Chapter 12
Weierstraß: The Last Effort Towards a *Substancial* Analysis

A Famous Man

Karl Theodor Wilhelm Weierstraß (1815–97) invented analysis as a precise mathematical doctrine. So, at least, goes the story told to every German student of analysis nowadays. According to that myth it was Weierstraß who was the first to establish clear concepts in analysis and who first carried out strict proofs with the help of "epsilontics" developed by him. Only in this way could the vague ideas of his predecessors be expressed clearly.

As with any commonplace, this is nonsense. To be sure, Weierstraß left a lasting legacy in analysis, but he did not achieve the marvels ascribed to him. As the foregoing chapters have shown, he could not have achieved this. *All* basic notions of analysis—function, differential, integral, continuity, convergence—had already been known for a long time while little Weierstraß learned to add. All, apart from a single notion: the concept of number. Here, indeed, Weierstraß produced a marvel: he invented a concept which none of his students was able to grasp and, in addition, not one mathematician after him has been able to reinvent, since about 1870!—this gradually became clear after a lecture note from winter semester 1880/81 emerged unexpectedly in the mathematics library of Frankfurt university in summer 2016 (see pp. 172f).

Having set out to study the development of the basic concepts of analysis, we shall refrain from celebrating Weierstraß' successes; instead, we shall concentrate on his contributions to the basic concepts of analysis. However, regarding the concept of number, both happened to coincide.

D. D. Spalt, *A Brief History of Analysis*,
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The Core of His Fame

It stands to reason that we shall not withhold a hint on the facts which constitute Weierstraß' renown as the "founder of rigorous analysis". It is based on Weierstraß' success in smashing the image of analysis of being—besides geometry—the second mathematical field which draws, by using clear concepts, convincing proofs from simple images. It was Weierstraß who (besides Bolzano, but this was unknown in the nineteenth century) designed analysis in such a way that it became a conceptual minefield, where each combination of notions produces surprises which render further investigations ever more difficult.

Earlier, analysts had already created grotesque structures. However, they were fenced in as extraordinary phenomena which were of no importance or could be controlled. In contrast to that, Weierstraß reversed things. He made these oddities the touchstone for the quality and the worth of the concepts instead of marginalizing them as meaningless anomalies.

Dirichlet in 1829: Analysis Has Frontiers!

In 1829, Dirichlet published a crazy idea: a "function" $f(x)$ which for each rational value of x has the "value" c and for each irrational value of x the "value" d, which is different from c.

There was *not one* concept of "function", including Dirichlet's own (p. 153), which was suitable for this creation. Although it has only two different "values" (in Dirichlet's thinking: equivalent to "ordinates"), namely c and d, this monster cannot be drawn. It cannot be drawn, because the rational numbers and the irrational numbers are intertwined with each other in such a way that they cannot be kept apart in a *drawing,* but only *as concepts.*

Each irrational number can be written as a decimal number with *infinitely many* digits which do not recur. If you stop this decimal number, say X, at *any* digit, you get a rational number X′. However, the *place* of the digit may be pushed as far out as one likes. The *difference* between X and X′ is of magnitude 10^{-k}, if the number is cut off at the k-th decimal place. In other words: the difference between X and X′ can be made as small as one wishes. In each case we have:

An irrational number has no nearest rational number.—The reverse is also true.

In the case of Dirichlet's so-called "function" this means: *near* each "value of the function" c there is a "value of the function" d—without the possibility of detecting the length of this "near". Consequently, it is *futile* to ask for the "jumps" of this "function" between its values c and d. Indeed, there is a distance between c and d (as they are assumed to be different), but the *smallest* difference of such values X and $\bar{\text{X}}$, which belong to c and d respectively, cannot be given. Or is it 0? So *no* difference?

It is true: Dirichlet published this so-called “function”, but he did not really care about it. He only *stated* it. Why? Dirichlet wanted to say: there are cases in which you cannot “integrate”—analysis has limits. Without integral, no calculus. Dirichlet gave this “function” to demonstrate the *limits of the theory of analysis.* This strange object was not worth examining for him. And this is no wonder, because *actually* it is—according to his own concepts—no “function” at all.

Riemann 1854: These Limits of Analysis Can Be Shifted!

Dirichlet’s student Bernhard Riemann was less humble. His idea was to fence in such extraordinary phenomena as proposed by his teacher. Riemann was not willing to delay the development of analysis. He tried to show that further progress was indeed possible and that it would not cause insurmountable problems.

In his habilitation thesis, written in 1854, Riemann constructed a function (and according to his concepts, see p. 155, it *actually is* a “function”!) which was not as unruly as that of his teacher, but which, nevertheless, was a hard nut to crack. Riemann presented a function which is “discontinuous” for incredibly many of its values, but nevertheless harmless—meaning that it can be integrated, and therefore qualifies as a legitimate object of calculus.

If Riemann’s construction is reversed, starting from his integral, one gets an actual oddity, namely a derivative—a function!—which is “discontinuous” for *a great plenitude* of values, in Dirichlet’s view: with *a great plenitude* of jumps. “A great plenitude” means: as many as there are fractions, and also ordered like them. The discontinuities of this derivative have the following properties:

1. They are infinitely many.
2. In between any two of them, there is another.

Such an enormous amount of discontinuities, that’s a bit much! Nevertheless, this function can be integrated. With the help of a trick, Riemann guaranteed for the discontinuities: the closer they get, the more harmless they become. No doubt, this is a smart idea!

