Chapter 10
Cauchy: The Bourgeois Revolutionary as Activist of the Restoration

Cauchy: The Atipode to Bolzano

Augustin-Louis Cauchy (1789–1857) was taught mathematics between the ages of 15 and 18 at the École Polytechnique in Paris, by first-rate teachers of his time. At the age of 26, he himself started teaching at this university. In the meantime, he worked as an engineer, for instance, in Paris on the construction of canals and an aqueduct and in Cherbourg on the construction of the new harbour.

This is an extraordinary biography: a singularly gifted mathematician works in his earliest years for several years in the material construction of the world *through* mathematics until he starts his creative mathematical career. This had unique consequences on the mathematical disposition of the man Cauchy as well as on the transformation of mathematics as a science through Cauchy.

For the sake of completeness, we may add: at the age of 26, after the downfall of Napoleon and the beginning of the Restoration, Cauchy (together with a physicist) was appointed by King Louis XVIII to the Académie des Sciences—this after the exclusion of 62-year-old Lazare Carnot and 69-year-old Gaspard Monge for political reasons.

The Heart of Cauchy's Revolution of Analysis

As a mathematician, Cauchy was influential not so much as a member of the Academy but as a university teacher. Accompanying his lectures in 1821, he published a textbook on Analysis (*Cours d'Analyse de l'École Royale Polytechnique*), in 1823 a textbook on differential calculus, and in 1829 one on integral calculus. (Cauchy counts as one of the most productive scientists. In addition, he wrote plenty of treatises; today's edition of his *Collected Works* comprises 27 large sized, thick volumes.)

D. D. Spalt, *A Brief History of Analysis*,
https://doi.org/10.1007/978-3-031-00650-0_10

In the introduction of his textbook on Analysis, he drew conclusions from his experience as an engineer who did practical work by writing the following two sentences (the *emphases* are added):

> We must also observe that [the arguments drawn from the generality of algebra] tend to grant a limitless scope to algebraic formulas, whereas, in reality, most of these formulas are valid only under certain conditions or *for certain values* of the quantities involved. In determining these conditions and *these values* and in establishing precisely the meaning of the notation that I will be using, I will make all uncertainty disappear, so that the different formulas present nothing but relations among real quantities, relations which will always be easy to verify *by substituting numbers for the quantities themselves.*

> There had never been a more radical declaration of war on traditional mathematics in modern times.

The radicalism of this challenge was so comprehensive that nobody was aware of it or even took it seriously, neither Cauchy's contemporaries nor later historians of mathematics.

This is true even though Cauchy said everything in these two sentences. For what do they say?

The *first sentence* says that the formulae of the Algebraic Analysis of Lagrange and Euler are completely general (they "tend to grant a limitless scope")—while practical mathematics (the "reality") shows that these formulae are valid *only* "for certain values". In short, the mathematics from the Academies is vague, aloof, and removed from reality—its application to practical problems requires the precise determination of their validity.

The *second* sentence says two things: (i) the meanings of the mathematical *terms* must be clear and unequivocal and (ii) through the *assignment of the* VALUES *in the expressions*, we obtain crystal-clear numbers which are suitable *in practice.* And the *numbers* are such "values" by all means.

In the years he was working with others as an engineer, Cauchy *experienced* at first hand:

> The analysis only becomes *useful and effective* when its general *formulae* are adapted in such a way that they result in *numerical values.*

What was Cauchy to do when he was appointed to teach Analysis at the École Polytechnique to future engineers? He taught them to adjust the general formulae by *substituting numbers* in order to make them *useful for the practice.*

Mathematical View of Cauchy's Revolution of Analysis

Euler and Lagrange founded Algebraic Analysis on a most *general* concept of "variable". By wanting to go beyond this, Cauchy, *as a matter of principle,* declared that he wished to *materialize* the "variables" by assigning "values" to them. However, this amounts to nothing else than that:

Cauchy wished to *add* to the original foundational concept of Algebraic Analysis "variable" a second and new one: the foundational concept "value".

Cauchy's *motive* for this innovation was a *non*-theoretical one: his practical experience in applying the formulae of Algebraic Analysis.

However, the *consequences* of Cauchy's move are deeply theoretical. Even the layman will understand that:

A change in the foundational concepts of a theory *necessarily* changes the theory itself.

The reason is that "theory" in mathematics involves proofs. Yet the foundational concepts are the starting points of those proofs. Subsequently, it follows that new foundational concepts must change the proofs!

In other words, by announcing, *as a matter of principle,* the making of general formulae of the "Algebraic Analysis" fit for praxis by introducing values into the expressions, Cauchy complements the foundation of the theory (the "variable") by concretizing it to take specific "values".

Such a foundational reconstruction of a theory is nothing other than a *conceptual revolution.* The use of the term "revolution" is not just empty talk or a way of attracting attention; it shows the significance of Cauchy's innovative idea.

Science needs clear concepts. A *clearly determined* fact needs a special, unique name.

This also holds for the science history of mathematics. The analysis of Euler and Lagrange has been called "Algebraic Analysis"—we propose the name "Calculus of Expressions" (see p. 94).

Cauchy abolishes this kind of analysis and establishes a new theory instead. He states this *explicitly* in the *Introduction* of his first textbook of analysis. Therefore, Cauchy's analysis clearly needs a new name of its own!

Fascinatingly, history of mathematics has not come up with such a name until today: the analysis as introduced by Cauchy has not been christened yet. Therefore, we are free to do so. My proposal is as follows: let us call it "Calculus of Values" or **"Analysis of Values",** a name that draws attention to the structural change in analysis. Our result is:

Cauchy transformed the "Calculus of Expressions" (or "Algebraic Analysis") into the "Calculus of Values" (or "Analysis of Values").

Cauchy's Concept of Variable Is Determined by "values"

Cauchy defines "variable" this way:

> We call a quantity *variable* if it is assumed to take on many different values.

This is not alarming. But it clearly differs from Euler's declaration (p. 71). Euler demanded that "all definite values whatsoever" should be taken by the variable. Thus, since Cauchy a "function" may be defined only within a finite interval. We now operate within a "Calculus of Values" and no longer within a "Calculus of Expressions".

The graduate of mathematics realizes that Fourier Analysis is only possible since Cauchy—i.e. the method of finding a suitable calculatory expression in the form of an infinite trigonometrical series for an arbitrarily given function. The reason is that the construction of this series requires the calculation of *definite integrals* and this is only possible if the domains of the respective functions are *finite* intervals.

Euler, the indefatigable calculating mind nevertheless managed to construct the aforementioned "Fourier Integrals". However, as Euler was unable to conceive of a "function" with a finite interval as its domain, he was unable to recognize the wider scope of his calculation. This interpretation goes back to Jean Baptiste Joseph de Fourier (1768–1830) who wrote it down in the first quarter of the nineteenth century and who subsequently lent his name to this technique.

What cannot be seen in this definition of "variable" is its actual *usage* by Cauchy. At first sight, Cauchy's *designation* of the "variable" seems to be *hardly* different from that of Euler—apart from the already noted (and very *important!*) aspect that Cauchy no longer demands the variable to take "all definite values whatsoever". But beyond that?

The difference cannot easily be seen from Cauchy's *definition* of the notion of variable. The meaning of "value" introduced by Cauchy has further importance and lends a greater degree of precision than it had for Euler or Lagrange and their "Calculus of Expressions".

Cauchy Derives "number" from "quantity"

"Quantity"

Modern analysis is founded on the concept of "number". Yet, Cauchy could not build on this concept because he did not have a suitable notion of "number". He knew that the decimals are suitable and useful for *practical calculations.* Through the lack of an alternative idea, Cauchy stuck to tradition and relied, as did his predecessors, on the notion of "quantity". (For the proofs, mathematicians need *concepts.*)

Cauchy starts, in a similar way to his teacher Sylvestre François Lacroix (1765–1843), with "magnitudes" and focuses on their *changes,* more precisely on their *increases* or *decreases.* These changes of the "magnitudes" he calls by the traditional name *"quantity".* (The word "Zahlgröße" was coined in the German language for this type of quantity at that time.) We summarize:

> The "quantity" is the increase or the decrease of a "magnitude". The "opposite" quantity is the decrease or the increase which the second quantity undergoes to reach the first.

That means that "quantity" is a *change,* which is a notion that comprises *motion.* This is clever because the *concept* itself is fixed, motionless. It only *comprises* motion (and we already know from p. 27 that according to the standards of classical mathematics motion is not accepted in mathematics).

Let us look at the philosophical aspect: Cauchy's definition, just like Euler's (see p. 70), does not say *what* "magnitude" essentially ought to be.

"Number"

Cauchy bases the concept "quantity" on the notion "magnitude". In the same way, he deals with the concept "number". He writes:

> The measure of the second magnitude, if compared to the first, is a *number* which is represented by the *geometrical ratio* of one to the other.

If, e.g., the two measures are "6 hours" and "2 hours", their geometrical ratio is the "number" 3:

$$\frac{6\text{ hours}}{2\text{ hours}} = \frac{6}{2} = 3\,.$$

This does not seem convincing: doesn't the measure "6 hours" already contain the number 6?—Strictly speaking, this is not the case! The ancient inhabitants of Mesopotamia arranged their civic and economic activities as well as their fortifications, canal planning, and accounting (the documents in clay have survived until today) *without knowledge of the abstract concept of "number"*. For anything and everything, they had their own system of measure; these systems were totally different from each other. The signs for "6 days" and for "6 goats" were, for instance, completely different.

