

Chapter 3

Lines and Variables



From Two to Infinity: Leibniz' Conception of the World

In Descartes we came to know a radical critic of contemporary rational thinking. With Gottfried Wilhelm Leibniz (1646–1716) we now turn to an equally radical diplomat who endeavoured to unite the world's antagonisms.

According to Leibniz, God created the “best possible world”. His world consists of prime elements or atoms which Leibniz called “Monads”. He says:

The Monad, of which we shall speak here, is nothing but a *simple* substance, which enters into compounds. By ‘simple’ is meant ‘without parts’.

All Monads differ from each other:

Indeed, each Monad must be different from every other. For in nature there are never two beings which are perfectly alike and in which it is not possible to find an internal difference, or at least a difference founded upon an intrinsic quality.

Monads are beings, simple beings. Fundamentally,

[...] nothing but this (namely, perceptions and their changes) can be found in a simple substance. It is also in this alone that all the *internal activities* of simple substances can consist.

Leibniz regarded it as a “metaphysical necessity”

that every created being, and consequently the created Monad, is subject to change, and further that this change is continuous in each.

These central statements are to be found in the small booklet written in 1714, which was published four years after Leibniz' death as *Monadology*.

This way of thinking is radically antagonistic to Descartes. Whereas Descartes presumed two *substances* (extension and thought), Leibniz presumed (actual) infinitely many (the Monads). In other words, whereas Descartes had *principles*, Leibniz had *individuals*.

Leibniz' wider conceptions of the world are not connected to his mathematical thinking. The principal idea is: Monads are nothing else than *perception* as well as their *change*, an important assumption which underlies his entire philosophy. Even if he does not mention this explicitly, he intends it to *be* a fundamental fact. We will soon recognize this.

Leibniz' Mathematical Writings

Leibniz never worked as a regular mathematician. Nonetheless he is, together with Isaac Newton, one of the two creators of that mathematical theory which developed into the most powerful of all, till today: calculus.

Leibniz documented his invention in a lengthy essay; the longest mathematical text he ever compiled. It was supposed to be printed in Paris after Leibniz' departure from the town. But this did not happen and eventually the manuscript was lost. Leibniz, much later, refused to rewrite the paper: in his eyes, too much time had passed already.

However, among the vast amount of papers left by Leibniz, some drafts of this essay were found. An extract from such a draft was first published by Lucie Scholz in her dissertation in 1934, a complete version in 1993 by Eberhard Knobloch and meanwhile (2012) also in Leibniz' *Schriften*; a French translation by Marc Parmentier appeared in 2004 and a German translation by Otto Hamborg has been available since 2007 via the internet and since 2016 as a book.

Reading Leibniz' papers (which is not very easy as he wrote mostly in Latin) repudiates all the prejudices of present-day scientists against their predecessors: they were less intelligent than we are, their ideas were more vague, their reasonings inconclusive—only today are we qualified to be exact and precise.

The point is this: many earlier scientists did not use “vague” notions and “diffuse” reasonings—but *completely different* ones. If one engages with those *different* concepts and follows these *other ways* of reasoning, one realizes that Leibniz' proofs are not incorrect at all. Quite the opposite, Leibniz' ideas are marked by great ingenuity. His proofs, judged by today's standards, are as precise as his concepts allowed for.

Especially the following three concepts: convergence, integral and differential stand out as great scientific achievements.

Leibniz' Theorem: Fresh from the Creator!

The Convergence of Infinite Series

Up until today the basic curriculum of higher mathematics contains “Leibniz’ Theorem”. It is about infinite series, such as

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots$$

or

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

As series are *infinite* sums, it is not possible to *calculate* them directly. Nevertheless, they often have a *value*, a *sum*. The two above happen to have a sum, but others do not. The series

$$1 - 2 + 3 - 4 + \dots$$

or

$$1 + 4 + 8 + 16 + \dots$$

do *not* have a sum, that is to say: no *finite* value.

If one ponders for a while perhaps the following idea may arise: promising candidates for series with a sum may be those which fulfil two conditions: (i) the summands, their terms *are* ALWAYS *decreasing* and can become *as small as one likes them to be*; (ii) their *signs* alternate.

Consequently, the first term is greater than the sum, as the second term is *taken away* from it; the sum of the first two terms is smaller than the sum, as the third term is *added to it*: condition ii., etc. Finally: the *difference* (a *change!*) between the successively calculated sums decreases and becomes arbitrarily small: condition i.

