Optimal Control of Piecewise Deterministic Markov Processes

O.L.V. Costa and F. Dufour

Abstract This chapter studies the infinite-horizon continuous-time optimal control problem of piecewise deterministic Markov processes (PDMPs) with the control acting continuously on the jump intensity λ and on the transition measure Q of the process. Two optimality criteria are considered, the discounted cost case and the long run average cost case. We provide conditions for the existence of a solution to an integro-differential optimality equality, the so called Hamilton-Jacobi-Bellman (HJB) equation, for the discounted cost case, and a solution to an HJB inequality for the long run average cost case, as well as conditions for the existence of a deterministic stationary optimal policy. From the results for the discounted cost case and under some continuity and compactness hypothesis on the parameters and non-explosive assumptions for the process, we derive the conditions for the long run average cost case by employing the so-called vanishing discount approach.

1 Introduction

Piecewise Deterministic Markov Processes (PDMPs) were introduced by M.H.A. Davis in the seminal paper [9] as a general family of nondiffusion stochastic models, suitable to formulate an enormous variety of applications in operations research,

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engineering systems and management science. The general theory of the PDMPs, including a full characterization of the extended generator as well as its applications in several stochastic control problems, were elegantly and comprehensively presented in the book [11]. PDMPs are characterized by three local parameters: the flow ϕ, the jump rate λ, and the transition measure Q. Roughly speaking, the motion of a PDMP starting at the initial state x_0 follows a deterministic flow $\phi(x_0,t)$ until the first jump time T_1, which occurs either spontaneously in a Poisson-like fashion with rate λ or when the flow $\phi(x_0,t)$ hits the boundary of the state space. In either case the post-jump location of the process is selected by the transition measure $Q(.|\phi(x,T_1))$ and the motion restarts from this new point afresh. As presented in [11], a suitable choice of the state space and the local characteristics ϕ, λ, and Q can cover a great deal of problems in operations research, engineering systems and management science. It is worth pointing out that the presence of the boundary is crucial for the modeling of some optimization problems as, for instance, in queueing and inventory systems or maintenance-replacement models (see, for instance, the capacity expansion problem in [9], item (21.13), in which the boundary represents that a project is completed, and the jump in this case represents that investment is channelled immediately into the next project).

Broadly speaking there are two types of control for PDMPs, as pointed out by Davis in [11, page 134]: *continuous control*, in which the control variable acts at all times on the process through the characteristics (ϕ,λ,Q), and *impulse control*, used to describe control actions that intervene in the process by moving it to a new point of the state space at some specific times. The focus of this chapter will be on the former case, but considering that the control acts only on (λ,Q). Two performance criteria will be considered along this chapter: the so-called infinite horizon discounted cost case and the long run average cost case. Other criteria that can be found in the literature for the PDMPs include, for instance, the risk-sensitive control problem, as analyzed in [20] and [22].

It is worth pointing out that the main difficulty in considering the control acting also on the flow ϕ relies on the fact that in this situation the time which the flow takes to hit the boundary as well as the first order differential operator associated to the flow ϕ would depend on the control. For the discounted cost criterion this problem was nicely studied in [10] by rewriting the integral cost as a sum of integrals between two consecutive jump times of the PDMP, which yields to the one step cost function for a discrete-time Markov decision model. However this decomposition for the long run average cost is not possible. When compared with the so-called continuous-time Markov decision processes (see, for instance, [18, 16, 17, 19, 26, 33, 34]), it should be highlighted that the PDMPs are characterized by a drift motion between jumps, and forced jumps whenever the process hits the boundary, so that the available results for the continuous-time Markov decision processes cannot be applied to the PDMPs case.

Two kinds of approach can be pointed out for dealing with the discounted and long run average control problems of PDMPs. The first one would be to characterize the value function as a solution to the so called Hamilton-Jacobi-Bellman (HJB) equation associated with an imbedded discrete-stage Markov decision model, with the stages defined by the jump times T_n of the process. As a sample of works along this direction we can refer to [2, 3, 5, 8, 10, 11, 15, 30, 31] and the references therein. The key idea behind this approach is to find, at each stage, a control function that solves an imbedded deterministic optimal control problem. Usually the control strategy is chosen among the set of piecewise open loop policies, that is, stochastic kernels or measurable functions that depend only on the last jump time and post jump location. The second approach for these problems, which we will call the infinitesimal approach, is to characterize the optimal value function as the viscosity solution of the corresponding integro-differential HJB equation. As a sample of works using this kind of approach we can mention [7, 11, 12, 13, 14, 32] and the references therein.

This chapter adopts the infinitesimal approach to study the discounted and long run average control problems of PDMPs. The results presented in this chapter were mainly drawn from [7] and [6]. The goal is to provide conditions for the existence of a solution to integro-differential HJB equality and inequality, and for the existence of a deterministic stationary optimal policy, associated to the discounted and long run average control problems. These conditions are essentially related to continuity and compactness assumptions on the parameters of the problem, as well as some non-explosive conditions for the controlled process. In order to derive the results for the long run average control problem we apply the so-called vanishing discounted approach by adapting and combining arguments used in the context of continuous-time Markov decision processes (see [33]), and the results obtained for the infinite-horizon discounted optimal control problem.

The chapter is organized as follows. In sections 2 and 3 we present the notation, some definitions, the parameters defining the model, the construction of the controlled process, the definition of the admissible strategies, and the problem formulation. In section 4 we give the main assumptions and some auxiliary results. In sections 5 and 6 we present the main results related to the discounted and long run average control problems (see Theorems 2, 3 and 4) that provide sufficient conditions for the existence of a solution to a HJB equality (for the discounted case) and inequality (for the long run average case) and for the existence of a deterministic stationary optimal policy. Some proofs of the auxiliary results are presented in the Appendix.

2 Notation and definition

In this section we present the notation and some definitions that will be used throughout the chapter as well as the definition of the generalized inferior limit and its properties. The generalized limit will be used for the results related to the vanishing discounted approach to be considered in section 6.

We will denote by $\mathbb{N}$ the set of natural numbers including 0, $\mathbb{N}^* = \mathbb{N} - \{0\}$, $\mathbb{R}$ the set of real numbers, $\mathbb{R}_+$ the set of non-negative real numbers, $\mathbb{R}_+^* = \mathbb{R}_+ - \{0\}$, $\widehat{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. By *measure* we will always refer to a countably additive, $\mathbb{R}_+$-valued set function. For X a Borel space (i.e. a Borel-measurable subset of a complete and separable metric space) we denote by $\mathcal{B}(X)$ its associated Borel σ-algebra, and by $\mathcal{M}(X)$ ($\mathcal{P}(X)$ respectively) the set of measures (probability measures) defined on $(X, \mathcal{B}(X))$, endowed with the weak topology. We represent by $\mathcal{P}(X|Y)$ the set of stochastic kernels on X given Y where Y denotes a Borel space. For any set A, I_A denotes the indicator function of the set A, and for any point $x \in X$, δ_x denotes the Dirac measure defined by $\delta_x(\Gamma) = I_\Gamma(x)$ for any $\Gamma \in \mathcal{B}(X)$.

The space of Borel-measurable (bounded, lower semicontinuous respectively) real-valued functions defined on the Borel space X will be denoted by $\mathbb{M}(X)$ ($\mathbb{B}(X)$, $\mathbb{L}(X)$ respectively) and we set $\mathbb{L}_b(X) = \mathbb{L}(X) \cap \mathbb{B}(X)$. Moreover, the space of Borel-measurable, lower semicontinuous, $\widehat{\mathbb{R}}$-valued functions defined on the Borel space X will be denoted by $\widehat{\mathbb{L}}(X)$. For all the previous space of functions the subscript $_+$ will indicate the case of non-negative functions. The infimum over an empty set is understood to be equal to $+\infty$, and $e^{-\infty} = 0$.

As in [29], the definition of the generalized inferior limit is as follows:

Definition 1 Let X be a Borel space and let $\{w_n\}$, be a family of functions in $\mathbb{M}(X)$. The generalized inferior limit of the sequence $\{w_n\}$, denoted by $\underline{\lim}^g_{n\to\infty} w_n$ is defined as

$$\underline{\lim}^g_{n\to\infty} w_n(x) = \sup_{k\geq 1} \sup_{\epsilon>0} \Big(\inf_{m\geq k} \inf_{\{y: d(y,x)<\epsilon\}} w_m(y) \Big) \tag{1}$$

where $d(.,.)$ is the metric in X. For notational convenience, $\underline{\lim}^g_{n\to\infty} w_n$ will be denoted by w_*.

The following properties from the generalized inferior limit will be used in section 6 for the vanishing discounted approach.

