Mathematical Modelling of Anomalous Dynamic Processes in Geomedia

V. K. Kazankov, S. E. Kholodova, and S. I. Peregudin

Abstract The problem of the possibility of the existence of a special nonlinear effect arising in the marine environment, called "rouge waves", is considered. Rouge waves are a phenomenon that cannot be described by means of the apparatus of linear wave theory, the existence of which is beyond doubt. There are various hypotheses explaining the occurrence of rouge waves, but there is no generally accepted point of view about the nature of their occurrence. The paper presents a formal apparatus that generalizes the concept of a dynamic system, in which it is possible to formulate the necessary conditions imposed on the system that determine the occurrence of a rouge wave.

Keywords Mathematical modelling · Dynamic processes in Geo-environments · Dynamic systems · Rouge waves · Time series

1 Introduction

It is known that abnormally large waves occur in the oceans, which have different shapes and profiles, but there are also characteristic features, such as sudden occurrence, a relatively short time span of life and a huge destructive potential. This phenomenon was called rouge waves [1, 2]. A few decades ago, rouge waves were perceived as a myth. Only the existence of "extreme waves" was allowed, as a phenomenon similar at first glance to a phenomenon whose occurrence does not go beyond the statistical distribution of random wind waves. The precedent for the independent study of rouge waves was the "New Year's Wave", registered on January 1, 1995 on the Drapner oil platform [1, 2], the appearance of which was an unexpected event, and the statistical data from the instruments did not agree with the theory of "extreme waves".

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Subsequently, several different scenarios were developed to explain the causes of rouge waves, namely, experimental studies were conducted that allowed for the first time to generate a rouge wave in the pool by colliding waves at a certain angle [3]. The classical method of describing the dynamics of waves is the use of the apparatus of partial differential equations, the system of which, as a rule, is a special case of the Cauchy-Kovalevskaya system, and, at first glance, does not contain a rouge wave, however, in [5–7], as a result of a computational experiment, it was possible to register the appearance of a rouge wave.

From the point of view of random processes, there is nothing surprising in the occurrence of abnormally large waves, even if their statistical indicators go beyond the framework of linear theory [4], especially with an unlimited time impact on the system. There is a need to build a formal apparatus that combines different scenarios of the occurrence of rouge waves, without losing sight of the very nature of the phenomenon, and each scenario of the occurrence of rouge waves requires taking into account the energy exchange between waves, which served as the basis for the creation of the corresponding axiomatic, which is a more general structure compared to dynamic systems, which allowed to prove the existence of rouge waves.

2 Problem Statement

2.1 Static

Consider a continuous medium V. Lets divide it into disjoint volumes v_i and fix them. We define for any v_i evaluation functionality $J_t : V \to K \subset \mathbb{R}$, where the parameter $t \in T \subset \mathbb{R}^+$ describes continuous time. Denote $w_i(t)$ energy assessment of each v_i at three time t, defined by the formula

$$J_t(v_i) = w_i(t),$$

where each number $w_i(t)$ displays the number of units of energy in the volume. Symbol W denotes the set of all estimates $w_i(t)$. And for anyone t the inequality is fulfilled

$$\sum_{v_i \in V} J_t(v_i) \leq \sup K < \infty.$$

Let's introduce the notation $H = (V, J_t)$ and we will call H as energy space. Finite sequences u off the form $u = (v_j, ..., v_i) = ji$

let's call the trajectory of energy from the volume v_i into volume v_j. Next, we assume that

$$ij = \begin{cases} ij, i \neq j, \\ e_{ii}, i = j. \end{cases}$$

Consider the set U of all finite sequences u,, by defining an equivalence relation on it $\sim$. We investigate the quantitative change of energy in the sequence at a fixed time moment t regardless of the intermediate volumes.

Definition 1.1 Finite sequences $a = (v_j, ..., v_i)$ и $b = (v_l, ..., v_k)$ belong to the same equivalence class $[u] \subset U$, if $v_j = v_l$ and $v_i = v_k$. Remark: In this case, we will consider that a and b indistinguishable from each other $a \sim b$.