This phenomenon that “the sum becomes increasingly *more accurate* if the differences between sums, which can truly be calculated, decrease below each given quantity” is what we call “convergence” today. Yet, during Leibniz’ time this name was not established.

Leibniz was the first ever to describe these novel objects of mathematics (“convergent series”) systematically.

Leibniz’ Formulation of His Theorem

In his manuscript Leibniz wrote:

If a quantity A is equal to a series $b - c + d - e + f - g$, etc.,

$$A = b - c + d - e + - \dots ,$$

which decreases infinitely in such a way that the terms eventually become smaller than an arbitrarily given quantity, it will be

$+b$	greater than A , so that the difference is smaller than	c
$+b - c$	smaller is smaller than d
$+b - c + d$	greater smaller than e
$+b - c + d - e$	smaller smaller than f .

And in general, the part of the decreasing series with alternating additions and subtractions, which ends with an addition, will be greater than the sum of the series, the part which ends with a subtraction will be smaller; but the error or the difference will always be smaller than the term of the series, which follows the part at once.

With the exception of the line “ $A = b - c + d - e + - \dots$ ” and the last three “is smaller than”, this is *exactly* what Leibniz wrote in his manuscript. It is as precise as possible. Leibniz describes *in all detail* what an “infinite series” with “alternating signs” and “steadily decreasing” terms which “decrease below each assumed quantity” is.

It is permissible *to read* this statement of Leibniz as follows:

Theorem. *If the terms of a series*

$$b - c + d - e + - \dots$$

decrease indefinitely (i.e. they eventually get less than any assumed quantity) than this series has a finite sum.

Leibniz did not leave it at this description but added a fairly detailed proof.

Leibniz’ Proof of His Theorem

Leibniz’ proof consists of a preliminary consideration and four steps.

Preliminary: as the amount of the terms steadily decreases, there is *altogether* more added than subtracted. Added were $b + d + f + \dots$, subtracted were $c + e + g + \dots$ and we have $b > c, d > e, f > g$, etc. In this manuscript Leibniz does not use a “greater” sign, but in others he does.

Especially we have for the sum A :

$$A < b .$$

1. The first step of the proof: using the last inequality we may consider the number

$$b - A .$$

(We see: Leibniz tries to deal with positive, *true* numbers!) Therefore,

$$b - A = b - (b - c + d - e + f - g + - \dots) = c - d + e - f + g - + \dots < c ,$$

where the last inequality $<$ holds for the same reason as was given for $A < b$ in the preliminary!

2. The second step of the proof: we found that $A < b$. *In the same way* we derive from (1) that

$$A > b - c .$$

Is it?—Really!—Therefore, it is allowed to take $A - (b - c)$ (which is also a true number) and we get

$$A - (b - c) = d - e + f - g + - \dots < d ,$$

by the same argument.

3. Leibniz' third step of the proof: as before, through rearranging the result above, we get as the next starting point:

$$A < b - c + d .$$

So we can build $(b - c + d) - A$. And by the same procedure we arrive at

$$b - c + d - A = e - f + g - + \dots < e .$$

Leibniz adds a fourth step in the same manner, but we need not repeat it here, as the pattern is clear by now.

As a result Leibniz gets a sequence of inequalities:

$$\begin{aligned} A &< b \\ b - A &< c \\ A - (b - c) &< d \\ (b - c + d) - A &< e \\ A - (b - c + d - e) &< f \\ &\dots \end{aligned}$$

From the second line onwards we have on the left side the *error* that arises, if instead of the whole series only its beginning till the n^{th} term is taken and the right hand side shows that this error is *always* less than the $(n + 1)^{\text{st}}$ term. But it was presupposed that the terms' magnitude "eventually decrease below any assumed magnitude", and, therefore, this assumption—as can be seen in the above inequalities—leads to the conclusion that the *the error caused by breaking off* the series also becomes less than any assumed magnitude.

What more do we want?

It is impossible to give a more precise, *stronger* argument.

And that even today. Leibniz did so in his manuscript of late summer 1676.

Reflection on Leibniz' Achievement

Through these notes Leibniz became the first person in the history of mathematics to describe, *as precisely as possible*, what is nowadays called the "convergence" of an "(infinite) series". (The attribute "infinite" is plainly superfluous but it sounds so breathtakingly bombastic.)