Proposition 1 *Let $\{w_n\}$ be a sequence of nonnegative functions in $\mathbb{M}(X)$ and consider an arbitrary $x \in X$. In this case, $w_*(x)$ as defined in (1) satisfies the following properties:*

(i) For any sequence $\{x_n\}$ such that $x_n \to x$, it follows that $\underline{\lim}_{n\to\infty} w_n(x_n) \geq w_*(x)$, and there exists a sequence $\{x_n\}$ such that $x_n \to x$ and $\lim_{n\to\infty} w_n(x_n) = w_*(x)$.

(ii) $w_* \in \mathbb{L}_+(X)$.

(iii) [Generalized Fatou's Lemma] Suppose that $\{\mu_n\}$ is a sequence of probability measures in $\mathcal{P}(X)$ and that $\{\mu_n\}$ converges weakly to a $\mu \in \mathcal{P}(X)$. Then

$$\underline{\lim}_{n\to\infty} \int_S w_n(x)\mu_n(dx) \geq \int_S w_*(x)\mu(dx). \tag{2}$$

Proof: For the proof of (i) see Lemma 4.1 in [4]. For (ii) see Lemma 3.1 in [25] and for (iii) see Lemma 3.2 in [25]. □

3 Problem formulation for the controlled PDMP

The goal of this section is to introduce the parameters defining the model, the construction of the controlled process, the definition of the admissible strategies, and the problem formulation. Since it follows closely sections 2 and 3 in [7] some details will be skipped.

3.1 Parameters of the model

We will consider the control model depending on the following elements:

- The state space $\mathbf{X}$, which we assume to be an open subset of $\mathbb{R}^d$ ($d \in \mathbb{N}^*$) with boundary represented by $\partial\mathbf{X}$.
- The flow $\phi(x,t) : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$, associated with a given Lipschitz continuous vector field in $\mathbb{R}^d$, that is, $\phi(x,0) = x$ and $\phi(x,t+s) = \phi(\phi(x,s),t)$ for all $x \in \mathbb{R}^d$ and $(t,s) \in \mathbb{R}^2$.
- The so called active boundary defined as $\Xi = \{x \in \partial\mathbf{X} : x = \phi(y,t)$ for some $y \in \mathbf{X}$ and $t \in \mathbb{R}_+^*\}$. With some abuse of notation, we set $\overline{\mathbf{X}}$ as $\mathbf{X} \cup \Xi$, and for $x \in \overline{\mathbf{X}}$, we define

$$t^*(x) = \inf\{t \in \mathbb{R}_+ : \ \phi(x,t) \in \Xi\}.$$

  The flow ϕ outside the space $\overline{\mathbf{X}}$ can be defined arbitrarily since it plays no role for the problem.

- The action space $\mathbf{A}$, assumed to be a Borel space, and the set of feasible actions in state $x \in \overline{\mathbf{X}}$, given by $\mathbf{A}(x)$, which is a nonempty measurable subset of $\mathbf{A}$. Define the set $\mathbf{K} = \mathbf{K}^i \cup \mathbf{K}^g$ with

$$\mathbf{K}^g = \{(x,a) \in \mathbf{X} \times \mathbf{A} : a \in \mathbf{A}(x)\} \in \mathcal{B}(\mathbf{X} \times \mathbf{A}),$$

$$\mathbf{K}^i = \{(x,a) \in \Xi \times \mathbf{A} : a \in \mathbf{A}(x)\} \in \mathcal{B}(\Xi \times \mathbf{A}).$$

  It is assumed that $\mathbf{K}^g$ (respectively, $\mathbf{K}^i$) contains the graph of a measurable function from $\mathbf{X}$ (respectively, Ξ) to $\mathbf{A}$.
- The controlled jumps intensity λ which is a $\mathbb{R}_+$-valued measurable function defined on $\mathbf{K}$.
- The stochastic kernel Q on $\mathbf{X}$ given $\mathbf{K}$ satisfying $Q(\mathbf{X} \setminus \{x\}|x,a) = 1$ for any $(x,a) \in \mathbf{K}$. It describes the state of the process after any jump. In other words, if a jump governed by the intensity λ occurs in the current state $x \in \mathbf{X}$ and with action $a \in \mathbf{A}(x)$, then $Q(\cdot|x,a)$ describes the distribution of the state immediately after the jump. If $z \in \Xi$, that is, the current state is at the boundary then an action $b \in \mathbf{A}(z)$ is applied and the state of the process changes instantly according to the stochastic kernel Q.

It should be noticed that in the framework of continuous-time MDPs, the signed kernel on $\mathbf{X}$ given $\mathbf{K}$, defined by

$$q(dy|x,a) = \lambda(x,a)[Q(dy|x,a) - \delta_x(dy)] \tag{3}$$

is the (controlled) infinitesimal generator of the jump process. For $V \in \mathbb{M}(\mathbf{X})$ we set,

$$QV(x,a) = \int_{\mathbf{X}} V(y)Q(dy|x,a), \ (x,a) \in \mathbf{K}, \tag{4}$$
$$\lambda QV(x,a) = \lambda(x,a)QV(x,a), \ (x,a) \in \mathbf{K}^i,$$

provided that the integral in (4) exists. From (3) we have that

$$qV(x,a) = \lambda(x,a)[QV(x,a) - V(x)], \ (x,a) \in \mathbf{K}^i. \tag{5}$$

We conclude this sub-section with the following definition that will be used in the sequel.

Definition 2 The set of functions $g \in \mathbb{M}(\mathbf{X})$ which are absolutely continuous with respect to the flow ϕ on $[0, t^*(x)[$ (that is, the function $g(\phi(x,\cdot))$ is absolutely continuous on $[0,t^*(x)] \cap \mathbb{R}_+$) and such that $\lim_{t \to t^*(x)} g(\phi(x,t))$ exists whenever $t^*(x) < \infty$ will be denoted by $\mathbb{A}(\overline{\mathbf{X}})$. In this case the domain of definition of the mapping g can be extended to $\overline{\mathbf{X}}$ by setting $g(z) = \lim_{t \to t^*(x)} g(\phi(x,t))$ where $z = \phi(x, t^*(x))) \in \mathbf{\Xi}$. Lemma 2.2 in [8] shows that, for $g \in \mathbb{A}(\overline{\mathbf{X}})$, there exists a real-valued measurable function $\mathcal{X}g$ defined on $\mathbf{X}$ satisfying

$$g(\phi(x,t)) = g(x) + \int_{[0,t]} \mathcal{X}g(\phi(x,s))ds, \tag{6}$$

for any $t \in [0, t^*(x)[$. Notice that for $g \in \mathbb{A}(\overline{\mathbf{X}})$ the function $\mathcal{X}g$ satisfying (6) is not necessarily unique. The case of bounded functions in $\mathbb{A}(\overline{\mathbf{X}})$ will be denoted, as before, by $\mathbb{A}_b(\overline{\mathbf{X}})$.

3.2 Construction of the controlled process ξ_t

The canonical space Ω is defined by $\Omega = \bigcup_{n=0}^{\infty} \Omega_n \bigcup (\mathbf{X} \times (\mathbb{R}_+^* \times \mathbf{X})^\infty)$ where $\Omega_n = \mathbf{X} \times (\mathbb{R}_+^* \times \mathbf{X})^n \times (\{\infty\} \times \{x^\infty\})^\infty$ and x^∞ is an isolated artificial point corresponding to the case when no jumps occur in the future, endowed with its Borel σ-algebra denoted by $\mathcal{F}$. In that case, the process stays forever in x^∞, and so $t^*(x^\infty) = +\infty$. Set $\mathbf{X}_\infty = \mathbf{X} \cup \{x^\infty\}$ and $\overline{\mathbf{X}}_\infty = \overline{\mathbf{X}} \cup \{x^\infty\}$. We also extend the definition of ϕ on $\mathbf{X}_\infty \times \widehat{\mathbb{R}}_+$ as $\phi(x^\infty, t) = x^\infty$ for any $t \in \widehat{\mathbb{R}}_+$ and also $\phi(x, t^*(x)) = x^\infty$ whenever $t^*(x) = \infty$ for $x \in \mathbf{X}$.

We set $\omega \in \Omega$ as

$$\omega = (x_0, \theta_1, x_1, \theta_2, x_2, \ldots),$$

where $x_0 \in \mathbf{X}$ represents the initial state of the controlled point process ξ, and for $n \in \mathbb{N}^*$, the components $\theta_n > 0$ and x_n correspond to the time interval between two consecutive jumps and the value of the process ξ immediately after the jump. For the case $\theta_n < \infty$ and $\theta_{n+1} = \infty$, the trajectory of the controlled point process has only n jumps, and we put $\theta_m = \infty$ and $x_m = x^\infty$ (artificial point) for all $m \geq n+1$. Between jumps, the state of the process ξ moves according to the flow ϕ. The path up to $n \in \mathbb{N}$ is denoted by $h_n = (x_0, \theta_1, x_1, \theta_2, x_2, \ldots \theta_n, x_n)$, and the collection of all such paths is denoted by $\mathbf{H}_n$. We denote by $H_n = (X_0, \Theta_1, X_1, \ldots, \Theta_n, X_n)$ the n-term random history process taking values in $\mathbf{H}_n$ for $n \in \mathbb{N}$.