Let's introduce the gluing operation «∘», such that for any $jk = (v_j, ..., v_k)$, $ki = (v_k, ..., v_i)$, $ji = (v_j, ..., v_i)$, $ij = (v_i, ..., v_j) \in U$ the relations are valid

$$jk \circ ki = ji,$$

$$ij \circ ji = e_{ii},$$

$$ij \circ e_{jj} = ij.$$

As a representative of any class $[u]$ we will consider the sequence u, consisting of two elements. By means G denote the set of all sequences consisting of two elements $\left(v_i, v_j\right)$. Note that the statement is true.

Statement If for any i and $c \neq e_{ii}$ there are $a \neq e_{ii}$ and $b \neq e_{ii}$, such that

$$c = a \circ b,$$

where $a, b, c \in G$, then $\langle V, G\rangle-$ the complete graph.

It follows from the presented statement that the system considered above is closed and described by a complete graph.

Definition 1.2 Sequence $u = (v_i, v_j)$ it is called connected at a time moment t, if between the elements v_i and v_j there is an exchange of energy at a time t.

3 Dynamics

Let in a closed dynamical system for any element $w_i(t) \in W$ there is a two—parameter family of closed operators $D = \left\{D_t^u : W \to W\right\}_{u\epsilon \mathrm{G}}$, such that when $t = 0$ $D_0^u = I$, and $D \subset (T)$. The result of the operator's action D_t^u characterizes the dynamics of energy in the energy space.

Definition 2.1 Let $\left(v_i, v_j\right)$— a related sequence at a point in time τ. If at $u = ji = (v_j, v_i)$ the equality is fulfilled.

$D_t^{ij} J_\tau(v_i) = J_i(v_j)$,

then v_j is called the valence point of the operator D_t^{ji} at the time moment t when $\tau \leq t$.

If for the operator $D_t^{ji} \epsilon \mathrm{D}$ there is a valence point v_i lets define a norm for it in the energy space H according to the formula

$$\|D_t^{ji}\|_H = \left|\int\limits_\tau^t F\left(D_t^{ji}\right) dJ_\tau(v_i)\right| = \left|J_t(v_i) - J_\tau(v_j)\right|,$$

where integration occurs by Lebesgue, the function F has the form

$$F\left(D_t^u\right) = \begin{cases} D_t^u, & \frac{dD_t^u}{dt} \geq 0, \\ -D_t^u, & \frac{dD_t^u}{dt} < 0. \end{cases}$$

Lemma For any sequence consisting of two elements v_i and v_j there is an operator D_t^u, and the only one with accuracy up to the choice of the valence point.

Proof Since the linked sequence consists of elements v_i and v_j, then there are two operators D_t^{ji} and D_t^{ij}, where $ji = (v_j, v_i)$ and $ij = (v_i, v_j)$.

Consider the norm of the difference of these operators in space H

$$\|D_t^{ji} - D_t^{ij}\|_H = \left|\int\limits_\tau^t F\left(D_t^{ji}\right) dJ_\tau(v_i) - \int\limits_\tau^t F\left(D_t^{ij}\right) dJ_\tau\left(v_j\right)\right| =$$

$$= \left|\left(J_t\left(v_j\right) - J_t(v_i)\right) - \left(J_\tau\left(v_j\right) - J_\tau(v_i)\right)\right| = \left|w_i - w_j\right| = \overline{w}$$

So $\overline{w}$ belongs to the set W, then there is an element $w^* \epsilon W$, to which the operator corresponds D_t^u such that the equality holds

$$D_t^u\left(w^*\right) = \overline{w}.$$

From existence $\overline{w}$ and the existence of a valence point $\overline{v}$, such that

$$J_t(\overline{v}) = \overline{w}.$$

According to the lemma, taking into account the statement about the completeness of the graph, it becomes a correct record of the energy exchange process in the space H for the volume v_i in the following form

$$D_t(w) = J_t(v_i),$$

in this case, the operator D_t describes energy exchange.

Definition 2.2 Let's call a class of operators a cycle $S \subset D$ such that under any $u \in$ G,$v \epsilon V$ and $D_t^u \epsilon$S the equality is fulfilled

$$\| D_t^u \|_H = 0,$$

where $\tau \leq t$.