Weierstraß’ Shocking Function

On 18 July 1872, Weierstraß presented a mathematical subject to the Royal Academy of Sciences in Berlin, which was suitable radically to transform the existing picture of analysis.

Two years after this event Leo Koenigsberger (1837–1921) still writes the following in his textbook with the title “Lectures on the Theory of Elliptic Functions”:

> One of the main doctrines of analysis is the fundamental theorem that to each function of a real variable there really belongs a differential quotient, i.e. that the ratio of the increments of the function and of the variable can only be zero or infinite or is discontinuous because

> of finite jumps in singular points of a finite length. However, for the rest it has finite values which are independent of the infinite small growth of the variable ...

According to Koenigsberger, basically *each* function can be differentiated ("to it belongs a differential quotient")—apart from *possibly* a few singular values (at these "singular values" the derivative may not be defined).

This "main doctrine" or this "fundamental theorem of analysis" is *totally wrong*. This was shown by Weierstraß in his momentous lecture on 18 July 1872, where he proved the following:

> Theorem. *There exist continuous functions which cannot be differentiated at any single value.*

That Bolzano had already given such a function 40 years earlier (p. 113), remained unknown for another 40 years.

This theorem states the contrary of what Koenigsberger still says in his textbook 2 years later and which he calls one of the "doctrines" of calculus.

A. Koenigsberger *claimed* in 1874 that a continuous[1] function can *generally*—that means: for nearly all values—be differentiated.
B. Weierstraß proved in 1872: there exist continuous functions which cannot be differentiated *for any single* value.

A greater contrast is hardly imaginable.

The Aftermath of Weierstraß' Construction

Weierstraß had presented a shocking function; soon functions like that were called "capricious". In retrospect, we may say: analysis was at a crossroads. (i) Should one say: this we did not mean, such crazy "functions" we do not want, this counter-intuitive stuff we bar from analysis? (ii) Or should one say: okay, up to now we had a wrong idea of analysis—we must change our attitude and think *much more carefully* about the relations of our analytical concepts?

We remember that the exotic function which Dirichlet presented, rather casually in 1829, did *not cause any reaction.* Neither Dirichlet nor any of his contemporaries (as far as is known) responded in some way or another. Dirichlet's capricious function (as it was called later) was simply ignored in 1829.

43 years later the world had changed. *Not a single* leading analyst of that time thought of putting Weierstraß' oddity aside and to return to do business as usual. Quite the reverse, immediately the analysts of that time set out to create other exceptional functions. (Even a name was coined: "capricious".)

This started the search for a version of a "Value Analysis" which was as general as possible. Surely it is to Weierstraß' great merit that he triggered this development.

[1] We can assume that his omission of "continuous" was quite unintentional.

What is Weierstraß' Understanding of a "Function"?

In 1854, Riemann proposed at the very beginning of his doctoral thesis that a "function" is nothing else than this: to each "value" of an independent variable there corresponds *exactly one* "value" of the function (which is "the value of the function"—see p. 155). Today this opinion of "function" is generally accepted.

However, the victory of Riemann's concept of function was by no means guaranteed. It was Weierstraß who fought nearly all his life against this, according to his own opinion, wrong understanding of the subject. Weierstraß justified his objection in clear words. In his lecture on the foundations of the theory of analytic functions, in 1874, he stated:

> This general definition of the function is the same as the following geometrical one: a curve is a line which is not straight in any of its parts. From such a definition there cannot be deduced any positive property of the defined.

The last sentence explains what Weierstraß demanded from a *correct* definition of a mathematical concept: it must be possible to *develop* the relevant properties from the definition of an object. In short: *the definition has to designate the essence of that object.*

This coincided with the theory of definitions which Western philosophy had taught since Antiquity, i.e. those started by Aristotle or by Leibniz. (And, very interestingly, even by Georg Cantor, in 1883.) For clarity, I am going to coin a word for this basic view of mathematical thinking and will speak of a *"substancial"* definition in this case. Here, "substancial", one of the archaic forms of "substantial", draws attention to the fact that the given definition has theoretical substance.

Clearly, such a "substancial" definition does not drop from the sky. It has to be acquired through mathematical working. This was explained by Weierstraß quite explicitly in his lecture of 1874:

> It is impossible to define the concept of an analytical function in a few words—instead, this has to be developed little by little and this is the task of this lecture.

Obviously, the result of this Weierstraßian lecture cannot be given in a few words. Instead, we shall at least name the most important concepts which Weierstraß needed in order to define his concept of function. They are: (1) "polynomial", (2) "infinite series", (3) "uniform convergence" as well as "domain of convergence" and (4) "development of a power series centred at a value".—All of them being far from elementary concepts.

A Sudden Change

For nearly all of his scientific life, Weierstraß was thoroughly convinced that Riemann's concept of function was of no worth—precisely because it is *devoid* *of substance* and it is impossible to derive *any* important property of a "function" therefrom.