Here is not the place to go into further detail, but those thousand years of early civilization show clearly: the operation with *measures* does not *necessarily* require the knowledge of the abstract object of "number". *Measuring* and (abstract) *counting* are completely different actions. Accounting does not need numbers! Historically, measuring preceded counting by specialists (with numbers) by thousands of years.

Cauchy dealt with the concept of number only to a certain degree. (He asked, for example, what about 0? Following the existing concept, zero is no number!) We are content with this answer: this way, Cauchy was at least able to give a clear concept of fractions (or rational numbers).

However, for doing analysis, Cauchy needs more, at least the "irrational" numbers. How did Cauchy think about them?—For this, we need some preliminaries.

> The central conceptual tool for the construction of analysis introduced by Cauchy is the concept of **"limit".**

The mathematical abbreviation is "lim", standing for the Latin "limes" and the French "limes" and "limite".

The Basic Definition of "limit"

The object is not new. We have already found it in Leibniz, with regard to his convergence criterion (as we call it today, p. 24) and his calculation of areas with a curved boundary (p. 31), then again in great detail in Bolzano (pp. 104 and 105).

These authors did not create a name for their new object. (Others did, e.g. d'Alembert.) Cauchy too, now gives it a name:

> When the values successively attributed to the same variable approach a fixed value indefinitely, in such a manner as to end up differing from it by as little as we will wish, this final value is called the *limit* of all the others. It is written
>
> $$\lim x = \mathrm{X}\,.$$

Undoubtedly, the concept "limit" is difficult. It is as difficult as the concepts "convergence" and "continuity"! Cauchy does not worry about these difficulties. (Pedagogical abilities do not always go hand in hand with subject knowledge.) His contemporary and textbook author Johann Tobias Mayer (1752–1830) tackles this problem in his book *Complete Teaching of Higher Analysis*, published in 1818, in quite another way. Mayer does not use the concept "limit" at all (which does not prevent him from writing at some point about a "ratio of limits"), but he treats the subject on no less than 20 printed pages!

This topic occupies Mayer's mind to such a degree that he even introduces *a special sign* for it. Interestingly, it is not the operator "lim" but the binary relation "≡":

> If one were to introduce a special sign in analysis to indicate an infinite approximation of some quantity to another, e.g. the sign ≡, nobody would take offence that, if the equation read
>
> $$T = \frac{1}{\log x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{2^x}$$
>
> where the quantity x increases without end, i.e. becomes infinite, we would get
>
> $$T \equiv \frac{1}{\log x}\,.$$

As we remember, Johann Bernoulli did not come up with the idea of *a second sign for another type of equality:* p. 61. Times have changed. (As a mathematician, Mayer cannot hold a candle to Johann Bernoulli!)

Cauchy needs less than four lines in his preface for the same fact. This is followed by another four to five lines with two examples: irrational numbers as limits of fractions and the circle as the limit of inscribed polygons. His students will have found this hard to swallow, but this is quite another matter.

The Unspoken Luxury Version of the Concept of Limit

Cauchy’s first example for his concept of limit, which was just mentioned above, is that of *the irrational numbers as limits of fractions.* However, there is a snag in this idea! *Strictly speaking,* Cauchy is not allowed to *introduce* an irrational number as the limit of an approaching sequence of fractions: it does not meet his notion of limit at all!

Just picture the problem: we do know what fractions are and we want to know what irrational numbers are. For us, at this stage, *the irrational numbers do not yet exist at all.*—Let us reread Cauchy’s text: “When the values successively attributed to the same variable approach a fixed value indefinitely ...” It clearly mentions a “fixed value” which is to be “approached” by a sequence of values. But if the irrational number does not yet *exist* (quite to the contrary, it is yet to be constructed), then Cauchy’s concept of limit is of no help—for there *does not yet exist* this “fixed value” which is to be approached by this sequence of fractions. This “fixed value” has, first of all, to be *created* by this construction.

However, Cauchy does not seem to be aware of this problem. *Directly afterwards,* just after having given his definition, he puts forward the irrational numbers as examples of “limits”.

And this is no slip, for later on when he defines the operations for the calculation of the numbers *in detail,* he proceeds in the same way. Cauchy takes yet again the irrational number B just to be $B = \lim b$ where the (variable) quantity b is only allowed to have fractional values.

Today we have a problem with this strategy. We strongly discriminate between whether the “limit” spoken of already *exists* and whether it does *not yet exist* and is only to be *constructed* or *defined* with the help of this sequence. For us, these are *two different* notions of “limit”. In the first case already existing facts are described: a given number is *called* a “limit”, whereas in the other case a new mathematical entity is created through *being* a “limit”. This second case could be named the “luxury version” of the notion of limit for it offers more than the other one, it *produces* a new mathematical entity (e.g. an irrational number).

> In the true sense of the luxury version, an “irrational number” *is* nothing else than just this sequence of fractions, for which it is the “limit” (which is approaching it *as nearly as we wish*).

They *exist only* in this way. For example, π is *nothing else than* the sequence:

$$3; \quad 3.1; \quad 3.14; \quad 3.141; \quad 3.1415; \quad \ldots$$

Cauchy’s contemporary Johann Traugott Müller was well aware of this (see pp. 200f).

What Is the Difference?

Cauchy writes as if there were no difference between the two (for us today) different versions. Was he conscious of this cheating?

I do not know. I did not find any formulation by him showing any awareness of this problem.

Let us remember Bolzano's view of this problem (see p. 104). According to him, a number is already *well-defined* if it could be calculated "as precisely as one wishes". As his contemporary, Cauchy could have thought in *just the same* way.

If this were the case, then for Cauchy the difference of these two versions of the notion of the limit was not discernible: it is the difference between those two "steps" which we made in presenting the modern notion of convergence (following Michael Spivak: p. 103)! We can state:

> Neither the brilliant thinker Bolzano nor the outstanding mathematician Cauchy in the first third of the nineteenth century made this distinction. However, Johann Traugott Müller did in 1838.

Consequently, this difference did not exist in their analysis. (But it *exists* today in our analysis.)

Some may judge this to be wrong. They might say that this difference between these two versions of the notion of limit (as well as the notion of convergence) is *eternal.*

But the proponents of this view are obliged to spell out what is meant by the "existence of a notion"—when even the most reknown mathematicians did not realize it. The hidden implication of this view is that mathematical notions (and truths) exist independently of their discovery and even if they are *not used* at all in *what is well-known* of mathematics.

"Function" and "value of a function" in Cauchy

Let us return to the foundational concepts of analysis, to one of the central notions. What is Cauchy's understanding of "function"?

The Concept of Function in Cauchy

Cauchy expresses very clearly and at length what he understands by a "function", namely a "changing" quantity:

> When variable quantities are so related among themselves that, the value of one of them being given, we are able to deduce the values of all the others, we usually consider these various quantities expressed by means of one among them, which then takes the name of the *independent variable,* and the other quantities, expressed by means of the independent variable, are what we call *functions* of this variable.

That is to say, given the four quantities x, x^3, $5e^{x+2}$, and $\sin x$, the last three are called "functions" of the first, which is the "independent variable" x.

Cauchy is very scrupulous and defines the same again—in case there are *more than one* "independent variable" (as in $5x^2 + 2y^3$). We can pass over this here.

The New in Cauchy's Concept of Function and a New Style of Notation

His detailed talk of "values" is easily discernible: the "independent variable" takes on "values" and from those the "values" of the variable quantity are "to be deduced". The whole is called a "function".

Not one mathematician before defined "function" in this way. This is definitely *new:* such detailed talk of the "values" of the variables. (We remember Bolzano's definition, which was given half a generation later: p. 111.)

Besides, a new entity deserves a new notation! Since Leibniz (and in the tradition of an idea from Descartes), the "variables" in mathematics are usually written as *small* and *italicized* letters, mostly "x", "y", "z". Now Cauchy introduces new and *important* entities. Consequently, it is utterly worthwhile that he introduces a new notation for these entities ("values"). (He clearly asks for this in his *Introduction!*) *Thus Cauchy consistently denotes the "values" of a "variable" in one of the following two ways: either he adds a* lower index *(*e.g. "x_0" *and* "x_1" *denote "values" of the "variable" x) or he uses the same letter written* in capitals and upright: "X", *too, denotes a "value" of x for Cauchy.* To sum up,

> In each case, Cauchy demands for an "independent" variable the precise specification *which* "values" it is allowed to take.

In general, Cauchy writes this: "Suppose x between x_0 and X." This means that the "variable" x takes *all* the "values" between the "values" x_0 and X. Today we write this as "$x_0 \leqq x \leqq \mathrm{X}$" or in the language of sets "$x \in [x_0, \mathrm{X}]$".

We will still have to think about the formulation "we are able to deduce" in his definition of function, but we postpone this to the section after the next. Let's ask first: is there anything else new with Cauchy's concept of function?