Let's split the set $\mathfrak{D} = \mathfrak{S} \cup \mathfrak{L}$, where $\mathfrak{S}-$ circles, and $\mathfrak{L}-$ other operators.

Definition 2.3 . Let's call a moment in time τ the end of the epoch, if for the operator $\overline{D_\tau} \epsilon$L equality is valid

$$\| \overline{D_\tau} \|_H = \sup_{\mathrm{t}\epsilon\mathrm{T}} \| D_t \|_H = \sup K,$$

where $D_t \epsilon$D.

Example Let $V = \{A, B, C\}$, then $G = \{AA, BB, CC, AB, BA, CB, BC, AC, CA\}$.. The amount of energy in the system $supK = 15$, and $J_\tau(A) = 5$, $J_\tau(B) = 8$, $J_\tau(C) = 2$, $J_t(A) = 7$, $J_t(B) = 3$, $J_t(C) = 5-$ energy estimates of volumes at time points τ and t correspondingly.

Then $\| D_t^{BA} \|_H = \| D_t^{BC} \|_H = \| D_t^{BB} \|_H = 5$, $\| D_t^{AB} \|_H = \| D_t^{AC} \|_H = \| D_t^{AA} \|_H = 2$,

$$\| D_t^{CC} \|_H = \| D_t^{CB} \|_H = \| D_t^{CA} \|_H = 3.$$

For any $\widehat{D}_t^a, \overline{D}_\tau^b \in \mathfrak{D}$ let 's define the composition of operators as

$$\widehat{D}_t^a \overline{D}_\tau^b = \tilde{D}_{t+\tau}^{a \circ b},$$

when $\tau \leq t$ and $a \circ b = c \sim u \epsilon$G.

Let $\Delta\tau-$ the time interval taken as a conditional unit, then the following entry describes a dynamic process

$$(D_{n\Delta\tau}(w))^n = J_t(v_i),$$

where $n \in N$ the number of time intervals. The value $\Delta\tau$ can also be interpreted as a time sampling parameter t, therefore, if, then and $n \to \infty$. Among the operators $D_t \in \mathfrak{L}$ an ordered hierarchy of operator classes arises $[D_t] = \mathfrak{L}_i$ regarding the growth rate of energy estimates. Let $2w \neq 0-$ is the conventional unit of energy measurement that can be registered is then the exact upper edge of the class $\mathfrak{S}$ there will be a representative of the operator class $\mathfrak{L}_1$ such that if $D_t \in \mathfrak{L}_1$, then

$$D_t \in \mathfrak{L}_1,$$

Since for each conditional period of time the amount of energy increases by a constant value $2w$, then

$$\sup S = O(t).$$

Consider $sup\mathfrak{L}_1$. Assuming that at a time moment t the amount of energy change is equal to $2w$, for any $D_t \in \mathfrak{L}_1$ the assessment will be valid

$$\|D_t\|_H \leq 2w + 2 \cdot 2w + \cdots + t \cdot 2w = (2w + w(t-1))t = w(t+1)t = O\left(t^2\right),$$

and for any operator $D_t \in \mathfrak{L}_2$, such that at each moment of time, the change in the amount of energy will be in $2w$ times more than at the previous time t the assessment will be valid

$$\|D_t\|_H \leq 2w + (2w)^2 + \cdots + (2w)^t = \frac{2w\left(1-(2w)^t\right)}{1-2w} \leq Ae^{\alpha t} = O\left(e^{\alpha t}\right),$$

where $A = \frac{2w}{2w-1}$ and $\alpha = \ln(2w)$. For an arbitrary operator D_t of the class $\mathfrak{L}_n$, when $n \geq 2$ the assessment will be valid

$$\|D_t\|_H \leq \sum_{k=1}^{t} 2w\uparrow^{n-1}k,$$

where $\uparrow^{n-1}$ is the hyperoperator designation in Knuth's annotation [8].

Note that the class of operators $\mathfrak{L}_2$ contains a set of semigroups of operators D_t, describing the evolutionary process, that is $D_t(w) = \partial w(t)/\partial t$, and the record is valid

$$\frac{\partial w(t)}{\partial t} = J_t(v_i).$$

Let the following be the behavior of the operator D_t, at the time moment t depends only on the moment $t - \Delta\tau$. Then the operator D_t can be interpreted as a Markov process.