However, Weierstraß *recasted* his position in the very last lecture of his life on this topic, in 1886. At the age of 75 this mathematical Nestor changed his mind and advocated the opposite of his previous position. "Continuity" was the only property which he sustained, but besides this he subscribed to the supposition "that the function is a so-called *arbitrary:* it is lawless".

How come? A late revelation?

Not at all! Even later, in 1886, Weierstraß was still a proponent of substancial thinking. Instead, his change of mind regarding the *proper* definition of a "function" was changed by a basic mathematical result he had reached in the preceding year.

In 1885 Weierstraß had formulated and proved a profound theorem of analysis that still bears his name: the "Weierstraß Approximation Theorem".

> Theorem. *Each function which is continuous within a closed interval can be uniformly approximated by polynomials.*

In other words: Weierstraß had succeeded in drawing from the following, seemingly *simple,* two assumptions

(a) the domain is a closed interval, and
(b) the "function" (following Riemann) is continuous in this domain,

the *essence* of a "function". According to his view, the *essence* of a "function" is constituted by a power series, at least nearly. (This *nearly* is specified with the help of the technical term "*uniformly* approximated".)

Weierstraß was now prepared to accept Riemann's "empty" concept of function—for meanwhile he had succeeded in deducing from two seemingly *simple* additional phrases (finite and closed domain, continuity therein) an important mathematical content (uniform approximate representation of a function through a polynomial sequence). On Friday, 25th of June 1886, he announced the following definition:

> If to any system of values $(u_1, u_2, \ldots, u_n)$ there corresponds one and only one value x of the function, we call this a *unique[ly determined]* function of the variables $u_1, u_2, \ldots, u_n$.

To sum up: although Weierstraß revised his *mathematical* view, he retained his basic *philosophical* conviction. He continued to insist on *substancial* foundational concepts.

Weierstraß' Concept of Number

We do not know any publication by Weierstraß on his concept of number. Nevertheless, his thinking about this concept is documented in some detail. This is due to two facts: first, he always stated his ideas on the concept of number at the beginning of his lecture cycle called "Preliminaries to the Theory of Analytic Functions" and secondly, there do exist several and sometimes very detailed notes taken from these lectures by students.

Nevertheless, it seemed impossible to figure out a clear-cut concept of (real and complex) numbers from these notes. That's why I proposed, in 1990, the view that Weierstraß worked on this concept all his life.

However, I have changed my mind since. Weierstraß does have a clear-cut concept of (real and complex) numbers. The real problem was: his students were unable to understand him and therefore they could not correctly report on it!

This is proved, for instance, by the lecture notes taken by Emil Strauß (1859–92) in winter 1880/81. They were discovered only recently and that by great chance: at the mathematical library at Frankfurt/Main university, in August 2016.

The Fundamental Hindrance That Obstructed the Understanding of Weierstraß' Concept of Number

Only after having completed the German original of this book in winter 2016/17, was I able finally to solve the riddle. The clue is this: Weierstraß relied on a concept not one historian of mathematics dared to think of: the concept of a (pure) set.

As we have noticed by now, Weierstraß was actually a conservative thinker, and set-theory was invented by his much younger pupil Georg Cantor (1845–1918) and by Richard Dedekind (1831–1916), from the second half of the 1870s onwards. Weierstraß never related to this theory.

Nevertheless, as Strauß' lecture notes clearly document, Weierstraß relied on the *concept* of set!

However, only on the *concept*—that means: Weierstraß did *not* make "set" the topic of mathematical reasoning. He did not invent set-theory. Weierstraß only *used* the concept of "set". Again and again he *proved* the associative and the commutative law for unions, as well as for products of sets *in special cases*—and that with the help of *analytical* arguments!

Nevertheless, Weierstraß' construction of the real as well as of the complex numbers is a true set-theoretical one. Which is to say: only *after* the invention of set-theory (in the twentieth century) Weierstraß' construction of real numbers *developed* into being *the most elementary we know today.*

Yet Weierstraß' students *had no chance* of understanding him. For, not surprisingly, Weierstraß never placed any emphasis on the concept of set. He did not even coin a name for it. Why should he? In his thinking it was a mere auxiliary concept and, indeed, a very *trivial* one. Why emphasize it? (Again, an excellent mathematician is not necessarily an excellent teacher.)

Consequently, this new concept *escaped* the understanding of his students—and what is not understood will not be adequately recorded. Normally.

The Peculiarity of the Student Emil Strauß

The student Emil Strauß was somewhat peculiar. In his lecture notes he did not (only) write down what he really understood (and valued as important)—but plainly *everything* he thought he might grasp. In one, very curious, case his writing actually amounts to *complete mathematical vacuity*—nevertheless this passage is highly valuable for the historian because it shows a sharp attack by Weierstraß on his colleagues; whereas parallel lecture notes, taken in the same lecture by other students (among them e.g. Adolf Kneser (1862–1930)) do not even show a hint of this attack.

Thus Emil Strauß' notes contain some sentences which clearly *prove* that Weierstraß operated with (pure) sets. You only have to take these single sentences *literally*—which is *not* an easy task! It took me a considerable amount of time to understand this. However, if the result finally adds up to a coherent mathematical statement, one can be fairly certain that one is embracing Weierstraß' idea.