For $n \in \mathbb{N}$, set the mappings $X_n : \ \Omega \to \mathbf{X}_\infty$ by $X_n(\omega) = x_n$ and, for $n \geq 1$, the mappings $\Theta_n : \ \Omega \to \overline{\mathbb{R}}_+^*$ by $\Theta_n(\omega) = \theta_n$; $\Theta_0(\omega) = 0$. The sequence $(T_n)_{n \in \mathbb{N}^*}$ of $\overline{\mathbb{R}}_+^*$-valued mappings is defined on Ω by $T_n(\omega) = \sum_{i=1}^n \Theta_i(\omega) = \sum_{i=1}^n \theta_i$ and $T_\infty(\omega) = \lim_{n \to \infty} T_n(\omega)$. The random measure μ associated with $(\Theta_n, X_n)_{n \in \mathbb{N}}$ is a measure defined on $\mathbb{R}_+^* \times \mathbf{X}$ by

$$\mu(\omega; dt, dx) = \sum_{n \geq 1} I_{\{T_n(\omega) < \infty\}} \delta_{(T_n(\omega), X_n(\omega))}(dt, dx).$$

The dependence on ω will be suppressed for notational convenience and it will be written $\mu(dt, dx)$ instead of $\mu(\omega; dt, dx)$. For $t \in \mathbb{R}_+$, define $\mathcal{F}_t = \sigma\{H_0\} \vee \sigma\{\mu(]0, s] \times B) : \ s \leq t, B \in \mathcal{B}(\mathbf{X})\}$. The controlled process $\{\xi_t\}_{t \in \mathbb{R}_+}$ is defined as:

$$\xi_t(\omega) = \begin{cases} \phi(X_n, t - T_n) & \text{if } T_n \leq t < T_{n+1} \text{ for } n \in \mathbb{N}; \\ x^\infty, & \text{if } T_\infty \leq t, \end{cases}$$

and it is easy to see that $(\xi_t)_{t \in \mathbb{R}_+}$ could be equivalently described by the sequence $(\Theta_n, X_n)_{n \in \mathbb{N}}$. As in [11], we set

$$p^*(dt) = I_{\{\xi_{t-} \in \Xi\}} \mu(dt, \mathbf{X})$$

which counts the number of jumps from the boundary of the controlled process ξ_t (see [11], sub-section 26).

3.3 Admissible strategies

Associated to the state x^∞ we consider a special action a^∞ and we set $\mathbf{A}_\infty = \mathbf{A} \cup \{a^\infty\}$; $\mathbf{A}_\infty(x^\infty) = \{a^\infty\}$ and $\mathbf{A}_\infty(x) = \mathbf{A}(x)$ for $x \in \overline{\mathbf{X}}$. We also extend the definition of λ and Q at the point (x^∞, a^∞) by defining $\lambda(x^\infty, a^\infty) = 0$ and $Q(\{x^\infty\} | x^\infty, a^\infty) = 1$. An admissible control strategy is a sequence $u = (\pi_n, \gamma_n)_{n \in \mathbb{N}}$ such that, for any $n \in \mathbb{N}$,

- $\pi_n \in \mathcal{P}(\mathbf{A}_\infty | \mathbf{H}_n \times \mathbb{R}_+^*)$ and satisfies $\pi_n(\mathbf{A}(\phi(x_n, t)) | h_n, t) = 1$ for $h_n = (x_0, \ldots, \theta_n, x_n) \in \mathbf{H}_n$ and $t \in]0, t^*(x_n)[$.
- $\gamma_n \in \mathcal{P}(\mathbf{A}_\infty | \mathbf{H}_n)$ and satisfies $\gamma_n(\mathbf{A}(\phi(x_n, t^*(x_n))) | h_n) = 1$ for $h_n = (x_0, \ldots, \theta_n, x_n) \in \mathbf{H}_n$ and $t^*(x_n) < \infty$.

We will denote by $\mathcal{U}$ the set of admissible control strategies, and for $u = (\pi_n, \gamma_n)_{n \in \mathbb{N}} \in \mathcal{U}$ we denote by π and γ the random processes with values in $\mathcal{P}(\mathbf{A}_\infty)$

correspondingly as

$$\pi(da|t) = \sum_{n\in\mathbb{N}} I_{\{T_n<t\leq T_{n+1}\}}\pi_n(da|H_n, t-T_n)$$

and

$$\gamma(da|t) = \sum_{n\in\mathbb{N}} I_{\{T_n<t\leq T_{n+1}\}}\gamma_n(da|H_n),$$

for $t \in \mathbb{R}_+^*$. The processes π and γ are $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$-predictable random processes with values in $\mathcal{P}(\mathbf{A}_\infty)$. The following class of admissible strategies will be considered along this chapter. A control strategy $u \in \mathcal{U}$ is called *deterministic stationary*, if $\pi_n(\cdot|h_n, t) = \delta_{\varphi^s(\phi(x_n,t))}(\cdot)$ and $\gamma_n(\cdot|h_n) = \delta_{\varphi^s(\phi(x_n,t^*(x_n)))}(\cdot)$, where $\varphi^s : \overline{\mathbf{X}}_\infty \to \mathbf{A}_\infty$ is a measurable mapping satisfying $\varphi^s(y) \in \mathbf{A}(y)$ for any $y \in \overline{\mathbf{X}}$. By a slight abuse of notation, such a strategy will be just denoted by $u = \varphi^s$.

From Theorem 3.6 in [23] (or Remark 3.43, page 87 in [24]) we have that, for any admissible strategy $u \in \mathcal{U}$ and an initial state $x_0 \in \mathbf{X}$, there exists a probability $\mathbb{P}_{x_0}^u$ on $(\Omega, \mathcal{F})$ such that the restriction of $\mathbb{P}_{x_0}^u$ to $(\Omega, \mathcal{F}_0)$ is given by (see [7] for further details) $\mathbb{P}_{x_0}^u(\{X_0 = x_0\}) = 1$, and (see Lemma 3.1 in [7]) the predictable projection of the random measure μ with respect to $\mathbb{P}_{x_0}^u$ is given by $\nu = \nu_0 + \nu_1$, where, for $\Gamma \in \mathcal{B}(\mathbb{R}_+^* \times \mathbf{X})$,

$$\nu_0(\Gamma) = \int_\Gamma \int_{\mathbf{A}(\xi_s)} Q(dx|\xi_s, a)\lambda(\xi_s, a)\pi(da|s)ds,$$

$$\nu_1(\Gamma) = \int_\Gamma \sum_{n\in\mathbb{N}^*} I_{\{\xi_{T_n-}\in\Xi\}} \int_{\mathbf{A}(\xi_{T_n-})} Q(dx|\xi_{T_n-}, a)\gamma(da|T_n-)\delta_{T_n}(ds).$$

3.4 Problems formulation

We introduce in this section the infinite-horizon expected discounted and long run average continuous-time optimal control problems we will consider in this chapter, with the control acting continuously on the jump intensity λ and on the transition measure Q of the process (but not on the deterministic flow ϕ).

In what follows the running cost rate C^g is a real-valued measurable mapping defined on $\mathbf{K}$ and the boundary cost C^i is a real-valued measurable mapping defined on $\mathbf{K}$. We set $C^g(x^\infty, a^\infty) = C^i(x^\infty, a^\infty) = 0$. The associated infinite-horizon discounted criterion corresponding to an admissible control strategy $u = (u_n)_{n\in\mathbb{N}} \in \mathcal{U}$, $u_n = (\pi_n, \gamma_n)$, is defined by

$$\mathcal{V}_\alpha(u,x_0) = \mathbb{E}^u_{x_0}\Big[\int_{]0,+\infty[} e^{-\alpha s}\int_{\mathbf{A}(\xi_s)} C^g(\xi_s,a)\pi(da|s)ds\Big] + \mathbb{E}^u_{x_0}\Big[\int_{]0,+\infty[} e^{-\alpha s}\int_{\mathbf{A}(\xi_{s-})} C^i(\xi_{s-},a)\gamma(da|s)p^*(ds)\Big], \quad (7)$$

where $\alpha > 0$ is the discount factor. Similarly, the associated long run average criterion corresponding to an admissible control strategy $u \in \mathcal{U}$ is defined by

$$\mathcal{A}(u,x_0) = \overline{\lim_{t\to\infty}}\frac{1}{t}\Big\{\mathbb{E}^u_{x_0}\Big[\int_{]0,t[}\int_{\mathbf{A}(\xi_s)} C^g(\xi_s,a)\pi(da|s)ds\Big] + \mathbb{E}^u_{x_0}\Big[\int_{]0,t[}\int_{\mathbf{A}(\xi_{s-})} C^i(\xi_{s-},a)\gamma(da|s)p^*(ds)\Big]\Big\}. \quad (8)$$

Definition 3 The optimization problems consist in minimizing the performance criterion $\mathcal{V}_\alpha(u,x_0)$ and $\mathcal{A}(u,x_0)$ within the class of admissible strategies $u \in \mathcal{U}$, where x_0 is the initial state. The optimal value functions will be denoted respectively by $\mathcal{V}^*_\alpha(x_0)$ and $\mathcal{A}^*(x_0)$, that is,

$$\mathcal{V}^*_\alpha(x_0) = \inf_{u\in\mathcal{U}}\mathcal{V}_\alpha(u,x_0), \quad \mathcal{A}^*(x_0) = \inf_{u\in\mathcal{U}}\mathcal{A}(u,x_0)$$

and $u \in \mathcal{U}$ will be an optimal strategy for the discounted (respectively, long run average) problem if $\mathcal{V}_\alpha(u,x_0) = \mathcal{V}^*_\alpha(x_0)$ (respectively, $\mathcal{A}(u,x_0) = \mathcal{A}^*(x_0)$).

