Selection of the Distribution of Sea Surface Elevations for Modeling the Reflected Pulse of the Radio Altimeter

A. S. Zapevalov

Abstract The influence of the choice of the probability density function (PDF) of sea surface elevations on the shape of the altimeter pulse reflected from the sea surface is analyzed. Currently, the main PDF model in problems of interaction of radio waves with the sea surface is a model based on the truncated Gram-Charlier series. This model allows us to describe changes in elevations only in a limited range, beyond which strong distortions occur, up to the appearance of negative values. The relatively small number of members of the Gram-Charlier series is due to the fact that in field experiments it is possible to determine the cumulants of elevations no older than the fourth order. Distortion at the tails of the distribution when calculating the reflected pulse of the altimeter leads to distortion of its shape, as a result, an error occurs in determining the level of the sea surface. For application of reflection of radio waves from the sea surface, it is proposed to use PDF in the form of a Gaussian mixture, free from the limitation of the truncated Gram-Charlier series. The parameters of the Gaussian mixture, as well as the parameters of the truncated Gram-Charlier series, are calculated based on the first four cumulants.

Keywords Remote sensing · Altimetry · Sea surface · Brown model · Distribution of surface elevations

1 Introduction

Improving the accuracy of altimetric measurements from spacecraft remains one of the main tasks of modern satellite oceanography [1, 2]. The main physical factor that affects the error of determining the sea level is the variability of the state of its surface [3, 4]. This factor is called sea state bias (SSB). There are three components of SSB. The first component SSB is due to the difference in the reflections of microwave radio waves from the crests and troughs of sea wave (electromagnetic bias) [5]. The second component is related to signal processing on board the spacecraft [6].

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The third component is caused by the deviation of the distribution of sea surface elevations from the Gaussian distribution (skewness bias) [7–9]. Deviations from the Gaussian distribution lead to the fact that the median of the distribution does not coincide with the average surface level, which leads to an error in determining it when radio sounding the sea surface from a spacecraft. In this paper, the skewness bias is analyzed.

Sea surface disturbance is a weakly nonlinear process [10, 11]. Currently, the main model describing the probability density function of elevations in application of the interaction of radio waves with the sea surface is the model built on the basis of the truncated Gram-Charlier series [12–14]. Modeling a distribution using truncated series has a significant limitation. Truncated series lead to distortions on the distribution tails [15]. Also, the question of how many members of the series should be left remains unresolved.

An alternative is to use the distributions of sea heights in the form of a Gaussian mixture [16]. Earlier this approach was proposed for modeling the slopes of the sea surface [17, 18]. Let's analyze how the shape of the reflected pulse depends on the choice of a model for the distribution of elevations of the sea surface.

2 Modeling the Shape of the Reflected Pulse of the Altimeter

At small angles of incidence, the pulse reflected from the sea surface is formed as a result of quasi-mirror reflection [19]. The shape of the reflected pulse of the altimeter is described by the Brown model [20]

$$V(t) = \chi(t) * s(t) * q(t), \tag{1}$$

where t is the time, $\chi(t)$ is the shape of the pulse reflected from a flat surface, $s(t)$ is the shape of the sounding pulse, $q(t)$ is the function associated with the probability density of the heights of the mirror reflection points; the symbol * means convolution. Initially, model (1) was constructed for the Gaussian distribution of sea surface elevations. Later, the scope of its use was expanded to the case when the distribution is quasi-Gaussian and can be described using truncated Gram-Charlier series [7].

Functions $\chi(t)$ and $s(t)$ set in the form [12]

$$\chi(t) = a\ \exp\left[-\frac{4\,c}{\gamma\,h}\cos(2\,\xi)\,t\right] I_0\left(2\sqrt{\frac{4\,c}{\gamma^2\,h}\sin^2(2\,\xi)\,t}\right) H(t), \tag{2}$$

$$s(t) = \frac{1}{\sqrt{2\pi\,D_r}}\exp\left(-\frac{t^2}{2\,D_r}\right), \tag{3}$$

where a is the amplitude, c is the speed of light, γ is the width of the antenna beam, h is the height of the spacecraft orbit, ξ is the absolute value of the incidence angle, I_0 is the modified Bessel function of the first kind, $H(t)$ is the Heaviside unit step function, parameter D_r determines the width of the radio pulse.