Weierstraß' idea is truly simple: he takes the utmost possible generalization of decimal numbers!

Preliminary

This generalization is carried out in three easy steps, which all follow this assumption:[2]

Preliminary: *We take the natural numbers as well as the fractions as given.*

This Preliminary comprises Weierstraß' utmost precise construction of the natural numbers as well as the fractions from scratch. This foundational part is a philosophical construction, and as our topic is analysis—not foundations—, we have to omit this here. However, I cannot resist the temptation of disclosing my judgement: Weierstraß' *philosophical* construction of the natural numbers is, in my opinion, the most precise which is known today, including that of Edmund Husserl (1859–1938), who attended Weierstraß' lecture in winter 1880/81.

The Construction

We proceed with the construction of Weierstraß' real numbers:

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[2] As already announced as well as justified in the introduction (see p. xxii), in the following I do not stick to my historiographical method, but instead I present a former idea completely in modern language.

1. Disregard the sign and start with the interpretation of a decimal number $d_0.d_1d_2d_3\ldots$ to be a (sometimes infinite) *set* of fractions: $\left\{\frac{d_0}{10^0}, \frac{d_1}{10^1}, \frac{d_2}{10^2}, \ldots\right\}$
   Now we go beyond this and allow that
2. the nominators and
3. the denominators can be *any* natural number.

That's all! These three steps give us the utmost general concept of "irrational" number, *together* with the two *direct* operations "addition" und "multiplication". We now have the following

Definitions (*addition, multiplication*).

1. The "addition" of two such numbers is their set-theoretical union— followed, of course, by the addition of each two elements with the same denominator.
2. The "multiplication" of two such numbers is, what we may call the "Weierstraß Product": multiply each element of one number with each element of the other number and finally add in the resulting set all fractions with equal denominator.

These two operations are well-defined. They are associative as well as commutative and they even obey the distributive law—because these laws hold for sets and the natural numbers and their operations.

This was truly easy, wasn't it? Unfortunately, we are not yet through!

What is missing? Well, we have the set of *all* imaginable (real) numbers (without sign) you may dream of. We are able to add and to multiply them—but do not know how to *compare* them yet! But of course, we *need* equality as well as comparisons like, e.g.:

$$\left\{\frac{1}{3}\right\} = \left\{\frac{3}{10^1}, \frac{3}{10^2}, \frac{3}{10^3}, \ldots\right\} < \left\{\frac{1}{2}\right\} \; !$$

Further Preliminaries for Tackling a Real Difficulty

How shall we deal with this plenitude of numbers? Where should we begin?

Since Weierstraß' numbers are finite as well as infinite sets, we shall start with the finite ones. Weierstraß himself gave *no one name* to his numbers, but we shall do. As the reader may expect:

Definition (*fraction*). Each number which is a finite set, will be called a "(general) fraction".

If we were pedantic we would choose the name "general fraction", for we already *have* constructed the "fractions" in our initial *Preliminary*. However, things seem fairly clear and therefore we might be allowed to dispense with this "general". Thus, let us stick for one more moment with the attribute "general", for we have also to state the following:

Definitions (*equality, transformation*).

1. Two (general) fractions are "equal", if one of them can be *transformed* into the other.
2. A "transformation" of a (general) fraction is any finite series of changes of one or more of its elements (a) either by cancelling or its opposite, or (b) by splitting it into two or more fractions or *vice versa.*

Therefore, we have e.g. $\left\{\frac{1}{2}, \frac{1}{4}\right\} = \left\{\frac{5}{7}, \frac{1}{28}\right\}$, since

$$\frac{1}{2} + \frac{1}{4} = \frac{3}{4} = \frac{21}{28} = \frac{5}{7} + \frac{1}{28}$$

from the *Preliminary.* Similarly, we state:

Definition (*less than*). A (general) fraction a is "less than" ($<$) another number b if b can be transformed such that $a \subset b$.

Caution: you need to transform b, for we need to have

$$\frac{1}{3} < \frac{1}{2} = \frac{2+1}{6} = \frac{1}{3} + \frac{1}{6}\,,$$

and this you cannot get by transforming a. Please, remember: this definition is not an arithmetical, but a set-theoretical one!—It is true: Weierstraß went for the arithmetical version, not for our set-theoretical one.

We opted in favour of the set-theoretical definition for "less than", because it is Weierstraß' solution in the case of the infinite sets. The real difficulty of this number concept is to define equality and to make comparisons for infinite sets—because the condition to permit only finitely many transformations is here not enough. We need to have

$$\left\{\frac{1}{1}\right\} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\right\} \quad \text{as well as} \quad \left\{\frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \ldots\right\} < \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\right\},$$

but one *cannot* carry out infinitely many transformations.

Solution of the (Perhaps) Real Difficulty

The concepts of equality and comparison in cases where numbers which are infinite sets are involved, seem to be the real difficulty of this number concept. These definitions do not come easily.

Really? Don't we have the fractions (the finite sets)? Doesn't it suggest that we should take them to be our standard?