4 Rouge Waves

The height of the waves in hydrodynamics correlates with the amount of energy, so the most important thing is to distinguish these physical indicators. Wave height v we will find by the formula

$$h_t(v) = x_{mx} - (x_{mf} - x_{ms})/2,$$

where $x_{mx}-$ is wave crest height, and x_{mf} and x_{ms} the nearest soles surrounding the crest.

To determine the rouge wave, it is necessary to introduce the concept of an amplitude criterion μ. Amplitude criterion μ for the wave v it is calculated as the ratio of the height of the wave itself to the average value among the third of the highest waves [9].

Definition 3 A rogue wave is the wave v, for which the amplitude criterion is $\mu(v) \geq 2.1$ [10].

Amplitude criterion $\mu(v)$ it can be represented in the following form

$$\mu(v) = \frac{h_t(v)}{s_t} = \frac{\|D_t\|_H}{\frac{1}{|M|}\sum_{\tau \epsilon \mathrm{M}} \|D_\tau\|_H}$$

where $\|D_t\|_H-$ displays the height of the wave at the time moment t, $M-$ a set consisting of a third of the highest waves up to the moment t. Let the operator $D_t \notin \mathfrak{S}$. Then the following inequality is true.

$$\frac{1}{|M|}\sum_{\tau \epsilon \mathrm{M}} \|D_\tau\|_H \leq \frac{\|D_{t-1}\|_H + \|D_{t-2}\|_H}{2}$$

taking into account the registration of at least two waves until t.

Then for the value of the amplitude criterion μ the assessment is valid

$$\mu \geq \frac{2\|D_t\|_H}{\|D_{t-1}\|_H + \|D_{t-2}\|_H}. \tag{1}$$

Let operator $D_t \in \mathfrak{L}_2$, and let's make an upper estimate of the right side of the inequality (1). Considering that up to the moment t there were at least two waves, so $t \geq 3$, and assuming $w = 1$, will get

$$\frac{2\|D_t\|_H}{\|D_{t-1}\|_H + \|D_{t-2}\|_H} \leq \frac{2\|D_t\|_H}{2\min(\|D_{t-1}\|_H, \|D_{t-2}\|_H)} \leq \frac{\|D_t\|_H}{\|D_{t-2}\|_H} \leq$$

$$\leq \left. \frac{2e^{t\ln 2w}}{(2w-1)(t-1)(t-2)} \right|_{t=3} = \frac{2e^{3\ln 2}}{2} = 8.$$

Next, we will make a lower estimate of the amplitude criterion μ :

$$\frac{2\|D_t\|_H}{\|D_{t-1}\|_H + \|D_{t-2}\|_H} \geq \frac{2\|D_t\|_H}{2\max(\|D_{t-1}\|_H, \|D_{t-2}\|_H)} \geq \frac{\|D_t\|_H}{\|D_{t-1}\|_H} \geq$$

$$\geq \left. \frac{(2w-1)(t+1)t}{2e^{(t-1)\ln 2w}} \right|_{t=3} = \frac{4 \cdot 3}{2e^{2\ln 2}} = 1.5.$$

Hence, there is an operator $D_t \in \mathfrak{L}_2$, describing the dynamic process in which rogue waves arise.

So, the analysis allows us to formulate a hypothesis: "The dynamic process is represented by repeating epochs, each of which is a combination of a cycle and a trend, which corresponds to a part of the epoch that arises as a result of filtering it from cycles, while rogue waves are born before the end of the epoch," for confirmation of which we turn to computational experiments.

