\phi^* \circ V_\phi, \tag{16}$$

provided suitable mapping properties of the (L^2-)adjoint V_ϕ^* of V_ϕ can be established. Obviously, V_ϕ^* fullfils

$$(V_\phi^* F, g)_{L^2(\mathbb{R}^d)} = (F, V_\phi g)_{L^2(\mathbb{R}^{2d})} \tag{17}$$

when $F \in \mathscr{S}(\mathbb{R}^{2d})$ and $g \in \mathscr{S}(\mathbb{R}^d)$.

By expressing the scalar product and the short-time Fourier transform in terms of integrals in (17), it follows by straight-forward manipulations that the adjoint in (17) is given by

$$(V_\phi^* F)(x) = (2\pi)^{-\frac{d}{2}} \iint_{\mathbb{R}^{2d}} F(y, \eta)\phi(x - y)e^{i\langle x,\eta\rangle}\, dy d\eta, \tag{18}$$

when $F \in \mathscr{S}(\mathbb{R}^{2d})$. We may now use mapping properties like (11)–(12) to extend the definition of $V_\phi^* F$ when F and ϕ belong to various classes of function and distribution spaces. For example, by (11), (10) and (12), it follows that the map

$$(F, g) \mapsto (F, V_\phi g)_{L^2(\mathbb{R}^{2d})}$$

defines a sesqui-linear form on $\mathscr{S}(\mathbb{R}^{2d}) \times \mathscr{S}'(\mathbb{R}^d)$, $\mathscr{S}'(\mathbb{R}^{2d}) \times \mathscr{S}(\mathbb{R}^d)$ and on $L^2(\mathbb{R}^{2d}) \times L^2(\mathbb{R}^d)$. This implies that if $\phi \in \mathscr{S}(\mathbb{R}^d)$, then V_ϕ^* in (17) is continuous from $\mathscr{S}(\mathbb{R}^{2d})$ to $\mathscr{S}(\mathbb{R}^d)$ which is uniquely extendable to a continuous map $\mathscr{S}'(\mathbb{R}^{2d})$ to $\mathscr{S}'(\mathbb{R}^d)$, and to $L^2(\mathbb{R}^{2d})$ to $L^2(\mathbb{R}^d)$. That is, the mappings

$$\begin{aligned} V_\phi^* : \mathscr{S}(\mathbb{R}^{2d}) \to \mathscr{S}(\mathbb{R}^d), \qquad & V_\phi^* : \mathscr{S}'(\mathbb{R}^{2d}) \to \mathscr{S}'(\mathbb{R}^d) \\ \text{and} \qquad & V_\phi^* : L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^d) \end{aligned} \tag{19}$$

are continuous.

2.2 STFT Projections and a Suitable Twisted Convolution

If $\phi \in \mathscr{S}(\mathbb{R}^d)$ satisfies $\|\phi\|_{L^2} = 1$, then (16) shows that $V_\phi^* \circ V_\phi$ is the identity operator on $\mathscr{S}'(\mathbb{R}^d)$. If we swap the order of this composition we get certain types of projections. In fact, for any $\phi \in \mathscr{S}(\mathbb{R}^d) \setminus \{0\}$, let P_ϕ be the operator given by

$$P_\phi \equiv \|\phi\|_{L^2}^{-2} \cdot V_\phi \circ V_\phi^*. \tag{20}$$

We observe that P_ϕ is continuous on $\mathscr{S}(\mathbb{R}^{2d})$, $L^2(\mathbb{R}^{2d})$ and $\mathscr{S}'(\mathbb{R}^{2d})$ due to the mapping properties for V_ϕ and V_ϕ^* above.

It is clear that $P_\phi^* = P_\phi$, i.e. P_ϕ is self-adjoint. Furthermore, P_ϕ is an projection:

$$P_\phi^2 = \|\phi\|_{L^2}^{-2} \cdot V_\phi \circ \underbrace{\left(\|\phi\|_{L^2}^{-2} \cdot V_\phi^* \circ V_\phi\right)}_{\text{The identity operator}} \circ V_\phi^* = \|\phi\|_{L^2}^{-2} \cdot V_\phi \circ V_\phi^* = P_\phi.$$

Hence,

$$P_\phi^* = P_\phi \quad \text{and} \quad P_\phi^2 = P_\phi, \tag{21}$$

which shows that P_ϕ is an orthonormal projection.

The ranks of P_ϕ are given by

$$\begin{aligned} P_\phi(\mathscr{S}(\mathbb{R}^{2d})) = V_\phi(\mathscr{S}(\mathbb{R}^d)), \qquad & P_\phi(L^2(\mathbb{R}^{2d})) = V_\phi(L^2(\mathbb{R}^d)), \\ \text{and} \qquad & P_\phi(\mathscr{S}'(\mathbb{R}^{2d})) = V_\phi(\mathscr{S}'(\mathbb{R}^d)). \end{aligned} \tag{22}$$

In fact, if $F \in \mathscr{S}'(\mathbb{R}^{2d})$, then

$$P_\phi F = V_\phi f,$$

where $f = \|\phi\|_{L^2}^{-2} V_\phi^* F \in \mathscr{S}'(\mathbb{R}^d)$. This shows that $P_\phi(\mathscr{S}'(\mathbb{R}^{2d})) \subseteq V_\phi(\mathscr{S}'(\mathbb{R}^d))$. On the other hand, if $f \in \mathscr{S}'(\mathbb{R}^d)$ and $F = V_\phi f$, then

$$P_\phi F = \left(V_\phi \circ \left(\|\phi\|_{L^2}^{-2} \cdot V_\phi^* \circ V_\phi\right)\right) f = V_\phi f,$$

which shows that any element in $V_\phi(\mathscr{S}'(\mathbb{R}^d))$ equals an element in $P_\phi(\mathscr{S}'(\mathbb{R}^{2d}))$, i.e. $P_\phi(\mathscr{S}'(\mathbb{R}^{2d})) = V_\phi(\mathscr{S}'(\mathbb{R}^d))$. This gives the last identity in (22). In the same way, the first two identities are obtained.

Remark 2.2 Let $F \in \mathscr{S}'(\mathbb{R}^{2d})$. Then it follows from the last identity in (22) that $F = V_\phi f$ for some $f \in \mathscr{S}'(\mathbb{R}^d)$, if and only if

$$F = P_\phi F. \tag{23}$$

Furthermore, if (23) holds, then $F = V_\phi f$ with

$$f = (\|\phi\|_{L^2}^{-2}) \cdot V_\phi^* F. \tag{24}$$

There is a twisted convolution which is linked to the projection in (20). In fact, if $F \in \mathscr{S}(\mathbb{R}^{2d})$ and $\phi \in \mathscr{S}(\mathbb{R}^d) \setminus \{0\}$, then it follows by expanding the integrals for V_ϕ and V_ϕ^* in (20), and performing some straight-forward manipulations that

$$P_\phi F = \|\phi\|_{L^2}^{-2} \cdot V_\phi \phi *_V F, \qquad F \in \mathscr{S}'(\mathbb{R}^{2d}), \tag{25}$$

where the *twisted convolution* $*_V$ is defined by

$$\begin{aligned}(F *_V G)(x,\xi) &= (2\pi)^{-\frac{d}{2}} \iint_{\mathbb{R}^{2d}} F(x-y,\xi-\eta)G(y,\eta)e^{-i\langle y,\xi-\eta\rangle}\,dyd\eta. \\ &= (2\pi)^{-\frac{d}{2}} \iint_{\mathbb{R}^{2d}} F(y,\eta)G(x-y,\xi-\eta)e^{-i\langle x-y,\eta\rangle}\,dyd\eta,\end{aligned} \tag{26}$$

when $F, G \in \mathscr{S}(\mathbb{R}^{2d})$. We observe that the definition of $*_V$ is uniquely extendable in different ways. For example, Young's inequality for ordinary convolution also holds for the twisted convolution. Moreover, the map $(F, G) \mapsto F *_V G$ extends uniquely to continuous mappings from $\mathscr{S}(\mathbb{R}^{2d}) \times \mathscr{S}'(\mathbb{R}^{2d})$ or $\mathscr{S}'(\mathbb{R}^{2d}) \times \mathscr{S}(\mathbb{R}^{2d})$ to $\mathscr{S}'(\mathbb{R}^{2d})$. By straight-forward computations it follows that

$$(F *_V G) *_V H = F *_V (G *_V H), \tag{27}$$

when $F, H \in \mathscr{S}(\mathbb{R}^{2d})$ and $G \in \mathscr{S}'(\mathbb{R}^{2d})$, or $F, H \in \mathscr{S}'(\mathbb{R}^{2d})$ and $G \in \mathscr{S}(\mathbb{R}^{2d})$.

Let $f \in \mathscr{S}'(\mathbb{R}^d)$ and $\phi_j \in \mathscr{S}(\mathbb{R}^d)$, $j = 1, 2, 3$. By straight-forward applications of Parseval's formula it follows that

$$\big((V_{\phi_2}\phi_3) *_V (V_{\phi_1} f)\big)(x,\xi) = (\phi_3, \phi_1)_{L^2} \cdot (V_{\phi_2} f)(x,\xi), \tag{28}$$

which is some sort of reproducing kernel of short-time Fourier transforms in the background of $*_V$.

2.3 Gelfand-Shilov Spaces

Before defining the Gelfand-Shilov spaces, we recall that the Schwartz space $\mathscr{S}(\mathbb{R}^d)$ consists of all (complex-valued) smooth functions $f \in C^\infty(\mathbb{R}^d)$ such that

$$\sup_{x\in\mathbb{R}^d} \big(|x^\beta \partial^\alpha f(x)|\big) \le C_{\alpha,\beta}, \tag{29}$$

for some constants $C_{\alpha,\beta} > 0$, which only depend on the multi-indices $\alpha, \beta \in \mathbb{N}^d$. The Schwartz space possess several convenient properties, and is heavily used in mathematics, science and technology. For example, the Schwartz space is invariant under Fourier transformation. By duality the same holds true for its (L^2-)dual $\mathscr{S}'(\mathbb{R}^d)$, the set of tempered distributions on $\mathbb{R}^d$.

On the other hand, we observe that there are no conditions on the growths of the constants $C_{\alpha,\beta}$ with respect to $\alpha, \beta \in \mathbb{N}^d$. This implies that in the context of the spaces $\mathscr{S}(\mathbb{R}^d)$ and $\mathscr{S}'(\mathbb{R}^d)$, it is almost impossible to investigate important properties like analyticity or related regularity properties which are stronger than pure smoothness. For investigating such stronger regularity properties, we need to modify $\mathscr{S}(\mathbb{R}^d)$ and the estimate (29) by imposing suitable growth conditions on the constants $C_{\alpha,\beta}$. This leads to the definition of Gelfand-Shilov spaces, [26, 40].

We only discuss Fourier invariant Gelfand-Shilov spaces and their properties. Let $0 < s \in \mathbb{R}$ be fixed. We have two different types of Gelfand-Shilov spaces. The Gelfand-Shilov space $\mathcal{S}_s(\mathbb{R}^d)$ of Roumieu type with parameter $s > 0$ consists of all $f \in C^\infty(\mathbb{R}^d)$ such that

$$\sup_{x\in\mathbb{R}^d} \big(|x^\beta \partial^\alpha f(x)|\big) \le C h^{|\alpha+\beta|}(\alpha!\beta!)^s, \tag{30}$$

for some constants $C, h > 0$. In the same way, the Gelfand-Shilov space $\Sigma_s(\mathbb{R}^d)$ of Beurling type with parameter $s > 0$ consists of all $f \in C^\infty(\mathbb{R}^d)$ such that for every $h > 0$, there is a constant $C = C_h > 0$ such that (30) holds. Hence, in comparison with the definition of Schwartz functions, we have limited ourself to constants $C_{\alpha,\beta}$ in (29) which are not allowed to grow faster than those of the form

$$C h^{|\alpha+\beta|}(\alpha!\beta!)^s$$

when dealing with Gelfand-Shilov spaces.

It can be proved that $\mathcal{S}_s(\mathbb{R}^d)$ and $\Sigma_t(\mathbb{R}^d)$ are dense in $\mathscr{S}(\mathbb{R}^d)$ when $s \ge \frac{1}{2}$ and $t > \frac{1}{2}$. We call such s and t admissible. On the other hand, for the other choices of s and t we have

$$\mathcal{S}_s(\mathbb{R}^d) = \Sigma_t(\mathbb{R}^d) = \{0\}, \quad \text{when} \quad s < \frac{1}{2},\ t \le \frac{1}{2}.$$

One has that $\mathcal{S}_1(\mathbb{R}^d)$ consists of real analytic functions, and that $\Sigma_1(\mathbb{R}^d)$ consists of smooth functions on $\mathbb{R}^d$ which are extendable to entire functions on $\mathbb{C}^d$. The topologies of $\mathcal{S}_s(\mathbb{R}^d)$ and $\Sigma_s(\mathbb{R}^d)$ are defined by the semi-norms

$$\|f\|_{\mathcal{S}_{s,h}} \equiv \sup \frac{|x^\beta \partial^\alpha f(x)|}{h^{|\alpha+\beta|}(\alpha!\beta!)^s}. \tag{31}$$

Here the supremum should be taken over all $\alpha, \beta \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$. We equip $\mathcal{S}_s(\mathbb{R}^d)$ and $\Sigma_s(\mathbb{R}^d)$ by the canonical inductive limit topology and projective limit topology, respectively, with respect to $h > 0$, which are induced by the semi-norms in (31).

Let $\mathcal{S}_{s,h}(\mathbb{R}^d)$ be the Banach space which consists of all $f \in C^\infty(\mathbb{R}^d)$ such that $\|f\|_{\mathcal{S}_{s,h}}$ in (31) is finite, and let $\mathcal{S}'_{s,h}(\mathbb{R}^d)$ be the (L^2-)dual of $\mathcal{S}_{s,h}(\mathbb{R}^d)$. If $s \geq \frac{1}{2}$, then the *Gelfand-Shilov distribution space* $\mathcal{S}'_s(\mathbb{R}^d)$ of *Roumieu type* is the projective limit of $\mathcal{S}'_{s,h}(\mathbb{R}^d)$ with respect to $h > 0$. If instead $s > \frac{1}{2}$, then the *Gelfand-Shilov distribution space* $\Sigma'_s(\mathbb{R}^d)$ of *Beurling type* is the inductive limit of $\mathcal{S}'_{s,h}(\mathbb{R}^d)$ with respect to $h > 0$. Consequently, for admissible s we have

$$\mathcal{S}'_s(\mathbb{R}^d) = \bigcap_{h>0} \mathcal{S}'_{s,h}(\mathbb{R}^d) \quad \text{and} \quad \Sigma'_s(\mathbb{R}^d) = \bigcup_{h>0} \mathcal{S}'_{s,h}(\mathbb{R}^d).$$

It can be proved that $\mathcal{S}'_s(\mathbb{R}^d)$ and $\Sigma'_s(\mathbb{R}^d)$ are the (strong) duals to $\mathcal{S}_s(\mathbb{R}^d)$ and $\Sigma_s(\mathbb{R}^d)$, respectively.

We have the following embeddings and density properties for Gelfand-Shilov and Schwartz spaces

$$\begin{aligned} &\mathcal{S}_s(\mathbb{R}^d) \hookrightarrow \Sigma_t(\mathbb{R}^d) \hookrightarrow \mathcal{S}_t(\mathbb{R}^d) \hookrightarrow \mathscr{S}(\mathbb{R}^d), \\ &\mathscr{S}'(\mathbb{R}^d) \hookrightarrow \mathcal{S}'_t(\mathbb{R}^d) \hookrightarrow \Sigma'_t(\mathbb{R}^d) \hookrightarrow \mathcal{S}'_s(\mathbb{R}^d), \qquad t > s \geq \frac{1}{2}, \end{aligned} \tag{32}$$

with dense embeddings. Here $A \hookrightarrow B$ means that the topological spaces A and B satisfy $A \subseteq B$ with continuous embeddings.

The Fourier transform possess convenient mapping properties on Gelfand-Shilov spaces and their distribution spaces. In fact, the Fourier transform extends uniquely to homeomorphisms on $\mathcal{S}'_s(\mathbb{R}^d)$ and on $\Sigma'_s(\mathbb{R}^d)$ for admissible s. Furthermore, $\mathscr{F}$ restricts to homeomorphisms on $\mathcal{S}_s(\mathbb{R}^d)$ and on $\Sigma_s(\mathbb{R}^d)$.

One of the most important characterizations of Gelfand-Shilov spaces is performed in terms of estimates of the functions and their Fourier transforms. More precisely, in [8, 15] it is proved that if $f \in \mathscr{S}'(\mathbb{R}^d)$ and $s > 0$, then $f \in \mathcal{S}_s(\mathbb{R}^d)$ ($f \in \Sigma_s(\mathbb{R}^d)$), if and only if

$$|f(x)| \lesssim e^{-r|x|^{\frac{1}{s}}} \quad \text{and} \quad |\widehat{f}(\xi)| \lesssim e^{-r|\xi|^{\frac{1}{s}}}, \tag{33}$$

for some $r > 0$ (for every $r > 0$). Here $g_1 \lesssim g_2$ means that $g_1(\theta) \leq c \cdot g_2(\theta)$ holds uniformly for all θ in the intersection of the domains of g_1 and g_2 and for some constant $c > 0$, and we write $g_1 \asymp g_2$ when $g_1 \lesssim g_2 \lesssim g_1$.

The analysis in [8, 15] can also be applied on the Schwartz space, from which it follows that an element $f \in \mathscr{S}'(\mathbb{R}^d)$ belongs to $\mathscr{S}(\mathbb{R}^d)$, if and only if

$$|f(x)| \lesssim \langle x\rangle^{-N} \quad \text{and} \quad |\widehat{f}(\xi)| \lesssim \langle \xi\rangle^{-N}, \tag{34}$$

for every $N \geq 0$. Here and in what follows we let

$$\langle x\rangle = (1 + |x|^2)^{\frac{1}{2}}.$$

Remark 2.3 Several properties in Sects. 2.1–2.3 in the background of $\mathscr{S}(\mathbb{R}^d)$ and $\mathscr{S}'(\mathbb{R}^d)$ also hold for the Gelfand-Shilov spaces and their distribution spaces. Let $s \geq \frac{1}{2}$. By similar arguments which lead to Proposition 2.1 and (13), it follows that

$$(f, \phi) \mapsto V_\phi f : \mathcal{S}_s(\mathbb{R}^d) \times \mathcal{S}_s(\mathbb{R}^d) \to \mathcal{S}_s(\mathbb{R}^{2d}) \tag{35}$$

is continuous, which extends uniquely to continuous mappings

$$(f, \phi) \mapsto V_\phi f : \mathcal{S}_s'(\mathbb{R}^d) \times \mathcal{S}_s(\mathbb{R}^d) \to \mathcal{S}_s'(\mathbb{R}^{2d}) \cap C^\infty(\mathbb{R}^{2d}) \tag{36}$$

and

$$(f, \phi) \mapsto V_\phi f : \mathcal{S}_s'(\mathbb{R}^d) \times \mathcal{S}_s'(\mathbb{R}^d) \to \mathcal{S}_s'(\mathbb{R}^{2d}). \tag{37}$$

It follows that (14) makes sense after each $\mathscr{S}$ in (15) are replaced by $\mathcal{S}_s$. Let $\phi \in \mathcal{S}_s(\mathbb{R}^d) \setminus \{0\}$ be fixed. Then by similar arguments which lead to (19) give that the mappings

$$V_\phi^* : \mathcal{S}_s(\mathbb{R}^{2d}) \to \mathcal{S}_s(\mathbb{R}^d), \qquad V_\phi^* : \mathcal{S}_s'(\mathbb{R}^{2d}) \to \mathcal{S}_s'(\mathbb{R}^d) \tag{19$'$}$$

are continuous. For P_ϕ in (20) we have that (21) still holds true and that (22) can be completed with

$$P_\phi(\mathcal{S}_s(\mathbb{R}^{2d})) = V_\phi(\mathcal{S}_s(\mathbb{R}^d)) \quad \text{and} \quad P_\phi(\mathcal{S}_s'(\mathbb{R}^{2d})) = V_\phi(\mathcal{S}_s'(\mathbb{R}^d)). \tag{38}$$

We also have that the twisted convolution in (26) is continuous from $\mathcal{S}_s(\mathbb{R}^{2d}) \times \mathcal{S}_s(\mathbb{R}^{2d})$ to $\mathcal{S}_s(\mathbb{R}^{2d})$ and uniquely extendable to a continuous map $\mathcal{S}_s(\mathbb{R}^{2d}) \times \mathcal{S}_s'(\mathbb{R}^{2d})$ or $\mathcal{S}_s'(\mathbb{R}^{2d}) \times \mathcal{S}_s(\mathbb{R}^{2d})$ to $\mathcal{S}_s'(\mathbb{R}^{2d})$, and that the formulae (25)–(28) still hold true after each $\mathscr{S}$ is replaced by $\mathcal{S}_s$ in the attached assumptions.

If instead $s > \frac{1}{2}$, then similar facts hold true with Σ_s in place of $\mathcal{S}_s$ above, at each occurrence.

Remark 2.4 In similar ways as characterizing Gelfand-Shilov spaces in terms of Fourier estimates (see (33)), we may also use the short-time Fourier transform to perform similar characterizations. Moreover, the short-time Fourier transform can in addition be used to characterize spaces of Gelfand-Shilov distributions.

In fact, let $\phi \in \mathcal{S}_s(\mathbb{R}^d) \setminus \{0\}$ ($\phi \in \Sigma_s(\mathbb{R}^d) \setminus \{0\}$) be fixed and let f be a Gelfand-Shilov distribution on $\mathbb{R}^d$. Then the following is true:

1. $f \in \mathcal{S}_s(\mathbb{R}^d)$ ($f \in \Sigma_s(\mathbb{R}^d)$), if and only if

$$|V_\phi f(x, \xi)| \lesssim e^{-r(|x|^{\frac{1}{s}} + |\xi|^{\frac{1}{s}})} \tag{39}$$

for some $r > 0$ (for every $r > 0$);

2. $f \in \mathcal{S}_s'(\mathbb{R}^d)$ ($f \in \Sigma_s'(\mathbb{R}^d)$), if and only if

$$|V_\phi f(x,\xi)| \lesssim e^{r(|x|^{\frac{1}{s}}+|\xi|^{\frac{1}{s}})} \tag{40}$$

for every $r > 0$ (for some $r > 0$).

We refer to [31, Theorem 2.7] for the characterization 1. concerning Gelfand-Shilov functions and to [51, Proposition 2.2]) for the characterization 2. concerning Gelfand-Shilov distributions.

2.4 Weight Functions

A *weight* or *weight function* on $\mathbb{R}^d$ is a positive function $\omega \in L^\infty_{loc}(\mathbb{R}^d)$ such that $1/\omega \in L^\infty_{loc}(\mathbb{R}^d)$. The weight ω is called *moderate*, if there is a positive weight v on $\mathbb{R}^d$ and a constant $C \geq 1$ such that

$$\omega(x+y) \leq C\omega(x)v(y), \qquad x, y \in \mathbb{R}^d. \tag{41}$$

If ω and v are weights on $\mathbb{R}^d$ such that (41) holds, then ω is also called v*-moderate*. We note that (41) implies that ω fulfills the estimates

$$C^{-1}v(-x)^{-1} \leq \omega(x) \leq Cv(x), \quad x \in \mathbb{R}^d. \tag{42}$$

We let $\mathscr{P}_E(\mathbb{R}^d)$ be the set of all moderate weights on $\mathbb{R}^d$.

We say that v is *submultiplicative* if

$$v(x+y) \leq v(x)v(y) \quad \text{and} \quad v(-x) = v(x), \qquad x, y \in \mathbb{R}^d. \tag{43}$$

We observe that if $v \in \mathscr{P}_E(\mathbb{R}^d)$ is even and satisfies

$$v(x+y) \leq Cv(x)v(y), \qquad x, y \in \mathbb{R}^d, \tag{44}$$

for some constant $C > 0$, then for $v_0 = C^{1/2}v$, one has that $v_0 \in \mathscr{P}_E(\mathbb{R}^d)$ is submultiplicative and $v \asymp v_0$ (see e.g. [17, 19, 28]).

We also recall from [29] that if v is positive and locally bounded and satisfies (44), then $v(x) \leq C_0 e^{r_0|x|}$ for some positive constants C_0 and r_0. In fact, if $x \in \mathbb{R}^d$,

$$r = \sup_{|x|\leq 1} \log v(x), \quad c = \log C$$

and n is an integer such that $n-1 \le |x| \le n$, then (44) gives

$$v(x) = v(n \cdot (x/n)) \le C^n v(x/n)^n \le C^n e^{rn} = e^{(r+c)n} \le e^{(r+c)(|x|+1)},$$

which gives the statement.

Therefore, if v is a submultiplicative weight, then

$$v(x) \lesssim e^{r|x|}, \qquad x \in \mathbb{R}^d, \tag{45}$$

for some $r \ge 0$. Hence, if $\omega \in \mathscr{P}_E(\mathbb{R}^d)$, then (41) and (45) imply

$$\omega(x+y) \lesssim \omega(x) e^{r|y|}, \quad x, y \in \mathbb{R}^d \tag{46}$$

for some $r > 0$. In particular, (42) shows that for any $\omega_0 \in \mathscr{P}_E(\mathbb{R}^d)$, there is a constant $r > 0$ such that

$$e^{-r|x|} \lesssim \omega_0(x) \lesssim e^{r|x|}, \quad x \in \mathbb{R}^d.$$

If (41) holds, then there is a smallest positive even function v_0 such that (41) holds with $C = 1$. We remark that this v_0 is given by

$$v_0(x) = \sup_{y \in \mathbb{R}^d} \left(\frac{\omega(x+y)}{\omega(y)}, \frac{\omega(-x+y)}{\omega(y)} \right),$$

and is submultiplicative (see e.g. [19, 27, 49]). Consequently, if ω is a moderate weight, then it is also moderated by a submultiplicative weight. In the sequel, v and v_j for $j \ge 0$, always stand for submultiplicative weights if nothing else is stated.

We also remark that in the literature it is common to define submultiplicative weights as (43) should hold, without the condition $v(-x) = v(x)$, i.e. that v does not have to be even (cf. e.g. [17, 19, 25, 28]). However, in the sequel it is convenient for us to include this property in the definition.

There are several subclasses of $\mathscr{P}_E(\mathbb{R}^d)$ which are interesting for different reasons. Though our results later on are formulated in background of weights in $\mathscr{P}_E(\mathbb{R}^d)$, we here mention some subclasses which especially appear in time-frequency analysis. First we observe the class $\mathscr{P}_E^0(\mathbb{R}^d)$, which consists of all $\omega \in \mathscr{P}_E(\mathbb{R}^d)$ such that (46) holds for every $r > 0$.

The class $\mathscr{P}_E^0(\mathbb{R}^d)$ is important when dealing with spectral invariance for matrix or convolution operators on $\ell^2(\mathbb{Z}^d)$ (see e.g. [30]). If $v \in \mathscr{P}_E(\mathbb{R}^d)$ is submultiplicative, then $v \in \mathscr{P}_E^0(\mathbb{R}^d)$, if and only if

$$\lim_{n \to \infty} v(nx)^{\frac{1}{n}} = 1 \tag{47}$$

(see e.g. [23]). The condition (47) is equivalent to

$$\lim_{n\to\infty} \frac{\log(v(nx))}{n} = 0, \tag{47$'$}$$

and is usually called the *GRS condition*, or *Gelfand-Raikov-Shilov condition*.

A more restrictive condition on v compared to (47)$'$ is given by the Beurling-Domar condition

$$\sum_{n=1}^{\infty} \frac{\log(v(nx))}{n^2} < \infty. \tag{48}$$

This condition is strongly linked to non quasi-analytic classes which contain non-trivial compactly supported elements (see e.g. [29]). Any subexponential submultiplicative weight satisfies the Beurling-Domar condition. That is, suppose that $\theta \in (0,1)$ and that $v(x) = e^{r|x|^\theta}$, $x \in \mathbb{R}^d$, then (48) is fulfilled. We let $\mathscr{P}_{\mathrm{BD}}(\mathbb{R}^d)$ be the set of all weights which are moderated by submultiplicative weights which satisfy the Beurling-Domar condition.

Finally we let $\mathscr{P}(\mathbb{R}^d)$ be the set of all weights on $\mathbb{R}^d$ which are moderated by polynomially bounded functions. That is, $\omega \in \mathscr{P}(\mathbb{R}^d)$, if and only if there are positive constants r and C such that

$$\omega(x+y) \le C\omega(x)(1+|y|)^r, \qquad x, y \in \mathbb{R}^d.$$

Here we observe that $v(x) = (1+|x|)^r$ is submultiplicative.

Among these weight classes we have

$$\mathscr{P}(\mathbb{R}^d) \subsetneq \mathscr{P}_{\mathrm{BD}}(\mathbb{R}^d) \subsetneq \mathscr{P}_E^0(\mathbb{R}^d) \subsetneq \mathscr{P}_E(\mathbb{R}^d). \tag{49}$$

In fact, it is clear that the ordering in (49) holds. On the other hand, if $r > 0$ and $\theta \in (0,1)$, then due to

$$\begin{gathered} e^{r|x|^\theta} \in \mathscr{P}_{\mathrm{BD}}(\mathbb{R}^d) \setminus \mathscr{P}(\mathbb{R}^d), \\ e^{r|x|/\log(e+|x|)} \in \mathscr{P}_E^0(\mathbb{R}^d) \setminus \mathscr{P}_{\mathrm{BD}}(\mathbb{R}^d), \\ \text{and} \quad e^{r|x|} \in \mathscr{P}_E(\mathbb{R}^d) \setminus \mathscr{P}_E^0(\mathbb{R}^d), \end{gathered} \tag{50}$$

it also follows that the inclusions in (49) are strict.

We refer to [16, 28, 29, 49] for more facts about weights in time-frequency analysis.

2.5 Mixed Norm Spaces of Lebesgue Type

For every $p, q \in (0, \infty]$ and weight ω on $\mathbb{R}^{2d}$, we set

$$\|F\|_{L^{p,q}_{(\omega)}(\mathbb{R}^{2d})} \equiv \|G_{F,\omega,p}\|_{L^q(\mathbb{R}^d)}, \quad \text{where} \quad G_{F,\omega,p}(\xi) = \|F(\,\cdot\,,\xi)\omega(\,\cdot\,,\xi)\|_{L^p(\mathbb{R}^d)}$$

and

$$\|F\|_{L^{p,q}_{*,(\omega)}(\mathbb{R}^{2d})} \equiv \|H_{F,\omega,q}\|_{L^p(\mathbb{R}^d)}, \quad \text{where} \quad H_{F,\omega,q}(x) = \|F(x,\,\cdot\,)\omega(x,\,\cdot\,)\|_{L^q(\mathbb{R}^d)},$$

when F is (complex-valued) measurable function on $\mathbb{R}^{2d}$. Then $L^{p,q}_{(\omega)}(\mathbb{R}^{2d})$ ($L^{p,q}_{*,(\omega)}(\mathbb{R}^{2d})$) consists of all measurable functions F such that $\|F\|_{L^{p,q}_{(\omega)}} < \infty$ ($\|F\|_{L^{p,q}_{*,(\omega)}} < \infty$).

In similar ways, let Ω_1, Ω_2 be discrete sets, ω be a positive function on $\Omega_1 \times \Omega_2$ and $\ell_0'(\Omega_1 \times \Omega_2)$ be the set of all formal (complex-valued) sequences $c = \{c(j,k)\}_{j\in\Omega_1, k\in\Omega_2}$. Then the discrete Lebesgue spaces, i.e. the Lebesgue sequence spaces

$$\ell^{p,q}_{(\omega)}(\Omega_1 \times \Omega_2) \quad \text{and} \quad \ell^{p,q}_{*,(\omega)}(\Omega_1 \times \Omega_2)$$

of mixed (quasi-)norm types consist of all $c \in \ell_0'(\Omega_1 \times \Omega_2)$ such that $\|c\|_{\ell^{p,q}_{(\omega)}(\Omega_1\times\Omega_2)} < \infty$ respectively $\|c\|_{\ell^{p,q}_{*,(\omega)}(\Omega_1\times\Omega_2)} < \infty$. Here

$$\|c\|_{\ell^{p,q}_{(\omega)}(\Omega_1\times\Omega_2)} \equiv \|G_{c,\omega,p}\|_{\ell^q(\Omega_2)}, \quad \text{where} \quad G_{c,\omega,p}(k) = \|F(\,\cdot\,,k)\omega(\,\cdot\,,k)\|_{\ell^p(\Omega_1)}$$

and

$$\|c\|_{\ell^{p,q}_{*,(\omega)}(\Omega_1\times\Omega_2)} \equiv \|H_{c,\omega,q}\|_{\ell^p(\Omega_1)}, \quad \text{where} \quad H_{c,\omega,q}(j) = \|c(j,\,\cdot\,)\omega(j,\,\cdot\,)\|_{\ell^q(\Omega_2)},$$

when $c \in \ell_0'(\Omega_1 \times \Omega_2)$.

2.6 Convolutions and Multiplications for Discrete Lebesgue Spaces

Next we discuss extended Hölder and Young relations for multiplications and convolutions on discrete Lebesgue spaces. The Hölder and Young conditions on Lebesgue exponent are then

$$\frac{1}{q_0} \le \frac{1}{q_1} + \frac{1}{q_2}, \tag{51}$$

respectively

$$\frac{1}{p_0} \le \frac{1}{p_1} + \frac{1}{p_2} - \max\left(1, \frac{1}{p_1}, \frac{1}{p_2}\right). \tag{52}$$

Notice that, when $p_1, p_2 \in (0, 1)$, then (52) becomes $p_0 \ge \max\{p_1, p_2\}$, while for $p_1, p_2 \ge 1$ it reduces to the common Young condition

$$1 + \frac{1}{p_0} \le \frac{1}{p_1} + \frac{1}{p_2}.$$

The conditions on the weight functions are

$$\omega_0(j) \le \omega_1(j)\omega_2(j), \qquad j \in \Lambda, \tag{53}$$

respectively

$$\omega_0(j_1 + j_2) \le \omega_1(j_1)\omega_2(j_2), \qquad j_1, j_2 \in \Lambda, \tag{54}$$

where Λ is a lattice of the form

$$\Lambda = \{ n_1 e_1 + \cdots + n_d e_d\, ;\ (n_1, \dots, n_d) \in \mathbb{Z}^d \},$$

where $e_1, \dots e_d$ is a basis for $\mathbb{R}^d$.

Proposition 2.5 *Let $p_j, q_j \in (0, \infty]$, $j = 0, 1, 2$, be such that (51) and (52) hold, let $\Lambda \subseteq \mathbb{R}^d$ be a lattice and let ω_j be weights on Λ, $j = 0, 1, 2$. Then the following is true:*

1. *if (53) holds, then the map $(a_1, a_2) \mapsto a_1 \cdot a_2$ from $\ell_0(\Lambda) \times \ell_0(\Lambda)$ to $\ell_0(\Lambda)$ extends uniquely to a continuous map from $\ell^{q_1}_{(\omega_1)}(\Lambda) \times \ell^{q_2}_{(\omega_2)}(\Lambda)$ to $\ell^{q_0}_{(\omega_0)}(\Lambda)$, and*

$$\|a_1 \cdot a_2\|_{\ell^{q_0}_{(\omega_0)}} \le \|a_1\|_{\ell^{q_1}_{(\omega_1)}} \|a_2\|_{\ell^{q_2}_{(\omega_2)}}, \qquad a_j \in \ell^{q_j}_{(\omega_j)}(\Lambda),\ j = 1, 2; \tag{55}$$

2. *if (54) holds, then the map $(a_1, a_2) \mapsto a_1 * a_2$ from $\ell_0(\Lambda) \times \ell_0(\Lambda)$ to $\ell_0(\Lambda)$ extends uniquely to a continuous map from $\ell^{p_1}_{(\omega_1)}(\Lambda) \times \ell^{p_2}_{(\omega_2)}(\Lambda)$ to $\ell^{p_0}_{(\omega_0)}(\Lambda)$, and*

$$\|a_1 * a_2\|_{\ell^{p_0}_{(\omega_0)}} \le \|a_1\|_{\ell^{p_1}_{(\omega_1)}} \|a_2\|_{\ell^{p_2}_{(\omega_2)}}, \qquad a_j \in \ell^{p_j}_{(\omega_j)}(\Lambda),\ j = 1, 2. \tag{56}$$

The assertion 1. in Proposition 2.5 is the standard Hölder's inequality for discrete Lebesgue spaces. The assertion 2. in that proposition is the usual Young's inequality for Lebesgue spaces on lattices in the case when $p_0, p_1, p_2 \in [1, \infty]$. A proof of Proposition 2.5 is given in Appendix A in [54].

3 Modulation Spaces, Multiplications and Convolutions

In this section we introduce modulation spaces, and recall their basic properties, in particular in the context of Gelfand-Shilov spaces. Notice that we permit the Lebesgue exponents to belong to the full interval $(0, \infty]$ instead of the most common choice $[1, \infty]$, and general moderate weights which may have a (sub)exponential growth. Here we also recall some facts on Gabor expansions for modulation spaces.

Then we deduce multiplication and convolution estimates on modulation spaces. There are several approaches to multiplication and convolution in the case when the involved Lebesgue exponents belong to $[1, \infty]$ (see [9, 17, 19, 32, 43, 48]). Here we consider the case when these exponents belong to $(0, \infty)$ (see also [1, 2, 25, 41, 42, 50]). In addition, and in order to keep the survey style of our exposition, we focus on the bilinear case, and refer to [54] for extension of these results to multi-linear products.

3.1 Modulation Spaces

The (classical) modulation spaces, essentially introduced in [17] by Feichtinger are given in the following. (See e.g. [18] for definition of more general modulation spaces.)

Definition 3.1 Let $p, q \in (0, \infty]$, $\omega \in \mathscr{P}_E(\mathbb{R}^{2d})$ and $\phi \in \Sigma_1(\mathbb{R}^d) \setminus \{0\}$.

1. The *modulation space* $M^{p,q}_{(\omega)}(\mathbb{R}^d)$ consists of all $f \in \Sigma_1'(\mathbb{R}^d)$ such that

$$\|f\|_{M^{p,q}_{(\omega)}} \equiv \|V_\phi f\|_{L^{p,q}_{(\omega)}}$$

   is finite. The topology of $M^{p,q}_{(\omega)}(\mathbb{R}^d)$ is defined by the (quasi-)norm $\|\cdot\|_{M^{p,q}_{(\omega)}}$;
2. The *modulation space (of Wiener amalgam type)* $W^{p,q}_{(\omega)}(\mathbb{R}^d)$ consists of all $f \in \Sigma_1'(\mathbb{R}^d)$ such that

$$\|f\|_{W^{p,q}_{(\omega)}} \equiv \|V_\phi f\|_{L^{p,q}_{*,(\omega)}}$$

   is finite. The topology of $W^{p,q}_{(\omega)}(\mathbb{R}^d)$ is defined by the (quasi-)norm $\|\cdot\|_{W^{p,q}_{(\omega)}}$.

For convenience we set $M^{p,q} = M^{p,q}_{(\omega)}$ and $W^{p,q} = W^{p,q}_{(\omega)}$ when the weight ω is trivial, i.e. when $\omega(x, \xi) = 1$ for every $x, \xi \in \mathbb{R}^d$. We also set

$$M^p_{(\omega)} \equiv M^{p,p}_{(\omega)} \; (= W^{p,p}_{(\omega)}) \quad \text{and} \quad M^p \equiv M^{p,p} \; (= W^{p,p}).$$

Remark 3.2 Modulation spaces possess several convenient properties. Let $p, q \in (0, \infty]$, $\omega \in \mathscr{P}_E(\mathbb{R}^{2d})$ and $\phi \in \Sigma_1(\mathbb{R}^d) \setminus \{0\}$. Then the following is true (see [17–20, 25, 28] and their analyses for verifications):

- the definitions of $M^{p,q}_{(\omega)}(\mathbb{R}^d)$ and $W^{p,q}_{(\omega)}(\mathbb{R}^d)$ are independent of the choices of $\phi \in \Sigma_1(\mathbb{R}^d) \setminus \{0\}$, and different choices give rise to equivalent quasi-norms;
- the spaces $M^{p,q}_{(\omega)}(\mathbb{R}^d)$ and $W^{p,q}_{(\omega)}(\mathbb{R}^d)$ are quasi-Banach spaces which increase with p and q, and decrease with ω. If in addition $p, q \geq 1$, then they are Banach spaces;
- If $p, q \geq 1$, then the $L^2(\mathbb{R}^d)$ scalar product, $(\,\cdot\,,\,\cdot\,)_{L^2(\mathbb{R}^d)}$, on $\Sigma_1(\mathbb{R}^d) \times \Sigma_1(\mathbb{R}^d)$ is uniquely extendable to dualities between $M^{p,q}_{(\omega)}(\mathbb{R}^d)$ and $M^{p',q'}_{(1/\omega)}(\mathbb{R}^d)$, and between $W^{p,q}_{(\omega)}(\mathbb{R}^d)$ and $W^{p',q'}_{(1/\omega)}(\mathbb{R}^d)$. If in addition $p, q < \infty$, then the dual spaces of $M^{p,q}_{(\omega)}(\mathbb{R}^d)$ and $W^{p,q}_{(\omega)}(\mathbb{R}^d)$ can be identified with $M^{p',q'}_{(1/\omega)}(\mathbb{R}^d)$ respectively $W^{p',q'}_{(1/\omega)}(\mathbb{R}^d)$, through the form $(\,\cdot\,,\,\cdot\,)_{L^2(\mathbb{R}^d)}$;
- if $\omega_0(x, \xi) = \omega(-\xi, x)$, then $\mathscr{F}$ on $\Sigma_1'(\mathbb{R}^d)$ restricts to a homeomorphism from $M^{p,q}_{(\omega)}(\mathbb{R}^d)$ to $W^{q,p}_{(\omega_0)}(\mathbb{R}^d)$.
- The inclusions

$$\Sigma_1(\mathbb{R}^d) \subseteq M^{p,q}_{(\omega)}(\mathbb{R}^d),\ W^{p,q}_{(\omega)}(\mathbb{R}^d) \subseteq \Sigma_1'(\mathbb{R}^d) \quad \text{when} \quad \omega \in \mathscr{P}_E(\mathbb{R}^{2d}), \tag{57}$$

$$\mathcal{S}_1(\mathbb{R}^d) \subseteq M^{p,q}_{(\omega)}(\mathbb{R}^d),\ W^{p,q}_{(\omega)}(\mathbb{R}^d) \subseteq \mathcal{S}_1'(\mathbb{R}^d) \quad \text{when} \quad \omega \in \mathscr{P}_E^0(\mathbb{R}^{2d}) \tag{58}$$

  and

$$\mathscr{S}(\mathbb{R}^d) \subseteq M^{p,q}_{(\omega)}(\mathbb{R}^d),\ W^{p,q}_{(\omega)}(\mathbb{R}^d) \subseteq \mathscr{S}'(\mathbb{R}^d) \quad \text{when} \quad \omega \in \mathscr{P}(\mathbb{R}^{2d}) \tag{59}$$

  are continuous. If in addition $p, q < \infty$, then these inclusions are dense.

We recall from [49] that the embeddings (57)–(59), are essentially special cases of certain characterizations of the Schwartz space, Gelfand-Shilov spaces and their distribution spaces in terms of suitable unions and intersections of modulation spaces. In fact, let $p, q \in (0, \infty]$ and $s \geq 1$ be fixed and set

$$v_{r,t}(x, \xi) = \begin{cases} e^{r(|x|^{\frac{1}{t}} + |\xi|^{\frac{1}{t}})}, & t \in \mathbb{R}_+ \\ (1 + |x| + |\xi|)^r, & t = \infty, \end{cases} \tag{60}$$

where $r > 0$. Then

$$\Sigma_s(\mathbb{R}^d) = \bigcap_{r>0} M^{p,q}_{(v_{r,s})}(\mathbb{R}^d) = \bigcap_{r>0} W^{p,q}_{(v_{r,s})}(\mathbb{R}^d), \tag{61}$$

$$\mathcal{S}_s(\mathbb{R}^d) = \bigcup_{r>0} M^{p,q}_{(v_{r,s})}(\mathbb{R}^d) = \bigcup_{r>0} W^{p,q}_{(v_{r,s})}(\mathbb{R}^d), \tag{62}$$

$$\mathscr{S}(\mathbb{R}^d) = \bigcap_{r>0} M^{p,q}_{(v_{r,\infty})}(\mathbb{R}^d) \quad = \bigcap_{r>0} W^{p,q}_{(v_{r,\infty})}(\mathbb{R}^d), \tag{63}$$

$$\mathscr{S}'(\mathbb{R}^d) = \bigcup_{r>0} M^{p,q}_{(1/v_{r,\infty})}(\mathbb{R}^d) = \bigcup_{r>0} W^{p,q}_{(1/v_{r,\infty})}(\mathbb{R}^d), \tag{64}$$

$$\mathcal{S}'_s(\mathbb{R}^d) = \bigcap_{r>0} M^{p,q}_{(1/v_{r,s})}(\mathbb{R}^d) \quad = \bigcap_{r>0} W^{p,q}_{(1/v_{r,s})}(\mathbb{R}^d) \tag{65}$$

and

$$\Sigma'_s(\mathbb{R}^d) = \bigcup_{r>0} M^{p,q}_{(1/v_{r,s})}(\mathbb{R}^d) \quad = \bigcup_{r>0} W^{p,q}_{(1/v_{r,s})}(\mathbb{R}^d). \tag{66}$$

The topologies of the spaces on the left-hand sides of (61)–(66) are obtained by replacing each intersection by projective limit with respect to $r > 0$ and each union with inductive limit with respect to $r > 0$.

The relations (61)–(66) are essentially special cases of [49, Theorem 3.9], see also [31, 45, 46]. In order to be self-contained we here give a proof of (62).

Proof of (62) Since

$$M^{\infty}_{(v_{2r,s})}(\mathbb{R}^d) \subseteq M^{p,q}_{(v_{r,s})}(\mathbb{R}^d),\ W^{p,q}_{(v_{r,s})}(\mathbb{R}^d) \subseteq M^{\infty}_{(v_{r,s})}(\mathbb{R}^d),$$

it suffices to prove the result for $p = q = \infty$. Let $\phi \in \Sigma_1(\mathbb{R}^d) \setminus \{0\}$ be fixed. First suppose that

$$f \in M^{\infty}_{(v_{r,s})}(\mathbb{R}^d) = W^{\infty}_{(v_{r,s})}(\mathbb{R}^d).$$

Then it follows from the definition of modulation space norm that (39) holds for some $r > 0$. By Remark 2.4 it follows that $f \in \mathcal{S}_s(\mathbb{R}^d)$, and we have proved

$$\bigcup_{r>0} M^{\infty}_{(v_{r,s})}(\mathbb{R}^d) \subseteq \mathcal{S}_s(\mathbb{R}^d). \tag{67}$$

Suppose instead that $f \in \mathcal{S}_s(\mathbb{R}^d)$. Then (39) holds for some $r > 0$, giving that $f \in M^{\infty}_{(v_{r,s})}(\mathbb{R}^d)$. Hence (67) holds with reversed inclusion, and the result follows. □

Example 3.3 Let $p = q = 1$ and $\omega = 1$. Then $M^{1,1}_{(\omega)}(\mathbb{R}^d) = M^1(\mathbb{R}^d)$ is *the Feichtinger algebra*, probably the most prominent example of a modulation space. We refer to a recent survey [34] for a detailed account on $M^1(\mathbb{R}^d)$, and to [14, Lemma 11] for a list of its basic properties.

Familiar examples arise when $p = q = 2$ and $\omega = 1$. Then $M^{2,2}_{(\omega)}(\mathbb{R}^d) = M^2(\mathbb{R}^d) = L^2(\mathbb{R}^d)$, and

$$M^{2,2}_{(\omega_s)}(\mathbb{R}^d) = H^s(\mathbb{R}^d), \qquad s \in \mathbb{R},$$

where $\omega_s(\xi) = \langle \xi \rangle^s$, and $H^s(\mathbb{R}^d)$ is the Sobolev space (also known as the Bessel potential space) of distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|^2_{H^s} := \int_{\mathbb{R}^d} \langle \xi \rangle^{2s} |\widehat{f}(\xi)|^2 d\xi < \infty,$$

cf. [28, Proposition 11.3.1]. Furthermore, if $v_s(x,\xi) = \langle (x,\xi) \rangle^s$, then $M^{2,2}_{(v_s)}(\mathbb{R}^d) = Q_s(\mathbb{R}^d)$, $s \in \mathbb{R}$, [7, Lemma 2.3]. Here Q_s denotes the Shubin-Sobolev space, [44].

Finally we remark that modulation spaces can be conveniently discretized in terms of Gabor expansions. In order for explaining some basic issues on this, in a similar way as in Subsection 1.5 in [54], we limit ourself to the case when the involved weights are moderated by subexponential functions. That is, we suppose that ω in $M^{p,q}_{(\omega)}(\mathbb{R}^d)$ satisfies

$$\omega(x+y, \xi+\eta) \lesssim \omega(x,\xi) e^{r(|x|^{\frac{1}{s}} + |\xi|^{\frac{1}{s}})}, \tag{68}$$

for some $s > 1$ and $r > 0$. We observe that this implies that

$$\Sigma_s(\mathbb{R}^d) \subseteq M^{p,q}_{(\omega)}(\mathbb{R}^d) \subseteq \Sigma_s'(\mathbb{R}^d), \tag{69}$$

in vew of (42), (61) and (66). For more general approaches we refer to [19, 27, 28, 42, 50].

Since $s > 1$, it follows from Sections 1.3 and 1.4 in [33] that there are $\phi, \psi \in \Sigma_s(\mathbb{R}^d)$ with values in $[0,1]$ such that

$$\operatorname{supp} \phi \subseteq \Big[-\frac{3}{4}, \frac{3}{4} \Big]^d, \quad \phi(x) = 1 \quad \text{when} \quad x \in \Big[-\frac{1}{4}, \frac{1}{4} \Big]^d \tag{70}$$

$$\operatorname{supp} \psi \subseteq [-1,1]^d, \qquad \psi(x) = 1 \quad \text{when} \quad x \in \Big[-\frac{3}{4}, \frac{3}{4} \Big]^d \tag{71}$$

and

$$\sum_{j \in \mathbb{Z}^d} \phi(\,\cdot - j) = 1. \tag{72}$$

Let $f \in \Sigma_s'(\mathbb{R}^d)$. Then $x \mapsto f(x)\phi(x-j)$ belongs to $\Sigma_s'(\mathbb{R}^d)$ and is supported in $j+[-\frac{3}{4},\frac{3}{4}]^d$. Hence, by periodization it follows from Fourier analysis that

$$f(x)\phi(x-j) = \sum_{\iota \in \pi\mathbb{Z}^d} c(j,\iota)e^{i\langle x,\iota\rangle}, \qquad x \in j+[-1,1]^d, \tag{73}$$

where

$$c(j,\iota) = 2^{-d}(f,\phi(\,\cdot\,-j)e^{i\langle\,\cdot\,,\iota\rangle}) = \left(\frac{\pi}{2}\right)^{\frac{d}{2}} V_\phi f(j,\iota), \qquad j \in \mathbb{Z}^d,\ \iota \in \pi\mathbb{Z}^d.$$

Since $\psi = 1$ on the support of ϕ, (73) gives

$$f(x)\phi(x-j) = \left(\frac{\pi}{2}\right)^{\frac{d}{2}} \sum_{\iota \in \pi\mathbb{Z}^d} V_\phi f(j,\iota)\psi(x-j)e^{i\langle x,\iota\rangle}, \qquad x \in \mathbb{R}^d, \tag{73$'$}$$

By (72) it now follows that

$$f(x) = \left(\frac{\pi}{2}\right)^{\frac{d}{2}} \sum_{(j,\iota)\in\Lambda} V_\phi f(j,\iota)\psi(x-j)e^{i\langle x,\iota\rangle}, \qquad x \in \mathbb{R}^d, \tag{74}$$

where

$$\Lambda = \mathbb{Z}^d \times (\pi\mathbb{Z}^d), \tag{75}$$

which is the *Gabor expansion* of f with respect to the *Gabor pair* (ϕ,ψ) and lattice Λ, i.e. with respect to the *Gabor atom* ϕ and the *dual Gabor atom* ψ. Here the series converges in $\Sigma_s'(\mathbb{R}^d)$. By duality and the fact that compactly supported elements in $\Sigma_s(\mathbb{R}^d)$ are dense in $\Sigma_s'(\mathbb{R}^d)$ we also have

$$f(x) = \left(\frac{\pi}{2}\right)^{\frac{d}{2}} \sum_{(j,\iota)\in\Lambda} V_\psi f(j,\iota)\phi(x-j)e^{i\langle x,\iota\rangle}, \qquad x \in \mathbb{R}^d, \tag{76}$$

with convergence in $\Sigma_s'(\mathbb{R}^d)$.

Let T be a linear continuous operator from $\Sigma_s(\mathbb{R}^d)$ to $\Sigma_s'(\mathbb{R}^d)$ and let $f \in \Sigma_s(\mathbb{R}^d)$. Then it follows from (74) that

$$(Tf)(x) = \left(\frac{\pi}{2}\right)^{\frac{d}{2}} \sum_{(j,\iota)\in\Lambda} V_\phi f(j,\iota)T(\psi(\,\cdot\,-j)e^{i\langle\,\cdot\,,\iota\rangle})(x)$$

and

$$T(\psi(\cdot - j)e^{i\langle \cdot ,\iota\rangle})(x) = \left(\frac{\pi}{2}\right)^{\frac{d}{2}} \sum_{(k,\kappa)\in\Lambda} (V_\phi(T(\psi(\cdot - j)e^{i\langle \cdot ,\iota\rangle})))(k,\kappa)\psi(x-k)e^{i\langle x,\kappa\rangle}.$$

A combination of these expansions show that

$$(Tf)(x) = \left(\frac{\pi}{2}\right)^{\frac{d}{2}} \sum_{(j,\iota)\in\Lambda} (A\cdot V_\phi f)(j,\iota)\psi(x-j)e^{i\langle x,\iota\rangle}, \tag{77}$$

where $A = (a(\boldsymbol{j},\boldsymbol{k}))_{\boldsymbol{j},\boldsymbol{k}\in\Lambda}$ is the $\Lambda\times\Lambda$-matrix, given by

$$a(\boldsymbol{j},\boldsymbol{k}) = \left(\frac{\pi}{2}\right)^{\frac{d}{2}} (T(\psi(\cdot - j)e^{i\langle \cdot ,\iota\rangle}), \phi(\cdot - k)e^{i\langle \cdot ,\kappa\rangle})_{L^2(\mathbb{R}^d)}$$
$$\text{when}\quad \boldsymbol{j} = (j,\iota) \text{ and } \boldsymbol{k} = (k,\kappa). \tag{78}$$

By the Gabor analysis for modulation spaces we get the following restatement of [54, Proposition 1.8]. We refer to [17, 19–21, 25, 27, 28, 50] for details.

Proposition 3.4 *Let $s>1$, $p,q\in(0,\infty]$, $\omega\in\mathscr{P}_E(\mathbb{R}^{2d})$ be such that (68) holds for some $r>0$, $\phi,\psi\in\Sigma_s(\mathbb{R}^d)$ with values in $[0,1]$ be such that (70), (71) and (72) hold true, and let $f\in\Sigma_s'(\mathbb{R}^d)$. Then the following is true:*

1. *$f\in M^{p,q}_{(\omega)}(\mathbb{R}^d)$, if and only if $\|V_\phi f\|_{\ell^{p,q}_{(\omega)}(\mathbb{Z}^d\times\pi\mathbb{Z}^d)}<\infty$;*
2. *$f\in M^{p,q}_{(\omega)}(\mathbb{R}^d)$, if and only if $\|V_\psi f\|_{\ell^{p,q}_{(\omega)}(\mathbb{Z}^d\times\pi\mathbb{Z}^d)}<\infty$;*
3. *the quasi-norms*

$$f\mapsto \|V_\phi f\|_{\ell^{p,q}_{(\omega)}(\mathbb{Z}^d\times\pi\mathbb{Z}^d)} \quad \textit{and} \quad f\mapsto \|V_\psi f\|_{\ell^{p,q}_{(\omega)}(\mathbb{Z}^d\times\pi\mathbb{Z}^d)}$$

are equivalent to $\|\cdot\|_{M^{p,q}_{(\omega)}}$.

The same holds true with $W^{p,q}_{(\omega)}$ and $\ell^{p,q}_{*,(\omega)}$ in place of $M^{p,q}_{(\omega)}$ respectively $\ell^{p,q}_{(\omega)}$ at each occurrence.

3.2 Multiplications and Convolutions in Modulation Spaces

As a first step for approaching multiplications and convolutions for elements in modulation spaces, we reformulate such products in terms of short-time Fourier transforms. Let $\phi_0,\phi_1,\phi_2\in\Sigma_1(\mathbb{R}^d)$ be fixed such that

$$\phi_0 = (2\pi)^{-\frac{d}{2}}\phi_1\phi_2 \tag{79}$$

and let $f_1, f_2 \in \Sigma_1(\mathbb{R}^d)$. Then the multiplication $f_0 = f_1 f_2$ can be expressed by

$$F_0(x,\xi) = \big(F_1(x,\,\cdot\,) * F_2(x,\,\cdot\,)\big)(\xi). \tag{80}$$

where

$$F_j = V_{\phi_j} f_j, \qquad j = 0, 1, 2. \tag{81}$$

In fact, by Fourier's inversion formula we get

$$\begin{aligned} &\big((V_{\phi_1} f_1)(x,\,\cdot\,) * (V_{\phi_2} f_2)(x,\,\cdot\,)\big)(\xi) \\ &= (2\pi)^{-d} \iiint f_1(y_1)\overline{\phi_1(y_1-x)} f_2(y_2)\overline{\phi_2(y_2-x)} e^{-i\langle y_1,\xi-\eta\rangle} e^{-i\langle y_2,\eta\rangle}\, dy_1 dy_2 d\eta \\ &= \int f_1(y)\overline{\phi_1(y-x)} f_2(y)\overline{\phi_2(y-x)} e^{-i\langle y,\xi\rangle}\, dy = (2\pi)^{\frac{d}{2}} (V_{\phi_1\phi_2}(f_1 f_2))(x,\xi). \end{aligned}$$

We also observe that we may extract $f_0 = f_1 f_2$ by the formula

$$f_0 = (\|\phi_0\|_{L^2})^{-1} V_{\phi_0}^* F_0, \tag{82}$$

provided ϕ_0 is not trivially equal to 0.

In the same way, let $\phi_0, \phi_1, \phi_2 \in \Sigma_1(\mathbb{R}^d)$ be fixed such that

$$\phi_0 = (2\pi)^{\frac{d}{2}} \phi_1 * \phi_2, \tag{83}$$

and let $f_1, f_2, g \in \Sigma_1(\mathbb{R}^d)$. Then the convolution $f_0 = f_1 * f_2$ can be expressed by

$$F_0(x,\xi) = \big(F_1(\,\cdot\,,\xi) * F_2(\,\cdot\,,\xi)\big)(x). \tag{84}$$

where F_j are given by (81), and that we may extract $f_0 = f_1 * f_2$ from (82).

Next we discuss convolutions and multiplications for modulation spaces, and start with the following convolution result for modulation spaces. For multiplications of elements in modulation spaces we need to swap the conditions for the involved Lebesgue exponents compared to (51) and (52). That is, these conditions become

$$\frac{1}{p_0} \le \frac{1}{p_1} + \frac{1}{p_2}, \qquad \frac{1}{q_0} \le \frac{1}{q_1} + \frac{1}{q_2} - \max\left(1, \frac{1}{p_0}, \frac{1}{q_1}, \frac{1}{q_2}\right), \tag{85}$$

or

$$\frac{1}{p_0} \le \frac{1}{p_1} + \frac{1}{p_2}, \qquad \frac{1}{q_0} \le \frac{1}{q_1} + \frac{1}{q_2} - \max\left(1, \frac{1}{q_1}, \frac{1}{q_2}\right). \tag{86}$$

The conditions on the weight functions are

$$\omega_0(x, \xi_1 + \xi_2) \le \omega_1(x, \xi_1)\omega_2(x, \xi_2), \qquad x, \xi_1, \xi_2 \in \mathbb{R}^d, \tag{87}$$

respectively

$$\omega_0(x_1 + x_2, \xi) \le \omega_1(x_1, \xi)\omega_2(x_2, \xi), \qquad x_1, x_2, \xi \in \mathbb{R}^d. \tag{88}$$

Theorem 3.5 *Let* $p_j, q_j \in (0, \infty)$ *and* $\omega_j \in \mathscr{P}_E(\mathbb{R}^{2d})$, $j = 0, 1, 2$, *be such that* (85) *and* (87) *hold. Then* $(f_1, f_2) \mapsto f_1 f_2$ *from* $\Sigma_1(\mathbb{R}^d) \times \Sigma_1(\mathbb{R}^d)$ *to* $\Sigma_1(\mathbb{R}^d)$ *is uniquely extendable to a continuous map from* $M^{p_1,q_1}_{(\omega_1)}(\mathbb{R}^d) \times M^{p_2,q_2}_{(\omega_2)}(\mathbb{R}^d)$ *to* $M^{p_0,q_0}_{(\omega_0)}(\mathbb{R}^d)$, *and*

$$\|f_1 f_2\|_{M^{p_0,q_0}_{(\omega_0)}} \lesssim \|f_1\|_{M^{p_1,q_1}_{(\omega_1)}} \|f_2\|_{M^{p_2,q_2}_{(\omega_2)}}, \quad f_j \in M^{p_j,q_j}_{(\omega_j)}(\mathbb{R}^d),\ j = 1, 2. \tag{89}$$

Theorem 3.6 *Let* $p_j, q_j \in (0, \infty)$ *and* $\omega_j \in \mathscr{P}_E(\mathbb{R}^{2d})$, $j = 0, 1, 2$, *be such that* (86) *and* (87) *hold. Then* $(f_1, f_2) \mapsto f_1 f_2$ *from* $\Sigma_1(\mathbb{R}^d) \times \Sigma_1(\mathbb{R}^d)$ *to* $\Sigma_1(\mathbb{R}^d)$ *is uniquely extendable to a continuous map from* $W^{p_1,q_1}_{(\omega_1)}(\mathbb{R}^d) \times W^{p_2,q_2}_{(\omega_2)}(\mathbb{R}^d)$ *to* $W^{p_0,q_0}_{(\omega_0)}(\mathbb{R}^d)$, *and*

$$\|f_1 f_2\|_{W^{p_0,q_0}_{(\omega_0)}} \lesssim \|f_1\|_{W^{p_1,q_1}_{(\omega_1)}} \|f_2\|_{W^{p_2,q_2}_{(\omega_2)}}, \quad f_j \in W^{p_j,q_j}_{(\omega_j)}(\mathbb{R}^d),\ j = 1, 2. \tag{90}$$

The corresponding results for convolutions are the following. Here the conditions on the involved Lebesgue exponents are swapped as

$$\frac{1}{p_0} \le \frac{1}{p_1} + \frac{1}{p_2} - \max\left(1, \frac{1}{q_0}, \frac{1}{p_1}, \frac{1}{p_2}\right), \qquad \frac{1}{q_0} \le \frac{1}{q_1} + \frac{1}{q_2} \tag{91}$$

or

$$\frac{1}{p_0} \le \frac{1}{p_1} + \frac{1}{p_2} - \max\left(1, \frac{1}{p_1}, \frac{1}{p_2}\right), \qquad \frac{1}{q_0} \le \frac{1}{q_1} + \frac{1}{q_2} \tag{92}$$

Theorem 3.7 *Let* $p_j, q_j \in (0, \infty)$ *and* $\omega_j \in \mathscr{P}_E(\mathbb{R}^{2d})$, $j = 0, 1, 2$, *be such that* (88) *and* (92) *hold. Then* $(f_1, f_2) \mapsto f_1 * f_2$ *from* $\Sigma_1(\mathbb{R}^d) \times \Sigma_1(\mathbb{R}^d)$ *to* $\Sigma_1(\mathbb{R}^d)$ *is uniquely extendable to a continuous map from* $M^{p_1,q_1}_{(\omega_1)}(\mathbb{R}^d) \times M^{p_2,q_2}_{(\omega_2)}(\mathbb{R}^d)$ *to* $M^{p_0,q_0}_{(\omega_0)}(\mathbb{R}^d)$, *and*

$$\|f_1 * f_2\|_{M^{p_0,q_0}_{(\omega_0)}} \lesssim \|f_1\|_{M^{p_1,q_1}_{(\omega_1)}} \|f_2\|_{M^{p_2,q_2}_{(\omega_2)}}, \quad f_j \in M^{p_j,q_j}_{(\omega_j)}(\mathbb{R}^d),\ j = 1, 2. \tag{93}$$

Theorem 3.8 *Let $p_j, q_j \in (0, \infty)$ and $\omega_j \in \mathscr{P}_E(\mathbb{R}^{2d})$, $j = 0, 1, 2$, be such that* (88) *and* (91) *hold. Then $(f_1, f_2) \mapsto f_1 * f_2$ from $\Sigma_1(\mathbb{R}^d) \times \Sigma_1(\mathbb{R}^d)$ to $\Sigma_1(\mathbb{R}^d)$ is uniquely extendable to a continuous map from $W^{p_1,q_1}_{(\omega_1)}(\mathbb{R}^d) \times W^{p_2,q_2}_{(\omega_2)}(\mathbb{R}^d)$ to $W^{p_0,q_0}_{(\omega_0)}(\mathbb{R}^d)$, and*

$$\|f_1 * f_2\|_{W^{p_0,q_0}_{(\omega_0)}} \lesssim \|f_1\|_{W^{p_1,q_1}_{(\omega_1)}} \|f_2\|_{W^{p_2,q_2}_{(\omega_2)}}, \qquad f_j \in W^{p_j,q_j}_{(\omega_j)}(\mathbb{R}^d),\ j = 1, 2. \tag{94}$$

We observe that Theorems 3.2–3.5 in [54] are multi-linear versions of the previous results. In particular, Theorems 3.5 and 3.6 are Fourier transformations of Theorems 3.7 and 3.8. Hence it suffices to prove the last two theorems, cf. [54]. To shed some ideas of the arguments, we give a proof in the unweighted case of Theorem 3.7. We will use Proposition A.1 from Appendix A, which is a special case of [54, Proposition 3.6].

Proof of Theorem 3.7 Suppose $f_j \in \mathscr{S}(\mathbb{R}^d)$, $\phi_j \in \mathscr{S}(\mathbb{R}^d)$, $j = 0, 1, 2$ be such that

$$f_0 = f_1 * f_2 \quad \text{and} \quad \phi_0 = (2\pi)^{\frac{d}{2}} \phi_1 * \phi_2 \neq 0,$$

and let F_j be the same as in (81). Then

$$F_0(x, \xi) = (V_{\phi_1} f_1(\,\cdot\,, \xi) * V_{\phi_2} f_2(\,\cdot\,, \xi))(x),$$

in view of (84).

We have

$$0 \le \chi_{k_1+Q} * \chi_{k_2+Q} \le \chi_{k_1+k_2+Q_{d,2}}, \qquad k_1, k_2 \in \mathbb{Z}^d,$$

where $Q_{d,r}$ is the cube

$$Q_{d,r} = [0, r]^d \quad \text{and} \quad Q = Q_{d,1} = [0, 1]^d,$$

and χ_E is the characteristic function with respect to the set E.