To make his students aware of the difficult notion of equality in the case of infinite objects (sets), Weierstraß presented the following "fallacy" (as he called it): starting with

$$x = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots ,$$

one gets

$$x + 1 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 2x ,$$

and this implies

$$x = 1 .$$

Weierstraß comments: "This conclusion uses a 'law of equality', which has not yet been stated." (Interestingly, only the lecture notes of Emil Strauß document this passage—and they show the (obvious) mistake "$1 + x$" instead of "$2x$"—, whereas e.g. the notes of the listener Adolf Kneser do not.)

Be that as it may, Weierstraß already had an idea how to solve this problem, and according to the sources, he had it already in, or (presumably) even before, 1868. That was at least 4 years earlier than Dedekind—who published the same idea, albeit with quite a different interpretation. The idea is this:

Definitions (*equal, less than*). A (general) fraction f is "less than" a number q, if there exists a finite subset $g \subset q$ with $f < g$. Two numbers are "equal", if each fraction less than one is also less than the other; if only one part of this condition is fulfilled, the relation "less than" ($<$) *resp.* "greater than" ($>$) is defined.

In 1872, this was Dedekind's idea of *defining* the (irrational) numbers (see p. 202). However, Weierstraß had used this idea to *compare* the (irrational) numbers already in 1868 or even earlier. We see the difference in their ways of thought in those approaches: Weierstraß used the idea of *comparing* his numbers, which he had previously defined (Weierstraß is a "substancial" thinker!), while Dedekind used this relation to *define* his numbers (therefore, I propose to call him a "relational" thinker).

The arithmetical laws for these relations ($=$, $<$) are easily proved.

Weierstraß abhorred infinite numbers (he sided with Leibniz and Cauchy against Johann Bernoulli and Euler) and proposed the

Definition (*finite*). A number a is called "finite," if there exists a natural number $n \ \left(:= \left\{\frac{n}{1}\right\}\right)$ with $a < n$.

(Weierstraß did not even dream of *infinite* natural numbers—at most in nightmares. However, what would Johann Bernoulli have said, see p. 52?)

And then Weierstraß banished the infinite numbers by calling them all together "∞".—Consequently, ∞ is not a true number, as it violates the arithmetical laws.

The Benefit of Structural Thinking

If we add the number $\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}$, these finite Weierstraßian numbers constitute what we call nowadays a *monoid.* Actually, we even have two monoids: one for each of the two operations, addition and multiplication.

Today, at least in our second semester at university, we are taught how to enlarge a monoid $(M, \circ)$ to a group $(G, \bullet, \star)$: we just take G to be $M \times M$ and operate this way:

$$
\begin{aligned}
(a, b) \bullet (c, d) \;&:=\; (a \circ c, b \circ d)\,, \\
(a, b) \star (c, d) \;&:=\; (a \circ d, c \circ b)\,, \\
(a, b) \;=\; (c, d) \;&:\Longleftrightarrow\; a \circ d \;=\; c \circ b\,,
\end{aligned}
$$

where "$\star$" is the "inverse" operation of "$\bullet$".

> Actually, this is just what Weierstraß did with his finite numbers, taking them to make up our set M! Exactly in this way he constructed his "real" numbers to be pairs of his "irrational" numbers. And since his "irrational" numbers M constitute two monoids, he actually got a *ring* $G = M \times M$ *with the unit* 1.

Heavy stuff, indeed! However, for those familiar with elementary modern algebra (*after* Bourbaki), It's mere child's play. Nonetheless, it's mere really astonishing that Weierstraß managed to come up with this construction, without *any* algebra to hand!

Clearly, not one of his students, not even the most talented, was able to grasp this construction. Besides, the concept of "ordered pair" had not yet been created (this only happened in the twentieth century). The same holds for the use of *arbitrary symbols* for operations. Inevitably, Weierstraß' students were unable to distinguish between our "$\circ$" and "$\bullet$" from above, and, even more dramatically, they confused the formal "$\star$" with the arithmetical ("true") "$-$"—because Weierstraß always used "$+$" as well as "$-$"! Weierstraß was not a protagonist of mathematical revolutions, but he was an enormously rigorous thinker.[3]

That is why we could not access this idea before Emil Strauß' lecture notes where found (in August 2016) and my subsequent interpretation. It was very lucky that Weierstraß disclosed his view of the numbers *in* (nearly) *all its elementary details* in winter 1880/81—and this in the presence of this ambitious student. Of course, Weierstraß did not go into all these details in each of his foundational lectures!

[3] In case he knew that $a = \bar{a} + c$ as well as $b = \bar{b} + c$, he easily concluded $(a, b) = (\bar{a}, \bar{b})$ for all irrational numbers a, b. The decisive point is, that he already *knew* the first two equations—for otherwise he would have used *general* subtraction of his irrational numbers, which is impossible.

The Benefit and the Disadvantage of Structural Thinking

Well, even a Weierstraß was not perfect and could make an error. Weierstraß taught procedures to divide his irrational as well as his real numbers—in fact, in different lectures he proposed two different methods.