The expression "convergence" was not used by Leibniz. However, the importance is the content of the finding, the name is quite irrelevant. What counts is that Leibniz stated his result in absolute clarity.

The philosophical question: what did Leibniz do? What was the object of his inquiry?

My answer: obviously, Leibniz explored a *variable quantity*, i.e. the *successively calculated sum* of a series. Today this is called the "partial sum" and we have a standard symbol for it: s_n . We write:

$$\begin{aligned} s_1 &= a \\ s_2 &= a - b \\ s_3 &= a - b + c \\ s_4 &= a - b + c - d \\ s_5 &= a - b + c - d + e \\ &\dots \\ s_n &= a - b + c - + \dots \pm n \end{aligned}$$

It is evident that this object, the "partial sum" or s_n , is a *variable* (i.e. changing) quantity. Leibniz only had the right hand sides of the above equations and, therefore,

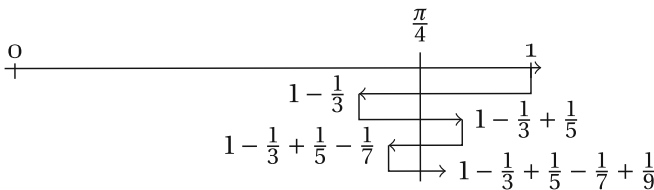
they can be named without any reservations (“ s_n ” will do as well as any other name).

Without explicitly saying so, Leibniz studied *variable* quantities.

Leibniz' way of arguing is so fascinating, so deeply mathematical that nobody dared to characterize it as “unmathematical” and to criticize it: According to the standards of classical mathematics this is not mathematics!—At least to our knowledge nobody dared to make such an accusation, written or otherwise.

An Idea Which Leibniz Could not Grasp and the Reason for His Inability

Today we illustrate the above facts by the following picture:



At a single glance we *grasp* the situation. However,

Leibniz was unable to draw this picture.

For this picture demands two things: (a) lengths have a *direction*, and (b) “negative” numbers *do exist*.

Actually we have seen earlier that a length has no direction! Moreover, if negative numbers were true “numbers”, some of the more than two thousand years old laws (known at least since Euclid!) would be invalidated.

One of these laws reads as follows:

$$\text{If } \frac{a}{b} = \frac{c}{d}, \quad \text{and if } a > c, \quad \text{then } b > d.$$

But if -1 is a “number”, this law requires:

$$\text{As } \frac{1}{-1} = \frac{-1}{1}, \quad \text{and if } 1 > -1 \quad \text{it follows } -1 > 1.$$

and thus a contradiction. Contradictions are *absolutely* forbidden in mathematics for otherwise *everything* whether false or correct can be proved.

The mathematicians of the late seventeenth century and the beginning of the eighteenth century had to make a *decision*: should those time-honoured laws be preserved, or should -1 become a true *number*?

Leibniz was an astoundingly creative thinker but not a revolutionary; he was a diplomat. He shunned revolutions (the dismissal of the validity of these classical laws) but praised the news by couching it in unctuous words:

Nevertheless, I do not want to deny [...] that -1 is a quantity smaller than nothing; this only has to be understood right-minded. Such statements are what I call *passable true* (following the renowned Joachim Jungius); [...] However, they would not bear a severe verification, but yet they are of great help for the calculation and of immense value for the inventive genius as well as for universal concepts.

The classical phraseology of diplomats: "... / as well as ..."; " -1 is smaller than 1 / but this has to be correctly understood"; "to be on the safe side, I cite an authority, however, unknown or vague, or hint at my obedience: strictly speaking this is forbidden / but it is of huge utility".

Thus, the paper published by Leibniz in a famous scientific journal in April 1712 can be understood as an act of great political diplomacy.

The quintessence is:

Accepting "negative" numbers as *true* ones goes hand in hand with the invalidity of some time-honoured laws.

If he or she may say something like: "this law is valid *only* for positive numbers, and all is fine", the known contradictions are outlawed. (Hopefully, no others will appear we have not yet thought of!)

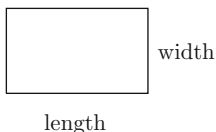
The Precise Calculation of Areas Bounded by Curves: The Integral

The Beginning Is Easy

Lasting for thousands of years, the Babylonians operated quite differently compared with classical Greek scholars, but in Greek culture the following art of planimetry was taught:

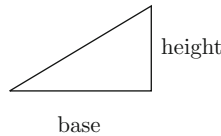
It only allowed for *rectangles*. The area is

length times width.