Cauchy's Concept of Function Is as Conservative as Possible for a Revolutionary

Apart from introducing the "values" of the variables (and consequently insisting on having a *restricted* domain for the "independent" variable), Cauchy's concept of function does not differ from that of Euler!

This we can *guess* from a comparison of the wordings of both definitions.

This can be *seen* in Cauchy's analysis: all his functions are given as *formulae—just as with Euler!*

And it can be *understood* by comparing Cauchy's definition with that given by Bolzano (some ten years later, p. 111). Bolzano operates *as generally as possible*—Cauchy *stays as closely as possible* to Euler's analysis.

In fact, Cauchy confines himself to transforming the "Calculus of Expressions" ("Algebraic Analysis") of his forerunners as closely as possible into his "Calculus of Values" (or "Analysis of Values"). He did not try to go beyond the intellectual horizon marked by the "Calculus of Expressions".

Considering the undisputed revolutionary nature of Cauchy's departure from the "Calculus of Expressions", one may judge this adherence to tradition to be appropriate or maybe even a clever diplomatic move. However, compared with Bolzano's way of thinking about analysis *and especially about the concept of a function*, Cauchy's thought is ultra-conservative—just as his personal convictions and his ideology.

Cauchy's Concept of the Value of a Function

Let us now return to the formulation in regard to the concept of a function which we have mentioned earlier: the "we are able to deduce". What could be meant by: we "deduce" the value of a function?

The problem is clear-cut. A "function" according to Cauchy (as well as to Euler) is e.g. $\frac{1}{x}$. Which values are there to be *deduced*? Clearly, all those which result from the instruction "Divide 1 by ...". This is simple—only $x = 0$ is excluded, for division by 0 is not possible. So what?

In this case, Cauchy is *forced* to *define* what is to be done—what the "value" of the function *in such a case of exclusion* ought to be.

And Cauchy does, of course! He is completely explicit and declares:

> If a particular case arises in which the given function cannot immediately give the value of the function under consideration, we seek the limit or limits towards which this function converges as the variables approach indefinitely the particular values assigned to them. If there exist one or more limits of this kind, they are regarded as the values of the function under the given hypotheses; however many they may be. We call *singular values* of the proposed function those values determined as we have just described.

Cauchy's Concept "value of a function": A First Example

We consider the function $\frac{1}{x}$. If $x = 0$, we do *not immediately* obtain a value. *Therefore* we will look for *all* the "limits" which we can deduce from $\lim x = 0$: $\lim_{x \to 0} \frac{1}{x} = ?$

This is not difficult: the smaller the value of x becomes, the larger the value of $\frac{1}{x}$ will be. And the sign of the value of $\frac{1}{x}$ is the same as the sign of x; both signs are possible. Cauchy writes an arbitrary large number (in other words, an infinite number) in the same way as Euler: " ∞".

Besides the numbers, ∞ is a "value", too.

The result is $\lim_{x \to 0} \frac{1}{x} = \pm\infty$. —Alright?

A Surprise: Cauchy's "limit" Is Not Unambiguous!

This example (it is given by Cauchy) shows that Cauchy's "limit" is not in the least uniquely defined! It is possible to get *different* "limits" by *different selections* of values of the independent variable(s). In other words,

Cauchy's concept of limit *generalizes* the concept of convergence.

Only in the case when the "limit" is uniquely defined, do the notions of limit and convergence coincide (see p. 110). And because "convergence" for the discrete is analogous to "continuity" for the continuum, we have:

The unique "limit" is a new formulation of "continuity".

The notion of "limit" highlights another aspect of "convergence". (It denotes the *aim* rather than the *way* to that aim.)

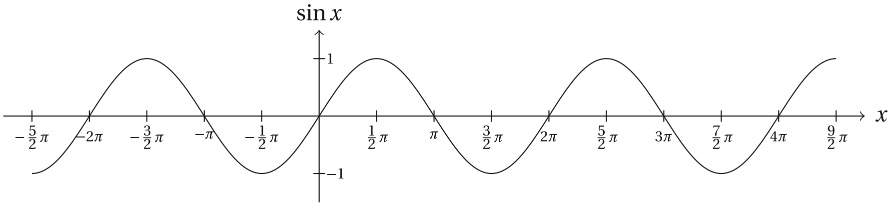
I have to confess: there has been a little bit of cheating in my argument! It presupposed that ∞ was a "limit". This is *commonplace,* also in textbooks of analysis, for it is very convenient.

However, the assumption is *false!* The "value" ∞ has different properties from the "numbers"! *At no time* does a *finite* "value", an ordinary number, approach the "value" ∞ as close as one wishes—which is indeed the very definition of the concept "limit"! Quite to the contrary, the distance of some number a from ∞ is *always infinitely large:* $\infty - a = \infty$.

Nevertheless, it is common practice to take $\lim_{x \to 0} \frac{1}{x} = \infty$, for some positive x, to be *really true.* In other words, when one considers the strict meaning of the concept of "limit", this equation is not *really true.* Instead, it is a—very convenient—*agreement.* Let us also stick to it.

A Second Example Relevant to Cauchy's Concept "value of a function"

For a better understanding of Cauchy's concept "value of a function", let us consider the sine function as a second example. We may visualize it by the following picture, a "curve" in the coordinate system:



The sine function

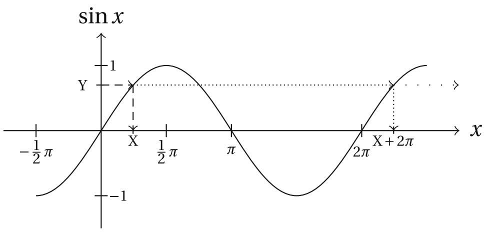
The "value of the function" $\sin x$ takes on every number between -1 and 1 for all the "values" of x taken from any interval of length 2π. The "value of the function" $\mathrm{Y} = \sin \mathrm{X}$ can be determined by a computer program (today's style) or looked up in a table (old-fashioned)—or computed with the help of an infinite series (Leibniz-style).

According to our agreement, ∞ is a "value" too. Therefore, we may ask now: what is the "value of the function" $\sin x$ for the "value" $x = \infty$?

What is $\sin\ \infty$?

The answer is not too hard, or is it? It is *each* value from -1 to $+1$.

A short proof: let Y be any value in the interval $[-1,\ 1]$.



The value of the sine function for $\mathrm{X} = \infty$

We are sure that there exists a value X between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$ with $\sin \mathrm{X} = \mathrm{Y}$. (For this is a property of the sine function.) If $\mathrm{Y} = 0$, we also have $\mathrm{X} = 0$, for we have $\sin 0 = 0$. If $\mathrm{Y} = +1$, we have $\mathrm{X} = \frac{1}{2}\pi$, for we have $\sin \frac{1}{2}\pi = +1$, etc. And we also have $\sin(\mathrm{X} + 2\pi) = \mathrm{Y}$, for 2π is the period of the sine function. Consequently, we have $\sin(\mathrm{X} + 4\pi) = \mathrm{Y}$, too, and so forth.

$$\sin(\mathrm{X} + 2k \cdot \pi) = \sin \mathrm{X} = \mathrm{Y}\ , \qquad \text{for all natural numbers } k.$$

Now we let k become infinitely large: $k \to \infty$, and consequently $2k \to \infty$ as well as $2k \cdot \pi \to \infty$. This amounts to

$$\sin\left(\lim_{k\to\infty}(\mathrm{X} + 2k \cdot \pi)\right) = \sin(\mathrm{X} + \infty) = \sin\infty = \mathrm{Y}\,.$$

However, Y was *any* arbitrary "value" between -1 and $+1$. *Hence,* $\sin\infty$ may be *any* of these values from -1 to $+1$ too. If we write *all values* from -1 to $+1$ as the interval "$[-1,\ +1]$", we get our result as a simple formula:

$$\boxed{\sin\ \infty = [-1, +1]}$$

And this was to be proved! (Cauchy invents another symbol for this, see p. 147, but it means the same.)

Some Very Surprising Consequences from Cauchy's Concept of "value of a function"

The Methodical Significance of Cauchy's Definition of This Concept

A crystal-clear definition of the concept "value of a function" is an absolute must for Cauchy. After all, he did announce: "in determining these conditions and *these values*, I will make all uncertainty disappear". Since Euler, "function" has been the central concept of analysis, the *most important entity* whatsoever. *Therefore,* Cauchy has to declare *unmistakably* which "values" the "function" ought to take without ifs and buts.

Cauchy has met this requirement entirely. We have already examined his definition of the concept "value of a function" (pp. 124f).

The Historical Significance of Cauchy's Definition of This Concept

We emphasize:

Cauchy is the first mathematician to define the concept of "value of a function".

From today's perspective, that seems to be surprising. However, if one is a little bit familiar with the kind of analysis *before* Cauchy, if one knows that the "Calculus of Expressions" set the academic standards before this time, the surprise will disappear. Those who know the "Calculus of Expressions" only a little, will be fully aware that the concept of value was of negligible relevance, playing only a minor role.

Cauchy is the very first mathematician who *explicitly* introduces the concept of "value" into the canon of fundamental concepts of analysis. This is a legacy of his long practice as an engineer during which Cauchy noticed: without numbers (as "values"), the algebraic expressions are of no use! This experience forced him to include the notion of value, which had been neglected until then, and place it at the centre of his theory.