However, our structural analysis of his construction genuinely proves: they are both wrong. The reason is that this idea of construction only produces a ring, not a field. And it is true: both of the methods lectured by Weierstraß are erroneous. They both rely on the possibility of *subtracting* "irrational" numbers—but this is *impossible.* (In hindsight: *if* it were possible, it would not have been necessary to construct the operation of subtraction in a *pure formal way*—as it is done by taking $M \times M$ which Weierstraß actually did.)

This line of thought inevitably leads to the question: *Do Weierstraß' real numbers form a ring (with* 1 *as unity) or a field?* We may put it this way: *can Weierstraß' real numbers be divided, yes or no?*

This question is very productive. It can be answered in two ways:

1. The real numbers are objects—it does not matter how they are constructed. These objects can be taken as subjects of a structural analysis. (You may construct the integers by adding the "signs" $+$ and $-$ to the natural numbers and zero—nevertheless, we obtain the structure of a group from this monoid.)
2. Whether some objects form a certain structure *depends* on the definition of "operation" taken.

This is a book neither on algebra nor on philosophy of mathematics, but solely one on the history of analysis. Therefore, we have to limit ourselves to elementary aspects of this topic and merely state two things:

- There is a fundamental difference between direct operations and their reverse, the inverse operations. The latter are methods of trial and error, they *cannot* be carried out directly.
  This fundamental difference between these two kinds of operation is smashed by taking the algebraic point of view. It makes a great difference whether we say: "An *operation* is a method of generating a new object from two given ones." or if we state: "An *operation* is defined whenever there exists an object which fulfils certain conditions." The first standpoint is Weierstraß', the second is Hilbert's.

- We all know how to subtract and how to divide decimal numbers (we just take suitable approximations). We have learnt this in school. Consequently, we are all able to subtract and to divide Weierstraß' real numbers—we only need to take them in their decimal (or binary or some equivalent) form.
  We may put this in higher algebraic language and say elaborately: (i) Equality is an equivalence relation for Weierstraßian real numbers. (ii) In each equivalence class there exists a decimal (or ...) number—which in Weierstraß' sense may be called the "value". (iii) The decimal (or ...) numbers are complete. (This proof is due to Hilbert.)

Therefore, we answer the above question in a personified way:

> In Weierstraß' view, his real numbers do not constitute a field, whereas according to Hilbert's view, they do.

(That Weierstraß himself, erroneously, held his real numbers to be a field, is a historical fact.)

Postscript

It was as late as 1994 when John Horton Conway (1937–2020) characterized the easiest way of constructing the real numbers:

> Proceed from the natural numbers to the non-negative rationals (or the strictly positive ones if you prefer), then construct the non-negative (or positive) reals from these, so having no sign-problem, and then construct signed reals from these in the way that we constructed the signed integers from the natural numbers. I think that this is in fact the simplest way to construct the real numbers along traditional lines.

Nevertheless, neither Conway nor any other mathematician ever since came up with this construction of the real numbers—although, as we have seen, it is *really* a straightforward generalization of the decimal numbers.

The inevitable conclusion is, yet again, the well-known truism: *Mathematics is* REALLY *difficult!*

I)

Infinite Series

With his construction of the *general* concept of "quantity"—which is our "real number"—Weierstraß was able to do analysis. In particular, he built the concept of "infinite series" or simply "series".

Weierstraß repeated the method used for his numbers step-by-step. He started with the study of "infinite series" of "fractions", followed by those of "irrational numbers". That is, he considered the series

$$a_1 + a_2 + a_3 + \dots ,$$

where a_k are irrational numbers. Each of it has the form

$$a_k = \alpha_0^{(k)} \cdot 1 + \alpha_1^{(k)} \cdot \tfrac{1}{n_1} + \alpha_2^{(k)} \cdot \tfrac{1}{n_2} + \dots$$

wherein the order does not matter. Taking σ_i to be the sum of all multiples $\alpha_i^{(k)}$ of the fractions $\frac{1}{n_i}$ for all the k terms of the series (i.e. $\sigma_i = \bigcup_k \alpha_i^{(k)}$) one gets

$$s = \bigcup_k a_k = \bigcup_i \sigma_i \cdot \frac{1}{n_i} .$$

(Of course, Weierstraß did not use the symbols of set-theory—this is our modern reading of his original concepts.) Therefore, to assure the finiteness of the series Weierstraß demands:

1. *All irrational numbers a_k* of the series *must be finite.* (Clearly, this demands that in all summands $a_k = \bigcup_i \alpha_i^{(k)} \cdot \frac{1}{n_i}$ the *numerators* $\alpha_i^{(k)}$ of the fractions with denominator n_i have to be finite; however, this condition is only necessary but not sufficient.)
2. Each "element" of s (which is to say, each $\sigma_i \cdot \frac{1}{n_i}$) has a finite multiplicity σ_i, or: *all σ_i are finite.*

Subsequently, this "sum" $s = \bigcup_k a_k$ is made up as follows: in each "element" the numerator of the fraction with denominator n_i has to be σ_i, which was built from all the irrational numbers a_k of the series.

The well-known example $1 \cdot 1 + 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{3} + \ldots$ shows, that the given conditions are not sufficient. Therefore, the relevant question is:

In which cases is the series $s = \bigcup_k a_k$ of the irrational numbers a_k finite?

