Chapter 8
Emphases in Algebraic Analysis After Euler

d'Alembert: Philosophical Legitimation of Algebraic Analysis as Well as His Critique of Euler's Concept of Function

Jean-Baptiste le Rond d'Alembert (1717–83), another first-rate mathematician, was a slightly younger contemporary of Euler. He was also active as a philosopher and involved in politics. His name, together with that of Denis Diderot (1713–84) personifies the French *Encyclopédie*, published in the years 1751–80, a work which greatly influenced the French Enlightenment. (However, d'Alembert left the scientific editorial staff of the *Encyclopédie* as early as 1758.)

d'Alembert's Reflections on the Notion of Quantity

In volume 7 of the *Encyclopédie*, published in 1757, d'Alembert starts his entry about *Quantity* with the following sentence: "It is one of those words, the entire world believes to have a clear idea of, but which is nevertheless very difficult to define precisely."

d'Alembert's Critique

D'Alembert starts with the very same notion of quantity which we have already found in Euler: quantity is, what "can be increased or decreased without end".

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At first, d'Alembert explains the importance of the "or" in this definition. Would it instead read "and", both zero and infinity would not meet the requirement of the definition: for zero cannot not be decreased and infinity cannot be increased. However, d'Alembert thinks it cannot be denied that both are "quantities".

We realize: d'Alembert's idea of the infinite is not like Euler's or Johann Bernoulli's, but more like Leibniz'—because Euler as well as Johann Bernoulli allows himself to increase the infinite (i): $i + 1$, $i + 2$, ...; in opposition to this, Leibniz and d'Alembert do not.

d'Alembert's Notion of Quantity

D'Alembert switches abruptly to his own understanding of "quantity" and declares: "It appears to me that *quantity* can be defined well as being something which is composed of parts."

This brings d'Alembert back to Euclid, who had declared two thousand years earlier: "A *number* is a multitude composed of units." And: "A number is *a part* of a number, the less of the greater, when it measures the greater."

And *by the way,* the philosopher d'Alembert now explains the foundational legitimacy of this Algebraic Analysis:

> The quantity[1] exists in each finite being and it expresses itself in an indefinite number which can only be known and understood with the help of a comparison and in relation to another homogene *magnitude.*

Descartes' universal unity is dismissed and the Law of Homogeneity restored, albeit with a new meaning.

Assessment: d'Alembert's Philosophical Legitimation of Algebraic Analysis

Unfortunately we cannot know d'Alembert's *intention* behind this formulation. Maybe he did not want to say something different from what umpteen generations of mathematicians have said already before him: only the fixing of a ruler (the "unit") allows the measuring of a quantity. However, closer inspection of the matter (which includes the historical moment of d'Alembert's writing) gives rise to read this sentence as the *philosophical legitimation of Algebraic Analysis:*

> It is not about *definite* values, but about *all possible* values, without any exception; only that counts.

This was Euler's demand, at the beginning of his textbook (p. 79).

[1] d'Alembert first uses "quantité" for "quantity" but in the end he uses "grandeur".

d'Alembert's Critique of Euler's Notion of Function

For many years Euler and d'Alembert competed for the best mathematical description of the vibrating string. Unable to agree—each of them thought his own answer to be superior. As so often in everyday life: they talked past each other. Neither of them understood the other. They had different ideas of the notion of a function.

What Euler understood by "function" *in this case,* was shown on p. 83, but what was d'Alembert's idea?

D'Alembert did not have Johann Bernoulli as his teacher (as Euler did) and was therefore further removed from the geometrical foundations of analysis. D'Alembert stuck to the algebraic notion of function, given by Euler: a function is described by a "calculatory expression". Nonetheless, it was clear to d'Alembert that this was not enough to deal with all the "curved lines" encountered in practical problems—but only with the nice ones: those which Euler had called continuous (p. 81).

D'Alembert tried to liberalize this algebraic notion of function in two steps. His principal idea was to replace the *one* calculatory expression by *two* of them.