The function $q(t)$ is calculated by replacing the variable probability density function (PDF) of the sea surface elevations $P(\eta)$ using the relationship between the spatial and temporal coordinates $t = \eta/(c/2)$. We get

$$P(\eta) = q[t(\eta)]\frac{d\,t(\eta)}{d\eta} \tag{4}$$

The relation $t = \eta/(c/2)$ is linear, respectively, the skewness A_η and kurtosis E_η of the sea surface elevations η and the coefficients A_q and E_q calculated for the function $q(t)$ are numerically equal to each other. Lowering the level of the sea surface leads to an increase in the time of passage of the radio pulse of the altimeter. Therefore, the asymmetry A_q should have a sign opposite to the sign of the skewness A_η.

3 Gram-Charlier Distribution

Gram-Charlier series are based on the well-known decomposition into a series of derivatives of the function [15]

$$PN(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right) \tag{5}$$

Derived functions are defined by the expression

$$\frac{d^n}{dx^n}PN(x) = (-1)^n H_n(x)\cdot PN(x) \tag{6}$$

where $H_n(x)$ is Hermite polynomial of order n.

The coefficients of the Gram-Charlier series are expressed in terms of the cumulants of a random variable x. If the average value of a random variable is zero and its variance is one, then the first seven cumulants are described by expressions

$$\begin{cases} \lambda_1 = 0 \\ \lambda_2 = 1 \\ \lambda_3 = \mu_3 \\ \lambda_4 = \mu_4 - 3 \\ \lambda_5 = \mu_5 - 10\,\mu_3 \\ \lambda_6 = \mu_6 - 15\,\mu_4 - 10\,\mu_3^2 + 30 \\ \lambda_7 = \mu_7 - 21\,\mu_5 - 35\,\mu_4\mu_3 + 210\mu_3 \end{cases} \tag{7}$$

where

$$\mu_n = \int\limits_{-\infty}^{\infty} x^n P(x)\,dx \tag{8}$$

is the statistical moment of the distribution random variable x. Usually, when modeling the PDF of sea surface elevations, the Edgeworth form of type A of Gram-Charlier series is used [10]

$$P_{GC}(x) = \frac{\exp\left(-\frac{x^2}{2}\right)}{\sqrt{2\pi}}\left[1 + \frac{\lambda_3}{6}H_3(x) + \frac{\lambda_4}{24}H_4(x) + \frac{\lambda_5}{120}H_5(x) + \right.$$
$$\left. + \frac{\lambda_6 + 10\,\lambda_3^2}{720}H_6(x) + \frac{\lambda_7 + 35\,\lambda_4\lambda_3}{5040}H_7(x) + \cdots\right] \tag{9}$$

In field experiments, cumulants not older than the fourth order are determined for elevations of the sea surface. Therefore, when constructing a PDF, it is usually assumed that $\lambda_n = 0$ if $n > 4$. According to (7), cumulants of the third and fourth order are, respectively, skewness and kurtosis.

When simulating the reflected pulse of a radio altimeter, two forms of PDF of sea surface elevations are used

$$P_{GC}^{(1)}(\tilde{\eta}) = \frac{\exp\left(-\frac{\tilde{\eta}^2}{2}\right)}{\sqrt{2\pi}}\left[1 + \frac{\lambda_3}{6}H_3(\tilde{\eta}) + \frac{\lambda_4}{24}H_4(\tilde{\eta})\right] \tag{10}$$

$$P_{GC}^{(2)}(\tilde{\eta}) = \frac{\exp\left(-\frac{\tilde{\eta}^2}{2}\right)}{\sqrt{2\pi}}\left[1 + \frac{\lambda_3}{6}H_3(\tilde{\eta}) + \frac{\lambda_4}{24}H_4(\tilde{\eta}) + \frac{\lambda_3^2}{72}H_6(\tilde{\eta})\right] \tag{11}$$

where $\tilde{\eta} = \eta/\sqrt{\mu_2}$,

$$\frac{\exp\left(-\frac{\tilde{\eta}^2}{2}\right)}{\sqrt{2\pi}} = P_G(\tilde{\eta}) \tag{12}$$

is PDF for the Gaussian distribution.