Set

$$G(x, \xi) = (|V_{\phi_1} f_1(\,\cdot\,, \xi)| * |V_{\phi_2} f_2(\,\cdot\,, \xi)|)(x),$$

$$a_j(k, \kappa) = \|V_{\phi_j} f_j\|_{L^\infty((k,\kappa)+Q_{2d,1})}, \quad j = 1, 2,$$

and

$$b(k, \kappa) = \|G\|_{L^\infty((k,\kappa)+Q_{2d,1})}$$

Then

$$\begin{aligned}\|V_{\phi_0}^* F_0\|_{M^{p_0,q_0}} &\asymp \|P_{\phi_0} F_0\|_{\mathsf{W}(\ell^{p_0,q_0})} \lesssim \|F_0\|_{\mathsf{W}(\ell^{p_0,q_0})}\\ &\le \|G\|_{\mathsf{W}(\ell^{p_0,q_0})} \asymp \|b\|_{\ell^{p_0,q_0}},\end{aligned} \tag{95}$$

and

$$\|f_j\|_{M^{p_j,q_j}} \asymp \|a_j\|_{\ell^{p_j,q_j}} \tag{96}$$

in view of (A.5) and Proposition A.1 in Appendix A (see also [25, Theorem 3.3])).

By (84) we have

$$\begin{aligned}G(x,\lambda) &\le \sum_{k_1,k_2\in\mathbb{Z}^d} a_1(k_1,\lambda)a_2(k_2,\lambda)(\chi_{k_1+Q} * \chi_{k_2+Q})(x)\\ &\le \sum_{k_1,k_2\in\mathbb{Z}^d} a_1(k_1,\lambda)a_2(k_2,\lambda)\chi_{k_1+k_2+Q_{d,2}}(x).\end{aligned} \tag{97}$$

We observe that

$$\chi_{k_1+k_2+Q_{d,2}}(x) = 0 \quad \text{when} \quad x \notin l + Q_d,\ (k_1,k_2) \notin \Omega_l,$$

where

$$\Omega_l = \{\,(k_1,k_2) \in \mathbb{Z}^{2d}\,;\, l_j - 2 \le k_{1,j} + k_{2,j} \le l_j + 1\,\},$$

and

$$k_j = (k_{j,1},\dots,k_{j,d}) \in \mathbb{Z}^d, \qquad j = 1,2, \quad \text{and} \quad l = (l_1,\dots,l_d) \in \mathbb{Z}^d.$$

Hence, if $x = l$ in (97), we get

$$\begin{aligned}b(l,\lambda) &\le \sum_{(k_1,k_2)\in\Omega_l} a_1(k_1,\lambda)a_2(k_2,\lambda)\\ &\le \sum_{m\in I} (a_1(\,\cdot\,,\lambda) * a_2(\,\cdot\,,\lambda))(l - 2e_0 + m),\end{aligned} \tag{98}$$

where $e_0 = (1,\dots,1) \in \mathbb{Z}^d$ and $I = \{0,1,2,3\}^d$.

If we apply the ℓ^{p_0} quasi-norm on (98) with respect to the l variable, then Proposition 2.5 (2) and the fact that I is finite set give

$$
\begin{aligned}
\|b(\,\cdot\,,\lambda)\|_{\ell^{p_0}} &\le \Big\| \sum_{m\in I} (a_1(\,\cdot\,,\lambda) * a_2(\,\cdot\,,\lambda))(\,\cdot\, - 2e_0 + m) \Big\|_{\ell^{p_0}} \\
&\le \sum_{m\in I} \|(a_1(\,\cdot\,,\lambda) * a_2(\,\cdot\,,\lambda))(\,\cdot\, - 2e_0 + m)\|_{\ell^{p_0}} \\
&\asymp \|a_1(\,\cdot\,,\lambda) * a_2(\,\cdot\,,\lambda)\|_{\ell^{p_0}} \\
&\le \|a_1(\,\cdot\,,\lambda)\|_{\ell^{p_1}} \|a_2(\,\cdot\,,\lambda)\|_{\ell^{p_2}}.
\end{aligned}
$$

By applying the ℓ^{q_0} quasi-norm and using Proposition 2.5 (1) we now get

$$
\|b\|_{\ell^{p_0,q_0}} \lesssim \|a_1\|_{\ell^{p_1,q_1}} \|a_2\|_{\ell^{p_2,q_2}}.
$$

This is the same as

$$
\|G\|_{L^{p_0,q_0}} \lesssim \|F_1\|_{L^{p_1,q_1}} \|F_2\|_{L^{p_2,q_2}}.
$$

A combination of this estimate with (95) and (96) gives that $f_1 * f_2$ is well-defined and that (93) holds.

The uniqueness now follows from that (93) holds for $f_1, f_2 \in \mathscr{S}(\mathbb{R}^d)$, and that $\mathscr{S}(\mathbb{R}^d)$ is dense in $M^{p,q}(\mathbb{R}^d)$ when $p, q < \infty$. □

4 Gabor Products and Modulation Spaces

In this section we give an illustration how the multiplication properties for modulation spaces can be used when treating certain nonlinear problems. We consider the Gabor product which is connected to such multiplication properties. It is introduced in [14] in order to derive a phase space analogue to the usual convolution identity for the Fourier transform. We will prove a formula related to (80), and then use results from previous section to extend the Gabor product initially defined on $M^1(\mathbb{R}^{2d}) \times M^1(\mathbb{R}^{2d})$ to some other spaces. Finally, we show how the Gabor product gives rise to a phase-space formulation of the qubic Schrödinger equation.

Definition 4.1 Let $\phi \in M^1(\mathbb{R}^d) \setminus \{0\}$, and let $F_1, F_2 \in M^1(\mathbb{R}^{2d})$. Then the Gabor product $\natural_\phi$ is given by

$$
\begin{aligned}
&\big(F_1 \natural_\phi F_2\big)(x,\xi) \\
&= (2\pi)^{-d} e^{-i\langle x,\xi\rangle} \iiint_{\mathbb{R}^{3d}} \overline{\widehat{\phi}(\zeta-\xi)} e^{i\langle x,\zeta\rangle} F_1(y,\eta) F_2(y,\zeta-\eta)\, dy\,d\eta\,d\zeta. \qquad (99)
\end{aligned}
$$

In the proof of [14, Lemma 13] it is justified that the Gabor product in (99) is well-defined, and that

$$\natural_\phi : M^1(\mathbb{R}^{2d}) \times M^1(\mathbb{R}^{2d}) \to M^1(\mathbb{R}^{2d})$$

is a continuous map.

The Gabor product is particularly well-suited in the context of the STFT.

Theorem 4.2 *Let $\phi, \phi_1, \phi_2 \in M^1(\mathbb{R}^d) \setminus \{0\}$. Then*

$$(\phi_2, \phi_1)_{L^2(\mathbb{R}^d)} V_\phi(f_1 \cdot f_2) = (V_{\phi_1} f_1) \natural_\phi (V_{\overline{\phi_2}} f_2), \quad f_1, f_2 \in M^1(\mathbb{R}^d). \tag{100}$$

Moreover, $V_\phi(f_1 \cdot f_2) \in M^1(\mathbb{R}^{2d})$.

Proof We have

$$((V_{\phi_1} f_1) \natural_\phi (V_{\overline{\phi_2}} f_2))(x, \xi) \tag{101}$$

$$= (2\pi)^{-d} e^{-i\langle x,\xi\rangle} \iint_{\mathbb{R}^{2d}} \overline{\widehat{\phi}(\zeta - \xi)} e^{i\langle x,\zeta\rangle} G(y, \zeta)\, dy d\zeta, \tag{102}$$

where

$$G(y, \zeta) = \int_{\mathbb{R}^d} (V_{\phi_1} f_1)(y, \eta)(V_{\overline{\phi_2}} f_2)(y, \zeta - \eta)\, d\eta.$$

By Parseval's formula we get

$$\begin{aligned} G(y, \zeta) &= \int_{\mathbb{R}^d} (V_{\phi_1} f_1)(y, \eta)(V_{\overline{\phi_2}} f_2)(y, \zeta - \eta)\, d\eta \\ &= \int_{\mathbb{R}^d} \mathscr{F}(f_1 \overline{\phi_1(\cdot - y)})(\eta) \mathscr{F}(f_2 \phi_2(\cdot - y))(\zeta - \eta)\, d\eta \\ &= (\mathscr{F}(f_1 \overline{\phi_1(\cdot - y)}), \mathscr{F}(\overline{f_2\, \phi_2(\cdot - y)} e^{i\langle \cdot, \zeta\rangle}))_{L^2(\mathbb{R}^d)} \\ &= (f_1 \overline{\phi_1(\cdot - y)}, \overline{f_2\, \phi_2(\cdot - y)} e^{i\langle \cdot, \zeta\rangle})_{L^2(\mathbb{R}^d)} \\ &= \int_{\mathbb{R}^d} f_1(z) \overline{\phi_1(z - y)} f_2(z) \phi_2(z - y) e^{-i\langle z, \zeta\rangle}\, dz. \end{aligned}$$

By inserting this into (102) and using Fubini's theorem we get

$$((V_{\phi_1} f_1)\natural_\phi (V_{\overline{\phi_2}} f_2))(x,\xi) \\ = (2\pi)^{-d} e^{-i\langle x,\xi\rangle} \iint_{\mathbb{R}^{2d}} \overline{\widehat{\phi}(\zeta-\xi)} e^{-i\langle z-x,\zeta\rangle} f_1(z) f_2(z) H(z)\, dz d\zeta,$$

where

$$H(z) = \int_{\mathbb{R}^d} \phi_2(z-y)\overline{\phi_1(z-y)}\, dy = (\phi_2,\phi_1)_{L^2}.$$

Hence, by evaluating the integral with respect to ζ, and using Fourier's inversion formula, we get

$$((V_{\phi_1} f_1)\natural_\phi ((V_{\overline{\phi_2}} f_2)))(x,\xi) \\ = (2\pi)^{-\frac{d}{2}} e^{-i\langle x,\xi\rangle} (\phi_2,\phi_1)_{L^2} \int_{\mathbb{R}^d} \overline{\phi(z-x)} e^{i\langle x-z,\xi\rangle} f_1(z) f_2(z)\, dz \\ = (\phi_2,\phi_1)_{L^2} V_\phi(f_1 f_2)(x,\xi),$$

which gives (100), and the result follows. □

The formula (100) is closely related to (80). In fact, the windows $\phi_j \in \Sigma_1(\mathbb{R}^d)$, $j = 0,1,2$, in (80) should satisfy the condition (79), while (100) is valid for arbitrary non-zero elements from $M^1(\mathbb{R}^d)$. For example, when $\phi = \phi_1 = \phi_2$ and $\|\phi\|_{L^2(\mathbb{R}^d)} = 1$, then (100) reduces to

$$V_\phi(f_1 \cdot f_2) = (V_\phi f_1)\natural_\phi (V_\phi f_2), \qquad f_1, f_2 \in M^1(\mathbb{R}^d), \tag{103}$$

while (80) does not allow such choice of windows.

One of the main goals of [14] are extensions of the Gabor product to some function spaces $\mathcal{F}_j(\mathbb{R}^{2d})$, $j = 0,1,2$, so that $\natural_\phi$ maps $\mathcal{F}_1 \times \mathcal{F}_2$ into $\mathcal{F}_0$, with:

$$\|F_1 \natural_\phi F_2\|_{\mathcal{F}_0} \le C \|F_1\|_{\mathcal{F}_1} \|F_2\|_{\mathcal{F}_2}. \tag{104}$$

This can be considered as a phase space form of the Young convolution inequality.

Next we discuss continuity of the Gabor product on certain spaces involving superpositions of short-time Fourier transforms. In the end we deduce properties similar to [14, Theorem 29]. Instead of modulation spaces of the form $M^{p,q}_{(\omega)}(\mathbb{R}^d)$, $p,q \in [1,\infty)$, $\omega \in \mathscr{P}_E(\mathbb{R}^{2d})$, here we consider modulation spaces of Wiener amalgam types $W^{p,q}_{(\omega)}(\mathbb{R}^d)$, and allow the "quasi-Banach" choice for Lebesgue parameters, i.e. p and q are allowed to be smaller than one.

Thus, in what follows we assume that $p, q \in (0,\infty)$, $\omega \in \mathscr{P}_E(\mathbb{R}^{2d})$ is v-moderate, and consider $L^{p,q}_{*,(\omega)}(\mathbb{R}^{2d})$ spaces rather than $L^{p,q}_{(\omega)}(\mathbb{R}^{2d})$ which are treated in [14].

We need some additional notation. Let $s > 1$, $N \in \mathbb{N}$ be given, and let

$$\mathscr{G} = \left\{\phi_n = \overline{\phi_n}\ ;\ n \in \mathbb{N}\right\} \subseteq \Sigma_s(\mathbb{R}^d),$$

be an orthonormal basis of $L^2(\mathbb{R}^d)$. Then let $\mathcal{V}^{(N),p,q}_{\mathscr{G},\omega}(\mathbb{R}^{2d})$ be the closure of

$$\mathcal{V}^{(N)}_{\mathscr{G}}(\mathbb{R}^{2d}) = \left\{\sum_{n=1}^{N} V_{\phi_n} f_n\ ;\ f_n \in \Sigma_1(\mathbb{R}^d)\right\} \tag{105}$$

with respect to the $L^{p,q}_{*,(\omega)}(\mathbb{R}^{2d})$ norm. In particular, if $N = 1$, $\phi = \phi_1$ and $p, q \geq 1$, then this reduces to the closure

$$P_\phi(L^{p,q}_{*,(\omega)}(\mathbb{R}^{2d})) = V_\phi(W^{p,q}_{(\omega)}(\mathbb{R}^d))$$

of

$$P_\phi(\Sigma_1(\mathbb{R}^{2d})) = V_\phi(\Sigma_1(\mathbb{R}^d))$$

with respect to the $L^{p,q}_{*,(\omega)}(\mathbb{R}^{2d})$ norm.

By [14, Theorem 26], it follows that for every $F \in \mathcal{V}^{(N),p,q}_{\mathscr{G},\omega}(\mathbb{R}^{2d})$ there exist $f_n \in W^{p,q}_{(\omega)}(\mathbb{R}^d)$, $n = 1, 2, \ldots, N$, and such that

$$F = \sum_{n=1}^{N} V_{\phi_n} f_n\ . \tag{106}$$

Theorem 4.3 *Let $p_j, q_j \in (0,\infty)$ and $\omega_j \in \mathscr{P}_E(\mathbb{R}^{2d})$ be v_j–moderate, $j = 0, 1, 2$, and such that (86) and (87) hold, and let $\phi \in \Sigma_s(\mathbb{R}^d)$, $s > 1$. Then the Gabor product $\natural_\phi$ from $\mathcal{V}^{(N)}_{\mathscr{G}}\left(\mathbb{R}^{2d}\right) \times \mathcal{V}^{(N)}_{\mathscr{G}}\left(\mathbb{R}^{2d}\right)$ to $W^{1,1}_{(v)}(\mathbb{R}^{2d})$, extends uniquely to a continuous map from $\mathcal{V}^{(N),p_1,q_1}_{\mathscr{G},\omega_1}(\mathbb{R}^{2d}) \times \mathcal{V}^{(N),p_2,q_2}_{\mathscr{G},\omega_2}(\mathbb{R}^{2d})$ to the closure of $P_\phi(L^{p_0,q_0}_{*,(\omega_0)}(\mathbb{R}^{2d}))$, and*

$$\|F_1 \natural_\phi F_2\|_{L^{p_0,q_0}_{*,(\omega_0)}} \lesssim \|F_1\|_{L^{p_1,q_1}_{*,(\omega_1)}} \|F_2\|_{L^{p_2,q_2}_{*,(\omega_2)}}\ , \tag{107}$$

for all $F_j \in \mathcal{V}^{(N),p_j,q_j}_{\mathscr{G},\omega_j}(\mathbb{R}^{2d})$, $j = 1, 2$.

In particular, if $F_j = V_\phi f_j$, $j = 1, 2$, and $\|\phi\|_{L^2} = 1$, then (107) reduces to

$$\|V_\phi f_1 \natural_\phi V_\phi f_2\|_{L^{p_0,q_0}_{*,(\omega_0)}} = \|f_1 f_2\|_{W^{p_0,q_0}_{(\omega_0)}} \lesssim \|f_1\|_{W^{p_1,q_1}_{(\omega_1)}} \|f_2\|_{W^{p_2,q_2}_{(\omega_2)}}\ . \tag{108}$$

We omit the proof which is a slight modification of the proof of Theorem 29 in [14].

We end the paper by formally demonstrating how the Gabor product arises in a phase space version of the cubic Schrödinger equation. Consider the elliptic nonlinear Schrödinger equation (NLSE) given by

$$i\frac{\partial \psi}{\partial t} + \Delta\psi + \lambda|\psi|^2\psi = 0, \tag{109}$$

subject to the initial condition:

$$\psi(x,0) = \varphi(x).$$

Here $\lambda = \pm 1$ stands for an attracting ($\lambda = +1$) or repulsive ($\lambda = -1$) power-law nonlinearity, and the Laplacian is given by

$$\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}.$$

Thus we consider $\psi = \phi(x,t)$ with $x \in \mathbb{R}^d$, and t in an open interval $I \subseteq \mathbb{R}$.

Using the following intertwining relations

$$V_\phi(x_j\psi) = -D_{\xi_j}V_\phi\psi, \qquad V_\phi(D_{x_j}\psi) = \left(\xi_j + D_{x_j}\right)V_\phi\psi,$$

$j = 1, \cdots, d$, and assuming that ϕ is a real-valued window, we obtain upon application of the STFT V_ϕ to (109) that

$$i\frac{\partial F}{\partial t} - \sum_{j=1}^{d}\left(\xi_j + D_{x_j}\right)^2 F + \lambda\widetilde{F}\natural_\phi F\natural_\phi F = 0. \tag{110}$$

Here, $D_{x_j} = -i\frac{\partial}{\partial x_j}$,

$$\begin{aligned} F(x,\xi,t) &= V_\phi(\psi(\,\cdot\,,t))(x,\xi) \\ &= (2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d}\psi(y,t)\overline{\phi(y-x)}e^{-i\langle y,\xi\rangle}\,dy, \quad x,\xi\in\mathbb{R}^d,\ t\in\mathbb{R}, \end{aligned}$$

and $\widetilde{F}$ is given by

$$\widetilde{F}(x,\xi) = \overline{F(x,-\xi)}. \tag{111}$$

By considering (110) the phase-space formulation of the initial value problem may be well-posed for more general initial distributions. This means that the phase-space formulation "contains" the solutions of the standard NLSE, but it is richer, as it admits other solutions. We refer to [11–13], where phase-space extensions are explored in several different contexts.

Let us conclude by noticing that (110) contains the triple product. Thus, its qualitative analysis calls for a multilinear extension of Theorems 3.6 and 4.3. Then the conditions (86) and (87) become more involved, see [54]. Such analysis demands a more technical tools and arguments and goes beyond the scope of this survey article.

Appendix A: Some Properties of Wiener Amalgam Spaces

There are convenient characterizations of modulation spaces in the framework of Gabor analysis.

Let $\omega_0 \in \mathscr{P}_E(\mathbb{R}^d)$, $\omega \in \mathscr{P}_E(\mathbb{R}^{2d})$, $p, q, r \in (0, \infty]$, $Q_d = [0, 1]^d$ be the unit cube, and set for measurable f on $\mathbb{R}^d$,

$$\|f\|_{\mathsf{W}^r(\omega_0,\ell^p)} \equiv \|a_0\|_{\ell^p(\mathbb{Z}^d)} \tag{A.1}$$

when

$$a_0(j) \equiv \|f \cdot \omega_0\|_{L^r(j+Q_d)}, \qquad j \in \mathbb{Z}^d,$$

and for measurable F on $\mathbb{R}^{2d}$,

$$\|F\|_{\mathsf{W}^r(\omega,\ell^{p,q})} \equiv \|a\|_{\ell^{p,q}(\mathbb{Z}^{2d})} \quad \text{and} \quad \|F\|_{\mathsf{W}(\omega,\ell^{p,q}_*)} \equiv \|a\|_{\ell^{p,q}_*(\mathbb{Z}^{2d})} \tag{A.2}$$

when

$$a(j,\iota) \equiv \|F \cdot \omega\|_{L^r((j,\iota)+Q_{2d})}, \qquad j, \iota \in \mathbb{Z}^d.$$

The Wiener amalgam space

$$\mathsf{W}^r(\omega_0, \ell^p) = \mathsf{W}^r(\omega_0, \ell^p(\mathbb{Z}^d))$$

consists of all measurable $f \in L^r_{loc}(\mathbb{R}^d)$ such that $\|F\|_{\mathsf{W}^r(\omega_0,\ell^p)}$ is finite, and the Wiener amalgam spaces

$$\mathsf{W}^r(\omega, \ell^{p,q}) = \mathsf{W}^r(\omega, \ell^{p,q}(\mathbb{Z}^{2d})) \quad \text{and} \quad \mathsf{W}^r(\omega, \ell^{p,q}_*) = \mathsf{W}^r(\omega, \ell^{p,q}_*(\mathbb{Z}^{2d}))$$

consist of all measurable $F \in L^r_{loc}(\mathbb{R}^{2d})$ such that $\|F\|_{\mathsf{W}^r(\omega,\ell^{p,q})}$ respectively $\|F\|_{\mathsf{W}^r(\omega,\ell^{p,q}_*)}$ are finite. We observe that $\mathsf{W}^r(\omega_0, \ell^p)$ is often denoted by $W(L^r, \ell^p_{(\omega)})$ in the literature (see e. e. [17, 19, 25, 41]).

The topologies are defined through their corresponding quasi-norms in (A.1) and (A.2). For conveniency we set

$$\mathsf{W}(\omega, \ell^{p,q}) = \mathsf{W}^\infty(\omega, \ell^{p,q}) \quad \text{and} \quad \mathsf{W}(\omega, \ell^{p,q}_*) = \mathsf{W}^\infty(\omega, \ell^{p,q}_*),$$

and if in addition $\omega = 1$, we set

$$\mathsf{W}(\ell^{p,q}) = \mathsf{W}(\omega, \ell^{p,q}) \quad \text{and} \quad \mathsf{W}(\ell^{p,q}_*) = \mathsf{W}(\omega, \ell^{p,q}_*).$$

Obviously, $\mathsf{W}^r(\omega_0, \ell^p)$ and $\mathsf{W}^r(\omega, \ell^{p,q})$ increase with p, q, decrease with r, and

$$\mathsf{W}(\omega, \ell^{p,q}) \hookrightarrow L^{p,q}_{(\omega)}(\mathbb{R}^{2d}) \cap \Sigma_1'(\mathbb{R}^{2d}) \hookrightarrow L^{p,q}_{(\omega)}(\mathbb{R}^{2d}) \hookrightarrow \mathsf{W}^r(\omega, \ell^{p,q}) \tag{A.3}$$

and

$$\| \cdot \|_{\mathsf{W}^r(\omega,\ell^{p,q})} \le \| \cdot \|_{L^{p,q}_{(\omega)}} \le \| \cdot \|_{\mathsf{W}(\omega,\ell^{p,q})}, \qquad r \le \min(1, p, q). \tag{A.4}$$

On the other hand, for modulation spaces we have

$$f \in M^{p,q}_{(\omega)}(\mathbb{R}^d) \quad \Leftrightarrow \quad V_\phi f \in L^{p,q}_{(\omega)}(\mathbb{R}^{2d}) \quad \Leftrightarrow \quad V_\phi f \in \mathsf{W}^r(\omega, \ell^{p,q}) \tag{A.5}$$

with

$$\|f\|_{M^{p,q}_{(\omega)}} = \|V_\phi f\|_{L^{p,q}_{(\omega)}} \asymp \|V_\phi f\|_{\mathsf{W}^r(\omega,\ell^{p,q})}. \tag{A.6}$$

The same holds true with $W^{p,q}_{(\omega)}$, $L^{p,q}_{*,(\omega)}$ and $\mathsf{W}(\omega, \ell^{p,q}_*)$ in place of $M^{p,q}_{(\omega)}$, $L^{p,q}_{(\omega)}$ and $\mathsf{W}(\omega, \ell^{p,q})$, respectively, at each occurrence. (For $r = \infty$, see [28] when $p, q \in [1, \infty]$, [25, 50] when $p, q \in (0, \infty]$, and for $r \in (0, \infty]$, see [53].)

We have now the following result on the projection operator P_ϕ in (20) when acting on Wiener amalgam spaces.

Proposition A.1 *Let $p, q \in (0, \infty]$ and $\phi \in \mathscr{S}(\mathbb{R}^d) \setminus \{0\}$. Then P_ϕ from $\mathscr{S}'(\mathbb{R}^{2d})$ to $\mathscr{S}'(\mathbb{R}^{2d})$, and V_ϕ^* from $\mathscr{S}'(\mathbb{R}^{2d})$ to $\mathscr{S}'(\mathbb{R}^d)$ restrict to continuous mappings*

$$P_\phi : \mathsf{W}(\ell^{p,q}(\mathbb{Z}^{2d})) \to V_\phi(M^{p,q}(\mathbb{R}^d)), \tag{A.7}$$

$$P_\phi : \mathsf{W}(\ell^{p,q}_*(\mathbb{Z}^{2d})) \to V_\phi(W^{p,q}(\mathbb{R}^d)), \tag{A.8}$$

$$V_\phi^* : \mathsf{W}(\ell^{p,q}(\mathbb{Z}^{2d})) \to M^{p,q}(\mathbb{R}^d) \tag{A.9}$$

and

$$V_\phi^* : \mathsf{W}(\ell^{p,q}_*(\mathbb{Z}^{2d})) \to W^{p,q}(\mathbb{R}^d). \tag{A.10}$$

We refer to [54, Proposition 3.6] for the proof of Proposition A.1 and to [19, 21, 28, 41, 42, 54] for some facts about the operators P_ϕ and V_ϕ^*,

For $p, q \geq 1$, i.e. the case when all spaces are Banach spaces, proofs of Proposition A.1 can be found in e.g. [28] as well as in abstract forms in [19]. In the general case when $p, q > 0$, we refer to [25, 42], since proofs of Proposition A.1 are essentially given there.

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The Hardy-Littlewood Inequalities in Sequence Spaces

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Abstract The investigation of the relation between the sums of coefficients of bilinear forms on $c_0 \times c_0$ and their supremum norms was initiated in 1930 by J.E. Littlewood. In 1934, in a joint paper with G.H. Hardy, Littlewood extended the previous results to ℓ_p spaces. The main goal of these notes is to present modern proofs of m-linear versions of the results of Hardy and Littlewood and the state-of-the-art of the subject. We also illustrate an application in a combinatorial game called Gale–Berlekamp switching game.

Keywords m-linear forms · Hardy–Littlewood inequalities

1 Introduction

G.H. Hardy and J.E. Littlewood have their names associated to dozens of inequalities and when we mention the Hardy–Littlewood inequalities it is natural that researchers of different fields conceive different results. In this work the Hardy–Littlewood inequalities are the main theorems of the paper [32] and their m-linear generalizations. In some sense the starting point of this cycle of ideas rests on the works of Orlicz, Littlewood, Bohnenblust and Hille in the beginning of the 1930's (see [17, 33]). These results show how the sums of the coefficients are dominated by the norms of m-linear forms in c_0 spaces and, in 1934, Hardy and Littlewood extended these inequalities to bilinear forms in ℓ_p spaces. We recall that an operator

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$T: E_1 \times \cdots \times E_m \to F$ between Banach spaces is called m-linear when it is linear in each coordinate. Considering the usual operations, the set of all continuous m-linear operators $T: E_1 \times \cdots \times E_m \to F$ is a Banach space when endowed with the norm

$$\|T\| := \sup\{\|T(x_1, \ldots, x_m)\| : \|x_1\|, \ldots, \|x_m\| \leq 1\}.$$

The space of all continuous m-linear operators from $E_1, \ldots, E_m$ to F is denoted by $\mathcal{L}(E_1, \ldots, E_m; F)$. It is obvious that when $E_1, \ldots, E_m$ are finite-dimensional every m-linear operator $T: E_1 \times \cdots \times E_m \to F$ is continuous. When $F = \mathbb{R}$ or $\mathbb{C}$, multilinear operators are simply called multilinear forms.

The aim of these notes is to provide a reasonably self-contained exposition of the subject up to the present date. For all $p_1, p_2 \in (1, \infty]$ satisfying $1/p_1 + 1/p_2 < 1$, let us define

$$\begin{cases} \dfrac{1}{\lambda} := 1 - \left(\dfrac{1}{p_1} + \dfrac{1}{p_2}\right); \\ \dfrac{1}{\mu} := \dfrac{3}{4} - \dfrac{1}{2}\left(\dfrac{1}{p_1} + \dfrac{1}{p_2}\right). \end{cases}$$

We consider, as usual, $1/\infty = 0$ and thus $\lambda = 1$ and $\mu = 4/3$ when $p_1 = p_2 = \infty$. Let us also settle other notations that shall be used throughout these notes. By $\mathbb{N}$ we represent the set of all positive integers, the symbol p^* denotes the conjugate of p, i.e., $1/p + 1/p^* = 1$ and $(e_k)_{k=1}^{\infty}$ denotes the sequence of canonical vectors in the sequence spaces. The vector spaces will be considered over the scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and, finally, $\mathbb{K}^n$, endowed with the ℓ_p norm, is denoted by ℓ_p^n.

In their seminal paper [32], Hardy and Littlewood prove five theorems, as follows:

Theorem 1.1 ([32, Theorems 1 and 4]) *Let $p_1, p_2 \in [2, \infty]$, with $1/p_1 + 1/p_2 \leq 1/2$. There is a constant C_{p_1,p_2} such that*

$$\left(\sum_{j_1=1}^{n}\left(\sum_{j_2=1}^{n}\left|T\left(e_{j_1}, e_{j_2}\right)\right|^2\right)^{\frac{\lambda}{2}}\right)^{\frac{1}{\lambda}} \leq C_{p_1,p_2}\|T\|, \tag{1}$$

$$\left(\sum_{j_2=1}^{n}\left(\sum_{j_1=1}^{n}\left|T\left(e_{j_1}, e_{j_2}\right)\right|^2\right)^{\frac{\lambda}{2}}\right)^{\frac{1}{\lambda}} \leq C_{p_1,p_2}\|T\|, \tag{2}$$

and

$$\left(\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\left|T\left(e_{j_1},e_{j_2}\right)\right|^{\mu}\right)^{\frac{1}{\mu}}\leq C_{p_1,p_2}\left\|T\right\|, \tag{3}$$

for all bilinear forms $T:\ell_{p_1}^{n}\times\ell_{p_2}^{n}\to\mathbb{K}$ *and all positive integers* n. *Moreover, the exponents* λ *and* μ *are optimal.*

Theorem 1.2 ([32, Theorems 2 and 4]) *Let* $p_1, p_2 \in [2,\infty]$, *with* $1/2 < 1/p_1 + 1/p_2 < 1$. *There is a constant* C_{p_1,p_2} *such that the inequalities* (1) *and* (2) *are still true, and*

$$\left(\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\left|T\left(e_{j_1},e_{j_2}\right)\right|^{\lambda}\right)^{\frac{1}{\lambda}}\leq C_{p_1,p_2}\left\|T\right\|, \tag{4}$$

for all bilinear forms $T:\ell_{p_1}^{n}\times\ell_{p_2}^{n}\to\mathbb{K}$ *and all positive integers* n. *Moreover,* λ *is optimal.*

Theorem 1.3 ([32, Theorem 5]) *Let* $p_1, p_2 \in (1,\infty]$ *be such that* $1/p_1 + 1/p_2 < 1$. *Then*

$$\left(\sum_{j_1=1}^{n}\left(\sum_{j_2=1}^{n}T\left(e_{j_1},e_{j_2}\right)^{p_2^*}\right)^{\frac{\lambda}{p_2^*}}\right)^{\frac{1}{\lambda}}\leq\left\|T\right\|$$

for all non-negative (i.e., $T\left(e_{j_1},e_{j_2}\right)\geq 0$ *for all* $(j_1,j_2)\in\mathbb{N}\times\mathbb{N}$) *bilinear forms* $T:\ell_{p_1}^{n}\times\ell_{p_2}^{n}\to\mathbb{K}$ *and all positive integers* n.

Theorem 1.4 ([32, Theorems 3 and 4]) *Let* $p_1, p_2 \in (1,\infty]$ *be such that* $1/p_1 + 1/p_2 < 1$. *If* $p_{k_1} \in (1,2]$ *and* $p_{k_2} \in (2,\infty]$, *for* $\{k_1,k_2\} = \{1,2\}$, *then there is a constant* C_{p_1,p_2} *such that*

$$\left(\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\left|T\left(e_{j_1},e_{j_2}\right)\right|^{\lambda}\right)^{\frac{1}{\lambda}}\leq C_{p_1,p_2}\left\|T\right\|, \tag{5}$$

for all bilinear forms $T:\ell_{p_1}^{n}\times\ell_{p_2}^{n}\to\mathbb{K}$ *and all positive integers* n. *Moreover,* λ *is optimal.*

The constant and, in some sense, the exponents in Theorem 1.4 were improved by Ozikiewicz and Tonge in [39]:

Theorem 1.5 ([39, Theorem 5]) *Let $p_1, p_2 \in (1, \infty]$ be such that $1/p_1 + 1/p_2 < 1$. If $p_{k_1} \in (1, 2]$ and $p_{k_2} \in (2, \infty]$, for $\{k_1, k_2\} = \{1, 2\}$, then*

$$\left(\sum_{j_{k_1}=1}^{n}\left(\sum_{j_{k_2}=1}^{n}\left|T\left(e_{j_1}, e_{j_2}\right)\right|^{p_{k_2}^*}\right)^{\frac{\lambda}{p_{k_2}^*}}\right)^{\frac{1}{\lambda}} \le \|T\|,$$

for all positive integers n and all bilinear forms $T : \ell_{p_1}^n \times \ell_{p_2}^n \to \mathbb{K}$.

Several natural questions related to the previous results arise:

- How do these inequalities behave for multilinear forms?
- What are the best constants C_{p_1,p_2}?
- What happens when $1/p_1 + 1/p_2 \ge 1$?

In the next sections we present modern proofs of the Hardy–Littlewood inequalities and discuss the three issues mentioned above. More precisely, in Sect. 2 we present some auxiliary results that will be used throughout the text. Sections 3–6 are devoted to present proofs of m-linear versions of Theorems 1.1–1.5. In Sects. 7 and 8 we try to present the state-of-the-art of the investigation related to the optimal constants of the Hardy–Littlewood inequalities and the case $1/p_1 + \cdots + 1/p_m \ge 1$. Finally, in the final section we present connections between the Hardy–Littlewood inequalities and the Gale–Berlekamp switching game.

2 Preliminary Results

From now on, E^* denotes the topological dual of a Banach space E and B_E denotes the closed unit ball of E. Moreover, $\sum_{\widehat{i_k}=1}^{n}$ denotes the multiple summation

$$\sum_{i_1=1}^{n}\cdots\sum_{i_{k-1}=1}^{n}\sum_{i_{k+1}=1}^{n}\cdots\sum_{i_m=1}^{n}$$

and, for all $\mathbf{p}_m = (p_1, \ldots, p_m) \in [1, \infty]^m$, and each $k \in \{1, \ldots, m\}$, let us define

$$\left|1/\mathbf{p}_m\right|_{\ge k} := \frac{1}{p_k} + \cdots + \frac{1}{p_m} \quad \text{and} \quad \left|1/\mathbf{p}_m\right| := \left|1/\mathbf{p}_m\right|_{\ge 1} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}.$$

When $\left|1/\mathbf{p}_m\right| < 1$ we define $1/\lambda_{\mathbf{p}_m} := 1 - \left|1/\mathbf{p}_m\right|$. We begin by proving two multilinear results that will be used several times.

The first one is due to Praciano-Pereira [45, Theorem A] and appears, in a slightly extended version, in [4, Proposition 4.1]; it is a multilinear version of the mixed inequalities (1) and (2). The proof presented here follows the lines of [4, Proposition 4.1].

Proposition 2.1 ([45, Theorem A]) *Let m, n be positive integers, with $m \geq 2$, $\mathbf{p}_m = (p_1, \ldots, p_m) \in [2, \infty]^m$ and let $A\colon \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$ be an m-linear form. Let us assume that*

(i) $\left|1/\mathbf{p}_m\right| < 1$;
(ii) $\left|1/\mathbf{p}_m\right| \leq 1/2 + 1/p_k$, *for any* $k \geq 1$.

Then, for any $k \in \{1, \ldots, m\}$,

$$\left(\sum_{i_k=1}^{n} \left(\sum_{\widehat{i_k}=1}^{n} \left| A(e_{i_1}, \ldots, e_{i_m}) \right|^2 \right)^{\frac{\lambda_{\mathbf{p}_m}}{2}} \right)^{\frac{1}{\lambda_{\mathbf{p}_m}}} \leq \left(\sqrt{2}\right)^{m-1} \|A\|.$$

Proof For the sake of convenience, we shall denote $M = \left(\sqrt{2}\right)^{m-1}$ and $a_{\mathbf{i}} = \left|A(e_{i_1}, \ldots, e_{i_m})\right|$. Let

$$l := \operatorname{card}\left\{ j : p_j \neq \infty \right\}.$$

When $l < m$ note that the conditions (i) and (ii) reduce to precisely $\left|1/\mathbf{p}_m\right| \leq 1/2$. We begin by proving by induction on $l \in \{0, \ldots, m-1\}$ that, for any $\mathbf{p}_m = (p_1, \ldots, p_m) \in [2, \infty]^m$ with $\left|1/\mathbf{p}_m\right| \leq 1/2$, we have

$$\left(\sum_{i_k=1}^{n} \left(\sum_{\widehat{i_k}=1}^{n} a_{\mathbf{i}}^2 \right)^{\frac{1}{2} \times \lambda_{\mathbf{p}_m}} \right)^{\frac{1}{\lambda_{\mathbf{p}_m}}} \leq M \|A\| \tag{6}$$

for all m-linear forms $A\colon \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$, and all $k \in \{1, \ldots, m\}$.

For $l = 0$, let $n \in \mathbb{N}$ and let $A\colon \ell_\infty^n \times \cdots \times \ell_\infty^n \to \mathbb{K}$ be an m-linear form. Then, $\lambda_{\mathbf{p}_m} = 1$ and, by the Khinchin inequality for multiple sums (see [24, p. 455]) we have

$$\left(\sum_{i_k=1}^{n} \left(\sum_{\widehat{i_k}=1}^{n} a_{\mathbf{i}}^2 \right)^{\frac{1}{2} \times \lambda_{\mathbf{p}_m}} \right)^{\frac{1}{\lambda_{\mathbf{p}_m}}} = \sum_{i_k=1}^{n} \left(\sum_{\widehat{i_k}=1}^{n} a_{\mathbf{i}}^2 \right)^{\frac{1}{2}} \leq M \|A\|$$

and, hence, the case $l = 0$ holds.

Let us assume that the result is valid for card $\left\{j : p_j \neq \infty\right\} = l - 1$ and let us prove that it is also valid when card $\left\{j : p_j \neq \infty\right\} = l$. If k is an index such that $p_k \neq \infty$, we fix $x \in \ell_{p_k}^n$ and consider

$$A_k \colon \ell_{p_1}^n \times \cdots \times \ell_{p_{k-1}}^n \times \ell_\infty^n \times \ell_{p_{k+1}}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$$

defined by

$$A_k(z^{(1)}, \ldots, z^{(m)}) = A(z^{(1)}, \ldots, z^{(k-1)}, xz^{(k)}, z^{(k+1)}, \ldots, z^{(m)}),$$

where $xz^{(k)} = \left(x_j z_j^{(k)}\right)_{j=1}^n$. By applying the induction hypothesis to A_k, we know that

$$\begin{aligned}
&\left(\sum_{i_k=1}^n \left|x_{i_k}\right|^{\lambda_k} \left(\sum_{\widehat{i_k}=1}^n a_{\mathbf{i}}^2\right)^{\frac{1}{2}\times\lambda_k}\right)^{\frac{1}{\lambda_k}} \\
&= \left(\sum_{i_k=1}^n \left(\sum_{\widehat{i_k}=1}^n \left|A(e_{i_1}, \ldots, e_{i_{k-1}}, xe_{i_k}, e_{i_{k+1}}, \ldots, e_{i_m})\right|^2\right)^{\frac{\lambda_k}{2}}\right)^{\frac{1}{\lambda_k}} \\
&= \left(\sum_{i_k=1}^n \left(\sum_{\widehat{i_k}=1}^n \left|A_k(e_{i_1}, \ldots, e_{i_m})\right|^2\right)^{\frac{\lambda_k}{2}}\right)^{\frac{1}{\lambda_k}} \\
&\leq M\|A_k\| \leq M\|A\|\|x\|_{\ell_{p_k}^n}, \qquad (7)
\end{aligned}$$

where we have set $1/\lambda_k := 1/\lambda_{\mathbf{p}_m} + 1/p_k$. Since $(p_k/\lambda_k)^* = \lambda_{\mathbf{p}_m}/\lambda_k$, we get

$$\begin{aligned}
\left(\sum_{i_k=1}^n \left(\sum_{\widehat{i_k}=1}^n a_{\mathbf{i}}^2\right)^{\frac{1}{2}\times\lambda_{\mathbf{p}_m}}\right)^{\frac{1}{\lambda_{\mathbf{p}_m}}} &= \left(\sum_{i_k=1}^n \left(\sum_{\widehat{i_k}=1}^n a_{\mathbf{i}}^2\right)^{\frac{1}{2}\times\lambda_k\times\left(\frac{p_k}{\lambda_k}\right)^*}\right)^{\frac{1}{\lambda_k}\times\frac{1}{\left(\frac{p_k}{\lambda_k}\right)^*}} \\
&= \left(\left\|\left(\left(\sum_{\widehat{i_k}=1}^n a_{\mathbf{i}}^2\right)^{\frac{1}{2}\times\lambda_k}\right)_{i_k=1}^n\right\|_{(p_k/\lambda_k)^*}\right)^{\frac{1}{\lambda_k}}
\end{aligned}$$

$$= \left(\sup_{y \in B_{\ell^n_{(p_k/\lambda_k)}}} \sum_{i_k=1}^{n} |y_{i_k}| \left(\sum_{\widehat{i_k}=1}^{n} a_{\mathbf{i}}^2 \right)^{\frac{1}{2} \times \lambda_k} \right)^{\frac{1}{\lambda_k}}$$

$$= \left(\sup_{x \in B_{\ell^n_{p_k}}} \sum_{i_k=1}^{n} |x_{i_k}|^{\lambda_k} \left(\sum_{\widehat{i_k}=1}^{n} a_{\mathbf{i}}^2 \right)^{\frac{1}{2} \times \lambda_k} \right)^{\frac{1}{\lambda_k}}$$

$$\leq M \|A\|,$$

where the last inequality holds by (7). This shows that (6) holds when k is an index such that $p_k \neq \infty$. Let us also show that it also holds when k is an index such that $p_k = \infty$. Without loss of generality, let us suppose that $p_1 \neq \infty$. The induction hypothesis applied to A_1 and

$$\frac{1}{\lambda_1} = \frac{1}{\lambda_{\mathbf{p}_m}} + \frac{1}{p_1}$$

now give

$$\forall x \in B_{\ell^n_{p_1}}, \quad \sum_{i_k=1}^{n} \left(\sum_{\widehat{i_k}=1}^{n} a_{\mathbf{i}}^2 |x_{i_1}|^2 \right)^{\frac{1}{2} \times \lambda_1} \leq M^{\lambda_1} \|A\|^{\lambda_1}. \tag{8}$$

Observe that $\lambda_1 < \lambda_{\mathbf{p}_m} \leq 2$. If $\lambda_{\mathbf{p}_m} = 2$, we have

$$\left(\sum_{i_k=1}^{n} \left(\sum_{\widehat{i_k}=1}^{n} a_{\mathbf{i}}^2 \right)^{\frac{1}{2} \times \lambda_{\mathbf{p}_m}} \right)^{\frac{1}{\lambda_{\mathbf{p}_m}}} = \left(\sum_{i_1=1}^{n} \left(\sum_{\widehat{i_1}=1}^{n} a_{\mathbf{i}}^2 \right)^{\frac{1}{2} \times \lambda_{\mathbf{p}_m}} \right)^{\frac{1}{\lambda_{\mathbf{p}_m}}} \leq M \|A\|.$$

If $\lambda_{\mathbf{p}_m} < 2$, let us denote, for $i_k \in \{1, \ldots, n\}$,

$$S_{i_k} = \left(\sum_{\widehat{i_k}=1}^{n} a_{\mathbf{i}}^2 \right)^{\frac{1}{2}}.$$

We shall show that $\sum_{i_k=1}^{n} S_{i_k}^{\lambda_{\mathbf{p}_m}} \leq M^{\lambda_{\mathbf{p}_m}} \|A\|^{\lambda_{\mathbf{p}_m}}$. We write

$$\begin{aligned}\sum_{i_k=1}^{n} S_{i_k}^{\lambda_{\mathbf{p}_m}} &= \sum_{i_k=1}^{n} S_{i_k}^{\lambda_{\mathbf{p}_m}-2} \sum_{\widehat{i_k}=1}^{n} a_{\mathbf{i}}^2 \\ &= \sum_{i_k=1}^{n} \sum_{\widehat{i_k}=1}^{n} \frac{a_{\mathbf{i}}^2}{S_{i_k}^{2-\lambda_{\mathbf{p}_m}}} \\ &= \sum_{i_1=1}^{n} \sum_{\widehat{i_1}=1}^{n} \frac{a_{\mathbf{i}}^{2/s}}{S_{i_k}^{2-\lambda_{\mathbf{p}_m}}} a_{\mathbf{i}}^{2/s^*}\end{aligned}$$

where (s, s^*) is a couple of conjugate exponents. We then apply the Hölder inequality twice to get

$$\begin{aligned}\sum_{i_k=1}^{n} S_{i_k}^{\lambda_{\mathbf{p}_m}} &\leq \sum_{i_1=1}^{n} \left(\sum_{\widehat{i_1}=1}^{n} \frac{a_{\mathbf{i}}^2}{S_{i_k}^{(2-\lambda_{\mathbf{p}_m})s}} \right)^{\frac{1}{s}} \left(\sum_{\widehat{i_1}=1}^{n} a_{\mathbf{i}}^2 \right)^{\frac{1}{s^*}} \\ &\leq \left(\sum_{i_1=1}^{n} \left(\sum_{\widehat{i_1}=1}^{n} \frac{a_{\mathbf{i}}^2}{S_{i_k}^{(2-\lambda_{\mathbf{p}_m})s}} \right)^{\frac{t}{s}} \right)^{\frac{1}{t}} \left(\sum_{i_1=1}^{n} \left(\sum_{\widehat{i_1}=1}^{n} a_{\mathbf{i}}^2 \right)^{\frac{t^*}{s^*}} \right)^{\frac{1}{t^*}}.\end{aligned}$$

Choosing

$$s = \frac{2-\lambda_1}{2-\lambda_{\mathbf{p}_m}} \quad \text{and} \quad \frac{t^*}{s^*} = \frac{\lambda_{\mathbf{p}_m}}{2}$$

so that $s/t = \lambda_1/\lambda_{\mathbf{p}_m}$, the inequality becomes

$$\begin{aligned}\sum_{i_k=1}^{n} S_{i_k}^{\lambda_{\mathbf{p}_m}} &\leq \left(\sum_{i_1=1}^{n} \left(\sum_{\widehat{i_1}=1}^{n} \frac{a_{\mathbf{i}}^2}{S_{i_k}^{(2-\lambda_1)}} \right)^{\frac{\lambda_{\mathbf{p}_m}}{\lambda_1}} \right)^{\frac{1}{t}} \left(\sum_{i_1=1}^{n} \left(\sum_{\widehat{i_1}=1}^{n} a_{\mathbf{i}}^2 \right)^{\frac{\lambda_{\mathbf{p}_m}}{2}} \right)^{\frac{1}{t^*}} \\ &\leq (M\|A\|)^{\frac{\lambda_{\mathbf{p}_m}}{t^*}} \left(\sum_{i_1=1}^{n} \left(\sum_{\widehat{i_1}=1}^{n} \frac{a_{\mathbf{i}}^2}{S_{i_k}^{(2-\lambda_1)}} \right)^{\frac{\lambda_{\mathbf{p}_m}}{\lambda_1}} \right)^{\frac{1}{t}}.\end{aligned}$$

It remains to control the last sum appearing on the right hand side of the previous inequality. By (the converse of) the Hölder inequality, it is sufficient to show that

$$\sum_{i_1=1}^{n}\left(\sum_{\widehat{i_1}=1}^{n}\frac{a_{\mathbf{i}}^2}{S_{i_k}^{(2-\lambda_1)}}\right)\left|y_{i_1}\right|\leq(M\|A\|)^{\lambda_1}$$

for any $y\in B_{\ell^n_{(\lambda_{\mathbf{p}_m}/\lambda_1)^*}}$. Since $\left(\lambda_{\mathbf{p}_m}/\lambda_1\right)^*=p_1/\lambda_1$, this means that we shall prove that

$$\sum_{i_1=1}^{n}\sum_{\widehat{i_1}=1}^{n}\frac{a_{\mathbf{i}}^2}{S_{i_k}^{2-\lambda_1}}\left|x_{i_1}\right|^{\lambda_1}\leq(M\|A\|)^{\lambda_1}$$

for any $x\in B_{\ell^n_{p_1}}$. Now, invoking the Hölder inequality with $(2-\lambda_1)/2+\lambda_1/2=1$ we obtain

$$\begin{aligned}\sum_{i_1=1}^{n}\sum_{\widehat{i_1}=1}^{n}\frac{a_{\mathbf{i}}^2}{S_{i_k}^{2-\lambda_1}}\left|x_{i_1}\right|^{\lambda_1}&=\sum_{i_k=1}^{n}\sum_{\widehat{i_k}=1}^{n}\frac{a_{\mathbf{i}}^2}{S_{i_k}^{2-\lambda_1}}\left|x_{i_1}\right|^{\lambda_1}\\&\leq\sum_{i_k=1}^{n}\left(\sum_{\widehat{i_k}=1}^{n}\frac{a_{\mathbf{i}}^2}{S_{i_k}^2}\right)^{\frac{2-\lambda_1}{2}}\left(\sum_{\widehat{i_k}=1}^{n}a_{\mathbf{i}}^2\left|x_{i_1}\right|^2\right)^{\frac{\lambda_1}{2}}.\end{aligned}$$

To conclude, it remains to observe that

$$\sum_{\widehat{i_k}=1}^{n}\frac{a_{\mathbf{i}}^2}{S_{i_k}^2}=\frac{\sum_{\widehat{i_k}}a_{\mathbf{i}}^2}{S_{i_k}^2}=1$$

and to use (8).

Finally, we can prove that the result is true when $\operatorname{card}\{j: p_j\neq\infty\}=m$ from the case $\operatorname{card}\{j: p_j\neq\infty\}=m-1$. The argument is exactly the same as we deduced the case $\operatorname{card}\{j: p_j\neq\infty\}=l$ from the case $\operatorname{card}\{j: p_j\neq\infty\}=l-1$, except that we do not need the second (and more difficult) part, since there is no index k such that $p_k\neq\infty$. This explains why we just need, for each $k\in\{1,\ldots,m\}$, $\left|1/\mathbf{p}_m\right|\leq 1/2+1/p_k$ and not $\left|1/\mathbf{p}_m\right|\leq 1/2$. □

The second result is a technical lemma that appears in [11, Lemma 3.1]:

Lemma 2.2 ([11, Lemma 3.1]) *Let $m \geq 1$, $\mathbf{q}_m = (q_1, \ldots, q_m) \in (0, \infty)^m$, and $\mathbf{p}_m = (p_1, \ldots, p_m) \in (1, \infty]^m$, with $\left|1/\mathbf{p}_m\right| < 1$. If there is a constant $C_{\mathbf{p}_m, \mathbf{q}_m}$ such that*

$$\left(\sum_{j_1=1}^{n} \left(\cdots \left(\sum_{j_m=1}^{n} T(e_{j_1}, \ldots, e_{j_m})^{q_m} \right)^{\frac{q_{m-1}}{q_m}} \cdots \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \leq C_{\mathbf{p}_m, \mathbf{q}_m} \|T\|$$

for all $n \in \mathbb{N}$ and all non-negative m-linear forms $T : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$, then

$$1/q_k \leq 1 - \left|1/\mathbf{p}_m\right|_{\geq k},$$

for all $k \in \{1, \ldots, m\}$.

Proof Let $p > 1$, $q > 0$ and suppose that there is a constant $C_{p,q}$ such that

$$\left(\sum_{j_1=1}^{n} T(e_{j_1})^q \right)^{\frac{1}{q}} \leq C_{p,q} \|T\|$$

for all non-negative linear forms $T : \ell_p^n \to \mathbb{K}$. For each n, consider the non-negative linear form $T_n(x) = \sum_{j=1}^{n} x_j$. By the Hölder inequality, we have $\|T_n\| \leq n^{1/p^*}$. On the other hand

$$\left(\sum_{j=1}^{n} T_n(e_j)^q \right)^{\frac{1}{q}} = n^{\frac{1}{q}}$$

and, since n is arbitrary, we conclude that $1/q \leq 1/p^*$ and the case $m = 1$ is done. Now, let us proceed by induction. Suppose that the result is valid for $m - 1$ and let $\left|1/\mathbf{p}_m\right| < 1$. Thus, $\left|1/\mathbf{p}_m\right|_{\geq 2} < 1$ and the induction hypothesis combined with a simple argument of summability tells us that, if there is a constant $C_{\mathbf{p}_m, \mathbf{q}_m}$ such that

$$\left(\sum_{j_1=1}^{n} \left(\cdots \left(\sum_{j_m=1}^{n} T(e_{j_1}, \ldots, e_{j_m})^{q_m} \right)^{\frac{q_{m-1}}{q_m}} \cdots \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \leq C_{\mathbf{p}_m, \mathbf{q}_m} \|T\|$$

for all $n \in \mathbb{N}$ and all non-negative m-linear forms $T : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$, then

$$1/q_k \leq 1 - \left|1/\mathbf{p}_m\right|_{\geq k}$$

for all $k \in \{2, \ldots, m\}$. So, we must only show that

$$1/q_1 \leq 1 - \left|1/\mathbf{p}_m\right| = 1/\lambda_{\mathbf{p}_m}.$$

For each n consider the non-negative m-linear form $B_n : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$ given by

$$B_n(x^{(1)}, \ldots, x^{(m)}) = \sum_{j=1}^{n} x_j^{(1)} x_j^{(2)} \cdots x_j^{(m)}.$$

By the Hölder inequality we obtain

$$\begin{aligned}\|B_n\| &= \sup_{\|x^{(1)}\|,\ldots,\|x^{(m)}\| \leq 1} \left| \sum_{j=1}^{n} x_j^{(1)} x_j^{(2)} \cdots x_j^{(m)} \right| \\ &\leq \sup_{\|x^{(1)}\|,\ldots,\|x^{(m)}\| \leq 1} \left(\prod_{k=1}^{m} \left\| \left(x_j^{(k)} \right)_{j=1}^{n} \right\|_{\ell_{p_k}^n} \left(\sum_{j=1}^{n} |1|^{\lambda_{\mathbf{p}_m}} \right)^{\frac{1}{\lambda_{\mathbf{p}_m}}} \right) \\ &\leq n^{1/\lambda_{\mathbf{p}_m}}.\end{aligned}$$

On the other hand,

$$\left(\sum_{j_1=1}^{n} \left(\cdots \left(\sum_{j_m=1}^{n} B_n(e_{j_1}, \ldots, e_{j_m})^{q_m} \right)^{\frac{q_{m-1}}{q_m}} \cdots \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} = n^{\frac{1}{q_1}},$$

and, since n is arbitrary, we conclude that $1/q_1 \leq 1/\lambda_{\mathbf{p}_m}$ and the proof is completed. □

3 Theorem 1.1 for m-Linear Forms

In this section we prove the following extended version of Theorem 1.1 for m-linear forms.

Theorem 3.1 ([5, Theorem 1.3]) *Let $m \geq 2$, let $\mathbf{p}_m = (p_1, \ldots, p_m) \in [2, \infty]^m$ be such that $0 \leq \left|1/\mathbf{p}_m\right| \leq 1/2$, and $\mathbf{q}_m = (q_1, \ldots, q_m) \in \left[\lambda_{\mathbf{p}_m}, 2\right]^m$. The following statements are equivalent:*

(i) *There is a constant $C_{\mathbf{p}_m,\mathbf{q}_m}$ such that, for each $n \in \mathbb{N}$ and each m-linear form $T: \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$ we have*

$$\left(\sum_{i_1=1}^{n}\left(\cdots\left(\sum_{i_m=1}^{n}\left|T\left(e_{i_1}, \ldots, e_{i_m}\right)\right|^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}} \leq C_{\mathbf{p}_m,\mathbf{q}_m}\|T\|. \tag{9}$$

(ii)

$$\left|1/\mathbf{q}_m\right| \leq \frac{m+1}{2} - \left|1/\mathbf{p}_m\right|. \tag{10}$$

Proof $(i) \Rightarrow (ii)$. The Kahane–Salem–Zygmund inequality (see, for instance, [3]) assures the existence of an m-linear form $T_0: \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$ with coefficients ± 1 satisfying

$$\|T_0\| \leq K_m n^{\frac{m+1}{2} - |1/\mathbf{p}_m|}$$

for a certain constant K_m. Plugging this m-linear form into (9) we arrive at (ii).

$(ii) \Rightarrow (i)$. Let $n \in \mathbb{N}$ and let $T: \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$ be an m-linear form. By Proposition 2.1, we have

$$\left(\sum_{i_k=1}^{n}\left(\sum_{\widehat{i_k}=1}^{n}\left|T(e_{i_1}, \ldots, e_{i_m})\right|^2\right)^{\frac{\lambda_{\mathbf{p}_m}}{2}}\right)^{\frac{1}{\lambda_{\mathbf{p}_m}}} \leq \left(\sqrt{2}\right)^{m-1}\|T\|.$$

for all n and all $k \in \{1, \ldots, m\}$. Using a well-known result of Minkowski (see [31, Corollary 5.4.2]), since $\lambda_{\mathbf{p}_m} \leq 2$, we can interchange the position of the exponent 2 and obtain m inequalities with the same constant $\left(\sqrt{2}\right)^{m-1}$ and exponents

$$\begin{cases} \left(q_{1,1}, q_{2,1}, q_{3,1}, \ldots, q_{m,1}\right) = \left(\lambda_{\mathbf{p}_m}, 2, 2, \ldots, 2\right), \\ \left(q_{1,2}, q_{2,2}, q_{3,2}, \ldots, q_{m,2}\right) = \left(2, \lambda_{\mathbf{p}_m}, 2, \ldots, 2\right) \\ \vdots \\ \left(q_{1,m}, q_{2,m}, \ldots, q_{m-1,m}, q_{m,m}\right) = \left(2, 2, \ldots, 2, \lambda_{\mathbf{p}_m}\right). \end{cases}$$

Thanks to the monotonicity of the ℓ_p-norms, the important case to be considered in (10) is the equality. If $\lambda_{\mathbf{p}_m} = 2$ we have that $q_1 = \cdots = q_m = 2$ and by Proposition 2.1 we arrive at (i). If $\lambda_{\mathbf{p}_m} < 2$, choosing

$$\theta_j = \frac{2\lambda_{\mathbf{p}_m} - \lambda_{\mathbf{p}_m} q_j}{2q_j - \lambda_{\mathbf{p}_m} q_j},$$

it is easy to see that

$$\frac{1}{q_j} = \sum_{k=1}^{m} \frac{\theta_k}{q_{j,k}}$$

and, by the Hölder inequality for mixed sums (see [1, Theorem 2.49]), we conclude that

$$\left(\sum_{i_1=1}^{n} \left(\cdots \left(\sum_{i_m=1}^{n} \left| T\left(e_{i_1}, \ldots, e_{i_m}\right) \right|^{q_m} \right)^{\frac{q_{m-1}}{q_m}} \cdots \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \leq \left(\sqrt{2}\right)^{m-1} \|T\| .$$

□

Observe that, if we take

$$q_1 = \cdots = q_m = \frac{2m}{m+1-2\left|1/\mathbf{p}_m\right|} \in \left[\lambda_{\mathbf{p}_m}, 2\right],$$

we obtain

$$\left|1/\mathbf{q}_m\right| = \frac{m+1}{2} - \left|1/\mathbf{p}_m\right|,$$

providing the following extension of (3) to the multilinear setting:

Theorem 3.2 ([45, Theorem B]) *Let $m \geq 2$, and let $\mathbf{p}_m = (p_1, \ldots, p_m) \in [2, \infty]^m$ be such that $0 \leq \left|1/\mathbf{p}_m\right| \leq 1/2$. There is a constant $C_{\mathbf{p}_m} \geq 1$ such that, for every positive integer n and each m-linear form $T : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$, we have*

$$\left(\sum_{i_1,\ldots,i_m=1}^{n} \left| T\left(e_{i_1}, \ldots, e_{i_m}\right) \right|^{\frac{2m}{m+1-2\left|1/\mathbf{p}_m\right|}} \right)^{\frac{m+1-2\left|1/\mathbf{p}_m\right|}{2m}} \leq C_{\mathbf{p}_m} \|T\| .$$

Moreover, the exponent $2m/\left(m+1-2\left|1/\mathbf{p}_m\right|\right)$ is optimal.

4 Theorem 1.2 for m-Linear Forms

An m-linear operator between Banach spaces $T: E_1 \times \cdots \times E_m \to F$ is multiple $(r; \mathbf{p}_m)$-summing (see e.g. [18, 34]), where $\mathbf{p}_m = (p_1, \ldots, p_m)$, if there is $C > 0$ such that

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} \left\| T\left(x_{j_1}^{(1)}, \ldots, x_{j_m}^{(m)}\right) \right\|^r \right)^{\frac{1}{r}} \leq C \prod_{k=1}^{m} \sup_{\varphi_k \in B_{E_k^*}} \left(\sum_{j=1}^{n} \left| \varphi_k \left(x_j^{(k)}\right) \right|^{p_k} \right)^{\frac{1}{p_k}}.$$

for every $\left(x_{i_j}^{(j)}\right)_{i_j=1}^{n} \subseteq E_j$, with $j = 1, \ldots, m$.

We denote by $\Pi_{(r;\mathbf{p}_m)}^m (E_1, \ldots, E_m; F)$ the space composed by all multiple $(r; \mathbf{p}_m)$-summing m-linear operators $T: E_1 \times \cdots \times E_m \to F$. In the case of linear operators, this notion reduces to the well-known concept of absolutely summing operators (see [28]).

The following extension of Theorem 1.2 is due to Dimant and Sevilla-Peris ([29, Proposition 4.1]); the proof presented here follows the lines of [9, Theorem 2.2].

Theorem 4.1 ([29, Proposition 4.1]) *Let $m \geq 2$, and let $\mathbf{p}_m = (p_1, \ldots, p_m) \in [2, \infty]^m$ be such that $1/2 \leq |1/\mathbf{p}_m| < 1$. There is a constant $C_{\mathbf{p}_m}$ such that*

$$\left(\sum_{i_1,\ldots,i_m=1}^{n} \left| T\left(e_{i_1}, \ldots, e_{i_m}\right) \right|^{\lambda_{\mathbf{p}_m}} \right)^{\frac{1}{\lambda_{\mathbf{p}_m}}} \leq C_{\mathbf{p}_m} \|T\|,$$

for all m-linear forms $T: \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$ and all positive integers n. Moreover, the exponent $\lambda_{\mathbf{p}_m}$ is optimal.