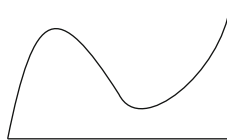
From this all further calculations had to be deduced, e.g.: the area of a *rectangular* triangle is

$$\frac{1}{2} \text{ times base times height.}$$



The Problem

What happens, if the boundary of one side is curved?



The Solution of Leibniz—The Original Way

For such an instance Leibniz has the following astounding idea:

I present Leibniz’ original figure, omitting what is not essential to the principal idea. Even so, the figure is still sufficiently complicate (Fig. 3.1).

The object of the construction is the area between the curve D_1, D_2, D_3, D_4 and the three line segments $\overline{D_1 B_1}, \overline{B_1 B_4}$, and $\overline{B_4 D_4}$. The points D_n are *any* points on the curve.

1. Leibniz takes the stairway $B_1, N_1, P_1, N_2, P_2, N_3, P_3, B_4, B_1$ as a *first approximation* of the area. In the following and for the sake of simplicity we will indicate the single steps of the stairway by their dotted “upper” lines, the “step zones”, i.e. $N_1 P_1$, etc.
2. Now let us assess the *error* of this approximation! It consists of the sum of *partial errors*. Compared with the actual area we get:
 - (a) The first step $N_1 P_1$ is *too big* by the (curved) triangle $D_1 N_1 F_1$ —as well as *too small* by the triangle $F_1 P_1 D_2$. Leibniz is saying that at any rate this *first partial error* is *less than the whole rectangle* in between D_1 and D_2 . This is *really* generous, isn’t it!

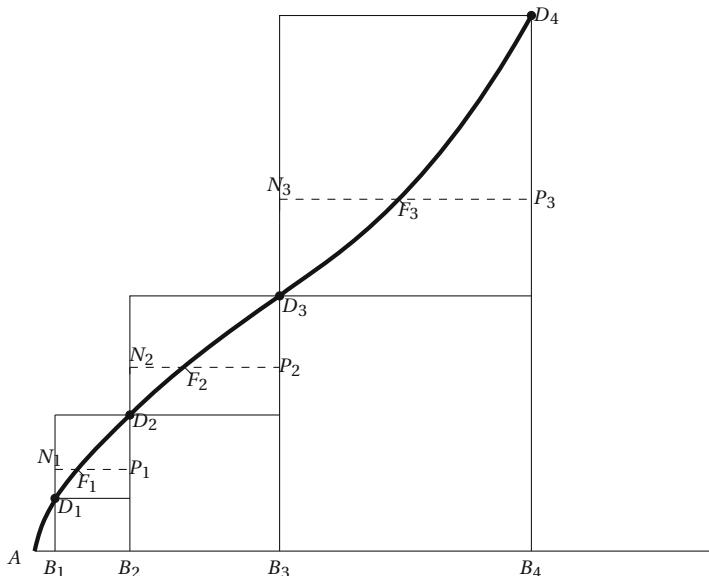


Fig. 3.1 Leibniz' figure to calculate the area (purified), 1676

- (b) The same with the following step $N_2 P_2$, again the *second* partial error caused by the approximation is clearly *less than the rectangle* in between D_2 and D_3 .
 - (c) And so forth.
3. So what is the *total error* at most? Most likely it is less than the *sum* of the partial errors; undoubtedly it is less than the sum of the rectangles $D_1, D_2; D_2, D_3$ and D_3, D_4 .
 4. Therefore, what is our estimate of the *total error*? The total *height* is obviously (if the curve is always ascending or always descending; otherwise it must be divided in such way) the height from D_1 to D_4 . And, to be on the safe side, Leibniz chooses as *width the maximal* width of the steps $\overline{B_1 B_2}, \overline{B_2 B_3}$ and $\overline{B_3 B_4}$. In the example it is $\overline{B_3 B_4}$.
Outcome: the *total error* of the first approximation is clearly *less than the product of the height from D_1 to D_4 and the maximal width of the $\overline{B_n B_{n+1}}$* .
 5. All boils down to the following conclusion:
The points D on the curve are completely arbitrary. We may choose as many of them as we please. Let's say, we choose k equidistant points such that the steps have width $\overline{B_1 B_k}/(k-1)$.
Consequently, the *total error* of this k^{th} approximation will be certainly *less than the rectangle with width $\overline{B_1 B_k}/(k-1)$ and height $\overline{D_1 D_k}$ (more precisely, $\overline{B_1 D_k}$)*.