The Political Significance of Cauchy's Definition of This Concept

Cauchy's manner of determining the concept of "value of a function" is of the greatest interest.

What does he say? This: the "value of a function" is *everything that is demanded by the existing conditions of the situation.* Cauchy says: if the situation is not *as clear as daylight by itself* (which means if the value cannot easily be calculated directly), we must take *all imaginable and* FITTING *values.* This is contrary to Bolzano's arbitrary requirement for all the values to be *somehow conceivable*—instead of only choosing those which *fit* to the *given* (existing) conditions. These are the "limits", *all of them.* Cauchy says: if we are not able to calculate the required "value" directly, then we *must exhaust all possibilities of the* EXISTING *conditions completely*—neither more nor less. This is still the generality of Euler's approach.

Do not question the existing limits of thought—this could threaten the order of the world! A greater contrast to Bolzano can hardly be conceived.

The Technical Significance of Cauchy's Definition of This Concept

Cauchy's definition of the concept of "value of a function" is also *in a purely technical sense* of greatest interest, for it is *of mathematical substance.* In other words, it allows for a very simple but powerful proposition, namely the following theorem:

> Fundamental Theorem of Functions. *If the function $f(x)$ with the independent variable x has a unique finite value for the finite value $x = \mathrm{X}$, then it is* continuous *at this value—and conversely.*

We learnt from Bolzano what continuity in the "Calculus of Values" meant (p. 108). Cauchy has precisely the same notion.

On p. 125, we already saw that the *unique* "limit" is the same as "continuity", and this is expressed exactly by the content of this theorem.—

A function is "continuous" for each value that is directly deduced from a calculatory expression. Today, the technical proof of this elementary fact is shown in every textbook of analysis. Its main idea is based on the fact that all values can be determined from a calculatory expression—as the operations are continuous.

Let me add two remarks.

1. The name of this theorem is new. It seems to be suitable.
2. Until today, nobody else has formulated this theorem, not even Cauchy.

What is the reason for that? Why did nobody formulate (and prove) this theorem which is true in Cauchy's analysis until today? Let me name three aspects.

(a) Within Cauchy's analysis, the theorem is *trivial,* a *mere truism.* One only has to recall the *strict analogy* of the concepts "convergence"—*resp. unique* "limit"—and "continuity". However, something which is trivial does not constitute a "theorem" for Cauchy!—Nevertheless, there is a formulation by Cauchy in a letter which is *exactly* the proposition of the theorem (letter to Coriolis from 13. February 1837).

(b) As all mathematicians might have noticed already,

In today's analysis this theorem is wrong!

The reason is that in today's analysis, we use a concept of "value of the function" which coincides with that of Bolzano. As Bolzano's concept of "value of the function" allows for a completely arbitrary determination of the value of a function, this value *cannot* have any regular properties or nature. Therefore the "Fundamental Theorem" *cannot* hold within the conceptual framework of modern day analysis. It is just the *conservative* character of Cauchy's analysis which makes this theorem valid—more precisely, it is the stipulation of Cauchy that the "values of the function" have to be *deduced* and that they are not allowed to be chosen *arbitrarily*. It was only due to the revolutionary Bolzano who mandated this, and we follow this conceptual framework until today, with all the consequences.

(c) Taking both points into account we can conclude: not necessarily Cauchy, but all those mathematicians, *who deal with Cauchy's analysis today,* have every reason for stating this theorem. It *shows explicitly* that and in which way analysis *has changed* from Cauchy until today. The mathematicians who deal with Cauchy's analysis today are the historians of mathematics. It is *they* who have to supply the information why they still refrain from pointing out this fine theoretical change. Since the 1990s (when I published my interpretation of this theorem), they have not risen to the challenge. More precisely, they ignore these facts. (This in turn raises the question of the current state of the history of mathematics as a science: what is its character today? Are the historians of mathematics afraid of mathematics?—We shall return to this.)

Excursus: Preview of a Failed Revolution of Analysis in the Years of 1958 and 1961

In the years 1958/61, analysis had the chance to revolutionize itself, to undergo a conceptual upheaval comparable with the one caused by Cauchy. It was proposed to found the new analysis on a *different concept of number.*

These ideas were published by two German authors and, quite independently, by an American author. The latter was born in Germany to Jewish parents who went into American exile in order to escape German fascism. He later became a mathematics lecturer in the USA.

Cauchy's example shows clearly that the detailed formulation of any analysis also depends on number and, consequently, on the accepted concept of number. The consequence is this:

> The introduction of *another* concept of number—i.e. of numbers with *other properties* than those used by Cauchy and ever since Cauchy—would change the configuration of analysis. The form of the "Analysis of Values" *would have to change* accordingly.

This was exactly the issue when in 1958 the article *An Enlargement of Calculus* by Curt Schmieden (1905–91) and Detlef Laugwitz (1932–2000) and in 1961 the paper *Non-Standard Analysis* by Abraham Robinson (1918–74) were published. Whereas Schmieden/Laugwitz spoke of a "proper enlargement" of calculus, Robinson declared a "proper enlargement of classical calculus" and at once came up with the name for his new theory: Nonstandard-analysis.

History Does Not Recur, Not Even in Mathematics

But 1958 was not 1821. One and a half centuries after Cauchy's textbook, it was *plainly impossible* to reorganize mathematics as completely as Cauchy had done (at least in large parts, with the exception of one aspect which will be mentioned on pp. 135f). Analysis was established far too deeply, in thousands of heads and hundreds of textbooks.

In 1821, the formation of analysis was the duty of a few prominent authorities. In those days, active mathematicians who dealt with analysis knew of each other. Nearly all of them lived in Paris or went there temporarily to do their years of apprenticeship. But by 1958, analysis was a worldwide established subject, at all scientific colleges and universities. Courses and syllabuses existed as well as textbooks and teachers—who were all *trained to think in the same analytical way.* Should all of them *change their minds?* Why? Who could direct or initiate this? Who should execute or control this? Such an expenditure, and for what purpose?—The traditional analysis was not *wrong* or *impracticable.* At worst, it was constructed in a somewhat complicated way (see Chap. 14). This is one of the main reasons used by the proponents of the nonstandard-analysis to persuade others to accept their theory. But *only to change the way of thinking,* maybe to simplify it, and this without producing new results? Of course, this is not enough to overthrow a worldwide church—moreover in a field where the mindset is canonized in all detail in an unparalleled manner. It would have required at least a new truth in order to give such a revolution at least a tiny chance of success. However, nonstandard-analysis did not offer any new truth—but only the old in a new guise. The hope of some pioneering rebels to persuade by pointing to *the existence of some new mathematical subjects* (e.g. to delta-functions, even rational ones!—see p. 240) remained unfulfilled. Delta-functions are a fascination only for specialists.

Under these circumstances a new revolution of analysis in the last third of the twentieth century did not occur. The tanker *Analysis* had grown far too sluggish and big and could not be *reversed* by individuals or by small groups.

> As soon as a system of thought is broadly established as well as institutionalized in a society—and in this way becomes *universally valid*—, it is impossible for a group of persons to take over. Majority is power when there is no absolute rule.

To put it ideologically, this is not a question of mathematical *truth* whatsoever. Nonstandard-analysis is neither more nor less "true" than standard-analysis (as the commonly used form of analysis is called now, for the sake of precision). Just as Cauchy's "Calculus of Values" is neither more nor less "true" than the "Calculus of Expressions" from Euler and Lagrange. The "Calculus of Values" is *more useful* in respect to the actual needs of the time (technical usage) than the former construction. However, it is not *"more true"* in any possible sense. The proofs of its theorems are by no means *more precise* than the proofs of the earlier theorems.

Unfortunately, historians of mathematics very often state the opposite of this, today more than ever before. (Exceptional authors like Henk Bos or Kirsti Andersen only confirm the rule.)

Of course, this is nonsense! Why should we, today, be qualified to think more precisely than our ancestors? There is no obvious reason for this. Majority is not law (or truth). Those who judge in this presumptuous manner are either too lazy to familiarize or not capable of familiarizing themselves with *another* way of thinking—as we are attempting in this book.

A Rebellion of Nonstandard-Analysis?

Many articles dealing with Cauchy's analysis suddenly appeared in the 1970s and 1980s. The reasons for this can only be guessed. One hypothesis is that this hype was a last attempt by the proponents of nonstandard-analysis to justify their theory by trying to place it within a historical tradition. And this in place of offering some new truth: history instead of mathematics?

The idea was: if it could be Nonstandard-analysis is the "true", the "right" understanding of analysis, then it would show its superiority—at least from a moral point of view.

This line of argumentation drew on history. The master plan was to *prove that some first-rate mathematicians were* ESSENTIALLY *doing nonstandard-analysis!*—an idea, no doubt.

The first step was very easy. *All* proponents of analysis living before the twentieth century liked to use the attribute “infinitely small”. However, this was exactly the novelty of Nonstandard-analysis, its characteristic distinction from standard-analysis:

> In Nonstandard-analysis there do exist “infinitely small” (and similarly also “infinitely large”) *numbers*—which do not exist in standard-analysis.