Proof The optimality of the exponent $\lambda_{\mathbf{p}_m}$ is a straightforward consequence of Lemma 2.2. Let

$$s = \min \left\{ r : \text{there exist } r \text{ indexes } k_1, \ldots, k_r \text{ such that } \frac{1}{2} \leq \sum_{i=1}^{r} \frac{1}{p_{k_i}} < 1 \right\}.$$

For the sake of simplicity let us suppose that $p_{k_1} = p_1, \ldots, p_{k_s} = p_s$. Observe that

$$\frac{1}{p_1} + \cdots + \frac{1}{p_{s-1}} < \frac{1}{2}.$$

Let $n \in \mathbb{N}$ and let $A\colon \ell_{p_1}^n \times \cdots \times \ell_{p_s}^n \to \mathbb{K}$ be s-linear. Let us fix $x \in \ell_{p_s}^n$ and consider

$$A_s\colon \ell_{p_1}^n \times \cdots \times \ell_{p_{s-1}}^n \times \ell_\infty^n \to \mathbb{K}$$
$$(z^{(1)}, \ldots, z^{(s)}) \mapsto A(z^{(1)}, \ldots, z^{(s-1)}, xz^{(s)}).$$

By Proposition 2.1, for A_s, we have

$$\left(\sum_{i_s=1}^n \left| x_{i_s} \right|^{\lambda_{s-1}} \left(\sum_{i_1,\ldots,i_{s-1}=1}^n \left| A(e_{i_1}, \ldots, e_{i_s}) \right|^2 \right)^{\frac{\lambda_{s-1}}{2}} \right)^{\frac{1}{\lambda_{s-1}}}$$
$$\leq \left(\sqrt{2}\right)^{s-1} \|A\| \|x\|_{\ell_{p_s}^n}$$

with $\lambda_{s-1} = [1 - (1/p_1 + \cdots + 1/p_{s-1})]^{-1}$. Since

$$\frac{1}{(p_s/\lambda_{s-1})^*} = 1 - \frac{\lambda_{s-1}}{p_s} = \frac{\lambda_{s-1}}{\lambda_s},$$

with $\lambda_s = [1 - (1/p_1 + \cdots + 1/p_s)]^{-1}$, we get

$$\left(\sum_{i_s=1}^n \left(\sum_{i_1,\ldots,i_{s-1}=1}^n \left| A(e_{i_1}, \ldots, e_{i_s}) \right|^2 \right)^{\frac{\lambda_s}{2}} \right)^{\frac{1}{\lambda_s}}$$
$$= \left(\sum_{i_s=1}^n \left(\sum_{i_1,\ldots,i_{s-1}=1}^n \left| A(e_{i_1}, \ldots, e_{i_s}) \right|^2 \right)^{\lambda_{s-1} \times (p_s/\lambda_{s-1})^*} \right)^{\frac{1}{\lambda_{s-1}} \times \frac{1}{(p_s/\lambda_{s-1})^*}}$$

$$= \left(\left\| \left(\left(\sum_{i_1,\ldots,i_{s-1}=1}^n \left| A(e_{i_1}, \ldots, e_{i_s}) \right|^2 \right)^{\lambda_{s-1}} \right)_{i_s=1}^n \right\|_{(p_s/\lambda_{s-1})^*} \right)^{\frac{1}{\lambda_{s-1}}}$$
$$= \left(\sup_{y \in B_{\ell^n_{\frac{p_s}{\lambda_{s-1}}}}} \sum_{i_s=1}^n \left| y_{i_s} \right| \left(\sum_{i_1,\ldots,i_{s-1}=1}^n \left| A(e_{i_1}, \ldots, e_{i_s}) \right|^2 \right)^{\lambda_{s-1}} \right)^{\frac{1}{\lambda_{s-1}}}$$

$$= \left(\sup_{x \in B_{\ell_{p_s}^n}} \sum_{i_s=1}^{n} \left| x_{i_s} \right|^{\lambda_{s-1}} \left(\sum_{i_1,\dots,i_{s-1}=1}^{n} \left| A(e_{i_1}, \dots, e_{i_s}) \right|^2 \right)^{\lambda_{s-1}} \right)^{\frac{1}{\lambda_{s-1}}}$$

$$\leq \left(\sqrt{2} \right)^{s-1} \|A\|.$$

Thanks to the monotonicity of the ℓ_p-norms, since $\lambda_s \geq 2$, the above estimate implies

$$\left(\sum_{i_1,\dots,i_s=1}^{n} \left| A(e_{i_1}, \dots, e_{i_s}) \right|^{\lambda_s} \right)^{1/\lambda_s} \leq \left(\sqrt{2} \right)^{s-1} \|A\|. \tag{11}$$

By the Khinchin inequality (see [28, Theorem 1.10]), with constant 1, because $\lambda_s \geq 2$, we have, for every n and all $(s+1)$-linear forms $T_{s+1} : \ell_{p_1}^n \times \cdots \times \ell_{p_s}^n \times \ell_\infty^n \to \mathbb{K}$,

$$\left(\sum_{i_1,\dots,i_s=1}^{n} \left(\sum_{i_{s+1}=1}^{n} \left| T_{s+1} \left(e_{i_1}, \dots, e_{i_{s+1}} \right) \right|^2 \right)^{\frac{\lambda_s}{2}} \right)^{\frac{1}{\lambda_s}}$$

$$\leq \left(\sum_{i_1,\dots,i_s=1}^{n} \left(\int_0^1 \left| \sum_{i_{s+1}=1}^{n} T_{s+1} \left(e_{i_1}, \dots, e_{i_{s+1}} \right) r_{i_{s+1}}(t) \right|^{\lambda_s} dt \right)^{\frac{\lambda_s}{\lambda_s}} \right)^{\frac{1}{\lambda_s}}$$

$$= \left(\int_0^1 \sum_{i_1,\dots,i_s=1}^{n} \left| T_{s+1} \left(e_{i_1}, \dots, e_{i_s}, \sum_{i_{s+1}=1}^{n} e_{i_{s+1}} r_{i_{s+1}}(t) \right) \right|^{\lambda_s} dt \right)^{\frac{1}{\lambda_s}}$$

$$\leq \left(\sup_{t \in [0,1]} \sum_{i_1,\dots,i_s=1}^{n} \left| T_{s+1} \left(e_{i_1}, \dots, e_{i_s}, \sum_{i_{s+1}=1}^{n} e_{i_{s+1}} r_{i_{s+1}}(t) \right) \right|^{\lambda_s} dt \right)^{\frac{1}{\lambda_s}}$$

$$= \sup_{t \in [0,1]} \left(\sum_{i_1,\dots,i_s=1}^{n} \left| T_{s+1} \left(e_{i_1}, \dots, e_{i_s}, \sum_{i_{s+1}=1}^{n} e_{i_{s+1}} r_{i_{s+1}}(t) \right) \right|^{\lambda_s} dt \right)^{\frac{1}{\lambda_s}}$$

$$\leq \left(\sqrt{2}\right)^{s-1} \sup_{t\in[0,1]} \left\| T_{s+1}\left(\cdot, \ldots, \cdot, \sum_{i_{s+1}=1}^{n} e_{i_{s+1}} r_{i_{s+1}}(t)\right) \right\|$$
$$= \left(\sqrt{2}\right)^{s-1} \|T_{s+1}\|.$$

Here and henceforth, $r_j(t)$ denotes the j-th Rademacher function. Thus, from the previous inequality together with canonical inclusion of ℓ_p spaces,

$$\left(\sum_{i_1,\ldots,i_{s+1}=1}^{n} \left|T_{s+1}\left(e_{i_1}, \ldots, e_{i_{s+1}}\right)\right|^{\lambda_s} \right)^{\frac{1}{\lambda_s}}$$
$$= \left(\sum_{i_1,\ldots,i_s=1}^{n} \left(\sum_{i_{s+1}=1}^{n} \left|T_{s+1}\left(e_{i_1}, \ldots, e_{i_{s+1}}\right)\right|^{\lambda_s} \right)^{\frac{\lambda_s}{\lambda_s}} \right)^{\frac{1}{\lambda_s}}$$
$$\leq \left(\sum_{i_1,\ldots,i_s=1}^{n} \left(\sum_{i_{s+1}=1}^{n} \left|T_{s+1}\left(e_{i_1}, \ldots, e_{i_{s+1}}\right)\right|^{2} \right)^{\frac{\lambda_s}{2}} \right)^{\frac{1}{\lambda_s}}$$
$$\leq \left(\sqrt{2}\right)^{s-1} \|T_{s+1}\|,$$

for every n and all $(s+1)$-linear forms $T_{s+1}\colon \ell_{p_1}^n \times \cdots \times \ell_{p_s}^n \times \ell_\infty^n \to \mathbb{K}$. Using the canonical isometric isomorphisms for the spaces of weakly summable sequences (see [28, Proposition 2.2]) we know that this is equivalent to assert that (see [29, p. 308])

$$\Pi^{s+1}_{(\lambda_s; p_1^*,\ldots,p_s^*,1)}(E_1, \ldots, E_{s+1}; \mathbb{K}) = \mathcal{L}(E_1, \ldots, E_{s+1}; \mathbb{K})$$

for all Banach spaces $E_1, \ldots, E_{s+1}$. Since

$$\frac{1}{\lambda_s} - \left(1 + \sum_{j=1}^{s} \frac{1}{p_j^*}\right) + \left(\sum_{j=1}^{s+1} \frac{1}{p_j^*}\right) = \frac{1}{\lambda_s} - \frac{1}{p_{s+1}} = 1 - \sum_{j=1}^{s+1} \frac{1}{p_j} > 0,$$

by the inclusion theorem for multiple summing forms proved in [2, 13], we have

$$\Pi^{s+1}_{(\lambda_{s+1}; p_1^*,\ldots,p_{s+1}^*)}(E_1, \ldots, E_{s+1}; \mathbb{K}) = \mathcal{L}(E_1, \ldots, E_{s+1}; \mathbb{K})$$

for all Banach spaces $E_1, \ldots, E_{s+1}$, with $1/\lambda_{s+1} = 1 - (1/p_1 + \cdots + 1/p_{s+1})$. Again (see [29, p. 308]), this is equivalent to say that there is a constant $C_{(p_1,\ldots,p_{s+1})}$ such that

$$\left(\sum_{i_1,\ldots,i_{s+1}=1}^{n} \left|T\left(e_{i_1},\ldots,e_{i_{s+1}}\right)\right|^{\lambda_{s+1}}\right)^{\frac{1}{\lambda_{s+1}}} \le C_{(p_1,\ldots,p_{s+1})}\|T\|,$$

for all $(s+1)$-linear forms $T\colon \ell_{p_1}^n \times \cdots \times \ell_{p_s}^n \times \ell_{p_{s+1}}^n \to \mathbb{K}$ and all positive integers n. The proof is completed by a standard induction argument. □

5 Theorem 1.3 for Non-negative *m*-Linear Forms

From now on, for all $\mathbf{p}_m = (p_1, \ldots, p_m) \in [1,\infty]^m$, with $\left|1/\mathbf{p}_m\right| < 1$, and each $k \in \{1,\ldots,m\}$, let us define

$$1/\lambda_{\mathbf{p}_m,k} = 1 - \left|1/\mathbf{p}_m\right|_{\ge k}.$$

Observe that $\lambda_{\mathbf{p}_m} = \lambda_{\mathbf{p}_m,1}$. Moreover, if $\sigma\colon \{1,\ldots,m\} \to \{1,\ldots,m\}$ is a bijection, then we will denote $\sigma_i = \sigma(i)$ for all $i = 1,\ldots,m$.

Theorem 5.1 ([37, Theorem 1.3]) *Let $m \in \mathbb{N}$, $\mathbf{p}_m = (p_1,\ldots,p_m) \in (1,\infty]^m$, with $\left|1/\mathbf{p}_m\right| < 1$, and $\mathbf{q}_m = (q_1,\ldots,q_m) \in (0,\infty)^m$. For any bijection $\sigma\colon \{1,\ldots,m\} \to \{1,\ldots,m\}$ the following assertions are equivalent:*

(i)

$$\left(\sum_{j_{\sigma_1}=1}^{n}\left(\sum_{j_{\sigma_2}=1}^{n}\cdots\left(\sum_{j_{\sigma_m}=1}^{n} T(e_{j_1},\ldots,e_{j_m})^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}} \le \|T\|$$

for all non-negative m-linear forms $T\colon \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$ and all positive integers n.

(ii) *There is a constant $C_{\mathbf{p}_m,\mathbf{q}_m}$ such that*

$$\left(\sum_{j_{\sigma_1}=1}^{n}\left(\sum_{j_{\sigma_2}=1}^{n}\cdots\left(\sum_{j_{\sigma_m}=1}^{n} T(e_{j_1},\ldots,e_{j_m})^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}} \le C_{\mathbf{p}_m,\mathbf{q}_m}\|T\|$$

for all non-negative m-linear forms $T : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$ *and all positive integers n.*

(iii) *The exponents* $q_1, \ldots, q_m$ *satisfy*

$$q_k \geq \lambda_{\mathbf{t}_m,k},$$

for all $k \in \{1, \ldots, m\}$, *with* $\boldsymbol{t}_m = \left(p_{\sigma_1}, \ldots, p_{\sigma_m}\right)$.

Proof To simplify the notation we will consider $\sigma_j = j$ for all j; the other cases are similar. We observe that, in this case, $\lambda_{\mathbf{t}_m,k} = \lambda_{\mathbf{p}_m,k}$ for all $k \in \{1, \ldots, m\}$.

$(i) \Rightarrow (ii)$ is obvious and $(ii) \Rightarrow (iii)$ follows from Lemma 2.2. So, it remains to prove $(iii) \Rightarrow (i)$. In the case $m = 1$ the result is immediate, it holds with constant 1 and does not need the non-negative assumption. Let us show the general case m, supposing that the result holds for $m - 1$; so we suppose that if $\mathbf{p}_{m-1} = (p_1, \ldots, p_{m-1}) \in (1, \infty]^{m-1}$ is such that $\left|1/\mathbf{p}_{m-1}\right| < 1$, then

$$\left(\sum_{j_1=1}^{n} \left(\cdots \left(\sum_{j_{m-1}=1}^{n} T(e_{j_1}, \ldots, e_{j_{m-1}})^{\mu_{m-1}} \right)^{\frac{\mu_{m-2}}{\mu_{m-1}}} \cdots \right)^{\frac{\mu_1}{\mu_2}} \right)^{\frac{1}{\mu_1}} \leq \|T\|,$$

where

$$(\mu_1, \ldots, \mu_{m-1}) = \left(\lambda_{\mathbf{p}_{m-1},1}, \ldots, \lambda_{\mathbf{p}_{m-1},m-1}\right),$$

for all positive integers n and all non negative $(m-1)$-linear forms $T : \ell_{p_1}^n \times \cdots \times \ell_{p_{m-1}}^n \to \mathbb{K}$. We recall that for an m-linear form $D : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$, we have

$$\begin{aligned}
\|D\| &= \sup_{\|x^{(i)}\|_{\ell_{p_i}^n} \leq 1;\ 1 \leq i \leq m} \left| D\left(x^{(1)}, \ldots, x^{(m)}\right) \right| \\
&= \sup_{\|x^{(i)}\|_{\ell_{p_i}^n} \leq 1;\ 1 \leq i \leq m} \left| \sum_{j_1, \ldots, j_m=1}^{n} D\left(e_{j_1}, \ldots, e_{j_m}\right) x_{j_1}^{(1)} \cdots x_{j_m}^{(m)} \right| \\
&= \sup_{\|x^{(i)}\|_{\ell_{p_i}^n} \leq 1;\ 1 \leq i \leq m-1} \left(\sum_{j_m=1}^{n} \left| \sum_{\widehat{j_m}=1}^{n} D\left(e_{j_1}, \ldots, e_{j_m}\right) x_{j_1}^{(1)} \cdots x_{j_{m-1}}^{(m-1)} \right|^{p_m^*} \right)^{\frac{1}{p_m^*}}.
\end{aligned} \tag{12}$$

Suppose that $\mathbf{p}_m = (p_1, \ldots, p_m) \in (1, \infty]^m$ satisfies $\left|1/\mathbf{p}_m\right| < 1$. In this case

$$\frac{1}{p_1} + \cdots + \frac{1}{p_{m-1}} < 1 - \frac{1}{p_m} = \frac{1}{\lambda_{\mathbf{p}_m,m}} = \frac{1}{p_m^*}$$

and thus we have $p_i > p_m^*$ for all $i \in \{1, \ldots, m-1\}$.

Let $D\colon \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$ be a non negative m-linear form. We define the non negative $(m-1)$-linear form $T\colon \ell_{r_1}^n \times \cdots \times \ell_{r_{m-1}}^n \to \mathbb{K}$ by

$$T(e_{j_1}, \ldots, e_{j_{m-1}}) = \sum_{j_m=1}^{n} D(e_{j_1}, \ldots, e_{j_m})^{p_m^*}, \tag{13}$$

with $r_i = p_i/p_m^*$ for each $i = 1, \ldots, m-1$. Note that $\mathbf{r}_{m-1} = (r_1, \ldots, r_{m-1}) \in (1, \infty]^{m-1}$ and, if we denote by ${}^+\ell_r^n$ the set of sequences $\left(x_j\right) \in \ell_r^n$, such that $x_j \geq 0$ for all j, we have

$$\begin{aligned}
&\sup_{\|x^{(i)}\|_{\ell_{r_i}^n} \leq 1;\ 1 \leq i \leq m-1} \sum_{\widehat{j_m}=1}^{n} \left| T(e_{j_1}, \ldots, e_{j_{m-1}}) x_{j_1}^{(1)} \cdots x_{j_{m-1}}^{(m-1)} \right| \\
&= \sup_{\|x^{(i)}\|_{+\ell_{r_i}^n} \leq 1;\ 1 \leq i \leq m-1} \sum_{\widehat{j_m}=1}^{n} T(e_{j_1}, \ldots, e_{j_{m-1}}) x_{j_1}^{(1)} \cdots x_{j_{m-1}}^{(m-1)} \\
&= \sup_{\|x^{(i)}\|_{+\ell_{p_i}^n} \leq 1;\ 1 \leq i \leq m-1} \sum_{\widehat{j_m}=1}^{n} T(e_{j_1}, \ldots, e_{j_{m-1}}) \left(x_{j_1}^{(1)} \cdots x_{j_{m-1}}^{(m-1)} \right)^{p_m^*}
\end{aligned}$$

$$\begin{aligned}
&= \sup_{\|x^{(i)}\|_{+\ell_{p_i}^n} \leq 1;\ 1 \leq i \leq m-1} \sum_{j_m=1}^{n} \sum_{\widehat{j_m}=1}^{n} D(e_{j_1}, \ldots, e_{j_m})^{p_m^*} \left(x_{j_1}^{(1)} \cdots x_{j_{m-1}}^{(m-1)} \right)^{p_m^*} \\
&= \sup_{\|x^{(i)}\|_{+\ell_{p_i}^n} \leq 1;\ 1 \leq i \leq m-1} \sum_{j_m=1}^{n} \left(\sum_{\widehat{j_m}=1}^{n} D(e_{j_1}, \ldots, e_{j_m}) x_{j_1}^{(1)} \cdots x_{j_{m-1}}^{(m-1)} \right)^{p_m^*} \\
&\leq \sup_{\|x^{(i)}\|_{\ell_{p_i}^n} \leq 1;\ 1 \leq i \leq m-1} \sum_{j_m=1}^{n} \left| \sum_{\widehat{j_m}=1}^{n} D\left(e_{j_1}, \ldots, e_{j_m}\right) x_{j_1}^{(1)} \cdots x_{j_{m-1}}^{(m-1)} \right|^{p_m^*} \\
&= \|D\|^{p_m^*},
\end{aligned} \tag{14}$$

for all $x^{(k)} \in B_{\ell_{r_k}^n}$, with $k = 1, \ldots, m-1$, where the last equality holds by (12).

Note that

$$\sum_{k=i}^{m-1}\frac{1}{r_k}=p_m^*\sum_{k=i}^{m-1}\frac{1}{p_k}=\left(1-\frac{1}{p_m}\right)^{-1}\sum_{k=i}^{m-1}\frac{1}{p_k}<1.$$

Hence, for each $i\in\{1,\dots,m-1\}$, a simple calculation shows that

$$p_m^*\lambda_{\mathbf{r}_{m-1},i}=\lambda_{\mathbf{p}_m,i}.$$

Therefore, by (13), making

$$(\alpha_1,\dots,\alpha_m)=\left(\lambda_{\mathbf{p}_m,1},\dots,\lambda_{\mathbf{p}_m,m}\right)$$

and

$$(\beta_1,\dots,\beta_{m-1})=\left(\lambda_{\mathbf{r}_{m-1},1},\dots,\lambda_{\mathbf{r}_{m-1},m-1}\right)$$

we have

$$\left(\sum_{j_1=1}^{n}\left(\cdots\left(\sum_{j_m=1}^{n}D(e_{j_1},\dots,e_{j_m})^{p_m^*}\right)^{\frac{\alpha_{m-1}}{\alpha_m}}\cdots\right)^{\frac{\alpha_1}{\alpha_2}}\right)^{\frac{1}{\alpha_1}\times p_m^*}$$

$$=\left(\sum_{j_1=1}^{n}\left(\cdots\left(\sum_{j_{m-1}=1}^{n}T(e_{j_1},\dots,e_{j_{m-1}})^{\frac{\alpha_{m-1}}{\alpha_m}}\right)^{\frac{\alpha_{m-2}}{\alpha_{m-1}}}\cdots\right)^{\frac{\alpha_1}{\alpha_2}}\right)^{\frac{1}{\alpha_1}\times p_m^*}$$

$$=\left(\sum_{j_1=1}^{n}\left(\cdots\left(\sum_{j_{m-1}=1}^{n}T(e_{j_1},\dots,e_{j_{m-1}})^{\beta_{m-1}}\right)^{\frac{\beta_{m-2}}{\beta_{m-1}}}\cdots\right)^{\frac{\beta_1}{\beta_2}}\right)^{\frac{1}{\beta_1}}.$$

By the last equality and the induction hypothesis we conclude that

$$\left(\sum_{j_1=1}^{n}\left(\cdots\left(\sum_{j_m=1}^{n}D(e_{j_1},\dots,e_{j_m})^{p_m^*}\right)^{\frac{\lambda_{\mathbf{p}_m,m-1}}{\lambda_{\mathbf{p}_m,m}}}\cdots\right)^{\frac{\lambda_{\mathbf{p}_m,1}}{\lambda_{\mathbf{p}_m,2}}}\right)^{\frac{1}{\lambda_{\mathbf{p}_m,1}}\times p_m^*}$$

$$\leq \sup_{\|x^{(i)}\|_{\ell_{r_i}^n} \leq 1;\ 1 \leq i \leq m-1} \left| \sum_{\widehat{j_m}=1}^{n} T(e_{j_1}, \ldots, e_{j_{m-1}}) x_{j_1}^{(1)} \cdots x_{j_{m-1}}^{(m-1)} \right|$$

$$\leq \|D\|^{p_m^*},$$

where in the last inequality we have used (14). □

6 Theorem 1.4 for *m*-Linear Forms

In this section we prove a multilinear version of Theorem 1.4. We start by proving a multilinear version of Theorem 1.5. The proof presented here is inspired by ideas of Ozikiewicz and Tonge [39].

Theorem 6.2 ([11, Theorem 3.2]) *Let $m \geq 2$, $\sigma: \{1, \ldots, m\} \to \{1, \ldots, m\}$ be a bijection, $\mathbf{q}_m = (q_1, \ldots, q_m) \in (0, \infty)^m$, and $\mathbf{p}_m = (p_1, \ldots, p_m) \in (1, \infty]^m$ be such that $p_{\sigma_m} \in (1, 2]$ and $\left|1/\mathbf{p}_m\right| < 1$. The following assertions are equivalent:*

(i)

$$\left(\sum_{j_{\sigma_1}=1}^{n} \left(\sum_{j_{\sigma_2}=1}^{n} \cdots \left(\sum_{j_{\sigma_m}=1}^{n} \left| T(e_{j_1}, \ldots, e_{j_m}) \right|^{q_m} \right)^{\frac{q_{m-1}}{q_m}} \cdots \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \leq \|T\|$$

for all m-linear forms $T: \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$ and all positive integers n.

(ii) *There is a constant $C_{\mathbf{p}_m, \mathbf{q}_m}$ such that*

$$\left(\sum_{j_{\sigma_1}=1}^{n} \left(\sum_{j_{\sigma_2}=1}^{n} \cdots \left(\sum_{j_{\sigma_m}=1}^{n} \left| T(e_{j_1}, \ldots, e_{j_m}) \right|^{q_m} \right)^{\frac{q_{m-1}}{q_m}} \cdots \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \leq C_{\mathbf{p}_m, \mathbf{q}_m} \|T\|$$

for all m-linear forms $T: \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$ and all positive integers n.

(iii) *The exponents $q_1, \ldots, q_m$ satisfy*

$$q_k \geq \lambda_{\mathbf{t}_m, k},$$

for all $k \in \{1, \ldots, m\}$, with $\mathbf{t}_m = \left(p_{\sigma_1}, \ldots, p_{\sigma_m}\right)$.

Proof To simplify the notation we will consider $\sigma_j = j$ for all j; the other cases are similar. $(i) \Rightarrow (ii)$ is immediate and $(ii) \Rightarrow (iii)$ follows from Lemma 2.2. Let us prove $(iii) \Rightarrow (i)$. Let $T: \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$ be an m-linear form. We define the non-negative m-linear form $A\colon \ell_{s_1}^n \times \cdots \times \ell_{s_m}^n \to \mathbb{K}$ by

$$A(e_{j_1},\ldots,e_{j_m}) = \left|T(e_{j_1},\ldots,e_{j_m})\right|^2,$$

with $s_m = p_m/(2-p_m)$ and $s_i = p_i/2$ for each $i \in \{1,\ldots,m-1\}$.

Note that $s_m^* = (p_m/(2-p_m))^* = p_m^*/2$, and then for all positive integers n, we have

$$\begin{aligned}
&\sup\left\{\left|\sum_{j_1,\ldots,j_m=1}^{n} A(e_{j_1},\ldots,e_{j_m})x_{j_1}^{(1)}\cdots x_{j_m}^{(m)}\right| : \left\|x^{(i)}\right\|_{\ell_{s_i}^n} \leq 1;\ 1\leq i\leq m\right\}\\
&= \sup\left\{\left|\sum_{j_1,\ldots,j_m=1}^{n} \left|T(e_{j_1},\ldots,e_{j_m})\right|^2 x_{j_1}^{(1)}\cdots x_{j_m}^{(m)}\right| : \left\|x^{(i)}\right\|_{\ell_{s_i}^n} \leq 1;\ 1\leq i\leq m\right\}\\
&= \sup_{\substack{x^{(i)}\in B_{\ell_{s_i}^n}\\ 1\leq i\leq m-1}} \left(\sum_{j_m=1}^{n}\left|\sum_{\widehat{j_m}=1}^{n}\left|T(e_{j_1},\ldots,e_{j_m})\right|^2 x_{j_1}^{(1)}\cdots x_{j_{m-1}}^{(m-1)}\right|^{\frac{p_m^*}{2}}\right)^{\frac{2}{p_m^*}}\\
&= \sup_{\substack{x^{(i)}\in B_{\ell_{p_i}^n}\\ 1\leq i\leq m-1}} \left(\sum_{j_m=1}^{n}\left|\sum_{\widehat{j_m}=1}^{n}\left|T(e_{j_1},\ldots,e_{j_m})\right|^2 \left(x_{j_1}^{(1)}\cdots x_{j_{m-1}}^{(m-1)}\right)^2\right|^{\frac{p_m^*}{2}}\right)^{\frac{2}{p_m^*}}\\
&\leq \sup_{\substack{x^{(i)}\in B_{\ell_{p_i}^n}\\ 1\leq i\leq m-1}} \left(\sum_{j_m=1}^{n}\left(\sum_{\widehat{j_m}=1}^{n}\left|T(e_{j_1},\ldots,e_{j_m})x_{j_1}^{(1)}\cdots x_{j_{m-1}}^{(m-1)}\right|^2\right)^{\frac{p_m^*}{2}}\right)^{\frac{2}{p_m^*}}.
\end{aligned}$$

We use the Khinchin inequality for multiple sums (see [24, p. 455]), with constant 1 because $p_m^* \geq 2$, to obtain

$$\begin{aligned}
&\sup_{\substack{x^{(i)}\in B_{\ell_{p_i}^n}\\ 1\leq i\leq m-1}} \left(\sum_{j_m=1}^{n}\left(\sum_{\widehat{j_m}=1}^{n}\left|T(e_{j_1},\ldots,e_{j_m})x_{j_1}^{(1)}\cdots x_{j_{m-1}}^{(m-1)}\right|^2\right)^{\frac{p_m^*}{2}}\right)^{\frac{2}{p_m^*}}\\
&\leq \sup_{\substack{x^{(i)}\in B_{\ell_{p_i}^n}\\ 1\leq i\leq m-1}} \left(\sum_{j_m=1}^{n}\left(\int_{I^{m-1}}\left|\sum_{\widehat{j_m}=1}^{n} T(e_{j_1},\ldots,e_{j_m})\prod_{k=1}^{m-1} r_{j_k}(t_k)\, x_{j_k}^{(k)}\right|^{p_m^*} \boldsymbol{dt}\right)^{\frac{p_m^*}{p_m^*}}\right)^{\frac{2}{p_m^*}}
\end{aligned}$$

$$\leq \sup_{\substack{x^{(i)} \in B_{\ell_{p_i}^n} \\ 1 \leq i \leq m-1}} \left(\int_{I^{m-1}} \left(\sum_{j_m=1}^{n} \left| \sum_{\widehat{j_m}=1}^{n} T(e_{j_1}, \ldots, e_{j_m}) \prod_{k=1}^{m-1} r_{j_k}(t_k) x_{j_k}^{(k)} \right|^{p_m^*} \right)^{\frac{p_m^*}{p_m^*}} \boldsymbol{dt} \right)^{\frac{2}{p_m^*}}$$

$$\leq \sup \left\{ \left(\int_{I^{m-1}} \|T\|^{p_m^*} \boldsymbol{dt} \right)^{\frac{2}{p_m^*}} : \left\| x^{(i)} \right\|_{\ell_{p_i}^n} \leq 1;\ 1 \leq i \leq m-1 \right\}$$

$$= \|T\|^2,$$

where $I = [0, 1]$ and $\boldsymbol{dt} := dt_1 \ldots dt_{m-1}$. Thus

$$\|A\| \leq \|T\|^2 .$$

On the other hand, observe that $\mathbf{s}_m = (s_1, \ldots, s_m) \in (1, \infty]^m$ and

$$\sum_{j=1}^{m} \frac{1}{s_j} = \frac{2}{p_1} + \cdots + \frac{2}{p_{m-1}} + \frac{2 - p_m}{p_m} = 2 \left(\sum_{j=1}^{m} \frac{1}{p_j} \right) - 1 < 1,$$

and

$$2 \left(\frac{1}{\lambda_{\mathbf{p}_m, i}} \right) = 2 \left(1 - \sum_{j=i}^{m} \frac{1}{p_j} \right) = 1 - \left(\frac{2}{p_i} + \cdots + \frac{2}{p_{m-1}} + \frac{2 - p_m}{p_m} \right) = \frac{1}{\lambda_{\mathbf{s}_m, i}}$$

for each $i \in \{1, \ldots, m\}$. Combining these facts with Theorem 5.1, we obtain

$$\left(\sum_{j_1=1}^{n} \left(\cdots \left(\sum_{j_m=1}^{n} \left| T(e_{j_1}, \ldots, e_{j_m}) \right|^{p_m^*} \right)^{\frac{\lambda_{\mathbf{p}_m, m-1}}{p_m^*}} \cdots \right)^{\frac{\lambda_{\mathbf{p}_m, 1}}{\lambda_{\mathbf{p}_m, 2}}} \right)^{\frac{2}{\lambda_{\mathbf{p}_m, 1}}}$$

$$= \left(\sum_{j_1=1}^{n} \left(\cdots \left(\sum_{j_m=1}^{n} \left| T(e_{j_1}, \ldots, e_{j_m}) \right|^{2 \times \frac{p_m^*}{2}} \right)^{\frac{2 \times \lambda_{\mathbf{p}_m, m-1}}{2 \times p_m^*}} \cdots \right)^{\frac{2 \times \lambda_{\mathbf{p}_m, 1}}{2 \times \lambda_{\mathbf{p}_m, 2}}} \right)^{\frac{2}{\lambda_{\mathbf{p}_m, 1}}}$$

$$= \left(\sum_{j_1=1}^{n} \left(\cdots \left(\sum_{j_m=1}^{n} A(e_{j_1}, \ldots, e_{j_m})^{\lambda_{\mathbf{s}_m,m}} \right)^{\frac{\lambda_{\mathbf{s}_m,m-1}}{\lambda_{\mathbf{s}_m,m}}} \cdots \right)^{\frac{\lambda_{\mathbf{s}_m,1}}{\lambda_{\mathbf{s}_m,2}}} \right)^{\frac{1}{\lambda_{\mathbf{s}_m,1}}}$$

$$\leq \|A\| \leq \|T\|^2 .$$

□

Since

$$\lambda_{\mathbf{t}_m,1} \geq \lambda_{\mathbf{t}_m,2} \geq \cdots \geq \lambda_{\mathbf{t}_m,m-1} \geq \lambda_{\mathbf{t}_m,m},$$

and

$$\lambda_{\mathbf{t}_m,1} = \lambda_{\mathbf{p}_m},$$

then, the extension of Theorem 1.4, with optimal exponent and optimal constant, follows immediately from the previous theorem:

Theorem 6.2 ([11, Theorem 3.2] and [29, Proposition 4.1]) *Let* $m \geq 2$, $\mathbf{p}_m = (p_1, \ldots, p_m) \in (1, \infty]^m$ *be such that* $p_i \in (1, 2]$ *for some* i *and* $\left|1/\mathbf{p}_m\right| < 1$. *Then*

$$\left(\sum_{j_1,\ldots,j_m=1}^{n} \left| T(e_{j_1}, \ldots, e_{j_m}) \right|^{\lambda_{\mathbf{p}_m}} \right)^{\frac{1}{\lambda_{\mathbf{p}_m}}} \leq \|T\| \tag{15}$$

for all m*-linear forms* $T : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$ *and all positive integers* n. *Moreover, the exponent* $\lambda_{\mathbf{p}_m}$ *is optimal.*

The optimal exponent $\lambda_{\mathbf{p}_m}$ in the above result was obtained in [29] and the optimal constant 1 was obtained in [11].

7 The Critical and Supercritical Cases

The paper of Hardy and Littlewood does not investigate summability of the coefficients of bilinear forms $T : \ell_{p_1}^n \times \ell_{p_2}^n \to \mathbb{K}$ when $1/p_1 + 1/p_2 \geq 1$. In fact, in this case it is simple to verify that there is no finite exponent q so that there is a constant C satisfying

$$\left(\sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \left| T\left(e_{j_1}, e_{j_2}\right) \right|^{q} \right)^{\frac{1}{q}} \leq C \|T\|$$

for all bilinear forms $T: \ell_{p_1}^n \times \ell_{p_2}^n \to \mathbb{K}$ and all positive integers n. This observation seems to trivialize the problem since it is obvious that

$$\sup_{j_1, j_2 \in \mathbb{N}} \left| T\left(e_{j_1}, e_{j_2}\right) \right| \leq \|T\|.$$

Until very recently the investigation of the m-linear case has followed this vein and ignored the case

$$1/p_1 + \cdots + 1/p_m \geq 1. \tag{16}$$

However, if we consider the problem from a broader perspective, we observe that the case (16) hides subtleties. In fact, under an anisotropic viewpoint (allowing different exponents for different indexes) the problem is no longer trivial since a Hardy–Littlewood type inequality

$$\left(\sum_{i_1=1}^{n} \left(\cdots \left(\sum_{i_m=1}^{n} \left| T\left(e_{i_1}, \ldots, e_{i_m}\right) \right|^{q_m} \right)^{\frac{q_{m-1}}{q_m}} \cdots \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \leq C \|T\|$$

for m-linear forms $T: \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$ and $1/p_1 + \cdots + 1/p_m \geq 1$ implies $q_1 = \infty$, but $q_2, \ldots, q_{m-1}$ may be finite. As far as we know the first work in this framework was [40], where the following result was proved for m-linear forms with $p_1 = \cdots = p_m = m$ (this case was called critical case):

Theorem 7.1 ([40, Theorem 1]) *Let $m \geq 2$. There is a constant C_m such that*

$$\sup_{j_1} \left(\sum_{j_2=1}^{n} \left(\cdots \left(\sum_{j_m=1}^{n} \left| T\left(e_{j_1}, \ldots, e_{j_m}\right) \right|^{s_m} \right)^{\frac{1}{s_m} s_{m-1}} \cdots \right)^{\frac{1}{s_3} s_2} \right)^{\frac{1}{s_2}} \leq C_m \|T\| \tag{17}$$

for all m-linear forms $T: \ell_m^n \times \cdots \times \ell_m^n \to \mathbb{K}$, and all positive integers n, with

$$s_k = \frac{2m(m-1)}{mk - 2k + 2}$$

for all $k = 2, \ldots, m$. Moreover, $s_1 = \infty$ and $s_2 = m$ are sharp and, for $m > 2$ the optimal exponents s_k satisfying (17) fulfill

$$s_k \geq \frac{m}{k-1}.$$

Very recently the configuration $1/p_1 + \cdots + 1/p_m > 1$ was investigated in [36–38] but there are still several open questions related to the subject.

8 On the Constants and Some Final Remarks

There are still several open problems related to the Hardy–Littlewood inequalities for m-linear forms. The first class of open problems are related to the optimal exponents. As we can easily observe, there are some cases not covered by the previous theorems and the optimal exponents of these cases are not known, in general. At least for bilinear forms and $p_1, p_2 \in [2, \infty]$, the optimal exponents are known for the non-critical cases:

Theorem 8.1 ([44, Theorem 5.1]) *Let $p_1, p_2 \in [2,\infty]$ with $1/p_1+1/p_2 < 1$, and $q_1, q_2 > 0$. For $\{k_1, k_2\} = \{1, 2\}$ the following assertions are equivalent:*

(i) *There is a constant C such that*

$$\left(\sum_{j_{k_1}=1}^{n} \left(\sum_{j_{k_2}=1}^{n} \left| T(e_{j_1}, e_{j_2}) \right|^{q_2} \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \leq C \left\| T \right\|,$$

for all bilinear forms $T : \ell_{p_1}^n \times \ell_{p_2}^n \to \mathbb{K}$ and all positive integers n.

(ii) *The exponents q_1, q_2 satisfy $(q_1, q_2) \in [\lambda, \infty) \times [p_{k_2}^*, \infty)$ and*

$$\frac{1}{q_1} + \frac{1}{q_2} \leq \frac{3}{2} - \left(\frac{1}{p_1} + \frac{1}{p_2} \right).$$

The optimal constants satisfying the Hardy–Littlewood inequalities are, in general, unknown. In this section we try to summarize what is known so far about the constants and the challenges ahead. The first important fact to note is that the optimal constants depend on the scalar field. It seems that the case $p_1 = \cdots = p_m = \infty$ with $\mathbb{K} = \mathbb{R}$ is the less difficult to deal with. For instance, for real scalars and $(m, p_1, p_2) = (2, \infty, \infty)$, the inequality (3) with $C_{p_1,p_2} = \sqrt{2}$ is a somewhat straightforward consequence of the Khinchin inequality. In this case, observing that the bilinear form $T : \ell_\infty^2 \times \ell_\infty^2 \to \mathbb{R}$ defined by

$$T(x, y) = x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2$$

has norm 2, we conclude that the constant $\sqrt{2}$ is sharp (see [30]). A similar argument shows that (for real scalars) $\sqrt{2}$ is also the optimal constant for (9) with $(m, p_1, p_2) = (2, \infty, \infty)$, regardless of the $q_1, q_2 \in [1, 2]$. The case $m > 2$, with $p_1 = \cdots = p_m = \infty$ and real scalars, with exponents $(q_1, \ldots, q_m) \in \{(1, 2, \ldots, 2), \ldots, (2, 2, \ldots, 2, 1)\}$ was solved in [41, 43] and the optimal constants for a more general family of exponents were obtained in [22]. The best known estimates for case for $m > 2$, with $\mathbf{p}_m = (\infty, \ldots, \infty)$ and $\mathbf{q}_m = \left(\frac{2m}{m+1}, \ldots, \frac{2m}{m+1}\right)$ can be found in [15] (upper bounds) and [30] (lower bounds):

$$\begin{cases} 2^{1-\frac{1}{m}} \leq C_{\mathbf{p}_m,\mathbf{q}_m} \leq 2^{\frac{446381}{55440}} \prod_{j=14}^{m} \left(\frac{\Gamma\left(\frac{3}{2}-\frac{1}{j}\right)}{\sqrt{\pi}} \right)^{\frac{j}{2-2j}} & \text{for } m \geq 14 \text{ and real scalars,} \\ 2^{1-\frac{1}{m}} \leq C_{\mathbf{p}_m,\mathbf{q}_m} \leq \prod_{j=2}^{m} 2^{\frac{1}{2j-2}} \text{ for } 2 \leq m < 14 \text{ and real scalars,} \\ 1 \leq C_{\mathbf{p}_m,\mathbf{q}_m} \leq \prod_{j=2}^{m} \Gamma\left(2-\frac{1}{j}\right)^{\frac{j}{2-2j}} & \text{for } m \geq 2 \text{ and complex scalars.} \end{cases}$$

The precise asymptotic growth of the upper bounds provided above is also calculated in [15]:

$$\begin{cases} C_{\mathbf{p}_m,\mathbf{q}_m} \leq \kappa m^{\frac{1-\gamma}{2}} \leq \kappa m^{0.212} \text{ for complex scalars,} \\ C_{\mathbf{p}_m,\mathbf{q}_m} \leq \kappa m^{\frac{2-\log 2-\gamma}{2}} \leq \kappa m^{0.365} \text{ for real scalars,} \end{cases}$$

where κ is a positive constant and γ is the Euler–Mascheroni constant. It is conjectured in [43] (universality conjecture) that the sharp constants $C_{\mathbf{p}_m,\mathbf{q}_m}$ for real scalars and $\mathbf{p}_m = (\infty, \ldots, \infty)$ and $\mathbf{q}_m = \left(\frac{2m}{m+1}, \ldots, \frac{2m}{m+1}\right)$ are $2^{1-\frac{1}{m}}$. Having good estimates for these constants play crucial role in applications as it can be seen in the papers [12, 35]. In [6] it is shown that replacing the sum

$$\left(\sum_{i_1,\ldots,i_m=1}^{n} \left| T\left(e_{i_1}, \ldots, e_{i_m}\right) \right|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}}$$

by

$$\left(\sum_{i_1,\ldots,i_k=1}^{n} \left| T\left(e_{i_1}^{n_1}, \ldots, e_{i_k}^{n_k}\right) \right|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}},$$

in (9), with $\mathbf{p}_m = (\infty, \ldots, \infty)$, where $e_i^n := \left(e_i, \overset{n \text{ times}}{\ldots}, e_i\right)$, then the new constant is contractive (i.e., tends to 1 as $m \to \infty$) when

$$\lim_{m \to \infty} \frac{k \log k}{m} = 0.$$

This happens, for instance if

$$k = \left\lfloor \frac{m}{(\log m)^{1+\frac{1}{\log\log\log m}}} \right\rfloor \text{ or } k = \left\lfloor m^{1-\frac{1}{\log\log m}} \right\rfloor.$$

It is important to mention that the case $m > 2$, with $\mathbf{p}_m = (\infty, \ldots, \infty)$ for real scalars was formally solved in [23] by means of an optimization technique that provides an algorithm furnishing all extreme points of the closed unit ball of the space of m-linear forms defined on $\ell_\infty^n \times \cdots \times \ell_\infty^n$. Although the result of [23] formally gives the optimal constants by means of a finite number of elementary operations, the effective calculation of these constants can only be done under strong computational assistance. An implementation of the algorithm can be found in [46]; the results obtained in [46] seem to bring evidence reinforcing the universality conjecture. An effective calculation of the constants provided by the algorithm of [23] still depend on better computational machinery.

When at least a p_j is not infinity, most of the constants of the Hardy–Littlewood inequalities are unknown. The most relevant exceptions are the constants of Theorems 5.1 and 6.2; in these cases the proof shows that the optimal constants are 1. The best known estimates for the constants of Theorem 3.2 can be found in [10] and the best known constants for Theorem 4.1 can be found in [9].

The Hardy–Littlewood inequalities for m-linear forms have a natural analogue for homogeneous polynomials and the constants for the polynomial case are also subject of investigation in the last decade, specially for ℓ_∞ spaces (in this case the Hardy–Littlewood inequalities are called Bohnenblust–Hille inequalities), with important applications in Complex Analysis (see [25, 26]). In [25] it was proved that (for m-homogeneous polynomials in ℓ_∞ and complex scalars) the constants D_m of the Bohnenblust–Hille inequality were dominated by C^m for a certain constant C, while the estimates of the original result of Bohnenblust and Hille just provided

$$D_m \leq \left(\sqrt{2}\right)^{m-1} \frac{m^{m/2} (m+1)^{(m+1)/2}}{2^m (m!)^{(m+1)/2}}. \tag{18}$$

The estimate presented in [25] has important applications in Analytic Number Theory and Complex Analysis. In [15] it was shown that the constant D_m has sub-exponential growth and, as a straightforward consequence, the authors succeeded in obtaining the exact asymptotic behavior of the n-dimensional Bohr radius K_n:

$$\lim_{n\to\infty} \frac{K_n}{\sqrt{(\log n)/n}} = 1.$$

The constants of the polynomial Hardy–Littlewood inequalities for real scalars were investigated in [20]. In particular, in [20] it was shown that for the case of real scalars the constants are no longer sub-exponential, and behave differently than in the complex case; applications can be found in [27].

Recently, a very interesting approach, related to the notion of fractional dimensions, was introduced by F. Bayart [14], investigating Hardy–Littlewood inequalities when the sums from $i_1, \dots, i_n = 1$ to n (i.e., sums in $\mathbb{N}^n$) are replaced by sums in arbitrary indexes, i.e., $i_1, \dots, i_n \in \Gamma$ for an arbitrary set $\Gamma \in \mathbb{N}^n$ (see also [8]).

9 The Gale-Berlekamp Switching Game

Designed independently by Elwyn Berlekamp and David Gale in the 1960's, the Gale–Berlekamp switching game consists of an $n \times n$ square matrix of light bulbs set-up at an initial light configuration. The goal is to turn off as many lights as possible using n row and n column switches, which invert the state of each bulb in the corresponding row or column.

For an initial pattern of lights Θ, let $i(\Theta)$ denote the smallest final number of on-lights achievable by row and column switches starting from Θ. We define

$$R_n := \max\{i(\Theta) : \Theta \text{ is an } n \times n \text{ light pattern}\},$$

which represents the smallest possible number of remaining on-lights, starting from "the worst" initial pattern. Sometimes the problem is posed as to find the maximum of the difference between the number of lights that are on and the number that are off, often denoted by G_n. Since

$$R_n = \frac{1}{2}\left(n^2 - G_n\right),$$

both formulations are equivalent.

Originally, the problem introduced by Berlekamp asks for the exact value of R_{10}. In 2004, Carlson and Stolarski proved that $R_{10} = 35$ (see [21]). Up to now, the exact value of R_n is known only for small values of n due to involving combinatorial arguments. The exact values of R_n for n up to 12 were obtained by Carlson and Stolarski [21] (see also [19]).

In this section we show how the Hardy–Littlewood cycle of ideas is associated to the Gale–Berlekamp switching game and how it is useful to provide asymptotic bounds for G_n. We initially notice that by associating $+1$ to the on-lights and -1 to the off-lights from the array of lights $\left(a_{ij}\right)_{i,j=1}^{n}$, we have

$$G_n = \min\left\{ \max_{x_i, y_j \in \{-1,1\}} \left| \sum_{i,j=1}^{n} a_{ij} x_i y_j \right| : a_{ij} = -1 \text{ or } +1 \right\}.$$

Now, we shall observe that G_n is precisely the norm of the bilinear form $A: \ell_\infty^n \times \ell_\infty^n \to \mathbb{R}$ defined by

$$A(x, y) = \sum_{i,j=1}^{n} a_{ij} x_i y_j. \tag{19}$$

In fact, it is obvious that $G_n \leq \|A\|$. In order to prove the converse inequality, we first recall that if E is a vector space and $A \subset E$, a vector $a \in A$ is an extreme point of A if

$$y, z \in A \text{ with } x = \frac{1}{2}(y+z) \Rightarrow x = y = z.$$

It is well-known that the extreme points of the closed unit ball of ℓ_∞^n are precisely $(x_j)_{j=1}^n$ such that $|x_j| = 1$ for all $j = 1, \ldots, n$. Finally, we recall that for all finite-dimensional Banach spaces E we have

$$\|T\| = \max\{|T(x, y)| : x, y \in \operatorname{ext}(B_E)\}$$

for all bilinear forms $T: E \times E \to \mathbb{R}$, where $\operatorname{ext}(B_E)$ represents the extreme points of the closed unit ball B_E of E. In fact, since B_E is compact, there exist $x, y \in B_E$ such that $|T(x, y)| = \|T\|$. By the Minkowski Theorem, we know that B_E is the convex hull of $\operatorname{ext}(B_E)$. Hence, there exist $\lambda_i^{(1)}, \lambda_j^{(2)} \in [0, 1]$ with $\sum_{i=1}^{k_1} \lambda_i^{(1)} = \sum_{j=1}^{k_2} \lambda_j^{(2)} = 1$ and $z_i^{(1)}, z_j^{(2)} \in \operatorname{ext}(B_E)$ such that

$$x = \sum_{i=1}^{k_1} \lambda_i^{(1)} z_i^{(1)} \quad \text{and} \quad y = \sum_{j=1}^{k_2} \lambda_j^{(2)} z_j^{(2)}.$$

Therefore

$$\begin{aligned} \|T\| = |T(x, y)| &= \left| T\left(\sum_{i=1}^{k_1} \lambda_i^{(1)} z_i^{(1)}, \sum_{j=1}^{k_2} \lambda_j^{(2)} z_j^{(2)} \right) \right| \\ &\leq \sum_{i=1}^{k_1} \sum_{j=1}^{k_2} \lambda_i^{(1)} \lambda_j^{(2)} \left| T\left(z_i^{(1)}, z_j^{(2)} \right) \right| \end{aligned}$$

and we conclude that there exist $z_{i_0}^{(1)}, z_{j_0}^{(2)} \in \operatorname{ext} B_E$ such that

$$\left| T\left(z_{i_0}^{(1)}, z_{j_0}^{(2)} \right) \right| = \|T\|.$$

The above results yield that $G_n = \|A\|$ for A as in (19). Now, as a straightforward consequence of the Hardy–Littlewood inequalities for $m = 2$ and $p_1 = p_2 = \infty$, we have

$$G_n \geq 2^{-1/2} n^{3/2}.$$

On the other hand, the Kahane–Salem–Zygmund inequality (see [16]) tells us that $G_n \leq \left(2\sqrt{5 \log 9}\right) n^{3/2}$ and we conclude that

$$\frac{1}{\sqrt{2}} \leq \frac{G_n}{n^{3/2}} \leq 2\sqrt{5 \log 9}.$$

Asymptotically, probabilistic techniques and an approximation scheme using Hadamard matrices provide better lower and upper bounds, respectively (see [7, 42]):

$$0.797 + o(1) \leq \frac{G_n}{n^{3/2}} \leq 1 + o(1).$$

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Symmetries of C^*-algebras and Jordan Morphisms

Jan Hamhalter and Ekaterina Turilova

Abstract They are many faces of C^*-algebras whose symmetries encode important aspects of their structures. We show that in surprisingly different situations these symmetries are implemented by Jordan *-isomorphisms and lead to full Jordan invariants. In this respect we study the following structures: 1. One dimensional projections in a Hilbert space with transition probability and orthogonality relation (Wigner type theorems). 2. Projection lattices of von Neumann algebras and AW*-algebras (Dye type theorems) 3. Abelian C*-subalgebras with set theoretic inclusion (Bohrification program in quantum theory) 4. Measures on state spaces endowed with the Choquet order.

Keywords C*-algebras · Jordan *-morphisms

1 Introduction

The aim of this chapter is to review selected results illustrating interrelations between symmetries of various structures attached to C^*-algebras and von Neumann algebras and Jordan *-isomorphisms. The full C^*-structure is captured by *-isomorphism. If we have two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, then *-isomorphism $\psi : \mathcal{A} \to \mathcal{B}$ is a linear bijective map satisfying the following conditions for all $a, b \in \mathcal{A}$.

$$\psi(ab) = \psi(a)\psi(b)$$

$$\psi(a^*) = \psi(a)^* .$$

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It turns out that many features of operator algebras relevant to their inner structure and applications in physics are preserved by Jordan *-isomorphism rather than *-isomorphisms. A Jordan *-isomorphism is a bijective linear map $\varphi : \mathcal{A} \to \mathcal{B}$ between C^*-algebras satisfying the following conditions for all $a \in \mathcal{A}$.

$$\varphi(a^2) = \varphi(a)^2$$

$$\varphi(a^*) = \varphi(a)^* .$$

As we can see Jordan morphisms are well behaved only with respect to individual elements and involve *-morphisms as a special case. They are ubiquitous in preserver theory of operator algebras. Even, the very Banach space structure of a C^*-algebra is in fact complete Jordan invariant. Indeed, famous Kadison isometry theorem and its ramifications say that if two C^*-algebras are isometric as Banach spaces then there is a Jordan *-isomorphism between them. Moreover, unital surjective linear isometries between C^*-algebras are precisely Jordan *-isomorphisms. We would like to review some aspects of this surprising universality of Jordan morphisms by reviewing some recent results the authors witnessed and have pleasure to deal with during last decade. We start at elementary level in the first section and then move to more advanced aspects of symmetries in operator algebras. The second section is devoted to famous Wigner theorem [18]. Based on our results in [1, 16], we provide elementary proof on symmetries on Hilbert spaces that preserve transition probability between one dimensional projections (rays). These symmetries are given by Jordan *-isomorphisms that are in this case implemented by unitary or antiunitary map. The core of our approach consists in elementary matrix algebra arguments proving Kadison's result to the effect that Jordan *-isomorphism on a matrix algebra is either *-isomorphism or *-antiisomorphism. We also show that quantum logic version of Wigner theorem due to Uhlhorn can be quickly derived from Gleason theorem. In the second section we analyze Dye theorem that is a generalization of Wigner theorem to a much general context of von Neumann algebras. In Jordan formulation Dye theorem says that any bijection between (almost all) projection lattices of von Neumann algebras preserving orthogonality in both directions extends to a Jordan *-isomorphism between algebras themselves. In other words, the projection lattice is a full Jordan invariant. We show that this deep result can be quickly derived from Generalized Gleason theorem due to Bunce and Wright. In the second part of Sect. 3 we outline the proof of Dye's theorem for AW*-algebras [7]. These algebras are algebraic generalizations of von Neumann algebras due to Kaplansky. They are more natural from the point of view of quantum theory. Also they seem to be useful from purely mathematical point of view. For example, there is one-one correspondence between complete Boolean algebras and projection lattices of abelian AW*-algebras. The technique of proving Dye theorem for AW*-algebras must be different in combining insights of von Neumann, Dye, and Heunen and Reyes [12] on one side and Gleason theorem for Type I finite AW*-algebras due to J. Hamhalter [7] on the other side. In Sect. 4 we consider yet another order structure associated with C^*-algebras. It is the poset of abelian subalgebras order by set theoretic inclusion. This structure has driven attention of many mathematical physicists in connection with Bohrification program in quantum theory [14]. In this approach quantum system can only be seen through its classical subsystems. In operator algebraic model of quantum theory quantum system is given by a C^*-algebra $\mathcal{A}$ and its classical subsystems are given by abelian C^*-subalgebras of $\mathcal{A}$. Therefore the poset $\mathcal{C}(\mathcal{A})$ of abelian C^*-subalgebras of $\mathcal{A}$ embodies the Bohr's doctrine. We show that $\mathcal{C}(\mathcal{A})$ is a complete Jordan invariant for many algebras and synthesize various results along this line. Finally, in concluding Sect. 5 we review recent results discovering new complete Jordan invariant—the Choquet order structure on orthogonal measures on state spaces of C^*-algebras. We show intimate connection of this structure with $\mathcal{C}(\mathcal{A})$ and describe preservers of the Choquet order in terms of Jordan *-isomorphisms.

We shall now recall basic notions and introduce the notation. For all details on operator algebras and their applications to physics the reader is advised to consult monographs [2, 13, 14, 17]. Let $\mathcal{A}$ be a C^*-algebra. If it has the unit it is called unital. We shall denote by $\mathcal{A}_s$, and $\mathcal{A}^+$ the set of all self-adjoint and positive elements, respectively. By the symbol $Z(\mathcal{A})$ we shall represent the center of $\mathcal{A}$. It is the set of all elements commuting with every element of $\mathcal{A}$. By the symbol $P(\mathcal{A})$ we shall denote the poset of projections in $\mathcal{A}$, that is the set of self-adjoint idempotents ordered by relation: $p \leq q$ if $p = pq = qp$. If the algebra $\mathcal{A}$ is unital with unit 1, then orthocomplement of projection p is the projection $1 - p$. Two projections p, q are called orthogonal, in symbols $p \perp q$, if $pq = 0$. The bijection between projections posets is called orthoisomorphism if it preserves the orthogonality in both directions. The central cover $c(p)$ of projection p is the smallest projection $c(p)$ that lies in the center and is bigger than p. Projection is called faithful if its central cover is 1. Finally, projection p is called abelian if $p\mathcal{A}p$ is an abelian C^*-algebra. Two projections p and q are said to be equivalent if $p = v^*v$ and $q = vv^*$ for some element $v \in \mathcal{A}$. In case of a unital C^*-algebra, the unitary element is an element u whose inverse is u^*. By the symbol $B(H)$ we shall denote the C^*-algebra of all bounded operators acting on a Hilbert space H. If $S \subset B(H)$, then S' will represent the commutant of S, that is the set of all operators commuting with all elements in S. A vector $\xi \in H$ is called separating for $S \subset B(H)$ if $a\xi = 0$ implies $a = 0$ for all $a \in S$. Moreover, ξ is called biseparating for S if it is separating both for S and S'. Von Neumann algebra is a C^*-algebra that has a predual. AW*-algebra $\mathcal{A}$ is a C^*-algebra that is Baer* ring, which means that for any nonempty set $S \subset \mathcal{A}$ there is unique projection $p \in \mathcal{A}$ such that $S^0 = \{x \in \mathcal{A} : sx = 0 \text{ for all } s \in S\} = p\mathcal{A}$.

Let us have two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$. A linear map $\varphi : \mathcal{A} \to \mathcal{B}$ that preserves the product and *-operation is called *-homomorphism. If it is a bijection we call it *-isomorphism. A linear map $\varphi : \mathcal{A} \to \mathcal{B}$ that reverses the product (that is $\varphi(ab) = \varphi(b)\varphi(a)$ for all $a, b \in \mathcal{A}$) and preserves *-operation is called *-antihomomorphism. If it is a bijection we call it *-antiisomorphism. A linear map $\varphi : \mathcal{A} \to \mathcal{B}$ that preserves the squares ($\varphi(a^2) = \varphi(a)^2$ for all $a \in \mathcal{A}$) and *-operation is called a Jordan *-homomorphism. If it is a bijection we call it a

Jordan *-isomorphism. We shall now collect a few folklore results about Jordan *-morphisms that will be used frequently in this work.

Proposition 1.1 *Let us have C^*-algebras $\mathcal{A}$ and $\mathcal{B}$. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a Jordan *-homomorphism. Then the following holds:*

(i) *φ preserves the triple products:*

$$\varphi(abc + cba) = \varphi(a)\varphi(b)\varphi(c) + \varphi(c)\varphi(b)\varphi(a) \textit{ for all } a, b, c \in \mathcal{A}.$$

(ii) *φ preserves commutativity.*

(iii) *If φ is a *-isomorphism between unital algebras then it preserves the unit.*

Moreover, the following well known fact will facilitate further discussion.

Proposition 1.2 *Let $\mathcal{A}$ be a C^*-algebra that is the closed linear span of its projections. Let $\mathcal{B}$ be another C^*-algebra. Then a bounded linear map $\psi : \mathcal{A} \to \mathcal{B}$ is a Jordan *-homomorphism if and only if it preserves projections.*

By a positive functional φ on a C^*-algebra $\mathcal{A}$ we mean the linear form that takes nonnegative values on $\mathcal{A}^+$. It is always a continuous map. It is called faithful if $\varphi(a^*a) = 0$ implies $a = 0$. We write $\varphi \leq \psi$ for positive functionals φ and ψ if $\psi - \varphi$ is positive. A state is a norm one positive functional. By the symbol $S(\mathcal{A})$ we shall denote the set of all states of $\mathcal{A}$. Extreme points of this set are called pure states. Two positive functionals ψ and φ are called orthogonal if $\|\varphi - \psi\| = \|\varphi\| + \|\psi\|$. For each state φ there is a Hilbert space H_φ, unit vector $\xi_\varphi \in H_\varphi$ and a *-homomorphism $\pi_\varphi : \mathcal{A} \to B(H_\varphi)$ such that the set $\pi_\varphi(\mathcal{A})\xi_\varphi$ is dense in H_φ and $\varphi(a) = \langle \pi_\varphi(a)\xi_\varphi, \xi_\varphi \rangle$ for all $a \in \mathcal{A}$. This is called Gelfand-Nairmark-Segal representation (GNS in short).

Having two posets $(P, \leq)$ and $(Q, \leq)$, we shall call a bijection $\varphi : P \to Q$ an order isomorphism if it preserves order in both directions: $a \leq b$ if and only if $\varphi(a) \leq \varphi(b)$.

After reviewing basic facts we recall Generalized Gleason theorem that plays an important role throughout the present treatment. Let $\mathcal{A}$ be a C^*-algebra and X a Banach space. By a finitely additive X-valued measure on $P(\mathcal{A})$ we mean a map $\mu : P(\mathcal{A}) \to X$ that satisfies: $\mu(p + q) = \mu(p) + \mu(q)$ whenever p and q are orthogonal projections. If X is $\mathbb{C}$ we are just talking about measure on $P(\mathcal{A})$. Gleason type theorems is a series of deep results that originated by Gleason theorem in 1957 and culminated in remarkable achievements of Bunce and Wright in the early 90s. For self-contained proof and history we refer the interested reader to [5] and references therein. Originally the following theorem has been proved by Gleason for completely additive probability measures.

Theorem 1.3 (Gleason) *Let μ be a finitely additive bounded measure on $P(B(H))$, where H is a Hilbert space of dimension not equal two. Then there is a bounded linear functional $f : B(H) \to \mathbb{C}$ extending μ, i.e.*

$$\mu(P) = f(P) \qquad \textit{for all } P \in P(B(H)).$$

The previous theorem has nontrivial extension to nearly all von Neumann algebras. Let us recall that von Neumann algebra has Type I_2 direct summand if it has a direct summand *-isomorphic to the algebra $C(X, M_2(\mathbb{C}))$ of all continuous functions on a hyperstonean space X with values in the algebra of two by two matrices.

Theorem 1.4 (Gleason-Bunce-Wright) *Let $\mathcal{M}$ be a von Neumann algebra not having Type I_2 direct summand and X be a Banach space. Then any bounded finitely additive measure $\mu : P(\mathcal{M}) \to X$ extends to a bounded linear functional $T : \mathcal{M} \to X$.*

We shall need the following n generalization of Jordan *-homomorphisms.

Definition 1.5 Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. A map $\psi : \mathcal{A} \to \mathcal{B}$ is called a quasi Jordan *-homomorphisms if the following conditions are satisfied for all $a, b \in \mathcal{A}$.

(i) ψ is homogeneous
(ii) $\psi(a + b) = \psi(a) + \psi(b)$ whenever a and b commute.
(iii) $\psi(a^*) = \psi(a)^*$.
(iv) $\psi(a^2) = \psi(a)^2$.

When ψ is a bijection such that both ψ and ψ^{-1} are quasi Jordan *-homomorphisms, we call ψ quasi Jordan *-isomorphism.

It is a consequence of Theorem 1.4 that any quasi Jordan *-homomorphism defined on a von Neumann algebra without Type I_2 direct summand is linear. This does not hold for Type I_2 von Neumann algebras (see [5]) for a more detailed discussion).

2 Wigner Theorem

2.1 Probability Version

Let H be a Hilbert space. By $P(H)$ we shall represent the set of all projections acting on H. Let $P_1(H)$ mean the set of all rank one projections in $P(H)$. Each projection in $P_1(H)$ is of the form P_ξ, where P_ξ is an orthogonal projection onto linear span of nonzero vector $\xi \in H$. By $F(H)$ we shall denote the set of finite rank operators acting on H. By the trace we understand the standard trace on $F(H)$, that is

$$\mathrm{tr}(A) = \sum_\alpha \langle T e_\alpha, e_\alpha \rangle ,$$

where (e_α) is an orthonormal basis of H. In probability structure of quantum mechanics projection P_ξ, $\|\xi\| = 1$, corresponds to the state of the system. Transition probability between two states P_ξ, P_μ is then given by

$$T(P_\xi, P_\mu) = \mathrm{tr}(P_\xi P_\mu) = |\langle \xi, \mu \rangle|^2 .$$

Wigner Theorem in its original form is concerned with symmetries preserving transition probability. It is not difficult to show that any map preserving transition probability is a restriction of a Jordan *-homomorphism.

Proposition 2.1 *Let H be a separable Hilbert space. Let $\varphi : P_1(H) \to P_1(H)$ be a map preserving transition probabilities, that is, φ satisfies*

$$\mathrm{tr}(PQ) = \mathrm{tr}(\varphi(P)\varphi(Q)) \qquad \textit{for all } P, Q \in P_1(H) .$$

Then φ extends to a Jordan $*$-homomorphism $\hat{\varphi} : F(H) \to F(H)$.

Proof We shall first extend φ to linear map on the real space $F_s(H)$ of self-adjoint finite rank operators. Given A in $F_s(H)$ we can suppose that

$$A = \sum_{i=1}^{n} \lambda_i P_i \qquad \lambda_i \in \mathbb{R},\ P_i \in P_1(H) . \tag{1}$$

Put

$$\hat{\varphi}(A) = \sum_{i=1}^{n} \lambda_i \varphi(P_i) .$$

The key argument of the proof is to show that the this definition is correct, that is, not depending on the expression (1). To this end let us take another decomposition

$$A = \sum_{j=1}^{m} \mu_j Q_j \qquad \mu_i \in \mathbb{R},\ Q_i \in P_1(H) .$$

Taking arbitrary $R \in P_1(H)$, we can compute

$$\mathrm{tr}\left(\sum_{i=1}^{n} \lambda_i \varphi(P_i)\varphi(R)\right) = \sum_{i=1}^{n} \lambda_i \mathrm{tr}(\varphi(P_i)\varphi(R)) = \tag{2}$$

$$= \sum_{i=1}^{n} \lambda_i \mathrm{tr}(P_i R) = \mathrm{tr}(AR) . \tag{3}$$

We shall use the following notation

$$T = \sum_{i=1}^{n} \lambda_i \varphi(P_i) \qquad S = \sum_{j=1}^{m} \mu_j \varphi(Q_j) .$$

From (2) we have, for each one dimensional projection R,

$$\mathrm{tr}\Big((S-T)\varphi(R)\Big) = 0\,.$$

By linearity $\mathrm{tr}\Big((S-T)B\Big) = 0$ for all B in the linear span of $\varphi(P_1(H))$. In particular, $\mathrm{tr}(S-T)^2 = 0$, giving $(S-T)^2 = 0$. Finally, by self-adjointness, $S-T=0$.

The map $\hat{\varphi}$ is obviously linear. Let us check that it preserves the squares. For this, each $A \in F_s(H)$ can be written in its spectral form as

$$A = \sum_{i=1}^{n} \lambda_i P_i\,,$$

where $\lambda_i \in \mathbb{R}$ and P_i are pairwise orthogonal one dimensional projections. Easy computation gives

$$\hat{\varphi}(A^2) = \hat{\varphi}\Big(\sum_{i=1}^{n} \lambda_i^2 P_i\Big) = \sum_{i=1}^{n} \lambda_i^2 \varphi(P_i) = \hat{\varphi}(A)^2\,.$$

Finally, φ can be canonically extended to the space $F(H)$ and it completes the proof. □

Now we shall demonstrate how one can express Jordan *-isomorphisms in terms of unitary and antiunitary maps. We start with illustrative situation of two by two matrices.

Proposition 2.2 *Let $\varphi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be a nonzero Jordan $*$-homomorphism. Then there is either a unitary operator U on $\mathbb{C}^2$ such that φ is of the form*

$$\varphi(A) = UAU^*\,, \qquad \textit{for all } A \in M_2(\mathbb{C})\,,$$

or there is an antiunitary operator U on $\mathbb{C}^2$ such that φ is of the form

$$\varphi(A) = UA^*U^*\,, \qquad \textit{for all } A \in M_2(\mathbb{C})\,.$$

(Let us note that φ is in fact a Jordan $*$-isomorphism)

Proof As φ is nonzero there must exist a projection P_ξ such that $\varphi(P_\xi)$ is nonzero. As all one dimensional projections are unitarily equivalent, we obtain that φ is nonzero at every one dimensional projection. This also implies that φ preserves one dimensional projections. Let e_1, e_2 be the standard orthonormal basis of $\mathbb{C}^2$. Then

$$P_{e_1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad P_{e_2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\,.$$

As we know $\varphi(P_{e_1})$ and $\varphi(P_{e_2})$ are orthogonal rank one projections projecting on linear span of unit orthogonal vectors f_1 and f_2, respectively. Let us take a unitary operator V satisfying conditions $Vf_1 = e_1$ and $Vf_2 = e_2$. Then, for $i = 1, 2$,

$$V\varphi(P_{e_i})V^* = VP_{f_i}V^* = P_{e_i} .$$

Therefore by passing to $V\varphi(\cdot)V^*$ we can suppose, without loss of generality, that φ fixes the projections P_{e_1} and P_{e_2}. Let us write $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the form

$$B = aP_{e_1} + P_{e_1}BP_{e_2} + P_{e_2}BP_{e_1} + dP_{e_2} .$$

By Proposition 1.1 we can see that

$$\varphi(B) = a\varphi(P_{e_1}) + \varphi(P_{e_1})\varphi(B)\varphi(P_{e_2}) + \varphi(P_{e_2})\varphi(B)\varphi(P_{e_1}) + d\varphi(P_{e_2}) =$$

$$aP_{e_1} + P_{e_1}\varphi(B)P_{e_2} + P_{e_2}\varphi(B)P_{e_1} + dP_{e_2} .$$

Consequently, φ preserves diagonal in the sense that

$$\varphi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & x \\ y & d \end{pmatrix}$$

for some $x, y \in \mathbb{C}$. Let us now consider the matrix

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} ,$$

having the property that $A^2 = 0$. Thanks to the properties of φ we can write

$$\varphi(A) = \begin{pmatrix} 0 & y \\ x & 0 \end{pmatrix} ,$$

for some unknown $x, y \in \mathbb{C}$. Then the condition $\varphi(A^2) = \varphi(A)^2 = 0$ gives

$$0 = \begin{pmatrix} xy & 0 \\ 0 & xy \end{pmatrix}$$

and so x or y must be zero. Let us examine at first the possibility $y = 0$. Then

$$\varphi\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}$$

for some $\alpha \in \mathbb{C}$. Consequently, since φ preserves the $*$-operation, it transforms the matrices in the following way:

$$\varphi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & \overline{\alpha} b \\ \alpha c & d \end{pmatrix} . \tag{4}$$

Since φ preserves spectra and sends matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to $\begin{pmatrix} 0 & \overline{\alpha} \\ \alpha & 0 \end{pmatrix}$, we conclude that $|\alpha| = 1$. Let us now take a unitary map

$$U : (z_1, z_2) \in \mathbb{C}^2 \to (\overline{\alpha} z_1, z_2) ,$$

that is

$$U = \begin{pmatrix} \overline{\alpha} & 0 \\ 0 & 1 \end{pmatrix} .$$

One can compute easily that

$$U \begin{pmatrix} a & b \\ c & d \end{pmatrix} U^* = \begin{pmatrix} a & \overline{\alpha} b \\ \alpha c & d \end{pmatrix} = \varphi \begin{pmatrix} a & b \\ c & d \end{pmatrix} .$$

Therefore, φ is implemented by a unitary map.