Whereas the height of the rectangle remains the same, its width *continuously decreases* if further points D are chosen. And the *total error* of the k^{th} estimate is clearly less than the area of the corresponding rectangle.

Subsequently, the product of these two values will also drop: if in the product $b_k \cdot h$ the factor b_k is constantly decreasing while h (the height) remains the same, the product decreases, too.

Thus the total error of the estimate reduces further.

6. How small does it become?

Obviously, there is no lower limit to the area's magnitude (actually, its "smallness"): with the exception of the limit zero, of course. According to Leibniz' own words:

The points D may be thought of as near and in such a great number that the straight-lined step-shaped area differs from the four-lined area $D_1 B_1 B_4 D_4 D_3$ etc. D_1 itself by a quantity which is less than an arbitrarily given.

This means: the *total error* which emerges from the calculation of the area below the steps instead of the area limited by the curve, can be *decreased* to any desired degree of exactitude.

Did we hit the jackpot? Do we have the area?—Yes and no. On the one hand, we have a *method of calculation*: Leibniz is capable of calculating the area as precisely as he wants to.

On the other hand, being able to calculate something does not mean to have a concept for doing it. Engineers may be satisfied with a method of calculation, but mathematics needs concepts! In this case, an appropriate concept of numbers. However, Leibniz could not offer one. Understandably so, as it turned out mathematics needed two further centuries to coin such concepts (Chaps. 13 and 12) and in a certain sense, these solutions contradict Leibniz' thinking, for they demand the acceptance of the "actual infinite" in mathematics (*see* Chap. 5).

Outlook

Leibniz' idea from 1676 provided the foundations for what was later called "integration", especially for the "integral" of a "function", the graph of which Leibniz still called "curved line".

However, the names "integral" as well as "function" were already used by Leibniz, but both with other meanings.

- (a) "Function" was Leibniz' title for a multitude of line segments which can be constructed to make up a curve when set in relation to straight coordinate axes: abscissa, ordinate, tangent, normal, sub-tangent, sub-normal, resecta, . . . , a lot of special geometrical constructions, which were frequently studied in his times.

Leibniz used the name "function" without a definite meaning.

- (b) The name “integral” was invented by Johann Bernoulli and in 1690 his brother Jacob used the notation for the first time in a printed paper. The first printed document in which Leibniz used the sign “ \int ” dates back to the year 1686. He used it in connection with his sign for the “differential”: “ dx ”—a concept which will be treated in the next section. As “ d ” is an operator, “ dx ” as well as “ dy ” are to be read as a *single* quantity in the following text.

If one transforms the differential equation $p \, dy = x \, dx$ into the “summatorial” [by building the sums on both sides], one has $\int p \, dy = \int x \, dx$. From what I have shown in the method of tangents, we clearly have $d(\frac{1}{2}xx) = x \, dx$; therefore, the reverse is $\frac{1}{2}xx = \int x \, dx$ (because like powers and roots in usual calculations, we have sums and differences or \int and d as reciprocal).

The first evidence of the integral sign, as it is still used today, “ \int ”, goes back to 1691. Leibniz had encountered the name “integral” for the first time in 1690, in an article by Jacob Bernoulli

However, the *topic* that was presented above, i.e. the exact calculation of an area with a curved boundary, was made the subject of mathematics, *as precisely as 1676*, only 178 years later. According to its later inventor Bernhard Riemann, the mathematical object is nowadays known as the “Riemann integral”. Shortly before Riemann, Cauchy had come up with a similar idea (pp. 145f).

To sum up: Leibniz had already developed this notion *as precisely as possible*—but without today’s concepts of “function”, “infinite series”, and “convergence”. It worked without these notions!

If Leibniz’ plan had succeeded and his manuscript had been printed, mathematics would have developed differently.

Leibniz’ Neat Construction of the Concept of a Differential

The First Publication: A False Start

Leibniz published his idea of “differential” from the 1670s in October 1684. Albeit, his *explanation* remained very vague. Even worse, Leibniz made a mistake such that the intelligent reader had to decide whether the whole treatise was wrong or merely the definition of its fundamental notion. In case of the latter, the reader had to reconsider the *correct* concept of “differential” all by himself.