(We will return to this, see pp. 226f, especially p. 227.)

That is why at first sight it was easy for the nonstandard-analysts to live off the reputation of their famous forerunners. In respect to Leibniz, the fine details of his concepts had still not been published, and his phraseology “infinitely small” could therefore be interpreted very freely: namely in a modern nonstandard-analytical way. With Euler and Johann Bernoulli this could be done effortlessly, for they *really* talked about such “infinite” *numbers*—we have read this! Consequently, they could easily be *co-opted* as being *essentially* proponents of Nonstandard-analysis. It did not matter here that neither Euler nor Johann Bernoulli made any attempt to explain how such a concept of number could be *explicated:* as *really great* mathematicians *must think in the right way*—there should be no problem!

Obviously, this practice deviates completely from doing history of science, as it is undertaken here. (By the way, even today there exist fundamentalists of history of mathematics who are unable to understand this way of thinking. A rational discussion with them is completely impossible. —Do you imagine such a controversy *in mathematics?*)

It was of great help when philosophers started to join the debate. Thus, the philosopher of science Imre Lakatos (1922–74) subscribed to the opinion that Cauchy’s style of thought was a nonstandard-analytical one. Lakatos’ in some way risqué arguments could be turned into merely stringent ones (which helped me to some renown after having written a book on it). As a result, the youngest of the triumvirate which started Nonstandard-analysis in the years 1958/61, Detlef Laugwitz, took the plunge in construing Cauchy as a *true* nonstandard-analyst, who actually thought within these concepts. Laugwitz succeeded in giving many beautiful mathematical arguments, but he failed in his *intent,* for:

> The interpretation of Cauchy’s “Calculus of Values” as an (early) state of Nonstandard-analysis can be proven to be false.

The reason of this will be explained shortly.

After Leibniz, Euler and Johann Bernoulli—as just stated—, there still remained Cauchy. Needless to say, Cauchy also uses the attribute “infinitely small”. Yet, there is something else!

A Digression Within the Excursus: Looking Back at a Criticism of Cauchy

Five years after Cauchy's textbook of analysis appeared, a claim of the young mathematician Niels Henrik Abel (1802–29) was printed which says that Cauchy's book contains a false theorem. (This will be spelled out in some detail on pp. 140f.) However, there was no direct reaction from Cauchy to this reproach.

Consequently, the analysis of the year 1826 was faced with a factual problem: is this theorem true or false? Cauchy had stated and proved it—Abel had pronounced it wrong. Two nowadays still very much appreciated mathematicians contradicted each other in their judgement.

24 years later, a treatise from Philipp Ludwig Seidel (1821–96) was published which also criticized that theorem of Cauchy. It also stated and proved an alternative theorem. Again, there was no reaction from Cauchy.

However, three years later (in 1853), Cauchy published a treatise which contained—in some hidden way, which we will examine later—his self-defence in this matter. Because Cauchy did not *explicitly* label this statement to be a "self-defence", it remained unnoticed. Only much later, when mathematicians finally realized the significance of Cauchy's treatise of 1853, they started to discuss it (p. 140).

Back to the Upheaval of Nonstandard-Analysis

With the birth of Nonstandard-analysis 1958/61, a quarrel had started which finally reached a climax in the 1970s and 1980s. Today this controversy is completely over and this to such an extent that it could not offer any contribution to a new idea concerning Cauchy's theorem. The controversy is:

> Which of the two alternatives is true?
>
> 1. Cauchy stated and proved in his textbook a (very elementary) false theorem.
> 2. Cauchy's theorem is true—because by "convergence" Cauchy meant not the same as we do today, but instead he meant what we call nowadays "uniform convergence".

The first alternative is not plausible at all. Inevitably, every mathematician makes mistakes. But a mistake such as this? A mistake regarding a theorem studied in the first course of analysis—which did not lead to a proper defence from its author, and this even after the publication of a criticism? Who should believe in this unlikely narrative?

Subsequently, the nonstandard-analyst s felt newly empowered: they simply changed the interpretation of the decisive concept of this theorem from Cauchy's (for him it is "convergence") to mean now "uniform convergence"—and that the theorem *in this new meaning* is correct is disputed.

However, the nonstandard-analysts clearly remained a clear minority. Therefore, the alternative (1) remained the dominant opinion and Cauchy became the most popular half-wit in the textbooks of analysis.

Walking on Very Thin Ice

The question for the nonstandard-analysts was: how could they justify changing the interpretation of Cauchy's concept? Their idea showed great promise. They argued: *Cauchy did not understand by "number" what everybody else (for at least one hundred years) thought of the concept but, quite contrarily, what we, the nonstandard-analyst s mean by it* TODAY!

From the mathematical point of view, they had succeeded. The reason being: the range of numbers within Nonstandard-analysis is considered to be definitely greater than the range of real numbers, say: the decimals (however, see chapter 14). And in the theorem under consideration Cauchy makes a certain supposition: he demands that a "series" is "converging" *for all values* (and he meant for all *real* numbers). However, if by the "for all numbers" he meant *more* than we thought until now (namely besides the "real" numbers also the "hyper-real" numbers), then this surplus in the assumption allows for a surplus in the conclusion, and that is, indeed, the claim of the theorem.

This mathematical argument is indubitably correct. Robinson had published this reasoning already in 1963, and Lakatos, this ingenious propagandist of his own ideas, exaggerated its importance, especially in philosophy of science. (To be honest, once I joined in this enterprise.)

Yet, the decoding and interpretation of former mathematical concepts is neither the duty nor a self-evident competence of a pure mathematician! Instead, it is a main task for the historians of mathematics.

However, a detailed historical investigation of the concepts used by Cauchy within his analysis was not forthcoming. Nobody undertook an inquiry of Cauchy's concept of number to check whether he *really* considered such exotic entities like "infinitely small" *numbers.* And, of course, Cauchy did not do that!

In any case, Nonstandard-analysis needs to perform some acts of conceptual acrobatics in order to construct "hyper-real" numbers in a mathematically acceptable way. Here Robinson's construction stands out, but without studying (at least) one semester of modern logic, one is not able to follow his construction. It is, however, unlikely that Cauchy should have anticipated such a stilted concept in 1821 in any possible sense. For this, we need a further supporting argument! But nobody has supplied one: instead there were plenty of rhetorical fireworks!

In short, this attempt by some nonstandard-analysts to justify their theory by recourse to the development of analysis remained without historical substantiation. (This is true, although at the time of my dissertation, around 1980/81, and some years later, I was convinced of the opposite—and so were the referees of my thesis, who were mathematicians.)

Not earlier than 1990, I undertook a thorough examination of those concepts which Cauchy *really* used. My result can be summarized as follows: if you have to choose between two possibilities, just take the third! Or, to put it somewhat more technically, the translation of Cauchy's concept of "convergence" in modern analysis is neither "convergence" nor "uniform convergence"—but a third notion. This will be explained in the next section.

Cauchy's Concept of Convergence: A Big Misunderstanding

Strictly speaking, the discussion of Cauchy's concept of convergence belongs to the section which is called "some very surprising consequences from Cauchy's concept of 'value of a function'", but because of its huge importance it will be considered separately.

A Mystery of History: Cauchy's Concept of Convergence

There is no other concept of analysis which caused such long, comprehensive, and vehement controversies and which also went beyond the field of history of mathematics, than Cauchy's concept of "convergence". Why is this?

The decisive point is: although Cauchy's analysis is shaped very revolutionarily, it remains firmly tied to established thinking—which is thoroughly alien to us today.

Let us return to Cauchy's concept of function (p. 123). Which basic notion does he use? It is "quantity". We should bear in mind:

Cauchy's first and unique foundational concept of analysis is "quantity". (From that he also deduced the concept of "number".)

As a consequence, Cauchy determined "convergence" for those objects, exclusively for quantities.

Originally Cauchy's concept of "convergence" applies to "quantities".

(To overlook this has been the decisive mistake in the mathematical–historical debate following the creation of Nonstandard-analysis and its efforts to co-opt Cauchy since the early 1960s. However, *today* no mathematician is able to give an account of this concept: "quantity". Cauchy's definition of "quantity" can be read on p. 118.)

Here is Cauchy's definition of "convergence":

We call a *series* an indefinite sequence of quantities,

$$u_0, \quad u_1, \quad u_2, \quad u_3, \quad \dots,$$

> which follow from one to another according to a determined law. These quantities themselves are the various *terms* of the series under consideration. Let
>
> $$s_n = u_0 + u_1 + u_2 + \ldots + u_{n-1}$$
>
> be the sum of the first n terms, where n denotes any integer number. If, for ever increasing values of n, the sum s_n approaches a certain limit s, the series is said to be *convergent,* and the limit in question is called the *sum* of the series.

The u_k are "quantities", e.g. "functions" $f_k(x)$ (Cauchy still uses the comma in its traditional meaning of the plus sign):

$$f_0(x) + f_1(x) + f_2(x) + f_3(x) + \ldots$$

We realize that, in the case of series, Cauchy deviates from his usual way of signification; in this case, the index does *not* indicate a "value" but instead belongs for practical reasons to the name of the "quantity". (Here the name "f"—in Cauchy: "u"—, supplied with an index, does not denote a variable! It is merely a "variable" *name.*) In regard to Euler and Lagrange we have already seen how difficult it is to deal with series without the index notation: pp. 75 and 95.