Let us now explore the second possibility when

$$\varphi \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}$$

for some $\alpha \in \mathbb{C}$. In the same way as before we can establish that α is a complex unit and

$$\varphi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & \alpha c \\ \overline{\alpha} b & d \end{pmatrix} . \tag{5}$$

Let us take an antiunitary map

$$U : (z_1, z_2) \in \mathbb{C}^2 \to (\alpha \overline{z_1}, \overline{z_2}) \in \mathbb{C}^2 .$$

It is easy to verify that, given a general matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ,$$

the action of UA^*U^* on (z_1, z_2) gives $(az_1 + \alpha c z_2, \overline{\alpha} b z_1 + d z_2)$ which is precisely the action of $\varphi(A)$ on (z_1, z_2). This complements the proof. □

The algebra $M_2(\mathbb{C})$ describes a two-level quantum system. Its state space has a natural geometric interpretation. Let us identify each state with a positive matrix of trace one (so called density matrix). Every density matrix T can be represented by a vector $\mathbf{r} = (r_1, r_2, r_3)$ in the three dimensional unit ball in the following way:

$$T = \frac{1}{2}(I + r_1\sigma_1 + r_2\sigma_2 + r_3\sigma_3)\,,$$

where $\sigma_1, \sigma_2, \sigma_3$ are Pauli spin matrices;

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Antiunitary case in Wigner Theorem given by (5) with specific $\alpha = 1$ corresponds to a reflection $(r_1, r_2, r_3) \to (r_1, -r_2, r_3)$.

Having a unitary or antiunitary map V on a Hilbert space H we can define a map $Ad\, V$ on $B(H)$ by

$$Ad\, V(T) = VTV^*\,, \qquad T \in B(H)\,.$$

We say that the map $Ad\, V$ is implemented by V.

Proposition 2.3 *Let $\varphi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map. Then the following holds:*

(i) *If φ is a $*$-isomorphisms, then it is implemented by a unitary V acting on $\mathbb{C}^n$.*
(ii) *If φ is a $*$-antiisomorphism, then it is implemented by an antiunitary map on $\mathbb{C}^n$.*

Proof We shall show (i). Fix a unit vector $\xi \in H$. Then $\varphi(P_\xi)$ is a rank-one projection. By composing with $Ad\, U \circ \varphi$, where U is a suitable unitary operator, we can suppose that $\varphi(P_\xi) = P_\xi$. Let us first observe that for any $A \in M_n(\mathbb{C})$ we have that $\|A\xi\| = \|\varphi(A)\xi\|$. Indeed,

$$\|\varphi(A)\xi\|^2 = \langle\varphi(A)\xi, \varphi(A)\xi\rangle = \langle\varphi(A^*A)\xi, \xi\rangle = \langle\varphi(A^*A)P_\xi\xi, P_\xi\xi\rangle =$$

$$= \langle\varphi(P_\xi A^*AP_\xi)\xi, \xi\rangle\,.$$

One can directly check that $P_\xi A^*AP_\xi = \|A\xi\|^2 P_\xi$. By this, we can continue the previous computation and obtain

$$\|\varphi(A)\xi\|^2 = \langle\|A\xi\|^2\xi, \xi\rangle = \|A\xi\|^2\,.$$

The foregoing identity allows us to define in a correct way the unitary map $V : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by

$$VA\xi = \varphi(A)\xi\,, \qquad A \in B(\mathcal{H})\,.$$

We shall prove that V implements φ. Any vector $z \in \mathbb{C}^n$ can be written in the form: $z = \varphi(B)\xi$ where $B \in M_n(\mathbb{C})$. Then we have

$$VAV^*z = VAV^*\varphi(B)\xi =$$

$$= VAB\xi = \varphi(AB)\xi = \varphi(A)\varphi(B)\xi = \varphi(A)z\,.$$

Hence, operators $\varphi(A)$ and VAV^* coincide. Therefore, φ is implemented by V. Case (ii) has the same proof. □

Let us now have complex numbers, α, β, γ of modulus one. We denote by $\Psi_{\alpha,\beta,\gamma}$ the map acting on $M_3(\mathbb{C})$ by

$$\Psi_{\alpha,\beta,\gamma}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & \overline{\alpha}a_{12} & \overline{\beta}a_{13} \\ \alpha a_{21} & a_{22} & \overline{\gamma}a_{23} \\ \beta a_{31} & \gamma a_{32} & a_{33} \end{pmatrix}.$$

Lemma 2.4 *The following conditions are equivalent*

(i) $\Psi_{\alpha,\beta,\gamma}$ *is a Jordan* $*$*-isomorphism.*
(ii) $\alpha\gamma = \beta$.
(iii) $\Psi_{\alpha,\beta,\gamma}$ *is implemented by a unitary operator.*
(iv) $\Psi_{\alpha,\beta,\gamma}$ *is a* $*$*-isomorphism.*

Proof (i) ⇒ (ii). Take the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}. \text{ Then } \Psi_{\alpha,\beta,\gamma}(A) = \begin{pmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ \beta & \gamma & 0 \end{pmatrix}.$$

As

$$A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \text{ we have that } \Psi_{\alpha,\beta,\gamma}(A^2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \beta & 0 & 0 \end{pmatrix}.$$

On the other hand,

$$[\Psi_{\alpha,\beta,\gamma}(A)]^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha\gamma & 0 & 0 \end{pmatrix},$$

giving immediately $\beta = \alpha\gamma$.

(ii) $\Rightarrow$ (iii). Suppose that $\alpha\gamma = \beta$. Put

$$U = \begin{pmatrix} \overline{\alpha} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \gamma \end{pmatrix} .$$

Then U is a unitary matrix and it can be verified by a direct calculation that

$$\Psi_{\alpha,\beta,\gamma} = AdU .$$

The remaining implications are obvious. □

In order to handle matrices of higher rank, we shall introduce some notation. Let A be a matrix in $M_n(\mathbb{C})$, $n > 1$, and $i, j \in \{1, \dots, n\}$, $i \neq j$. By A^{ij} we shall denote the matrix in $M_n(\mathbb{C})$ having the same entries as A in positions (k, l), where $k, l \in \{i, j\}$ and zeros elsewhere. It is easy to verify that

$$M_n^{ij}(\mathbb{C}) := \{A^{ij} \ : \ A \in M_n(\mathbb{C})\}$$

is a $*$-subalgebra of $M_n(\mathbb{C})$ isomorphic to $M_2(\mathbb{C})$. Further we shall denote by $D(A)$ the diagonal matrix having the same diagonal as A.

Theorem 2.5 *Let $\varphi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a nonzero Jordan $*$-homomorphism. Then φ is implemented by either unitary or antiunitary operator.*

Proof The theorem is true for $n = 1$ (trivial reason) and we have established it for $n = 2$ in Proposition 2.2. First suppose that $n = 3$. Without loss of generality we can assume that φ fixes projections P_{e_i}, $i = 1, 2, 3$. Then it fixes all subalgebras $M_3^{ij}(\mathbb{C})$ and diagonal matrices. We say that the algebra $M_3^{ij}(\mathbb{C})$ has positive (resp. negative) orientation if the restriction of φ to $M_3^{ij}(\mathbb{C})$ is implemented by a unitary (resp. antiunitary) operator. We prove that all subalgebras have the same orientation. First consider the pair $M_3^{12}(\mathbb{C})$, $M_3^{13}(\mathbb{C})$. Suppose that the first algebra has positive orientation while the second algebra has negative one. Consider the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} .$$

We have that $A^2 = 0$. By Proposition 2.2 (and its proof) there are complex numbers α, β of modulus one such that

$$\varphi(A) = \begin{pmatrix} 0 & 0 & \overline{\beta} \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

Then

$$0 = \varphi(A^2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha\overline{\beta} \\ 0 & 0 & 0 \end{pmatrix},$$

which is a contradiction. Suppose that $M_3^{12}(\mathbb{C})$ has negative orientation and $M_3^{13}(\mathbb{C})$ has positive orientation. Then, similarly, there are complex numbers α and β of modulus one such that

$$\varphi(A) = \begin{pmatrix} 0 & \overline{\alpha} & 0 \\ 0 & 0 & 0 \\ \beta & 0 & 0 \end{pmatrix}.$$

Then

$$0 = \varphi(A^2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \beta\overline{\alpha} & 0 \end{pmatrix},$$

which is a contradiction. Now we shall prove that $M_3^{12}(\mathbb{C})$ and $M_3^{23}(\mathbb{C})$ have the same orientation. For this we consider action of φ on the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and its square } A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Suppose that $M_3^{12}(\mathbb{C})$ has positive and $M_3^{23}(\mathbb{C})$ has negative orientation. Then there are complex numbers α, γ of modulus one such that

$$\varphi(A) = \begin{pmatrix} 0 & 0 & 0 \\ \alpha & 0 & \overline{\gamma} \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $\varphi(A^2) \neq 0$, however $\varphi(A)^2 = 0$, which is not possible. If the orientation of $M_3^{12}(\mathbb{C})$ is negative and orientation of $M_3^{23}(\mathbb{C})$ is positive, then one can reach a contradiction in the same way by computing that $\varphi(A^2) = 0$ again. If all orientations on the corresponding 2 by 2 matrix subalgebras are positive, then φ must be a map $\Psi_{\alpha,\alpha\gamma,\gamma}$ (see Lemma 2.4) that is a $*$-isomorphism. On the other hand, if all orientations are negative, then by composing φ with the transpose map we obtain another Jordan $*$-isomorphism that is of the form above. In that case φ is a $*$-antihomomorphism. Now it suffices to apply Proposition 2.3.

This way the result is established for $n \leq 3$. Let us now tackle general case of $n \geq 3$. We shall show that one of the following statements are true

$$\varphi(PQ) = \varphi(P)\varphi(Q) \text{ for all rank-one projections } P \text{ and } Q \tag{6}$$

$$\varphi(PQ) = \varphi(Q)\varphi(P) \text{ for all rank-one projections } P \text{ and } Q\,. \tag{7}$$

To this end let us fix a rank-one projection P. There exists a rank-one projection Q that does not commute with P. The projection $E = P \vee Q$ is a rank-two projection and the same holds for $\varphi(E)$. As φ acts on $EM_n(\mathbb{C})E$ as a Jordan $*$-isomorphism (that must preserve commutativity in both directions), we have that $\varphi(P)\varphi(Q) \neq \varphi(Q)\varphi(P)$. Besides, we know from the case $n = 2$, that there are two (mutually exclusive) possibilities: $\varphi(PQ) = \varphi(P)\varphi(Q)$ and $\varphi(PQ) = \varphi(Q)\varphi(P)$. Suppose the first one holds. Take any rank-one projection R that is not orthogonal to P. Then the projection $F = P \vee Q \vee R$ has rank at most three, and the same holds for its image under φ. Based on our result for $n \leq 3$, φ must act as $*$-isomorphisms on $FM_n(\mathbb{C})F$. Therefore $\varphi(PR) = \varphi(P)\varphi(R)$. As the same holds for rank-one projections R orthogonal to P, we conclude that $\varphi(PR) = \varphi(P)\varphi(R)$ for all rank-one projections R. If $\varphi(PQ) = \varphi(Q)\varphi(P)$, then, similarly, we can show that $\varphi(PR) = \varphi(R)\varphi(P)$ for all rank-one projections R.

In summary, we have proved that for any rank-one projection P one of the following two statements is true.

$$\varphi(PQ) = \varphi(P)\varphi(Q) \text{ for all rank one projections } Q. \tag{8}$$

$$\varphi(PQ) = \varphi(Q)\varphi(P) \text{ for all rank one projections } Q. \tag{9}$$

We call P to have a positive orientation if (8) holds and negative orientation if (9) holds. We shall show that (6) holds or (7) holds by demonstrating that all rank-one projections have the same orientation. Suppose, for a contradiction, that there is a rank-one projection P with positive orientation and rank-one projection Q with negative orientation. As $P \neq Q$, we can find a rank-one projection R not commuting either with P or Q. Then $\varphi(RQ) = \varphi(QR)^* = (\varphi(R)\varphi(Q))^* = \varphi(Q)\varphi(R) \neq \varphi(R)\varphi(Q)$. Therefore R has negative orientation. Considering now the pair P, R, we have for the same reason that P has negative orientation—a contradiction.

Finally, as rank-one projections span the whole algebra, we can see that φ is either $*$-isomorphism (case (6)) or $*$-antiisomorphism (case (7)). Now Proposition 2.3 concludes the proof. □

The foregoing results on the structure of Jordan *-isomorphisms allows us to establish Wigner theorem both for finite and infinite dimensional Hilbert space.

Theorem 2.6 (Wigner Theorem for Finite Quantum Systems) *Let* $\varphi : P_1(H) \to P_1(H)$, *where* H *is a finite dimensional Hilbert space, be a map preserving* *transition probabilities, that is, φ satisfies*

$$\mathrm{tr}(PQ) = \mathrm{tr}(\varphi(P)\varphi(Q)) \qquad P, Q \in P_1(H)\,.$$

Then there is either a unitary or an antiunitary map U acting on H such that

$$\varphi(P) = UPU^* \qquad \text{for all } P \in P_1(H)\,.$$

Proof Let us identify H with $\mathbb{C}^n$. By Proposition 2.1 φ extends to a nonzero Jordan $*$-homomorphism acting on $M_n(\mathbb{C})$. By the virtue of Theorem 2.5 φ is implemented by either a unitary or an antiunitary map U such that

$$\varphi(P) = UPU^* \qquad \text{for all } P \in P(H)\,.$$

□

The infinite-dimensional variant of the previous theorem can be found in [1].

2.2 Logical Version

Later version of Wigner theorem is about preserving logical structure of projections. Projections impose mutually exclusive quantum operations if they are orthogonal. It turns out that symmetries of this orthogonality structure are the same as for transition probabilities except for the algebra $M_2(\mathbb{C})$.

Theorem 2.7 *Let H be a Hilbert space with $dim H \geq 3$. Let $\varphi : P(H) \to P(H)$ be an orthoisomorphism. Then there is either a unitary or an antiunitary operator U acting on H such that*

$$\varphi(P) = UPU^* \textit{ for all } P \in P(H)\,.$$

Proof The crucial idea is to find a bounded linear extension of φ. For this consider two identical linear combinations of projections in $B(H)$, i.e.

$$\sum_{i=1}^{n} \lambda_i P_i = \sum_{j=1}^{m} \mu_j Q_j\,,$$

where $P_1, \ldots, P_n, Q_1, \ldots, Q_m$ are projections and $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m$ are complex numbers. Let us take a functional f on $B(H)$. The composition $f \circ \varphi$ is a finitely additive bounded measure on $P(H)$ that has a bounded linear extension to $B(H)$ by Theorem 1.3. By applying this extension to the equality above we obtain

$$f\left(\sum_{i=1}^{n}\lambda_i\varphi(P_i)\right)=f\left(\sum_{j=1}^{m}\mu_j\varphi(Q_j)\right).$$

Employing now Hahn-Banach theorem, we can conclude that

$$\sum_{i=1}^{n}\lambda_i\varphi(P_i)=\sum_{j=1}^{m}\mu_j\varphi(Q_j).$$

This enables us to define a linear operator T acting on linear span of projections by

$$T\left(\sum_{i=1}^{n}\lambda_i P_i\right)=\sum_{i=1}^{n}\lambda_i\varphi(P_i).$$

It can be verified easily that T is bounded and so can be extended to a bounded linear operator (denoted by the same letter) $T : B(H) \to B(H)$. As this map preserves projection, it has to be Jordan *-homomorphism by Proposition 1.2. Arguing in the same way for the inverse map φ^{-1} gives us that T is a Jordan *-isomorphism. By the previous discussion (see e.g. Theorem 2.5 for a finite dimensional case) this map is implemented by unitary or antiunitary operator. □

Theorem 2.8 (Uhlhorn) *Let H be a Hilbert space with $dim H \geq 3$. Let $\varphi : P(H) \to P(H)$ be an orthoisomorphism. Then there is either a unitary or an antiunitary operator U acting on H implementing φ.*

Proof It is enough to show that φ extends to an orthoisomorphism of $P(H)$. Let us take any projection P acting on $B(H)$. Suppose that

$$P=\sup Q_\alpha=\sup R_\beta$$

where (Q_α) and (R_β) are one-dimensional projections. Then

$$\sup_\alpha \varphi(Q_\alpha)=\sup_\beta \varphi(R_\beta)$$

because both these suprema have the same orthocomplement. This allows to extend φ to an orthoisomorphism on $P(H)$ by putting

$$\varphi(P)=\sup\{\varphi(R)\ :\ R\leq P, dim P=1\}$$

By evoking Theorem 2.7 we can conclude the proof. □

3 Dye Theorem

3.1 Dye Theorem for von Neumann Algebras

The following celebrated theorem has been proved in [4].

Theorem 3.1 (Dye) *Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras, where $\mathcal{M}$ has no Type I_2 direct summand. Then any orthoisomorphism $\varphi : P(\mathcal{N}) \to P(\mathcal{N})$ extends to a Jordan *-isomorphism $\Phi : \mathcal{M} \to \mathcal{N}$.*

Proof Even if this is much general situation, the proof is the same as for Theorem 2.7. Using Theorem 1.4, we can extend φ to a bounded linear map $\Phi : \mathcal{M} \to \mathcal{N}$ preserving projections and so being a Jordan *-homomorphism. The argument involving the inverse φ^{-1} says that Φ is in fact a Jordan *-isomorphism. □

3.2 Dye Theorem for AW*-algebras

Main result of this part is taken from [7].

Theorem 3.2 (Hamhalter) *Let $\mathcal{A}$ be an AW*-algebra without Type I_2 direct summand and $\mathcal{B}$ be an AW*-algebra. Let $\varphi : P(A) \to P(B)$ be a map preserving all suprema and orthocomplements, i.e.*

$$\varphi(\sup_\alpha p_\alpha) = \sup_\alpha \varphi(p_\alpha)$$

and

$$\varphi(1 - p) = 1 - \varphi(p)\,.$$

Then φ extends to a Jordan $*$-homomorphism $\Phi : \mathcal{A} \to \mathcal{B}$.

One of major problems in theory of AW*-algebras is whether Generalised Gleason theorem holds also for AW*-algebras without Type I_2. Since the positive answer would resolve many other difficult problems in the theory of operator algebras (see [7] for details) this problem is expected to be extremely difficult. Therefore the idea of proving Dye theorem for AW*-algebras must be different from the case of von Neumann algebras. Fortunately, we can combine local Gleason theorem and matrix approach initiated by J. von Neumann and Dye [4] and developed further by Heunen and Reyes [12].

We shall need the following notation.

Let $M_n(\mathcal{A})$ be the C^*-algebra of all n by n matrices over the C^*-algebra $\mathcal{A}$. Let us take distinct integers $1 \leq i, j \leq n$ and $a \in \mathcal{A}$. We shall consider the matrix $p_{ij}(a)$ in $M_n(\mathcal{A})$ such that all entries are zero except for positions (i, i), $(i, j)(j, i)$, (j, j)

which give the submatrix

$$\begin{pmatrix} (1+aa^*)^{-1} & (1+aa^*)^{-1}a \\ a^*(1+aa^*)^{-1} & a^*(1+aa^*)^{-1}a \end{pmatrix}.$$

Further by e_{ii} we shall denote the matrix having at position (i, i) the unit and zeros elsewhere. It turns out that each $p_{ij}(a)$ is a projection and even that projections of this form generate the projection lattice of $M_n(\mathcal{A})$ as a complete orthomodular lattice (see e.g. [12, Lemma 4.1]).

There is an elegant bridge between linear structure of $\mathcal{A}$ and lattice operations in $M_n(\mathcal{A})$, which is a great discovery going back to J. von Neumann (see [4, Lemma 4, Lemma 3(i)]). Let us recall that by a lattice polynomial in variables $p_1, \ldots, p_k$ we mean a formal expressions that is a result of finitely many lattice operations $(\vee, \wedge)$ performed on elements in $\{p_1, \ldots, p_k\}$.

Lemma 3.3 *There exist lattice polynomials P, Q and R such that for any elements a, b, c of a C^*-algebra $\mathcal{A}$, where c is invertible, any integer $n \geq 3$, and any distinct indices $1 \leq i, j, k \leq n$, the following holds.*

(i) $p_{ij}(a+b) = P(p_{ij}(a), p_{ij}(b), p_{ik}(c), e_{ii}, e_{jj}, e_{kk})$.
(ii) $p_{ij}(-ab) = Q(p_{ik}(a), p_{kj}(b), e_{ii}, e_{jj})$
(iii) $p_{ij}(-a^*) = R(p_{ji}(a), e_{ii}, e_{jj})$.

As a consequence of Lemma 3.3 any lattice morphism of $M_n(\mathcal{A})$ that preserves projections of type $p_{ij}(a)$ and e_{ii}, induces a *-ring morphism on the underlying algebra $\mathcal{A}$. This is an important ingredient in proving Dye theorem as well in establishing the following deep result of Heunen and Reyes [12, Theorem 4.6].

Theorem 3.4 (Heunen and Reyes) *Let $\mathcal{A}$ and $\mathcal{B}$ be AW^*-algebras. Let*

$$f : P(\mathcal{A}) \to P(\mathcal{B})$$

*be a map preserving arbitrary suprema and orthocomplements. Then f extends to a Jordan *-homomorphisms if and only the following condition holds*

$$f((1-2p)q(1-2p)) = (1-2f(p))f(q)(1-2f(p)), \tag{10}$$

for all projections $p, q \in \mathcal{A}$.

The previous theorem says that when proving Dye theorem one has to check (10), which is an identity involving projections $1, e, f$. Therefore, one can restrict himself to a unital AW*-algebra generated by e and f. A thorough analysis of position of two projections in AW*-algebras has led to the following structural result proved in [7] which has independent meaning.

Proposition 3.5 *Let e and f be projections in a AW*-algebra $\mathcal{A}$. Then the smallest AW*-subalgebra, $AW^*(e, f)$ of $\mathcal{A}$ that contains e and f, is *-isomorphic to the* *direct sum*

$$\mathcal{C} \oplus M_2(\mathcal{D}),$$

where $\mathcal{C}$ *and* $\mathcal{D}$ *are abelian* C^**-algebras.*

The algebra $M_2(\mathcal{D})$ in the previous Proposition is a Type I_2 AW*-algebra. For this algebra the Gleason theorem does not hold. However we have succeeded in proving that for any matrix algebra of higher rank the generalized Gleason theorem does hold [7, Theorem 3.8].

Theorem 3.6 (Hamhalter) *Let* $\mathcal{A}$ *be an AW*-subalgebra of type* I_n*, where* $n \neq 2, n < \infty$*, and* X *a Banach space. Then any bounded finitely additive measure*

$$\mu : P(\mathcal{A}) \to X,$$

extends to a bounded linear functional

$$T : \mathcal{A} \to X.$$

Unfortunately the studied algebra $AW^*(1, e, f)$ generated by projections e and f is not covered by the previous theorem. However, the fact that this algebra is sitting inside the algebra with no summand of Type I_2 allows one, after nontrivial arguments using geometry of projections and their angles, to show that the Generalized Gleason theorem does hold on subalgebras generated by two projections.

Theorem 3.7 (Hamhalter) *Let* e, f *be projections in a AW*-algebra* $\mathcal{A}$ *without Type* I_2*. Let* X *be a Banach space. Let* $\mathcal{B}$ *be an* AW^**-subalgebra generated by projections* $e, f, 1$*. Then any bounded finitely additive measure*

$$\mu : P(\mathcal{B}) \to X,$$

extends to a bounded linear functional

$$T : \mathcal{B} \to X.$$

Proof of Theorem 3.2 We have now all ingredients to prove the (nonbijective) Dye theorem for AW*-algebras. Let us have an AW*-algebra $\mathcal{A}$, not having Type I_2 direct summand, and another AW*-algebra $\mathcal{B}$. Consider a map $\varphi : P(\mathcal{A}) \to P(\mathcal{B})$ that preserves suprema of arbitrary projections and orthocomplements. Fix now two projections $e, f \in \mathcal{A}$. Theorem 3.7 assures us that there is a Jordan map J mapping the algebra $\mathcal{C} = AW^*(1, e, f)$ to $\mathcal{B}$ that coincides with φ on $P(\mathcal{C})$. By the algebraic properties of the Jordan morphism we can conclude that (10) is satisfied. Now we can use Theorem 3.4 to prove Theorem 3.2.

4 Structure of Abelian Subalgebras

Let $\mathcal{A}$ be a unital C^*-algebra. By the symbol $\mathcal{C}(\mathcal{A})$ we shall denote the structure of all abelian C^*-subalgebras of $\mathcal{A}$ containing the unit of $\mathcal{A}$. When endowed with set theoretic inclusion, $\mathcal{C}(\mathcal{A})$ becomes the poset with the least element span$\{1\}$. Infima in this poset are given by set theoretic intersections of subalgebras. On the other hand, supremum of two elements $\mathcal{E}$ and $\mathcal{F}$ exists in $\mathcal{C}(\mathcal{A})$ if and only if $\mathcal{E}$ and $\mathcal{F}$ mutually commute. Similarly we shall denote by $\mathcal{C}_0(\mathcal{A})$ the poset of all unital abelian C^*-subalgebras (not necessarily containing the unit of $\mathcal{A}$). Let $\mathcal{P}$ be a subposet of $\mathcal{C}_0(\mathcal{A})$ and $\mathcal{Q}$ be a subposet of $\mathcal{C}_0(\mathcal{B})$. The map $\varphi : \mathcal{P} \to \mathcal{Q}$ is said to be implemented by a map $\Phi : \mathcal{A} \to \mathcal{B}$ if

$$\varphi(\mathcal{C}) = \Phi[\mathcal{C}] = \{\Phi(x) \ : \ x \in \mathcal{C}\} \qquad \text{for all } \mathcal{C} \in \mathcal{P}.$$

Certainly the poset $\mathcal{C}(\mathcal{A})$ is a C*-invariant in category of unital C^*-algebras. It is not a complete invariant in general, as the opposite algebra $\mathcal{A}^o$ has precisely the same poset $\mathcal{C}(\mathcal{A}^o) = \mathcal{C}(\mathcal{A})$, while $\mathcal{A}$ and $\mathcal{A}^o$ may be not isomorphic as C^*-algebras as celebrated result on Type III factors due to Connes shows. (The opposite algebra $\mathcal{A}^0$ is the same Banach space $\mathcal{A}$ with the same *-operation and reversed product $(a, b) \to ba$.)

4.1 *Abelian C*-subalgebras*

Albeit the poset $\mathcal{C}(\mathcal{A})$ is not a complete invariant in category of C^*-algebras, it has been shown by Mendivil, that $\mathcal{C}(\mathcal{A})$ is a complete invariant in the category of abelian unital C^*-algebras. We have shown that more is true by establishing a one-to-one correspondence between *-isomorphisms of abelian C^*-algebras and order isomorphisms of posets of unital abelian C^*-subalgebras. This is the content of the following theorem proved in [6].

Theorem 4.1 (Hamhalter) *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian unital C^*-algebras. Let*

$$\varphi : \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{B})$$

*be an order isomorphism. Then there is a *-isomorphism*

$$\psi : \mathcal{A} \to \mathcal{B}$$

such that

$$\varphi(\mathcal{C}) = \psi[\mathcal{C}] \qquad \text{for all } \mathcal{C} \in \mathcal{C}(\mathcal{A})\,.$$

Moreover, if $\dim \mathcal{A} > 2$, *then ψ is uniquely determined by φ.*

This result has a topological background. In fact, any unital abelian C^*-algebra is *-isomorphic to $C(X)$, where X is a compact Hausdorff space. Moreover, by Banach-Stone theorem any *-isomorphism $\psi : C(X) \to C(Y)$ is given by the homeomorphism $\tau : Y \to X$ in the following way

$$\psi(f) = f \circ \tau \qquad \text{for all } f \in C(X)\,.$$

There are interrelations between algebraic structure of abelian C^*-algebra $C(X)$ and topology of X. For example, for any closed ideal I of $C(X)$ there is a unique closed subset F of X such that

$$I = \{f \in C(X) \;:\; f \text{ is zero on F}\}$$

The key role in studying the poset $\mathcal{C}(C(X))$ is played by so called ideal subalgebras, that is by C^*-subalgebras of $C(X)$ generated by a closed proper ideal I of $C(X)$ and the unit **1**. More precisely, for each proper ideal algebra $\mathcal{C}$ of $C(X)$ there is a closed subset F of X with at least two points such that

$$\mathcal{C} = \{f \in C(X) \;:\; f \text{ is constant on } F\}\,.$$

It can be shown that any order isomorphism of subalgebras structures preserves ideal subalgebras. Therefore it induces an order isomorphism between closed subsets of X. An important part of proving Theorem 4.1 is therefore establishing the form of isomorphisms of the poset of closed subsets. This is achieved in the following theorem (see [6, Theorem 2.3]). Let us denote by the symbol $\mathcal{F}(X)$ the poset of all closed subsets of X with at least two points ordered by set theoretic inclusion.

Theorem 4.2 *Let X and Y be compact Hausdorff spaces. Suppose that X is not a singleton. Let*

$$\psi : \mathcal{F}(X) \to \mathcal{F}(Y)$$

be an order isomorphism. Then there is a homeomorphism

$$\tau : X \to Y$$

such that

$$\psi(F) = \tau[F] \qquad \text{for all } F \in \mathcal{F}(X)\,.$$

The homeomorphism τ in the previous theorem is then used in the proof of Theorem 4.1.

In case of abelian algebra the poset $\mathcal{C}(\mathcal{A})$ is a complete lattice. This is not true in noncommutative case where suprema of elements do not exist in general. One cannot hope for Theorem 4.1 to be valid in this wider context. Indeed, based on the fact that Jordan *-isomorphisms preserve commutativity, it can be shown that a Jordan *-isomorphism ψ between unital C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ implements an order isomorphism φ. Since there are many Jordan *-isomorphisms that are not *-isomorphisms we can see that, in the light of the previous observation, not all order isomorphisms of the posets of abelian subalgebras are induced by *-isomorphisms. Even more is true, one can realize that every quasi Jordan *-isomorphisms implements order isomorphisms. This is the content of the following Proposition.

Proposition 4.3 *Let $\psi : \mathcal{A} \to \mathcal{B}$ be a unital quasi Jordan *-isomorphism. Then the map $\varphi : \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{B})$ given by*

$$\varphi(\mathcal{C}) = \psi[\mathcal{C}] \qquad \textit{for all } \mathcal{C} \in \mathcal{C}(\mathcal{A})$$

is an order isomorphisms.

There is a natural question whether the converse holds as well. The answer is in the positive. It was proved by the first author in [6, Theorem 3.4]. For further treatment and alternative proofs we recommend [14, 15].

Theorem 4.4 (Hamhalter) *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. Then for any order isomorphism*

$$\varphi : \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{B})$$

*there is a unital quasi Jordan *-isomorphism*

$$\psi : \mathcal{A} \to \mathcal{B}$$

such that

$$\varphi(\mathcal{C}) = \psi[\mathcal{C}] \qquad \textit{for all } \mathcal{C} \in \mathcal{C}(\mathcal{A})\,.$$

Moreover, if $\mathcal{A}$ is not isomorphic neither to $\mathbb{C}^2$ nor $M_2(\mathbb{C})$, then ψ is uniquely determined by φ.

This theorem is a generalization of Theorem 4.1. Let us note that one cannot have uniqueness of the induced Jordan map in case of $\mathbb{C}^2$ or $M_2(\mathbb{C})$. To explain it, let us consider the latter case $\mathcal{A} = M_2(\mathbb{C})$. In that situation the poset $\mathcal{C}(\mathcal{A})$ has the least element span $\{1\}$ and uncountably many 2-dimensional abelian C^*-subalgebras that are atoms and maximal elements simultaneously. On each atom let us choose separately a unital Jordan *-automorphism. The union of such automorphisms (and its canonical extension to the whole algebra) now gives a quasi Jordan *-isomorphism implementing the identical order automorphism of $M_2(\mathbb{C})$. We can get this way uncountably many quasi Jordan *-isomorphisms implementing the same (identical) automorphism of $\mathcal{C}(\mathcal{A})$).

Except for the poset $\mathcal{C}(\mathcal{A})$, it is also natural to consider the poset $\mathcal{C}_0(\mathcal{A})$ of all abelian C^*-subalgebras (unital or not) of $\mathcal{A}$. The poset $\mathcal{C}_0(\mathcal{A})$ has different properties. For example, let us consider $\mathcal{A} = \mathbb{C}^2$. The poset $\mathcal{C}(\mathcal{A})$ consists of two elements span $\{(1, 1)\}$ and $\mathcal{A}$, related by span $\{(1, 1)\} \leq \mathcal{A}$. In case of $\mathcal{C}_0(\mathcal{A})$ we have the largest element $\mathcal{A}$, the smallest element $\{0\}$ and three incomparable elements span $\{(1, 1)\}$, span $\{(1, 0)\}$, and span $\{(0, 1)\}$. The group of order isomorphisms is just the group of permutations of these three point set. As Jordan *-isomorphisms are unital, they implement only those order automorphisms of $\mathcal{C}_0(\mathcal{A})$ that leave the element span$\{(1, 1)\}$ fixed. Therefore one cannot hope that order isomorphisms are implemented by (quasi) Jordan automorphisms in this case. However careful analysis shows that implementation is possible if we assume that order isomorphisms preserve the subalgebra generated by the unit. Given a unital C^*-algebra $\mathcal{A}$ we shall denote by $\mathbb{O}$ the one dimensional subalgebra span$\{1\}$.

The authors have obtained the following result [8]:

Theorem 4.5 (Hamhalter and Turilova) *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras isomorphic neither to $\mathbb{C} \oplus \mathbb{C}$ nor to $M_2(\mathbb{C})$. Let*

$$\varphi : \mathcal{C}_0(\mathcal{A}) \to \mathcal{C}_0(\mathcal{B})$$

be an order isomorphism such that

$$\varphi(\mathbb{O}) = \mathbb{O}\,.$$

*Then there is a unique unital quasi Jordan *-isomorphism*

$$\psi : \mathcal{A} \to \mathcal{B}$$

such that

$$\varphi(\mathcal{C}) = \psi[\mathcal{C}] \qquad \textit{for all } \mathcal{C} \in \mathcal{C}_0(\mathcal{A})\,.$$

Example In general, one cannot replace quasi Jordan *-morphisms by Jordan *-morphisms in the previous results. Consider the algebra $\mathcal{A} = M_2(\mathbb{C})$. Then each element that is neither the greatest nor the least element in $\mathcal{C}(\mathcal{A})$ is a two dimensional algebra of the form

$$V_\xi = \text{span}\{P_\xi, 1 - P_\xi\}\,,$$

where ξ is a unit vector in $\mathbb{C}^2$ and P_ξ is an orthogonal projection onto its linear span. Denote the set of such subalgebras by S. Let us have an order automorphism φ of $\mathcal{C}(\mathcal{A})$ implemented by a Jordan *-automorphism ψ. Let P_{V_ξ} be the projection of the Banach space $\mathcal{A}$ onto its closed subspace V_ξ. As ψ is a bounded linear map on $\mathcal{A}$, the assignment $\xi \to P_{\psi[V_\xi]}$ is a continuous map from the unit sphere $\mathbb{C}^2$ into the space of bounded operators acting on $\mathcal{A}$. Let us now take a sequence (ξ_n) of unit vectors in $\mathbb{C}^2$ converging to a vector $\xi \notin (\xi_n)$ in $\mathbb{C}^2$. Denote by Γ the bijection of the set S such that $\Gamma(V_{\xi_n}) = V_{\xi_n}$ for all n and $\Gamma(V_\xi) = V_\nu \neq V_\xi$. Let us now consider the order automorphism φ of $\mathcal{C}(\mathcal{A})$ that coincides with Γ on S. Then the map $\xi \to P_{\varphi(V_\xi)}$ is not continuous and so it cannot be implemented by any Jordan *-isomorphism.

Nevertheless, we can see that the previous counterexample is quite special as the poset $\mathcal{C}(M_2(\mathbb{C}))$ has all nontrivial elements as atoms. This cannot happen in $M_3(\mathbb{C})$ and higher dimensions. In fact, when applying the generalized Gleason theorem (Theorem 1.4), we can see that any quasi Jordan $*$-homomorphism on a von Neumann algebra without Type I_2 direct summand is a Jordan *-homomorphism. We then have the following description of symmetries of $\mathcal{C}(\mathcal{A})$ for nearly all von Neumann algebras. The following result was proved in [6].

Theorem 4.6 (Hamhalter) *Let $\mathcal{M}$ be a von Neumann algebra without Type I_2 direct summand. Let $\mathcal{N}$ be another von Neumann algebra. Let*

$$\varphi : \mathcal{C}(\mathcal{M}) \to \mathcal{C}(\mathcal{N})$$

*be an order isomorphisms. Then there is a Jordan *-isomorphism*

$$\psi : \mathcal{M} \to \mathcal{N}$$

such that

$$\varphi(\mathcal{C}) = \psi[\mathcal{C}] \qquad \textit{for all } \mathcal{C} \in \mathcal{C}(\mathcal{M}) .$$

Based on Theorem 4.5 we can now obtain the nonunital version of Theorem 4.6 proved in [8].

Theorem 4.7 (Hamhalter and Turilova) *Let $\mathcal{M}$ be a von Neumann algebra without Type I_2 direct summand. Let $\mathcal{N}$ be another von Neumann algebra. Let*

$$\varphi : \mathcal{C}_0(\mathcal{M}) \to \mathcal{C}_0(\mathcal{N})$$

be an order isomorphism such that

$$\varphi(\mathbb{O}) = \mathbb{O} .$$

*Then there is a Jordan *-isomorphism*

$$\psi : \mathcal{M} \to \mathcal{N}$$

such that

$$\varphi(\mathcal{C}) = \psi[\mathcal{C}] \qquad \textit{for all } \mathcal{C} \in \mathcal{C}(\mathcal{M}) .$$

4.2 Abelian von Neumann Subalgebras

In case of von Neumann algebras, another possibility how to embody Bohr's doctrine is to consider the poset of abelian von Neumann subalgebras instead of all abelian C^*-subalgebras. Let $\mathcal{M}$ be a von Neumann algebra. By the symbol $\mathcal{V}(\mathcal{M})$ we shall denote the poset of all abelian von Neumann subalgebras of $\mathcal{M}$ containing the unit of $\mathcal{M}$ and ordered by set theoretic inclusion. Of course, $\mathcal{V}(\mathcal{M})$ is a subposet of $\mathcal{C}(\mathcal{M})$. The first result in this context has been proved by Döring and Harding in [3].

Theorem 4.8 (Döring and Harding) *Let $\mathcal{M}$ be a von Neumann algebra without Type I_2 direct summand and $\mathcal{N}$ be another von Neumann algebra. Then for any order isomorphism*

$$\varphi : \mathcal{V}(\mathcal{M}) \to \mathcal{V}(N)$$

*there is a unique Jordan *-isomorphism*

$$\psi : \mathcal{M} \to \mathcal{N}$$

implementing φ:

$$\varphi(\mathcal{C}) = \psi[\mathcal{C}] \qquad \textit{for all } \mathcal{C} \in \mathcal{V}(\mathcal{N}) \,.$$

Original proof of this result is based on Dye Theorem. It can be also proved quickly by using Generalised Gleason theorem as in the previous subsection. We have generalized this result to AW*-algebras. Since we do not have Gleason type theorem to our disposal in this case we have to rely fully on Dye theorem for AW*-algebras discussed in Sect. 2. The following result may be found in [7, Theorem 4.6] for $\mathcal{C}(\mathcal{M})$ or in [15, Theorem 9.2.8] for $\mathcal{V}(\mathcal{N})$. Having an AW*-algebra $\mathcal{M}$ we shall denote by $\mathcal{V}(\mathcal{M})$ the poset of abelian AW*-subalgebras of $\mathcal{M}$ containing the unit of $\mathcal{M}$.

Theorem 4.9 (Hamhalter, Lindenhovous) *Let $\mathcal{M}$ be an AW*-algebra without Type I_2 direct summand and $\mathcal{N}$ be another AW*-algebra.*

(i) *Let*

$$\varphi : \mathcal{V}(\mathcal{M}) \to \mathcal{V}(\mathcal{N})$$

*be an order isomorphism. Then there exists a Jordan *-isomorphism*

$$\psi : \mathcal{M} \to \mathcal{N}$$

implementing φ*:*

$$\varphi(\mathcal{C}) = \psi[\mathcal{C}] \qquad \textit{for all } \mathcal{C} \in \mathcal{V}(\mathcal{N})\,.$$

(ii) *Let*

$$\varphi : \mathcal{C}(\mathcal{M}) \to \mathcal{C}(\mathcal{N})$$

*be an order isomorphism. Then there exists a Jordan *-isomorphism*

$$\psi : \mathcal{M} \to \mathcal{N}$$

implementing φ*:*

$$\varphi(\mathcal{C}) = \psi[\mathcal{C}] \qquad \textit{for all } \mathcal{C} \in \mathcal{C}(\mathcal{N})\,.$$

Besides the order structure of all abelian C^*-subalgebras and abelian von Neumann subalgebras we can also consider the simplest structure of finite dimensional abelian subalgebras. Each algebra of this type is isomorphic to the power $\mathbb{C}^n$ and corresponds to decomposition of the unit into sum of orthogonal projections. Let $\mathcal{C}_{fin}(\mathcal{A})$ be the set of all finite dimensional abelian C^*-subalgebras of a unital C^*-algebra $\mathcal{A}$ containing the unit and ordered by set theoretic inclusion. Using Dye theorem one can show once more that order isomorphism of this structure is implemented by Jordan *-isomorphism (see [9, Proposition 3.5]).

Theorem 4.10 (Hamhalter and Turilova) *Let* $\mathcal{M}$ *be a von Neumann algebra without Type* I_2 *direct summand. Let* $\mathcal{N}$ *be another von Neumann algebra. Let*

$$\varphi : \mathcal{C}_{fin}(\mathcal{M}) \to \mathcal{C}_{fin}(\mathcal{A})$$

*be an order isomorphism. Then there is a unique Jordan *-isomorphism* $\psi : \mathcal{M} \to \mathcal{N}$ *such that*

$$\varphi(\mathcal{C}) = \psi[\mathcal{C}] \qquad \textit{for all } \ \mathcal{C} \in \mathcal{C}_{fin}(\mathcal{A})\,.$$

4.3 Abelian Subalgebras as Invariants

In the previous section we could see that for algebras not containing Type I_2 part, the structure of abelian subalgebras implies that given algebras are isomorphic as Jordan algebras. For algebras of Type I_2 we know that not all order isomorphisms between the structure of abelian subalgebras are implemented by Jordan maps. However, it is surprising that such algebras are even *-isomorphic if they have isomorphic posets of abelian subalgebras. In fact, we have shown that $\mathcal{C}(\mathcal{A})$ is a complete Jordan invariant for all von Neumann algebras (see Theorem 2.3 in [11].)

Theorem 4.11 (Hamhalter and Turilova) *Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras. The following assertions are equivalent*

(i) *$\mathcal{V}(\mathcal{M})$ and $\mathcal{V}(\mathcal{N})$ are isomorphic.*
(ii) *$\mathcal{C}_{fin}(\mathcal{M})$ and $\mathcal{C}_{fin}(\mathcal{N})$ are isomorphic.*
(iii) *$\mathcal{P}(\mathcal{M})$ and $\mathcal{P}(\mathcal{N})$ are orthoisomorphic.*
(iv) *$\mathcal{M}$ and $\mathcal{N}$ are isomorphic as Jordan algebras.*

Proof First observe that (i) implies (ii). This is due to the fact that finite dimensional abelian subalgebras can be characterized as those elements in $\mathcal{V}(\mathcal{M})$ that have only finitely many elements beneath. This is of course preserved by any order isomorphism. Therefore any order isomorphism between $\mathcal{V}(\mathcal{M})$ and $\mathcal{V}(\mathcal{N})$ restricts to an order isomorphism between $\mathcal{C}_{fin}(\mathcal{M})$ and $\mathcal{C}_{fin}(\mathcal{N})$.

Now we focus on proving that (ii) implies (iii). It has been proved in [3, Lemma 3.1] that $\mathcal{C}_{fin}(\mathcal{M})$ is isomorphic to the poset of finite Boolean subalgebras $B^{fin}(\mathcal{P}(\mathcal{M}))$ of $\mathcal{P}(\mathcal{M})$ by the map

$$X \in \mathcal{C}_{fin}(\mathcal{M}) \to \mathcal{P}(X) \in B^{fin}(\mathcal{P}(\mathcal{M}))\,.$$

Therefore, condition (ii) implies an order isomorphism between $B^{fin}(\mathcal{P}(\mathcal{M}))$ and $B^{fin}(\mathcal{P}(\mathcal{N}))$. By Döring and Harding [3, Lemma 3.3.] this isomorphism extends naturally to an order isomorphism between the structures of all Boolean subalgebras $B(\mathcal{P}(\mathcal{M}))$ and $B(\mathcal{P}(\mathcal{N}))$. However, according to a nice result of Harding and Navara (see e.g. [14]), any isomorphism between structures of Boolean subalgebras of orthomodular lattices induces an orthoisomorphism (may be not in a unique way) between orthomodular lattices themselves. Therefore $\mathcal{P}(\mathcal{M})$ is orthoisomorphic to $\mathcal{P}(\mathcal{N})$.

Let us now show that (iii) implies (iv). Suppose that $\varphi : \mathcal{P}(\mathcal{M}) \to \mathcal{P}(\mathcal{M})$ is an orthoisomorphism. Let z be the central projection in $\mathcal{M}$ such that $z\mathcal{M}$ is either zero or of Type I_2 and $(1-z)\mathcal{M}$ is either zero or does not contain any direct summand of Type I_2. If $z = 0$ then the proof follows from Theorem 3.1. Therefore, let us suppose that z is non-zero. By the properties of orthoisomorphism we know that $w = \varphi(z)$ is a central projection in $\mathcal{N}$. Hence,

$$\mathcal{N} = w\mathcal{N} \oplus (1-w)\mathcal{N}\,.$$

Moreover, the restriction of φ gives orthoisomorphism between $\mathcal{P}(z\mathcal{M})$ and $\mathcal{P}(w\mathcal{N})$ and also between $\mathcal{P}((1-z)\mathcal{M})$ and $\mathcal{P}((1-w)\mathcal{N})$. By Theorem 3.1 there is a Jordan *-isomorphism between $(1-z)\mathcal{M}$ and $(1-w)\mathcal{N}$. It remains to show that $z\mathcal{M}$ and $w\mathcal{N}$ are Jordan $*$-isomorphic. We shall prove even stronger statement that these algebras are isomorphic as C^*-algebras. First we verify that $w\mathcal{N}$ is of Type I_2. Let e be a faithful abelian projection in $z\mathcal{M}$ such that $z-e$ is also a faithful abelian projection in $z\mathcal{M}$. (A projection is faithful if its central cover is the unity.) As φ preserves commutativity in both directions and consequently it preserves the central covers, we infer that $\varphi(e)$ and $w-\varphi(e)$ are orthogonal faithful abelian projections in $w\mathcal{M}$. Therefore $w\mathcal{N}$ is of Type I_2. Further φ gives an orthoisomorphism between

$\mathcal{P}(Z(z\mathcal{M}))$ and $\mathcal{P}(Z(w\mathcal{N}))$. Therefore $Z(z\mathcal{M})$ and $Z(w\mathcal{N})$ are isomorphic as C^*-algebras (it follows from the fact than any orthoisomorphism between projection lattices of abelian von Neumann algebras extends to a *-isomorphism). According to the structure theory of finite homogeneous von Neumann algebras of Type I_n such algebras are $*$-isomorphic if they have $*$-isomorphic centers. This concludes the proof of given implication.

Finally, the implication (iv) $\Rightarrow$ (i) is easy. □

5 Choquet Order Structure

In this section we shall present our main results on complete Jordan invariants based on Choquet order structure of decompositions of states. Let us first recall basic definitions. For details on Choquet theory on state spaces we refer the reader to monograph [17]. Let us have a compact Hausdorff space X. By a *Radon measure* μ on X we mean an element of the dual space $C(X)^*$. By celebrated Riesz representation theorem there is a bijection between Radon measures and regular Borel measures on X. In this correspondence $\mu \in C(X)^*$ can be canonically identified with a regular Borel measure μ on X in the sense of the formula

$$\mu(f) = \int_X f(\omega)\, d\mu(\omega) \qquad f \in C(X)\,.$$

The set of all positive Radon measures on X will be denoted by $M^+(X)$. A probability Radon measure ν is a positive measure for which $\nu(X) = 1$. The symbol $\mathcal{P}(X)$ will be reserved for the set of all probability Radon measures on X.

Let now K be a non-empty compact convex set in a locally convex Hausdorff vector topological space E. Let $A(K)$ and $CC(K)$ represent the set of all continuous affine functions on K and all continuous convex real functions on K, respectively. Take $\mu \in \mathcal{P}(K)$. A point $b(\mu) \in K$ is called the *barycenter* of μ if, for each $a \in A(K)$,

$$a(b(\mu)) = \mu(a) = \int_K a(\omega)\, d\mu(\omega)\,.$$

Every probability Radon measure admits a (unique) barycenter. To see an example, let us have a finite convex combination of Dirac measures $\mu = \sum_{i=1}^{n} \lambda_i \delta_{x_i}$, $x_i \in K$. Then $b(\mu) = \sum_{i=1}^{n} \lambda_i x_i$. A measure $\mu \in \mathcal{P}(X)$ is called *representing* for a given point $x \in K$ if x is the barycenter of μ. The set of all representing measures of x will be denoted by $M_x(K)$. Note that the Dirac measure, δ_x, is one of the representing measures for x. Let us recall that convex combinations of Dirac measures are just probability measures with finite support.

Let μ and ν be positive Radon measures on a compact convex set K. We define the relation $\mu \prec_C \nu$ as follows:

$$\mu \prec_C \nu \quad \text{if} \;\; \mu(f) \le \nu(f) \text{ for all } f \in CC(K)\,.$$

It is known that the relation $\prec_C$ is a partial order on the set of positive Radon measures (see e.g. [2, Proposition 4.1.3, p.325], [17, Definition 6.5,p. 233]). The order $\prec_C$ is called the *Choquet order*.

Now we turn to the situation when K is a state space of a C^*-algebra and establish some new results for Choquet order in this situation. Let φ be a state on a C^*-algebra $\mathcal{A}$. The triple $(\pi_\varphi, \xi_\varphi, \mathcal{H}_\varphi)$ will represent the GNS data of φ. By $\mathcal{M}_\varphi$ we shall denote the von Neumann algebra generated by $\pi_\varphi(\mathcal{A})$. Then $\mathcal{M}'_\varphi = \pi_\varphi(\mathcal{A})'$. Let C_φ be the (real) space of all functionals in $\mathcal{A}^*$ spanned by positive functionals dominated by φ. In other words,

$$C_\varphi = \operatorname{span}\{\psi \;:\; 0 \le \psi \le \varphi\}\,.$$

It is well known that there is a bijective positive map between C_φ and $\pi_\varphi(\mathcal{A})'$, sending each element $\psi \in C_\varphi$ to an operator $a'_\psi \in \mathcal{M}'_\varphi$ such that, for each $a \in \mathcal{A}$,

$$\psi(a) = \langle a'_\psi \pi_\varphi(a)\xi_\varphi, \xi_\varphi\rangle$$

(see e.g. [17, Proposition IV 3.10, p. 201]). Let $\mu \in M^+_\varphi(S(\mathcal{A}))$. Take $f \in L^\infty(S(\mathcal{A}), \mu)$. Then, according to the previous discussion, there is a unique element, $\theta_\mu(f) \in \mathcal{M}'_\varphi$ such that, for each $a \in \mathcal{A}$,

$$\langle \theta_\mu(f)\pi_\varphi(a)\xi_\varphi, \xi_\varphi\rangle = \int_{S(A)} f(\omega)a(\omega)\,d\mu(\omega)\,.$$

The map θ_μ is a unital weak* to weak* continuous map from $L^\infty(S(\mathcal{A}), \mu)$ into von Neumann algebra $\mathcal{M}'_\varphi$ (see e.g. [17, Proposition 6.18, p. 238]). The measure $\mu \in M^+_\varphi(S(\mathcal{A}))$ is called *orthogonal* if, for each Borel set $E \subset S(\mathcal{A})$, the positive functionals φ_E and φ_{E^c} on $\mathcal{A}$ given by

$$\varphi_E(a) = \int_E a(\omega)\,d\mu(\omega) \qquad \varphi_{S(\mathcal{A})\setminus E}(a) = \int_{S(\mathcal{A})\setminus E} a(\omega)\,d\mu(\omega)$$

are orthogonal. It is known that μ is an orthogonal measure if and only if θ_μ is a $*$-isomorphism that maps $L^\infty(S(\mathcal{A}), \mu)$ onto the von Neumann abelian subalgebra

$$C_\mu = \theta_\mu(L^\infty(S(\mathcal{A}), \mu))$$

of $\mathcal{M}'_\varphi$ (see [17, Theorem 6.19, p. 239]).

Let us denote by $O_\varphi(\mathcal{A})$ the set of all orthogonal measures having barycenter φ. Of course, $\delta_x \in O_\varphi(\mathcal{A})$. Let us denote by $O_\varphi^{fin}(\mathcal{A})$ the set of all finitely supported orthogonal measures in $O_\varphi(\mathcal{A})$.

As an example, let us look at $O_\varphi(M_2(\mathbb{C}))$. The state space of $M_2(\mathbb{C})$ can be identified with all matrices of the form

$$\frac{1}{2}\begin{pmatrix} 1+\beta_1 & \beta_2+i\beta_3 \\ \beta_2-i\beta_3 & 1-\beta_1 \end{pmatrix},$$

where $(\beta_1, \beta_2, \beta_3)$ is a point in the three dimensional unit ball. (See also example after Proposition 2.2.) In this way the state space is affine isomorphic to the unit three dimensional ball. For simplicity, let φ be the normalized trace. It corresponds to the origin. Any orthogonal measure on the state space that has barycenter φ is a convex combination of two Dirac measures concentrated at vector states (pure states) corresponding to orthogonal unit vectors in $\mathbb{C}^2$. When using the ball representation of the state space, we can see that these orthogonal states correspond to antipodal points on the unit sphere. The set $O_\varphi(M_2(\mathbb{C}))$ can be then viewed as a set of measures on the unit ball that are concentrated at two antipodal points on the unit sphere and that assign mass 1/2 to each of them.

Going back to general situation, we shall denote by Θ_φ the map

$$\Theta_\varphi : O_\varphi(\mathcal{A}) \to \mathcal{V}(\mathcal{M}'_\varphi) : \mu \to C_\mu .$$

One of the basic theorems we shall use in this note is the following Tomita theorem (see e.g. [17, Prop. 6.23, p. 241, Theorem 6.25 p.244]). It establishes a one-to-one correspondence between orthogonal measures and abelian subalgebras that preserves the Choquet order.

Theorem 5.1 (Tomita Theorem) *The map $\Theta_\varphi : \mu \to C_\mu$ is a bijection of $O_\varphi(\mathcal{A})$ onto $\mathcal{V}(\mathcal{M}'_\varphi)$. Moreover, the following conditions are equivalent for $\mu, \nu \in O_\varphi(\mathcal{A})$:*

(i) $\mu \prec \nu$
(ii) $C_\mu \subset C_\nu$.

In particular, the posets $(O_\varphi(\mathcal{A}), \prec)$ and $(\mathcal{V}(\mathcal{M}'_\varphi), \subset)$ are order isomorphic.

In [9] we showed that discrete version of Tomita's theorem holds for finitely supported measures as well.

Theorem 5.2 (Hamhalter and Turilova) *The map $\Theta_\varphi : \mu \to C_\mu$ is an order isomorphism of $O_\varphi^{fin}(\mathcal{A})$ onto $\mathcal{C}_{fin}(\mathcal{M}'_\varphi)$.*

The following theorem has been proved in [10, Theorem 6].

Theorem 5.3 (Hamhalter and Turilova) *Let φ and ψ be states on C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Let one of the following statements be true*

1. *$\mathcal{M}'_\varphi$ is a von Neumann algebra without type I_2 direct summand.*
2. *$\mathcal{M}_\varphi$ has no nonzero Type I direct summand.*

Then the following two statements hold:

(i) *For each order isomorphism $F : O_\varphi(\mathcal{A}) \to O_\psi(\mathcal{B})$ there is a unique Jordan *-isomorphism $J : \mathcal{M}'_\varphi \to \mathcal{M}'_\psi$ such that*

$$F(\mu) = \Theta_\psi^{-1} J[\Theta_\varphi(\mu)] .$$

for each $\mu \in O_\varphi(\mathcal{A})$.

(ii) *For each order isomorphism $F : O_\varphi^{fin}(\mathcal{A}) \to O_\psi^{fin}(\mathcal{B})$ there is a unique Jordan $*$-isomorphism $J : \mathcal{M}'_\varphi \to \mathcal{M}'_\psi$ such that*

$$F(\mu) = \Theta_\psi^{-1} J[\Theta_\varphi(\mu)]$$

for each $\mu \in O_\varphi^{fin}(\mathcal{A})$.

Proof Let us prove (i), the statement (ii) can be proved analogously. By Theorem 5.1 we know that order isomorphism F induces order isomorphism between $\mathcal{V}(\mathcal{M}'_\varphi)$ and $\mathcal{V}(\mathcal{M}'_\psi)$. Suppose that $\mathcal{M}'_\varphi$ has no Type I_2 direct summand. By employing Theorem 4.8, we see that this isomorphism is implemented by a Jordan *-isomorphism $J : \mathcal{M}'_\varphi \to \mathcal{M}'_\psi$. This shows the form of F. By Theorem 9.1.3 in [13] if a von Neumann algebra is of type I, then the same holds for its commutant. Therefore, if $\mathcal{M}_\varphi$ has no nonzero Type I direct summand, then $\mathcal{M}'_\varphi$ has no Type I_2 direct summand and we apply the previous reasoning.

□

The assumption on Type I_2 direct summand in the previous theorem is essential as the following example demonstrates.

Example Let $\mathcal{A} = M_2(\mathbb{C})$ and let φ be a faithful state on $\mathcal{A}$. Then there is a Choquet order isomorphism on $O_\varphi(\mathcal{A})$ that is not induced by any Jordan $*$-isomorphism on $\mathcal{M}'_\varphi$.

Proof We can write $\varphi = \lambda_1\omega_{e_1} + \lambda_2\omega_{e_2}$, where e_1, e_2 is a standard orthonormal basis of $\mathbb{C}^2$ and λ_1 and λ_2 are positive non-zero numbers with sum one. It can be verified easily that GNS data reads as follows:

$$H_\varphi = \mathbb{C}^2 \oplus \mathbb{C}^2 ,$$

$$\xi_\varphi = \lambda_1^{1/2} e_1 \oplus \lambda_2^{1/2} e_2 ,$$

and π_φ sends a 2 by 2 matrix a to block 4 by 4 matrix in the following way

$$\pi_\varphi(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

It is clear that $\pi_\varphi(\mathcal{A})'$ consists of all matrices of the form

$$\begin{pmatrix} \alpha I & \beta I \\ \gamma I & \delta I \end{pmatrix},$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and I is the identity 2 by 2 matrix. It is apparent that this algebra is isomorphic to $M_2(\mathbb{C})$. Using now Example 4.1 we can find an order isomorphism of $\mathcal{V}(\pi_\varphi(\mathcal{A})')$ that is not implemented by any Jordan $*$-isomorphism. □

So far the structure of measures with the Choquet order has been identified with the structure of abelian subalgebras of the commutant resulting in the GNS representation. However, in some important cases we can identify Choquet order structure directly with the poset of abelian subalgebras of a given algebra. Let us consider the following situation. Let φ be a faithful normal state on a von Neumann algebra $\mathcal{M}$. It is known that the GNS representation π_φ is a normal faithful representation. Therefore we can identify $\mathcal{M}$ with $\mathcal{M}_\varphi$ and suppose that $\mathcal{M}$ acting on a Hilbert space H has a biseparating vector ξ. According to deep Tomita-Takesaki modular theory of von Neumann algebras there is a *-antiisomorphism between $\mathcal{M}$ and $\mathcal{M}'$. This is the content of celebrated Tomita-Takesaki theorem.

Theorem 5.4 (Tomita-Takesaki Theorem) *Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space H and having a biseparating vector $\xi \in H$. Then there is a conjugate linear isometry J acting on H such that the map*

$$j(x) = Jx^*J \qquad x \in \mathcal{M}$$

*is an *-antiisomorphism between $\mathcal{M}$ and $\mathcal{M}'$.*

As a consequence of the previous theorem combined with the previous discussion we obtain that the posets $O_\varphi(\mathcal{M})$ and $\mathcal{V}(\mathcal{M})$ are isomorphic whenever φ is a faithful normal state on a von Neumann algebra $\mathcal{M}$. Based on this, we can show that Choquet order on representing measures, the barycenter of which is a faithful normal state, is a complete Jordan invariant for σ-finite algebras (see [8]).

Theorem 5.5 *Let $\mathcal{M}$ and $\mathcal{N}$ be (σ-finite) von Neumann algebras with faithful normal states φ and ψ, respectively. The following statements are equivalent.*

(i) *$O_\varphi(\mathcal{M})$ and $O_\psi(\mathcal{M})$ are isomorphic.*
(ii) *$O_\varphi^{fin}(\mathcal{M})$ and $O_\psi^{fin}(\mathcal{M})$ are isomorphic.*
(iii) *$\mathcal{M}$ and $\mathcal{N}$ are isomorphic as Jordan algebras.*

Proof As φ is faithful, the representation π_φ is a $*$-isomorphism. Therefore, $\mathcal{M}$ is $*$-isomorphic to $\pi_\varphi(\mathcal{M})$. Since the algebra $\pi_\varphi(\mathcal{M})$ has separating and generating vector, by Theorem 5.4 we have that $\pi_\varphi(\mathcal{M})'$ and $\pi_\varphi(\mathcal{M})$ are $*$-antiisomorphic and thereby Jordan $*$-isomorphic. Now by Theorem 4.11 and Theorem 5.1 we can see that conditions (i) and (ii) are equivalent to the fact that $\pi_\varphi(\mathcal{M})'$ and $\pi_\varphi(\mathcal{N})'$ are Jordan *-isomorphic. However, the previous reasoning tells us that this is equivalent to (iii). □

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Part V
Inequalities in Commutative and Noncommutative Probability Spaces

Mixed Norm Martingale Hardy Spaces and Applications in Fourier Analysis

Ferenc Weisz

Abstract We consider martingale Hardy spaces defined with the help of mixed $L_{\vec{p}}$-norm. Five mixed normed martingale Hardy spaces will be investigated: $H^s_{\vec{p}}$, $H^S_{\vec{p}}$, $H^M_{\vec{p}}$, $P_{\vec{p}}$ and $Q_{\vec{p}}$. We give two different generalizations of Doob's maximal inequality for mixed-norm $L_{\vec{p}}$ spaces. We prove also two versions of atomic decompositions. Several martingale inequalities and the generalization of the well-known Burkholder-Davis-Gundy inequality are also presented. The dual spaces of the mixed-norm martingale Hardy spaces are given as the mixed-norm $BMO_{\vec{r}}(\vec{\alpha})$ spaces. This implies the John-Nirenberg inequality $BMO_1(\vec{\alpha}) \sim BMO_{\vec{r}}(\vec{\alpha})$ for $1 < \vec{r} < \infty$. As an application in Fourier-analysis, we verify the boundedness of the Fejér maximal operator from $H_{\vec{p}}$ to $L_{\vec{p}}$, whenever $1/2 < \vec{p} < \infty$. As a consequence of the boundedness, we get some almost everywhere and norm convergence results.

Keywords Mixed Lebesgue spaces · Mixed normed martingale Hardy spaces · Atomic decomposition · Doob's inequality · Martingale inequalities · Burkholder-Davis-Gundy inequality · BMO spaces · John-Nirenberg inequality · Walsh system · Fejér means · Fejér maximal operator · Boundedness

1 Introduction

Since 1970, the theory of Hardy spaces has been developed very quickly (see e.g. Fefferman and Stein [19], Stein [81], Grafakos [34]). Fefferman [18] proved that the dual space of the Hardy space is equivalent to the space of functions of bounded mean oscillation (BMO). John and Nirenberg [55] obtained their famous inequality, i.e., that the BMO_p spaces are equivalent. One year later, Fefferman and Stein [19] characterized the dual space of H_p $(0 < p < 1)$ as a Lipschitz space. The most powerful technique in the theory of Hardy spaces, the so-called

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atomic decomposition was given in Coifman and Weiss [14, 15]. Recently several papers were published about the generalization of Hardy spaces. For example, Hardy spaces with variable exponents were considered in Nakai and Sawano [68], Yan, et al. [99], Jiao et al. [54], Liu et al. [63] and [64]. Moreover Musielak-Orlicz-Hardy spaces were studied in Yang et al. [100]. The mixed norm classical Hardy spaces have been developed in Cleanthous et al. [11] and intensively studied by Huang et al. in [41–44, 46].

Parallel, a similar theory was evolved for different types of martingale Hardy spaces H_p^s, H_p^S, H_p^M, P_p and Q_p (see e.g. Garsia [23], Long [65] and Weisz [87]). In the celebrated work of Burkholder and Gundy [6], it was proved that the L_p norms of the maximal function and the quadratic variation, that is the spaces H_p^M and H_p^S, are equivalent for $1 < p < \infty$. In the same year, Davis [16] extended this result for $p = 1$. For martingale Hardy spaces, Weisz [87] worked out the theory of atomic decomposition. Some boundedness results, duality theorems, martingale inequalities and interpolation results can be proved with the help of the atomic decomposition. A martingale analogue of H_1-BMO duality can be found in the books Garsia [23], Long [65] and Weisz [87]. For dyadic martingales, Herz [37] obtained the dual space of H_p $(0 < p < 1)$. In 1990, Weisz [86] characterized the dual space of H_p $(0 < p < 1)$ for general martingales via atomic decomposition. For a regular stochastic basis, the BMO_p spaces are equivalent in the martingale case, too. Recently, these results were extended to more general cases. Jiao et al. investigated martingale Hardy-Lorentz spaces in [51, 52] and variable martingale Hardy spaces in [49, 50, 53]. Martingale Musielak–Orlicz Hardy spaces were investigated in Xie et al. [96–98]. The theory of martingale Hardy spaces can be well applied in Fourier analysis (see Gát [25, 26], Goginava [30, 31] or Weisz [87, 90]).