4 Main assumptions and auxiliary results

The objective of this section is to introduce the assumptions and present some technical results that will be used along this chapter.

4.1 Main assumptions

Our approach requires that the process must be non-explosive and that the expected value of the number of jumps at the boundary up to a time $t \in \mathbb{R}_+$ must be bounded from above by an *affine* function in the variable t. One of the main goals of Assumption A is to ensure these properties.

Assumption A. There are constants $K \geq 0$ and $\varepsilon_1 > 0$ such that

(A1) For any $(x,a) \in \mathbf{K}^g$, $\lambda(x,a) \leq K$.
(A2) For any $(z,b) \in \mathbf{K}^i$, $Q(A_{\varepsilon_1}|z,b) = 1$ *where*

$$A_{\varepsilon_1} = \{x \in \mathbf{X} : t^*(x) > \varepsilon_1\}.$$

(A3) *For any* $(x,a) \in \mathbf{K}^g$, $Q(A(x)|x,a) = 1$ *where*

$$A(x) = \{y \in \mathbf{X} : t^*(y) \geq \min\{t^*(x), \varepsilon_1\}\}.$$

Assumptions B and C are classical hypotheses. They mainly ensure the existence of an optimal selector.

Assumption B.

- *(B1)* *For every* $y \in \overline{\mathbf{X}}$ *the set* $\mathbf{A}(y)$ *is compact.*
- *(B2)* *The kernel* Q *is weakly continuous (also called weak-Feller Markov kernel) on* $\mathbf{K}^g$.
- *(B3)* *The function* λ *is continuous on* $\mathbf{K}^g$.
- *(B4)* *The flow* ϕ *is continuous on* $\mathbb{R}_+ \times \mathbb{R}^p$.
- *(B5)* *The function* t^* *is continuous on* $\overline{\mathbf{X}}$.

Assumption C.

- *(C1)* *The multifunction* Ψ^g *from* $\mathbf{X}$ *to* $\mathbf{A}$ *defined by* $\Psi^g(x) = \mathbf{A}(x)$ *is upper semicontinous. The multifunction* Ψ^i *from* Ξ *to* $\mathbf{A}$ *defined by* $\Psi^i(z) = \mathbf{A}(z)$ *is upper semicontinous.*
- *(C2)* *The cost function* C^g *(respectively,* C^i*) is bounded and lower semicontinuous on* $\mathbf{K}^g$ *(respectively,* $\mathbf{K}^i$*).*

Without loss of generality, we assume, from Assumption (C2), that the inequalities $|C^g| \leq K$ and $|C^i| \leq K$ are valid, where K is the same constant as in Assumption (A1).

4.2 Auxiliary results

We present in this subsection some auxiliary results that will be useful to study both the infinite-horizon discounted control problem as well as the long-run average cost control problem. The first result of this subsection, Lemma 1, shows that the controlled process is non-explosive and provides an upper bound for the sum of the expected values of $e^{-\alpha T_n}$ as well as an affine upperbound on t for the expected value on the number of jumps from the frontier up to a time t. This result requires only Assumption A.

Lemma 1 *If Assumption A is satisfied then there exist positive numbers* $M < \infty$, $c_0 < \infty$ *such that, for any control strategy* $u \in \mathcal{U}$ *and initial state* $x_0 \in \mathbf{X}$,

$$\mathbb{E}^u_{x_0}\Big[\sum_{n\in\mathbb{N}^*} e^{-\alpha T_n}\Big] \leq M,\ \mathbb{P}^u_{x_0}(T_\infty < +\infty) = 0. \tag{9}$$

Furthermore for any $t \in \mathbb{R}_+$,

$$\mathbb{E}^u_{x_0}\Big[\sum_{n\in\mathbb{N}^*} I_{\{T_n \leq t, \xi_{T_n^-} \in \Xi\}}\Big] \leq Mt + c_0. \tag{10}$$

Proof: For the proof of (9), see Lemma 4.1 in [7] and, for the proof of (10), see Lemma 3.1 in [6]. □

Recalling the definitions of $\mathcal{V}_\alpha$ and $\mathcal{A}$ (see equations (7) and (8) respectively), it is easy to get that for any control strategy $u \in \mathcal{U}$

$$|\mathcal{V}_\alpha(u,x_0)| \leq K(\frac{1}{\alpha} + \mathbb{E}^u_{x_0}\Big[\sum_{n\in\mathbb{N}^*} e^{-\alpha T_n}\Big]) \leq K(\frac{1}{\alpha} + M)$$

and

$$|\mathcal{A}(u,x_0)| \leq K\Big(1 + \overline{\lim_{t\to\infty}}\frac{1}{t}\mathbb{E}^u_{x_0}\Big[\sum_{n\in\mathbb{N}^*} I_{\{T_n\leq t, \xi_{T_n^-}\in\Xi\}}\Big]\Big) \leq K(1+M),$$

by using Lemma 1 and the fact that $|C^g| \leq K$ and $|C^i| \leq K$ (see Assumption (C2)). Therefore, the mappings $\mathcal{V}_\alpha(u,\cdot)$ and $\mathcal{A}(u,\cdot)$ are well defined.

The next lemma will be useful to obtain the characterization of the value functions in terms of integro differential equations.

Lemma 2 *Consider a bounded from below real-valued measurable function F defined on $\mathbf{X}$ such that, for a real number $\beta > 0$, it satisfies*

$$\int_{[0,t^*(x)[} e^{-\beta s} F(\phi(x,s))ds < +\infty$$

for any $x \in \overline{\mathbf{X}}$, and a bounded from below real-valued measurable function G defined on Ξ. Then the real-valued mapping V defined on $\overline{\mathbf{X}}$ by

$$V(x) = \int_{[0,t^*(x)[} e^{-\beta s} F(\phi(x,s))ds + e^{-\beta t^*(x)} G(\phi(x,t^*(x)))$$

belongs to $\mathbb{A}(\overline{\mathbf{X}})$. Moreover there exists a bounded from below measurable function $\mathcal{X}V$ satisfying

$$-\beta V(x) + \mathcal{X}V(x) = -F(x),$$

for any $x \in \mathbf{X}$ and, furthermore, $V(z) = G(z)$ for any $z \in \Xi$.

Proof: See the Appendix.

For any function V in $\mathbb{M}(\overline{\mathbf{X}})$ bounded from below let us introduce the $\widehat{\mathbb{R}}$-valued mappings $\mathfrak{R}V$ and $\mathfrak{T}V$ defined on $\mathbf{X}$ and Ξ respectively by

$$\mathfrak{R}V(x) = \inf_{a\in\mathbf{A}(x)} \Big\{C^g(x,a) + qV(x,a) + KV(x)\Big\}, \tag{11}$$

$$\mathfrak{T}V(z) = \inf_{b\in\mathbf{A}(z)} \Big\{C^i(z,b) + QV(z,b)\Big\}, \tag{12}$$

where the constant K has been defined in Assumption (A1) and the transition kernel q in equation (3). Observe that qV and QV are well defined since by hypothesis V is bounded from below. Note also qV and QV may take the value $+\infty$. Finally, for any $\alpha \in [0,1]$, let us introduce the $\widehat{\mathbb{R}}$-valued function $\mathfrak{B}_\alpha V$ defined on $\overline{\mathbf{X}}$ by

$$\mathfrak{B}_\alpha V(y) = \int_{[0,t^*(y)[} e^{-(K+\alpha)t} \mathfrak{R}V(\phi(y,t))dt + e^{-(K+\alpha)t^*(y)} \mathfrak{T}V(\phi(y,t^*(y))). \quad (13)$$

Again, remark the integral term in (13) is well defined but may take the value $+\infty$. Moreover, since $|C^g| \leq K$ and $|C^i| \leq K$, we have clearly that $\mathfrak{R}V(x) \geq -Kc_0$ and $\mathfrak{T}V(z) \geq -c_0$ for some constant $c_0 > 0$. By using the definition of $\mathfrak{B}_\alpha V$

$$\mathfrak{B}_\alpha V(y) \geq -c_0(1-e^{-Kt^*(y)}) - c_0 e^{-Kt^*(y)} = -c_0$$

for any $\alpha \in [0,1]$.