To answer this question, we need to know Weierstraß' concept of the "summability" of a series.

Summability

Since Cauchy we are used to defining the concept of "convergence" of an infinite series

$$u_0 + u_1 + u_2 + \ldots = s_n + r_n$$

to be the condition

$$\lim r_n = 0 .$$

Weierstraß gave another concept, and, to make things more clear, he also chose another name: "summability" ("Summierbarkeit"). His notion is this:

Definition (*summability*). An infinite series

$$a_1 + a_2 + a_3 + \ldots$$

is called "summable" if its sum is finite (*see* p. 177).

As we already know: the sum s of the series $\bigcup_k a_k$ is $\bigcup_i \sigma_i \cdot \frac{1}{n_i}$.

Unfortunately, Weierstraß did not state this definition. We have to deduce it from the text or, more precisely: from the name he used for the concept.

Summability for Series of Irrational Numbers

Weierstraß gives a *criterion* for the sum of an infinite series to be finite. In the case of a series of irrational numbers, he says (his style is moderately updated):

> Theorem. *The infinite series $\bigcup_k a_k$ of irrational numbers a_k is finite if and only if it is possible to determine a number g, which is larger than each sum of finitely many a_k.*

Weierstraß proves this theorem by demonstrating it to be "correct". That means: if and only if the stated criterion is fulfilled, then $s = \bigcup_k a_k$ is finite.

Clearly, if the sum of the infinite series is finite, the given criterion is valid. But what about the opposite direction?

If the specified criterion is fulfilled, we shall get the sum s of the series to be $s = \underline{\lim}\, g$, i.e. the least upper bound of $\bigcup_i a_i$. However, Weierstraß did not say a single word about how this sum can be defined. That is why his proof of the above theorem cannot be understood that easily. It plainly lacks the following definition:

Definition (*sum of an infinite series, summability*). The *sum* of an infinite series is a (finite) number which is the least upper bound of any finite sum of its terms.

Nowadays we are to take this definition to be the specification of Weierstraß' definition of "summability".

Weierstraß even proves the following:

Theorem. *If the infinitely many irrational numbers a_k are arbitrarily grouped, and the sums of these groups are built, the result will always be the same.*

In the course of the long proof, some important theorems on irrational numbers are revealed.

Theorems on Irrational Numbers

Theorem. *Each finite irrational number a, with infinitely many elements $\alpha_i \cdot \frac{1}{n_i}$ can be represented in this way:*

$$a = a_0 + a_1\,,$$

where (i) a_0 *has finitely many elements;* (ii) a_1 *can be made arbitrarily small.*

This statement may be expressed this way: each finite irrational number a can be approximated by a fraction (a_0) to any degree of accuracy (a_1).

This leads Weierstraß to his theorem for the "equality" of two finite irrational numbers:

Theorem. *Two finite irrational numbers a and b are equal if and only if it is possible to split each of them into two:* $a = a_0 + a_1$ *and* $b = b_0 + b_1$ *(*a_0 *and* b_0 *being fractions) such that* $|a_0 - b_0| < \varepsilon$ *as well as* $a_1 < \varepsilon$ *and* $b_1 < \varepsilon$ *are valid for any previously given* ε.

(It is easy to see that the subtraction of a (general) fraction is possible.) Stated without formulae:

Two finite irrational numbers are equal if and only if it is possible to split each of the quantities into two, an approximating fraction and an (irrational) excess, such that the difference of these approximating fractions as well as each of the two remainders in itself is less than any previously given tolerance ε.

Formulated this way, the theorem seems to be clear. The technical proof in the lecture notes comprises one large page. Relying on this theorem, Weierstraß "easily" proves the following

Theorem. *In an infinite sum, equal irrational numbers can replace each other.*

Summability for Series of Real Numbers

The addition of real numbers is different from that of irrational numbers. (Conway did not ponder on that!) Therefore, in case of series of real numbers another *criterion* for summability is required. Weierstraß presents it as follows:

Main Theorem. *If the series* $x_1,\ x_2,\ x_3,\ \ldots$ *has the property that there exists a finite number g such that the absolute sum of arbitrarily many and arbitrarily chosen terms is less than g, then the series is summable.—The reverse also holds.*

To prove this, Weierstraß argues thus:

(a) Take the positive of the chosen terms x_k (i.e. the pairs $x_k = (a_k, b_k)$ of irrational numbers a_k, b_k with $a_k > b_k$). One can change each of them in such a manner ($(a_k, b_k) = (\overline{a_k}, \overline{b_k})$) that its "negative" part ($\overline{b_k}$) is as small as one wishes.
(b) The same can be done with the negative terms x_k (i.e. the pairs $x_k = (c_k, d_k)$ with $c_k < d_k$) such that $x_k = (\overline{c_k}, \overline{d_k})$ where $\overline{c_k}$ is as small as one wishes.
(c) Consequently, it is possible to change the positive of the chosen terms in such a manner that their "negative" parts ($\overline{b_k}$) are less than the terms f_k of a converging series $\sum f_k = f$.
(d) The same can be done with the negative of the chosen terms: their "positive" parts ($\overline{c_k}$) can be made less than the terms h_k of a converging series $\sum h_k = h$.
(e) Consequently, the series can be replaced by (i) a sum of positive ($\overline{a_k}$) (ii) as well as "negative" fractions ($\overline{d_k}$) plus (iii) a sum of the "negative" irrational numbers ($\overline{b_k}$), which if taken absolutely ($\overline{b_k}$), have a finite value ($< f$), as well as (iv) a sum of irrational numbers ($\overline{c_k}$), which is also finite ($< h$). (This is true even if the series is not summable!)
(f) However, the criterion demands that there exists a finite number g such that the absolute sum of any finite set of terms x_k is less than g. Consequently, each of these sums is less than $(g + f) + (g + h)$.