The mixed Lebesgue spaces were introduced in 1961 by Benedek and Panzone [3] (see also Hörmander [40]). They considered the Descartes product $(\Omega, \mathcal{F}, \mathbb{P})$ of the probability spaces $(\Omega^i, \mathcal{F}^i, \mathbb{P}^i)$, where $\Omega = \prod_{i=1}^d \Omega^i$, $\mathcal{F}$ is generated by $\prod_{i=1}^d \mathcal{F}^i$ and $\mathbb{P}$ is generated by $\prod_{i=1}^d \mathbb{P}^i$. The mixed $L_{\vec{p}}$-norm of the measurable function f is defined as a number obtained after taking successively the L_{p_1}-norm of f in the variable x_1, the L_{p_2}-norm in the variable x_2, ..., the L_{p_d}-norm in the variable x_d. Some basic properties of the spaces $L_{\vec{p}}$ were proved in [3], such as the well known Hölder's inequality and the duality theorem. Mixed-norm Lebesgue and Hardy spaces were investigated in a great number of papers (e.g. in [1, 9–13, 27, 28, 35, 38, 39, 41–44, 46, 48, 56–60, 80]).

In this paper we will introduce five mixed normed martingale Hardy spaces: $H_{\vec{p}}^s$, $H_{\vec{p}}^S$, $H_{\vec{p}}^M$, $P_{\vec{p}}$ and $Q_{\vec{p}}$. In Sect. 3, Doob's inequality will be proved, that is, we will show that

$$\left\| \sup_{n\in\mathbb{N}} |\mathbb{E}_n f| \right\|_{\vec{p}} \le C \, \|f\|_{\vec{p}}$$

for all $f \in L_{\vec{p}}$, where $1 < \vec{p} < \infty$. We present also another version of Doob's inequality. In Sect. 5, we give the atomic decomposition for the five mixed normed martingale Hardy spaces. Using the atomic decomposition and Doob's inequality, several martingale inequalities will be proved in Sect. 6. We will show that, if the stochastic basis $(\mathcal{F}_n)$ is regular, then the five martingale Hardy spaces are equivalent. As a consequence of Doob's inequality, the generalization of the well-known Burkholder-Davis-Gundy inequality can be shown. In the next section, we prove that the dual of $H_{\vec{p}}^M$ is $BMO_2(\vec{\alpha})$ and, if the stochastic basis is regular, then the dual of $H_{\vec{p}}^M$ is $BMO_{\vec{r}}(\vec{\alpha})$, where $0 < \vec{p} \le 1$, $\vec{\alpha} = 1/\vec{p} - 1$ and $1 < \vec{r} < \infty$. Consequently, we obtain the generalization of the John and Nirenberg theorem for mixed normed martingale spaces: if $0 \le \vec{\alpha} < \infty$ and $1 < \vec{r} < \infty$, then $BMO_1(\vec{\alpha}) = BMO_{\vec{r}}(\vec{\alpha})$ with equivalent norms.

In the one-dimensional case, Paley [71] (see also Schipp, Wade, and Simon [75] and Weisz [90]) proved the L_p-norm convergence of the partial sums of the Walsh-Fourier series of f in case of $1 < p < \infty$. There is no convergence result for $p \le 1$ (see [33, 75]). Using summability methods, such as Fejér means, the L_1-norm convergence can be reached for functions in L_1, too. It was proved by Fine [21] that the Fejér means of the one-dimensional Walsh-Fourier series converges almost everywhere to the function if $f \in L_1$. Schipp [72] obtained the same result by proving the weak type inequality of the maximal operator σ_* of the Fejér means. By interpolation, this implies that σ_* is bounded on L_p $(1 < p < \infty)$. Next Fujii [22] extended this and showed that σ_* is bounded from the dyadic Hardy space H_1 to L_1 (see also Schipp and Simon [74]). Later the author (see [88]) generalized this further and proved that σ_* is bounded from H_p to L_p for $1/2 < p < \infty$. The boundedness does not hold for $0 < p \le 1/2$ (see [78]).

In the two-dimensional case, Weisz considered the Fejér maximal operator over a cone and he proved in [89] that σ_* is bounded from H_p to L_p for $1/2 < p < \infty$. Gát [24] and Weisz [89] proved that the Fejér means of the two-dimensional Walsh-Fourier series converge to the function almost everywhere if we consider the convergence over the diagonal, or more generally, over a cone. This result was proved for trigonometric Fourier series by Marcinkievicz and Zygmund [67] and Weisz [92]. Similar results were obtained in numerous other papers (see, e.g., Gát [25, 26] and Goginava [29–31]).

In this paper, we generalize the previous results for mixed normed martingale Hardy spaces. We will prove that the Fejér maximal operator defined over a cone is bounded from $H_{\vec{p}}$ to $L_{\vec{p}}$ $(1/2 < \vec{p} < \infty)$. As a consequence, we get some convergence results, such as almost everywhere and norm convergence of the multi-dimensional Fejér means defined over a cone. This result generalizes the well-known theorem of Gát [24] and Weisz [89]. Some summability results for classical mixed norm Hardy spaces $H_{\vec{p}}(\mathbb{R}^d)$ and for Fourier transforms can be found in [45, 93].

We denote by C a positive constant, which can vary from line to line, and denote by C_p a constant depending only on p. The symbol $A \sim B$ means that there exist constants $\alpha, \beta > 0$ such that $\alpha A \le B \le \beta A$ and $A \lesssim B$ means that there exist $C > 0$ such that $A \le CB$.

2 Mixed Lebesgue Spaces

For $1 \le d \in \mathbb{N}$ and $i = 1, \ldots, d$, let $(\Omega^i, \mathcal{F}^i, \mathbb{P}^i)$ be probability spaces and $\vec{p} := (p_1, \ldots, p_d)$ with $0 < p_i \le \infty$. Consider the product space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \prod_{i=1}^d \Omega^i$, the σ-algebra $\mathcal{F}$ is generated by $\prod_{i=1}^d \mathcal{F}^i$ and the probability measure $\mathbb{P}$ is generated by $\prod_{i=1}^d \mathbb{P}^i$. For a constant p, the L_p space is equipped with the quasi-norm

$$\|f\|_p := \left(\int_\Omega |f(\vec{x})|^p \, d\mathbb{P}(\vec{x}) \right)^{1/p} \qquad (0 < p < \infty),$$

with the usual modification for $p = \infty$, where $\vec{x} = (x_1, \ldots, x_d)$. We generalize this space as follows. A measurable function $f : \Omega \to \mathbb{R}$ belongs to the mixed $L_{\vec{p}}$ space if

$$\begin{aligned} \|f\|_{\vec{p}} &:= \Big\| \ldots \|f\|_{L_{p_1}(dx_1)} \cdots \Big\|_{L_{p_d}(dx_d)} \\ &= \left(\int_{\Omega_d} \cdots \left(\int_{\Omega_1} |f(x_1, \ldots, x_d)|^{p_1} \, d\mathbb{P}^1(x_1) \right)^{p_2/p_1} \ldots d\mathbb{P}^d(x_d) \right)^{1/p_d} \end{aligned}$$

is finite, with the usual modification if $p_j = \infty$ for some $j \in \{1, \ldots, d\}$. If for some $0 < p \le \infty$, $\vec{p} = (p, \ldots, p)$, then we get back the classical Lebesgue space L_p. Under $r < \vec{p} \le q$, we mean that for all $i = 1, \ldots, d$, $r < p_i \le q$, where $0 \le r < q \le \infty$. For a vector $\vec{p}$, we will use the notations

$$p_- := \min\{p_1, \ldots, p_d\}.$$

The conjugate exponent vector of $\vec{p}$ will be denoted by $(\vec{p})'$, that is, $(\vec{p})' = \left(p_1', \ldots, p_d'\right)$, where $1/p_i + 1/p_i' = 1$ $(i = 1, \ldots, d)$. For $\alpha > 0$, $\vec{p}/\alpha := (p_1/\alpha, \ldots, p_d/\alpha)$. Benedek and Panzone [3] proved the next two basic results for the mixed Lebesgue space.

Theorem 2.1 *If $1 \le \vec{p} \le \infty$, then for all $f \in L_{\vec{p}}$ and $g \in L_{(\vec{p})'}$,*

$$\int_\Omega |fg| \, d\mathbb{P} \le \|f\|_{\vec{p}} \, \|g\|_{(\vec{p})'}.$$

Moreover,

$$\|f\|_{\vec{p}} = \sup_{\|g\|_{(\vec{p})'} \le 1} \left| \int_\Omega fg \, d\mathbb{P} \right|.$$

Similarly to the Lebesgue spaces, the following result holds for the dual of $L_{\vec{p}}$.

Theorem 2.2 *If* $1 < \vec{p} < \infty$, *then*

$$\left(L_{\vec{p}}\right)^* = L_{(\vec{p})'}$$

with equivalent norms.

3 Doob's Inequality

Suppose that the σ-algebra $\mathcal{F}_n^i \subset \mathcal{F}^i$ $(n \in \mathbb{N},\ i = 1, \ldots, d)$, $(\mathcal{F}_n^i)_{n\in\mathbb{N}}$ is increasing and $\mathcal{F}^i = \sigma\left(\cup_{n\in\mathbb{N}}\mathcal{F}_n^i\right)$. Let $\mathcal{F}_n = \sigma\left(\prod_{i=1}^d \mathcal{F}_n^i\right)$. The expectation and conditional expectation operators relative to Ω, Ω^i, $\mathcal{F}_n$ and $\mathcal{F}_n^i$ are denoted by $\mathbb{E}$, $\mathbb{E}^i$, $\mathbb{E}_n$ and $\mathbb{E}_n^i$ $(i = 1, \ldots, d, n \in \mathbb{N})$, respectively. Obviously, $\mathbb{E}_n f = \mathbb{E}_n^1 \circ \ldots \circ \mathbb{E}_n^d f$. An integrable sequence $f = (f_n)_{n\in\mathbb{N}}$ is called a martingale if

(i) $(f_n)_{n\in\mathbb{N}}$ is adapted, that is for all $n \in \mathbb{N}$, f_n is $\mathcal{F}_n$-measurable;
(ii) $\mathbb{E}_n f_m = f_n$ in case $n \leq m$.

Definition 3.1 The stochastic basis $(\mathcal{F}_n)$ is said to be regular, if there exists $R > 0$ such that for all nonnegative martingales (f_n),

$$f_n \leq R f_{n-1}.$$

If for all $n \in \mathbb{N}$, $f_n \in L_{\vec{p}}$, then f is called an $L_{\vec{p}}$-martingale. Moreover, if

$$\|f\|_{\vec{p}} := \sup_{n\in\mathbb{N}} \|f_n\|_{\vec{p}} < \infty,$$

then f is an $L_{\vec{p}}$-bounded martingale, briefly $f \in L_{\vec{p}}$. We define the Doob's maximal function by

$$M(f) := \sup_{n\in\mathbb{N}} |f_n|.$$

Of course,

$$M(f) \leq M_d \circ M_{d-1} \circ \ldots \circ M_1(f),$$

where, for any $f \in L_1$ and $i = 1, \ldots, d$,

$$M_i(f) := \sup_{n\in\mathbb{N}} \left|\mathbb{E}_n^i f\right|.$$

Doob's inequality is well known:

Theorem 3.2 *If* $1 < p < \infty$ *and* $f \in L_p$, *then*

$$\|M(f)\|_p \leq C_p \|f\|_p.$$

There is also a weak type inequality for $p = 1$, however, we do not use it so we omit it. Theorem 3.2 can be found in Doob [17] and Burkholder and Gundy [5, 6] (see also Garsia [23], Long [65] or Weisz [87]) and for the classical Hardy-Littlewood maximal operator in Stein [81]. Now we generalize this inequality. To this end, we have to use the following result (proved in [84]), that is interesting in itself and is a crucial point in the proof of the Theorem 3.4.

Theorem 3.3 *Let* φ *be a positive function. Then for all* $1 < r < \infty$, *we have*

$$\int_\Omega |M(f)|^r \varphi \, d\mathbb{P} \leq C_r \int_\Omega |f|^r \, M(\varphi) \, d\mathbb{P}.$$

The first generalization of Doob's inequality is

Theorem 3.4 ([84]) *Suppose that* $1 < \vec{p} < \infty$ *or*

$$\vec{p} = (\infty, \infty, \ldots, \infty, p_{k+1}, \ldots, p_d), \quad 1 < p_{k+1}, \ldots, p_d < \infty \tag{1}$$

for some $k \in \{1, \ldots, d\}$. *Then, for all* $f \in L_{\vec{p}}$,

$$\|M_d(f)\|_{\vec{p}} \leq C\|f\|_{\vec{p}}.$$

For $1 < \vec{p} < \infty$ and for the classical Hardy-Littlewood inequality, Theorem 3.4 was shown in Bagby [2]. This theorem implies easily the next generalization of Doob's inequality, that is to say, the maximal operator M is bounded on $L_{\vec{p}}$ in case $1 < \vec{p} < \infty$ (see [84]).

Theorem 3.5 *Under the same conditions as in Theorem* 3.4, *for all* $f \in L_{\vec{p}}$,

$$\|M(f)\|_{\vec{p}} \leq C \, \|f\|_{\vec{p}} \, .$$

Proof Since $Mf \leq M_d \circ \cdots \circ M_1 f$, it follows from Theorem 3.4 that

$$\begin{aligned} \|M(f)\|_{\vec{p}} &\leq \|M_d \circ M_{d-1} \circ \ldots \circ M_1 f\|_{\vec{p}} \leq C \, \|M_{d-1} \circ \ldots \circ M_1 f\|_{\vec{p}} \\ &\leq C \, \|M_{d-2} \circ \ldots \circ M_1 f\|_{\vec{p}} \leq \cdots \leq C \, \|f\|_{\vec{p}} \end{aligned}$$

and the proof is complete. □

If $\vec{p}$ is a constant, then we get back Theorem 3.2. Note that this theorem is not true for all $1 < \vec{p} \leq \infty$ (see [84]). The counterexample in [84] proves also that M_2 is not bounded on $L_{(p_1,\infty)}$ $(1 < p_1 < \infty)$. Moreover, the classical Hardy-Littlewood maximal operator considered in Huang et al. [41] is not bounded on

$L_{(p_1,\infty)}$ (cf. Lemma 3.5 in [41] and Lemma 4.8 in [69]). A weighted version of Doob's inequality can be found in Chen et al. [9].

We can easily modify the definition of the maximal operator. For a constant q and $f \in L_q$, let

$$M_q(f) := \sup_{n\in\mathbb{N}} \left(\mathbb{E}_n(|f|^q)\right)^{1/q}.$$

The next result immediately follows from Theorem 3.2.

Theorem 3.6 *If* $0 < q < p < \infty$ *and* $f \in L_p$*, then*

$$\|M_q(f)\|_p \le C_p \|f\|_p.$$

The generalization to mixed norm spaces is much more complicated. Let us introduce the new maximal function

$$M_{\vec{q}}(f) := \sup_{n\in\mathbb{N}} \left(\mathbb{E}_n^d \left(\mathbb{E}_n^{d-1} \cdots \left(\mathbb{E}_n^2 \left(\mathbb{E}_n^1 |f|^{q_1}\right)^{\frac{q_2}{q_1}}\right)^{\frac{q_3}{q_2}} \cdots\right)^{\frac{q_d}{q_{d-1}}}\right)^{\frac{1}{q_d}},$$

where $0 < \vec{q} < \infty$. Now, we can show that under some conditions, this operator is bounded on $L_{\vec{p}}$, too.

Theorem 3.7 ([94]) *Let* $0 < \vec{q} < \infty$ *and* $0 < \vec{p} < \infty$ *or*

$$\vec{p} = (\infty, \infty, \ldots, \infty, p_{k+1}, \ldots, p_d), \quad 0 < p_{k+1}, \ldots, p_d < \infty$$

for some $k \in \{1, \ldots, d\}$. *Suppose that*

$$\begin{cases} p_1 > q_1, q_2, \ldots, q_d, \\ p_2 > q_2, \ldots, q_d, \\ \cdots \\ p_d > q_d. \end{cases}$$

Then, for all $f \in L_{\vec{p}}$,

$$\|M_{\vec{q}}(f)\|_{\vec{p}} \le C\|f\|_{\vec{p}}.$$

This theorem was proved in [45] for the classical maximal function (for a part of this theorem see also [70]).

4 Mixed Martingale Hardy Spaces

For $n \in \mathbb{N}$ and a martingale $f = (f_n)_{n\in\mathbb{N}}$, the martingale differences are defined by

$$d_n f := f_n - f_{n-1}, \qquad f_0 := f_{-1} := 0.$$

The map $\nu : \Omega \to \mathbb{N} \cup \{\infty\}$ is called a stopping time relative to $(\mathcal{F}_n)$ if for all $n \in \mathbb{N}$, $\{\nu = n\} \in \mathcal{F}_n$. For a martingale $f = (f_n)$ and a stopping time ν, the stopped martingale is defined by

$$f_n^\nu = \sum_{m=0}^{n} d_m f \, \chi_{\{\nu \geq m\}}.$$

Let us define the quadratic variation and the conditional quadratic variation of a martingale f relative to $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_n)_{n\in\mathbb{N}})$ by

$$S_m(f) := \left(\sum_{n=0}^{m} |d_n f|^2\right)^{1/2}, \quad S(f) := \left(\sum_{n=0}^{\infty} |d_n f|^2\right)^{1/2}$$

$$s_m(f) := \left(\sum_{n=0}^{m} \mathbb{E}_{n-1} |d_n f|^2\right)^{1/2}, \quad s(f) := \left(\sum_{n=0}^{\infty} \mathbb{E}_{n-1} |d_n f|^2\right)^{1/2}.$$

The set of the sequences $(\lambda_n)_{n\in\mathbb{N}}$ of non-decreasing, non-negative and adapted functions with $\lambda_\infty := \lim_{n\to\infty} \lambda_n$ is denoted by Λ. With the help of the previous operators, we introduce five mixed normed martingale Hardy as follows:

$$H_{\vec{p}}^M := \left\{ f = (f_n)_{n\in\mathbb{N}} : \|f\|_{H_{\vec{p}}^M} := \|M(f)\|_{\vec{p}} < \infty \right\};$$

$$H_{\vec{p}}^S := \left\{ f = (f_n)_{n\in\mathbb{N}} : \|f\|_{H_{\vec{p}}^S} := \|S(f)\|_{\vec{p}} < \infty \right\};$$

$$H_{\vec{p}}^s := \left\{ f = (f_n)_{n\in\mathbb{N}} : \|f\|_{H_{\vec{p}}^s} := \|s(f)\|_{\vec{p}} < \infty \right\};$$

$$Q_{\vec{p}} := \Big\{ f = (f_n)_{n\in\mathbb{N}} : \exists\, (\lambda_n)_{n\in\mathbb{N}} \in \Lambda, \text{ such that } S_n(f) \leq \lambda_{n-1},\ \lambda_\infty \in L_{\vec{p}} \Big\},$$

$$P_{\vec{p}} := \Big\{ f = (f_n)_{n\in\mathbb{N}} : \exists\, (\lambda_n)_{n\in\mathbb{N}} \in \Lambda, \text{ such that } |f_n| \leq \lambda_{n-1},\ \lambda_\infty \in L_{\vec{p}} \Big\}.$$

Define

$$\|f\|_{Q_{\vec{p}}} := \inf_{(\lambda_n)\in\Lambda} \|\lambda_\infty\|_{\vec{p}}, \qquad \|f\|_{P_{\vec{p}}} := \inf_{(\lambda_n)\in\Lambda} \|\lambda_\infty\|_{\vec{p}}.$$

For a constant p, we get back the well known martingale Hardy spaces H_p^M, H_p^S, H_p^s, Q_p and P_p investigated exhaustively in [87].

The following corollary comes from Theorem 3.5. It is well-known for martingale Hardy spaces with $\vec{p} = (p, \ldots, p)$ (see e.g. [87]).

Corollary 4.1 *If* $1 < \vec{p} < \infty$ *or* $\vec{p}$ *satisfies* (1)*, then* $H_{\vec{p}}^M$ *is equivalent to* $L_{\vec{p}}$.

5 Atomic Decomposition

In this section, we consider two atomic characterizations of mixed Hardy spaces. The atomic decomposition is a useful characterization of the Hardy spaces by the help of which some inequalities, duality theorems and boundedness results can be proved. The next atomic decomposition of martingale Hardy spaces with constant p was proved by Herz [36] and the author [87]. For classical Hardy spaces see Latter [61], Lu [66], Stein [81] and Weisz [92].

Definition 5.1 A measurable function a is called an $(s, \vec{p}, \infty)$-atom if there exists a stopping time τ such that

(i) $\mathbb{E}_n a = 0$ for all $n \leq \tau$,
(ii) $\left\| s(a)\chi_{\{\tau<\infty\}} \right\|_\infty \leq \frac{1}{\|\chi_{\{\tau<\infty\}}\|_{\vec{p}}}$.

If $s(a)$ in (ii) is replaced by $S(a)$ (resp. $M(a)$), then the function a is called $(S, \vec{p}, \infty)$-atom (resp. $(M, \vec{p}, \infty)$-atom).

If p is a constant, then (ii) reads as follows:

$$\left\| s(a)\chi_{\{\tau<\infty\}} \right\|_\infty \leq \mathbb{P}(\tau < \infty)^{-1/p}.$$

Every function from the Hardy space H_p^s $(0 < p \leq 1)$ can be decomposed into the sum of atoms.

Theorem 5.2 ([87]) *Let p be a constant with $0 < p \leq 1$. A martingale $f = (f_n)_{n\in\mathbb{N}} \in H_p^s$ if and only if there exist a sequence $(a^k)_{k\in\mathbb{Z}}$ of (s, p, ∞)-atoms and a sequence $(\mu_k)_{k\in\mathbb{Z}}$ of real numbers such that*

$$f_n = \sum_{k\in\mathbb{Z}} \mu_k \mathbb{E}_n a^k \qquad a.\ e. \quad (n \in \mathbb{N})$$

and

$$\|f\|_{H_p^s} \sim \inf \left(\sum_{k\in\mathbb{N}} \mu_k^p \right)^{1/p}, \tag{2}$$

where the infimum is taken over all decompositions of f as above.

In the present form the theorem does not hold for $1 < p < \infty$ and it cannot be extended to mixed norm Hardy spaces. It is easy to see that for $0 < p \le 1$, (2) can be written as

$$\|f\|_{H_p^s} \sim \inf \left(\sum_{k\in\mathbb{N}} \mu_k^p \right)^{1/p} = \inf \left\| \left(\sum_{k\in\mathbb{N}} \left(\frac{\mu_k \chi_{\{\tau_k<\infty\}}}{\|\chi_{\{\tau_k<\infty\}}\|_p} \right)^p \right)^{1/p} \right\|_p .$$

Writing the $\vec{p}$-norm instead of the p-norm, we can generalize this form of the atomic decomposition to mixed norm Hardy spaces (see [84]).

Theorem 5.3 *Let $0 < \vec{p} < \infty$. A martingale $f = (f_n)_{n\in\mathbb{N}} \in H_{\vec{p}}^s$ if and only if there exist a sequence $(a^k)_{k\in\mathbb{Z}}$ of $(s, \vec{p}, \infty)$-atoms and a sequence $(\mu_k)_{k\in\mathbb{Z}}$ of real numbers such that*

$$f_n = \sum_{k\in\mathbb{Z}} \mu_k \mathbb{E}_n a^k \qquad a.\ e. \quad (n \in \mathbb{N}) \tag{3}$$

and

$$\|f\|_{H_{\vec{p}}^s} \sim \inf \left\| \left(\sum_{k\in\mathbb{Z}} \left(\frac{\mu_k \chi_{\{\tau_k<\infty\}}}{\left\|\chi_{\{\tau_k<\infty\}}\right\|_{\vec{p}}} \right)^t \right)^{1/t} \right\|_{\vec{p}} ,$$

where $0 < t \le \min\{p_-, 1\}$ and the infimum is taken over all decompositions of the form (3).

If we replace the space $H_{\vec{p}}^s$ by $P_{\vec{p}}$ (resp. by $Q_{\vec{p}}$) and the $(s, \vec{p}, \infty)$-atoms by $(M, \vec{p}, \infty)$-atoms (resp. by $(S, \vec{p}, \infty)$-atoms), then the theorem holds, too.

If the stochastic basis $(\mathcal{F}_n)$ is regular and $0 < t < \min\{p_-, 1\}$, then the same holds for the space $H_{\vec{p}}^M$ as for $P_{\vec{p}}$ and the same for $H_{\vec{p}}^S$ as for $Q_{\vec{p}}$.

Proof We will sketch the proof for $H_{\vec{p}}^s$, only. Assume that $f \in H_{\vec{p}}^s$ and let us define the following stopping times:

$$\tau_k := \inf \left\{ n \in \mathbb{N} : s_{n+1}(f) > 2^k \right\}.$$

Obviously f_n can be written in the form

$$f_n = \sum_{k\in\mathbb{Z}} \left(f_n^{\tau_{k+1}} - f_n^{\tau_k} \right) = \sum_{k\in\mathbb{Z}} \mu_k a_n^k,$$

where

$$\mu_k := 3 \cdot 2^k \left\| \chi_{\{\tau_k<\infty\}} \right\|_{\vec{p}} \quad \text{and} \quad a_n^k := \frac{f_n^{\tau_{k+1}} - f_n^{\tau_k}}{\mu_k}.$$

Moreover, there exists $a^k \in L_2$ such that $\mathbb{E}_n a^k = a_n^k$. Because of $s\left(f^{\tau_k}\right) = s_{\tau_k}(f) \leq 2^k$, we have that

$$s\left(a^k\right) \leq \frac{s\left(f^{\tau_{k+1}}\right) + s\left(f^{\tau_k}\right)}{\mu_k} \leq \left\|\chi_{\{\tau_k<\infty\}}\right\|_{\vec{p}}^{-1},$$

thus a^k is an $(s, \vec{p}, \infty)$-atom.

Since

$$\lim_{k\to\infty} s\left(f - f^{\tau_k}\right) = \lim_{k\to-\infty} s\left(f^{\tau_k}\right) = 0$$

almost everywhere, by the dominated convergence theorem (see e.g. [3]) we get that

$$\left\|f - \sum_{k=-l}^{m} \mu_k a^k\right\|_{H_{\vec{p}}^s} \leq \left\|f - f^{\tau_{m+1}}\right\|_{H_{\vec{p}}^s} + \left\|f^{\tau_{-l}}\right\|_{H_{\vec{p}}^s} \to 0$$

as $l, m \to \infty$. From this it follows that

$$f = \sum_{k\in\mathbb{Z}} \mu_k a^k \qquad \text{in the } H_{\vec{p}}^s\text{-norm.}$$

Denote by

$$\mathcal{O}_k := \{\tau_k < \infty\} = \left\{s(f) > 2^k\right\}.$$

Then for all $k \in \mathbb{Z}$, $\mathcal{O}_{k+1} \subset \mathcal{O}_k$. Moreover, for all $x \in \Omega$ and for all $0 < t \leq 1$,

$$\sum_{k\in\mathbb{Z}} \left(3 \cdot 2^k \chi_{\mathcal{O}_k}(x)\right)^t \leq C \left(\sum_{k\in\mathbb{Z}} 3 \cdot 2^k \chi_{\mathcal{O}_k\setminus\mathcal{O}_{k+1}}(x)\right)^t.$$

Since the sets $\mathcal{O}_k \setminus \mathcal{O}_{k+1}$ are disjoint, we have

$$\begin{aligned}\left\|\left(\sum_{k\in\mathbb{Z}}\left(\frac{\mu_k \chi_{\{\tau_k<\infty\}}}{\left\|\chi_{\{\tau_k<\infty\}}\right\|_{\vec{p}}}\right)^t\right)^{1/t}\right\|_{\vec{p}} &= \left\|\left(\sum_{k\in\mathbb{Z}}\left(3 \cdot 2^k \chi_{\{\tau_k<\infty\}}\right)^t\right)^{1/t}\right\|_{\vec{p}} \\ &\leq C \left\|\sum_{k\in\mathbb{Z}} 3 \cdot 2^k \chi_{\mathcal{O}_k\setminus\mathcal{O}_{k+1}}\right\|_{\vec{p}}\end{aligned}$$

$$\leq C \left\| \sum_{k\in\mathbb{Z}} s(f)\chi_{\mathcal{O}_k\setminus\mathcal{O}_{k+1}} \right\|_{\vec{p}}$$
$$= C \left\| s(f) \right\|_{\vec{p}} .$$

Conversely, if f has a decomposition of the form (3), then

$$s(f) \leq \sum_{k\in\mathbb{Z}} \mu_k s(a^k) \leq \sum_{k\in\mathbb{Z}} \mu_k \frac{\chi_{\{\tau_k<\infty\}}}{\left\|\chi_{\{\tau_k<\infty\}}\right\|_{\vec{p}}},$$

and so for all $0 < t \leq 1$,

$$\|f\|_{H^s_{\vec{p}}} \leq \left\| \sum_{k\in\mathbb{Z}} \mu_k \frac{\chi_{\{\tau_k<\infty\}}}{\left\|\chi_{\{\tau_k<\infty\}}\right\|_{\vec{p}}} \right\|_{\vec{p}} \leq \left\| \left(\sum_{k\in\mathbb{Z}} \left(\frac{\mu_k \chi_{\{\tau_k<\infty\}}}{\left\|\chi_{\{\tau_k<\infty\}}\right\|_{\vec{p}}} \right)^t \right)^{1/t} \right\|_{\vec{p}},$$

which proves the theorem. □

It follows from the proof of this theorem that

$$\|f\|_{H^s_{\vec{p}}} \sim \inf \left\| \left(\sum_{k\in\mathbb{Z}} \left(3 \cdot 2^k\right)^t \chi_{\{\tau_k<\infty\}} \right)^{1/t} \right\|_{\vec{p}}, \tag{4}$$

where the infimum is taken over all atomic decompositions of the form (3). There are also corresponding equivalences for the other Hardy spaces. From Theorem 5.3, we get immediately the next corollary.

Corollary 5.4 *If the stochastic basis $(\mathcal{F}_n)$ is regular, then*

$$H^S_{\vec{p}} = Q_{\vec{p}} \quad \textit{and} \quad H^M_{\vec{p}} = P_{\vec{p}} \quad (0 < \vec{p} < \infty)$$

with equivalent quasi-norms.

For the duality results in Sect. 7, we need a finer atomic decomposition. For this, we assume that every σ-algebra $(\mathcal{F}_n)_n$ is generated by countably many atoms. We denote by $A(\mathcal{F}_n)$ the set of all atoms in $\mathcal{F}_n$. We introduce the concept of simple atoms.

Definition 5.5 Suppose that every σ-algebra $(\mathcal{F}_n)_n$ is generated by countably many atoms and $1 < \vec{r} \leq \infty$. A measurable function a is called a simple $(s, \vec{p}, \vec{r})$-atom if there exist $j \in \mathbb{N}$, $I \in A(\mathcal{F}_j)$ such that

(i) the support of a is contained in I,
(ii) $\|s(a)\|_{\vec{r}} \leq \frac{\|\chi_I\|_{\vec{r}}}{\|\chi_I\|_{\vec{p}}}$,
(iii) $\mathbb{E}_j(a) = 0$.

If $s(a)$ in (ii) is replaced by $S(a)$ (resp. $M(a)$), then the function a is called simple $(S, \vec{p}, \vec{r})$-atom (resp. simple $(M, \vec{p}, \vec{r})$-atom).

The atomic decomposition via simple $(s, \vec{p}, \vec{r})$-atoms are more complicated than the atomic decomposition via $(s, \vec{p}, \infty)$-atoms. To this, we need the condition that every σ-algebra is generated by countably many atoms.

Theorem 5.6 ([94]) *Suppose that every σ-algebra $(\mathcal{F}_n)_n$ is generated by countably many atoms. Let $1 < \vec{r} \leq \infty$ and*

$$\begin{cases} p_1 < r_1, r_2, \ldots, r_d, \\ p_2 < \quad r_2, \ldots, r_d, \\ \cdots \\ p_d < \qquad\qquad r_d. \end{cases}$$

A martingale $f = (f_n)_{n \in \mathbb{N}} \in H^s_{\vec{p}}$ if and only if there exist a sequence $(a^{k,j,i})_{k,j,i}$ of simple $(s, \vec{p}, \vec{r})$-atoms associated with $(I_{k,j,i})_{k,j,i} \subset A(\mathcal{F}_j)$, which are disjoint for fixed k, and a sequence $(\mu_{k,j,i})_{k \in \mathbb{Z}, j \in \mathbb{N}, i}$ of positive real numbers such that

$$f_n = \sum_{k \in \mathbb{Z}} \sum_{j=0}^{\infty} \sum_i \mu_{k,j,i} \mathbb{E}_n a^{k,j,i} \qquad a.e. \quad (n \in \mathbb{N}) \tag{5}$$

and

$$\|f\|_{H^s_{\vec{p}}} \sim \inf \left\| \left(\sum_{k \in \mathbb{Z}} \left(\sum_{j=0}^{\infty} \sum_i \frac{\mu_{k,j,i} \chi_{I_{k,j,i}}}{\|\chi_{I_{k,j,i}}\|_{\vec{p}}} \right)^t \right)^{1/t} \right\|_{\vec{p}},$$

where $0 < t < \min\{p_-, 1\}$ and the infimum is taken over all decompositions of the form (5).

If we replace the space $H^s_{\vec{p}}$ by $P_{\vec{p}}$ (resp. by $Q_{\vec{p}}$) and the simple $(s, \vec{p}, \vec{r})$-atoms by simple $(M, \vec{p}, \vec{r})$-atoms (resp. by simple $(S, \vec{p}, \vec{r})$-atoms), then the theorem holds, too.

If the stochastic basis $(\mathcal{F}_n)$ is regular, then the same holds for the space $H^M_{\vec{p}}$ as for $P_{\vec{p}}$ and the same for $H^S_{\vec{p}}$ as for $Q_{\vec{p}}$.

Proof We sketch the proof, only. Besides Theorem 3.7, the basic idea of the proof is to decompose the sets $\{\tau_k = j\}$ into the union of atoms $(I_{k,j,i})_i \subset \mathcal{F}_j$ such that

$$\bigcup_i I_{k,j,i} = \{\tau_k = j\} \in \mathcal{F}_j,$$

where the stopping times τ_k were defined in the proof of Theorem 5.3. Note that for fixed k, j, the atoms $(I_{k,j,i})_i \subset \mathcal{F}_j$ are disjoint. We can show that

$$f_n = \sum_{k\in\mathbb{Z}} (f_n^{\tau_{k+1}} - f_n^{\tau_k}) = \sum_{k\in\mathbb{Z}} \sum_{j=0}^{n-1} \sum_i \chi_{I_{k,j,i}} (f_n^{\tau_{k+1}} - f_n^{\tau_k})$$
$$= \sum_{k\in\mathbb{Z}} \sum_{j=0}^{n-1} \sum_i \mu_{k,j,i} a_n^{k,j,i},$$

where

$$\mu_{k,j,i} = 3 \cdot 2^k \left\| \chi_{I_{k,j,i}} \right\|_{\vec{p}} \qquad \text{and} \qquad a_n^{k,j,i} = \chi_{I_{k,j,i}} \frac{f_n^{\tau_{k+1}} - f_n^{\tau_k}}{\mu_{k,j,i}}.$$

□

Similarly to (4), we obtain that

$$\|f\|_{H_{\vec{p}}^s} \sim \inf \left\| \left(\sum_{k\in\mathbb{Z}} \left(3 \cdot 2^k\right)^t \sum_{j=0}^{\infty} \sum_i \chi_{I_{k,j,i}} \right)^{1/t} \right\|_{\vec{p}}, \tag{6}$$

where the infimum is taken over all atomic decompositions of the form (5). The corresponding equivalences for the other Hardy spaces hold, too.

6 Martingale Inequalities

In this section, we present the generalization of some classical martingale inequalities (see e.g., Weisz [87]) for the five mixed normed martingale Hardy spaces. To this end, we need the following definition and boundedness results.

Let X be a martingale space, Y be a measurable function space. Then the operator $U : X \to Y$ is called σ-sublinear operator if for any $\alpha \in \mathbb{C}$,

$$\left| U \left(\sum_{k=1}^{\infty} f_k \right) \right| \le \sum_{k=1}^{\infty} |U(f_k)| \quad \text{and} \quad |U(\alpha f)| = |\alpha||U(f)|.$$

The σ-algebra generated by the stopping time τ is denoted by

$$\mathcal{F}_\tau = \{F \in \mathcal{F} : F \cap \{\tau \le n\} \in \mathcal{F}_n, \quad n \ge 1\}.$$

Of course, $\mathcal{F}_\tau$ is a sub-σ-algebra of $\mathcal{F}$. The conditional expectation with respect to $\mathcal{F}_\tau$ is denoted by $\mathbb{E}_\tau$.

Theorem 6.1 ([84]) *Let* $0 < \vec{p} < \infty$ *and suppose that the* σ*-sublinear operator* $T : H_r^s \to L_r$ *is bounded, where* $\vec{p} = (p_1, \ldots, p_d)$ *and* $r > p_i$ $(i = 1, \ldots, d)$*. If for all* $(s, \vec{p}, \infty)$*-atom a*

$$(Ta)\chi_A = T(a\chi_A) \quad (A \in \mathcal{F}_\tau), \tag{7}$$

where τ *is the stopping time associated with the* $(s, \vec{p}, \infty)$*-atom a, then for all* $f \in H_{\vec{p}}^s$,

$$\|Tf\|_{\vec{p}} \le C \|f\|_{H_{\vec{p}}^s}.$$

If we replace the spaces H_r^s *and* $H_{\vec{p}}^s$ *by* H_r^M *and* $P_{\vec{p}}$ *(resp. by* H_r^S *and* $Q_{\vec{p}}$*) and the* $(s, \vec{p}, \infty)$*-atoms by* $(M, \vec{p}, \infty)$*-atoms (resp. by* $(S, \vec{p}, \infty)$*-atoms), then the theorem holds, too.*

It is easy to see that for all $(s, \vec{p}, \infty)$-atoms a, $(S, \vec{p}, \infty)$-atoms a or $(M, \vec{p}, \infty)$-atoms a and $A \in \mathcal{F}_\tau$, $s(a\chi_A) = s(a)\chi_A$, $S(a\chi_A) = S(a)\chi_A$ and $M(a\chi_A) = M(a)\chi_A$. This means that the operators s, S and M satisfy condition (7). Applying the preceding theorem to these operators, we [84] obtain

Theorem 6.2 *We have the following martingale inequalities:*

(i)
$$\|f\|_{H_{\vec{p}}^M} \le C \|f\|_{H_{\vec{p}}^s}, \quad \|f\|_{H_{\vec{p}}^S} \le C \|f\|_{H_{\vec{p}}^s} \quad (0 < \vec{p} < 2).$$

(ii)
$$\|f\|_{H_{\vec{p}}^M} \le \|f\|_{P_{\vec{p}}}, \quad \|f\|_{H_{\vec{p}}^S} \le \|f\|_{Q_{\vec{p}}} \quad (0 < \vec{p} < \infty).$$

(iii)
$$\|f\|_{H_{\vec{p}}^S} \le C \|f\|_{P_{\vec{p}}}, \quad \|f\|_{H_{\vec{p}}^M} \le C \|f\|_{Q_{\vec{p}}} \quad (0 < \vec{p} < \infty).$$

(iv)
$$\|f\|_{P_{\vec{p}}} \le C \|f\|_{Q_{\vec{p}}}, \quad \|f\|_{Q_{\vec{p}}} \le C \|f\|_{P_{\vec{p}}} \quad (0 < \vec{p} < \infty).$$

(v)
$$\|f\|_{H_{\vec{p}}^s} \le C \|f\|_{P_{\vec{p}}} \quad \textit{and} \quad \|f\|_{H_{\vec{p}}^s} \le C \|f\|_{Q_{\vec{p}}} \quad (0 < \vec{p} < \infty).$$

Proof Let $f \in H_{\vec{p}}^s$. The σ-sublinear operator M is bounded from H_2^s to L_2 (see e.g. Weisz [87]), that is $\|Mf\|_2 \le C\,\|f\|_{H_2^s}$. So we can apply Theorem 6.1 with the choice $r = 2$ and $\vec{p} := (p_1, \ldots, p_d)$, where $p_i < 2$ and we get that

$$\|f\|_{H_{\vec{p}}^M} = \|M(f)\|_{\vec{p}} \le C\,\|f\|_{H_{\vec{p}}^s} \quad (0 < \vec{p} < 2)\,.$$

The operator S is also bounded from H_2^s to L_2 (see [87]), hence using Theorem 6.1 we obtain

$$\|f\|_{H_{\vec{p}}^S} \le C\,\|f\|_{H_{\vec{p}}^s} \quad (0 < \vec{p} < 2)\,.$$

From the definition of the Hardy spaces it follows immediately that

$$\|f\|_{H_{\vec{p}}^M} \le \|f\|_{P_{\vec{p}}}\,, \quad \|f\|_{H_{\vec{p}}^S} \le \|f\|_{Q_{\vec{p}}} \quad (0 < \vec{p} < \infty)\,.$$

By Burkholder-Gundy and Doob's inequality, for all $1 < r < \infty$, $\|S(f)\|_r \approx \|M(f)\|_r \approx \|f\|_r$ (see Weisz [87]). Using this, the previous inequality and Theorem 6.1, we have

$$\|f\|_{H_{\vec{p}}^S} \le C\,\|f\|_{P_{\vec{p}}} \quad \text{and} \quad \|f\|_{H_{\vec{p}}^M} \le C\,\|f\|_{Q_{\vec{p}}} \quad (0 < \vec{p} < \infty)\,.$$

For $f = (f_n)_{n\in\mathbb{N}} \in Q_{\vec{p}}$ there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ for which $S_n(f) \le \lambda_{n-1}$ and $\lambda_\infty \in L_{\vec{p}}$. Using the inequality $|f_n| \le M_{n-1}(f) + \lambda_{n-1}$ and the preceding inequality, we get that

$$\|f\|_{P_{\vec{p}}} \le \|M(f)\|_{\vec{p}} + \|\lambda_\infty\|_{\vec{p}} \le \|f\|_{H_{\vec{p}}^M} + C\,\|f\|_{Q_{\vec{p}}} \le C\,\|f\|_{Q_{\vec{p}}}\,.$$

Similarly, if $f = (f_n)_{n\in\mathbb{N}} \in P_{\vec{p}}$, then $|f_n| \le \lambda_{n-1}$ with a suitable sequence $(\lambda_n)_{n\in\mathbb{N}}$ for which $\lambda_\infty \in L_{\vec{p}}$. Since

$$S_n(f) = \left(\sum_{k=0}^{n} |d_k f|^2\right)^{1/2} \le S_{n-1}(f) + |d_n f| \le S_{n-1}(f) + 2\lambda_{n-1},$$

we have that

$$\|f\|_{Q_{\vec{p}}} \le \|S(f)\|_{\vec{p}} + 2\,\|\lambda_\infty\|_{\vec{p}} = \|f\|_{H_{\vec{p}}^S} + 2\,\|f\|_{P_{\vec{p}}} \le C\,\|f\|_{P_{\vec{p}}}$$

for all $0 < \vec{p} < \infty$.

From [87] Proposition 2.11 (ii), we get that the operator s is bounded from H_r^M to L_r and from H_r^S to L_r if $2 \le r < \infty$. Again, using Theorem 6.1, we obtain

$$\|f\|_{H_{\vec{p}}^s} \le C\,\|f\|_{P_{\vec{p}}} \quad \text{and} \quad \|f\|_{H_{\vec{p}}^s} \le C\,\|f\|_{Q_{\vec{p}}} \quad (0 < \vec{p} < \infty)\,.$$

□

We know (see e.g. Weisz [87]) that $S_n(f) \leq R^{1/2} s_n(f)$ if the stochastic basis is regular. Using the definition of $Q_{\vec{p}}$ and the fact that $s_n(f) \in \mathcal{F}_{n-1}$ we get

$$\|f\|_{Q_{\vec{p}}} \leq C\|s(f)\|_{\vec{p}} = C\|f\|_{H^s_{\vec{p}}}.$$

By the last inequality of Theorem 6.2, we obtain that $Q_{\vec{p}} = H^s_{\vec{p}}$. The next corollary follows from Theorem 6.2 and Corollary 5.4 (see [84]).

Corollary 6.3 *If the stochastic basis $(\mathcal{F}_n)$ is regular, then the five Hardy spaces are equivalent, that is*

$$H^S_{\vec{p}} = Q_{\vec{p}} = P_{\vec{p}} = H^M_{\vec{p}} = H^s_{\vec{p}} \qquad (0 < \vec{p} < \infty)$$

with equivalent quasi-norms.

Using Theorems 2.1 and 3.5 and a duality argument, we can prove

Theorem 6.4 ([84]) *Suppose that $1 < \vec{p} < \infty$ or*

$$\vec{p} = (1, \ldots, 1, p_{k+1}, \ldots, p_d), \quad 1 < p_{k+1}, \ldots, p_d < \infty \tag{8}$$

for some $k \in \{1, \ldots, d\}$. Then for all non-negative, measurable function sequence $(f_n)_{n\in\mathbb{N}}$,

$$\left\|\sum_{n\in\mathbb{N}} \mathbb{E}_n (f_n)\right\|_{\vec{p}} \leq C \left\|\sum_{n\in\mathbb{N}} f_n\right\|_{\vec{p}}.$$

As an application of the previous theorem with $f_n := |d_{n+1} f|^2$, we get the following martingale inequality.

Corollary 6.5 *If $2 < \vec{p} < \infty$ or $\vec{p}/2$ satisfies (8), then*

$$\|f\|_{H^s_{\vec{p}}} \leq C\,\|f\|_{H^S_{\vec{p}}}.$$

To generalize the well known Burkholder-Davis-Gundy inequality, we introduce a new space with the norm

$$\|f\|_{\mathcal{G}_{\vec{p}}} := \left\|\sum_{n\in\mathbb{N}} |d_n f|\right\|_{\vec{p}}.$$

The so called Davis decomposition (Lemma 6.6) holds also for mixed norm spaces. The main idea of the proof of this decomposition is Theorem 6.4 (see [84]).

Lemma 6.6 *Suppose that* $1 < \vec{p} < \infty$ *or* $\vec{p}$ *satisfies* (8)*. If* $f \in H^S_{\vec{p}}$*, then there exists* $h \in \mathcal{G}_{\vec{p}}$ *and* $g \in Q_{\vec{p}}$ *such that* $f = h + g$ *and*

$$\|h\|_{\mathcal{G}_{\vec{p}}} \leq C \|f\|_{H^S_{\vec{p}}} \qquad \text{and} \qquad \|g\|_{Q_{\vec{p}}} \leq C \|f\|_{H^S_{\vec{p}}}.$$

If $f \in H^M_{\vec{p}}$*, then there exists* $h \in \mathcal{G}_{\vec{p}}$ *and* $g \in P_{\vec{p}}$ *such that* $f = h + g$ *and*

$$\|h\|_{\mathcal{G}_{\vec{p}}} \leq C \|f\|_{H^M_{\vec{p}}} \qquad \text{and} \qquad \|g\|_{P_{\vec{p}}} \leq C \|f\|_{H^M_{\vec{p}}}.$$

Now the generalization of the Burkholder-Davis-Gundy inequality can be proved for mixed norm spaces.

Theorem 6.7 ([84]) *If* $1 < \vec{p} < \infty$ *or* $\vec{p}$ *satisfies* (8)*, then the spaces* $H^S_{\vec{p}}$ *and* $H^M_{\vec{p}}$ *are equivalent, that is*

$$H^S_{\vec{p}} = H^M_{\vec{p}}$$

with equivalent norms.

7 Dual Spaces of Mixed Hardy Spaces

In this section, we study the dual spaces of mixed normed martingale Hardy spaces. To this end, we have to suppose that every σ-algebra $(\mathcal{F}_n)_n$ is generated by countably many atoms.

Definition 7.1 Suppose that every σ-algebra $(\mathcal{F}_n)_n$ is generated by countably many atoms. Let $0 \leq \vec{\alpha} < \infty$ and $1 \leq \vec{q} < \infty$. Define $BMO_{\vec{q}}(\vec{\alpha})$ as the space of functions $f \in L_{\vec{q}}$ for which

$$\|f\|_{BMO_{\vec{q}}(\vec{\alpha})} = \sup_{n \geq 0} \sup_{I \in A(\mathcal{F}_n)} \|\chi_I\|^{-1}_{\frac{1}{\vec{\alpha}+1}} \|\chi_I\|_{(\vec{q})'} \|(f - f_n)\chi_I\|_{\vec{q}} < \infty.$$

If q is a constant and $\vec{\alpha} = 0$, then this definition goes back to the classical martingale BMO_q space. If both q and $\vec{\alpha}$ are non-zero constants, then this definition becomes the classical martingale Lipschitz space investigated in Weisz [86, 87].

Now we are ready to characterize the dual of $H^s_{\vec{p}}$ (see [94]).

Theorem 7.2 *Suppose that every* σ*-algebra* $(\mathcal{F}_n)_n$ *is generated by countably many atoms. If* $0 < \vec{p} \leq 1$ *and* $\vec{\alpha} = 1/\vec{p} - 1$*, then*

$$\left(H^s_{\vec{p}}\right)^* = BMO_2(\vec{\alpha})$$

with equivalent norms.

Proof For $\varphi \in BMO_2(\vec{\alpha}) \subset L_2$, define a linear functional by

$$l_\varphi(f) = \mathbb{E}(f\varphi) \qquad (f \in L_2).$$

L_2 can be embedded continuously in $H^s_{\vec{p}}$, since by Hölder's inequality,

$$\|f\|_{H^s_{\vec{p}}} = \|s(f)\|_{\vec{p}} \le \|s(f)\|_2 = \|f\|_2 \qquad (f \in L_2).$$

Theorem 5.6 implies that for each $f \in L_2$,

$$f = \sum_{k\in\mathbb{Z}} \sum_{j=0}^{\infty} \sum_i \mu_{k,j,i} a^{k,j,i}$$

and the convergence holds also in the L_2-norm, where $a^{k,j,i}$ is a simple $(s, \vec{p}, 2)$-atom and $\mu_{k,j,i} = 3 \cdot 2^k \|\chi_{I_{k,j,i}}\|_{\vec{p}}$. Hence

$$l_\varphi(f) = \mathbb{E}(f\varphi) = \sum_{k\in\mathbb{Z}} \sum_{j=0}^{\infty} \sum_i \mu_{k,j,i} \mathbb{E}(a^{k,j,i}\varphi).$$

Observe that

$$\mathbb{E}(a^{k,j,i}\varphi) = \mathbb{E}(a^{k,j,i}(\varphi - \varphi_j)).$$

Using this, we conclude that

$$\begin{aligned}
|l_\varphi(f)| &\le \sum_{k\in\mathbb{Z}} \sum_{j=0}^{\infty} \sum_i \mu_{k,j,i} \left| \int_\Omega a^{k,j,i}(\varphi - \varphi_j) d\mathbb{P} \right| \\
&\le \sum_{k\in\mathbb{Z}} \sum_{j=0}^{\infty} \sum_i \mu_{k,j,i} \|a^{k,j,i}\|_2 \|(\varphi - \varphi_j)\chi_{I_{k,j,i}}\|_2 \\
&\le \sum_{k\in\mathbb{Z}} \sum_{j=0}^{\infty} \sum_i \mu_{k,j,i} \frac{\mathbb{P}(I_{k,j,i})^{1/2}}{\left\|\chi_{I_{k,j,i}}\right\|_{\vec{p}}} \|(\varphi - \varphi_j)\chi_{I_{k,j,i}}\|_2 \\
&\lesssim \sum_{k\in\mathbb{Z}} \sum_{j=0}^{\infty} \sum_i \mu_{k,j,i} \|\varphi\|_{BMO_2(\vec{\alpha})}.
\end{aligned} \tag{9}$$

If $g, h \in L_{\vec{p}}$ are two positive functions, then

$$\|g\|_{\vec{p}} + \|h\|_{\vec{p}} \le \|g + h\|_{\vec{p}}.$$

Indeed, since $0 < \vec{p} \le 1$, the inequality holds for all L_{p_i} spaces ($i = 1, \dots, d$), hence it holds also for $L_{\vec{p}}$ spaces. Taking into account this inequality, (6) and Theorem 5.6, we conclude that

$$\sum_{k\in\mathbb{Z}}\sum_{j=0}^{\infty}\sum_{i}\mu_{k,j,i} \lesssim \sum_{k\in\mathbb{Z}}2^k\sum_{j=0}^{\infty}\sum_{i}\|\chi_{I_{k,j,i}}\|_{\vec{p}}$$

$$\le \left\|\sum_{k\in\mathbb{Z}}2^k\sum_{j=0}^{\infty}\sum_{i}\chi_{I_{k,j,i}}\right\|_{\vec{p}} \lesssim \|f\|_{H^s_{\vec{p}}}.$$

Then (9) implies

$$|l_\varphi(f)| \lesssim \|f\|_{H^s_{\vec{p}}}\|\varphi\|_{BMO_2(\vec{\alpha})}.$$

Since by Theorem 5.6, L_2 is dense in $H^s_{\vec{p}}$, thus l_φ can be uniquely extended to a linear functional on $H^s_{\vec{p}}$.

Conversely, let l be an arbitrary bounded linear functional on $H^s_{\vec{p}}$. Since L_2 can be embedded continuously to $H^s_{\vec{p}}$, there exists $\varphi \in L_2$ such that

$$l(f) = l_\varphi(f) = \mathbb{E}(f\varphi) \qquad (f \in L_2).$$

For $I \in A(\mathcal{F}_j)$, set

$$a = \frac{(\varphi-\varphi_j)\chi_I}{\|(\varphi-\varphi_j)\chi_I\|_2\,\|\chi_I\|_{\frac{1}{\vec{\alpha}+1}}\,\|\chi_I\|_2^{-1}}.$$

Then the function a is a simple $(s, \vec{p}, 2)$-atom and so $a \in H^s_{\vec{p}}$ with $\|a\|_{H^s_{\vec{p}}} \lesssim 1$. Finally,

$$\|l\| \gtrsim l(a) = \mathbb{E}\left(a(\varphi-\varphi_j)\right) = \|\chi_I\|^{-1}_{\frac{1}{\vec{\alpha}+1}}\,\|\chi_I\|_2\,\|(\varphi-\varphi_j)\chi_I\|_2.$$

This means that

$$\|\varphi\|_{BMO_2(\vec{\alpha})} \lesssim \|l\|$$

and the theorem is shown. □

Let us denote by $(P_{\vec{p}})^*_1$ those elements l from $(P_{\vec{p}})^*$ for which there exists $\varphi \in L_1$ such that

$$l(f) = \mathbb{E}(f\varphi) \qquad (f \in L_\infty).$$

We can verify the following result similarly to Theorem 7.2 (see [94]).

Theorem 7.3 *Suppose that every σ-algebra $(\mathcal{F}_n)_n$ is generated by countably many atoms. If $0 < \vec{p} \le 1$ and $\vec{\alpha} = 1/\vec{p} - 1$, then*

$$\left(P_{\vec{p}}\right)_1^* = BMO_1(\vec{\alpha})$$

with equivalent norms.

If $(\mathcal{F}_n)$ is regular, then $P_{\vec{p}}$ is equivalent to $H_{\vec{p}}^s$ (see Corollary 6.3). We know that L_2 can be embedded continuously to $H_{\vec{p}}^s$. Thus, for $l \in P_{\vec{p}}$, there exists $\varphi \in L_2 \subset L_1$ such that $l(f) = \mathbb{E}(f\varphi)$ for any $f \in L_2 \supset L_\infty$. Hence $\left(P_{\vec{p}}\right)_1^* = \left(P_{\vec{p}}\right)^*$ and Theorem 7.3 imply the next corollary.

Corollary 7.4 *Suppose that every σ-algebra $(\mathcal{F}_n)_n$ is generated by countably many atoms. If $0 < \vec{p} \le 1$, $\vec{\alpha} = 1/\vec{p} - 1$ and $(\mathcal{F}_n)$ is regular, then*

$$\left(P_{\vec{p}}\right)^* = BMO_1(\vec{\alpha})$$

with equivalent norms.

For a regular stochastic basis $(\mathcal{F}_n)$, we can prove sharper results.

Theorem 7.5 *Suppose that every σ-algebra $(\mathcal{F}_n)_n$ is generated by countably many atoms. If $0 < \vec{p} \le 1$, $\vec{\alpha} = 1/\vec{p} - 1$, $1 < \vec{r} < \infty$ and $(\mathcal{F}_n)$ is regular, then*

$$\left(H_{\vec{p}}^M\right)^* = BMO_{\vec{r}}(\vec{\alpha})$$

with equivalent norms.

From Theorem 7.5, we get a generalization of the well known John-Nirenberg inequality.

Corollary 7.6 *Suppose that every σ-algebra $(\mathcal{F}_n)_n$ is generated by countably many atoms. If $0 \le \vec{\alpha} < \infty$, $1 < \vec{r} < \infty$ and $(\mathcal{F}_n)$ is regular, then*

$$BMO_1(\vec{\alpha}) = BMO_{\vec{r}}(\vec{\alpha})$$

with equivalent norms.

Note that these results for a constant p and α are also due to the author [87].

8 One-dimensional Walsh-Fourier Series

Now we turn to some applications in Walsh-Fourier analysis and present some summability results of one-dimensional Walsh-Fourier series. Let $\Omega := [0, 1)$ and consider the dyadic intervals $I_{k,n} := [k2^{-n}, (k+1)2^{-n})$ $(n \in \mathbb{N},\ k = 0, \ldots, 2^n-1)$.

The dyadic σ-algebras $\mathcal{F}_n$ are generated by $I_{k,n}, k = 0, \ldots, 2^n - 1$ $(n \in \mathbb{N})$. $\mathcal{F}$ (resp. $\mathbb{P}$) denotes the one-dimensional Lebesgue σ-algebra (resp. Lebesgue measure).

To introduce the Walsh orthonormal system, let us define first the Rademacher functions by

$$r_n(x) := r(2^n x) \qquad (x \in [0,1), n \in \mathbb{N}),$$

where r is a 1-periodic function and

$$r(x) := \begin{cases} 1, & \text{if } x \in [0, \frac{1}{2}); \\ -1, & \text{if } x \in [\frac{1}{2}, 1). \end{cases}$$

It is clear that, for any $n \in \mathbb{N}$, r_n is $\mathcal{F}_{n+1}$ measurable. The product system generated by the Rademacher functions is the Walsh system:

$$w_n := \prod_{k=0}^{\infty} r_k{}^{n_k} \qquad (n \in \mathbb{N}),$$

where

$$n = \sum_{k=0}^{\infty} n_k 2^k, \qquad (0 \le n_k < 2).$$

For a one-dimensional function $f \in L_1$ and for any $n \in \mathbb{N}$, the number

$$\widehat{f}(n) := \mathbb{E}(f w_n) \qquad (n \in \mathbb{N})$$

is said to be the nth Walsh-Fourier coefficient of f. We can extend this definition to martingales as follows. If $f = (f_k)_{k \ge 0}$ is a martingale, then let

$$\widehat{f}(n) := \lim_{k \to \infty} \mathbb{E}(f_k w_n) \qquad (n \in \mathbb{N}).$$

Since w_n is $\mathcal{F}_k$ measurable for $n < 2^k$, it can immediately be seen that this limit does exist. We remember that if $f \in L_1$, then $\mathbb{E}_k f \to f$ in the L_1-norm as $k \to \infty$, hence

$$\widehat{f}(n) = \lim_{k \to \infty} \mathbb{E}((\mathbb{E}_k f) w_n) \qquad (n \in \mathbb{N}).$$

Thus the Walsh-Fourier coefficients of $f \in L_1$ are the same as the ones of the martingale $(\mathbb{E}_k f)_{k \ge 0}$ obtained from f.

Denote by $s_n f$ the nth partial sum of the Walsh–Fourier series of a martingale f, namely,

$$s_n f := \sum_{k=0}^{n-1} \widehat{f}(k) w_k \qquad (n \in \mathbb{N}).$$

It is a basic question, whether the function f can be reconstructed from the partial sums of its Fourier series. It is easy to see that, for any martingale $f = (f_n)$,

$$s_{2^n} f = f_n \qquad (n \in \mathbb{N}).$$

Then the martingale convergence theorem implies that

$$\lim_{n\to\infty} s_{2^n} f = f \qquad \text{in the } L_p\text{-norm},$$

where $1 \leq p < \infty$ and $f \in L_p$. This result was generalized by Paley [71] and Schipp et al. [75, Theorem 4.1].

Theorem 8.1 *If $f \in L_p$ for some $1 < p < \infty$, then*

$$\sup_{n\in\mathbb{N}} \|s_n f\|_{L_p} \leq C_p \|f\|_{L_p}$$

and

$$\lim_{n\to\infty} s_n f = f \qquad \textit{in the } L_p\text{-norm.}$$

One of the deepest results in harmonic analysis is Carleson's result (see Carleson [8], Hunt [47], Billard [4], Sjölin [79]). Using tree martingales, Schipp [73] gave a nice proof for the theorem (see also [76, 87]).

Theorem 8.2 *If $f \in L_p$ for some $1 < p < \infty$, then*

$$\left\| \sup_{n\in\mathbb{N}} |s_n f| \right\|_{L_p} \leq C_p \|f\|_{L_p}$$

and

$$\lim_{n\to\infty} s_n f = f \qquad \text{a.e.}$$

Though Theorems 8.1 and 8.2 are not true for $p = 1$, with the help of some summability methods they can be generalized for these endpoint cases. Obviously, summability means have better convergence properties than the original Fourier series. Summability is intensively studied in the literature. We refer at this time only to the books Stein and Weiss [82], Butzer and Nessel [7], Trigub and Belinsky [85], Grafakos [34] and Weisz [90–92, 95] and the references therein.

The best known summability method is the Fejér method. In 1904 Fejér [20] investigated the arithmetic means of the partial sums of the trigonometric Fourier series, the so called Fejér means and proved that if the left and right limits $f(x-0)$ and $f(x+0)$ exist at a point x, then the Fejér means converge to $(f(x-0)+f(x+0))/2$. One year later Lebesgue [62] extended this theorem and obtained that every integrable function is Fejér summable almost everywhere.

We define the Fejér summability means by the arithmetic means of the partial sums:

$$\sigma_n f := \frac{1}{n}\sum_{k=0}^{n-1} s_k f = \sum_{j=0}^{n}\left(1-\frac{j}{n}\right)\widehat{f}(j)w_j.$$

The following theorem improves Theorem 8.1 (see Paley [71]).

Theorem 8.3 *If $f \in L_p$ for some $1 \le p < \infty$, then*

$$\sup_{n\in\mathbb{N}} \|\sigma_n f\|_{L_p} \le C_p \|f\|_{L_p}$$

and

$$\lim_{n\to\infty} \sigma_n f = f \qquad \text{in the } L_p\text{-norm.}$$

To obtain almost everywhere convergence for the Fejér means, we introduce the maximal operator of the Fejér means:

$$\sigma_* f := \sup_{n\in\mathbb{N}} |\sigma_n f|.$$

Fujii [22] proved that σ_* is bounded from H_1 to L_1 (see also Schipp and Simon [74]). Later, using the atomic decomposition, the author [88] (see also [90]) generalized this result to all $1/2 < p < \infty$:

Theorem 8.4 *If $1/2 < p \le \infty$ and $f \in H_p$, then*

$$\|\sigma_* f\|_{L_p} \le C_p\|f\|_{H_p}.$$

For $p \le 1/2$, the theorem does not hold (see Simon and Weisz [78], Simon [77]). We get the next weak type $(1,1)$ inequality from Theorem 8.4 by interpolation (Weisz [88, 90]). It was originally proved by Schipp [72].

Corollary 8.5 *If $f \in L_1$, then*

$$\sup_{\rho>0} \rho\lambda(\sigma_* f > \rho) \le C\|f\|_1.$$

This weak type (1, 1) inequality and the density argument of Marcinkiewicz and Zygmund [67] imply the next corollary, which was proved by Fine [21] and later Schipp [72].

Corollary 8.6 *If $f \in L_1$, then*

$$\lim_{n\to\infty} \sigma_n f = f \qquad a.e.$$

9 Higher Dimensional Walsh-Fourier Series

In this section, we generalize the summability results to higher dimensional Walsh-Fourier series and to mixed norm spaces. For all $i =, \ldots, d$, let $\Omega^i := [0, 1)$ and the dyadic σ-algebras $\mathcal{F}_n^i$ be generated by $I_{k,n}$, $k = 0, \ldots, 2^n - 1$ $(n \in \mathbb{N})$. $\mathcal{F}^i$ and $\mathbb{P}^i$ denote again the one-dimensional Lebesgue σ-algebra and the Lebesgue measure. Note that these definitions are independent of i. Consider the product space $(\Omega, \mathcal{F}, \mathbb{P})$ as defined in Sect. 2.

The d-dimensional Walsh-system is defined by

$$w_{\vec{n}}(\vec{x}) := \prod_{k=1}^{d} w_{n_k}(x_k)$$

where $\vec{x} = (x_1, \ldots, x_d) \in [0, 1)^d$ and $\vec{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$. If $f \in L_1$, then the $\vec{n}$-th Walsh-Fourier coefficient of f are defined by

$$\widehat{f}(\vec{n}) := \mathbb{E}(f w_{\vec{n}}) \quad \left(\vec{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d\right).$$

This definition can be extended to martingales as before in the one-dimensional case.

The $\vec{n}$-th partial sum and the $\vec{n}$-th Fejér mean of the Walsh-Fourier series of a martingale f are defined by

$$s_{\vec{n}} f := \sum_{k_1=0}^{n_1-1} \cdots \sum_{k_d=0}^{n_d-1} \widehat{f}(\vec{k})\, w_{\vec{k}} \qquad (\vec{n} \in \mathbb{N}^d)$$

and

$$\sigma_{\vec{n}} f := \frac{1}{\prod_{k=1}^{d} n_k} \sum_{k_1=1}^{n_1} \cdots \sum_{k_d=1}^{n_d} s_{\vec{k}} f \qquad (\vec{n} \in \mathbb{N}^d),$$

respectively. We will investigate the convergence of the Fejér means over the diagonal, or more generally, over cones. For $\alpha \geq 0$ let us define the cone

$$\Gamma_\alpha := \left\{ \vec{n} = (n_1, \ldots, n_d) \in \mathbb{N} : 2^{-\alpha} \leq \frac{n_i}{n_j} \leq 2^\alpha \ (i, j = 1, \ldots, d) \right\}.$$

For the almost everywhere convergence, we have to investigate the Fejér maximal operator,

$$\sigma_* f := \sup_{\vec{n} \in \Gamma_\alpha} |\sigma_n f| .$$

Using Theorem 5.3, we can verify that σ_* is bounded from $H_{\vec{p}}$ to $L_{\vec{p}}$ for $1/2 < \vec{p} < \infty$.

Theorem 9.1 ([83]) *If $\alpha \geq 0$ and $1/2 < \vec{p} < \infty$, then for all $f \in H_{\vec{p}}$,*

$$\|\sigma_* f\|_{\vec{p}} \leq C \, \|f\|_{H_{\vec{p}}}.$$

For a constant p, this theorem is due to the author [89, 90]. There are counterexamples for the boundedness of σ_* if $p \leq 1/2$ (Goginava and Nagy [32]). Since the Walsh polynomials are dense in $H_{\vec{p}}$, the following consequences of Theorem 9.1 can be proved by a density argument in the usual way (see [83]).