Clearly, this publication was almost a complete failure. Where to find a clever and astute reader? Jacob Bernoulli (1654–1705) had tried hard to understand this text since 1687. In the end he needed two or three years to understand the main idea in order to use the new method himself. What’s more, he was able to develop it further, in dialogue with his younger brother Johann—and with Leibniz.

Another False Start: The New Edition

The differential calculus consists of the *concept* of “differential” as well as *laws of calculation* for those differentials. Without these laws the concept is of no use.

Leibniz owed the explicit formulation and *the detailed foundation* of these laws to the public of his time. Just as before, he wrote a manuscript thereon but did not publish it. In 1846, when it was finally published, together with a lot of other manuscripts on this topic, nobody took notice. The same happened in 1920, when the English translation appeared. This fact has been documented only fairly recently: by Henk Bos in 1972 and by Richard T. W. Arthur in 2013, who are both historians of mathematics.

The Neat Construction, Part I

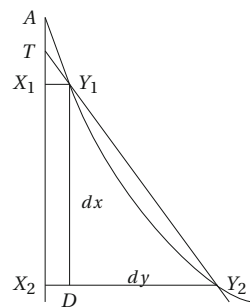
In regard to the above presentation it comes as no surprise that Leibniz took also the “differential” to be a *geometrical* notion. More precisely, with help of the “differential” it should become possible to draw a *tangent* to an arbitrary curved line (Fig. 3.2).

A “tangent” is a straight line that *touches* the curve, i.e. that snuggles up to it. *To touch* usually means there is *no cut*, the curve remains on the one side of the straight line. However, there are unusual curves and *sometimes* the “tangent” still cuts those curves.

Leibniz came up with the following geometrical construction of the differential. The abscissa x is drawn upwards, the ordinate y to the right (today we usually do it the other way round). Take a parabola

$$y = \frac{x^2}{a} ,$$

Fig. 3.2 Leibniz' calculation of the tangent



which is represented by the curved line in the figure. Leibniz chooses $AX_1 = x$ and $X_1Y_1 = y$. From the point Y_1 the perpendicular line Y_1D to the larger horizontal line X_2Y_2 (the ordinate) is drawn. The *difference* of AX_2 and AX_1 is called by Leibniz the “differential” dx , similarly, the *difference* of X_2Y_2 and X_1Y_1 is called the “differential” dy . These are the notations, now the calculations:

The equation of the curve reads

$$y = \frac{x^2}{a}.$$

Leibniz starts by *changing* x to $x + dx$ and subsequently y to $y + dy$. This modifies the equation to:

$$y + dy = \frac{(x + dx)^2}{a} = \frac{x^2 + 2x dx + dx^2}{a}.$$

We subtract the original equation and get:

$$dy = \frac{2x dx + dx^2}{a} = \frac{2x + dx}{a} \cdot dx \quad \text{or} \quad \frac{dy}{dx} = \frac{2x + dx}{a}.$$

Of course, dx and dy are *changing* quantities. And as we would expect they *decrease indefinitely: below any given quantity*.

Therefore, the numerator of the fraction will *approach* $2x$. But then, as zero is not allowed as denominator, we encounter a problem regarding the dx on the left side of the equation. Yet, if dx does not *truly* reach its limit zero, the numerator on the right will not become $= 2x$.

What are we to do?

Interlude: The General Rule: The Law of Continuity

Leibniz needs an argument that enables him to *extend* the validity for cases only holding for $dx \neq 0$ to the case $dx = 0$.

And indeed, Leibniz really had such an argument at his disposal: his Law of Continuity. According to the historian of mathematics Herbert Breger, this cognitive law has a similar significance and power for Leibniz as later the Method of Dialectics has for Hegel. It is a “universal scheme of thought and cognition”. And as we might expect, coming from Leibniz, its main characteristic is a diplomatic rather than a logical one. The Law of Continuity *unites opposites* instead of separating them as logic does.

Generally speaking, Leibniz’ Law of Continuity says: different things, even if they are contradictory, can emerge (or be understood) from a common principle.

This is the principal idea. Naturally, the law may be given some more precise formulation, in accordance with the concrete requirement.