Starting from a "sequence", Cauchy now constructs *finite* sums $s_n(x)$:

$$s_n(x) = f_0(x) + f_1(x) + f_2(x) + \ldots + f_{n-1}(x) .$$

If the $s_n(x)$ approach a "limit", this is called "convergence". Written in Cauchy's way, we have

$$\lim s_n(x) = s(x) .$$

Before we proceed with the technicalities, we need to discuss an emerging problem: Cauchy speaks of a "limit". However, a "limit" is a value. Yet this $s(x)$ is a *quantity (as can be seen by the* "x"), *no value.* Notwithstanding, Cauchy *speaks* of the *"limit"!* The only possible explanation is that Cauchy means "the *value* of $\lim s_n(x)$ for a $x = \mathrm{X}$"!

Consequently, Cauchy's definition of "convergence" is this:

> A series of functions
>
> $$f_0(x) + f_1(x) + f_2(x) + f_3(x) + \ldots$$
>
> is called "convergent" for the value $x = \mathrm{X}$, if for increasing n the sum
>
> $$s_n(x) = f_0(x) + f_1(x) + f_2(x) + \ldots + f_{n-1}(x)$$
>
> has for $x = \mathrm{X}$ a unique "limit".

It is possible to reformulate this definition. Instead of "$\lim s_n(x) = s(x)$", we may write "$\lim(s_n(x) - s(x)) = 0$", couldn't we? Using Cauchy's notation for this difference, which is still in use today, "$r_n(x)$" ("reste" = "rest"), we get:

> A series of functions
>
> $$f_0(x) + f_1(x) + f_2(x) + f_3(x) + \ldots$$
>
> is called "convergent" for the value $x = \mathrm{X}$, if for that value the special sum (the "rest")
>
> $$r_n(x) = f_n(x) + f_{n+1}(x) + f_{n+2}(x) + \ldots$$
>
> we have
>
> $$\lim r_n(x) = 0\,.$$

The Solution of the Mystery

Until 1990, there has been a consensus among all Cauchy specialists about this formulation. But since 1990, there has existed a new idea, arising from *two open questions:*

1. *What* IS " $r_n(x)$" *for Cauchy?*
   The only possible answer is a *"variable",* depending on the TWO "independent" variables n AND x.
2. *What* IS " $\lim r_n(x)$" *for Cauchy?*
   Again the answer is simple: $\lim r_n(x)$ as a mathematical notion in Cauchy's analysis is a *"value of a function"* and, consequently, "the limit or the limits"—to Cauchy: *all* of them!—of the variable $r_n(x)$ (for the value $x = \mathrm{X}$).

And that is the point! By "$\lim r_n(x)$ for the value $x = \mathrm{X}$", Cauchy *not only* means

$$\lim_{n\to\infty} r_n(\mathrm{X})\,,$$

but moreover, there are also *all* "limits" to be included:

$$\lim_{\substack{N\to\infty \\ x_k\to\mathrm{X}}} r_N(x_k)\,, \qquad (\|)$$

for $r_n(x)$ is a function of the *two* variables n and x.

Cauchy *never* writes subscripts when he uses "lim". Only later did it become customary. The example above shows *why it was superfluous for Cauchy:* he always means *all* possible limits. Our example also shows that this notation—which today is a *must*—is generally somewhat laborious. Thus, as Cauchy usually means all limits, he can use "lim" unindexed.

The Mathematical Significance of This Solution

We remember Cauchy's theorem and that it has been considered false (except by nonstandard-analysts) until today. It reads literally (only the sign "X" is added two times):

Theorem I. *When the various terms of the series*

$$u_0, \quad u_1, \quad u_2, \quad \ldots, \quad u_n, \quad u_{n+1}, \quad \ldots$$

are functions of the same variable x, continuous with respect to this variable in the neighbourhood of a particular value X *for which the series converges, the sum s of the series is also a continuous function of x in the neighbourhood of this particular value* X.

In the special literature, sometimes the name "Cauchy's Sum Theorem" is used.

The decisive assumption of this theorem is that the series of functions "converges" and this also "in the neighbourhood" of the particular value X.

Today we can offer this assumption in the terms of all *three* rivalling interpretations:

1. The standard-analysts understand Cauchy using the concepts which are customary today. Which means they say "$\lim_{n\to\infty} r_n(\mathrm{X}) = 0$".
2. The nonstandard-analysts understand Cauchy using their concepts and say "$\lim_{n\to\infty} r_n(\mathrm{X}') = 0$ *for all values of* X' *in the neighbourhood of the value* X *in question*."
3. According to the new interpretation, obtained from an investigation of Cauchy's concepts, we reach the conclusion that Cauchy means what is formulated in line ||. (By "neighbourhood", Cauchy means "$\mathrm{X} + \alpha$" instead of other "values" $\mathrm{X}' \neq \mathrm{X}$.)

Stated in the terminology of *modern* analysis, this amounts to the following:

1. In his theorem, Cauchy presupposes the "convergence" of the series and consequently the theorem is *wrong*.
2. In his theorem, Cauchy assumes the "uniform convergence" of the series and consequently the theorem is *true*.
3. In his theorem, Cauchy demands the "continuous convergence" of the series and consequently the theorem is *true*.

The *exact* definition of "continuous convergence" can be found in some textbooks of the theory of functions, published since 1921. For the understanding of the historical development, it is not necessary to understand this definition; it is enough to accept the following box.

In case (1), Cauchy looks bad. In case (2), the standard-analysts look bad. In case (3), after having investigated Cauchy's concepts, the historian of mathematics reaches the judgement: Cauchy did not make a mistake here, but he proves something other than the standard- as well as the nonstandard-analysts think.

In the terminology of modern analysis, we state the following ("$\Rightarrow$" means "leads logically to", and *not one* of the following inversions is true):

continuous convergence	$\Rightarrow$	uniform convergence	$\Rightarrow$	convergence

The *weakest* presupposition (on the right side) is not strong enough to prove the theorem. The presupposition *in the middle* is stronger and suffices—however, there is no clear evidence in Cauchy's text that he might have meant this. The *strongest* presupposition (on the left) is Cauchy's *true* condition—and it is all the more sufficient to prove his assertion.

Cauchy's Proof of His Theorem

We finish our argument by replicating Cauchy's proof of his theorem. The proof is simple.

The claim is that the sum $s(x)$ is continuous at a value $x = \mathrm{X}$. Continuity means $s(x + \alpha) - s(x)$ decreases (for the value $x = \mathrm{X}$) together with α and becomes arbitrarily small. This has to be proven:

$$\lim \big(s(x + \alpha) - s(x)\big) = 0 \qquad \text{for} \qquad \lim \alpha = 0 \quad \textit{and} \quad \text{for the value} \quad x = \mathrm{X}\,.$$

This is a consequence which follows directly from the usual representation of s as $s = s_n + r_n$:

$$s(x + \alpha) - s(x) = s_n(x + \alpha) - s_n(x) + r_n(x + \alpha) - r_n(x)\,. \tag{**}$$

Since the s_n are finite sums of continuous functions, $s_n(x + \alpha) - s_n(x)$ is a finite sum of infinitely small quantities and, consequently, infinitely small. Only the two other summands remain $r_n(x + \alpha)$ and $r_n(x)$—not to forget, in each case for the value $x = \mathrm{X}$.

However, we have $\lim r_n(x) = 0$ (for the value $x = \mathrm{X}$): this is *exactly* the presupposed "convergence" of the series of functions for the value $x = \mathrm{X}$.

Similarly, we know $\lim r_n(x + \alpha) = 0$ for the value $x = \mathrm{X}$, *because of Cauchy's concept of "value of a function"* —here for the value X! For we know "$f(\mathrm{X})$" are *all* limits $\lim f(\mathrm{X}')$ for $\mathrm{X}' \to \mathrm{X}$, and this can be written as $\lim f(\mathrm{X}+\alpha)$ for $\alpha \to 0$, or just for our function $r_n(x)$: $\lim r_n(\mathrm{X} + \alpha)$ for $\alpha \to 0$. This is required to be $= 0$ *because of the assumed "convergence"—in Cauchy's sense!*—FOR THE VALUE X. End of the proof.

The very last step has always been controversial:

1. The standard-analysts read "$x+\alpha$" as VALUES *different from* X and say Cauchy cheats by assuming "convergence" for values *different* from the value X—but this is not permitted. However, for Cauchy "$x + \alpha$" is *always* a "variable" and *never* a "value"!

2. The nonstandard-analysts argue no problem, α is an "infinitely small" *number*—and these are included in Cauchy's presupposition of "convergence" *even in the neighbourhood*. And this fits!—however, not to Cauchy's way of thinking.

It is impossible clearly to demonstrate the concept of "infinitely small" *number* within Cauchy's texts. But Cauchy offers a concept of "value of a function" (in detail, *all* the limits!) nobody had ever thought of. Today this concept is unknown, and before 1990 nobody read Cauchy's textbook *very carefully.*[1] And because of Cauchy's concept of "value of a function", "$\lim r_n(x+\alpha) = 0$ for the value $x = \mathrm{X}$". This is guaranteed by the presupposition of "convergence (at the value $x = \mathrm{X}$)". —Bingo!