Corollary 9.2 *If $\alpha \geq 0$ and $1/2 < \vec{p} < \infty$, then for all $f \in H_{\vec{p}}$, $\sigma_{\vec{n}} f(\vec{x})$ converges for almost every $\vec{x} \in [0, 1)^d$ and in the $L_{\vec{p}}$-norm as $\vec{n} \to \infty$ and $\vec{n} \in \Gamma_\alpha$.*

If $I \in \mathcal{F}_k$ is a dyadic cube with length 2^{-dk}, then the restriction of the martingale f to I is defined by

$$f \chi_I := (f_n \chi_I : n \geq k) .$$

Corollary 9.3 *Let $\alpha \geq 0$, $1/2 < \vec{p} < \infty$ and $f \in H_{\vec{p}}$. If there exists a dyadic cube I, such that the restricted martingale $f \chi_I \in L_1(I)$, then*

$$\lim_{\vec{n} \in \Gamma_\alpha, \vec{n} \to \infty} \sigma_{\vec{n}} f(\vec{x}) = f(\vec{x}) \quad \text{for a.e. } \vec{x} \in I \text{ and in the } L_{\vec{p}}(I)\text{-norm.}$$

If $1 \leq \vec{p} < \infty$ and $f \in H_{\vec{p}}$, then $f \in L_1$. So we have

Corollary 9.4 *If $\alpha \geq 0$ and $1 \leq \vec{p} < \infty$, then for all $f \in H_{\vec{p}}$,*

$$\lim_{\vec{n} \in \Gamma_\alpha, \vec{n} \to \infty} \sigma_{\vec{n}} f(\vec{x}) = f(\vec{x}) \quad \text{for a.e. } \vec{x} \in [0, 1)^2 \text{ and in the } L_{\vec{p}}\text{-norm.}$$

Recall that $H_1 \subset L_1$ and $H_{\vec{p}} \sim L_{\vec{p}}$ for $1 < \vec{p} < \infty$. Next we generalize the preceding corollary. Theorem 9.1 and interpolation imply (Weisz [89, 90])

Corollary 9.5 *If* $\alpha \geq 0$ *and* $f \in L_1$*, then*

$$\sup_{\rho>0} \rho\lambda(\sigma_* f > \rho) \leq C\|f\|_1.$$

Corollary 9.6 *If* $\alpha \geq 0$ *and* $f \in L_1$*, then*

$$\lim_{\vec{n}\in\Gamma_\alpha, \vec{n}\to\infty} \sigma_{\vec{n}} f = f \qquad a.e.$$

This corollary was proved independently by Gát [24] and Weisz [89].

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The First Eigenvalue for Nonlocal Operators

Julio D. Rossi

To the memory of Ireneo Peral, a great mathematician and friend

Abstract In this chapter we present some results concerning the first eigenvalue for a nonlocal operator in convolution form with a smooth kernel. Given a bounded domain $\Omega \subset \mathbb{R}^N$ and a smooth kernel J, we deal with the eigenvalue problem

$$\int_A J(x-y)(u(y)-u(x))\,dy = -\lambda_1 u(x), \qquad x \in \Omega,$$

both with Dirichlet boundary conditions (take $A = \mathbb{R}^N$ and prescribe that $u = 0$ in $\mathbb{R}^N \setminus \Omega$) and Neumann boundary conditions (now $A = \Omega$, in this case we study the first nontrivial eigenvalue).

Keywords Eigenvalues · Nonlocal equations · Smooth kernels

1 Nonlocal Diffusion Problems

First, let us briefly introduce the prototype of nonlocal problem that will be considered along this chapter. Let $J : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative, radial, continuous function with

$$\int_{\mathbb{R}^N} J(z)\,dz = 1.$$

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Nonlocal evolution equations of the form

$$\frac{\partial u}{\partial t}(x,t) = (J * u - u)(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)\,dy - u(x,t), \tag{1}$$

and variations of it, have been recently widely used to model diffusion processes. More precisely, as stated in [2, 20], if $u(x,t)$ is thought of as a density at the point x at time t and $J(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then $\int_{\mathbb{R}^N} J(y-x)u(y,t)\,dy = (J * u)(x,t)$ is the rate at which individuals are arriving at position x from all other places and $-u(x,t) = -\int_{\mathbb{R}^N} J(y-x)u(x,t)\,dy$ is the rate at which they are leaving location x to travel to all other sites. This consideration, in the absence of external or internal sources, leads immediately to the fact that the density u satisfies Eq. (1).

Equation (1) is called nonlocal diffusion equation since the to evaluate the right hand side at a point x and time t one needs to know u in a neighborhood of x to compute the convolution term $J * u$. This equation shares many properties with the classical heat equation, $u_t = \Delta u$, such as: bounded stationary solutions are constant, a maximum principle holds for both of them and, even if J is compactly supported, perturbations propagate with infinite speed [20]. However, there is no regularizing effect in general.

Let us fix Ω a bounded domain in $\mathbb{R}^N$. For local problems the two most common boundary conditions for local PDEs are Dirichlet and Neumann. When looking at boundary conditions for nonlocal problems, one has to modify the usual formulations for local problems.

Concerning the homogeneous Dirichlet boundary conditions for nonlocal problems we consider

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x-y)u(y,t)\,dy - u(x,t), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \notin \Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{2}$$

In this model we have that diffusion takes place in the whole $\mathbb{R}^N$ but we impose that u vanishes outside Ω. In the biological interpretation, we have a hostile environment outside Ω, any individual that jumps outside dies instantaneously. This is the analogous of what is called homogeneous Dirichlet boundary conditions for the heat equation. However, the boundary datum is not understood in the usual sense, since we are not imposing that u is continuous up to $\partial\Omega$.

For an analogous to Neumann boundary conditions for nonlocal problems we propose

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x-y)(u(y,t)-u(x,t))\,dy, & x \in \Omega,\ t>0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{3}$$

In this model we have that the integral term takes into account the diffusion inside Ω. In fact, as we have explained, the integral $\int J(x-y)(u(y,t)-u(x,t))\,dy$ takes into account the individuals arriving or leaving position x from other places. Since we are integrating in Ω, we are imposing that diffusion takes place only in Ω. The individuals may not enter nor leave the domain. This is the analogous of what is called homogeneous Neumann boundary conditions in the literature.

These nonlocal problems has been used to model very different applied situations, for example in biology [12, 23, 29], image processing [22, 27], particle systems [9], coagulation models [21], nonlocal anisotropic models for phase transition [1], mathematical finances [26], etc. Besides the interest for the applications there is also a great amount of work dealing with purely mathematical issues. For example, see [3, 4, 6, 7, 10–13, 15–20, 24, 25], and references therein.

Associated with these evolution problems there are two eigenvalue problems that play a central role in determining the asymptotic behaviour of the solutions as $t \to +\infty$. They are given by

$$\begin{cases} \phi(x) - \displaystyle\int_\Omega J(x-y)\phi(y)\,dy = \lambda_1(x), & x \in \Omega; \\ \phi(x) = 0, \qquad x \notin \Omega, \end{cases}$$

and

$$-\int_\Omega J(x-y)(\varphi(y)-\varphi(x))\,dy = \beta_1\varphi(x), \quad x \in \Omega.$$

The analysis of these eigenvalue problems is our main concern in this chapter. The proofs of the main results are taken from the references [2, 14, 28], but we include here some details to make this chapter self-contained.

We have to mention the close relation between this kind of evolution problems and probability theory. In fact, when one looks at a Levy process [8] the nonlocal operator that appears naturally is a fractional power of the Laplacian. This is out of the scope of this chapter and we refer to [5] for a reference concerning the interplay between nonlocal partial differential equations and probability. Although we are not dealing with probability issues, let us explain briefly why the concrete problem (1) has a clear probabilistic interpretation. Let $(E, \mathcal{E})$ be a measurable space and $P : E \times \mathcal{E} \to [0,1]$ be a probability transition on E. Then, we define a transition function, for any $x \in E$, $\mathcal{A} \in \mathcal{E}$, let

$$P_t(x, \mathcal{A}) = e^{-t} \sum_{n=0}^{+\infty} \frac{t^n}{n!} P^{(n)}(x, \mathcal{A}) \qquad t \in \mathbb{R}_+,$$

where $P^{(n)}$ denotes the n-ieth iterate of P. The associated family of Markovian operators, $P_t f(x) = \int f(y) P_t(x, dy)$ satisfies

$$\frac{\partial}{\partial t} P_t f(x) = \int P_t f(y) P(x, dy) - P_t f(x).$$

Now, consider the Markov process $(Z_t)_{t \geq 0}$ associated to the transition function $(P_t)_{t \geq 0}$, and denote by μ_t its distribution. Then the family $(\mu_t)_{t \geq 0}$ satisfies a linear equation of the form

$$\frac{\partial}{\partial t} \mu_t = \int P(y, \cdot) \mu_t(dy) - \mu_t.$$

In particular, for $E = \mathbb{R}^N$, if the probability transition $P(x, dy)$ has a density $y \mapsto J(x, y)$, and μ_t has a density $y \mapsto u(y, t)$, then we get the following equation

$$\frac{\partial}{\partial t} u(x, t) = \int J(x, y) u(y, t)\, d\lambda(y) - u(x, t). \tag{4}$$

With different particular choices of P we recover the equation studied in the Dirichlet and the Neumann cases. For example, if $P(x, dy) = J(y - x) dy$ is the transition probability of a random walk, Eq. (4) is just Eq. (1). The results described here give interesting information on the asymptotic behaviour of some natural Markov processes.

2 The First Eigenvalue with Dirichlet or Neumann Boundary Conditions

Now, let us introduce with some detail the two eigenvalue problems that we discuss in this chapter (Dirichlet or Neumann boundary conditions). We will deal mainly with the L^2 formulation that gives linear eigenvalue problems, and at the end of the chapter we will comment briefly on the results and difficulties for nonlinear eigenvalues.

2.1 Dirichlet Boundary Conditions

Let $\lambda_1 = \lambda_1(\Omega)$ be given by

$$\lambda_1 = \inf_{u\in L^2(\Omega)} \frac{\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(\overline{u}(x)-\overline{u}(y))^2\,dx\,dy}{\int_\Omega (u(x))^2\,dx}. \quad (5)$$

Here and in what follows we denote by $\overline{u}$ the extension by zero of u to the whole $\mathbb{R}^N$, that is,

$$\overline{u}(x) = \begin{cases} u(x) & x \in \Omega, \\ 0 & x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

If the minimum is attained at a function ϕ_1 we get, just by differentiation, that it is a solution of

$$\phi_1(x) - \int_{\mathbb{R}^N} J(x-y)\overline{\phi}_1(y)\,dy = \lambda_1(x), \quad x \in \Omega; \quad (6)$$

conversely, it is easy to check that if $\phi_1 > 0$ is a solution to (6) (with λ_1 the smallest eigenvalue) then it is a minimizer of (5). Hence, we look for the first eigenvalue of (6), which is equivalent to

$$(1-\lambda_1)\phi_1(x) = \int_{\mathbb{R}^N} J(x-y)\overline{\phi}_1(y)\,dy, \quad x \in \Omega. \quad (7)$$

Let $S : L^2(\Omega) \to L^2(\Omega)$ be the operator given by

$$S(u)(x) := \int_{\mathbb{R}^N} J(x-y)\overline{u}(y)\,dy = \int_\Omega J(x-y)u(y)\,dy, \quad x \in \Omega.$$

Hence we are looking for the largest eigenvalue, $\mu = 1 - \lambda_1$, of S. Since the kernel is smooth, S is compact and then this eigenvalue is attained at some function $\phi_1(x)$ that turns out to be an eigenfunction for our original problem (6). By taking $|\phi_1|$ instead of ϕ_1 in (5) we may assume that $\phi_1 \geq 0$ in Ω. Indeed, one simply has to use the fact that $(a-b)^2 \geq (|a|-|b|)^2$.

Let us present some properties of the eigenvalue problem (6).

Proposition 2.1 ([2, 14]) *Let λ_1 the first eigenvalue of (6) and denote by $\phi_1(x)$ a corresponding non-negative eigenfunction. Then $\phi_1(x)$ is strictly positive in Ω and λ_1 is a positive simple eigenvalue with $\lambda_1 < 1$.*

Proof Since $J(0) > 0$ and J is continuous we have that $B(0,d) \subset \text{supp}(J)$ for some $d > 0$. Let us assume, for simplicity, that $\text{supp}(J) = \overline{B}(0,1)$. First, observe that $\lambda_1 = 1$ can not be an eigenvalue since then

$$\int_{\mathbb{R}^N} J(x-y)\overline{\phi}_1(y)\,dy = 0, \qquad \phi_1(x) \geq 0,$$

which implies $\phi_1 = 0$. Consequently, we have that

$$(1-\lambda_1)\phi_1(x) = \int_{\mathbb{R}^N} J(x-y)\overline{\phi}_1(y)\,dy, \quad x \in \Omega, \quad \lambda_1 \neq 1,$$

which implies that ϕ_1 is uniformly continuous in Ω. In what follows, we consider bounded continuous functions in Ω extended in the natural way to $\overline{\Omega}$. We begin with the positivity of the eigenfunction ϕ_1. Assume for contradiction that the $\mathbf{B} = \{x \in \Omega : \phi_1(x) = 0\} \neq \emptyset$. Then, from the continuity of ϕ_1 in Ω, we have that $\mathbf{B}$ is closed. We next prove that $\mathbf{B}$ is also open, and hence, since Ω is connected, by standard topological arguments we conclude that $\Omega \equiv \mathbf{B}$, a contradiction. Consider $x_0 \in \mathbf{B}$. Since $\phi_1 \geq 0$, we obtain from (7) that $\Omega \cap B(x_0, 1) \in \mathbf{B}$ (we use here that $\text{supp}(J) = \overline{B}(0,1)$). Hence $\mathbf{B}$ is open and the result follows.

Next, let us prove that $\lambda_1 > 0$. Assume by contradiction that $\lambda_1 \leq 0$ and denote by M^* the maximum of ϕ_1 in $\overline{\Omega}$ and by x^* a point where such maximum is attained. Assume for the moment that $x^* \in \Omega$. One can choose x^* in such a way that $\phi_1(x) \neq M^*$ in $\Omega \cap B(x^*, 1)$. By using (7) we get that,

$$M^* \leq (1-\lambda_1)\phi_1(x^*) = \int_{\mathbb{R}^N} J(x^*-y)\overline{\phi}_1(y) < M^*$$

and a contradiction follows. If $x^* \in \partial\Omega$, we obtain a similar contradiction after substituting and passing to the limit in (7) on a sequence $\{x_n\} \in \Omega$, $x_n \to x^*$ as $n \to \infty$. To obtain the upper bound, assume that $\lambda_1 \geq 1$. Then, from (7) we have for every $x \in \Omega$ that

$$0 \geq (1-\lambda_1)\phi_1(x^*) = \int_{\mathbb{R}^N} J(x^*-y)\overline{\phi}_1(y)dy$$

a contradiction with the positivity of ϕ_1.

Finally, to prove that λ_1 is a simple eigenvalue, let $\phi_1 \neq \phi_2$ be two different eigenfunctions associated to λ_1 and define

$$C^* = \inf\{C > 0 : \phi_2(x) \leq C\phi_1(x),\ x \in \overline{\Omega}\}.$$

The regularity of the eigenfunctions and the previous analysis shows that C^* is nontrivial and bounded. Moreover from its definition, there must exists $x^* \in \overline{\Omega}$ such that $\phi_2(x^*) = C^*\phi_1(x^*)$. Define $\phi(x) = C^*\phi_1(x) - \phi_2(x)$. From the linearity of (6), ϕ is a non-negative eigenfunction associated to λ_1 with $\phi(x^*) = 0$. From the positivity of the eigenfunctions stated above, it must be $\phi \equiv 0$. Therefore, $\phi_2(x) = C^*\phi_1(x)$ and the result follows. This completes the proof. □

Observe that the first eigenfunction ϕ_1 is strictly positive in Ω (with a positive continuous extension to $\overline{\Omega}$) and vanishes outside Ω. Therefore a discontinuity occurs on $\partial\Omega$ and the boundary value is not taken in the classical sense.

Now, let us prove that the first eigenvalue λ_1 gives an exponential decay for the solutions to the evolution problem (2).

Theorem 2.2 ([2, 14]) *If $u_0 \in L^2(\Omega)$, then the solution u of (2) decays to zero as $t \to \infty$ with an exponential rate,*

$$\|u(\cdot,t)\|_{L^2(\Omega)} \le \|u_0\|_{L^2(\Omega)} e^{-\lambda_1 t}. \tag{8}$$

If u_0 is continuous, positive and bounded, then there exist positive constants C and C^* such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C\, e^{-\lambda_1 t} \tag{9}$$

and

$$\lim_{t\to\infty} \|e^{\lambda_1 t} u(\cdot,t) - C^*\phi_1(\cdot)\|_{L^\infty(\Omega)} = 0. \tag{10}$$

Proof Using the symmetry of J, we have

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2}\int_\Omega u^2(x,t)\,dx\right) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\,(u(y,t)-u(x,t))\,u(x,t)\,dy\,dx\\ &= -\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\,(u(y,t)-u(x,t))^2\,dy\,dx.\end{aligned}$$

From the definition of λ_1, (5), we get

$$\frac{\partial}{\partial t}\int_\Omega u^2(x,t)\,dx \le -2\lambda_1\int_\Omega u^2(x,t)\,dx.$$

Therefore

$$\int_\Omega u^2(x,t)\,dx \le e^{-2\lambda_1 t}\int_\Omega u_0^2(x)\,dx$$

and (8) is obtained.

We now establish the decay rate and the convergence stated in (9) and (10) respectively. Consider a nontrivial and non-negative continuous initial datum $u_0(x)$ and let $u(x,t)$ be the corresponding solution to (2). We first note that $u(x,t)$ is a continuous function satisfying $u(x,t) > 0$ for every $x \in \Omega$ and $t > 0$, and the same holds in $\overline{\Omega}$. This instantaneous positivity can be obtained by using analogous topological arguments to those used in Proposition 2.1.

In order to deal with the asymptotic analysis, it is more convenient to introduce the rescaled function $v(x,t) = e^{\lambda_1 t}u(x,t)$. We have that the function $v(x,t)$ satisfies

$$v_t(x,t) = \int_{\mathbb{R}^N} J(x-y)v(y,t)\,dy - (1-\lambda_1)v(x,t) \quad x \in \Omega. \tag{11}$$

On the other hand, we have that $C\phi_1(x)$ is a solution of (11) for every $C \in \mathbb{R}$ and moreover, it follows from the above eigenfunction analysis that the set of stationary solutions of (11) is given by $\mathbf{S}^* = \{C\phi_1,\ C \in \mathbb{R}\}$.

Define now for every $t > 0$, the function

$$C^*(t) = \inf\{C > 0 : v(x,t) \le C\phi_1(x),\ x \in \overline{\Omega}\}.$$

By definition and by using the linearity of Eq. (11), $C^*(t)$ is a non-increasing function. In fact, this is a consequence of the comparison principle applied to the solutions $C^*(t_1)\phi_1(x)$ and $v(x,t)$ for t larger than any fixed $t_1 > 0$. It implies that $C^*(t_1)\phi_1(x) \ge v(x,t)$ for every $t \ge t_1$, and therefore, $C^*(t_1) \ge C^*(t)$ for every $t \ge t_1$. In an analogous way, one can see that the function

$$C_*(t) = \sup\{C > 0 : v(x,t) \ge C\phi_1(x),\ x \in \overline{\Omega}\},$$

is non-decreasing. These properties imply that the following two limits exist,

$$\lim_{t\to\infty} C^*(t) = K^* \quad \text{and} \quad \lim_{t\to\infty} C_*(t) = K_*,$$

and also provide the compactness of the orbits which is necessary to pass to the limit (after extracting subsequences if needed) in order to obtain that $v(\cdot, t+t_n) \to w(\cdot, t)$ as $t_n \to \infty$ uniformly on compact subsets in $\overline{\Omega} \times \mathbb{R}_+$ and $w(x,t)$ is a continuous function which satisfies (11). Let us recall the concept of ω-limit set of the trajectory $u(t)$ that begins with $u(0) = u_0$,

$$\omega(u_0) := \left\{ g \in L^2(\Omega) \ : \ \exists t_n \to \infty \text{ with } u(t_n) \to g \text{ in } L^2(\Omega) \right\}.$$

For every $g \in \omega(u_0)$ we also have

$$K_*\phi_1(x) \le g(x) \le K^*\phi_1(x).$$

Moreover, $C^*(t)$ plays a role of a Lyapunov function and this fact allows to conclude that $\omega(u_0) \subset \mathbf{S}^*$, the set of stationary solutions of (11), and the uniqueness of the profile. In more detail, assume that $g \in \omega(u_0)$ does not belong to $\mathbf{S}^*$ and consider $w(x,t)$ the solution of (11) with initial data $g(x)$ and define

$$C^*(w)(t) = \inf\{C > 0 : w(x,t) \le C\phi_1(x),\ x \in \overline{\Omega}\}.$$

It is clear that $W(x,t) = K^*\phi_1(x) - w(x,t)$ is a non-negative continuous solution of (11) and it becomes strictly positive for every $t > 0$. This implies that there exists $t^* > 0$ such that $C^*(w)(t^*) < K^*$ and by the convergence, the same holds before passing to the limit. Hence, $C^*(t^* + t_j) < K^*$ if j is large enough, which is a contradiction with the properties of $C^*(t)$. The same arguments allow us to establish the uniqueness of the profile. □

We summarize in the next result different versions of the variational characterization of the principal eigenvalue.

Theorem 2.3 ([28]) *The first eigenvalue, $\lambda_1(\Omega)$, can be variationally characterized as*

$$\lambda_1(\Omega) = 1 - \left(\sup_{\substack{u \in L^2(\Omega) \\ u \neq 0}} \frac{\displaystyle\int_\Omega \left(\int_\Omega J(x-y)u(y)\,dy \right)^2 dx}{\displaystyle\int_\Omega u^2(x)\,dx} \right)^{1/2} \tag{12}$$

or

$$\lambda_1(\Omega) = 1 - \sup_{\substack{u \in L^2(\Omega) \\ u \neq 0}} \frac{\displaystyle\int_\Omega \int_\Omega J(x-y)u(x)u(y)\,dy\,dx}{\displaystyle\int_\Omega u^2(x)\,dx} \tag{13}$$

or

$$\lambda_1(\Omega) = \frac{1}{2} \inf_{\substack{u \in L^2(\Omega) \\ u \neq 0}} \frac{\displaystyle\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x-y)(\overline{u}(x) - \overline{u}(y))^2\,dy\,dx}{\displaystyle\int_\Omega u^2(x)\,dx}. \tag{14}$$

Proof Recall that we introduced the compact operator $S : L^2(\Omega) \to L^2(\Omega)$ given by

$$S(u)(x) := \int_{\mathbb{R}^N} J(x-y)\overline{u}(y)\,dy = \int_\Omega J(x-y)u(y)\,dy, \quad x \in \Omega$$

and that we have

$$\lambda_1 = 1 - \|S\|.$$

Then, the variational characterizations (12) and (13) are can be obtained at once since

$$\|S\|^2 = \sup_{\substack{u\in L^2(\Omega)\\ u\neq 0}} \frac{|Su|^2_{L^2(\Omega)}}{|u|^2_{L^2(\Omega)}} = \sup_{\substack{u\in L^2(\Omega)\\ u\neq 0}} \frac{\displaystyle\int_\Omega \left(\int_\Omega J(x-y)u(y)\,dy\right)^2 dx}{\displaystyle\int_\Omega u^2(x)\,dx},$$

and

$$\|S\| = \sup_{\substack{u\in L^2(\Omega)\\ u\neq 0}} \frac{|\langle Su,u\rangle|}{|u|^2_{L^2(\Omega)}} = \sup_{\substack{u\in L^2(\Omega)\\ u\neq 0}} \frac{\displaystyle\int_\Omega\int_\Omega J(x-y)u(x)u(y)\,dy\,dx}{\displaystyle\int_\Omega u^2(x)\,dx},$$

since S is self-adjoint. Finally, by expanding the square in the numerator and applying Fubini's theorem, it is easily seen that

$$\frac{\displaystyle\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(\overline{u}(x)-\overline{u}(y))^2 dy\,dx}{\displaystyle\int_\Omega u^2(x)\,dx} = 1 - \frac{\displaystyle\int_\Omega\int_\Omega J(x-y)u(x)u(y)dy\,dx}{\displaystyle\int_\Omega u^2(x)\,dx},$$

since J is even and $\overline{u} = 0$ in $\mathbb{R}^N \setminus \Omega$. Thus (14) follows. □

As an immediate consequence of the variational characterizations (12) and (13), we have an estimate for $\lambda_1(\Omega)$, which will be useful when dealing with the asymptotic behavior of $\lambda_1(\Omega)$ in large and small domains.

Corollary 2.4 *For the principal eigenvalue $\lambda_1(\Omega)$ we have the estimates:*

$$\left(\frac{1}{|\Omega|}\int_\Omega A^2(x)\,dx\right)^{1/2} \le 1-\lambda_1(\Omega) \le \sup_{y\in\Omega}\left(\int_\Omega A(x)J(x-y)\,dx\right), \qquad (15)$$

where $A(x) = \int_\Omega J(x-y)\,dy$.

Proof Taking $u \equiv 1$ as test function in (12), we obtain

$$1-\lambda_1(\Omega) \ge \left(\frac{1}{|\Omega|}\int_\Omega A^2(x)\,dx\right)^{1/2}. \qquad (16)$$

On the other hand, thanks to Cauchy-Schwartz inequality, we have:

$$\int_\Omega\left(\int_\Omega J(x-y)u(y)\,dy\right)^2 dx \le \int_\Omega A(x)\left(\int_\Omega J(x-y)u^2(y)\,dy\right)dx.$$

Using Fubini's theorem, we get

$$\begin{aligned}\int_\Omega \left(\int_\Omega J(x-y)u(y)\,dy \right)^2 dx &\le \int_\Omega u^2(y) \left(\int_\Omega A(x)J(x-y)dx \right) dy \\ &\le \sup_{y\in\Omega} \left(\int_\Omega A(x)J(x-y)dx \right) \int_\Omega u^2(y)\,dy,\end{aligned}$$

and this implies that

$$1-\lambda_1(\Omega_0) \le \sup_{y\in\Omega} \left(\int_\Omega A(x)J(x-y)dx \right). \tag{17}$$

Finally, (15) follows from (16) and (17). This concludes the proof of the corollary. □

The estimates (15) obtained in Corollary 2.4 are not sharp: if the domain Ω contains a ball of radius 2, say, then the right-hand side in (15) equals one, so the estimate is useless.

Now, we turn our attention to the dependence of the first eigenvalue on the domain Ω. A first consequence of the variational characterization, (5), is the strict monotonicity of $\lambda_1(\Omega)$.

Theorem 2.5 ([28]) *The principal eigenvalue, $\lambda_1(\Omega)$, is decreasing with respect to the domain, that is, if $\Omega_1 \subset \Omega_2$, then $\lambda_1(\Omega_1) > \lambda_1(\Omega_2)$.*

Proof of Theorem 2.5 We notice that $L^2(\Omega_1) \subset L^2(\Omega_2)$, provided we extend all functions of the first space by zero outside Ω_1. Hence we have, thanks to the characterization (12), that $\lambda_1(\Omega_1) \ge \lambda_1(\Omega_2)$. To show that the inequality is strict, we notice that if $\lambda_1(\Omega_1) = \lambda_1(\Omega_2)$, then we obtain an associated eigenfunction which is positive in $\overline{\Omega}_1$, but zero in $\Omega_2 \setminus \overline{\Omega}_1$, which contradicts the strong maximum principle. □

Next, we analyze perturbations Ω_δ of a fixed domain Ω, where δ is a small parameter, and consider the issues of continuity and differentiability of $\lambda_1(\Omega_\delta)$ with respect to δ. We assume that the perturbed domain verifies $\Omega_\delta = \Psi(\delta, \Omega)$, where $\Psi : (-\varepsilon, \varepsilon) \times \overline{\Omega} \to \mathbb{R}^N$ takes the form

$$\Psi(\delta, x) = x + \Phi(\delta, x), \tag{18}$$

with $\Phi(0, \cdot) = 0$. The continuity of $\lambda_1(\Omega_\delta)$ is a more or less simple consequence of the continuity of Φ with respect to δ. We denote by $D\Phi$ the differential of Φ with respect to x.

Theorem 2.6 ([28]) *Let $\lambda_1(\Omega_\delta)$ be the principal eigenvalue in Ω_δ, and assume $\Omega_\delta = \Psi(\delta, \Omega)$, where Ψ has the form (18) with $\Phi, D\Phi \in C((-\varepsilon, \varepsilon) \times \overline{\Omega})$ for some $\varepsilon > 0$ and $\Phi(0, \cdot) = 0$. Then, $\lambda_1(\Omega_\delta) \to \lambda_1(\Omega)$ as $\delta \to 0$.*

Proof of Theorem 2.6 We first notice that for small δ we can always assume $\Omega_1 \subset \Omega_\delta \subset \Omega_2$ for some smooth domains Ω_1 and Ω_2 not depending on δ. Thanks to Theorem 2.5 this implies

$$0 < \lambda_1(\Omega_2) < \lambda_1(\Omega_\delta) < \lambda_1(\Omega_1) < 1. \tag{19}$$

Now let u_δ be a positive eigenfunction associated to $\lambda_1(\Omega_\delta)$:

$$\int_{\Omega_\delta} J(x-y)u_\delta(x)\,dx = (1-\lambda_1(\Omega_\delta))u_\delta(x), \quad x \in \Omega_\delta.$$

We make the change of variables $x = z + \Phi(\delta, z)$, $y = w + \Phi(\delta, w)$ with $x, w \in \Omega$ to obtain

$$\int_\Omega J(z-w+\Phi(\delta, z)-\Phi(\delta, w))v_\delta(w)\Delta(\delta, w)\,dw = (1-\lambda_1(\Omega_\delta))v_\delta(z), \tag{20}$$

for $z \in \Omega$, where $v_\delta(w) = u_\delta(w+\Phi(\delta, w))$ and $\Delta(\delta, w) = \det(I+D\Phi(\delta, w))$. We select v_δ with the normalization $|v_\delta|_{L^2(\Omega)} = 1$. Then, for every sequence $\delta_n \to 0$, we have a subsequence—still denoted by δ_n—such that $v_{\delta_n} \rightharpoonup v$ weakly in $L^2(\Omega)$. Since

$$J(z-w+\Phi(\delta_n, z)-\Phi(\delta_n, w))\ \Delta(\delta_n, w) \to J(z-w)$$

uniformly in $z, w \in \Omega$, we obtain thanks to weak convergence

$$\int_\Omega J(z-w+\Phi(\delta_n, z)-\Phi(\delta_n, w))v_{\delta_n}(w)\ \Delta(\delta, w)\,dw \to \int_\Omega J(z-w)v(w)\,dw$$

for almost every $w \in \Omega$. Using the dominated convergence theorem, we also have the convergence in $L^2(\Omega)$.

On the other hand, since $\lambda_1(\Omega_{\delta_n})$ is bounded, we may pass to a further subsequence to have $\lambda_1(\Omega_{\delta_n}) \to \mu$, where $0 < \mu < 1$, thanks to (19). Then, setting $\delta = \delta_n$ in (20) and passing to the limit we have that the convergence of v_{δ_n} to v_0 is strong in $L^2(\Omega)$. Therefore, $|v_0|_{L^2(\Omega)} = 1$. By (20) we finally have

$$\int_\Omega J(x-y)v_0(y)\,dy = (1-\mu)v_0(x), \quad x \in \Omega,$$

with $v_0 \geq 0$, $v_0 \not\equiv 0$. According to Theorem 2.3, we obtain that $\mu = \lambda_1(\Omega)$, that is, $\lambda_1(\Omega_{\delta_n}) \to \lambda_1(\Omega)$. Since δ_n was arbitrary, this shows that $\lambda_1(\Omega_\delta) \to \lambda_1(\Omega)$ as $\delta \to 0$, as we wanted to prove. □

We now consider the question of differentiability of $\lambda_1(\Omega_\delta)$. We assume the function Ψ in (18) is differentiable and prove that $\lambda_1(\Omega_\delta)$ is differentiable at $\delta = 0$, providing in addition an explicit formula for the derivative.

Theorem 2.7 *Let $\lambda(\delta) = \lambda_1(\Omega_\delta)$ be the principal eigenvalue in Ω_δ, and assume $\Omega_\delta = \Psi(\delta, \Omega)$, where Ψ is of the form (18) with $\Phi \in C^1((-\varepsilon, \varepsilon) \times \overline{\Omega})$ for some $\varepsilon > 0$ and $\Phi(0, \cdot) = 0$. Then $\lambda(\delta)$ is differentiable with respect to δ at $\delta = 0$, and*

$$\lambda'(0) = -(1 - \lambda_1(\Omega)) \int_{\partial\Omega} u_0^2(x) \left\langle \frac{\partial \Phi}{\partial \delta}(0, x), \nu(x) \right\rangle dS(x), \tag{21}$$

where u_0 is the positive eigenfunction associated to $\lambda_1(\Omega)$ normalized as $|u_0|_{L^2(\Omega)} = 1$ and $\nu(x)$ is the outward unit normal to $\partial\Omega$.

Note that the eigenfunction u_0 is strictly positive on $\partial\Omega$, see [14]. Thus, the integral in (21) is not necessarily zero.

Proof of Theorem 2.7 We use the variational characterization (13) to estimate the incremental quotients of $\lambda_1(\Omega_\delta)$. For simplicity, let us write $\mu(\delta) = 1 - \lambda_1(\Omega_\delta)$. If we denote

$$H_\delta(u) = \frac{\displaystyle\int_{\Omega_\delta} \int_{\Omega_\delta} J(x - y) u(x) u(y)\, dx\, dy}{\displaystyle\int_{\Omega_\delta} u^2(x)\, dx},$$

we have, thanks to (13), that

$$\frac{\mu(\delta) - \mu(0)}{\delta} \geq \frac{H_\delta(u_0) - \mu(0)}{\delta} \tag{22}$$

for $\delta > 0$ (recall that $u_0 = 0$ outside Ω). Now, we perform the change of variables $x = z + \Phi(\delta, z)$, $y = w + \Phi(\delta, w)$ in the integrals in H_δ and we obtain

$$\begin{aligned} H_\delta(u_0) &= \frac{\displaystyle\int_{\Omega_\delta} \int_{\Omega_\delta} J(x - y) u_0(x) u_0(y)\, dx\, dy}{\displaystyle\int_{\Omega_\delta} u_0^2(x)\, dx} \\ &= \frac{\displaystyle\int_{\Omega} \int_{\Omega} A(z, w)\ \Delta(z)\Delta(w)\, dz dw}{\displaystyle\int_{\Omega} u_0^2(z + \Phi(\delta, z))\Delta(z)\, dz} \end{aligned} \tag{23}$$

where

$$A(z, w) = J(z - w + \Phi(\delta, z) - \Phi(\delta, w)) u_0(z + \Phi(\delta, z)) u_0(w + \Phi(\delta, w))$$

and $\Delta(z) = \det(I + D\Phi(\delta, z))$ and D stands for differentiation with respect to the second variable. By our regularity assumptions we have that

$$A(z, w)\ \Delta(z)\Delta(w) = J(z - w)u_0(z)u_0(w) + K(z, w)\delta + o(\delta), \tag{24}$$

where

$$\begin{aligned} K(z, w) = &\left\langle \nabla J(z - w), \Phi'(0, z) - \Phi'(0, w)\right\rangle u_0(z)u_0(w) \\ &+ J(z - w)u_0(w)\left\langle \nabla u_0(z), \Phi'(0, z)\right\rangle \\ &+ J(z - w)u_0(z)\left\langle \nabla u_0(w), \Phi'(0, w)\right\rangle \\ &+ J(z - w)u_0(z)u_0(w)\ \mathrm{div}(\Phi'(0, z)) \\ &+ J(z - w)u_0(z)u_0(w)\ \mathrm{div}(\Phi'(0, w)), \end{aligned}$$

and $'$ stands for differentiation with respect to δ. Integrating (24) with respect to z and w in Ω, we get,

$$\begin{aligned} &\int_\Omega\int_\Omega A(z, w)\ \Delta(z)\Delta(w)\,dz\,dw \\ &\quad = \int_\Omega\int_\Omega J(z - w)u_0(z)u_0(w)\,dz\,dw + \delta\int_\Omega\int_\Omega K(z, w)\,dz\,dw + o(\delta) \\ &\quad = \mu(0) + \delta\int_\Omega\int_\Omega K(z, w)\,dz\,dw + o(\delta). \end{aligned} \tag{25}$$

Taking into account that J is even—and hence ∇J is odd—and using Fubini's theorem we have that

$$\begin{aligned} \int_\Omega\int_\Omega K(z, w)\,dz\,dw = &\,2\int_\Omega\int_\Omega \left\langle \nabla J(z - w), \Phi'(0, z)\right\rangle u_0(z)u_0(w)\,dz\,dw \\ &+ 2\int_\Omega\int_\Omega J(z - w)u_0(w)\left\langle \nabla u_0(z), \Phi'(0, z)\right\rangle dz\,dw \\ &+ 2\int_\Omega\int_\Omega J(z - w)u_0(z)u_0(w)\ \mathrm{div}(\Phi'(0, z))\,dz\,dw. \end{aligned}$$

Integrating by parts in the last integral, we arrive to

$$\int_\Omega\int_\Omega K(z, w)\,dz\,dw = 2\int_\Omega\int_{\partial\Omega} J(z - w)u_0(z)u_0(w)\langle\Phi'(0, z), \nu(z)\rangle dS(z)\,dw.$$

Noticing that u_0 is an eigenfunction, this expression can be further transformed into:

$$\int_\Omega\int_\Omega K(z, w)\,dz\,dw = 2\mu(0)\int_{\partial\Omega} u_0^2(z)\ \left\langle \Phi'(0, z), \nu(z)\right\rangle dS(z)\,dw.$$

Hence, from (25) we obtain

$$\begin{aligned}\int_\Omega\int_\Omega A(z,w)\ \Delta(z)\Delta(w)\,dz\,dw \\ = \mu(0)+2\mu(0)\delta\int_{\partial\Omega} u_0^2(z)\ \left\langle\Phi'(0,z),\nu(z)\right\rangle\,dS(z)+o(\delta).\end{aligned} \tag{26}$$

On the other hand, with a similar procedure, we obtain:

$$\int_\Omega u_0^2(z+\Phi(\delta,z))\Delta(z)\,dz = 1+\delta\int_{\partial\Omega} u_0^2(z)\left\langle\Phi'(0,z),\nu(z)\right\rangle\,dS(z)+o(\delta). \tag{27}$$

Taking into account (26) and (27), we obtain from (23):

$$H_\delta(u_0)=\mu(0)+\mu(0)\delta\int_{\partial\Omega} u_0^2(z)\left\langle\Phi'(0,z),\nu(z)\right\rangle dS(z)+o(\delta).$$

Hence (22) gives:

$$\frac{\mu(\delta)-\mu(0)}{\delta}\geq\mu(0)\int_{\partial\Omega} u_0^2(z)\left\langle\Phi'(0,z),\nu(z)\right\rangle dS(z)+o(1),$$

and thus

$$\liminf_{\delta\to 0+}\frac{\mu(\delta)-\mu(0)}{\delta}\geq\mu(0)\int_{\partial\Omega} u_0^2(z)\left\langle\Phi'(0,z),\nu(z)\right\rangle dS(z).$$

The remaining limits, $\limsup_{\delta\to 0+}$, $\liminf_{\delta\to 0-}$ and $\limsup_{\delta\to 0-}$ of the incremental quotients $\frac{\mu(\delta)-\mu(0)}{\delta}$ can be proved with similar calculations (we only remark that for the upper estimate the continuity of u_δ is needed), and therefore we finally conclude that

$$\lim_{\delta\to 0}\frac{\mu(\delta)-\mu(0)}{\delta}=\mu(0)\int_{\partial\Omega} u_0^2(z)\left\langle\Phi'(0,z),\nu(z)\right\rangle dS(z).$$

This proves (21), and concludes the proof of the theorem. □

An important example of perturbation of a domain is provided when Ω is enlarged in the direction of the unit normal an amount δ. To make this precise, assume $\partial\Omega$ splits into m connected components, and select k of these components $\Gamma_1,\ldots,\Gamma_k$. Set

$$\Omega_\delta=\Omega\bigcup_{i=1}^{k}\{x\in\mathbb{R}^N:\ \operatorname{dist}(x,\Gamma_i)<\delta\}. \tag{28}$$

We have $\Omega_\delta = \Psi(\delta, \Omega)$, where $\Psi(\delta, x) = x + \delta\widetilde{\Phi}(x)$. Moreover, the derivative with respect to δ, $\widetilde{\Phi} = \frac{\partial \Phi}{\partial \delta}(0, \cdot)$. verifies $\widetilde{\Phi} = \nu$ on the components Γ_i while $\widetilde{\Phi} = 0$ on the remaining components of the boundary. Hence, we obtain that $\lambda_1(\Omega_\delta)$ decreases linearly as δ goes to zero.

Corollary 2.8 *Let Ω be a bounded C^1 domain of $\mathbb{R}^N$, and assume Ω_δ is the perturbation of Ω given by (28). Then $\lambda(\delta) = \lambda_1(\Omega_\delta)$ is differentiable with respect to δ at $\delta = 0$, and*

$$\lambda'(0) = -(1 - \lambda_1(\Omega)) \sum_{i=1}^{k} \int_{\Gamma_i} u_0^2(x)\, dS(x) < 0,$$

where u_0 is the positive eigenfunction of $\lambda_1(\Omega)$ normalized with $|u_0|_{L^2(\Omega)} = 1$.

Having established the smoothness and monotonicity of $\lambda_1(\Omega)$, we proceed with the analysis of its asymptotic behavior both for small and large domains Ω. In this context $\Omega_n \to \mathbb{R}^N$ means that the sequence of sets Ω_n contains balls B_{R_n} (centered at a fixed point) with radii $R_n \to +\infty$.

Theorem 2.9 ([28]) *For the principal eigenvalue $\lambda_1(\Omega)$ we have $\lambda_1(\Omega) \to 1$ when $|\Omega| \to 0$ and $\lambda_1(\Omega_n) \to 0$ when $\Omega_n \to \mathbb{R}^N$.*

Proof of Theorem 2.9 We make use of Corollary 2.4. First, notice that if $|\Omega| \to 0$, the integral in the second inequality in (15) goes to zero, and thus $\lambda_1(\Omega) \to 1$.

To prove that $\lambda_1(\Omega_n) \to 0$ when $\Omega_n \to \mathbb{R}^N$, we first show that $\lambda_1(B_R) \to 0$ when $R \to \infty$. According to (15) we have

$$\lambda_1(B_R) \leq 1 - \left(\frac{1}{|B_R|} \int_{B_R} \left(\int_{B_R} J(x-y)\, dy \right)^2 dx \right)^{1/2},$$

hence we need to prove

$$\frac{1}{|B_R|} \int_{B_R} \left(\int_{B_R} J(x-y)\, dy \right)^2 dx \to 1 \tag{29}$$

as $R \to \infty$. We set in the inner integral $y = x - z$, and then $x = Rw$, and arrive at

$$\frac{1}{|B_R|} \int_{B_R} \left(\int_{B_R} J(x-y)\, dy \right)^2 dx = \frac{1}{|B_1|} \int_{B_1} \left(\int_{|z-Rw|<R} J(z)\, dz \right)^2 dw.$$

Now observe that for fixed w with $|w| < 1$ it holds

$$\int_{|z-Rw|<R} J(z)\, dz \to \int_{\mathbb{R}^N} J(z)\, dz = 1,$$

as $R \to \infty$, and (29) follows thanks to the dominated convergence theorem.

Finally, let us show that $\lambda_1(\Omega_n) \to 0$ as $\Omega_n \to \mathbb{R}^N$. We can assume $0 \in \Omega_n$ and that there exists balls B_{R_n} such that $B_{R_n} \subset \Omega_n$ with $R_n \to \infty$, and hence $\lambda_1(\Omega_n) < \lambda_1(B_{R_n})$. It follows that

$$\limsup_{n\to\infty} \lambda_1(\Omega_n) \leq \lim_{n\to\infty} \lambda_1(B_{R_n}) = 0,$$

which concludes the proof. □

To make more precise the information given by Theorem 2.9, we fix a C^1 bounded domain Ω and consider dilatations of it, $\Omega_\gamma = \gamma\Omega$, where $\gamma > 0$ is the dilatation parameter. As a consequence of the previous theorems, we have that $\lambda_1(\Omega_\gamma)$ is a decreasing function of γ and $\lambda_1(\Omega_\gamma) \to 1$ when $\gamma \to 0$, $\lambda_1(\Omega_\gamma) \to 0$ as $\gamma \to +\infty$. Our last theorem describes precisely the asymptotic behavior of $\lambda_1(\Omega_\gamma)$ both when $\gamma \to 0$ and when $\gamma \to \infty$.

Theorem 2.10 *Let Ω be a smooth bounded domain of $\mathbb{R}^N$, and for $\gamma > 0$ denote $\Omega_\gamma = \gamma\Omega$. Then*

$$\lambda_1(\Omega_\gamma) \sim 1 - J(0)|\Omega|\gamma^N \quad \text{as } \gamma \to 0+. \tag{30}$$

If in addition J is radially symmetric and radially decreasing, then

$$\lambda_1(\Omega_\gamma) \sim A(J)\sigma_1(\Omega)\gamma^{-2} \quad \text{as } \gamma \to +\infty, \tag{31}$$

where $\sigma_1(\Omega)$ is the principal eigenvalue of the Laplacian in Ω with Dirichlet boundary conditions,

$$\begin{cases} -\Delta v(x) = \sigma_1(\Omega)v(x), & x \in \Omega, \\ v(x) = 0, & x \in \partial\Omega \end{cases} \tag{32}$$

and the constant $A(J)$ is given by

$$A(J) = \frac{1}{2N}\int_{\mathbb{R}^N} J(z)|z|^2\,dz.$$

Roughly speaking, when conveniently scaled to a large domain, our nonlocal problem resembles a local one. Indeed, for the first eigenvalue of the Laplacian it is well known that $\sigma_1(\Omega_\gamma) = \sigma_1(\Omega)\gamma^{-2}$, therefore the asymptotic behavior as $\gamma \to \infty$ for both problems coincide (up to a factor that depends on J, $A(J)$). Notice that the vanishing rate of $1 - \lambda_1(\Omega_\gamma)$ at $\gamma = 0$ and of $\lambda_1(\Omega_\gamma)$ at $\gamma = +\infty$ is different, which is in contrast with the already mentioned scaling invariance of the Laplacian. This phenomenon is caused by the lack of homogeneity of the convolution term $J * u$. Hence, there is a strong difference between the behaviour of the first eigenvalue for local diffusion and for nonlocal diffusion when the domain is small (case $\gamma \sim 0$) but there is no big difference for large domains (case $\gamma \sim \infty$).

***Proof of Theorem* 2.10** First, we prove (30). Let u_γ be an arbitrary positive eigenfunction associated to $\lambda_1(\Omega_\gamma)$. Choose an arbitrary $\mathbf{e} > 0$. Now, for γ small enough we have

$$J(x-y) \le J(0) + \mathbf{e}$$

if $x, y \in \Omega_\gamma$. Then

$$\begin{aligned}(1-\lambda_1(\Omega_\gamma)) \int_{\Omega_\gamma} u_\gamma(x)\,dx &= \int_{\Omega_\gamma}\int_{\Omega_\gamma} J(x-y)u_\gamma(y)\,dy\,dx \\ &\le (J(0)+\mathbf{e}) \int_{\Omega_\gamma}\int_{\Omega_\gamma} u_\gamma(y)\,dy\,dx = (J(0)+\mathbf{e})|\Omega|\gamma^N \int_{\Omega_\gamma} u_\gamma(y)\,dy.\end{aligned}$$

It follows that

$$\limsup_{\gamma\to 0+} \frac{1-\lambda_1(\Omega_\gamma)}{\gamma^N} \le J(0)|\Omega|.$$

The reverse inequality for the liminf can be proved in an analogous way. This completes the proof of (30).

Let us prove now (31), which is much more involved. The first step is to show that $\lambda_1(\Omega_\gamma) \le C\gamma^{-2}$ for a certain positive constant. Indeed, we will show the more precise estimate,

$$\limsup_{\gamma\to+\infty} \gamma^2 \lambda_1(\Omega_\gamma) \le \sigma_1(\Omega) A(J). \tag{33}$$

Let ϕ be the positive eigenfunction of the Laplacian in Ω, normalized by $\int_\Omega \phi^2(x)\,dx = 1$ and extended by zero outside Ω. Taking as a test function $\phi_\gamma(x) = \phi(x/\gamma)$ in the variational characterization (14), we obtain

$$\lambda_1(\Omega_\gamma) \le \frac{\displaystyle\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\left(\phi\left(\frac{x}{\gamma}\right) - \phi\left(\frac{y}{\gamma}\right)\right)^2 dy\,dx}{\displaystyle\int_{\Omega_\gamma} \phi\left(\frac{x}{\gamma}\right)^2 dx}.$$

Setting $x = y + z$ and $y = \gamma w$ in the integrals of the numerator, and $x = \gamma\theta$ in the integral of the denominator, we obtain

$$\begin{aligned}\lambda_1(\Omega_\gamma) &\le \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(z)\left(\phi\left(w + \frac{z}{\gamma}\right) - \phi(w)\right)^2 dw\,dz \\ &= \frac{1}{2}\int_{B_1}\int_{\mathbb{R}^N} J(z)\left(\phi\left(w + \frac{z}{\gamma}\right) - \phi(w)\right)^2 dw\,dz.\end{aligned}$$

Taking into account that the function ϕ belongs to $W^{1,\infty}(\mathbb{R}^N)$, we have

$$\phi\left(w+\frac{z}{\gamma}\right)-\phi(w)=\frac{1}{\gamma}\int_0^1\left\langle\nabla\phi\left(w+s\frac{z}{\gamma}\right),z\right\rangle ds$$

for every $w\in\mathbb{R}^N$, $z\in B_1$. Hence,

$$\gamma^2\lambda_1(\Omega_\gamma)\le\frac{1}{2}\int_{B_1}\int_{\mathbb{R}^N}J(z)\left(\int_0^1\left\langle\nabla\phi\left(w+s\frac{z}{\gamma}\right),z\right\rangle ds\right)^2 dw\,dz. \tag{34}$$

Thanks to dominated convergence theorem, we can pass to the limit in (34) as $\gamma\to+\infty$ to obtain,

$$\limsup_{\gamma\to+\infty}\gamma^2\lambda_1(\Omega_\gamma)\le\frac{1}{2}\int_{B_1}\int_{\mathbb{R}^N}J(z)\left\langle\nabla\phi(w),z\right\rangle^2 dw\,dz. \tag{35}$$

In the last integral, we apply Fubini's theorem to obtain

$$\begin{aligned}\int_{B_1}\int_{\mathbb{R}^N}J(z)\left\langle\nabla\phi(w),z\right\rangle^2 dw\,dz&=\int_{\mathbb{R}^N}\int_{B_1}J(z)\left\langle\nabla\phi(w),z\right\rangle^2 dz\,dw\\&=\sum_{i,j=1}^N\int_{\mathbb{R}^N}\frac{\partial\phi}{\partial x_i}(w)\frac{\partial\phi}{\partial x_j}(w)\left(\int_{B_1}J(z)z_iz_j\,dz\right)dw.\end{aligned}$$

We notice that the integrals $\int_{B_1}J(z)z_iz_j\,dz$ vanish by symmetry when $i\neq j$, while they are all equal to $2A(J)$ when $i=j$. Thus (35) implies (33).

Now let φ_γ be a positive eigenfunction associated to $\lambda_1(\Omega_\gamma)$, and set $\psi_\gamma(x)=\varphi_\gamma(\gamma x)$, $x\in\Omega$. We normalize ψ_γ by $\int_\Omega\psi_\gamma^2(x)\,dx=1$. According to the variational characterization (14), we have

$$2\lambda_1(\Omega_\gamma)=\int_{\widetilde{\Omega}}\int_{\widetilde{\Omega}}J_\gamma(x-y)(\psi_\gamma(x)-\psi_\gamma(y))^2dxdy,$$

where $J_\gamma(x)=\gamma^NJ(\gamma x)$, and $\widetilde{\Omega}$ is a smooth bounded domain such that $\Omega\subset\subset\widetilde{\Omega}$.

Now let $\gamma_n\to+\infty$ be an arbitrary sequence. By passing to a subsequence, we may assume $\psi_n:=\psi_{\gamma_n}$ converges weakly in $L^2(\widetilde{\Omega})$ to a function ψ. Since J is radially decreasing and $\lambda_1(\Omega_{\gamma_n})\le C\gamma_n^{-2}$, thanks to (33), we may apply Proposition 3.2 of [4], which implies that $\psi_n\to\psi$ strongly in $L^2(\widetilde{\Omega})$ with $\psi\in H^1(\widetilde{\Omega})$. Since $\psi=0$ in $\widetilde{\Omega}\setminus\Omega$, we obtain

$$\psi\in H_0^1(\Omega)\ \text{ and }\ \int_\Omega\psi^2(x)\,dx=1. \tag{36}$$

We claim that ψ is the principal eigenfunction of a multiple of the Laplacian in Ω with Dirichlet boundary conditions, and this implies

$$\lim_{n\to\infty} \gamma_n^2 \lambda_1(\Omega_{\gamma_n}) = A(J)\sigma_1(\Omega).$$

Indeed, thanks to (33), we may assume that $\gamma_n^2\lambda_1(\Omega_{\gamma_n}) \to \lambda_0 \geq 0$. We notice that ψ_n satisfies

$$J_{\gamma_n} * \psi_n - \psi_n = -\lambda_1(\Omega_{\gamma_n})\psi_n. \tag{37}$$

Choose an arbitrary function $v \in C_0^\infty(\Omega)$. Multiply (37) by v and integrate in Ω to obtain

$$\begin{aligned} &\gamma^N \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(\gamma(x-y))\psi_n(y)v(x)\,dy\,dx - \int_{\mathbb{R}^N} \psi_n(x)v(x)\,dx \\ &\quad = -\lambda_1(\Omega_{\gamma_n}) \int_{\mathbb{R}^N} \psi_n(x)v(x)\,dx. \end{aligned} \tag{38}$$

Note that all the integrals in what follows may be considered in $\mathbb{R}^N$, since v and ψ_n vanish outside Ω. Thanks to Fubini's theorem, the integrals in the left-hand side of (38) can be rewritten to obtain

$$\begin{aligned} &\gamma_n^N \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(\gamma_n(x-y))(v(y)-v(x))\psi_n(x)\,dx\,dy \\ &\quad = -\lambda_1(\Omega_{\gamma_n}) \int_{\mathbb{R}^N} \psi_n(x)v(x)\,dx, \end{aligned} \tag{39}$$

since J has unit integral. Letting $z = -\gamma_n(x-y)$ in the first integral of (39), we get

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(z)\left(v\left(x+\frac{z}{\gamma_n}\right) - v(x)\right)\psi_n(x)\,dx\,dz \\ &\quad = -\lambda_1(\Omega_{\gamma_n}) \int_{\mathbb{R}^N} \psi_n(x)v(x)\,dx. \end{aligned} \tag{40}$$

We now use Taylor expansion up to the second order in v:

$$\begin{aligned} &v\left(x+\frac{z}{\gamma_n}\right) - v(x) \\ &= \frac{1}{\gamma_n}\sum_{i=1}^N \frac{\partial v}{\partial x_i}(x) z_i + \frac{1}{\gamma_n^2}\sum_{i,j=1}^N \int_0^1 (1-s)\frac{\partial^2 v}{\partial x_i \partial x_j}\left(x+\frac{sz}{\gamma_n}\right) z_i z_j\, ds, \end{aligned}$$

which, when plugged into (40), gives

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(z)\left(\gamma_n \sum_{i=1}^N \frac{\partial v}{\partial x_i}(x) z_i + \sum_{i,j=1}^N \int_0^1 (1-s)\frac{\partial^2 v}{\partial x_i \partial x_j}\left(x+\frac{sz}{\gamma_n}\right) z_i z_j ds\right)\psi_n(x)\,dxdz$$
$$= -\gamma_n^2 \lambda_1(\Omega_{\gamma_n}) \int \psi_n(x) v(x) dx.$$

Next we analyze the integrals involving the first derivatives of v. Notice that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(z)\frac{\partial v}{\partial x_i}(x) z_i \psi_n(x) dxdz = \int_{\mathbb{R}^N} \frac{\partial v}{\partial x_i}(x)\psi_n(x)\left(\int_{\mathbb{R}^N} J(z) z_i dz\right) dx = 0$$

by the symmetry of J. Hence,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(z)\left(\sum_{i,j=1}^N \int_0^1 (1-s)\frac{\partial^2 v}{\partial x_i \partial x_j}\left(x+\frac{sz}{\gamma_n}\right) z_i z_j\, ds\right)\psi_n(x)\,dx\,dz$$
$$= -\gamma_n^2 \lambda_1(\Omega_{\gamma_n}) \int_{\mathbb{R}^N} \psi_n(x) v(x)\,dx. \tag{41}$$

Now we pass to the limit as $n \to \infty$ in (41). Notice that

$$\frac{\partial^2 v}{\partial x_i \partial x_j}\left(x+\frac{sz}{\gamma_n}\right) \to \frac{\partial^2 v}{\partial x_i \partial x_j}(x)$$

uniformly for $x \in \Omega$, $z \in B_1$, and hence the first term in (41) converges to

$$\frac{1}{2}\int_{\mathbb{R}^N} \sum_{i,j=1}^N \frac{\partial^2 v}{\partial x_i \partial x_j}(x)\psi(x)\left(\int_{\mathbb{R}^N} J(z) z_i z_j dz\right) dx = A(J)\Delta v(x)\psi(x).$$

Thus

$$A(J)\int_{\mathbb{R}^N} \Delta v(x)\psi(x)\,dx = -\lambda_0 \int_{\mathbb{R}^N} \psi(x) v(x)\,dx. \tag{42}$$

According to (36), we may integrate by parts in the integral of the left-hand side in (42) to obtain

$$A(J)\int_{\mathbb{R}^N} \nabla v(x)\nabla\psi(x) dx = \lambda_0 \int_{\mathbb{R}^N} \psi(x) v(x)\,dx.$$

Since $v \in C_0^\infty(\Omega)$ is arbitrary, and $\psi \in H_0^1(\Omega)$ with $\psi \not\equiv 0$, we have that ψ is a positive eigenfunction associated to $-\Delta$ in Ω. Thus $\lambda_0 = A(J)\sigma_1(\Omega)$, and since the sequence γ_n was arbitrary, the theorem is proved. □

2.2 Neumann Boundary Conditions

For the Neumann problem (3) the first eigenvalue is zero (with an eigenfunction given by a constant). Hence we look for the first nontrivial eigenvalue, given by

$$\beta_1(J,\Omega) = \inf_{u\in L^2(\Omega),\int_\Omega u=0} \frac{\frac{1}{2}\int_\Omega\int_\Omega J(x-y)(u(y)-u(x))^2\,dy\,dx}{\int_\Omega (u(x))^2\,dx} \tag{43}$$

In this case the associated equation reads as

$$\int_\Omega J(x-y)(\varphi(x)-\varphi(y))\,dy = \beta_1\varphi(x), \quad x\in\Omega. \tag{44}$$

Notice that a minimizer of (43) is a solution to (44).

Our first result shows that $\beta_1(J,\Omega)$ is indeed nontrivial.

Proposition 2.11 *The quantity $\beta_1(J,\Omega)$ defined by (43) is strictly positive.*

Proof It is clear that $\beta_1 \geq 0$. Let us prove that β_1 is in fact strictly positive. To this end, consider the subspace H of $L^2(\Omega)$ given by the orthogonal to the constants, and the symmetric (self-adjoint) operator $T : H \mapsto H$ given by

$$T(u)(x) = \int_\Omega J(x-y)(u(x)-u(y))\,dy = -\int_\Omega J(x-y)u(y)\,dy + A(x)u(x),$$

where $A(x) = \int_\Omega J(x-y)dy$. Note that T is the sum of an invertible operator and a compact operator. Since T is symmetric, its spectrum verifies $\sigma(T) \subset [m, M]$, where

$$m = \inf_{u\in H,\ \|u\|_{L^2(\Omega)}=1} \langle Tu, u\rangle \qquad \text{and} \qquad M = \sup_{u\in H,\ \|u\|_{L^2(\Omega)}=1} \langle Tu, u\rangle.$$

Remark that

$$m = \inf_{u\in H,\ \|u\|_{L^2(\Omega)}=1} \int_\Omega\int_\Omega J(x-y)(u(x)-u(y))\,dy\,u(x)\,dx = \beta_1.$$

Then $m \geq 0$. Let us show now that

$$m > 0.$$

If not, since $m \in \sigma(T)$, $T : H \mapsto H$ is not invertible. Using Fredholm's alternative, this implies that there exists a nontrivial $u \in H$ such that $T(u) = 0$, but then, a simple computation shows that u must be constant in Ω, which is a contradiction.

□

To study the asymptotic behaviour of the solutions, an upper estimate on β_1 is needed. Here and in what follows, χ_D denotes the characteristic function of the set D.

Lemma 2.12 ([2]) *Let β_1 be given by (43) then*

$$\beta_1 \leq \min_{x \in \overline{\Omega}} \int_\Omega J(x-y)\,dy. \tag{45}$$

Proof Let

$$A(x) = \int_\Omega J(x-y)\,dy.$$

Since $\overline{\Omega}$ is compact and A is continuous there exists a point $x_0 \in \overline{\Omega}$ such that

$$A(x_0) = \min_{x \in \overline{\Omega}} A(x).$$

For every ε small let us choose two disjoint balls of radius ε contained in Ω, $B(x_{1,\varepsilon}, \varepsilon)$ and $B(x_{2,\varepsilon}, \varepsilon)$ in such a way that $x_{i,\varepsilon} \to x_0$ as $\varepsilon \to 0$. By using

$$u_\varepsilon(x) = \chi_{B(x_{1,\varepsilon},\varepsilon)}(x) - \chi_{B(x_{2,\varepsilon},\varepsilon)}(x)$$

as a test function in the definition of β_1 for ε small, it holds

$$\begin{aligned}
\beta_1 &\leq \frac{\dfrac{1}{2}\displaystyle\int_\Omega\int_\Omega J(x-y)(u_\varepsilon(y)-u_\varepsilon(x))^2\,dy\,dx}{\displaystyle\int_\Omega (u_\varepsilon(x))^2\,dx} \\
&= \frac{\displaystyle\int_\Omega A(x)u_\varepsilon^2(x)\,dx - \int_\Omega\int_\Omega J(x-y)u_\varepsilon(y)\,u_\varepsilon(x)\,dy\,dx}{\displaystyle\int_\Omega (u_\varepsilon(x))^2\,dx} \\
&= \frac{\displaystyle\int_\Omega A(x)u_\varepsilon^2(x)\,dx - \int_\Omega\int_\Omega J(x-y)u_\varepsilon(y)\,u_\varepsilon(x)\,dy\,dx}{2|B(0,\varepsilon)|}.
\end{aligned}$$

Using the continuity of A and the explicit form of u_ε we obtain

$$\lim_{\varepsilon\to 0} \frac{\displaystyle\int_\Omega A(x)u_\varepsilon^2(x)\,dx}{2|B(0,\varepsilon)|} = A(x_0)$$

and

$$\lim_{\varepsilon\to 0} \frac{\displaystyle\int_\Omega\int_\Omega J(x-y)u_\varepsilon(y)\,u_\varepsilon(x)\,dy\,dx}{2|B(0,\varepsilon)|} = 0.$$

Therefore, (45) follows. □

In the Neumann case we find an asymptotic behaviour analogous to the one that holds for the heat equation. The solution $u(x,t)$ of (3) converge exponentially to the mean value of the initial datum, and the decay is determined by the eigenvalue β_1.

First we show that the solution u of (3) preserves the total mass.

Proposition 2.13 *For every $u_0 \in L^1(\Omega)$ the unique solution u of (3) preserves the total mass in Ω, that is,*

$$\int_\Omega u(y,t)\,dy = \int_\Omega u_0(y)\,dy.$$

Proof Since

$$u(x,t) - u_0(x) = \int_0^t \int_\Omega J\,(x-y)\,(u(y,s) - u(x,s))\,dy\,ds,$$

integrating in x and applying Fubini's theorem, it follows

$$\int_\Omega u(x,t)\,dx - \int_\Omega u_0(x)\,dx = 0.$$

□

The corresponding stationary problem to (3) is described by the equation

$$0 = \int_\Omega J(x-y)(\varphi(y) - \varphi(x))\,dy. \qquad (46)$$

The only solutions for this equation are the constants.

Proposition 2.14 *Every stationary solution of* (3) *is constant in* Ω*, and, since the total mass is preserved, the unique stationary solution with the same mass as* u_0 *is*

$$\varphi = \frac{1}{|\Omega|}\int_\Omega u_0(x)\,dx.$$

Proof Observe (46) implies that φ is a continuous function. Set

$$K = \max_{x\in\overline{\Omega}} \varphi(x)$$

and consider the set $\mathcal{A} = \left\{x \in \overline{\Omega} \ : \ \varphi(x) = K\right\}$. The set $\mathcal{A}$ is clearly closed and non empty. We claim that it is also open in $\overline{\Omega}$. Let $x_0 \in \mathcal{A}$, then

$$0 = \int_\Omega J(x_0 - y)(\varphi(y) - \varphi(x_0))\,dy,$$

therefore, since $\varphi(y) \le \varphi(x_0)$, this implies $\varphi(y) = \varphi(x_0)$ for all $y \in \Omega \cap B(x_0, d)$, for any $B(0, d) \subset \text{supp}(J)$. Hence $\mathcal{A}$ is open as claimed. Consequently, as Ω is connected, $\mathcal{A} = \overline{\Omega}$ and φ is constant. □

Theorem 2.15 ([2]) *For every* $u_0 \in L^2(\Omega)$ *the solution* $u(x, t)$ *of* (3) *satisfies*

$$\left\| u(\cdot, t) - \frac{1}{|\Omega|}\int_\Omega u_0 \right\|_{L^2(\Omega)} \le e^{-\beta_1 t}\left\| u_0 - \frac{1}{|\Omega|}\int_\Omega u_0 \right\|_{L^2(\Omega)}, \tag{47}$$

where β_1 *is given by* (43)*. Moreover, if* u_0 *is continuous and bounded, there exists a positive constant* $C > 0$ *such that*

$$\left\| u(\cdot, t) - \frac{1}{|\Omega|}\int_\Omega u_0 \right\|_{L^\infty(\Omega)} \le Ce^{-\beta_1 t}. \tag{48}$$

Proof Let

$$H(t) = \frac{1}{2}\int_\Omega \left(u(x, t) - \frac{1}{|\Omega|}\int_\Omega u_0 \right)^2 dx.$$

Differentiating, using (43) and the conservation of the total mass, we obtain

$$\begin{aligned} H'(t) &= -\frac{1}{2}\int_\Omega\int_\Omega J(x - y)(u(y, t) - u(x, t))^2\,dy\,dx \\ &\le -\beta_1 \int_\Omega \left(u(x, t) - \frac{1}{|\Omega|}\int_\Omega u_0 \right)^2 dx. \end{aligned}$$

Hence

$$H'(t) \le -2\beta_1 H(t).$$

Therefore, integrating,

$$H(t) \le e^{-2\beta_1 t} H(0),$$

and (47) follows.

In order to prove (48) let $w(x,t)$ denote the difference

$$w(x,t) = u(x,t) - \frac{1}{|\Omega|} \int_\Omega u_0.$$

We seek for an exponential estimate in L^∞ of the decay of $w(x,t)$. The linearity of the equation implies that $w(x,t)$ is a solution of (3) and satisfies

$$w(x,t) = e^{-A(x)t} w_0(x) + e^{-A(x)t} \int_0^t e^{A(x)s} \int_\Omega J(x-y) w(y,s)\, dy\, ds,$$

where $A(x) = \int_\Omega J(x-y)dx$. By using (47) and Hölder's inequality it follows that

$$|w(x,t)| \le e^{-A(x)t} |w_0(x)| + C e^{-A(x)t} \int_0^t e^{A(x)s - \beta_1 s}\, ds.$$

Therefore, $w(x,t)$ decays to zero exponentially fast and, moreover, (48) holds thanks to Lemma 2.12. □

An analysis of the dependence of the first eigenvalue β_1 with respect to the domain is left open.