Thus, in order to give his concept of “differential” a rigorous foundation, Leibniz formulated his Law of Continuity in his manuscript as follows:

If some continuous transition ends in some limit, it should be allowed to state a common law of thought that includes the last limit.

This bails him out.

The Neat Construction, Part II

Leibniz takes the differential triangle $Y_1 D Y_2$. It is his idea to oppose it to an auxiliary triangle with two features: (a) It is *similar* to the differential triangle.

(b) One of its sides is *fixed*.

This auxiliary triangle originates by producing the side $Y_2 Y_1$ to T on the axis $A X_2$.

The auxiliary triangle $T X_1 Y_1$ has the same angles as the differential triangle and is, thus, similar to it.

Then point Y_2 moves on the curved line (in our example: the parabola) toward the point Y_1 . What is going to happen?

Point T moves along the vertical axis $A X_2$ up and down, whereas the side $X_1 Y_1$ remains fixed. However, the differential triangle $Y_1 D Y_2$ and the auxiliary triangle $T X_1 Y_1$ remain similar (if the latter triangle degenerates, one has to think anew).

Next, if point Y_2 coincides with point Y_1 , we have exactly the situation which is covered by the Law of Continuity in its above specification: the coinciding of the points Y_2 and Y_1 represents the *limit*; the case in which the differential triangle has *vanished*. However, the auxiliary triangle survives the coinciding of Y_2 and Y_1 —for its side $X_1 Y_1$ is *fixed* and cannot vanish. Therefore, *this auxiliary triangle $T X_1 Y_1$ represents the COMMON PRINCIPLE which covers BOTH CASES the vanishing differential triangle AND its disappearance.*

Both triangles are similar. Consequently, we have

$$\frac{D Y_2}{Y_1 D} = \frac{dy}{dx} = \frac{X_1 Y_1}{T X_1}.$$

Leibniz already knows: $\frac{dy}{dx} = \frac{2x+dx}{a}$. So all together he has:

$$\frac{dy}{dx} = \frac{X_1 Y_1}{T X_1} = \frac{2x + dx}{a}.$$

Actually, Leibniz can apply his Law of Continuity: on the left there is a fraction in which numerator and denominator vanish together; the middle is a fraction with changing value but both, numerator and denominator, *remain* and stay *finite*; on the

right there is a fraction with *vanishing* dx in the numerator but all other quantities remain fixed.

Next, Leibniz lets the dx vanish, i.e. $dx \rightarrow 0$. The outcome is obvious:

$$\frac{dy}{dx} = \frac{2x}{a},$$

which is with the help of his Law of Continuity absolutely neatly derived—although on the left is a fraction where numerator as well as denominator tend toward zero!

This is exactly the point! Leibniz does not divide 0 by 0 but analyzes a ratio $\frac{dx}{dy}$ with *simultaneously* (or, at the same time—but we should keep time out of mathematics!) *vanishing* terms. In today’s notation: $\lim_{dx \rightarrow 0} \frac{dy}{dx} = \frac{2x}{a}$. The auxiliary triangle provides Leibniz with a fixed point to unhinge the world. The Law of Continuity, invented by him, allows him the precise calculation of this ratio dy/dx even in the case where $dx \rightarrow 0$ as well as $dy \rightarrow 0$.

What Is x (and What Is dx) for Leibniz?

Leibniz develops the differential as a geometrical concept. He denotes the respective *length* on the x -axes by “ x ”. Of course, this length is not invariable—just the opposite! For Leibniz the length x is *changing* and this change is *described* by “ $x + dx$ ”.

For Leibniz *both* x and dx are *varying lengths*.

We know, Descartes denoted by “ x ” a certain fixed number or length. Leibniz revolutionized Descartes’ world of concepts completely! Only the letter “ x ” survived—as if nothing had happened. But an upheaval took place: the *continuum* was introduced in the calculations on the quiet.

Of equal importance is that Leibniz created dx as *a changing quantity which decreases below any given quantity*. This circumstance was named by him an “infinitely small quantity”.

Consequently, an “infinitely small quantity” is for Leibniz nothing supernatural, inconceivable—but only a *special case* of a commonly used changing quantity: just one which *decreases indefinitely* (although Leibniz had no “negative” numbers).

Using the concept “limit” one may say: for Leibniz *an “infinitely small quantity” is a changing quantity with zero as its “limit”*.

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