1. At first d'Alembert allowed a "function" to be determined by an *equation,* that is to say by two calculatory expressions, related by an equal sign.
   Unfortunately this idea is of no great help. If we have an unknown on *both* sides of the equation it is very doubtful whether we will be able to make y the subject of the equation. (Since the nineteenth century it has been known that this can already fail with equations of the fifth degree. During the eighteenth century one was far more optimistic.) If one does not succeed in isolating the unknown? How should the "function" be further inspected, if it is not given explicitly?

2. D'Alembert tried to modify the notion of function after 1761 and from 1780 he became more specific.
   His writing is not terribly clear, but two points can be discerned:

   a. It is clear that he connects two functions and their "equations" with adjoining domains to make up one "continuous" (because of the *two* expressions!) function. That is to say, he changes his original notion of function decisively by *limiting* its *scope.* For Euler this was *unthinkable!*
   b. He carries over the notion of "continuous" from "curved line" to "function". Contrary to modern concepts, d'Alembert calls a *compound* "function" "continuous", if it describes a *single* "curve" and if these two *different* "functions" at their meeting point (i) take the same "value" and (ii) have the same (one-sided) derived function (increase or decrease of the tangent).

   Thus, d'Alembert aims to give the notion of function more flexibility *within the conceptual realm of Algebraic Analysis,* because the algebraic notion of function as demanded by Euler is not very useful in practical circumstances.

d'Alembert's Impulse: Condorcet

If one digs even further into the mathematical sources of the late eighteenth century, more precisely into the treatise *On the continuity of arbitrary functions* of Marie Jean Antoine Nicolas de Caritat, Marquis de Condorcet (1743–94) from 1774, one gets some evidence that d'Alembert's last step was inspired by a much younger contemporary with that illustrious name. However, such subtleties exceed the scope of this book.

Lagrange: Making Algebra the Sole Foundation of Analysis

The last first-rate mathematician who completely subscribed to the spirit of Euler's analysis was Joseph Louis Lagrange (1736–1813). His textbook was published in 1797, the "year V" following the calendar of the revolution, and in 1813 in a second edition. A German translation by Johann Philipp Grüson (1768–1857) appeared as soon as 1798 under the title (in German) *Theory of analytic functions, wherein the principles of differential calculus are given, independently from considerations of the infinitely small or vanishing quantities, the limits or fluxions, and grounded in Algebraic Analysis.* Eventually this text led to the establishment of "Algebraic Analysis" as a title for the new theory. It is an apt name.

Lagrange's New Foundation of Analysis: The Base

The lengthy title of the work encapsulates its content and plan. Lagrange aims to present the theory of functions and the differential calculus as simply as possible—and, consequently, *independently of* such complicated notions as "infinitely small" quantities, "limit" or (as with Newton) "fluxion". Analysis as easy as possible, a commendable plan.

Lagrange's idea is very consequential and can be understood in three steps.

First Step. Lagrange starts at the very beginning, with Euler's notion of function. Accordingly to Euler, a "function" is a "variable" quantity, which is described by a calculatory expression. Lagrange radicalizes this and says:

A function is a calculatory expression.

Following this, *any* calculatory expression is a "function" for Lagrange—be it the description of a variable quantity or not.

Here, he differs from Euler. Euler only considers a calculatory expression as a "function" which describe a *changing* quantity. Lagrange no longer cares whether the calculatory expression can take on only one or more values. For him *every* calculatory expression is a "function".

Let's recapitulate Euler's notion of "calculatory expression".

1. The simplest calculatory expressions are the *sums*, finite ones like $2 + 3x - 5x^2$ or infinite ones like $1 - x + x^2 - x^3 + x^4 - + \dots$
2. However, a simple fraction can also be written as an (infinite) sum (see formula $*$ on p. 73):

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

or (see formula † on p. 87)

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - + \dots$$

3. Therefore Euler is convinced that *every* function can be described as such an (infinite) *sum.* This he noted in § 59 of his textbook on differential calculus:

> Thereupon it should be beyond doubt that every function can be transformed into such an expression that runs towards the infinite
>
> $$Ax^\alpha + Bx^\beta + Cx^\gamma + Dx^\delta + \cdots, \qquad (\ddagger)$$
>
> wherein the exponents α, β, γ, $\delta \dots$ stand for any numbers whatsoever.