2.3 Optimal Constants in L^q

Here we briefly comment on the L^q case. We refer to [2] for details. The main difficulty is to show that there exists an eigenfunction associated with the usual minimization of the corresponding Raleigh quotients. Below, we just show that the associated optimal constants are positive.

2.3.1 The Dirichlet Case

Proposition 2.16 ([2]) *Given $q \geq 1$, Ω a bounded domain in $\mathbb{R}^N$, there exists $\lambda = \lambda(J, \Omega, q) > 0$ such that*

$$\lambda \int_{\Omega} |u(x)|^q \, dx \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x-y)|\overline{u}(y) - \overline{u}(x)|^q \, dy \, dx \tag{49}$$

for all $u \in L^q(\Omega)$.

Proof Let $r, \alpha > 0$ such that $J(x) \geq \alpha$ in $B(0, r)$. Let

$$\begin{aligned}
B_0 &= \{x \in \Omega_J \setminus \overline{\Omega} \ : \ d(x, \Omega) \leq r/2\}, \\
B_1 &= \{x \in \Omega \ : \ d(x, B_0) \leq r/2\}, \\
B_j &= \{x \in \Omega \setminus \cup_{k=1}^{j-1} B_k \ : \ d(x, B_{j-1}) \leq r/2\}, \quad j = 2, 3, \ldots
\end{aligned}$$

Observe that we can cover Ω by a finite number of non null sets $\{B_j\}_{j=1}^{l_r}$. Now

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x-y)|\overline{u}(y) - \overline{u}(x)|^q \, dy \, dx \geq \int_{B_j} \int_{B_{j-1}} J(x-y)|\overline{u}(y) - \overline{u}(x)|^q \, dy \, dx,$$

for $j = 1, \ldots, l_r$, and

$$\begin{aligned}
&\int_{B_j} \int_{B_{j-1}} J(x-y)|\overline{u}(y) - \overline{u}(x)|^q \, dy \, dx \\
&\geq \frac{1}{2^q} \int_{B_j} \int_{B_{j-1}} J(x-y)|u(x)|^q \, dy \, dx - \int_{B_j} \int_{B_{j-1}} J(x-y)|\overline{u}(y)|^q \, dy \, dx \\
&= \frac{1}{2^q} \int_{B_j} \left(\int_{B_{j-1}} J(x-y) \, dy \right) |u(x)|^q dx - \int_{B_{j-1}} \left(\int_{B_j} J(x-y) \, dx \right) |\overline{u}(y)|^q dy \\
&\geq \alpha_j \int_{B_j} |u(x)|^q \, dx - \beta \int_{B_{j-1}} |\overline{u}(y)|^q \, dy,
\end{aligned}$$

where

$$\alpha_j = \frac{1}{2^q} \min_{x \in \overline{B_j}} \int_{B_{j-1}} J(x-y) \, dy > 0$$

(since $J(x) \geq \alpha$ in $B(0, r)$) and

$$\beta = \int_{\mathbb{R}^N} J(x) \, dx.$$

Hence

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)|\overline{u}(y)-\overline{u}(x)|^q dydx \geq \alpha_j \int_{B_j} |u(x)|^q dx - \beta \int_{B_{j-1}} |\overline{u}(y)|^q dy.$$

Therefore, since $\overline{u}(y) = 0$ if $y \in B_0$, $\overline{u}(y) = u(y)$ if $y \in B_j$, $j = 1, \ldots, l_r$, $B_j \cap B_i = \emptyset$, for all $i \neq j$ and $|\Omega \setminus \cup_{j=1}^{j_r} B_j| = 0$, it is easy to see, by cancelation, that there exists $\lambda = \lambda(J, \Omega, q) > 0$ such that

$$\lambda \int_\Omega |u|^q \leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)|\overline{u}(y)-u(x)|^q \, dy \, dx.$$

This ends the proof. □

2.3.2 The Neumann Case

Theorem 2.17 ([2]) *Given $q \geq 1$, J as above and Ω a bounded domain in $\mathbb{R}^N$, the quantity*

$$\beta_q(J, \Omega, q) = \inf_{u \in L^q(\Omega), \int_\Omega u = 0} \frac{\dfrac{1}{2}\displaystyle\int_\Omega\int_\Omega J(x-y)|u(y)-u(x)|^q \, dy \, dx}{\displaystyle\int_\Omega |u(x)|^q \, dx}$$

is strictly positive. Consequently

$$\beta_q \int_\Omega \left| u - \frac{1}{|\Omega|}\int_\Omega u \right|^q \leq \frac{1}{2}\int_\Omega\int_\Omega J(x-y)|u(y)-u(x)|^q \, dy \, dx, \tag{50}$$

for every $u \in L^q(\Omega)$.

Proof It is enough to prove that there exists a constant c such that

$$\|u\|_q \leq c\left(\left(\int_\Omega\int_\Omega J(x-y)|u(y)-u(x)|^q dydx\right)^{1/q} + \left|\int_\Omega u\right|\right), \tag{51}$$

for every $u \in L^q(\Omega)$.

Let $r > 0$ such that $J(z) \geq \alpha > 0$ in $B(0, r)$. Since $\overline{\Omega} \subset \cup_{x\in\Omega} B(x, r/2)$, there exists $\{x_i\}_{i=1}^m \subset \Omega$ such that $\Omega \subset \cup_{i=1}^m B(x_i, r/2)$. Let $0 < \delta < r/2$ such that $B(x_i, \delta) \subset \Omega$ for all $i = 1, \ldots, m$. Then, for any $\hat{x}_i \in B(x_i, \delta)$, $i = 1, \ldots, m$,

$$\Omega = \bigcup_{i=1}^m (B(\hat{x}_i, r) \cap \Omega). \tag{52}$$

Let us argue by contradiction. Suppose that (51) is false. Then, there exists $u_n \in L^q(\Omega)$, with $\|u_n\|_{L^q(\Omega)} = 1$, and satisfying

$$1 \geq n\left(\left(\int_\Omega \int_\Omega J(x-y)|u_n(y)-u_n(x)|^q dydx\right)^{1/q} + \left|\int_\Omega u_n\right|\right) \quad \forall n \in \mathbb{N}.$$

Consequently,

$$\lim_n \int_\Omega \int_\Omega J(x-y)|u_n(y)-u_n(x)|^q \, dy \, dx = 0 \tag{53}$$

and

$$\lim_n \int_\Omega u_n = 0. \tag{54}$$

Let

$$F_n(x,y) = J(x-y)^{1/q}|u_n(y)-u_n(x)|$$

and

$$f_n(x) = \int_\Omega J(x-y)|u_n(y)-u_n(x)|^q \, dy.$$

From (54), it follows that

$$f_n \to 0 \quad \text{in } L^1(\Omega).$$

Passing to a subsequence if necessary, we can assume that

$$f_n(x) \to 0 \quad \forall x \in \Omega \setminus B_1, \quad B_1 \text{ null.} \tag{55}$$

On the other hand, by (53), we also have that

$$F_n \to 0 \quad \text{en } L^q(\Omega \times \Omega).$$

So we can suppose, up to a subsequence,

$$F_n(x,y) \to 0 \quad \forall (x,y) \in \Omega \times \Omega \setminus C, \quad C \text{ null.} \tag{56}$$

Let $B_2 \subset \Omega$ a null set satisfying that,

$$\text{for all } x \in \Omega \setminus B_2, \text{ the section } C_x \text{ of } C \text{ is null.} \tag{57}$$

Let $\hat{x}_1 \in B(x_1, \delta) \setminus (B_1 \cup B_2)$, then there exists a subsequence, denoted equal, such that

$$u_n(\hat{x}_1) \to \lambda_1 \in [-\infty, +\infty].$$

Consider now $\hat{x}_2 \in B(x_2, \delta) \setminus (B_1 \cup B_2)$, then up to a subsequence, we can assume

$$u_n(\hat{x}_2) \to \lambda_2 \in [-\infty, +\infty].$$

So, successively, for $\hat{x}_m \in B(x_m, \delta) \setminus (B_1 \cup B_2)$, there exists a subsequence, again denoted equal, such that

$$u_n(\hat{x}_m) \to \lambda_m \in [-\infty, +\infty].$$

By (56) and (57),

$$u_n(y) \to \lambda_i \quad \forall y \in (B(\hat{x}_i, r) \cap \Omega) \setminus C_{\hat{x}_i}.$$

Now, by (52),

$$\Omega = (B(\hat{x}_1, r) \cap \Omega) \cup (\cup_{i=2}^{m} (B(\hat{x}_i, r) \cap \Omega)).$$

Hence, since Ω is a bounded domain, there exists $i_2 \in \{2, .., m\}$ such that

$$(B(\hat{x}_1, r) \cap \Omega) \cap (B(\hat{x}_{i_2}, r) \cap \Omega) \neq \emptyset.$$

Therefore, $\lambda_1 = \lambda_{i_2}$. Let us call $i_1 := 1$. Again, since

$$\Omega = ((B(\hat{x}_{i_1}, r) \cap \Omega) \cup ((B(\hat{x}_{i_1}, r) \cap \Omega)) \cup (\cup_{i \in \{1,\dots,m\} \setminus \{i_1, i_2\}} (B(\hat{x}_i, r) \cap \Omega)),$$

and there exists $i_3 \in \{1, \dots, m\} \setminus \{i_1, i_2\}$ such that

$$((B(\hat{x}_{i_1}, r) \cap \Omega) \cup ((B(\hat{x}_{i_1}, r) \cap \Omega)) \cap (B(\hat{x}_{i_3}, r) \cap \Omega) \neq \emptyset.$$

Consequently

$$\lambda_{i_1} = \lambda_{i_2} = \lambda_{i_3}.$$

Using the same argument we get

$$\lambda_1 = \lambda_2 = \dots = \lambda_m = \lambda.$$

If $|\lambda| = +\infty$, we have shown that

$$|u_n(y)|^q \to +\infty \quad \text{for almost every } y \in \Omega,$$

which contradicts $\|u_n\|_{L^q(\Omega)} = 1$ for all $n \in \mathbb{N}$. Hence λ is finite.

On the other hand, by (55), $f_n(\hat{x}_i) \to 0$, $i = 1, \ldots, m$. Hence,

$$F_n(\hat{x}_1, .) \to 0 \quad \text{in } L^q(\Omega).$$

Since $u_n(\hat{x}_1) \to \lambda$, from the above we conclude that

$$u_n \to \lambda \quad \text{in } L^q(B(\hat{x}_i, r) \cap \Omega).$$

Using again a compactness argument we get

$$u_n \to \lambda \quad \text{in } L^q(\Omega).$$

By (54), $\lambda = 0$, so

$$u_n \to 0 \quad \text{in } L^q(\Omega),$$

which contradicts $\|u_n\|_{L^q(\Omega)} = 1$. □

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Comparing Banach Spaces for Systems of Free Random Variables Followed by the Semicircular Law

Ilwoo Cho and Palle Jorgensen

Abstract We study certain Banach-space operators from noncommutative free probability, acting on systems of free random variables whose free distributions are followed by the semicircular law. In particular, we consider (i) non-self-adjoint free random variables T of a C^*-probability space followed by the semicircular law in the sense that: all n-th joint free moments of $\{T, T^*\}$ are identical to the $\frac{n}{2}$-th Catalan numbers $c_{\frac{n}{2}}$, for all $n \in \mathbb{N}$, with axiomatization: $c_{\frac{n}{2}} = 0$, whenever $\frac{n}{2} \notin \mathbb{N}$, (ii) some structure theorems of C^*-probability spaces generated by countable-infinitely many free random variables followed by the semicircular law, and (iii) certain Banach-space operators acting on free random variables of (i) and (ii).

Keywords Free probability · Semicircular elements · Free random variables followed by the semicircular law · Banach-space operators

1 Introduction

As the counterpart of measure spaces in commutative function theory, let (B, ψ) be a topological (noncommutative free) $*$-probability space (e.g., a C^*-probability space, or, a W^*-probability space, or, a Banach $*$-probability space, etc.) of a topological $*$-algebra B (a C^*-algebra, respectively, a von Neumann algebra, respectively, a Banach $*$-algebra, etc.), and a bounded linear functional ψ. Moreover, as the noncommutative version of probability spaces in commutative theory, we assume that (B, ψ) is unital in the sense that it contains the unity (or, the

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multiplication-identity) $1_B \in B$, satisfying

$$1_B \cdot T = T = T \cdot 1_B, \ \forall T \in B,$$

and

$$\psi(1_B) = 1 = \psi\left(1_B^n\right), \ \forall n \in \mathbb{N}.$$

Note that, even though a given topological $*$-probability space (B, ψ) is not unital, our main results of the text hold under non-unitality. Throughout this paper, all given (noncommutative) free probability spaces are assumed to be unital just for convenience.

An element $T \in B$ is said to be a free random variable if we understand it as an element of (B, ψ). For example, a self-adjoint free random variable $S \in (B, \psi)$ is a self-adjoint element $S \in B$ satisfying $S = S^*$ in B, where S^* is the adjoint of S in B.

For any arbitrary free random variables $T_1, \ldots, T_s \in (B, \psi)$, for $s \in \mathbb{N}$, the free distribution of $T_1, \ldots, T_s$ is characterized by the joint free moments,

$$\psi\left(\prod_{l=1}^{n} T_{i_l}^{r_l}\right) = \psi\left(T_{i_1}^{r_1} T_{i_2}^{r_2} \ldots T_{i_n}^{r_n}\right),$$

or, the joint free cumulants,

$$k_n^{\psi}\left(T_{i_1}^{r_1}, \ldots, T_{i_n}^{r_n}\right) = \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} \psi\left(\prod_{l \in V} T_{i_l}^{r_l}\right)\right) \mu(\pi, 1_n), \tag{1}$$

of $\{T, T^*\}$, for all $(i_1, \ldots, i_n) \in \{1, \ldots, s\}^n$ and $(r_1, \ldots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$, where $k_{\bullet}^{\psi}(.)$ is the free cumulant on B in terms of the linear functional ψ, by the Möbius inversion (e.g., [17, 22–25]). In (1), the set $NC(n)$ is the lattice of all "noncrossing" partitions over a discrete set $\{1, \ldots, n\}$, with its maximal partition,

$$1_n = \{(1, \ldots, n)\},$$

the single-block partition with its block $(1, \ldots, n)$, for all $n \in \mathbb{N}$; and μ is the Möbius functional of [23] satisfying

$$\mu(0_n, 1_n) = (-1)^{n+1} c_n, \ \forall n \in \mathbb{N},$$

and

$$\sum_{\theta \in NC(n)} \mu(\theta, 1_n) = 0,$$

where

$$c_n = \frac{(2n)!}{n!(n+1)!}, \ \forall n \in \mathbb{N},$$

and $0_n = \{(1), (2), \ldots, (n)\}$is the minimal partition of $NC(n)$, having its n-many blocks (1), (2), ..., (n), and $[\theta_1, \theta_2]$ is the interval in $NC(n)$ under the partial ordering,

$$\theta_1 \leq \theta_2, \Longleftrightarrow \forall U \in \theta_1, \ \exists V \in \theta_2, \text{ s.t., } U \subseteq V,$$

where "$U \in \theta_1$" means "U is a block of θ_1."

By (1), the free distribution of a single free random variable $T \in (B, \psi)$ is characterized by the joint free moments,

$$\psi\left(\prod_{l=1}^{n} T^{r_l}\right) = \psi\left(T^{r_1} T^{r_2} ... T^{r_n}\right),$$

or, the joint free cumulants,

$$k_n^{\psi}\left(T^{r_1}, T^{r_2}, \ldots, T^{r_n}\right),$$

of $\{T, T^*\}$, for all $(r_1, \ldots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$. As a special case, if a free random variable $S \in (B, \psi)$ is self-adjoint, then the free distribution of S is characterized by the free moment sequence,

$$\left(\psi\left(S^n\right)\right)_{n=1}^{\infty},$$

equivalently, by the free cumulant sequence,

$$\left(k_n^{\psi}\left(\underbrace{S, S, \ldots\ldots, S}_{n\text{-times}}\right)\right)_{n=1}^{\infty}, \tag{2}$$

since $S = S^*$ in (B, ψ).

The main purposes of this paper are (i) to show the existence of "non-self-adjoint" free random variables T in a certain unital topological $*$-probability space (B, ψ), whose free distributions are followed by the semicircular law in the sense that:

$$\psi\left(\prod_{l=1}^{n} T^{r_l}\right) = \omega_n c_{\frac{n}{2}},$$

for all $(r_1, \ldots, r_n) \in \{1, *\}^n$, where

$$\omega_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ \\ 0 & \text{if } n \text{ is odd,} \end{cases} \tag{3}$$

for all $n \in \mathbb{N}$, and

$$c_k = \frac{1}{k+1}\binom{2k}{k} = \left(\frac{1}{k+1}\right)\left(\frac{(2k)!}{k!(2k-k)!}\right) = \frac{(2k)!}{k!(k+1)!}, \tag{4}$$

are the k-th Catalan numbers for all $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$; (ii) to study free-distributional data in (B, ψ) induced by the free random variables of (i); (iii) to investigate structure theorems of a unital C^*-probability space generated by countable-infinitely many free random variables of (i), (iv) to consider certain actions of Banach-space operator on the free random variables of (iii); and (v) to characterize how the actions of (iv) deform the free distributions of (ii).

1.1 Background

In both classical and free probability theory, *the semicircular law* is an important topic (e.g., [1, 2, 5–8, 10–12, 20, 21, 28, 30]). The (classical, or free) distributions of semicircular elements, called the semicircular law, are well-known in statistical language. In particular, *operators* satisfying the semicircular law (under a fixed linear functional) have been studied in free probability, and they are well-characterized in analytic, or combinatorial free-probabilistic senses (e.g., [1, 17, 18, 21, 28–30]).

Semicircular elements play a key role in operator-algebraic free theory by the (*free*) *central limit theorem*(*s*) (e.g., see [2, 17, 19, 28–30]). Roughly speaking, the semicircular law is the noncommutative counterpart of the classical Gaussian (or, the normal) distribution in commutative theory. From combinatorial approaches (e.g., [17, 22, 23, 25]), the *free distributions* of semicircular elements are universally characterized by the *Catalan numbers* $\{c_k\}_{k=1}^{\infty}$ of (4). More precisely, the semicircular law is characterized by the *free-moment sequence*,

$$\left(\omega_n c_{\frac{n}{2}}\right)_{n=1}^{\infty} = (0,\ c_1,\ 0,\ c_2,\ 0,\ c_3,\ \ldots).$$

1.2 Motivation

Recently, from the analysis on p-adic number fields $\mathbb{Q}_p$, semicircular elements are constructed (e.g., [5, 10]), for primes p. It shows connections among number theory, operator algebra and quantum statistical physics (e.g., [26, 27]) via free probability. Motivated by the constructions of [5, 10], semicircular elements are generated, as *Banach-space operators* (e.g., [13, 14]), by $|\mathbb{Z}|$-many *orthogonal projections* in a C^*-algebra (e.g., [6–8, 11, 12]), different from classical approaches. Independently, the joint free distributions of mutually free, multi semicircular elements are re-characterized both combinatorially and analytically in [9]. Such re-characterizations and the combinatorial techniques of [9] are applied in this paper (See Sect. 3 below).

1.3 Overview

Section 2 is devoted to introduce basic concepts of this paper. In Sects. 3 through 6, we construct, and study the frames for our works. We show that (i) there are suitably many non-self-adjoint free random variables whose free distributions are followed by the semicircular law (See Sects. 7.1–7.3), (ii) a C^*-probability space generated by countable-infinitely many free random variables followed by the semicircular law is well-defined under tensor product, and the structure theorems of this free-probabilistic structure are characterized (See Sect. 7.4), and (iii) there are certain Banach-space operators acting on the free random variables of (i) and (ii), deforming the original free-distributional data (followed by the semicircular law), and the deformations are characterized (See Sect. 8).

2 Preliminaries

For fundamental free probability theory, see e.g., [3, 15–17, 28–30]. *Free probability* is a noncommutative operator-algebraic version of classical *measure theory* and *statistical analysis*. Roughly, a topological $*$-probability space (B, ψ) is a noncommutative free-probabilistic analogue of a classical measure space (X, ρ) of a measurable space X and its (bounded) measure ρ. In particular, if (B, ψ) is unital, satisfying $\varphi(1_B) = 1$, then it is a noncommutative counterpart of a classical probability space (X, ρ), equipped with the probability measure ρ, satisfying $\rho(X) = 1$.

Free probability is an important branch of operator algebra theory (e.g., [9, 17, 20–22, 28, 29]), and it provides interesting applications in both mathematical and scientific fields (e.g., [4–8, 10–12, 24, 25]). Here, we use combinatorial free probability of e.g., [17, 22, 23, 25].

Let (A, φ) be a *unital topological $*$-probability space*. As we discussed at the beginning, the free distribution of a self-adjoint free random variable a is characterized by

$$\text{the free-moment sequence } \left(\varphi(a^n)\right)_{n=1}^{\infty},$$

or,

$$\text{the free-cumulant sequence } \left(k_n(a,\ \ldots, a)\right)_{n=1}^{\infty}, \tag{5}$$

by (1) and (2) (e.g., [17, 22, 23, 25]), where $k_{\bullet}(.)$ is the *free cumulant on A in terms of* φ under the *Möbius inversion* of [22].

Definition 1 A self-adjoint free random variable $x \in (A, \varphi)$ is said to be semicircular, if

$$\varphi(x^n) = \omega_n c_{\frac{n}{2}},\ \forall n \in \mathbb{N}, \tag{6}$$

where ω_n are in the sense of (3), and c_k are the k-th Catalan numbers (4) for all $k \in \mathbb{N}_0$.

By the Möbius inversion, a free random variable x is semicircular in (A, φ), if and only if

$$k_n\,(x,\ \ldots, x) = \delta_{n,2} \tag{7}$$

for all $n \in \mathbb{N}$, where δ is the Kronecker delta. So, one can use the definition (6) and the characterization (7) alternatively. i.e., *the semicircular law*, which is the free distributions of semicircular elements, is characterized by the free-moment sequence,

$$(0,\ c_1,\ 0,\ c_2,\ 0,\ c_3,\ 0,\ c_4,\ \ldots), \tag{8}$$

equivalently, by the free-cumulant sequence,

$$(0, 1, 0, 0, 0, 0, \ldots), \tag{9}$$

by (6) and (7), respectively.

By (8) and (9), all semicircular elements are "identically" free-distributed from each other, and hence, the free distributions of all semicircular elements are said to be "the" semicircular law.

3 Free Distributions of Multi Semicircular Elements

In this section, we study joint free distributions of mutually free, multi semicircular elements in a unital C^*-probability space (A, φ). Suppose there are N-many semicircular elements $x_1, \ldots, x_N$ in (A, φ), for $N \in \mathbb{N}$, and assume that they are mutually free in (A, φ). By the self-adjointness of $x_1, \ldots, x_N$ in A, the (joint) free distribution, say

$$\rho \overset{denote}{=} \rho_{x_1,\ldots,x_N}, \tag{10}$$

of them are characterized by the *joint free-moments*,

$$\underset{n=1}{\overset{\infty}{\cup}} \left(\underset{(i_1,\ldots,i_n)\in\{1,\ldots,N\}^n}{\cup} \left\{ \varphi \left(x_{i_1} x_{i_2} \ldots x_{i_n} \right) \right\} \right) \tag{11}$$

by (1) (e.g., [17, 22, 23]).

Throughout this section, we fix $s \in \mathbb{N}$, and an s-tuple I_s,

$$I_s \overset{denote}{=} (i_1, \ldots, i_s) \in \{1, \ldots, N\}^s, \tag{12}$$

in $\{1, \ldots, N\}$. From the sequence I_s of (12), define a set,

$$[I_s] = \{i_1, i_2, \ldots, i_s\}, \tag{13}$$

with its cardinality s, without considering repetition. i.e., even though $i_{j_1} = i_{j_2}$ as entries of the sequence I_s of (12) for $j_1 \neq j_2 \in \{1, \ldots, s\}$, regard them as distinct elements of the set $[I_s]$ of (13).

Then, from the set $[I_s]$ of (13), define a "noncrossing" partition $\pi(i_1)$ in the noncrossing-partition lattice $NC([I_s])$ (e.g., [17, 22, 23, 25]) as follows; (i) starting from the very first entry i_1 of I_s, construct the maximal block U_1 satisfying

$$U_1 = \left(i_1 = i_{j_1}, i_{j_2}, \ldots, i_{j_{|U_1|}} \right) \in \pi_{(I_s)},$$

with the rule:

$$i_1 = i_{j_1} = i_{j_2} = \ldots = i_{j_{|U_1|}}, \tag{14}$$

in I_s, (ii) and then, by fixing the very next entry of

$$[I_s] \setminus U_1,$$

construct the second maximal block U_2 of $\pi(i_1)$ containing the entry, as in (14), and do these processes until end to have the noncrossing partition $\pi(i_1)$, and (iii) such a resulted partition $\pi(i_1)$ must be "maximal" in $NC([I_s])$, under the partial ordering on $NC([I_s])$ (e.g., see [22, 23, 25]), satisfying both (i) and (ii). For example, if

$$I_{10} = (1, 1, 2, 2, 1, 1, 1, 2, 1, 2)$$

and

$$[I_{10}] = \{i_1, i_2, \ldots, i_{10}\},$$

with

$$i_1 = i_2 = i_5 = i_6 = i_7 = i_9 = 1 \text{ and } i_3 = i_4 = i_8 = i_{10} = 2,$$

then there exists a noncrossing partition,

$$\pi\,(i_1) = \{(i_1, i_2, i_5, i_6, i_7, i_9),\ (i_3, i_4),\ (i_8),\ (i_{10})\}$$
$$= \{(1, 1, 1, 1, 1, 1),\ (2, 2),\ (2),\ (2)\},$$

in $NC([I_8])$, satisfying the conditions (i), (ii) and (iii). Remark here that, even though $i_3 = i_4 = i_8 = i_{10} = 2$, one cannot construct the block (i_3, i_4, i_8, i_{10}) in $\pi\,(i_1)$, because this block has two crossings with the first block $(i_1, i_2, i_5, i_6, i_7, i_9)$, so, to avoid the crossings, we need separated blocks (i_3, i_4), (i_8) and (i_{10}).

Now, similar to the noncrossing partition $\pi\,(i_1)$ for the first entry i_1 of I_s, construct noncrossing partitions,

$$\pi\,(i_2)\,, \ldots, \pi\,(i_s) \text{ in } NC\,([I_s])\,,$$

similarly satisfying the above conditions (i), (ii) and (iii) by replacing i_1 to i_l, for all $l = 2, \ldots, s$. i.e., $\pi\,(i_l)$ are the maximal partitions containing the block containing all identical entries of I_s with i_l, for all $l = 1, \ldots, s$. It is not hard to check that if $i_{l_1} = i_{l_2}$ in $\{1, \ldots, N\}$, as entries of I_s, then $\pi\left(i_{l_1}\right) = \pi\left(i_{l_2}\right)$ in $NC\,([I_s])$. Thus, if $i_{k_1}, \ldots, i_{k_n}$ are mutually distinct in $\{1, \ldots, N\}$ as entries of I_s, for $n \leq s$, then the corresponding partitions,

$$\pi\left(i_{k_1}\right), \ldots, \pi\left(i_{k_n}\right)$$

"can" be distinct. Remark that, sometimes, some of them can be identically same in $NC\,([I_s])$; for instance, if

$$J = (1, 1, 1, 1, 2, 2) \overset{\text{let}}{=} (i_1, \ldots, i_6)\,,$$

then

$$\pi\,(1) = \{(i_1, i_2, i_3, i_4)\,, (i_5, i_6)\} = \pi\,(2)\,,$$

in $NC\,(\{i_1, \ldots, i_6\})$.

Now, suppose $\pi\,(i_l) \in NC\,([I_s])$, for $l = 1, \ldots, s$, is the noncrossing partition induced by the s-tuple I_s of (12),

$$\pi\,(i_l) = \{U_1, \ldots, U_t\},$$

where $t \leq s$ and $U_k \in \pi_{(I_s)}$ are the blocks of (i) and (ii), satisfying (iii), for $k = 1, \ldots, t$. Then the partition $\pi(i_l)$ is regarded as the join partition (e.g., [17, 22, 23]),

$$\pi(i_l) = 1_{|U_1|} \vee 1_{|U_2|} \vee \ldots \vee 1_{|U_t|},$$

where $1_{|U_k|}$ are the maximal elements of $NC(U_k)$, for all $k = 1, \ldots, t$, by regarding each block U_k as an independent discrete finite sets.

By collecting all such partitions, define the subset $\Pi([I_s])$ of $NC([I_s])$ by

$$\Pi([I_s]) = \{\pi(i_l) : l = 1, \ldots, s\}.$$

Also, define a subset $\Pi_e([I_s])$ of $\Pi([I_s])$ by

$$\Pi_e([I_s]) = \{\theta \in \Pi([I_s]) : |V| \in 2\mathbb{N}, \ \forall V \in \theta\}, \tag{15}$$

i.e., all partitions of $\Pi_e([I_s])$ have their blocks with even cardinalities, where $2\mathbb{N} = \{2n : n \in \mathbb{N}\}$.

Let I_s be in the sense of (12), and let $x_{i_1}, \ldots, x_{i_s}$ be the corresponding semicircular elements of (A, φ) induced by I_s, without considering repetition in the free semicircular family $\{x_1, \ldots, x_N\}$. Define a free random variable $X[I_s]$ by

$$X[I_s] \overset{def}{=} \prod_{l=1}^{s} x_{i_l} \in (A, \varphi). \tag{16}$$

Theorem 2 *Let I_s be an s-tuple (12), and let $X[I_s] = \prod_{l=1}^{s} x_{i_l}$ be the corresponding free random variable (16) of (A, φ). Then*

$$\varphi(X[I_s]) = \sum_{\theta \in \Pi_e([I_s])} \varphi_\theta\left(x_{i_1}, \ldots, x_{i_s}\right),$$

with

$$\varphi_\theta\left(x_{i_1}, \ldots, x_{i_s}\right) = \prod_{V \in \theta} c_{\frac{|V|}{2}}, \tag{17}$$

where c_k are the k-th Catalan numbers (4). Clearly,

$$\varphi(X[I_s]) = 0 \Longleftrightarrow \Pi_e([I_s]) = \emptyset,$$

where $\emptyset$ is the empty set.

Proof The formula (17) is proven in [9] by the Möbius inversion of [23]. □

4 A C^*-Probability Space Generated by $|\mathbb{N}|$-Many Semicircular Elements

In this section, we study a C^*-algebra $\mathfrak{X}$ generated by mutually free, $|\mathbb{N}|$-many semicircular elements. Let (A, φ) be a unital C^*-probability space, and assume that it contains a family $X = \{s_n\}_{n=1}^{\infty}$ of mutually free semicircular elements. By [17, 22, 23], all mixed free cumulants of $\{s_n\}_{n=1}^{\infty}$ vanish with respect to φ by the freeness on X. For a notational convenience, we re-index the free semicircular family X,

$$\{s_n\}_{n=1}^{\infty} = \{s_1, s_2, s_3, ...\},$$

to

$$\{x_n\}_{n=0}^{\infty} = \{x_0, x_1, x_2, ...\},$$

by identifying s_n to x_{n-1} in A, for all $n \in \mathbb{N}$. i.e., from below, we will let

$$X = \{x_n\}_{n=0}^{\infty} = \{x_n\}_{n \in \mathbb{N}_0},$$

without loss of generality.

4.1 Free-Isomorphic Relations

A unital C^*-probability space (A_1, φ_1) is *free-homomorphic* to a unital C^*-probability space (A_2, φ_2), if there is a $*$-homomorphism $\Omega : A_1 \to A_2$, such that,

$$\varphi_2 (\Omega(a)) = \varphi_1(a), \text{ for all } a \in (A_1, \varphi_1).$$

In such a case, the $*$-homomorphism Ω is called a *free-homomorphism from* (A_1, φ_1) *to* (A_2, φ_2). We write this free-homomorphic relation by

$$(A_1, \varphi_1) \overset{\text{free-homo}}{\longrightarrow} (A_2, \varphi_2). \tag{18}$$

Definition 3 Suppose $(A_1, \varphi_1) \overset{\text{free-homo}}{\longrightarrow} (A_2, \varphi_2)$ in the sense of (18), by a free-homomorphism $\Omega : A_1 \to A_2$. If Ω is a $*$-isomorphism, then it is called a free-isomorphism, and (A_1, φ_1) is said to be free-isomorphic to (A_2, φ_2). We denote this relation by

$$(A_1, \varphi_1) \overset{\text{free-iso}}{=} (A_2, \varphi_2). \tag{19}$$

By the definitions (18) and (19), if two C^*-probability spaces are free-isomorphic, then they are understood as a same unital C^*-probability space.

4.2 A C^*-Probability Space $\mathfrak{X}_\varphi$

Let (A, φ) be a unital C^*-probability space containing a free semicircular family $X = \{x_n\}_{n=0}^{\infty}$. Construct the C^*-subalgebra $\mathfrak{X} = C^*(X)$ of A, generated by the family X, where $C^*(Y)$ are the C^*-subalgebras of A generated by

$$Y \cup Y^* \text{ of } A, \text{ with } Y^* = \{y^* \in A : y \in Y\}.$$

Then one can obtain a C^*-probabilistic sub-structure,

$$\mathfrak{X}_\varphi \overset{denote}{=} (\mathfrak{X},\ \varphi = \varphi\mid_{\mathfrak{X}}) \tag{20}$$

in (A, φ).

Now, let (B, ψ) be a unital C^*-probability space, containing a family $S = \{y_n\}_{n\in\mathbb{Z}}$ of mutually free, $|\mathbb{Z}|$-many semicircular elements, and let

$$\mathfrak{S}_\psi \overset{denote}{=} (\mathfrak{S},\ \psi = \psi\mid_{\mathfrak{S}}) \tag{21}$$

be the corresponding C^*-probabilistic sub-structure of (B, ψ), as in (20), where $\mathfrak{S} = C^*(S)$ is the C^*-subalgebra of B generated by the family S. Such a C^*-probability space (21) does exist naturally (e.g., [5, 12, 20, 21]), or artificially-but-canonically (e.g., [6–8]).

Proposition 4 *Let $\mathfrak{X} = C^*(X)$ be a C^*-subalgebra (20) of A. Then*

$$\mathfrak{X} \overset{*\text{-}iso}{=} \underset{n=0}{\overset{\infty}{\star}} \left(C^*(\{x_n\})\right) \overset{*\text{-}iso}{=} C^*\left(\underset{n=0}{\overset{\infty}{\star}} \{x_n\}\right), \tag{22}$$

in (A, φ), where $(\star)$ in the first $*$-isomorphic relation of (22) is the free-probabilistic free product of [17, 22, 29, 30], and the $(\star)$ in the second $*$-isomorphic relation of (22) is the pure-algebraic free product inducing the noncommutative free words in $\bigcup_{n=0}^{\infty} \{x_n\} = X$.

Proof Let $\mathfrak{X} = C^*(X)$ be a fixed C^*-subalgebra of A. Since X is a free family consisting of mutually free semicircular elements $\{x_n\}_{n\in\mathbb{N}_0}$ in (A, φ), one has that

$$\mathfrak{X} \overset{def}{=} C^*(X) = C^*\left(\{x_n\}_{n\in\mathbb{N}_0}\right) \overset{*\text{-iso}}{=} \underset{n\in\mathbb{N}_0}{\star} \left(C^*(\{x_n\})\right). \tag{23}$$

Therefore, the first $*$-isomorphic relation of (22) holds by (23). i.e., all elements of $\mathfrak{X}$ are the limits of linear combinations of free reduced words (under operator multiplication on A) in X by [17, 22, 23, 29, 30].

So, if we consider all noncommutative free words in the family $X = \{x_n\}_{n \in \mathbb{N}_0}$, then they have their unique operator forms in $\mathfrak{X}$, which are the free reduced words by (23). It shows that the second $*$-isomorphic relation of (22) holds, too. □

By (22), one can understand the C^*-probability space $\mathfrak{X}_\varphi$ of (20) as an independent free-probabilistic structure,

$$\mathfrak{X}_\varphi = \left(\underset{n \in \mathbb{N}_0}{\star} C^* \left(\{x_n\} \right), \underset{n \in \mathbb{N}_0}{\star} \varphi \mid_{C^*(\{x_n\})} \right). \tag{24}$$

Proposition 5 *Let $\mathfrak{S}_\psi$ be the C^*-probability space (21) in (B, ψ). Then*

$$\mathfrak{S}_\psi = \left(\underset{j \in \mathbb{Z}}{\star} C^* \left(\{s_j\} \right), \underset{j \in \mathbb{Z}}{\star} \psi \mid_{C^*(\{s_j\})} \right), \tag{25}$$

where $C^*(Z)$, here, mean the C^*-subalgebras of B generated by the subsets Z of B.

Proof The proof of (25) is similar to that of (22). □

Now, let's partition $\mathbb{N}_0$ and $\mathbb{Z}$ as follows:

$$\mathbb{N}_0 = \{0\} \sqcup (2\mathbb{N}) \sqcup (2\mathbb{N} - 1),$$

and

$$\mathbb{Z} = (-\mathbb{N}) \sqcup \{0\} \sqcup \mathbb{N}, \tag{26}$$

where

$$2\mathbb{N} = \{2n : n \in \mathbb{N}\}, 2\mathbb{N} - 1 = \{2n - 1 : n \in \mathbb{N}\},$$

and

$$-\mathbb{N} = \{-n : n \in \mathbb{N}\}.$$

Motivated by (26), define a bijection $g : \mathbb{N}_0 \to \mathbb{Z}$ by

$$g(n) = \begin{cases} 0 & \text{if } n = 0 \\ \frac{n+1}{2} & \text{if } n \in 2\mathbb{N} - 1 \\ -\frac{n}{2} & \text{if } n \in 2\mathbb{N}, \end{cases} \tag{27}$$

in $\mathbb{Z}$, for all $n \in \mathbb{N}_0$. From this bijection g, one can define a bijection,

$$G : X \to S,$$

by

$$G(x_n) = y_{g(n)}, \text{ for all } n \in \mathbb{N}_0. \tag{28}$$

where X is the generating family (20) of $\mathfrak{X}$, S is the generating family (21) of $\mathfrak{S}$, and g is the bijection (27).

By (28), one can define the corresponding "multiplicative" linear transformation,

$$\Psi : \mathfrak{X} \to \mathfrak{S}$$

satisfying

$$\Psi\,(x_n) = G(x_n) = s_{g(n)} \in S, \forall x_n \in X, \tag{29}$$

in $\mathfrak{S}$, where G is in the sense of (28). i.e., for an alternating N-tuple $(n_1, \ldots, n_N) \in \mathbb{N}_0^N$, satisfying

$$n_1 \neq n_2, n_2 \neq n_3, \ldots, n_{N-1} \neq n_N \text{ in } \mathbb{N}_0,$$

if one has a free reduced word $T = \prod_{l=1}^{N} x_{n_l}^{k_l} \in \mathfrak{X}_\varphi$, where $x_{n_1}, \ldots, x_{n_N} \in X$, for $k_1, \ldots, k_N \in \mathbb{N}$, for $N \in \mathbb{N}$, then

$$\Psi(T) = \Psi\left(\prod_{l=1}^{N} x_{n_l}^{k_l}\right) = \prod_{l=1}^{N} \Psi\left(x_{n_l}^{k_l}\right)$$

by the multiplicativity of Ψ

$$= \prod_{l=1}^{N} \left(\Psi(x_{n_l})\right)^{k_l}$$

by the multiplicativity of Ψ

$$= \prod_{l=1}^{N} s_{g(n_l)}^{k_l}, \tag{30}$$

in $\mathfrak{S}_\psi$, by (29).

Since $(n_1, \ldots, n_N) \in \mathbb{N}_0^N$ is an alternating N-tuple in $\mathbb{N}_0$, the N-tuple,

$$(g(n_1),\ \ldots,\ g(n_N)) \in \mathbb{Z}^N,$$

is an alternating N-tuple in $\mathbb{Z}$, too, by (27) and (28). i.e., the formula (30) shows that the images $\Psi(T)$ of all free reduced words $T \in \mathfrak{X}_{\varphi}$ with their lengths-N become free reduced words of $\mathfrak{S}_{\psi}$ with the same lengths-N.

Lemma 6 *The multiplicative linear transformation $\Psi : \mathfrak{X} \to \mathfrak{S}$ of (29) is a $*$-isomorphism. i.e.,*

$$\mathfrak{X} \overset{*\text{-}iso}{=} \mathfrak{S}, \tag{31}$$

where $\overset{*\text{-}iso}{=}$ means "being $*$-isomorphic to."

Proof Remark that all elements of $\mathfrak{X}$ (or, of $\mathfrak{S}$) are the limits of linear combinations of free reduced words in X (resp., in S) by (22) (resp., by (25)). So, the multiplicative linear transformation Ψ of (29) is bijective and bounded. Observe that, for any $x_n \in X \subset \mathfrak{X}$, and $t \in \mathbb{C}$,

$$\Psi\left((tx_n)^*\right) = \Psi\left(\bar{t}\, x_n\right)$$

since $x_n^* = x_n$, under the semicircularity

$$= \bar{t}\Psi(x_n) = \bar{t}\, s_{g(n)} = \bar{t} s_{g(n)}^*$$

since $s_{g(n)}^* = s_{g(n)}$, by the semicircularity

$$= \left(t s_{g(n)}\right)^* = \left(\Psi(tx_n)\right)^*,$$

in $\mathfrak{S}$. So, by (22), (25), and the linearity of Ψ,

$$\Psi\left(T^*\right) = \left(\Psi(T)\right)^* \text{ in } \mathfrak{S}, \tag{32}$$

for all $T \in \mathfrak{X}$. Therefore, Ψ is a $*$-isomorphism by (32). □

By (31), we obtain the following free-isomorphic relation.

Theorem 7 *The C^*-probability spaces $\mathfrak{X}_{\varphi}$ and $\mathfrak{S}_{\psi}$ are free-isomorphic, i.e.,*

$$\mathfrak{X}_{\varphi} \overset{\textit{free-iso}}{=} \mathfrak{S}_{\psi}. \tag{33}$$

Proof By (31), there exists a $*$-isomorphism Ψ of (29) from $\mathfrak{X}$ onto $\mathfrak{S}$. By (22) and (25), it suffices to show that the $*$-isomorphism Ψ preserves the free distributions of generators of $\mathfrak{X}_{\varphi}$ to those of generators of $\mathfrak{S}_{\psi}$.

Let $x_n \in X \subset \mathfrak{X}_{\varphi}$. Then

$$\psi\left((\Psi(x_n))^k\right) = \psi\left(s_{g(n)}^k\right) = \omega_k c_{\frac{k}{2}} = \varphi\left(x_n^k\right),$$

for all $k \in \mathbb{N}$, by the semicircularity (6) of $X \cup S$.

It shows that Ψ preserves the free probability on $\mathfrak{X}_{\varphi}$ to that on $\mathfrak{S}_{\psi}$ by (17), and hence, it is a free-isomorphism. Therefore, two C^*-probability space $\mathfrak{X}_{\varphi}$ and $\mathfrak{S}_{\psi}$ are free-isomorphic. □

Assumption and Notation From below, we will identify $\mathfrak{X}_{\varphi}$ and $\mathfrak{S}_{\psi}$ as the same C^*-probability space, and denote it by $\mathfrak{X}_{\varphi}$, by (33). □

5 Free-Distributional Data on $\mathfrak{X}_{\varphi}$

Let $\mathfrak{X}_{\varphi} = (\mathfrak{X}, \varphi)$ be the C^*-probability space (25), "identified with (24) by (33)," generated by the free semicircular family $X = \{x_j\}_{j \in \mathbb{Z}}$.

Theorem 8 *Let $I_s = (i_1, \ldots, i_s)$ be an arbitrary s-tuple in $\mathbb{Z}^s$, for $s \in \mathbb{N}$, like in (12), and let $\pi_{(I_s)} \in NC(\{i_1, \ldots, i_s\})$ be the noncrossing partition (15) induced by I_s. If $X[I_s]$ be a free random variable (16) of $\mathfrak{X}_{\varphi}$, then the free-distributional data $\varphi(X[I_s])$ is characterized by the formula (17).*

Proof Under hypothesis, the free-distributional data $\varphi(X[I_s])$ on $\mathfrak{X}_{\varphi}$ are obtained by (17), since all elements of the generator set $X = \left\{x_j\right\}_{j \in \mathbb{Z}}$ of $\mathfrak{X}_{\varphi}$ are mutually free, semicircular elements. □

6 Certain Free-Isomorphisms on $\mathfrak{X}_{\varphi}$

By (24), (25) and (33), our C^*-probability space $\mathfrak{X}_{\varphi}$ is a representative of all unital C^*-probability spaces generated by mutually free, $|\mathbb{N}|$-many semicircular elements. As we assumed in Sects. 4 and 5, we let $\mathfrak{X}_{\varphi}$ be the C^*-probability space (25) generated by a free semicircular family $X = \{x_j\}_{j \in \mathbb{Z}}$ of mutually free, $|\mathbb{Z}|$-many semicircular elements.

6.1 Shifts on $\mathbb{Z}$

Define bijection h on the set $\mathbb{Z}$ of all integers by

$$h(j) = j + 1, \tag{34}$$

for all $j \in \mathbb{Z}$.

Remark that, by (34), one can define a function h' on $\mathbb{N}_0$ by

$$h' = g^{-1} \circ h \circ g \text{ on } \mathbb{N}_0, \tag{34'}$$

where g is the bijection (27) from $\mathbb{N}_0$ onto $\mathbb{Z}$, and $g^{-1} : \mathbb{Z} \to \mathbb{N}_0$ is the inverse function of g. Thus, the well-defined bijection h of (34) on $\mathbb{Z}$ implies the existence of bijections h' of (34)' on $\mathbb{N}_0$. We now concentrate on h of (34).

Define the bijections $h^{(n)}$ on $\mathbb{Z}$, by

$$h^{(n)} \overset{def}{=} \begin{cases} id_{\mathbb{Z}}, \text{ the identity function on } \mathbb{Z} & \text{if } n = 0 \\ \underbrace{h \circ h \circ h \circ \ldots\ldots\ldots\ldots \circ h}_{n\text{-times}} & \text{if } n > 0 \\ \underbrace{h^{-1} \circ h^{-1} \circ \ldots \circ h^{-1}}_{|n|\text{-times}} & \text{if } n < 0, \end{cases} \tag{35}$$

for all $n \in \mathbb{Z}$, where $(\circ)$ is the usual functional composition, and h^{-1} is the inverse of h,

$$h^{-1}(j) = j - 1, \ \forall j \in \mathbb{Z}.$$

By (34) and (35),

$$h^{(n)}(j) = j + n,$$

for all $j, n \in \mathbb{Z}$. And $h^{(n)}$ are invertible with their inverse $h^{(-n)}$, for all $n \in \mathbb{Z}$.

Definition 9 We call the bijections $h^{(n)}$ of (35), the n-th shifts on $\mathbb{Z}$, for $n \in \mathbb{Z}$.

6.2 Integer Shifts on $\mathfrak{X}_\varphi$

Let $h^{(n)}$ be the n-th shifts (35) on $\mathbb{Z}$, for all $n \in \mathbb{Z}$. Let $k \in \mathbb{Z}$, and define a "multiplicative" linear transformation λ^k acting on $\mathfrak{X}_\varphi$ by the morphism satisfying

$$\lambda^k \left(x_j\right) = x_{h^{(k)}(j)} = x_{j+k}, \ \forall x_j \in X \subset \mathfrak{X}_\varphi. \tag{36}$$

By the multiplicativity of the morphism λ^k of (36), if $T = \prod_{l=1}^{N} x_{j_l}^{n_l}$ is a free reduced words of $\mathfrak{X}_\varphi$ with its length-N in $X = \left\{x_j\right\}_{j \in \mathbb{Z}}$, then

$$\lambda^k(T) = \prod_{l=1}^{N} \lambda^k \left(x_{j_l}\right)^{n_l} = \prod_{l=1}^{N} x_{j_l+k}^{n_l}, \tag{37}$$

in $\mathfrak{X}_\varphi$, where $(j_1, \ldots, j_N) \in \mathbb{Z}^N$ is alternating, and $n_1, \ldots, n_N \in \mathbb{N}$.

Note that, if $(j_1, \ldots, j_N) \in \mathbb{Z}^N$ is alternating, then

$$(j_1 + k, \ldots, j_N + k) \in \mathbb{Z}^N$$

is alternating in $\mathbb{Z}$, too. So, the computation (37) says that the morphism λ^k of (36) assign free reduced words to free reduced words preserving lengths in $\mathfrak{X}_\varphi$.

Also, by (36) and (37), one has

$$\lambda^k = \begin{cases} 1_{\mathfrak{X}_\varphi}, \text{ the identity map on } \mathfrak{X}_\varphi & \text{if } k = 0 \\ \underbrace{\lambda \cdot \lambda \cdot \lambda \cdot \ldots\ldots \cdot \lambda}_{k\text{-times}} & \text{if } k > 0 \\ \underbrace{\lambda^{-1} \cdot \lambda^{-1} \cdot \ldots\ldots \cdot \lambda^{-1}}_{|k|\text{-times}} & \text{if } k < 0, \end{cases}$$

for all $k \in \mathbb{Z}$, by (35) and (36), where $(\cdot)$ is the multiplication (or, composition) of linear transformations.

Observe that, for $t \in \mathbb{C}$, and $x_j \in X \subset \mathfrak{X}$,

$$\lambda^k \left((t x_j)^* \right) = \overline{t}\, x_{j+k} = \overline{t} x_{j+k}^* = \left(\lambda^k (t x_j) \right)^*,$$

implying that

$$\lambda^k \left(T^* \right) = \left(\lambda^k (T) \right)^*, \text{ for all } T \in \mathfrak{X}_\varphi, \tag{38}$$

in $\mathfrak{X}_\varphi$, by (25) and (37).

Theorem 10 *A multiplicative linear transformation λ^k of (36) is a free-isomorphism on $\mathfrak{X}_\varphi$, for all $k \in \mathbb{Z}$.*

Proof By (38), the morphism λ^k of (36) is a well-defined $*$-homomorphism on $\mathfrak{X}_\varphi$. And, by the bijectivity of the k-th shift $h^{(k)}$ on $\mathbb{Z}$, the restriction $\lambda^k\mid_X$ is a bijection on the free-generator set X of $\mathfrak{X}_\varphi$. Thus, by (25) and (37), it is a $*$-isomorphism on $\mathfrak{X}_\varphi$.

Observe now that

$$\varphi\left(\left(\lambda^k (x_j) \right)^n \right) = \varphi\left(x_{j+k}^n \right) = \omega_n c_{\frac{n}{2}} = \varphi\left(x_j^n \right), \tag{39}$$

for all $n \in \mathbb{N}$, for all $x_j \in X$.

Therefore, for all s-tuple $I_s \in \mathbb{Z}^s$,

$$\varphi\left(X[I_s] \right) = \varphi\left(\lambda^k \left(X[I_s] \right) \right) \text{ in } \mathfrak{X}_\varphi,$$

by Theorem 8 (or, (17)) and (39), where $X[I_s]$ are in the sense of (16). Thus,

$$\varphi(T) = \varphi\left(\lambda^k(T)\right), \text{ for all } T \in \mathfrak{X}_\varphi,$$

in $\mathfrak{X}_\varphi$, by (25). Therefore, λ^k is a free-isomorphism on $\mathfrak{X}_\varphi$. □

Let $Aut\left(\mathfrak{X}_\varphi\right)$ be the *automorphism group of* $\mathfrak{X}_\varphi$,

$$Aut\left(\mathfrak{X}_\varphi\right) \overset{def}{=} \left(\left\{\alpha \,\middle|\, \begin{array}{c} \alpha \text{ is a} \\ *\text{-isomorphism} \\ \text{on } \mathfrak{X}_\varphi \end{array}\right\}, \cdot\right),$$

where $(\cdot)$ is the product (or composition) on $*$-isomorphisms. Define a subset λ of $Aut\left(\mathfrak{X}_\varphi\right)$ by

$$\lambda = \{\lambda^k : k \in \mathbb{Z}\}, \tag{40}$$

where λ^k are the free-isomorphisms (36) on $\mathfrak{X}_\varphi$.

Theorem 11 *The subset λ of (40) is an abelian subgroup of $Aut(\mathfrak{X}_\varphi)$, satisfying*

$$\lambda \overset{\text{Group}}{=} (\mathbb{Z}, +), \text{ the infinite abelian cyclic group,} \tag{41}$$

where "$\overset{\text{Group}}{=}$" means "being group-isomorphic."

Proof Let $\lambda^{k_1}, \lambda^{k_2} \in \lambda$. Then, by (36) and (37),

$$\lambda^{k_1}\lambda^{k_2} = \lambda^{k_1+k_2}, \text{ in } \lambda.$$

So, the algebraic structure $(\lambda, \cdot)$ is well-determined in $Aut\left(\mathfrak{X}_\varphi\right)$. And hence,

$$\left(\lambda^{k_1}\lambda^{k_2}\right)\lambda^{k_3} = \lambda^{k_1+k_2+k_3} = \lambda^{k_1}\left(\lambda^{k_2}\lambda^{k_3}\right),$$

in λ, for all $k_1, k_2, k_3 \in \mathbb{Z}$.

Observe that the set λ contains $\lambda^0 = 1_{\mathfrak{X}_\varphi}$, the identity map on $\mathfrak{X}_\varphi$, by (34) and (36), satisfying

$$1_{\mathfrak{X}_\varphi} \cdot \lambda^k = \lambda^{0+k} = \lambda^k = \lambda^k \cdot 1_{\mathfrak{X}_\varphi}, \text{ on } \mathfrak{X}_\varphi,$$

for all $k \in \mathbb{Z}$. And, for any $k \in \mathbb{Z}$,

$$\lambda^k\lambda^{-k} = \lambda^{k+(-k)} = \lambda^0 = 1_{\mathfrak{X}_\varphi} = \lambda^{-k}\lambda^k, \text{ in } \lambda,$$

showing that every element $\lambda^k \in \lambda$ has its unique $(\cdot)$-inverse λ^{-k}. Thus, the set λ of (40) forms a subgroup of $Aut\left(\mathfrak{X}_{\varphi}\right)$. Clearly,

$$\lambda^{k_1}\lambda^{k_2} = \lambda^{k_1+k_2} = \lambda^{k_2+k_1} = \lambda^{k_2}\lambda^{k_1},$$

in λ. Therefore, the subgroup $(\lambda, \cdot)$ is commutative in $Aut\left(\mathfrak{X}_{\varphi}\right)$.

Define now a map $\Phi : \mathbb{Z} \to \lambda$ by

$$\Phi\left(j\right) = \lambda^{j}, \text{ for all } j \in \mathbb{Z},$$

Then it is a group-isomorphism, satisfying

$$\Phi\left(j_1 + j_2\right) = \lambda^{j_1+j_2} = \lambda^{j_1}\lambda^{j_2} = \Phi(j_1)\Phi(j_2),$$

in λ, for all $j_1, j_2 \in \mathbb{Z}$. Therefore, the group-isomorphic relation (41) holds. □

The above theorem shows that the subgroup λ of (40) is an infinite abelian cyclic group $\left\langle\lambda^1\right\rangle$ embedded in $Aut(\mathfrak{X}_{\varphi})$, where $\langle g\rangle$ means the cyclic (sub)group generated by $\left\{g, g^{-1}\right\}$ (in a group).

By (37) and (41), there is a natural group-action θ of λ acting on our C^*-probability space $\mathfrak{X}_{\varphi}$, satisfying

$$\theta\left(\lambda^k\right)(T) = \lambda^k(T), \text{ for all } T \in \mathfrak{X}_{\varphi}, \tag{42}$$

for all $k \in \mathbb{Z}$.

Definition 12 The group λ of (40), acting on $\mathfrak{X}_{\varphi}$ via the group-action θ of (42), is called the integer-shift group on $\mathfrak{X}_{\varphi}$.

6.3 *Free-Isomorphic Relations on $\mathfrak{X}_{\varphi}$*

We here study how the integer-shift group λ of (40) affects the free probability on $\mathfrak{X}_{\varphi}$, under the group-action θ of (42).

Theorem 13 *Let λ be the integer-shift group, and let θ be the group-action (42) of λ acting on $\mathfrak{X}_{\varphi}$. Then the free probability on $\mathfrak{X}_{\varphi}$ is preserved by θ, in the sense that:*

$$\varphi\left(\theta\left(\lambda^k\right)(T)\right) = \varphi\left(T\right), \textit{ for all } T \in \mathfrak{X}_{\varphi}, \tag{43}$$

for all $\lambda_e^k \in \lambda$.

Proof By (42), for any $T \in \mathfrak{X}_\varphi$,

$$\theta\left(\lambda^k\right)(T) = \lambda^k(T), \text{ in } \mathfrak{X}_\varphi,$$

and hence,

$$\varphi\left(\theta\left(\lambda^k\right)(T)\right) = \varphi\left(\lambda^k(T)\right) = \varphi(T),$$

for all $k \in \mathbb{Z}$, by Theorem 10. So, the action θ preserves the free probability on $\mathfrak{X}_\varphi$. □

Notation From below, we denote the images $\theta\left(\lambda^k\right)(T) \in \mathfrak{X}_\varphi$ of $T \in \mathfrak{X}_\varphi$ simply by $\lambda^k(T)$, for all $\lambda^k \in \lambda$. □

7 Free Random Variables followed by the Semicircular Law

Let $\lambda \subset Aut(\mathfrak{X}_\varphi)$ be the integer-shift group (40) acting on the C^*-probability space $\mathfrak{X}_\varphi$ (via the canonical action θ of (42)) generated by the free family $\{x_j\}_{j\in\mathbb{Z}}$ of semicircular elements x_j's. In this section, we construct some free random variables in a certain C^*-probability space, containing $\mathfrak{X}_\varphi$, whose free distributions are followed by the semicircular law.

Definition 14 Let (B, ψ) be an arbitrary topological $*$-probability space. A free random variable $y \in (B, \psi)$ is followed by the semicircular law, if

$$\psi\left(\prod_{l=1}^{n} y^{r_l}\right) = \omega_n c_{\frac{n}{2}}, \tag{44}$$

for all $(r_1, \ldots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$, where ω_n are in the sense of (3) for all $n \in \mathbb{N}$, and c_k are the k-th Catalan numbers (4) for all $k \in \mathbb{N}_0$.

By the definition (44), if a self-adjoint free random variable y is followed by the semicircular law in (B, ψ), then it is nothing but a semicircular element. i.e., all semicircular elements are followed by the semicircular law in the sense of (44), but not all such free random variables are semicircular. In the text, we focus on studying "non-self-adjoint" free random variables followed by the semicircular law.

7.1 *Group C*-Algebra* Λ *of* λ

Let Γ be an arbitrary discrete group, and let H be the group Hilbert space,

$$H = l^2(\Gamma), \text{ the } l^2\text{-space},$$

with its orthonormal basis,

$$\mathcal{B} = \left\{\xi_g : g \in \Gamma \setminus \{e\}\right\}, \tag{45}$$

where $e \in \Gamma$ is the group-identity, satisfying $\xi_e = 1_H$, the identity vector, and

$$\left\langle \xi_{g_1}, \ \xi_{g_2} \right\rangle_2 = \delta_{g_1, g_2},$$

and

$$\left\| \xi_g \right\|_2 = \sqrt{\left\langle \xi_g, \xi_g \right\rangle_2} = 1, \tag{46}$$

for all $g_1, g_2, g \in \Gamma$, where δ is the Kronecker delta, and $\langle , \rangle_2$ is the canonical l^2-inner product inducing the l^2-norm $\|.\|_2$ on H.

Every Hilbert-space vector $\xi \in H$ is expressed by

$$\xi = \sum_{g \in \Gamma} t_g \xi_g, \ \text{for } t_g \in \mathbb{C},$$

where $\sum$ is the infinite sum under l^2-topology induced by (46).

Note that, for any Hilbert-space vectors $\left\{\xi_g\right\}_{g \in \Gamma} = \mathcal{B} \cup \{e\}$, the following multiplication-rule holds;

$$\xi_{g_1} \xi_{g_2} = \xi_{g_1 g_2} \text{ in } H, \ \forall g_1, g_2 \in \Gamma, \tag{47}$$

where $\mathcal{B}$ is the orthonormal basis of (45).

In the operator algebra $B(H)$ of all (bounded linear) operators on the group-Hilbert space H of (45), every group-element $g \in \Gamma$ forms a (left) multiplication operator m_g with its symbol ξ_g,

$$m_g \left(\sum_{u \in \Gamma} t_u \xi_u \right) = \sum_{u \in \Gamma} t_u \xi_g \xi_u = \sum_{u \in \Gamma} t_u \xi_{gu}, \tag{48}$$

in H by (47).

The relation (48) shows that the group Γ is acting on the operator algebra $B(H)$ via a group-action m,

$$m(g) = m_g \in B(H), \ \forall g \in \Gamma, \tag{49}$$

where m_g are the multiplication operators (48).

Define a set,

$$\mathcal{M} \overset{\text{def}}{=} \left\{ m_g \in B(H) : g \in \Gamma \right\},$$

of all multiplication operators (49), and construct the C^*-subalgebra $\mathscr{M}$,

$$\mathscr{M} \overset{\text{def}}{=} C^* (\mathcal{M}) \text{ of } B(H), \tag{50}$$

under the operator-norm on $B(H)$ (e.g., see [14]).

Definition 15 We call the C^*-algebra $\mathscr{M}$ of (50), the group (C^*-)algebra of Γ.

Let λ be the integer-shift group (40) acting on the C^*-probability space $\mathfrak{X}_\varphi$ of (25). Then, by (50), one can have the corresponding group algebra,

$$\Lambda \overset{\text{def}}{=} C^* (\lambda) \text{ in } B(H_\lambda),$$

where

$$H_\lambda = l^2 (\lambda) \tag{51}$$

is the group-Hilbert space (45).

Definition 16 The group algebra Λ of λ is called the integer-shift(-group) algebra. All elements of Λ are said to be (integer-)shift operators.

By (51), all shift operators T of Λ are expressed by

$$T = \sum_{\lambda^k \in \lambda} t_{\lambda^k} m_{\lambda^k} = \sum_{k \in \mathbb{Z}} t_k m_{\lambda^k}, \text{ with } t_{\lambda^k} = t_k,$$

because λ is isomorphic to $(\mathbb{Z}, +)$ by (41), and $\sum$ is an infinite sum under the C^*-topology for Λ.

Let Λ be the integer-shift algebra (51) of λ. Then this C^*-algebra Λ is acting on the C^*-probability space $\mathfrak{X}_\varphi$, via an $*$-algebra-action,

$$\Theta : \Lambda \to B\left(\mathfrak{X}_\varphi\right),$$

defined by

$$\Theta\left(\sum_{k \in \mathbb{Z}} t_k m_{\lambda^k}\right)(S) = \sum_{k \in \mathbb{Z}} t_k \lambda^k (S), \tag{52}$$

for all $S \in \mathfrak{X}_\varphi$, where $B\left(\mathfrak{X}_\varphi\right)$ is the operator space (in the sense of [13]), consisting of all bounded linear transformations on $\mathfrak{X}_\varphi$, by regarding $\mathfrak{X}_\varphi$ as a Banach space with its C^*-norm. Indeed, Θ is a well-defined algebra-action of Λ acting on (the Banach space) $\mathfrak{X}_\varphi$, since

$$\Theta (S_1 S_2) = \Theta (S_1) \Theta (S_2),$$

and

$$\Theta\left(S^{*}\right)=(\Theta\,(S))^{*}, \tag{53}$$

by (42), (48) and (52), for all $S_1, S_2, S \in \mathfrak{X}_{\varphi}$. i.e., by (52) and (53), all operators of Λ are understood to be Banach-space operators acting on the Banach space, our C^*-probability space $\mathfrak{X}_{\varphi}$.

Proposition 17 *If* $T = \sum_{k\in\mathbb{Z}} t_k m_{\lambda^k} \in \Lambda$ *is a shift operator, and* $x_j \in X$ *is a generating semicircular element of* $\mathfrak{X}_{\varphi}$*, then*

$$\varphi\left(\Theta(T)\left(x_j^n\right)\right)=\left(\omega_n c_{\frac{n}{2}}\right)\left(\sum_{k\in\mathbb{Z}} t_k\right), \tag{54}$$

for all $n \in \mathbb{N}$.

Proof For any $n \in \mathbb{N}$, and $x_j \in X \subset \mathfrak{X}_{\varphi}$, one has

$$\varphi\left(\Theta\left(\sum_{k\in\mathbb{Z}} t_k m_{\lambda^k}\right)\left(x_j^n\right)\right)=\varphi\left(\sum_{k\in\mathbb{Z}} t_k \lambda^k\left(x_j^n\right)\right)$$

by (52)

$$=\sum_{k\in\mathbb{Z}} t_k \varphi\left(x_{j+k}^n\right)=\sum_{k\in\mathbb{Z}} t_k \varphi\left(x_j^n\right)$$

by (43)

$$=\left(\omega_n c_{\frac{n}{2}}\right)\left(\sum_{k\in\mathbb{Z}} t_k\right).$$

Therefore, the free-distributional data (54) holds. □

The above proposition shows how the algebra-action Θ of the integer-shift algebra Λ affects the original free probability on $\mathfrak{X}_{\varphi}$ by (54).

7.2 *The Tensor Product C*-Algebra* $\Lambda \otimes \mathfrak{X}$

In this section, we construct the tensor product C^*-algebra,

$$\mathscr{X} \overset{\text{def}}{=} \Lambda \otimes \mathfrak{X} \tag{55}$$

of the integer-shift algebra Λ of (51) generated by the integer-shift group λ, and the C^*-algebra $\mathfrak{X}$ generated by the free semicircular family $X = \{x_j\}_{j\in\mathbb{Z}}$, where $\otimes$ is the tensor product of C^*-algebras.

Define a linear functional τ on this C^*-algebra $\mathscr{X}$ of (55) by the morphism satisfying

$$\tau\,(S \otimes T) \overset{\text{def}}{=} \varphi\,(\Theta\,(S)\,(T)) = \varphi\,(S\,(T))\,, \tag{56}$$

for all $S \otimes T \in \mathscr{X}$, with $S \in \Lambda$ and $T \in \mathfrak{X}$.

Definition 18 Let $\mathscr{X}$ be the tensor product C^*-algebra (55), and τ, the linear functional (56) on $\mathscr{X}$. Then the C^*-probability space,

$$\mathscr{X}_\tau \overset{\text{denote}}{=} (\mathscr{X},\ \tau)\,,$$

is called the (integer-)shift-semicircular C^*-probability space.