Cauchy's Self-Defence

As stated earlier, Cauchy later published a treatise in which he answered his critics. He did so in a very polite manner and did not name them (nor their criticism) explicitly. Niels Henrik Abel had criticized Cauchy's theorem in all detail in 1826 (see p. 133). Abel declared the theorem to be wrong. However, he did not *prove* this, but merely *claimed* that a particular function is a counter-example (see p. 156).

In a treatise from 1853 Cauchy offers a calculation which *proves* Abel's claim to be false.

In place of Cauchy's somewhat demanding calculation,[2] we transcribe it and Abel's counter-example into a simpler version.

Abel says something like the following. The series

$$x^0 + (x^1 - x^0) + (x^2 - x^1) + (x^3 - x^2) + \ldots$$

converges for each value of x between 0 and 1 inclusively. Its single terms $x^{n+1} - x^n$ are continuous. However, the whole sum is not, and therefore it is a counter-example to Cauchy's Sum Theorem of p. 138!

Let us examine this argument. Abel is correct in saying that the terms $x^{n+1} - x^n$ of this series are continuous, but that the sum is not. Why the latter? If we let $0 \leqq x < 1$, this sum has the value 0, for we have

$$s_n(x) = x^0 + (x^1 - x^0) + (x^2 - x^1) + \ldots + (x^n - x^{n-1}) = x^n \,,$$

and consequently (for $x = \mathrm{X}$ and since $0 \leqq \mathrm{X} < 1$)

[1] Not even the translators Robert E. Bradley and C. Edward Sandifer in 2009 did!

[2] See *Die Analysis im Wandel und im Widerstreit*, p. 352.

$$s(\mathrm{X}) = \lim s_n(\mathrm{X}) = \lim_{n\to\infty} \mathrm{X}^n = 0\,. \tag{††}$$

However, for $x = 1$, the series is plainly one:

$$s(1) = 1 + (1-1) + (1-1) + (1-1) + \ldots = 1\,.$$

Subsequently, *this* function $s(x)$ is discontinuous for the value $x = 1$, because for $\mathrm{X} < 1$ we have from ††:

$$\lim_{\mathrm{X}\to 1} s(\mathrm{X}) = \lim 0 = 0\,,$$

whereas we have $s(1) = 1 \neq 0$. In Abel's mindset, this disproves Cauchy's theorem.

Nevertheless, this does not follow from Cauchy's way of thinking! Quite contrarily: Cauchy is able to *prove* that this series of functions does *not* "converge" for the value $x = 1$—namely in the *special* sense in which Cauchy had established the concept of "convergence"! But if this series *does not meet the presuppositions of his theorem,* his theorem cannot state anything about it.

To conclude, we will show that our example of a series for the value $x = 1$ does not "converge" in Cauchy's understanding of that notion. This is quite easy. We only have to study $\lim r_n(x)$ for the value $x = 1$. We have

$$r_n(x) = (x^{n+1} - x^n) + (x^{n+2} - x^{n+1}) + (x^{n+3} - x^{n+2}) + \ldots = -x^n\,.$$

Is $\lim r_n(x) = -x^n$ for the value $x = 1$ also $= 0$?—No, for the value $x = 1$, we get $\lim r_n(1) = \lim -1^n = -1 \neq 0$. Quite simple!

Abel's counter-example (which is technically somewhat more complicated) cannot be refuted that easily. An additional consideration is needed. This additional consideration can also be explained by using our simple example (the original argument was given by Cauchy in 1853). It runs like this. Instead of taking $x = 1$, we investigate the $\lim r_n(x)$ for $x = 1 - \alpha$ and $\lim \alpha = 0$. All the values $1 - 1/n$ are < 1 and we have $\lim_{n\to\infty}(1 - 1/n) = 1$. Therefore $\lim_{n\to\infty} r_n(1 - 1/n)$ is also a "value of the function" $r_n(x)$ for $x = 1$. However, it is (compare with the formula on p. 77, for $k = 1$, we get $a = e$, and also, choose $x = -1$ *resp.* $i = -\frac{1}{\omega} = \mathrm{N}$):

$$\lim_{\mathrm{N}\to\infty} r_{\mathrm{N}}(1 - 1/\mathrm{N}) = \lim_{\mathrm{N}\to\infty} -\left(1 - \tfrac{1}{\mathrm{N}}\right)^{\mathrm{N}} = -(e^{-1}) \neq 0\,.$$

A completely flawless calculation—which constitutes in Cauchy's analysis the *proof* that this chosen series, *in his understanding* does *not* "converge" for the value $x = 1$! (Cauchy only skipped the part "$\lim r_n(x) =$" in his calculation—this step he assumed to be clear to his learned readers.) Clearly, the meaning of Cauchy's calculation can *only* be understood if one relies on his concept of "convergence" *as well as*—and at least equally importantly!—his concept of "value of a function".

However, his critics have refused to do this: in the nineteenth, in the twentieth, and (until now) in the twenty-first century.

By the way, this example of a series (just like that which Cauchy had provided) is *not "uniformly convergent"*. That is why it (as well as Cauchy's series), sadly, is not suited to refute the claim of the nonstandard-analysts, that Cauchy had thought of his "convergence" in the same way as them, mathematically. That is a pity, because a *technical* argument would be considered by many as mathematically more conclusive than a merely *philosophical* one.

What Are the Reasons for the Prevailing Misunderstanding of Cauchy's Notion of Convergence?

Why do modern mathematicians not understand Cauchy's concept of "convergence"? The brief answer is: because they are no historians of mathematics. A more detailed answer consisting of two parts is given below:

1. Cauchy has defined this concept for "quantities"—whereas we (in analysis) define it *exclusively* for "numbers".
2. Cauchy's concept of "value of a function" is quite different from that which we use today (p. 124), a fact that has stayed unnoticed until now.

We have reached the result: if we take Cauchy at *his* word, the quarrel about his theorem which has now lasted for more than 96 years has finally been resolved. Strangely enough, today nobody is interested: each publication of this explication remained without professional resonance. Will this change now?

Today it is even possible to accuse Cauchy—by ignoring his conceptual limits—of *new "errors"* in his analysis and to publish this in a scientific journal viewed to be first rate, like *Historia Mathematica*. And even worse, the named journal refused to publish a criticism of such an "erroneous" article. (The editor then decided, relying on the—clearly contra-factual—argument, this journal would not publish "Letters to the Editor". However, I did write an article, not a letter, but it was not even reviewed.) From this, we may conclude that history of mathematics is not a separate subject, but only the servant of ideological warfare. It cannot be part of an independent scientific discourse.

Cauchy's Concept of Derivative—Again a Misunderstanding

The notion "derivative" was introduced by Lagrange. Lagrange defined it as a "function" $f'(x)$ which can be determined from the series expansion of the function $f(x)$ by changing the argument, i.e. from the series expansion of $f(x+a)$ (formula § on p. 96).

Instead of working with "derivatives", Euler operated with "differential quotients", which he took to be *true* quotients made from two "infinitely small" quantities. This is a complicated notion and was therefore avoided by Lagrange.

As we have read in the preface of his textbook on analysis, Cauchy made a point of "making all uncertainty" of the formulae "disappear" by the "determination of the values". Cauchy follows this principle also in case of the derivative. He introduces a formulation which is still, today, presented to the beginners—even though, again, in another meaning.

Cauchy refers to the fact that for a "continuous" function $f(x)$, an "infinitely small" increase α of the variable x (i.e. he takes $x + \alpha$ and demands $\lim \alpha = 0$) produces an "infinitely small" increase of the function. Cauchy likes to denote these "increases" by capital delta (Δ), i.e. $\alpha = \Delta x$ as well as $f(x + \Delta x) - f(x) = \Delta y$. We read Cauchy (by "the two terms", he always means numerator and denominator of the fractions):

> By consequence, the two terms of the *ratio of differences*
>
> $$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$
>
> will be infinitely small quantities. But, while these two terms indefinitely and simultaneously will approach the limit zero, the ratio itself may be able to converge to another limit, either positive or negative. This limit, when it exists, has a determined value for each particular value of x; but, it varies along with x.

Cauchy defines this "value" of $\lim \frac{\Delta y}{\Delta x}$ for $\lim \Delta x = 0$ to be the "value" of a new "function":

> The form of the new function of the variable x, which will serve as the limit of the ratio $\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$, will depend on the form of the proposed function $y = f(x)$. To indicate this dependence, we give to the new function the name of *derived function,* and we represent it, with the help of an accent mark, by the notation
>
> $$y' \qquad \text{or} \qquad f'(x)\,.$$

As already stated above, even today the "derivative" (as we call the "derived function" of Lagrange and Cauchy today) is introduced to beginners in this way. However, as Cauchy's thinking *basically* differs from that of ours today, we must add two things.

1. To say only the least, in Cauchy's way of thinking, this manner of defining a "function" is problematical because Cauchy establishes a "function" to be a "variable" which has certain properties (see p. 123). However, the "derived function" is *not* defined to be a certain variable, but it is constructed as a value-to-value relation. (At most one may, *in hindsight, call* that value-to-value relation, defined by Cauchy, a "variable".)