5 Processing of Computational Experiments

Computational experiments were carried out in [6, 7]. The resulting waveform is shown in Fig. 1. We will interpret it as a time series X_t. We introduce an estimate of the height of each wave according to the following formula

$$J_t = x_{mx} - \left(x_{mf} + x_{ms}\right)/2.$$

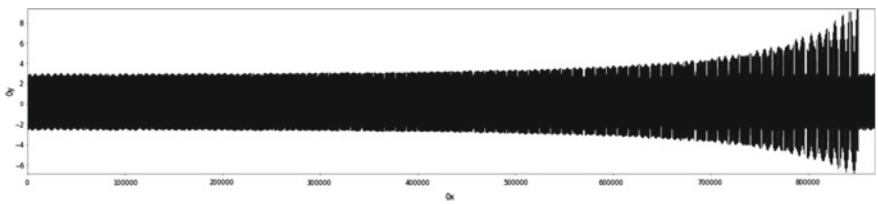

Fig. 1 Visualization of a computational experiment

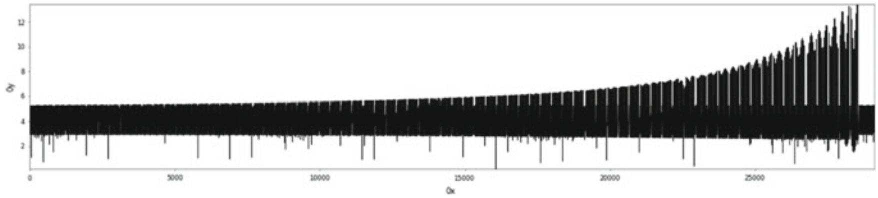

Fig. 2 Wave height

Figure 2 shows the result of the time series transformation X_t into Y_t, which displays the change in wave height over time.

According to the amplitude criterion, in the time series Y_t 25 rouge waves have been recorded. For a time series Y_t we define an estimate of local uniformity $\mu(U_n)$, as the standard deviation calculated for the set $U_n \subset Y_t$, where $n-$ the number of elements in the set U_n.

5.1 Frequency Response

Time series Y_t consist of 29,103 elements, rouger waves occurred only at moments in time

$$t = [26925; 27164; 27198; 27232; 27403; ...; 28429; 28463; 28497; 28531].$$

Denote T_r the set of all the moments of the occurrence of rouge waves. Lets introduce the set P, consisting of time intervals during which rogue waves occur in the next control.

$$p_i = t_{i+1} - t_i$$

where $t_i, t_i + 1 \in T_r$. We assume that the dynamic process contains cycles, which means that the occurrence of a rouge wave must be systematic. Moreover, it is more important that there is only one rouge wave in each cycle, or there are none. There are several possible options for choosing the value of the number n. First, let's assume that $n = 67$, since the averaging of all elements of the set P up to three digits is equal to 66.917, where $n \in N$. The number 34 is often found in the set P, therefore, it may be $n = 34$. Note that the average value of the elements of the set P close to twice the value of the most encountered element, decomposing it into prime numbers, we get 17 and 2, therefore $n = 17$.

The graphs are shown below in Figs. 3, 4, 5 for $\mu(U_n)$ when $n = 67, 34, 17$. The blue color shows the change in the value $\mu(U_n)$, denoted as «Origin». Bya «Trend» the result of exponential smoothing of «Origin» is indicated for $\alpha = 0.02$. The red