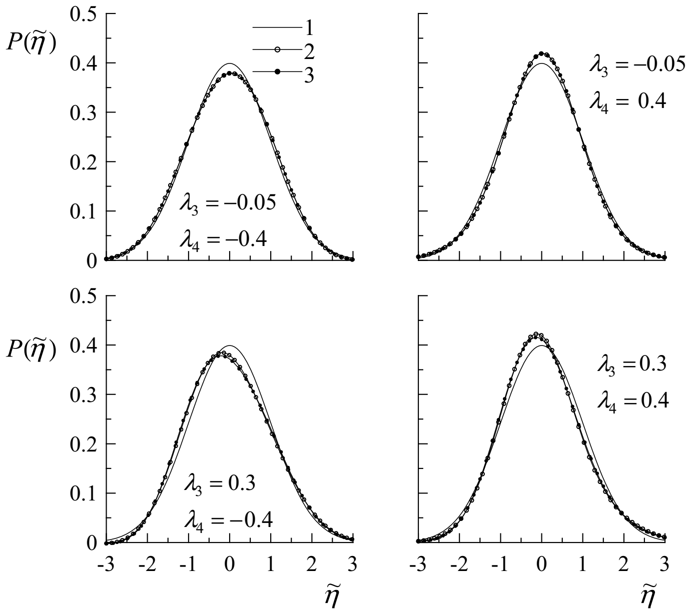

Fig. 1 PDF of sea surface elevations, curve 1 is $P_G(\tilde{\eta})$, curve 2 is $P_{GC}^{(1)}(\tilde{\eta})$, curve 3 is $P_{GC}^{(2)}(\tilde{\eta})$

According to the data of wave measurements carried out on a stationary oceanographic platform of the Marine Hydrophysical Institute of the Russian Academy of Sciences located on the Black Sea, the values of skewness and kurtosis mainly lie within range $-0.05 < \lambda_3 < 0.3$ and $-0.4 < \lambda_4 < 0.4$ [21]. The correlation between λ_3 and λ_4 is weak. PDF $P_{GC}^{(1)}(\tilde{\eta})$ and $P_{GC}^{(2)}(\tilde{\eta})$, calculated for the limit values of cumulants λ_3 and λ_4, are shown in Fig. 1.

The features of PDF behavior in the region of large values $|\tilde{\eta}|$ are shown in Fig. 2. It can be seen that with some combinations λ_3 and λ_4, negative PDF values are observed. Negative PDFs are observed both in the area of negative values $\tilde{\eta}$ and in the area of positive values $\tilde{\eta}$. An earlier analysis of the possibility of using the Gram–Charlier distribution to describe the distributions of sea surface slopes showed that this distribution can be used if the value of the normalized random variable does not exceed in absolute value 2.5 [22]. Since the values of cumulants λ_3 and λ_4 elevations and slopes of the sea surface are close, it can be assumed that for surface elevations, a possible area of use of the Gram–Charlier distribution. This assumption is consistent with the results of [23], in which the Gram–Charlier model distribution was compared with the data of direct wave measurements.

PDF $P_{GC}^{(1)}(\tilde{\eta})$ and $P_{GC}^{(2)}(\tilde{\eta})$ turned out to be very close to each other. This allows us to further consider only one PDF when calculating the reflected pulse of the radio altimeter. We will use PDF in the form (10).

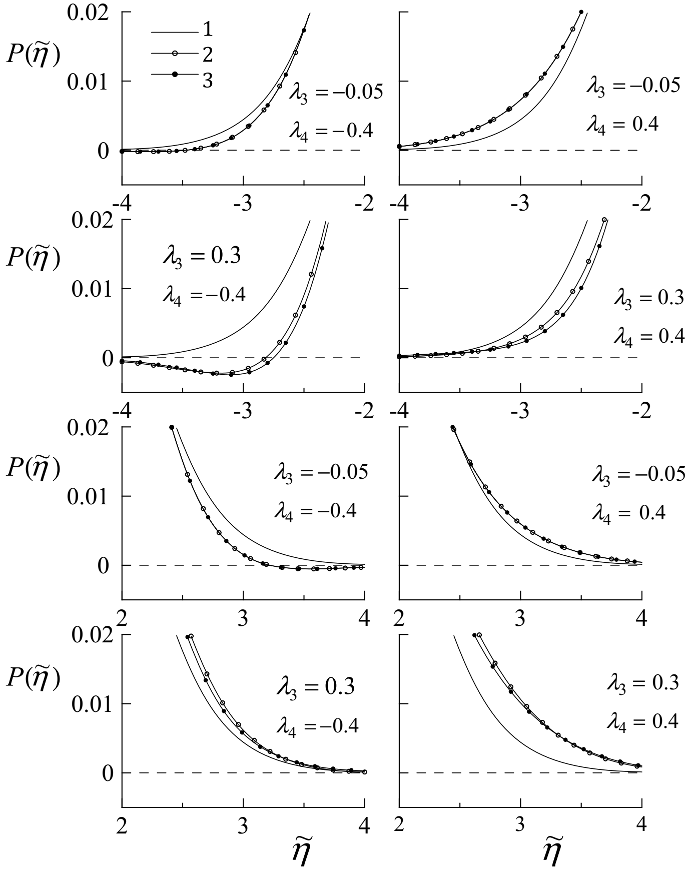

Fig. 2 PDF fragments of sea surface elevations at high values $|\tilde{\eta}|$, curve 1 is $P_G(\tilde{\eta})$, curve 2 is $P_{GC}^{(1)}(\tilde{\eta})$, curve 3 is $P_{GC}^{(2)}(\tilde{\eta})$