The next lemma provides important properties of the operators $\mathfrak{R}$, $\mathfrak{T}$ and $\mathfrak{B}_\alpha$.

Lemma 3 *Suppose that Assumptions A, B and C are satisfied. If $V \in \mathbb{L}(\overline{\mathbf{X}})$ is bounded from below then for any $\alpha \in [0,1]$ we have that*

$$\mathfrak{R}V \in \widehat{\mathbb{L}}(\mathbf{X}),\ \mathfrak{T}V \in \widehat{\mathbb{L}}(\mathbf{\Xi}),\ \mathfrak{B}_\alpha V \in \widehat{\mathbb{L}}(\overline{\mathbf{X}})$$

and all these functions are bounded from below.

Proof: See the Appendix.

For any $0 < \alpha < 1$, let us introduce

$$K_\alpha = \frac{K(1+K)(1-e^{-(K+\alpha)\varepsilon_1}) + (K+\alpha)Ke^{-(K+\alpha)\varepsilon_1}}{\alpha(1-e^{-(K+\alpha)\varepsilon_1})},$$
$$K_C = \frac{2K(1+K)}{1-e^{-K\varepsilon_1}},$$

where K and ϵ_1 have been defined in Assumption A. Clearly, for any $0 < \alpha < 1$

$$0 < \alpha K_\alpha \leq K_C. \quad (14)$$

The next lemma provides upper bounds and absolutely continuity properties of the operator $\mathfrak{B}_\alpha$.

Lemma 4 *Suppose that Assumptions A, B and C hold. Consider $V \in \mathbb{L}_b(\overline{\mathbf{X}})$ satisfying, for any $y \in \overline{\mathbf{X}}$,*

$$|V(y)| \leq K_\alpha I_{A_{\varepsilon_1}}(y) + (K_\alpha + K) I_{A^c_{\varepsilon_1}}(y).$$

Then $\mathfrak{B}_\alpha V \in \mathbb{A}_b(\overline{\mathbf{X}})$ and for any $y \in \overline{\mathbf{X}}$,

$$|\mathfrak{B}_\alpha V(y)| \leq K_\alpha I_{A_{\varepsilon_1}}(y) + (K_\alpha + K) I_{A^c_{\varepsilon_1}}(y).$$

Proof: See Lemma 5.4 in [7]. □

We conclude this section with the following result, which is a consequence of the so-called Dynkin formula associated with the controlled process $(\xi_t)_{t\in\mathbb{R}_+}$.

Theorem 1 *Suppose that Assumption A is satisfied and that the cost functions C^g and C^i are bounded (below or above). Then we have, for any strategy $u=(\pi_n,\gamma_n)\in\mathcal{U}$ and $(W,\mathcal{X}W)\in\mathbb{A}_b(\overline{\mathbf{X}})\times\mathbb{B}(\mathbf{X})$, that*

$$\begin{aligned}
&\mathcal{V}_\alpha(u,x_0) = W(x_0) + \mathbb{E}^u_{x_0}\Big[\int_{]0,+\infty[} e^{-\alpha s}\,[\mathcal{X}W(\xi_s)-\alpha W(\xi_s)]\,ds\Big]\\
&+\mathbb{E}^u_{x_0}\Big[\int_{]0,+\infty[} e^{-\alpha s}\int_{\mathbf{A}^g}\{C^g(\xi_s,a)\\
&+\int_{\mathbf{X}} W(y)Q(dy|\xi_s,a)\lambda(\xi_s,a)-W(\xi_s)\lambda(\xi_s,a)\}\pi(da|s)]ds\Big]\\
&+\mathbb{E}^u_{x_0}\Big[\sum_{n\in\mathbb{N}^*} I_{\{\xi_{T_n-}\in\Xi\}}e^{-\alpha T_n}\Big[\int_{\mathbf{A}^i}\{C^i(\xi_{T_n-},a)\\
&+\int_{\mathbf{X}} W(y)Q(dy|\xi_{T_n-},a)\}\gamma(da|T_n-)-W(\xi_{T_n-})\Big]\Big].
\end{aligned}\qquad(15)$$

Proof: See Corollary 4.3 in [7]. □

5 The discounted control problem

Theorem 2 below presents sufficient conditions based on the three local characteristics of the process ϕ, λ, Q, and the semi-continuity properties of the set valued action space, for the existence of a solution for an integro-differential HJB optimality equation associated with the discounted control problem as well as conditions for the existence of an optimal selector. Moreover it shows that the solution of the integro-differential HJB optimality equation is in fact unique and coincides with the optimal value for the α-discounted problem, and the optimal selector derived in Theorem 2 yields an optimal deterministic stationary strategy for the discounted control problem.

Theorem 2 *Suppose Assumptions A, B and C are satisfied. Then there exist $W\in\mathbb{A}_b(\overline{\mathbf{X}})$ and $\mathcal{X}W\in\mathbb{B}(\mathbf{X})$ satisfying, for any $x\in\mathbf{X}$,*

$$-\alpha W(x)+\mathcal{X}W(x)+\inf_{a\in A^g(x)}\Big\{C^g(x,a)+qW(x,a)\Big\}=0, \qquad (16)$$

and, for any $z\in\mathbf{\Xi}$,

$$W(z)=\inf_{b\in A^i(z)}\Big\{C^i(z,b)+QW(z,b)\Big\}. \qquad (17)$$

Moreover there is a measurable mapping $\widehat{\varphi}_\alpha : \overline{\mathbf{X}} \to \mathbf{A}$ such that $\widehat{\varphi}_\alpha(y) \in \mathbf{A}(y)$ for any $y \in \overline{\mathbf{X}}$ and satisfying, for any $x \in \mathbf{X}$,

$$C^g(x, \widehat{\varphi}_\alpha(x)) + qW(x, \widehat{\varphi}_\alpha(x)) = \inf_{a \in \mathbf{A}(x)} \Big\{ C^g(x,a) + qW(x,a) \Big\}, \tag{18}$$

and, for any $z \in \Xi$,

$$C^i(z, \widehat{\varphi}_\alpha(z)) + QW(z, \widehat{\varphi}_\alpha(z)) = \inf_{b \in \mathbf{A}(z)} \Big\{ C^i(z,b) + QW(z,b) \Big\}. \tag{19}$$

Furthermore we have that

a) the deterministic stationary strategy $\widehat{\varphi}_\alpha$ is optimal for the α-discounted problem,
b) the function $W \in \mathbb{A}_b(\overline{\mathbf{X}})$, solution of (16)-(17), is unique and coincides with $\mathcal{V}^*_\alpha(x) = inf_{u \in \mathcal{U}} \mathcal{V}_\alpha(u,x)$, and
c) $\mathcal{V}^*_\alpha(x)$ satisfies

$$|\mathcal{V}^*_\alpha(x)| \leq K_\alpha + K I_{A^c_{\varepsilon_1}}(x). \tag{20}$$

Proof: By Lemma 3, one can define recursively the sequence of functions $\{W_i\}_{i \in \mathbb{N}}$ in $\mathbb{L}_b(\overline{\mathbf{X}})$ as follows: $W_{i+1}(y) = \mathfrak{B}_\alpha W_i(y)$, for $i \in \mathbb{N}$ and $W_0(y) = -K_\alpha I_{A_{\varepsilon_1}}(y) - (K_\alpha + K) I_{A^c_{\varepsilon_1}}(y)$ for any $y \in \overline{\mathbf{X}}$. By using Lemma 4 and the definition of W_0, we obtain that $W_1(y) \geq W_0(y)$ for any $y \in \overline{\mathbf{X}}$. Now, note that the operator $\mathfrak{B}_\alpha$ is monotone, that is, $V_1 \leq V_2$ implies $\mathfrak{B}_\alpha V_1 \leq \mathfrak{B}_\alpha V_2$. Consequently, it can be shown by induction on i that the sequence $\{W_i\}_{i \in \mathbb{N}}$ is increasing and, from Lemma 4 and the definition of W_0, that for every $i \in \mathbb{N}$,

$$|W_{i+1}(x)| = |\mathfrak{B}_\alpha W_i(x)| \leq K_\alpha I_{A_{\varepsilon_1}}(x) + (K_\alpha + K) I_{A^c_{\varepsilon_1}}(x). \tag{21}$$