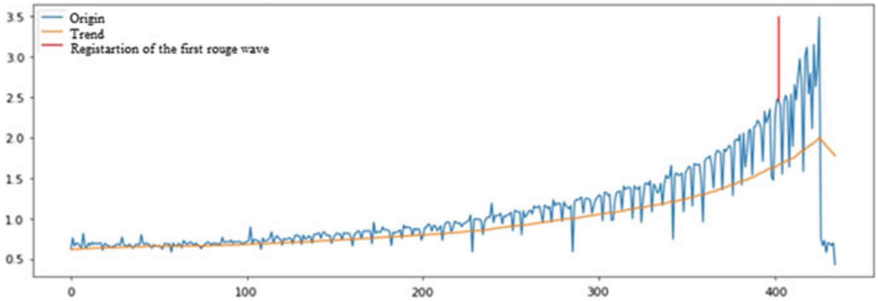

Fig. 3 The variation of $\mu(U_n)$ over time, when $n = 67$

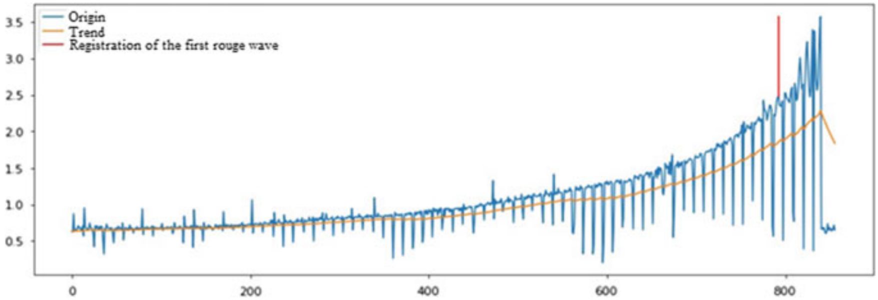

Fig. 4 The variation of $\mu(U_n)$ over time, when $n = 34$

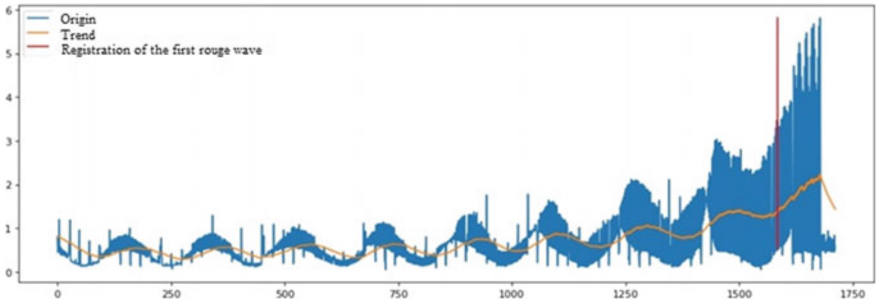

Fig. 5 The variation of $\mu(U_n)$ over time, when $n = 17$

line marks the moment of registration of the first rouge wave, while it begins with the value of the indicator $\mu(U_n)$ and ends at the maximum possible value $\mu(U_n)$.

Figure 5 shows an epoch consisting explicitly of a cyclic component in the form of a sine wave and a trend in the form of an exponent.

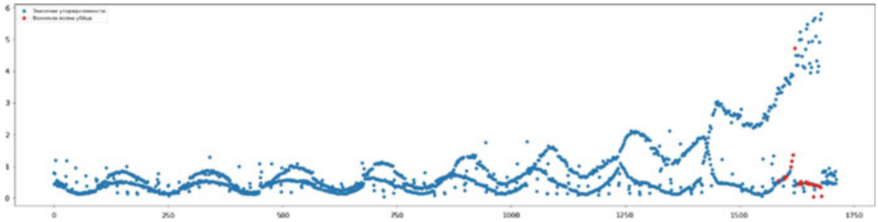

Fig. 6 Point-by-point graph $\mu(U_n)$

5.2 *Minimization of Functionality*

According to the hypothesis, the occurrence of a rouge wave should be a systematic phenomenon. And if this is the case, then you can choose one in which the value $\mu(U_n)$ it will be minimal at all points in the time of the occurrence of the rouge wave. The solution to this problem is reduced to the construction of the objective function $\rho(n)$ and its minimization. Let the objective function have the form

$$\rho(n) = \frac{1}{25}\sum_{t\epsilon \mathrm{T}} (\mu_t(U_n))^2 \to min.$$

Before starting optimization, it is necessary to determine the range of acceptable values. Let $n \in [3, 500]$. By a particle we will understand the implementation of the annealing simulation method for the objective function $\rho(n)$. The smallest value of the objective function $\rho(n)$ achieved when $n = 17$. This value was found most often and had the smallest value of all those obtained as a result of launching a swarm of 60 particles. Figure 6 shows the function $\mu_t(U_n)$ when $n = 17$.

So, the analysis of the presented computational experiment allows us to conclude about the validity of the hypothesis formulated above.

6 Conclusions

The presented method of mathematical modeling of anomalous dynamic processes in continuous media, being more general in comparison with the methods of the theory of dynamical systems, but more specialized than the methods of the theory of random processes, whose elements can be introduced and used without any difficulties, allowed us to prove the existence for some differential operators of a special nonlinear effect arising in a continuous medium, called “rouge waves”.

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