> *The manner of defining a "function",*
>
> $$X \longmapsto f'(X)\,,$$
>
> *is very modern* (to be precise, this *is* the method we use today), *but it definitely does not fit to the concept of "function" given by Cauchy!*

In other words, by giving this notion of the "derived" function, Cauchy violates his own conceptual framework. Strictly speaking, in Cauchy's thinking, his "derived function" is no "function" at all!

2. What is Cauchy's *precise* definition of "derivation"? It is that he determines which is the (uniquely defined!) "value" $f'(X)$ for the "value" $x = X$—namely the value which is the "limit of $\frac{f(x+\Delta x)-f(x)}{\Delta x}$".

> The value $x = X$ produces the value $\lim \frac{f(x+\Delta x)-f(x)}{\Delta x}$ for $\lim \Delta x = 0$.

However, we must also take into account that Cauchy takes each value X (also) as a limit $\lim x_n = X$! In other words, Cauchy does mean the following:

> The value $x = X$ produces the (unique!) value $\lim \frac{f(x_n)-f(\bar{x}_n)}{x_n-\bar{x}_n}$ for $x_n \neq \bar{x}_n$, $\lim (x_n - \bar{x}_n) = 0$ (with $\lim x_n = X = \lim \bar{x}_n$).

Nonetheless, in today's language, this is the definition of *"continuous derivability"!* This entails that Cauchy's definition of derivability *demands* that the derived function $f'(x)$ is continuous. (And that even though his *formula* is the same as ours!—However, we already know that Cauchy's concept of "value of a function" differs from that of today—and this must have mathematical consequences. Mathematics involves not just formulae but also the *interpretation* of the formulae. In Cauchy's words: we must make the "uncertainty" of the formulae "disappear".)

After having realized that Cauchy's concept of convergence is equivalent to today's concept of continuous convergence, we shall find it evident that Cauchy uses also the modern concept of "continuous derivability" *instead* of our "derivability".

Let me add two remarks, a historical one and a mathematical one.

(a) The historical one: Laugwitz argued already in 1987 that Cauchy's "derivability" is our modern "continuous derivability". His argument was different and I was unable to accept it. Unfortunately, even in *Die Analysis im Wandel und im Widerstreit*, pp. 333–335, I had not understood Cauchy correctly.

(b) The mathematical one: Strangely, in today's lectures for beginners, it is not mentioned that the formula

$$\lim_{\substack{x_n \to x_0 \\ \bar{x}_n \to x_0 \\ x_n \neq \bar{x}_n}} \frac{f(x_n) - f(\bar{x}_n)}{x_n - \bar{x}_n}$$

determines the "continuous derivability" of the function f at the value x_0. In 1965, Paul Lorenzen (1915–94) called this concept, which was introduced by Guiseppe Peano (1858–1932) in 1892, "free derivability".

Cauchy's Concept of the Integral

In the "Algebraic Analysis", the integral was introduced as the inverse operation of differentiation, i.e. as the "indefinite integral"—and, consequently, as a "function".

Cauchy ends this tradition by introducing the "definite integral"—which is to say the integral as a "value".

We have already studied Leibniz' method of calculation of the area when one side is curved (pp. 29f) and seen that this method of Leibniz has only been known since the late twentieth century. Subsequently, if Cauchy proceeds in a similar manner, he could not have been influenced by Leibniz (or one of his followers).

This similarity is due to the visualization of the geometrical construction, not to the concepts.

- Where Leibniz takes a "curved line", Cauchy deals with a "function" $f(x)$ of an independent variable x.
- Where Leibniz introduces "points of division" B_k on the horizontal below, Cauchy sets forth "values of division" x_k in the assumed interval between x_0 and X.
- Where Leibniz uses "straight lines" $D_k\ B_k$, Cauchy relies on "values of a function" $f(x_k)$.
- Both take different quantities for their approximation:
  - Leibniz takes an "area" which *nearly fits,* namely the rectangles $B_k\ P_k$.
  - Cauchy decides to choose the suitable approximation $(x_{k+1} - x_k) \cdot f(x_k)$, which is a "value".
- And of course, where Leibniz *proves* that his approximate area differs from the true area by a quantity which can be made smaller than any given quantity, Cauchy *defines* the analytical concept of the "definite integral" as a "limit":

$$\int_{x_0}^{\mathrm{X}} f(x)\,dx = \lim_{n\to\infty} \sum_{k=0}^{n-1} (x_{k+1} - x_k) \cdot f(x_k)\,.$$

  Thereby Cauchy assumes $f(x)$ to be a "continuous" function. (The subscript "$n \to \infty$" is of course missing in Cauchy.)

To define something is an easy exercise. The real difficulty is to prove that the definition *makes sense.* In our case, "making sense" is to show that the defined "limit" is well-defined and unique—in other words, that this limit (a value) does not depend on the method of calculation, that there are many ways to arrive at that limit or, to be precise, that each division $x_0, x_1, x_2, \dots\ x_n = \mathrm{X}$ of the interval from $x = x_0$ to $x = \mathrm{X}$ produces the same value.

In other words, there are two different approaches with their own intricacies. Whereas Leibniz had to deal with the estimation of the greatest possible error, Cauchy is concerned with comparing those limits that result from different divisions of the interval and subsequently has to prove that they are all equal.

We may skip the details here, for they can be looked up in any modern textbook of analysis. We shall only examine the *structure* of Cauchy's argumentation.[3]

Cauchy's Basic Idea in His Proof of the Existence of the Definite Integral

The decisive step of Cauchy's proof is the validity of this equation:

$$\sum_{k=0}^{n-1}(x_{k+1} - x_k) \cdot f(x_k) = (\mathrm{X} - x_0) \cdot f(x_0 + \theta \cdot (\mathrm{X} - x_0)) .$$

- On the left, we have the sum of products $(x_{k+1} - x_k) \cdot f(x_k)$. Geometrically, these are the "steps" *below* the graph of the function, in case of "increasing" functions as those by Leibniz.
- On the right, we have just one product, consisting of the length of the whole interval and the value of the function at the value $x = x_0 + \theta \cdot (\mathrm{X} - x_0)$, where θ is an unknown value between 0 and 1. For such a θ, $x = x_0 + \theta \cdot (\mathrm{X} - x_0)$ is *any* value inside the interval from x_0 to X. —That this is possible, i.e. that a value with this property really exists, is guaranteed by the so-called (and very famous) *"Intermediate Value Theorem"*. Bolzano formulated this theorem very precisely and proved it in 1817. We did not touch on this part of Bolzano's treatise. The decisive point is that the Intermediate Value Theorem is only valid for "continuous" functions. That is why Cauchy demands the continuity of $f(x)$ in his definition of the definite integral $\int\limits_{x_0}^{\mathrm{X}} f(x)\,dx$.

Let us summarize. A (*finite*) sum of products $(x_{k+1} - x_k) \cdot f(x_k)$ may be replaced by a single product which is made from the length of the interval and some value of the function inside of the interval.

The problem with this proof is the comparison of *different* divisions of the interval such as $x_0, x_1, x_2, \ldots\ x_n = \mathrm{X}$ and $x_0, x_1', x_2', \ldots\ x_m' = \mathrm{X}$. The idea is to "refine" each of these divisions in such a way that both refinements coincide. This is not hard but a little bit technical.

For each part of such a common "refinement"—say $x_0, x_1'', x_2'', \ldots\ x_l'' = \mathrm{X}$—we have the above equation and in this way is the sum of products with x_i''-intervals transformed to a single product of length $(x_{k+1} - x_k)$.

The rest goes without saying.

[3] It is shown in *Die Analysis im Wandel und im Widerstreit*, pp. 339–343.

One cannot deny that Cauchy's proof of that useful equation which transforms a sum of products into a single product (with the length of the interval as one factor) is remarkable. To do so, Cauchy relies on a very general theorem regarding the estimation of quantities which he had already stated in his first textbook of analysis. (His textbook on the calculus of integrals was printed only eight years later, in 1829.)

When adapted to a form which is required for the proof of the existence of the definite integral, this theorem reads:

> Theorem. *Let* $b, b', b'' \ldots$ *denote* n *quantities of the same sign and* $a, a', a'', \ldots$ *be the same number of arbitrary quantities, then we have*
>
> $$\alpha b + \alpha' b' + \alpha'' b'' + \ldots = (\alpha + \alpha' + \alpha'' + \ldots) \cdot \mathrm{M}(b, b', b'', \ldots),$$
>
> *where* $\mathrm{M}(b, b', b'', \ldots)$ *denotes any quantity in between the greatest and the smallest of the* $b, b', b'', \ldots$.

It is very clear that "quantity" is a forceful basic concept in Cauchy 's analysis.

What Is Cauchy's "*x*"?

A final summary:

> Cauchy splits the formerly *unique* "*x*" of the old "Calculus of Expressions" for his new "Calculus of Values" into *two*:
>
> 1. a *small* "*x*" *without index* to denote the *"independent" variable,*
> 2. a *small and indexed* "*x*" or a *capital (and* upright*)* "X" to denote a *"value"* of the independent variable.
>
> In "determining precisely" these conditions he makes "all uncertainty disappear".

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