Therefore from (21) the sequence of functions $\{W_i\}_{i \in \mathbb{N}}$ is uniformly bounded, that is, for any $i \in \mathbb{N}$, $\sup_{y \in \overline{\mathbf{X}}} |W_i(y)| \leq K_\alpha + K$. As a result, $\{W_i\}_{i \in \mathbb{N}}$ converges to a mapping $W \in \mathbb{B}(\overline{\mathbf{X}})$. Since $\{W_i\}_{i \in \mathbb{N}}$ is an increasing sequence of lower semicontinuous functions, $W \in \mathbb{L}_b(\overline{\mathbf{X}})$, $KW_i + qW_i \in \mathbb{L}_b(\mathbf{K}^g)$, and so, $C^g + KW_i + qW_i \in \mathbb{L}_b(\mathbf{K}^g)$ by Assumption (C2). Therefore, combining Assumptions (B1) and (C1) and Lemma 2.1 in [28], it follows that $\lim_{i \to \infty} \mathfrak{R} W_i(x) = \mathfrak{R} W(x)$ for any $x \in \mathbf{X}$ and $\lim_{i \to \infty} \mathfrak{T} W_i(z) = \mathfrak{T} W(z)$ for any $z \in \Xi$. By using the bounded convergence Theorem, it implies that the mapping W satisfies the following equations

$$\begin{aligned} W(y) &= \mathfrak{B}_\alpha W(y) \\ &= \int_{[0,t^*(y)[} e^{-(K+\alpha)t} \mathfrak{R} W(\phi(y,t)) dt + e^{-(K+\alpha)t^*(y)} \mathfrak{T} W(\phi(y,t^*(y))), \end{aligned} \tag{22}$$

where $y \in \overline{\mathbf{X}}$. Applying Lemma 2 to the mapping W where the function F (respectively G) is given by $\mathfrak{R} W$ (respectively, $\mathfrak{T} W$), it yields that the function $W \in \mathbb{A}_b(\overline{\mathbf{X}})$ and satisfies

$$-(\alpha + K) W(x) + \mathcal{X} W(x) = - \inf_{a \in A^g(x)} \Big\{ C^g(x,a) + qW(x,a) + KW(x) \Big\},$$

for any $x \in \mathbf{X}$ and

$$W(z) = \inf_{b \in A^i(z)} \Big\{ C^i(z,b) + QW(z,b) \Big\},$$

for any $z \in \boldsymbol{\Xi}$. This shows the existence of $W \in \mathbb{A}_b(\overline{\mathbf{X}})$ and $\mathcal{X}W \in \mathbb{B}(\mathbf{X})$ satisfying equations (16) and (17).

Now, under Assumptions B and C, for any $x \in \mathbf{X}$ the mapping defined on $\mathbf{A}(x)$ by

$$a \to C^g(x,a) + \lambda(x,a)[QW(x,a) - W(x)] + KW(x)$$

is lower semicontinuous and since Ψ^g is upper semicontinuous, it follows from Proposition D.5 in [21] that there exists a measurable mapping $\varphi_\alpha^g : \mathbf{X} \to \mathbf{A}^g$ such that $\forall x \in \mathbf{X}\ \varphi_\alpha^g(x) \in \mathbf{A}(x)$ and equation (18) holds. Similar arguments can be used to show the existence of a measurable mapping $\varphi_\alpha^i : \boldsymbol{\Xi} \to \mathbf{A}^i$ satisfying $\varphi_\alpha^i(z) \in \mathbf{A}(z)$ for any $z \in \boldsymbol{\Xi}$ and equation (19) holds. Therefore, the measurable mapping $\widehat{\varphi}_\alpha$ defined by $\widehat{\varphi}_\alpha(x) = \varphi_\alpha^i(x)$ for any $x \in \mathbf{X}$ and $\widehat{\varphi}_\alpha(z) = \varphi_\alpha^i(z)$ for any $z \in \boldsymbol{\Xi}$ satisfies the claim.

To show a) and b), notice that for an arbitrary control strategy $u \in \mathcal{U}$ we have, by using Theorem 1, that $\mathcal{V}_\alpha(u,x) \geq W(x)$ for any $x \in \mathbf{X}$ and also that $\mathcal{V}_\alpha(\widehat{\varphi},x) = W(x)$ for any $x \in \mathbf{X}$. Indeed from (16) and (17) we have that

$$\begin{aligned} \mathcal{X}W(\xi_s) - \alpha W(\xi_s) + \int_{\mathbf{A}^g} \{ & C^g(\xi_s,a) \\ & + \int_{\mathbf{X}} W(y) Q(dy|\xi_s,a)\lambda(\xi_s,a) - W(\xi_s)\lambda(\xi_s,a)\}\pi(da|s)] \geq 0 \end{aligned}$$

and, for any $z \in \boldsymbol{\Xi}$,

$$\int_{\mathbf{A}^i} \{C^i(\xi_{T_n-},a) + \int_{\mathbf{X}} W(y)Q(dy|\xi_{T_n-},a)\}\gamma(da|T_n-) - W(\xi_{T_n-}) \geq 0$$

with equality whenever the strategy $\widehat{\varphi}$ is used. From (15) the terms inside the expected value are positive, being zero whenever the strategy $\widehat{\varphi}$ is used, which shows that $\mathcal{V}_\alpha(u,x) \geq W(x)$ and $\mathcal{V}_\alpha(\widehat{\varphi},x) = W(x)$ as desired. Finally from (21) we have c) since $\mathcal{V}_\alpha^*(x) = W(x) = \sup_{i \in \mathbb{N}} W_i(x)$. □

6 The average control problem

The objective of this section is to provide sufficient conditions to show the existence of a solution to an integro-differential HJB inequality as well as the existence on optimal selector. This results is proved by using the so-called vanishing discount approach. The second main result of this section (see Theorem 4) gives the existence of a deterministic stationary optimal policy for the infinite-horizon long run average continuous-time control problem according to Definition 3.

Let us introduce

$$m_\alpha = \inf_{x\in\overline{\mathbf{X}}} \mathcal{V}^*_\alpha(x),\ \ \rho_\alpha = \alpha m_\alpha, \tag{23}$$

$$h_\alpha(x) = \mathcal{V}^*_\alpha(x) - m_\alpha \geq 0, \tag{24}$$

where $x \in \overline{\mathbf{X}}$. In what follows we refer to section 2 for the definition of the generalized inferior limit $\underline{\lim}^g$. The following final assumption will be required.

Assumption D. $\underline{\lim}^g_{\alpha\to 0} h_\alpha(x) < \infty$ for all $x \in \overline{\mathbf{X}}$.

It is easy to show that there exist a sequence $\{\alpha_n\}$ satisfying $\lim_{n\to\infty} \alpha_n = 0$ and such that $\lim_{n\to\infty} \rho_{\alpha_n} = \rho$ for some $|\rho| \leq K_C + K$. To see this, observe that by combining equations (14), (20) and (23) we obtain that for any $0 < \alpha < 1$

$$|\rho_\alpha| = |\alpha m_\alpha| \leq \alpha |\inf_{x\in\overline{\mathbf{X}}} \mathcal{V}^*_\alpha(x)| \leq \alpha \sup_{x\in\overline{\mathbf{X}}} |\mathcal{V}^*_\alpha(x)| \leq \alpha K_\alpha + K \leq K_C + K. \tag{25}$$

Let us introduce the function h_* given by

$$h_*(x) = \underline{\lim}^g_{n\to\infty} h_{\alpha_n}(x). \tag{26}$$

It is easy to see that $h_*(x) \geq 0$ since $h_\alpha(x) \geq 0$. Clearly,$h_*(x) < \infty$ by Assumption D and $h_* \in \mathbb{L}_+(\overline{\mathbf{X}})$ by using Proposition 1.

Before showing the main results of this section, we need the following technical result.

Lemma 5 *The function h_* defined in* (26) *satisfies the following inequality:*

$$h_*(x) \geq \int_{[0,t^*(x)[} e^{-Ks}(\Re h_*(\phi(x,s)) - \rho)ds + e^{-Kt^*(x)} \mathfrak{T} h_*(\phi(x,t^*(x))). \tag{27}$$

Proof: See the Appendix. □

The following theorem provides sufficient conditions for the existence of a solution and optimal selector to an integro-differential HJB inequality, associated to the long run average control problem.

Theorem 3 *Suppose that Assumptions A, B, C and D are satisfied. Then the following holds:*

a) There exist $H \in \mathbb{A}(\overline{\mathbf{X}}) \cap \mathbb{L}(\overline{\mathbf{X}})$ bounded from below satisfying

$$\rho \geq \mathcal{X} H(x) + \inf_{a\in A^g(x)} \Big\{ C^g(x,a) + qH(x,a) \Big\}, \tag{28}$$

for any $x \in \mathbf{X}$, and

$$H(z) \geq \inf_{b\in A^i(z)} \Big\{ C^i(z,b) + QH(z,b) \Big\}, \tag{29}$$

for any $z \in \Xi$.

b) There is a measurable mapping $\widehat{\varphi} : \overline{\mathbf{X}} \to \mathbf{A}$ *such that* $\widehat{\varphi}(y) \in \mathbf{A}(y)$ *for any* $y \in \overline{\mathbf{X}}$ *and satisfying*

$$C^g(x, \widehat{\varphi}(x)) + qH(x, \widehat{\varphi}(x)) = \inf_{a \in \mathbf{A}(x)} \left\{ C^g(x,a) + qH(x,a) \right\}, \tag{30}$$

for any $x \in \mathbf{X}$, *and*

$$C^i(z, \widehat{\varphi}(z)) + QH(z, \widehat{\varphi}(z)) = \inf_{b \in \mathbf{A}(z)} \left\{ C^i(z,b) + QH(z,b) \right\}, \tag{31}$$

for any $z \in \Xi$.