Euler chooses Greek letters (alpha, beta, gamma, delta) as exponents. They can also be *arbitrary* numbers.

The Idea of Lagrange

To implement his plan, Lagrange—in his *second step*—seizes on a fact which was originally advanced by Johann Bernoulli (in the year 1694) and again, very clearly and repeatedly, by Euler. In his *Integral Calculus*, Euler restated a theorem, which he had already proven in his *Differential Calculus* (and earlier, see p. 75):

> *Theorem.* If y denotes a function of x, which changes to b, if $z = a$, and if we put $\frac{dy}{dz} = P$, $\frac{dP}{dz} = Q$, $\frac{dQ}{dz} = R$, $\frac{dR}{dz} = S$ etc., then we obtain the general expression:
>
> $$y = b + P(z-a) - \tfrac{1}{2}Q(z-a)^2 + \tfrac{1}{6}R(z-a)^3 - \tfrac{1}{24}S(z-a)^4 + \tfrac{1}{120}T(z-a)^5 - \text{etc.}$$

For a better understanding of the main idea we simplify this calculatory expression. If $(z - a)$ is replaced by x and the differential quotients P, Q, R, etc. are shortened to $1P = p$, $-\frac{1}{2}Q = q$, $\frac{1}{6}R = r$, $-\frac{1}{24}S = s, \dots$ we get the following condensed expression:

$$f(a+x) = f(a) + p \cdot x + q \cdot x^2 + r \cdot x^3 + s \cdot x^4 + t \cdot x^5 + \text{etc.} \qquad (\S)$$

Euler had shown that *each* function, each calculatory expression can be represented in this way. The left side of the equation is new, instead of "$f(x)$" we now have "$f(a+x)$".

It is now Lagrange's idea—his *third step*—to prove that the scheme ‡ is nothing other than Eq. §. In other words, Lagrange proves the following theorem:

Theorem. EACH *function* $f(x)$ *can be written as a calculatory expression of the form* §.

The proof of this theorem is Lagrange's opening to his textbook on analysis.

If Lagrange were really able to prove this theorem, he would stage a coup d'état. Because of Euler's theorem which we have cited above, one *then* was able to deduce that the coefficient p of the second summand in § is the first differential quotient of the function (*viz* $\frac{dy}{dz}$); the coefficient q of the third summand on the right in § is the second differential quotient (in Euler: $\frac{dP}{dz}$) inclusive of the factor $\frac{1}{1\cdot 2}$ and the sign; the coefficient r of the forth summand on the right in § is the third differential quotient of the function ($\frac{dQ}{dz}$) inclusive of the factor $\frac{1}{1\cdot 2\cdot 3}$; etc. In other words: if Lagrange can prove this theorem he would be able to obtain for every function all differential quotients. *In this case, the differential calculus is founded all at once and entirely without the use of the commonly applied notions like "infinitely small" quantities* etc.

Lagrange introduces also a new notation which is still used today. Instead of the "first differential quotient" $\frac{dy}{dz}$ of the function f Lagrange simply writes "f'"; instead of the second differential quotient $\frac{dP}{dz}$ he writes "f''" etc. Until today, f', f'' etc. are called the "first", "second", ... "derivatives" of the function f.

A Contemporary Criticism on Lagrange's Plan

August Leopold Crelle (1780–1855), who does not count as a first-rate mathematician, edited Lagrange's textbook in a new German translation in 1823. Crelle saw fit to launch a fundamental criticism of Lagrange. He wrote:

> In my opinion, the proof that the series expansion of an arbitrary function $f(a+x)$ only contains positive integer powers of the quantity x, is firstly defective or at least very weak and much too complicated to found the principles of a whole science; and, secondly, I think such a proof is completely *superfluous.*

That is strong stuff, and the latter judgement is very noteworthy: why is the proof of this theorem in Crelle's eyes "superfluous"?