4 Two-Component Gaussian Mixture

The PDF approximation in the form of a two-component Gaussian mixture has the form [24]

$$P_S(\tilde{\eta}) = \frac{\alpha_1}{\sqrt{2\pi}\sigma_1} \exp\left(-\frac{(\tilde{\eta} - m_1)^2}{2\,\sigma_1^2}\right) + \frac{\alpha_2}{\sqrt{2\pi}\sigma_2} \exp\left(-\frac{(\tilde{\eta} - m_2)^2}{2\,\sigma_2^2}\right) \tag{13}$$

where α_i is the weight of the i-th component, $(i = 1,\ 2)$, $\alpha_i \in (0, 1)$, m_i is the mathematical expectation, σ_i^2 is variance. The following conditions are met for the weight coefficients

$$\alpha_i > 0 \text{ and } \alpha_1 + \alpha_2 = 1 \tag{14}$$

The statistical moments of a Gaussian mixture are related to the statistical moments of its components by the equation

$$\mu_j = \alpha_1 \mu_{j,1} + \alpha_2 \mu_{j,2} \tag{15}$$

where $\mu_{j,i}$ is the statistical moment of the i-th component ($i = 1,\ 2$).

When develop a PDF $P_S(\tilde{\eta})$, taking into account the condition for the weighting coefficients, it is necessary to determine five parameters m_1, m_2, σ_1^2, σ_2^2 and α_1. To calculate the parameters m_1, m_2, σ_1^2, and σ_2^2 the first four statistical moments of sea surface elevation can be used. The parameter α_1 is selected from the unimodal condition of the distribution [18].

The PDF $P_s(\tilde{\eta})$ approximation parameters are calculated by solving a system of four equations for statistical moments [16, 18]

$$\begin{cases} \mu_1 = 0 \\ \mu_2 = 1 \\ \mu_3 = \lambda_3 \\ \mu_4 = \lambda_4 + 3 \end{cases} \tag{16}$$

Cumulants λ_3 and λ_4 are set according to measurement data [21] or as a result of mathematical modeling obtained within the framework of various nonlinear wave modeling models [25, 26].

PDF $P_s(\tilde{\eta})$ constructed with the same values of cumulants λ_3 and λ_4 that were used in the construction $P_{GC}^{(1)}(\tilde{\eta})$ are shown in Fig. 3.

Unlike the distribution constructed on the basis of truncated Gram-Charlier series, the Gaussian mixture has no negative values.

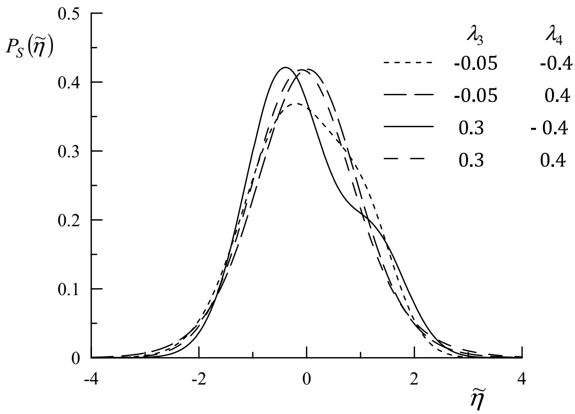

Fig. 3 PDF approximation in the form of a Gaussian mixture

Any PDF approximation of sea surface elevations must meet several requirements. It should be single-mode, and not have more than two inflection points. It turned out that these requirements may not be met for all values of cumulants λ_3 and λ_4. No solution for the system of Eq. (16) has been found in which the condition for the number of inflection points at $\lambda_3 = 0.3$ and $\lambda_4 = -0.4$ is satisfied. The approximation PDF described by third curve in Fig. 3 is not completely correct.

The main problem when develop PDF approximation in the form of a Gaussian mixture is that the procedure for calculating its parameters is very complex. It can be assumed that it is this circumstance that so far prevents the widespread use of this approximation for describing the statistical characteristics of the sea surface.

5 Numerical Simulation of the Reflected Pulse of a Altimeter

For numerical calculations, the values of the parameters included in Eqs. (2) and (3) are assumed to be equal to the corresponding parameters of the altimeter located on SEASAT-1: $\gamma = 1.6\,^{\circ}$, $\sqrt{D_r} = 1.327$ ns and $h = 8 \times 10^5$ m [12].