Proof: Let us introduce $H(x)$ as

$$H(x) = \int_{[0,t^*(x)[} e^{-Ks}(\Re h_*(\phi(x,s)) - \rho)ds + e^{-Kt^*(x)}\mathfrak{T}h_*(\phi(x,t^*(x))), \tag{32}$$

for all $x \in \overline{\mathbf{X}}$.

We will prove first item a). Observe that $H(x) = \mathfrak{B}_0 h_*(x) - \rho \int_{[0,t^*(x)[} e^{-Ks} ds$. Now by Lemma 3 it follows that H is bounded below and that $H \in \widehat{\mathbb{L}}(\overline{\mathbf{X}})$ since $h_* \in \mathbb{L}(\overline{\mathbf{X}})$ and t^* is continuous by Assumption (B5). Observe that equation (27) implies that $H(x) \leq h_*(x)$ showing that $H \in \mathbb{L}(\overline{\mathbf{X}})$.

A straightforward application of Lemma 2 shows that $H(x) \in \mathbb{A}(\overline{\mathbf{X}})$ and it also follows that there exists a bounded from below measurable function $\mathcal{X}H$ satisfying

$$-KH(x) + \mathcal{X}H(x) + \inf_{a \in \mathbf{A}(x)} \left\{ C^g(x,a) + qh_*(x,a) + Kh_*(x) \right\} = \rho \tag{33}$$

for any $x \in \mathbf{X}$ and

$$H(z) = \inf_{b \in \mathbf{A}(z)} \left\{ C^i(z,b) + Qh_*(z,b) \right\}, \tag{34}$$

for any $z \in \Xi$. Recalling that $h_*(x) \geq H(x)$, we obtainfrom (33) and (34) that for any $x \in \mathbf{X}$,

$$\begin{aligned} &\mathcal{X}H(x) + \inf_{a \in \mathbf{A}(x)} \left\{ C^g(x,a) + qH(x,a) \right\} \\ &\qquad \leq -KH(x) + \mathcal{X}H(x) + \inf_{a \in \mathbf{A}(x)} \left\{ C^g(x,a) + qh_*(x,a) + Kh_*(x) \right\} = \rho \end{aligned} \tag{35}$$

and for any $z \in \Xi$,

$$\inf_{b \in \mathbf{A}(z)} \left\{ C^i(z,b) + QH(z,b) \right\} \leq \inf_{b \in \mathbf{A}(z)} \left\{ C^i(z,b) + Qh_*(z,b) \right\} = H(z). \tag{36}$$

Combining equations (35), (36), we finally get that $H \in \mathbb{A}(\overline{\mathbf{X}}) \cap \mathbb{L}(\overline{\mathbf{X}})$ and satisfies equations (28) and (29) giving item a).

Item b) is an easy consequence of the fact that H is lower semicontinuous on $\overline{\mathbf{X}}$, Assumptions A, B, C and Proposition D.5 in [21]. □

The goal now is to establish a deterministic stationary optimal policy for the long run average control problem as defined in Definition 3, based on a solution for the integro-differential HJB inequality (28), (29) and its associated optimal selector (30), (31). In order to do that we introduce the following notation for a measurable selector φ, a function $W \in \mathbb{M}(\overline{\mathbf{X}})$ bounded from below, and any $x \in \mathbf{X}$,

$$\begin{aligned}
\lambda^{\varphi}(x) &= \lambda(x,\varphi(x)), \;\; \Lambda^{\varphi}(x,t) = \int_0^t \lambda^{\varphi}(\phi(x,s))ds, \\
Q^{\varphi}W(x) &= QW(x,\varphi(x)), \;\; q^{\varphi}W(x) = qW(x,\varphi(x)), \\
\lambda^{\varphi}Q^{\varphi}W(x) &= \lambda(x,\varphi(x))QW(x,\varphi(x)), \\
C^{g,\varphi}(x) &= C^{g}(x,\varphi(x)), \;\; C^{i,\varphi}(z) = C^{z}(x,\varphi(z)), \;\; z \in \mathbf{\Xi}
\end{aligned}$$

and for ρ, $\widehat{\varphi}$ as in Theorem 3,

$$\begin{aligned}
G^{\widehat{\varphi}}W(x) &= \int_{]0,t^*(x)[} e^{-\Lambda^{\widehat{\varphi}}(x,s)} \lambda^{\widehat{\varphi}} Q^{\widehat{\varphi}} W(\phi(x,s)))ds + e^{\Lambda^{\widehat{\varphi}}(x,t^*(x))} Q^{\widehat{\varphi}} W(\phi(x,t^*(x))), \\
L^{\widehat{\varphi}}W(x) &= \int_{]0,t^*(x)[} e^{-\Lambda^{\widehat{\varphi}}(x,s)} W(\phi(x,s))ds, \\
\mathcal{L}^{\widehat{\varphi}}(x) &= \int_{]0,t^*(x)[} e^{-\Lambda^{\widehat{\varphi}}(x,s)} ds, \\
P^{\widehat{\varphi}}W(x) &= e^{-\Lambda^{\widehat{\varphi}}(x,t^*(x))} W(\phi(x,t^*(x)), \\
\mathcal{T}^{\widehat{\varphi}}(\rho,W)(x) &= -\rho\mathcal{L}^{\widehat{\varphi}}(x) + L^{\widehat{\varphi}}C^{g,\widehat{\varphi}}(x) + P^{\widehat{\varphi}}C^{i,\widehat{\varphi}}(x) + G^{\widehat{\varphi}}W(x).
\end{aligned}$$

We have the following auxiliary result.

Lemma 6 *For H and ρ, $\widehat{\varphi}$ as in Theorem 3 we have that*

$$H(x) \geq \mathcal{T}^{\varphi}(\rho,H)(x) \tag{37}$$

$$J_m^{\widehat{\varphi}}(t,x) \leq H(x) \tag{38}$$

where

$$\begin{aligned}
J_m^{\widehat{\varphi}}(t,x) =& \mathbb{E}_x^{\widehat{\varphi}}\Big[\int_{]0,t\wedge T_m[} \Big[C^g(\xi_s,\widehat{\varphi})) - \rho\Big]ds\Big] \\
&+ \mathbb{E}_x^{\widehat{\varphi}}\Big[\int_{]0,t\wedge T_m[} C^i((\xi_{s-},\widehat{\varphi}))dp^*(s) + \mathcal{T}^{\widehat{\varphi}}(\rho,H)(\xi_{t\wedge T_m})\Big].
\end{aligned}$$

Proof: See the Appendix. □

Theorem 4 *Suppose that Assumptions A, B, C and D are satisfied and consider $\widehat{\varphi}$ as in (30), (31). Then the deterministic stationary strategy $\widehat{\varphi}$ is optimal for the average cost problem and for any $x \in \mathbf{X}$,*

$$\rho = \mathcal{A}(\widehat{\varphi}, x) = \mathcal{A}^*(x). \tag{39}$$

Proof: Applying Proposition 4.6 in [8] it follows that $\overline{\lim}_{\alpha\downarrow 0}\alpha\mathcal{V}_\alpha^*(x) \le \mathcal{A}^*(x)$. Therefore,

$$\rho = \lim_{n\to\infty}\alpha_n \inf_{x\in\overline{\mathbf{X}}}\mathcal{V}_{\alpha_n}^*(x) \le \overline{\lim_{n\to\infty}}\,\alpha_n\mathcal{V}_{\alpha_n}^*(x) \le \mathcal{A}^*(x).$$

To get the reverse inequality, first observe that, since $\mathcal{T}^{\widehat{\varphi}}(\rho, H)$ is bounded from below by, say, $-c_0$, we obtain from Lemma 6 that

$$-c_0 + \mathbb{E}_x^{\widehat{\varphi}}\Big[\int_{]0,t\wedge T_m[}\Big[C^g(\xi_s,\widehat{\varphi}))\Big]ds + \int_{]0,t\wedge T_m[} C^i((\xi_{s-},\widehat{\varphi}))dp^*(s)\Big] \\ \le H(x) + \rho\mathbb{E}_x^{\widehat{\varphi}}(t\wedge T_m).$$

Taking the limit as m goes to infinity, this yields

$$-c_0 + \mathbb{E}_x^{\widehat{\varphi}}\Big[\int_{]0,t[}\Big[C^g(\xi_s,\widehat{\varphi}))\Big]ds + \int_{]0,t[} C^i((\xi_{s-},\widehat{\varphi}))dp^*(s)\Big] \le H(x) + \rho t,$$

and so,

$$\mathcal{A}(x,\widehat{\varphi})(x) \le \rho.$$

However, $\mathcal{A}^*(x) \le \mathcal{A}(x,\widehat{\varphi})(x)$ giving the results. □

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Appendix

In this appendix we present the proof of some auxiliary results needed along this chapter.