By definition, the shift-semicircular C^*-probability space $\mathscr{X}_\tau$ is unital equipped with its unity $I = m_{\lambda^0} \otimes 1_{\mathfrak{X}}$, satisfying

$$\tau\,(I) = \varphi\left(\lambda^0\,(1_{\mathfrak{X}})\right) = \varphi\,(1_{\mathfrak{X}}) = 1.$$

And the operators,

$$u_{k,j} \overset{\text{denote}}{=} \lambda^k \otimes x_j \in \mathscr{X}_\tau,\ \text{ for } k, j \in \mathbb{Z}, \tag{57}$$

generate $\mathscr{X}_\tau$. Observe that

$$\begin{aligned}\tau\left((u_{k,j})^n\right) &= \tau\left(\left(\lambda^k \otimes x_j\right)^n\right) = \tau\left(\lambda^{kn} \otimes x_j^n\right)\\ &= \varphi\left(\lambda^{kn}\left(x_j^n\right)\right) = \varphi\left(x_{j+kn}^n\right) = \varphi\left(x_j^n\right) = \omega_n c_{\frac{n}{2}},\end{aligned} \tag{58}$$

on $\mathfrak{X}_\varphi$, by (43), (54) and (56), since all generating shift operators $\lambda^k \in \Lambda$ are free-isomorphisms on $\mathfrak{X}_\varphi$.

Lemma 19 *Let $u_{k,j} = \lambda^k \otimes x_j$ be a generating operator (57) of the shift-semicircular C^*-probability space $\mathscr{X}_\tau$, for $k, j \in \mathbb{Z}$. Then*

$$\tau\left((u_{k,j})^n\right) = \omega_n c_{\frac{n}{2}} = \varphi\left(x_j^n\right), \tag{59}$$

for all $n \in \mathbb{N}$.

Proof The free-distributional data (59) is obtained by (58). □

By (59), one can verify that if $k = 0$ in $\mathbb{Z}$, then the generating operator $u_{k,j} = u_{0,j}$ is semicircular in $\mathscr{X}_\tau$.

Lemma 20 *A generating operator $u_{0,j} = \lambda^0 \otimes x_j$ of (57) is semicircular in the shift-semicircular C^*-probability space $\mathscr{X}_\tau$, for all $j \in \mathbb{Z}$. And hence, they are followed by the semicircular law in the sense of (44).*

Proof For any fixed $j \in \mathbb{Z}$, the corresponding generating operator $u_{0,j}$ of $\mathscr{X}_\tau$ satisfies that

$$\left(u_{0,j}\right)^* = \left(\lambda^0\right)^* \otimes x_j^* = id_\Lambda \otimes x_j = \lambda^0 \otimes x_j = u_{0,j},$$

in $\mathscr{X}$. So, the generating operators $u_{0,j}$ are self-adjoint in $\mathscr{X}_\tau$, for all $j \in \mathbb{Z}$.

Such a self-adjoint free random variable $u_{0,j} \in \mathscr{X}_\tau$ satisfies

$$\tau\left(\left(u_{0,j}\right)^n\right) = \omega_n c_{\frac{n}{2}} = \varphi\left(x_j^n\right),$$

for all $n \in \mathbb{N}$, by (59). So, it is semicircular in $\mathscr{X}_\tau$, for all $j \in \mathbb{Z}$. Since it is semicircular in $\mathscr{X}_\tau$, it is followed by the semicircular law in the sense of (44). □

The above lemma shows that all generating operators of the shift-semicircular C^*-probability space $\mathscr{X}_\tau$, formed by

$$u_{0,j} = \lambda^0 \otimes x_j \text{ in } \mathscr{X}_\tau,$$

are followed by the semicircular law in the sense of (44), because they are semicircular in $\mathscr{X}_\tau$. So, we are now interested in the cases where

$$k \neq 0 \text{ in } \mathbb{Z}.$$

Let $u_{k,j} \in \mathscr{X}$ be a generating free random variable for $k \neq 0$ in $\mathbb{Z}$. Then

$$\left(u_{k,j}\right)^* = \left(\lambda^k\right)^* \otimes x_j^* = \lambda^{-k} \otimes x_j = u_{-k,j}, \text{ in } \mathscr{X}_\tau. \tag{60}$$

i.e., if $k \neq 0$, then the generating operators $u_{k,j}$ are not self-adjoint in $\mathscr{X}_\tau$, and hence, they cannot be semicircular in $\mathscr{X}_\tau$, for all $j \in \mathbb{Z}$.

Lemma 21 *Let $u_{k,j} \in \mathscr{X}_\tau$ be a generating free random variable for $k \in \mathbb{Z}^\times = \mathbb{Z} \setminus \{0\}$, and $j \in \mathbb{Z}$. Then*

$$\tau\left(\left(u_{k,j}^*\right)^n\right) = \omega_n c_{\frac{n}{2}} = \varphi\left(x_j^n\right), \tag{61}$$

for all $n \in \mathbb{N}$.

Proof If $k \neq 0$ in $\mathbb{Z}^{\times}$ and $j \in \mathbb{Z}$, and hence, if $\left(u_j^k\right)^* = u_j^{-k}$ in $\mathscr{X}_\tau$, then

$$\tau\left(\left(\left(u_j^k\right)^*\right)^n\right) = \tau\left(\left(u_j^{-k}\right)^n\right) = \varphi\left(x_{j-kn}^n\right) = \omega_n c_{\frac{n}{2}} = \varphi\left(x_j^n\right),$$

for all $n \in \mathbb{N}$, by (60). So, the free-distributional data (61) holds. □

Now, let $k \in \mathbb{Z}^{\times}$, and $j \in \mathbb{Z}$, and $u_{k,j}$, the corresponding generating free random variable (57) of the shift-semicircular C^*-probability space $\mathscr{X}_\tau$. Consider the joint free moments of

$$\left\{u_{k,j},\ \left(u_{k,j}\right)^* = u_{-k,j}\right\},$$

in $\mathscr{X}_\tau$. First, observe that, if $(r_1, \ldots, r_n) \in \{1, *\}^n$ is a "mixed" n-tuple of $\{1, *\}$, for $n \in \mathbb{N}_{>1} = \mathbb{N} \setminus \{1\}$, in the sense that: there exists at least one $i_0 \in \{r_1, \ldots, r_n\}$, such that $i_0 \neq r_m$ in $\{1, *\}$, for some $m \in \{1, \ldots, n\}$, then

$$\prod_{l=1}^{n} \left(u_{k,j}\right)^{r_l} = \lambda^{\#(1)k - \#(*)k} \otimes x_j^n,$$

where

$$\#(1) = \text{the number of 1's in } (r_1, \ldots, r_n), \tag{62}$$

and

$$\#(*) = \text{the number of } * \text{'s in } (r_1, \ldots, r_n),$$

in $\mathscr{X}$. Thus, by (56) and (62),

$$\tau\left(\lambda^{(\#(1)-\#(*))k} \otimes x_j^n\right) = \varphi\left(\lambda^{(\#(1)-\#(*))k}\left(x_j^n\right)\right). \tag{63}$$

Lemma 22 *Let $u_{k,j} \in \mathscr{X}_\tau$ be a generating free random variable for $k \in \mathbb{Z}^{\times}$, and $j \in \mathbb{Z}$, and let*

$$(r_1, \ldots, r_n) \in \{1, *\}^n, \text{ for } n \in \mathbb{N}_{>1},$$

be a "mixed" n-tuple of $\{1, *\}$. Then

$$\tau\left(\prod_{l=1}^{n} \left(u_{k,j}\right)^{r_l}\right) = \omega_n c_{\frac{n}{2}} = \varphi\left(x_j^n\right), \ \forall n \in \mathbb{N}. \tag{64}$$

Proof Under hypothesis,

$$\prod_{l=1}^{n} \left(u_{k,j}\right)^{r_l} = \lambda^{(\#(1)-\#(*))k} \otimes x_j^n,$$

in $\mathscr{X}_\tau$ by (62), implying that

$$\tau \left(\prod_{l=1}^{n} \left(u_j^k\right)^{r_l} \right) = \varphi \left(\lambda^{(\#(1)-\#(*))k} \left(x_j^n \right) \right)$$

by (63)

$$= \varphi \left(x^n_{j+(\#(1)-\#(*))k} \right) = \varphi \left(x_j^n \right) = \omega_n c_{\frac{n}{2}},$$

for all $n \in \mathbb{N}$, by (59) and (61). Therefore, the free-distributional data (64) holds. □

By (59), (61) and (64), we obtain the following result.

Theorem 23 *Every generating free random variable $u_{k,j}$ of (57) are followed by the semicircular law in the sense of (44) in the shift-semicircular C^*-probability space $\mathscr{X}_\tau$, for all $k, j \in \mathbb{Z}$. i.e.,*

$$\tau \left(\prod_{l=1}^{n} \left(u_{k,j}\right)^{r_l} \right) = \omega_n c_{\frac{n}{2}} = \varphi \left(x_j^n \right), \tag{65}$$

for all $(r_1, \ldots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$.

Proof Let $(r_1, \ldots, r_n) \in \{1, *\}^n$ be non-mixed for $n \in \mathbb{N}$, i.e., either

$$(1, 1, \ldots, 1), \text{ or } (*, *, \ldots, *).$$

Then, by (59) and (61), the free-distributional data (65) holds. Meanwhile, if $(r_1, \ldots, r_n) \in \{1, *\}^n$ is mixed for $n \in \mathbb{N}_{>1}$, then the formula (65) holds too, by (64).

Therefore, the free-distributional data (65) holds as the joint free moments of $\left\{u_{k,j}, \left(u_{k,j}\right)^* = u_{-k,j}\right\}$, on $\mathscr{X}_\tau$. Equivalently, the generating free random variable $u_{k,j}$ is followed by the semicircular law in $\mathscr{X}_\tau$, for all $k, j \in \mathbb{Z}$. □

The above theorem shows that there do exist free random variables in a C^*-probability space followed by the semicircular law in the sense of (44).

Theorem 24 *Let (B, ψ) be a unital C^*-probability space containing mutually free, semicircular elements $\{y_1, \ldots, y_N\}$, for $N \in \mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$. Then there exists a*

C^*-probability space $(\mathscr{B}, \tau)$ and a free random variable $y \in (\mathscr{B}, \tau)$, such that y is followed by the semicircular law.

Proof Suppose a unital C^*-probability space (B, ψ) contains mutually free $|\mathbb{N}|$-many semicircular elements, $\mathcal{Y} = \{y_1, y_2, y_3, ...\}$. Then the C^*-subalgebra $C^*(\mathcal{Y})$ of B induces a C^*-probability space $\left(C^*(\mathcal{Y}), \psi = \psi\mid_{C^*(\mathcal{Y})}\right)$, which is free-isomorphic to our C^*-probability space $\mathfrak{X}_\varphi = (\mathfrak{X}, \varphi)$ generated by the free semicircular family $X = \left\{x_j\right\}_{j\in\mathbb{Z}}$, by (33). And hence, there exists the corresponding shift-semicircular C^*-probability space $\mathscr{X}_\tau = (\mathscr{X}, \tau)$, containing infinitely many generating free random variables $u_{k,j}$ of (57) followed by the semicircular law by (65). i.e., if

$$N = |\mathbb{N}| = \infty, \text{ in } \mathbb{N}^\infty,$$

then this theorem holds true.

Assume now that a unital C^*-probability space (B, ψ) contains mutually free, N-many semicircular elements,

$$\mathcal{Y}_N = \{y_1, \ldots, y_N\}, \text{ for } N < \infty.$$

Then the C^*-subalgebra $B_N = C^*(\mathcal{Y}_N \cup \{1_B\})$ of B induces the C^*-probability space (B_N, ψ). From (B_N, ψ), one can construct a C^*-probability space $(\mathcal{B}, \tau)$, with

$$\mathcal{B} = \mathop{\star}_{i=1}^{\infty} B[i], \text{ with } B[i] = B_N, \ \forall i \in \mathbb{N},$$

and

$$\tau = \psi^{\star\infty}, \text{ on } \mathcal{B},$$

where $(\star)$ is the free product of C^*-algebras (e.g., [17, 30]). Remark that all free factors $\{B[i]\}_{i=1}^\infty$, identified with B_N, are free from each other (e.g., [30]) in $(\mathcal{B}, \tau)$, and hence, the C^*-probability space $(\mathcal{B}, \tau)$ contains its free semicircular family,

$$Y = \mathop{\sqcup}_{i=1}^{\infty} \{y_{i1}, \ldots, y_{iN}\},$$

where $\sqcup$ is the disjoint union, and

$$\{y_{i1}, \ldots, y_{iN}\}, \text{ with } y_{i1} = y_1, \ldots, y_{iN} = y_N,$$

in a free factor $B[i] = B_N$, for all $i \in \mathbb{N}$. Under the possible rearrangement, one can let

$$Y = \{y_j\}_{j \in \mathbb{N}_0}.$$

Therefore, by (33) and (65), this theorem also holds even if $N < \infty$ in $\mathbb{N}^\infty$.

In conclusion, if a unital C^*-probability space (B, ψ) contains mutually free N-many semicircular elements for $N \in \mathbb{N}^\infty$, then one can have free random variables in a certain C^*-probability space $(\mathcal{B}, \tau)$, followed by the semicircular law. □

The above theorem shows not only that there do exist free random variables followed by the semicircular law, but also how to construct them. It also shows there are sufficiently many such free random variables. Motivated by Theorem 24, one can verify that, whenever a semicircular element x, and a free-isomorphism β exist for a unital C^*-probability space containing x, one can construct the free random variables,

$$\left\{\beta^k \otimes x\right\}_{k \in \mathbb{Z}}$$

followed by the semicircular law, in a certain tensor-product C^*-probability space.

7.3 Free-Distributional Information on $\mathscr{X}_\tau$

In this section, we study free-distributional data on the shift-semicircular C^*-probability space,

$$\mathscr{X}_\tau = (\Lambda \otimes \mathfrak{X},\ \tau),$$

generated by the free random variables followed by the semicircular law. Throughout this section, we let

$$u_l \overset{\text{denote}}{=} u_{k_l, j_l} = \lambda^{k_l} \otimes x_{j_l} \in \mathscr{X}_\tau \tag{66}$$

be the generating free random variables of $\mathscr{X}_\tau$, for $l = 1, \ldots, N$, for $N \in \mathbb{N}_{>1}$. Recall that

$$u_l^* = \lambda^{-k_l} \otimes x_{j_l} = u_{-k_l, j_l},\ \forall l = 1, \ldots, N, \tag{67}$$

in $\mathscr{X}_\tau$.

Theorem 25 *Let $x_{j_1}, \ldots, x_{j_N} \in X$ be generating semicircular elements of the C^*-probability space $\mathfrak{X}_\varphi$ (where $j_1, \ldots, j_N$ are not necessarily distinct in $\mathbb{Z}$), and let $\lambda^{k_1}, \ldots, \lambda^{k_N} \in \lambda$ be integer-shifts generating the shift algebra Λ (where $k_1, \ldots, k_N$*

are not necessarily distinct in $\mathbb{Z}$*), inducing the generating free random variables* $u_1, \ldots, u_N$ *of (66) in the shift-semicircular* C^**-probability space* $\mathscr{X}_\tau$*. If* $w = \prod_{l=1}^{N} x_{j_l}$ *is a free random variable of* $\mathfrak{X}_\varphi$*, and if* $W_{(r_1,\ldots,r_N)} = \prod_{l=1}^{N} u_l^{r_l}$ *is a free random variable of* $\mathscr{X}_\tau$*, where* $u_l^{r_l} \in \mathscr{X}_\tau$ *are in the sense of (66), or (67), for* $l = 1, \ldots, N$*, then*

$$\tau\left(W_{(r_1,\ldots,r_N)}\right) = \varphi\left(w\right), \tag{68}$$

for all $(r_1, \ldots, r_N) \in \{1, *\}^N$.

Proof Assume that $w = \prod_{l=1}^{N} x_{j_l} \in \mathfrak{X}_\varphi$ is a free random variable satisfying

$$\varphi\left(w\right) = \varpi, \text{ in } \mathbb{C},$$

determined by Theorem 8. Suppose

$$W_{(r_1,\ldots,r_N)} = \prod_{l=1}^{N} u_l^{r_l} \in \mathscr{X}_\tau,$$

for an arbitrarily fixed $(r_1, \ldots, r_N) \in \{1, *\}^N$. Then

$$W_{(r_1,\ldots,r_N)} = \left(\prod_{l=1}^{N} \lambda^{i_l}\right) \otimes \left(\prod_{l=1}^{N} x_{j_l}^{r_l}\right) = \left(\prod_{l=1}^{N} \lambda^{i_l}\right) \otimes w, \tag{69}$$

in $\mathscr{X}_\tau$, by the self-adjointness of $x_{j_1}, \ldots, x_{j_N} \in X$ in $\mathfrak{X}_\varphi$, where

$$i_l = \begin{cases} k_l & \text{if } r_l = 1 \\ -k_l & \text{if } r_l = *, \end{cases}$$

in $\mathbb{Z}$, for all $l = 1, \ldots, N$.

Then, there exists $k_{(r_1,\ldots,r_N)} \in \mathbb{Z}$, such that

$$\prod_{l=1}^{N} \lambda^{i_l} = \lambda^{k_{(r_1,\ldots,r_N)}} \in \lambda,$$

in Λ, by (41) and (51), i.e., one has

$$W_{(r_1,\ldots,r_N)} = \lambda^{k_{(r_1,\ldots,r_N)}} \otimes w \text{ in } \mathscr{X}_\tau,$$

by (69), implying that

$$\tau\left(W_{(r_1,\ldots,r_N)}\right)=\varphi\left(\lambda^{k_{(r_1,\ldots,r_N)}}(w)\right)=\varpi=\varphi(w),$$

by (43). Therefore, the formula (68) holds. □

By (68), one can obtain the following generalized results.

Theorem 26 *Let $y_1,\ldots,y_N$ be mutually free semicircular elements in a unital C^*-probability space (B,ψ), and let $y_{i_l}\in\{y_1,\ldots,y_N\}$, for $l=1,\ldots,n$, for $n\in\mathbb{N}$, where $i_1,\ldots,i_n$ are not necessarily distinct from each other in $\{1,\ldots,N\}$, and $\beta\in Aut(B)$, a free-isomorphism on (B,ψ). Let*

$$u_{k_l,j}\overset{\text{denote}}{=}\beta^{k_l}\otimes y_{i_l}\in(\Lambda_B\otimes B,\ \tau)$$

be a free random variable of a C^*-probability space $(\Lambda_B\otimes B,\ \tau)$, for $k_l\in\mathbb{Z}$, for $l=1,\ldots,n$, where Λ_B is the group algebra of the cyclic group $\langle\beta\rangle$, and τ is the linear functional on $\Lambda_B\otimes B$, satisfying

$$\tau(S\otimes T)=\psi(S(T)),$$

for all $S\otimes T\in\Lambda_B\otimes B$, where β^0 is the identity map on B, and β^{-1} is the inverse of β on B. Then

$$\tau\left(\prod_{l=1}^{n}u_{k_l,i_l}^{r_l}\right)=\psi\left(\prod_{j=1}^{N}y_{i_l}\right),\tag{70}$$

for all $(r_1,\ldots,r_n)\in\{1,*\}^n$, for all $n\in\mathbb{N}$, where the right-hand side of (70) is determined by (17).

Proof The proof of the free-distributional data (70) is similar to that of (68) by Theorem 24. Indeed, there exists $k_0\in\mathbb{Z}$, such that

$$\prod_{l=1}^{n}u_{k_l,i_l}^{r_l}=\left(\prod_{j=1}^{n}\beta^{\varepsilon_l k_l}\right)\otimes\left(\prod_{j=1}^{n}y_{i_l}\right)=\beta^{k_0}\otimes\left(\prod_{j=1}^{n}y_{i_l}\right),$$

in $(\Lambda_B\otimes B,\ \tau)$, where

$$\varepsilon_l=\begin{cases}1 & \text{if } r_l=1\\ -1 & \text{if } r_l=*,\end{cases}$$

for all $l = 1, \ldots, n$, implying that

$$\tau\left(\prod_{l=1}^{n} u_{k_l,l}^{r_l}\right) = \psi\left(\beta^{k_0}\left(\prod_{j=1}^{n} y_{i_l}\right)\right) = \psi\left(\prod_{j=1}^{n} y_{i_l}\right),$$

since β^{k_0} is a free-isomorphism on (B, ψ), by assumption. □

7.4 A Structure Theorem of $\mathscr{X}_\tau$

In this section, we consider some structure theorems of our shift-semicircular C^*-probability space,

$$\mathscr{X}_\tau = (\Lambda \otimes \mathfrak{X},\ \tau),$$

generated by the generating free random variables,

$$\mathcal{X} = \left\{ u_{k,j} = \lambda^k \otimes x_j : k, j \in \mathbb{Z} \right\},$$

followed by the semicircular law by (65), where $\lambda^k \in \lambda \subset \Lambda$, and $x_j \in X \subset \mathfrak{X}_\varphi$.

Suppose $u_l \overset{\text{denote}}{=} u_{k_l, j_l} = \lambda^{k_l} \otimes x_{j_l} \in \mathcal{X}$ are arbitrary generating free random variables of our shift-semicircular C^*-probability space $\mathscr{X}_\tau$, for $k_l, j_l \in \mathbb{Z}$, for $l = 1, \ldots, s$, for $s \in \mathbb{N}$. By (68), such operators are followed by the semicircular law in $\mathscr{X}_\tau$. Observe that

$$k_n^\tau\left(u_{i_1}^{r_1}, \ldots, u_{i_n}^{r_n}\right) = \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} \tau\left(\prod_{i_{j_t} \in V} u_{i_{j_t}}^{r_{j_t}}\right)\right) \mu\left(\pi, 1_n\right)$$

by the Möbius inversion

$$= \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} \varphi\left(\prod_{i_{j_t} \in V} x_{i_{j_t}}\right)\right) \mu\left(\pi, 1_n\right)$$

by (68) (or, (70))

$$= k_n^\varphi\left(\underbrace{x_{i_1}, x_{i_2}, \ldots \ldots, x_{i_n}}_{n\text{-times}}\right), \tag{71}$$

for all $(r_1, \ldots, r_n) \in \{1, *\}^n$ and $(i_1, \ldots, i_n) \in \{1, \ldots, s\}^n$, for all $n \in \mathbb{N}$, where $k_\bullet^\tau(.)$ (respectively, $k_\bullet^\varphi(.)$) is the free cumulant on $\mathscr{X}_\tau$ (respectively, on $\mathfrak{X}_\varphi$) in terms of the linear functional τ on $\mathscr{X}_\tau$ (respectively, φ on $\mathfrak{X}_\varphi$), by the semicircularity (7) of the generating operators $x_j \in X$ of $\mathfrak{X}_\varphi$.

Proposition 27 *Let $u_{k,j} \in \mathcal{X}$ be a generating free random variable of $\mathscr{X}_\tau$, followed by the semicircular law, for $k, j \in \mathbb{Z}$. Then*

$$k_n^\tau \left(u_{k,j}^{r_1}, \ldots, u_{k,j}^{r_n}\right) = \delta_{n,2} = k_n^\varphi \left(x_j, \ldots, x_j\right), \tag{72}$$

for all $(r_1, \ldots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$.

Proof The free-distributional data (72) is obtained by the general formula (71). □

The above free-distributional data (72) is equivalent to (65). i.e., Proposition 27 re-characterizes the free distributions followed by the semicircular law, i.e., the formula (72) characterizes (44).

Theorem 28 *Let $u_l = u_{k_l, j_l} \in \mathcal{X}$ be generating free random variables of $\mathscr{X}_\tau$, for $l = 1, 2$. Then $j_1 \neq j_2$ in $\mathbb{Z}$, if and only if u_1 and u_2 are free in $\mathscr{X}_\tau$.*

Proof ($\Rightarrow$) Suppose $j_1 \neq j_2$ in $\mathbb{Z}$, and hence, the generating operators u_1 and u_2 are distinct in $\mathcal{X} \subset \mathscr{X}_\tau$. Observe that, for any "mixed" n-tuple $(l_1, \ldots, l_n) \in \{1, 2\}^n$, for $n \in \mathbb{N}_{>1}$, we have that

$$k_n^\tau \left(u_{l_1}^{r_1}, \ldots, u_{l_n}^{r_n}\right) = k_n^\varphi \left(x_{j_{l_1}}, \ldots, x_{j_{l_n}}\right) = 0,$$

by (71) and (68), for all $(r_1, \ldots, r_n) \in \{1, *\}^n$. In particular, the second equality holds by the freeness of x_{j_1} and x_{j_2} in $\mathfrak{X}_\varphi$. Indeed, by assumption, the semicircular elements x_{j_1} and x_{j_2} are distinct in the generating free semicircular family $X = \{x_j\}_{j \in \mathbb{Z}}$ in $\mathfrak{X}_\varphi$, implying the freeness of them (e.g., [22, 23]). Therefore,

$$k_n^\tau \left(u_{l_1}^{r_1}, \ldots, u_{l_n}^{r_n}\right) = 0,$$

for all mixed n-tuples $(l_1, \ldots, l_n) \in \{1, 2\}^n$, for all $(r_1, \ldots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}_{>1}$. Equivalently, two subsets,

$$\{u_1, u_1^*\} \text{ and } \{u_2, u_2^*\}$$

are free in $\mathscr{X}_\tau$, i.e., if $j_1 \neq j_2$ in $\mathbb{Z}$, then two free random variables u_1 and u_2 are free in $\mathscr{X}_\tau$ (e.g., [30]).

($\Leftarrow$) Assume now that $j_1 = j = j_2$ in $\mathbb{Z}$, and hence, $u_l = u_{k_l, j} = \lambda^{k_l} \otimes x_j \in \mathcal{X}$ in $\mathscr{X}_\tau$. Then, for any (mixed, or non-mixed) $(l_1, \ldots, l_n) \in \{1, 2\}^n$, for $n \in \mathbb{N}$, we have

$$k_n^{\tau}\left(u_{l_1}^{r_1}, \ldots, u_{l_n}^{r_n}\right) = k_n^{\varphi}\left(\underbrace{x_j, x_j, \ldots, x_j}_{n\text{-times}}\right) = \delta_{n,2},$$

for all $(r_1, \ldots, r_n) \in \{1, *\}^n$ by (72), implying that, if $n = 2$ in $\mathbb{N}$, then

$$k_n^{\tau}\left(u_{l_1}^{r_1}, u_{l_2}^{r_2}\right) = k_n^{\varphi}\left(x_j, x_j\right) = 1,$$

even though $l_1 \neq l_2$ in $\{1, 2\}$. It shows that such two free random variables u_1 and u_2 are not free in $\mathfrak{X}_\tau$. i.e., if $j_1 = j_2$ in $\mathbb{Z}$, then u_1 and u_2 are not free in $\mathfrak{X}_\tau$. □

The above theorem shows that the generator set $\mathcal{X}$ of $\mathfrak{X}_\tau$ is decomposed to be

$$\mathcal{X} = \underset{j\in\mathbb{Z}}{\sqcup} \mathcal{X}_j \text{ in } \mathfrak{X}_\tau,$$

with

$$\mathcal{X}_j = \left\{u_{k,j} = \lambda^k \otimes x_j \in \mathcal{X} : k \in \mathbb{Z}\right\}, \ \forall j \in \mathbb{Z}, \tag{73}$$

where $\mathcal{X}_j$ are mutually free from each other in $\mathfrak{X}_\tau$, for all $j \in \mathbb{Z}$.

Corollary 29 *The generator set $\mathcal{X}$ of our shift-semicircular C^*-probability space $\mathfrak{X}_\tau$ is decomposed by (73), and the blocks $\mathcal{X}_j$ of $\mathcal{X}$ are mutually free from each other in $\mathfrak{X}_\tau$.*

Proof It is proven by Theorem 28 and (73). □

By Corollary 29, we obtain the following structure theorem of $\mathfrak{X}_\tau$.

Theorem 30 *The shift-semicircular C^*-probability space $\mathfrak{X}_\tau$ satisfies*

$$\mathfrak{X}_\tau \overset{*\text{-iso}}{=} \underset{j\in\mathbb{Z}}{\star} \left(C^*\left(\mathcal{X}_j\right)\right),$$

where

$$\mathcal{X}_j = \left\{u_{k,j} = \lambda^k \otimes x_j \in \mathcal{X} : k \in \mathbb{Z}\right\}, \ \forall j \in \mathbb{Z}, \tag{74}$$

and $C^*(Y)$ are the C^*-subalgebras of $\mathfrak{X}_\tau$ generated by subsets $Y \cup Y^*$ of $\mathfrak{X}_\tau$.

Proof Since $\mathcal{X}$ is the generator set of $\mathfrak{X}_\tau$, we have that

$$\mathfrak{X}_\tau = C^*(\mathcal{X}) = C^*\left(\underset{j\in\mathbb{Z}}{\sqcup} \mathcal{X}_j\right) = \underset{j\in\mathbb{Z}}{\star} \left(C^*\left(\mathcal{X}_j\right)\right),$$

by (73) and Corollary 29. □

By the structure theorem (74), one obtains the following result, too.

Theorem 31 *The shift-semicircular C^*-probability space $\mathscr{X}_\tau$ satisfies that*

$$\mathscr{X}_\tau \overset{*\text{-iso}}{=} \underset{j \in \mathbb{Z}}{\star} \left(\Lambda \otimes C^*_{\mathfrak{X}}\left(\{x_j\}\right)\right),$$

and hence,

$$\mathscr{X}_\tau \overset{*\text{-iso}}{=} \underset{j \in \mathbb{Z}}{\star_\Lambda} \left(\Lambda \otimes C^*_{\mathfrak{X}}\left(\{x_j\}\right)\right), \tag{75}$$

where $C^*_{\mathfrak{X}}(Z)$ are the C^*-subalgebras of $\mathfrak{X}_\varphi$ generated by the subsets $Z \cup Z^*$ of $\mathfrak{X}_\varphi$, and where "$\star_\Lambda$" is the amalgamated free product with its amalgamation over the group C^*-algebra Λ (in the sense of [22]).

Proof By (74), we have $\mathscr{X}_\tau \overset{*\text{-iso}}{=} \underset{j \in \mathbb{Z}}{\star} \left(C^*\left(\mathcal{X}_j\right)\right)$, where $\mathcal{X}_j$ are in the sense of (73) for all $j \in \mathbb{Z}$. So, by definition,

$$C^*\left(\mathcal{X}_j\right) \overset{*\text{-iso}}{=} C^*_{B(H)}(\lambda) \otimes C^*_{\mathfrak{X}}\left(\left\{x_j\right\}\right) = \Lambda \otimes C^*_{\mathfrak{X}}\left(\left\{x_j\right\}\right),$$

where $\lambda = \left\{\lambda^k\right\}_{k \in \mathbb{Z}}$ is the integer-shift group, generating the group algebra Λ in the operator algebra $B(H)$, where H is the group Hilbert space (45) of λ. Therefore, the first $*$-isomorphic relation of (75) holds.

By the definition of amalgamated free products with amalgamations of [22], the second $*$-isomorphic relation (75) holds, too, since

$$\mathscr{X}_\tau \overset{*\text{-iso}}{=} \underset{j \in \mathbb{Z}}{\star} \left(\Lambda \otimes C^*_{\mathfrak{X}}\left(\{x_j\}\right)\right) \overset{*\text{-iso}}{=} \Lambda \otimes \left(\underset{j \in \mathbb{Z}}{\star} \left(C^*_{\mathfrak{X}}\left(\{x_j\}\right)\right)\right),$$

by understanding Λ as the common C^*-subalgebras $\left\{\Lambda \otimes C^*_{\mathfrak{X}}\left(\{x_j\}\right)\right\}_{j \in \mathbb{Z}}$ of $\mathscr{X}_\tau$. □

8 Certain Banach-Space Operators Acting on $\mathscr{X}_\tau$

Let $\mathscr{X}_\tau = (\mathscr{X}, \tau)$ be our shift-semicircular C^*-probability space of the unital C^*-algebra,

$$\mathscr{X} = \Lambda \otimes \mathfrak{X},$$

where Λ is the shift-operator algebra of the integer-shift group λ, acting on the unital C^*-probability space $\mathfrak{X}_\varphi = (\mathfrak{X}, \varphi)$ generated by the free semicircular family

$X = \{x_j\}_{j \in \mathbb{Z}}$, equipped with the linear functional τ, satisfying

$$\tau (T \otimes S) = \varphi (T (S)),$$

for all $T \otimes S \in \mathscr{X}$. Recall that $\mathscr{X}_\tau$ is generated by the free random variables,

$$u_{k,j} \overset{\text{denote}}{=} \lambda^k \otimes x_j \in \mathscr{X}_\tau, \text{ for all } k, j \in \mathbb{Z}, \tag{76}$$

followed by the semicircular law by (65) and (68).

In this section, we consider certain Banach-space operators, bounded linear transformations, acting on $\mathscr{X}_\tau$, by regarding $\mathscr{X}$ as a Banach space equipped with its tensor-product C^*-norm (e.g., [13]). i.e., we are interested in some elements of the operator space $B(\mathscr{X}_\tau)$ (e.g., [13, 14]).

8.1 Banach-Space Operators $T^t_{s,l} \in B(\mathscr{X}_\tau)$

Let $B(\mathscr{X}_\tau)$ be the operator space consisting of all Banach-space operators acting on our shift-semicircular C^*-probability space $\mathscr{X}_\tau$, by regarding $\mathscr{X} = \Lambda \otimes \mathfrak{X}$ as a Banach space. Define an element $T^t_{s,l} \in B(\mathscr{X}_\tau)$ by

$$T^t_{s,l} \overset{\text{def}}{=} M_{t\lambda^s} \otimes \lambda^l, \text{ on } \mathscr{X}_\tau,$$

satisfying

$$\begin{aligned} T^t_{s,l}(u_{k,j}) &= (M_{t\lambda^s} \otimes \lambda^l)(\lambda^k \otimes x_j) \\ &= (t\lambda^s\lambda^k \otimes \lambda^l(x_j)) = t\lambda^{s+k} \otimes x_{j+l}, \end{aligned} \tag{77}$$

i.e.,

$$T^t_{s,l}(u_{k,j}) = t\left(\lambda^{k+s} \otimes x_{j+l}\right) = tu_{k+s,j+l}, \tag{78}$$

in $\mathscr{X}_\tau$, for all $u_{k,j} \in \mathcal{X}$, and for all $t \in \mathbb{C}$, and $s, l \in \mathbb{Z}$, where

$$\mathcal{X} = \{u_{k,j} : k, j \in \mathbb{Z}\}$$

is the generator set of all generating free random variables (76) of $\mathscr{X}_\tau$. In (77), the tensor factor $M_{t\lambda^s} \in B(\Lambda)$ of $T^t_{s,l}$ is a multiplication operator acting on the shift-operator algebra Λ (by regarding it as a Banach space), defined by

$$M_{t\lambda^s}(S) = t\lambda^s S, \text{ in } \Lambda, \ \forall S \in \Lambda,$$

and the other tensor factor $\lambda^l \in \lambda$ of $T_{s,l}^t$ is our generating shift operator of Λ acting on $\mathfrak{X}_\varphi$. So, the Banach-space operator $T_{s,l}^t$ of (77) is well-defined in $B(\mathscr{X}_\tau)$.

Theorem 32 *For $t \in \mathbb{C}^\times$, and $s, l \in \mathbb{Z}$, let $T_{s,l}^t \in B(\mathscr{X}_\tau)$ be the Banach-space operator (77), and let $u_{k,j} \in \mathcal{X}$ be a generating operator (76) of $\mathscr{X}_\tau$. Then the image $\boldsymbol{u} \overset{\text{denote}}{=} T_{s,l}^t(u_{k,j})$ satisfies the free-distributional data,*

$$\tau\left(\prod_{i=1}^{n} \boldsymbol{u}^{r_i}\right) = \omega_n\left(t^{\#(1)}\overline{t}^{\#(*)}\right) c_{\frac{n}{2}}, \text{ on } \mathscr{X}_\tau, \tag{79}$$

where $\overline{t}$ is the conjugates of t, and

$$\#(1) = \text{ the number of } 1\text{'s in } (r_1, \ldots, r_n),$$

and

$$\#(*) = \text{ the number of } *\text{'s in } (r_1, \ldots, r_n),$$

for all $(r_1, \ldots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$,

Proof Under hypothesis, one has that

$$\mathbf{u} = T_{s,l}^t(u_{k,j}) = t\lambda^{k+s} \otimes x_{j+l} = t u_{k+s,j+l},$$

by (77) and (78). Note that an operator $u_{k+s,j+l} \in \mathcal{X}$, in the far-right-hand side is a generating free random variable of $\mathscr{X}_\tau$, followed by the semicircular law, by (65). So, the element $\mathbf{u} \in \mathscr{X}_\tau$ is a scalar multiple of $u_{k+n,j+l}$, and hence, the free distribution of $\mathbf{u}$ may be affected by the semicircularity. Indeed, observe that, for any

$$(r_1, \ldots, r_n) \in \{1, *\}^n, \text{ for } n \in \mathbb{N},$$

one has

$$\tau\left(\prod_{i=1}^{n} \mathbf{u}^{r_i}\right) = \tau\left(\left(\prod_{i=1}^{n} t^{r_i}\right)\left(\prod_{i=1}^{n} u_{k+s,j+l}^{r_i}\right)\right)$$

where

$$t^{r_i} = \begin{cases} t & \text{if } r_i = 1 \\ \overline{t} & \text{if } r_i = *, \end{cases}$$

in $\mathbb{C}$, where $\overline{t}$ is the conjugate of t, and hence, it goes to

$$= \left(\prod_{i=1}^{n} t^{r_i}\right)\left(\omega_n c_{\frac{n}{2}}\right)$$

since the generating free random variable $u_{k+s,j+l} \in \mathcal{X}$ is followed by the semicircular law in $\mathscr{X}_\tau$

$$= \left(t^{\#(1)}\overline{t}^{\#(*)}\right)\left(\omega_n c_{\frac{n}{2}}\right),$$

where

$$\#(1) = \text{ the number of 1's in } (r_1, \ldots, r_n),$$

and

$$\#(*) = \text{ the number of } * \text{'s in } (r_1, \ldots, r_n).$$

Therefore, the free-distributional data (79) holds. □

The above theorem shows how our Banach-space operator $T_{s,l}^{t} \in B(\mathscr{X}_\tau)$ of (77) affects the original free-distributional data on $\mathscr{X}_\tau$ by (79). It distorts the free distributions followed by the semicircular law to the free random variables satisfying (79), deformed by $\{t, \overline{t}\} \subset \mathbb{C}$.

Now, recall the following concept, introduced in [5–8, 12].

Definition 33 Let (B, ψ) be an arbitrary topological $*$-probability space. A "self-adjoint" free random variable $y \in (B, \psi)$ is said to be weighted-semicircular with its weight $t_0 \in \mathbb{C}^{\times}$ (or, in short, t_0-semicircular), if

$$\psi\left(y^n\right) = \omega_n t_0^{\frac{n}{2}} c_{\frac{n}{2}}, \text{ for all } n \in \mathbb{N}. \tag{80}$$

The free distributions of weighted-semicircular elements are called weighted-semicircular laws.

By definition, all semicircular elements are 1-semicircular in the sense of (80), and hence, the semicircular law is a 1-semicircular law in terms of Definition 33. Also, by (80), even though the semicircular law is universal by (8) and (9), weighted semicircular laws are not universally determined because they are dictated by their weights, i.e., they are depending on choices of weights. Such free random variables whose free distributions are weighted-semicircular laws do exist and have interesting properties up to weight (e.g., see [5] through [6, 10–12]).

Definition 34 Let (B, ψ) be a topological $*$-probability space. A free random variable $y \in (B, \psi)$ is said to be followed by a t_0-semicircular law, if the joint free moments of $\{y, y^*\}$ satisfy

$$\psi\left(\prod_{i=1}^{n} y^{r_i}\right) = \omega_n t_0^{\frac{n}{2}} c_{\frac{n}{2}}, \text{ for all } n \in \mathbb{N}, \tag{81}$$

for all $(r_1, \ldots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$.

By (80), all weighted-semicircular elements are followed by the corresponding weighted-semicircular laws in the sense of (81). Clearly, not all free random variables followed by weighted-semicircular laws are weighted-semicircular. Then, similar to Theorems 23 and 24, are there sufficiently many free random variables followed by weighted-semicircular laws? The answer is positive by (79).

Corollary 35 *Let $T_{s,l}^{t} \in B(\mathscr{X}_\tau)$ be a Banach-space operator (77) for $t \in \mathbb{C}^\times$, and $s, l \in \mathbb{Z}$, and let $u_{k,j} \in \mathscr{X}_\tau$ be a generating free random variable for $k, j \in \mathbb{Z}$. If "$t \in \mathbb{R}$" in $\mathbb{C}^\times$, then the image $\mathbf{u} = T_{s,l}^{t}(u_{k,j}) \in \mathscr{X}_\tau$ is followed by the t^2-semicircular law in the sense of (81).*

Proof Let $\mathbf{u} = T_{s,l}^{t}(u_{k,j}) = t u_{k+s,j+l} \in \mathscr{X}_\tau$ be the corresponding free random variable, where $t \in \mathbb{R}^\times = \mathbb{R} \setminus \{0\}$. Then

$$\tau\left(\prod_{i=1}^{n} \mathbf{u}^{r_i}\right) = \omega_n \left(t^{\#(1)} \overline{t}^{\#(*)}\right) c_{\frac{n}{2}},$$

for all $(r_1, \ldots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$, by (79). Moreover, since $t \in \mathbb{R}^\times$,

$$\tau\left(\prod_{i=1}^{n} \mathbf{u}^{r_i}\right) = \omega_n \left(t^{\#(1)+\#(*)}\right) c_{\frac{n}{2}} = \omega_n t^n c_{\frac{n}{2}},$$

implying that

$$\tau\left(\prod_{i=1}^{n} \mathbf{u}^{r_i}\right) = \omega_n \left(t^2\right)^{\frac{n}{2}} c_{\frac{n}{2}}, \tag{82}$$

for all $(r_1, \ldots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$.

Therefore, if $t \in \mathbb{R}^\times$, then the free random variable $T_{s,l}^{t}(u_{k,j}) \in \mathscr{X}_\tau$ is followed by the t^2-semicircular law by (82). □

The above corollary shows not only that there do exist free random variables followed by weighted-semicircular laws, but also that there are sufficiently many such free random variables. Also, it shows how our Banach-space operator $T_{s,l}^{t} \in B(\mathscr{X}_\tau)$ deforms the free distributions followed by the semicircular law to free distributions followed by t^2-semicircular laws, for all $t \in \mathbb{R}^\times$.

Let $T_{s_i,l_i}^{t_i} \in B(\mathscr{X}_\tau)$ be Banach-space operators (77), where $t_i \in \mathbb{C}^\times$, and $s_i, l_i \in \mathbb{Z}$, for $i = 1, \ldots, N$, for $N \in \mathbb{N}$. Observe that

$$\prod_{i=1}^{N} T_{s_i,l_i}^{t_i} = \left(\prod_{i=1}^{N} t_i M_{\lambda^{s_i}}\right) \otimes \left(\prod_{i=1}^{N} \lambda^{l_i}\right)$$

since $M_{t\lambda^s} = tM_{\lambda^s} \in B(\Lambda)$, for all $t \in \mathbb{C}^\times$, and $s, l \in \mathbb{Z}$

$$= \left(\prod_{i=1}^{N} t_i\right)\left(M_{\prod\limits_{i=1}^{N} \lambda^{s_i}}\right) \otimes \left(\prod_{i=1}^{N} \lambda^{l_i}\right)$$

since $M_{\lambda^{s_1}} M_{\lambda^{s_2}} = M_{\lambda^{s_1}\lambda^{s_2}}$ on Λ by the very definition

$$= \left(\prod_{i=1}^{N} t_i\right) M_{\lambda^{s_o}} \otimes \lambda^{l_o}$$

where

$$s_o = \sum_{i=1}^{N} s_i, \text{ and } l_o = \sum_{i=1}^{N} l_i, \text{ in } \mathbb{Z},$$

and hence, it goes to

$$= t_o M_{\lambda_o} \otimes \lambda^{l_o} = M_{t_o\lambda^o} \otimes \lambda^{l_o} = T_{s_o,l_o}^{t_o},$$

in $B(\mathscr{X}_\tau)$. i.e.,

$$\prod_{i=1}^{N} T_{s_i,l_i}^{t_i} = T_{s_o,l_o}^{t_o} \text{ in } B(\mathscr{X}_\tau),$$

with

$$t_o = \prod_{i=1}^{N} t_i, \ s_o = \sum_{i=1}^{N} s_i, \text{ and } l_o = \sum_{i=1}^{N} l_i. \tag{83}$$

By (79) and (83), we obtain the following result.

Theorem 36 *Let* $T_{s_i,l_i}^{t_i} \in B(\mathscr{X}_\tau)$ *be the Banach-space operators (77), for* $i = 1, \ldots, N$, *for* $N \in \mathbb{N}$, *and let* $\boldsymbol{T}_N \overset{\text{denote}}{=} \prod_{i=1}^{N} T_{s_i,l_i}^{t_i} \in B(\mathscr{X}_\tau)$. *Then there exists*

$$t_o = \prod_{i=1}^{N} t_i \in \mathbb{C}^\times,$$

such that

$$\tau\left(\prod_{i=1}^{n}\left(\boldsymbol{T}_N\left(u_{k,j}\right)\right)^{r_i}\right) = \omega_n\left(t_o^{\#(1)}\overline{t_o}^{\#(*)}\right)c_{\frac{n}{2}}, \tag{84}$$

for all $(r_1, \ldots, r_n) \in \{1, *\}^n$, *for all* $n \in \mathbb{N}$, *for any fixed generating free random variables* $u_{k,j} \in \mathcal{X}$ *of* $\mathscr{X}_\tau$, *for all* $k, j \in \mathbb{Z}$, *where* $\#(1)$ *and* $\#(*)$ *are in the sense of (79).*

Proof Under hypothesis, we have

$$\mathbf{T}_N = T_{s_o,l_o}^{t_o} \in B(\mathscr{X}_\tau),$$

in the sense of (77), where $t_o \in \mathbb{C}^\times$, and $s_o, l_o \in \mathbb{Z}$ are in the sense of (83).

For any generating operator $u_{k,j} \in \mathcal{X}$ of $\mathscr{X}_\tau$, we have

$$\mathbf{T}_N\left(u_{k,j}\right) = T_{s_o,l_o}^{t_o}\left(u_{k,j}\right) = t_o u_{k+s_o,j+l_o} \text{ in } \mathscr{X}_\tau.$$

Therefore, the free-distributional data (84) holds by (79). □

The above theorem generalizes (79) by (84). It is interesting that finite products of Banach-space operators (77) become Banach-space operators again in the sense of (77) in $B(\mathscr{X}_\tau)$ by (83).

Corollary 37 *Under the same conditions of Theorem 36, if* $t_o = \prod_{i=1}^{N} t_i \in \mathbb{R}^\times$, *then the free random variable* $\boldsymbol{T}_N\left(u_{k,j}\right) \in \mathscr{X}_\tau$ *is followed by the* t_o^2*-semicircular law, for all generating operators* $u_{k,j} \in \mathcal{X}$ *of* $\mathscr{X}_\tau$.

Proof It is proven by (84) and Corollary 37. □

8.2 Multiplication Operators $M_{u_{k_0,j_0}}$ of $B(\mathscr{X}_\tau)$

In this section, we consider a different type of Banach-space operators acting on the shift-semicircular C^*-probability space $\mathscr{X}_\tau$, and consider how such operators deform the original free-distributional data on $\mathscr{X}_\tau$. Recall again that all generating free random variables

$$\mathcal{X} = \left\{ u_{k,j} = \lambda^k \otimes x_j : k, j \in \mathbb{Z} \right\}$$

of $\mathscr{X}_\tau$ are followed by the semicircular law, for all $\lambda^k \in \lambda \subset \Lambda$, and $x_j \in X \subset \mathfrak{X}_\varphi$.

Fix an arbitrary generating element

$$u_0 \overset{\text{denote}}{=} u_{k_0, j_0} \in \mathcal{X} \tag{85}$$

of $\mathscr{X}_\tau$, and define a Banach-space operator $M_{u_0} \in B\left(\mathscr{X}_\tau\right)$ by

$$M_{u_0}(T) = u_0 T \text{ in } \mathscr{X}_\tau, \ \forall T \in \mathscr{X}_\tau. \tag{86}$$

Indeed, this morphism M_{u_0} of (86) is a well-defined bounded linear transformation acting on the Banach space $\mathscr{X}_\tau$, understood to be a multiplication Banach-space operator on $\mathscr{X}_\tau$ with its symbol u_0 of (85).

Definition 38 We call the Banach-space operator $M_{u_0} \in B\left(\mathscr{X}_\tau\right)$ of (86), the multiplication operator with its symbol $u_0 \in \mathscr{X}_\tau$. More generally, if $w \in \mathscr{X}_\tau$, and if M_w is a Banach-space operator of $B\left(\mathscr{X}_\tau\right)$,

$$M_w(y) = wy, \ \forall y \in \mathscr{X}_\tau,$$

then it is called the multiplication operator with its symbol w.

Observe that, for any generators $u_{k,j} \in \mathcal{X}$ of $\mathscr{X}_\tau$, if $M_{u_0} \in B\left(\mathscr{X}_\tau\right)$ is a multiplication operator (86), then

$$M_{u_0}\left(u_{k,j}\right) = u_{k_0,j_0} u_{k,j} = \left(\lambda^{k_0}\lambda^k\right) \otimes \left(x_{j_0} x_j\right) = \lambda^{k_0+k} \otimes x_{j_0} x_j, \tag{87}$$

in $\mathscr{X}_\tau$, by (85).

Let's consider the tensor-factor $x_{j_0} x_j \in \mathfrak{X}_\varphi$ of the image (87) of $M_{u_0}\left(u_{k,j}\right) \in \mathscr{X}_\tau$. Suppose first that $j = j_0$ in $\mathbb{Z}$. Then

$$x_{j_0} x_j = x_{j_0}^2 \in \mathfrak{X}_\varphi,$$

as a self-adjoint free random variable. By the structure theorem (25) of $\mathfrak{X}_\varphi$ (which is free-isomorphic to (22) by (33)), this element $x_{j_0}^2$ is a free reduced word with its length-1 in $\mathfrak{X}_\varphi$, satisfying its free-distributional data,

$$\varphi\left(\left(x_{j_0}^2\right)^n\right) = \varphi\left(x_{j_0}^{2n}\right) = \omega_{2n} c_{\frac{2n}{2}} = c_n, \tag{88}$$

the n-th Catalan number, for all $n \in \mathbb{N}$.

So, if $\mathbf{u}_{j_0} = M_{u_0}\left(u_{k,j_0}\right) = \lambda^{k_0+k} \otimes x_{j_0}^2 \in \mathscr{X}_\tau$, then

$$\begin{aligned}\tau\left(\prod_{i=1}^{n} \mathbf{u}_{j_0}^{r_i}\right) &= \tau\left(\lambda^{\#(1)-\#(*)} \otimes x_{j_0}^{2n}\right) \\ &= \varphi\left(\lambda^{\#(1)-\#(*)}\left(x_{j_0}^{2n}\right)\right) = \varphi\left(x_{j_0}^{2n}\right) = c_n,\end{aligned} \tag{89}$$

by (88), for all $(r_1, \ldots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$, since $\lambda^l \in \lambda \subset \Lambda$ are free-isomorphisms on $\mathfrak{X}_\varphi$, for all $l \in \mathbb{Z}$.

Lemma 39 *Let $M_{u_0} \in B(\mathscr{X}_\tau)$ be the multiplication operator (86) with its symbol $u_0 \in \mathscr{X}_\tau$ of (85). Then, for any generating free random variables $u_{k,j_0} \in \mathcal{X} \subset \mathscr{X}_\tau$, for all $k \in \mathbb{Z}$, and fixed $j_0 \in \mathbb{Z}$, we have*

$$\tau\left(\prod_{i=1}^{n}\left(M_{u_0}\left(u_{k,j_0}\right)\right)^{r_i}\right) = c_n, \tag{90}$$

for all $(r_1, \ldots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$.

Proof The free-distributional data (90) is obtained by (89). □

Now, suppose $j \neq j_0$ in $\mathbb{Z}$ in (87). Then the tensor-factor $x_{j_0}x_j \in \mathfrak{X}_\varphi$ of $\mathbf{u}_0 \overset{\text{denote}}{=} M_{u_0}\left(u_{k,j}\right) \in \mathscr{X}_\tau$ is a free reduced word with its length-2, since two generating semicircular elements x_{j_0} and x_j (are mutually distinct in the generating free semicircular family $X = \{x_k\}_{k \in \mathbb{Z}}$, and hence, they) are free in $\mathfrak{X}_\varphi$.

For $(r_1, \ldots, r_n) \in \{1, *\}^n$, for $n \in \mathbb{N}$, one has that

$$\tau\left(\prod_{i=1}^{n} \mathbf{u}_0^{r_i}\right) = \tau\left(\prod_{i=1}^{n}\left(\lambda^{k_0+k} \otimes x_{j_0}x_j\right)^{r_i}\right)$$

by (87)

$$= \tau\left(\prod_{i=1}^{n}\left(\lambda^{r_i(k_0+k)} \otimes \left(x_{j_0}x_j\right)^{r_i}\right)\right)$$

where

$$r_i\left(k_0+k\right) = \begin{cases} k_0+k & \text{if } r_i = 1 \\ -\left(k_0+k\right) & \text{if } r_i = *, \end{cases}$$

and

$$\left(x_{j_0}x_j\right)^{r_i} = \begin{cases} x_{j_0}x_j & \text{if } r_i = 1 \\ & \\ x_j x_{j_0} & \text{if } r_i = *, \end{cases}$$

for all $i = 1, \ldots, n$, so, it goes to

$$= \varphi \left(\lambda^{N_{(r_1,\ldots,r_n)}} \otimes \left(\prod_{i=1}^{n} \left(x_{j_0} x_j\right)^{r_i} \right) \right)$$

where

$$N_{(r_1,\ldots,r_n)} = (k_0 + k)\,(\#(1) - \#(*)) \in \mathbb{Z},$$

and hence, it goes to

$$= \varphi \left(\prod_{i=1}^{n} \left(x_{j_0} x_j\right)^{r_i} \right), \tag{91}$$

since $\lambda^{N_{(r_1,\ldots,r_n)}}$ is a free-isomorphism on $\mathfrak{X}_{\varphi}$. Note here that the $\mathbb{C}$-quantity (91) is characterized by Theorem 8, or (17).

Lemma 40 *Let $\boldsymbol{u}_0 = M_{u_0}\left(u_{k,j}\right) \in \mathscr{X}_\tau$, where $M_{u_0} \in B\left(\mathscr{X}_\tau\right)$ is the multiplication operator (86) with its symbol $u_0 = u_{k_0,j_0} \in \mathcal{X} \subset \mathscr{X}_\tau$ of (77), and suppose $j \neq j_0$ in $\mathbb{Z}$. Then*

$$\tau \left(\prod_{i=1}^{n} \mathbf{u}_0^{r_i} \right) = \varphi \left(\prod_{i=1}^{n} \left(x_{j_0} x_j\right)^{r_i} \right), \tag{92}$$

and the right-hand side of (92) is characterized by Theorem 8.

Proof The free-distributional data (92) is obtained by (91). □

By the previous two lemmas, we obtain the following result showing how our multiplications operators of $B\left(\mathscr{X}_\tau\right)$ affect the free-distributional data on $\mathscr{X}_\tau$.

Theorem 41 *Let $M_{u_0} \in B\left(\mathscr{X}_\tau\right)$ be a multiplication operator (86), and $u_{k,j} \in \mathcal{X}$, a generating free random variable of $\mathscr{X}_\tau$ followed by the semicircular law. Let*

$$\boldsymbol{u}_{j_0} = M_{u_0}\left(u_{k,j_0}\right) \text{ and } \boldsymbol{u}_0 = M_{u_0}\left(u_{k,j}\right), \text{ in } \mathscr{X}_\tau,$$

where $j \neq j_0$ in $\mathbb{Z}$. Then the images $\boldsymbol{u}_{j_0}$ and $\boldsymbol{u}_0$ are no longer followed by the semicircular law. In particular,

$$\tau \left(\prod_{i=1}^{n} \boldsymbol{u}_{j_0}^{r_i} \right) = c_n, \text{ the } n\text{-th Catalan number,} \tag{93}$$

and

$$\tau\left(\prod_{i=1}^{n}\boldsymbol{u}_0^{r_i}\right)=\varphi\left(\prod_{i=1}^{n}\left(x_{j_0}x_j\right)^{r_i}\right),$$

characterized by Theorem 8 (or, (17)), for all $(r_1,\ldots,r_n)\in\{1,*\}^n$, *for all* $n\in\mathbb{N}$.

Proof The free-distributional data (93) are obtained by (90) and (92). Therefore, the free random variables $\mathbf{u}_{j_0}$ and $\mathbf{u}_0$ are not followed by the semicircular law in $\mathscr{X}_\tau$. □

The above theorem illustrates how our multiplication operators deform the free probability on the shift-semicircular C^*-probability space $\mathscr{X}_\tau$. In particular, the generating free random variables of $\mathscr{X}_\tau$, followed by the semicircular law, are no longer followed by the semicircular law, under the action of our multiplications operators by (93). For example, if

$$\mathbf{u}_0=M_{u_0}\left(u_{k,j}\right)=\lambda^{k_0+k}\otimes x_{j_0}x_j\in\mathscr{X}_\tau$$

(with $j\neq j_0$) is as above in Theorem 41, then

$$\begin{aligned}\tau\left(\mathbf{u}_0^n\right)&=\tau\left(\left(\lambda^{k_0+k}\otimes x_{j_0}x_j\right)^n\right)=\tau\left(\lambda^{n(k_0+k)}\otimes\left(x_{j_0}x_j\right)^n\right)\\&=\varphi\left(\lambda^{n(k_0+k)}\left(\left(x_{j_0}x_j\right)^n\right)\right)=\varphi\left(\left(x_{j_0}x_j\right)^n\right)\\&=\varphi\left(x_{j_0}x_jx_{j_0}x_j...x_{j_0}x_j\right)\\&=\left(\varphi\left(x_{j_0}^n\right)\right)\left(\varphi\left(x_j\right)\right)^n=0,\end{aligned}\tag{94}$$

since $\varphi(x_j)=\omega_1c_{\frac{1}{2}}=0$, for all $n\in\mathbb{N}$, by (93). Note that, the formula (94) is obtained by Theorem 8. Indeed, the free reduced word,

$$W=\left(x_{j_0}x_j\right)^n=x_{j_0}x_jx_{j_0}x_j...x_{j_0}x_j\in\mathfrak{X}_\varphi$$

with its length-$(2n)$ induces the corresponding noncrossing partition,

$$\pi_W=\{(i_1,i_3,\ldots,i_{2n-1}),(i_2),(i_4),\ldots,(i_{2n})\},$$

in the lattice $NC\left(\{i_1,\ldots,i_{2n}\}\right)$ of all noncrossing partitions over $\{i_1,\ldots,i_{2n}\}$.

Similar to (94), under the same hypothesis, we have

$$\tau\left(\left(\mathbf{u}_0^*\right)^n\right)=0,\ \text{ for all } n\in\mathbb{N},$$

since

$$\left(\mathbf{u}_0^*\right)^n = \left(x_j x_{j_0}\right)^n = x_j x_{j_0} x_j x_{j_0} ... x_j x_{j_0} \in \mathfrak{X}_\varphi$$

is a free reduced word with its length-$(2n)$.

Also, one can have, for example, that

$$\begin{aligned}\tau\left(\mathbf{u}_0^*\mathbf{u}_0\right) &= \tau\left(\left(\lambda^{-(k_0+k)} \otimes x_j x_{j_0}\right)\left(\lambda^{k_0+k} \otimes x_{j_0} x_j\right)\right) \\ &= \tau\left(\lambda^0 \otimes x_j x_{j_0}^2 x_j\right) = \varphi\left(x_j x_{j_0}^2 x_j\right)\end{aligned}$$

where $x_j x_{j_0}^2 x_j = \left(x_j x_{j_0}\right)\left(x_{j_0} x_j\right) \in \mathfrak{X}_\varphi$ is a free reduced word with its length-3

$$= \varphi\left(x_j^2\right)\varphi\left(x_{j_0}^2\right) = \left(\omega_2 c_{\frac{2}{2}}\right)\left(\omega_2 c_{\frac{2}{2}}\right) = c_1^2 = 1,$$

by (93), because the free random variable $w = x_j x_{j_0}^2 x_j \in \mathfrak{X}_\varphi$ induces its corresponding noncrossing partition,

$$\pi_w = \{(i_1, i_4), (i_2, i_3)\},$$

in $NC(\{i_1, i_2, i_3, i_4\})$, by (17).

Let $M_{u_i} = M_{u_{k_i, j_i}} \in B(\mathscr{X}_\tau)$ be the multiplication operators (87) with their symbols $u_{k_i, j_i} = \lambda^{k_i} \otimes x_{j_i} \in \mathcal{X} \subset \mathscr{X}_\tau$, for $i = 1, 2$. Then it is not difficult to check that

$$M_{u_1} M_{u_2} = M_{u_1 u_2} \text{ on } \mathscr{X}_\tau,$$

since

$$\begin{aligned}M_{u_1} M_{u_2}\left(u_{k,j}\right) &= M_{u_1}\left(\lambda^{k_2}\lambda^k \otimes x_{j_l} x_j\right) \\ &= \lambda^{k_1}\left(\lambda^{k_2}\lambda^k\right) \otimes x_{j_1}\left(x_{j_2} x_j\right) = \left(\lambda^{k_1}\lambda^{k_2}\right)\lambda^k \otimes \left(x_{j_1} x_{j_2}\right) x_j \\ &= \left(\lambda^{k_1}\lambda^{k_2} \otimes x_{j_1} x_{j_2}\right)\left(\lambda^k \otimes x_j\right) = (u_1 u_2)\left(u_{k,j}\right) \\ &= M_{u_1 u_2}\left(u_{k,j}\right),\end{aligned}$$

for all generating operators $u_{k,j} \in \mathcal{X}$ of $\mathscr{X}_\tau$, implying that

$$M_{u_1} M_{u_2} = M_{u_1 u_2} \text{ in } B(\mathscr{X}_\tau). \tag{95}$$

Let $M_{u_i} = M_{u_{k_i,j_i}} \in B(\mathscr{X}_\tau)$ be the multiplication operators with their symbols $u_i = u_{k_i,j_i} \in \mathcal{X}$, for $i = 1, \ldots, N$, for $N \in \mathbb{N}$. Then, by (95),

$$\prod_{i=1}^{N} M_{u_i} = M_{\prod_{i=1}^{N} u_i} = M_{\left(\lambda^{\sum_{i=1}^{N} k_i}\right) \otimes \left(\prod_{i=1}^{N} x_{j_i}\right)}. \tag{96}$$

If $u_{k,j} \in \mathcal{X}$ is a generating operator of $\mathscr{X}_\tau$, then one has

$$\left(\prod_{i=1}^{N} M_{u_i}\right)\left(u_{k,j}\right) = \left(M_{\left(\lambda^{\sum_{i=1}^{N} k_i}\right) \otimes \left(\prod_{i=1}^{N} x_{j_i}\right)}\right)\left(\lambda^k \otimes x_j\right)$$

by (96)

$$= \left(\lambda^{k+\sum_{i=1}^{N} k_i}\right) \otimes \left(\left(\prod_{i=1}^{N} x_{j_i}\right) x_j\right), \tag{97}$$

in $\mathscr{X}_\tau$.

Theorem 42 *Let $M_{u_i} = M_{u_{k_i,j_i}} \in B(\mathscr{X}_\tau)$ be the multiplication operators with their symbols $u_{k_i,j_i} \in \mathcal{X}$, for $i = 1, \ldots, N$, for $N \in \mathbb{N}$, and let $u_{k,j} \in \mathcal{X}$ be an arbitrary generating free random variable of $\mathscr{X}_\tau$. If $\boldsymbol{M} = \prod_{i=1}^{N} M_{u_i} \in B(\mathscr{X}_\tau)$, and if $\boldsymbol{u} = \boldsymbol{M}\left(u_{k,j}\right) \in \mathscr{X}_\tau$, then*

$$\tau(\boldsymbol{u}) = \varphi\left(\left(\prod_{i=1}^{N} x_{j_1}\right) x_j\right) = \varphi\left(x_{j_1} x_{j_2} \ldots x_{j_N} x_j\right), \tag{98}$$

where the far-right-hand side of (98) is characterized by Theorem 8 (or (17)).

Proof Under hypothesis, one has

$$\mathbf{u} = \mathbf{M}\left(u_{k,j}\right) = \lambda^{k_o} \otimes w_o \in \mathscr{X}_\tau,$$

where

$$k_o = k + \sum_{i=1}^{N} k_i \in \mathbb{Z},$$

and

$$w_o = x_{j_1} x_{j_2} ... x_{j_N} x_j \in \mathfrak{X}_{\varphi},$$

by (97). Therefore, the free-distributional data (98) holds by (93). □

As a special case of (98), we obtain the following corollary.

Corollary 43 *Let $M_i = M_{u_{k_i,j}} \in B(\mathscr{X}_\tau)$ be the multiplication operators with their symbols $u_{k_i,j} \in \mathcal{X}$ in $\mathscr{X}_\tau$, for a fixed $j \in \mathbb{Z}$, and $k_i \in \mathbb{Z}$, for $i = 1, \ldots, N$, for $N \in \mathbb{N}$. For a generating free random variable $u_{k,j} \in \mathcal{X}$ of $\mathscr{X}_\tau$, for $k \in \mathbb{Z}$, and a fixed $j \in \mathbb{Z}$. If $\boldsymbol{u} = \left(\prod_{i=1}^{N} M_i\right)\left(u_{k,j}\right) \in \mathscr{X}_\tau$, then*

$$\tau\left(\prod_{i=1}^{n} \mathbf{u}^{r_i}\right) = \omega_{n(N+1)} c_{\frac{n(N+1)}{2}} = \varphi\left(x_j^{n(N+1)}\right), \tag{99}$$

for all $(r_1, \ldots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$.

Proof Under hypothesis, let $\mathbf{M} = \prod_{i=1}^{N} M_i \in B(\mathscr{X}_\tau)$. Then, by (96),

$$\mathbf{M} = M_{\left(\lambda^{\sum_{i=1}^{N} k_i}\right) \otimes x_j^N} \overset{\text{denote}}{=} M_{\lambda^{k_o} \otimes x_j^N}, \text{ on } \mathscr{X}_\tau,$$

with

$$\lambda^{k_o} \in \Lambda, \text{ with } k_o = \sum_{i=1}^{N} k_i \in \mathbb{Z}, \tag{100}$$

and $x_j^N \in \mathfrak{X}_\varphi$ is the free reduced word with its length-1 for a fixed generating semicircular element x_j of $\mathfrak{X}_\varphi$.

So, one has

$$\mathbf{u} = \mathbf{M}\left(u_{k,j}\right) = \left(\lambda^{k_o} \lambda^k\right) \otimes \left(x_j^N x_j\right) = \lambda^{k_o+k} \otimes x_j^{N+1},$$

in $\mathscr{X}_\tau$, by (100), for $u_{k,j} \in \mathcal{X} \subset \mathscr{X}_\tau$, for $k \in \mathbb{Z}$. Thus,

$$\tau\left(\prod_{i=1}^{n} \mathbf{u}^{r_i}\right) = \varphi\left(\lambda^K\left(x_j^{n(N+1)}\right)\right),$$

for $K = \sum_{i=1}^{n} \varepsilon_i (k_o + k) \in \mathbb{Z}$, with

$$\varepsilon_i = \begin{cases} 1 & \text{if } r_i = 1 \\ -1 & \text{if } r_i = *, \end{cases}$$

for all $i = 1, \ldots, n$, implying that

$$\tau \left(\prod_{i=1}^{n} \mathbf{u}^{r_i} \right) = \varphi \left(x_j^{n(N+1)} \right) = \omega_{n(N+1)} c_{\frac{n(N+1)}{2}},$$

for all $(r_1, \ldots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$.

Therefore, the free-distributional data (100) holds. □

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