(g) Therefore, each absolute sum (of finitely many) terms of the series is guaranteed to be less than $2g + f + h$.
(h) From this, Weierstraß concludes that the series is summable—which proves the Main Theorem.

As Weierstraß operates with the absolute value of the partial sums, his last conclusion relies again on the (unsaid) concept of the least upper bound, as it has been in the case of series of irrational numbers.

The Concept of "Convergence"

Weierstraß uses the notion of "convergence" only towards the end of his lecture (on p. 155 of Strauß' notes). There he even *identifies* it with his concept of "summability".

When speaking of "convergence", Weierstraß relies of course on the *order* of the terms in the series. After having introduced the "complex" numbers, Weierstraß proves the following:

Theorem. *If the (complex) series is summable and has the sum s, it is possible to determine to any given* ε [4] *a natural number n such that*

$$|s - s_n| < \varepsilon\,.$$

And he adds: "Because of this theorem one calls a summable series also convergent and says that the sum s_n converges to the value s."

Weierstraß continues: if the concept of "limit" of a series is defined, this notion is taken to be equivalent to the concept of its "sum". However, the value of this "sum" may depend on the order of the terms. Series with this property Weierstraß suspends "for the time being", closing with the following

Definition (*(un)conditional convergence*). If the order of the terms influences the value of the sum, the series is called "conditionally convergent"; otherwise it is called "unconditionally convergent" (or "absolutely convergent"—in case of real numbers both notions are equivalent).

Upshot of Weierstraß' Concept of Number

Weierstraß takes the concept of "series" to be a generalization of the concept of "sum". That is why he dispenses with the idea of order when dealing with series.

Therefore, his notion of the finiteness of the sum of a series, namely "summability", is narrower then our concept of "convergence" (which demands *ordered* series). Consequently, Weierstraß' "summability" coincides with our concept of "unconditional convergence".

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[4] Weierstraß, indeed, uses "δ" ...!

Results:

1. Weierstraß succeeds in founding analysis on a well-established concept of (irrational, real and complex) number.
2. Additionally, Weierstraß succeeds in formulating a substancial concept of "equality" for "irrational" numbers.
3. In his untimely anticipation of an algebraic construction Weierstraß completes the monoid of his irrational numbers to a group, thereby giving a unique elementary construction of the real numbers, which has not been repeated since.
4. Because Weierstraß took the concept of "series" to be a generalization of the concept of "sum", his notion of finiteness for the sum of a series (called "summability") is narrower than the usual concept of "convergence" used nowadays: "summability" is "unconditional" convergence.

Weierstraß in Retrospect

Weierstraß was not the first to define clear concepts of analysis, nor did he invent epsilontics. He learnt all of that.

Nevertheless, he pioneered other foundational landmarks in analysis:

1. In his "Approximation Theorem" Weierstraß succeeded in developing calculatory expressions ("polynoms") as a rather good approximation for any "continuous" function, which has a "compact" domain. This paved the way for Riemann's concept of a function in analysis.
2. Weierstraß actually developed his concept of the real numbers *constructively.* Albeit, this simple and elegant construction came far too early and therefore could not be expressed in a suitable language. This is also why his students, even the most gifted, were unable to grasp it. Mysteriously, this concept was not re-invented in fourteen decades.
3. Weierstraß' idea of turning "capricious" functions into legitimate objects of analysis paved the way for "Value Analysis" finally to become a very general theory of "Functional Analysis".

Weierstraß' thinking was clearly traditional—I proposed the name "substancial". He was not keen on manipulating formulae: quite contrarily, he substantiated every step within his construction of the numbers.

And it is a real paradox of history that Weierstraß was (to my knowledge) the very first mathematician to give a strictly formal construction of a group by forming the Cartesian product of a monoid with itself—even decades before the concept of "ordered pair" was created, not to speak of the concept of a "monoid".

Today's judgement is inevitably this: Weierstraß has given a precise concept of real number; (i) it is totally systematic, (ii) it does not rely on the concept of "negative" number and (iii) it is the most elementary known to this day.

However, nothing originates from nothing. Weierstraß has to pay a price for his concepts to be of such an elementary character. This price is the lack of inverse operations: Weierstraß' irrational numbers *cannot* be subtracted nor divided (see p. 179). To implement the inverse operations for Weierstraß' irrational numbers one has to change from his most elementary concept to a more ambitious one, to a "positional system" of numbers. But the way of subtracting and dividing decimal numbers we are all taught in school.

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