A shift of the leading edge by 0.1 ns corresponds to a change in the calculated surface level by 1.5 cm.

The shape of the leading edge of the reflected altimeter pulse calculated for the PDF $P_{GC}^{(1)}(\tilde{\eta})$ of sea surface elevations in form (2) is shown in Fig. 4. For clarity, when constructing Fig. 4, anormalization is introduced, according to which the amplitude of the reflected pulse is equal to unity. Significant wave height is taken equal to 3 m. Calculations were carried out for the same values λ_3 and λ_4 for which PDF approximations are constructed in Figs. 1 and 2.

Note that with some combinations of values λ_3 and λ_4, a non-physical effect is observed. It manifests itself as the occurrence of a negative intensity of the reflected signal. It can be assumed that negative values of the PDF approximation $P_{GC}^{(1)}(\tilde{\eta})$ also distort the shape of the pulse in the vicinity of its maximum.

The main factors that determine the accuracy of the representation of the shape of the reflected radio pulse is the correct approximation of the PDF of the sea surface elevations. The use of PDF in the form (10) or (11) in the problems of radio altimetric measurements can lead to the appearance of negative values in the models describing the shape of the reflected radio pulse. Negative values $V(t)$ are small in comparison with the amplitude of the reflected pulse, but their appearance indicates the need to construct an PDF approximation that allows to correctly describe the distribution of sea surface elevations.

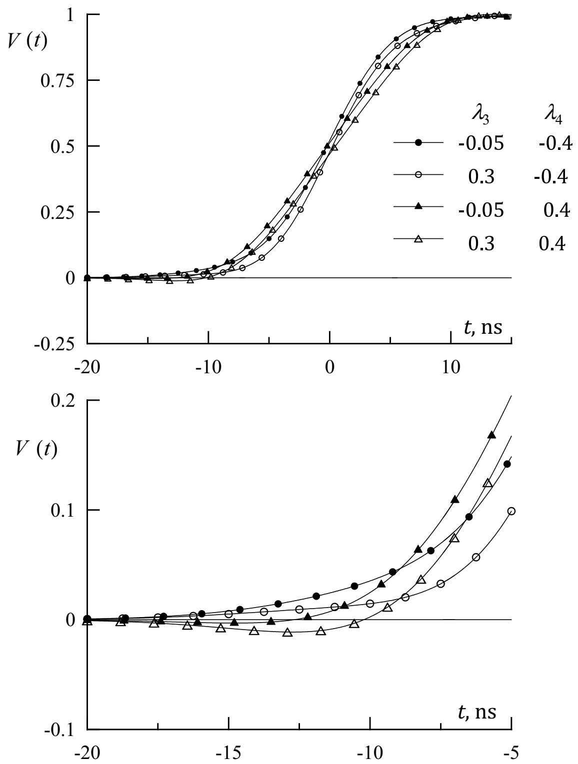

Fig. 4 The leading edge of the altimeter pulse reflected from the sea surface, built in accordance with the PDF approximation of the sea surface elevations in the form (9)

6 Conclusion

Sea disturbance is a random quasi-Gaussian process. Uncertainty of the state of the sea surface is the main source of error in determining the mean surface level from the data of measurements with a radio altimeter installed on the spacecraft. The mean level is calculated from the transit time of a radio pulse from the spacecraft to the sea surface and back. This parameter depends on the PDF elevation of the surface, which is not known a priori.

Currently, the main PDF model of the sea surface in problems of interaction of radio waves with the sea surface is a model based on the truncated Gram-Charlier series. This model allows us to describe the distribution of elevations only in a limited range, beyond which strong distortions occur, up to the appearance of negative values. The limitation of the number of members of the Gram-Charlier series is caused by the fact that in field experiments it is possible to determine the cumulants of elevations no older than the fourth order. Cumulants of the fifth and higher orders are assumed to be equal to zero. Distortion at the tails of the elevation distribution when calculating the reflected pulse of the altimeter leads to distortion of its shape, as a result, an error occurs in determining the level of the sea surface. Currently, two approximations in the form (10) and (11) are used to simulate the reflected pulse of the altimeter using Gram–Charlier series. Numerical modeling has shown that both approximations lead to similar results. Both approximations have the same disadvantages.

As an alternative to approximations (10) and (11), it is proposed to use the PDF of sea surface elevations in the form of a Gaussian mixture. To calculate the parameters of the Gaussian mixture, as well as for the truncated Gram-Charlier series, the first four cumulants of the surface elevations are used. Unlike the truncated Gram-Charlier series, the Gaussian mixture does not have negative values.

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