Crelle makes it easy for himself and says: later in the book it is shown, how it works—but if you show *how* it works, you need not prove *that it works.*

This is obviously self-deception. Of course, Lagrange does not show in his book how to expand *all* functions in a series: the book is finite (the original has 296 pages), but there are infinitely many functions! And of course, Lagrange (without any doubt being a first-rate mathematician) did not resort to such kind of argument. He was sure to have a watertight, general proof for his theorem.

How Does Lagrange Proceed?

Lagrange's idea that his theorem is correct is simple. It goes as follows: the essential argument is to show that alpha, beta, gamma, delta, ... in Euler's calculatory expression ‡ (p. 95) *can only* be taken by positive integers 1, 2, 3, ... However, this is self-evident, for (and that's the point!) in case of an exponent different from a positive integer the corresponding expression is *one-to-many*, whereas the given function, clearly, is one-to-one.—And this is all there is to it!

That is to say, Lagrange's argument is this: $x^{\frac{1}{2}} = \sqrt{x}$ has two value s ($\sqrt{4} = \pm 2$); $x^{\frac{1}{3}} = \sqrt[3]{x}$ has three values ($\sqrt[3]{1}$ has the value 1 and the two values $\frac{1}{2} \pm \frac{\sqrt{-3}}{2}$) etc. This is a real fact—if one is to accept the "complex" numbers.

The Fundamental Gap in Lagrange's Proof

Nevertheless, Lagrange's proof has a gap. This gap is of a fundamental nature and can be described as follows: Lagrange resorts to the argument that the "function" $x^{\frac{1}{2}}$ has *two* "values". However, only the following is true: there exist two "values" X to be discovered from the instruction $x^{\frac{1}{2}}$ ($= \sqrt{x}$), i.e. $X^2 = x$. (Example: we have $2^2 = 4$ as well as $(-2)^2 = 4$—and *consequently* two values for X.) But calculating is one thing—and mathematics another!

Lagrange did not set out to present an elementary *calculation,* but to *prove* a theorem! And within his proof he clearly *uses* the notion "value of a function"—e.g. by arguing that $x^{\frac{1}{3}}$ has *three* values. This argument is *wrong,* if one only considers "real" functions—as we do today in our first course at university. Three *real* numbers (i.e. those without the component $\sqrt{-1}$), which can be calculated following the instruction $x^{\frac{1}{3}} = \sqrt[3]{x}$ do not exist.

Well, you might object and say: Lagrange does not restrict himself to *real* functions—and, consequently, this objection is irrelevant.

The objection is only true in so far as Lagrange, indeed, did not write a theory of *real* functions, but (as we say today:) a theory of *complex* functions. Nevertheless, Crelle's argumentation is too weak, as it only deals with the given example (restriction to a *real* analysis) but not the essence. The essence remains:

Nowhere does Lagrange specify the concept "value of a function".

However, he implicitly *uses* the concept. This is definitely a deficiency in mathematical rigour.

You may say: what a tiresome thing, "value of a function"—isn't it obvious, what this means?

Counter-question: is it really so obvious? What is the "value of the function" for $\frac{1}{x}$ in case of $x = 0$? Or in case of $x = \infty$? That is to say: what is $\frac{1}{0}$, what is $\frac{1}{\infty}$? And moreover: what is the value of the function $\log\ 0$, or $\tan\frac{\pi}{2}$? Or of $\sin\ \infty$? *This we do not know beforehand or through calculating,* these answers need a theory—and that means: we are to do some mathematics!

That is why we have to launch some criticism at Lagrange: nowhere do you declare what you take the notion "value of a function" to be! And it is unacceptable in mathematics to base a proof on an *undefined* notion!

Usually, Lagrange is criticized for something quite different. He is accused of making a technical mathematical error: that he has not taken into consideration that the representation § does not work *in particular cases.*

However, this criticism is completely unfounded. A first-rate mathematician such as Lagrange does not make such a technical mistake. If one reads his book *carefully,* one can see that he treated the said technical problem extensively and explained what he thought about it. Therefore, he knew about this problem—but this aspect does not touch his theorem and, in particular, it does not refute it. Because, just like with Euler, Lagrange's theorem is about "quantities", but not about particular "values". Unfortunately, today's mathematicians (as well as historians of mathematics) often do not follow this argumentation.

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