Proof of Lemma 2: Write $V_n(x) = \int_{[0,t^*(x)[} e^{-\beta s}F_n(\phi(x,s))ds + e^{-\beta t^*(x)}G_n(\phi(x,t^*(x)))$ for $x \in \overline{\mathbf{X}}$ with $F_n(x) = \min\{F(x), n\}$ and $G_n(x) = \min\{G(x), n\}$ on $\mathbf{X}$ (respectively, Ξ). Now, observe that for any $x \in \mathbf{X}$, $t^*(\phi(x,t)) = t^*(x) - t$, $\phi(\phi(x,t), t^*(\phi(x,t))) = \phi(x, t^*(x))$ and $\phi(\phi(x,t), s) = \phi(x, t+s)$, for any $(t,s) \in \mathbb{R}_+^2$ with $t + s \le t^*(x)$.Then, it can be easily shown by a change of variable that for any $x \in \mathbf{X}$ and $t \in [0, t^*(x)[$,

$$V_n(\phi(x,t)) = e^{\beta t}\int_{[t,t^*(x)[} e^{-\beta s}F_n(\phi(x,s))ds + e^{\beta t}e^{-\beta t^*(x)}G_n(\phi(x,t^*(x))),$$

and so,

$$V(\phi(x,t)) = e^{\beta t} \int_{[t,t^*(x)[} e^{-\beta s} F(\phi(x,s))ds + e^{\beta t} e^{-\beta t^*(x)} G(\phi(x,t^*(x))) \tag{40}$$

by the monotone convergence theorem. Consequently, the function $V(\phi(x,\cdot))$ is absolutely continuous on $[0,t^*(x)]\cap\mathbb{R}_+$ and so, $V \in \mathbb{A}(\overline{\mathbf{X}})$. Equation (40) implies that for any $x \in \mathbf{X}$ $\mathcal{X}V(\phi(x,t)) = \beta V(\phi(x,t)) - F(\phi(x,t))$, almost everywhere w.r.t. the Lebesgue measure on $[0,t^*(x)[$. This implies that $-\beta V(x) + \mathcal{X}V(x) = -F(x)$ for any $x \in \mathbf{X}$. Moreover, we have $V(z) = G(z)$ for any $z \in \Xi$, showing the result. □

Proof of the Lemma 3: Define $V_n(x) = \min\{V(x), n\}$ so that $V_n \in \mathbb{L}_b(\overline{\mathbf{X}})$. By using hypotheses (B2)-(B3) and the fact that λ is bounded by K on $\mathbf{K}^g$, we obtain that $qV_n + KV_n \in \mathbb{L}(\mathbf{K}^g)$, and so, by Assumption (C2) $C_0^g + qV_n + KV_n \in \mathbb{L}(\mathbf{K}^g)$. Therefore, combining Lemma 17.30 in [1] with Assumptions (B1) and (C1), it yields that $\mathfrak{R}V_n \in \mathbb{L}(\mathbf{X})$. By using the same arguments, it can be shown that $\mathfrak{T}V_n \in \mathbb{L}(\Xi)$.

Now consider $y \in \overline{\mathbf{X}}$ and a sequence $\{y_n\}_{n\in\mathbb{N}}$ in $\overline{\mathbf{X}}$ converging to y. By a slight abuse of notation, for any $y \in \mathbf{X}$, $I_{[0,t^*(y)[}(t)\ e^{-(K+\alpha)t}\ \mathfrak{R}V_n(\phi(y,t))$ denotes the function defined on $\mathbb{R}_+$ which is equal to $e^{-(K+\alpha)t}\mathfrak{R}V_n(\phi(y,t))$ on $[0,t^*(y)[$ and zero elsewhere. It can be shown easily by using the lower semicontinuity of the function $\mathfrak{R}V_n$ and the continuity of the flow ϕ that $\underline{\lim}_{n\to\infty} I_{[0,t^*(y_n)[}(t)\ e^{-(K+\alpha)t}\mathfrak{R}V_n(\phi(y_n,t)) \geq I_{[0,t^*(y)[}(t)\ e^{-(K+\alpha)t}\mathfrak{R}V_n(\phi(y,t))$, for any $t \in [0,t^*(y)[$. An application of Fatou's Lemma gives that

$$\underline{\lim}_{n\to\infty} \int_{[0,t^*(y_n)[} e^{-(K+\alpha)t}\mathfrak{R}V_n(\phi(y_n,t))dt \geq \int_{[0,t^*(y)[} e^{-(K+\alpha)t}\mathfrak{R}V_n(\phi(y,t))dt.$$

The case $t^*(y) = \infty$ is trivial. Now, if $t^*(y) < \infty$ then combining the lower semicontinuity of the function $\mathfrak{T}V$ with the continuity of the flow ϕ and t^* (see Assumptions (B4)-(B5)), it gives easily that

$$\underline{\lim}_{n\to\infty} e^{-(K+\alpha)t^*(y_n)}\mathfrak{T}V_n(\phi(y_n,t^*(y_n))) \geq e^{-(K+\alpha)t^*(y)}\mathfrak{T}V_n(\phi(y,t^*(y))),$$

showing the results hold for V_n, that is, $\mathfrak{R}V_n \in \mathbb{L}_b(\mathbf{X})$, $\mathfrak{T}V_n \in \mathbb{L}_b(\Xi)$, and $\mathfrak{B}_\alpha V_n \in \mathbb{L}_b(\overline{\mathbf{X}})$. From Proposition 10.1 in [27], it follows that $\mathfrak{R}V = \lim_{n\to\infty}\mathfrak{R}V_n \in \widehat{\mathbb{L}}(\mathbf{X})$ and similarly, $\mathfrak{T}V = \lim_{n\to\infty}\mathfrak{T}V_n \in \widehat{\mathbb{L}}(\Xi)$. Now, from the monotone convergence theorem, we have $\mathfrak{B}_\alpha V = \lim_{n\to\infty}\mathfrak{B}_\alpha V_n$, and so $\mathfrak{B}_\alpha V \in \widehat{\mathbb{L}}(\overline{\mathbf{X}})$. Clearly, these functions are bounded from below, giving the result. □

Proof of the Lemma 5: From Theorem 2 we have that $W(x) = \mathcal{V}_\alpha^*(x)$ satisfies (16) and (17), and thus from (23), (24) and after some algebraic manipulations we obtain that

$$-(\alpha+K)h_\alpha(x) + \mathcal{X}h_\alpha(x) + \inf_{a\in A^g(x)}\left\{C^g(x,a) + qh_\alpha(x,a) + Kh_\alpha(x)\right\} - \rho_\alpha = 0, \tag{41}$$

for any $x \in \mathbf{X}$,

$$h_\alpha(z) = \inf_{b\in A^i(z)} \Big\{C^i(z,b) + Qh_\alpha(z,b)\Big\}, \tag{42}$$

for any $z \in \boldsymbol{\Xi}$. Moreover, according to Theorem 2 there exists a measurable selector $\widehat{\varphi}_\alpha : \overline{\mathbf{X}} \to \mathbf{A}$ satisfying $\widehat{\varphi}_\alpha(y) \in \mathbf{A}(y)$ for any $y \in \overline{\mathbf{X}}$ reaching the infimum in (41) and (42). Thus,

$$-(\alpha+K)h_\alpha(x) + \mathcal{X}h_\alpha(x) + C^g(x,\widehat{\varphi}_\alpha(x)) + qh_\alpha(x,\widehat{\varphi}_\alpha(x)) + Kh_\alpha(x) - \rho_\alpha = 0, \tag{43}$$

for any $x \in \mathbf{X}$,

$$h_\alpha(z) = C^i(z,b) + Qh_\alpha(z,\widehat{\varphi}_\alpha(z)) \tag{44}$$

for any $z \in \boldsymbol{\Xi}$. Taking the integral of (43) along the flow $\phi(x,t)$, we get from (43) and (44) (see [8]) that for any $x \in \mathbf{X}$,

$$h_\alpha(x) = \int_{[0,t^*(x)[} e^{-(K+\alpha)t}\left(\mathfrak{R}h_\alpha(\phi(x,t)) - \rho_\alpha\right)dt + e^{-(K+\alpha)t^*(x)}\mathfrak{T}h_\alpha(\phi(x,t^*(x))), \tag{45}$$

where we recall that

$$\mathfrak{R}h_\alpha(y) = C^g(y,\widehat{\varphi}_\alpha(y)) + qh_\alpha(y,\widehat{\varphi}_\alpha(y)) + Kh_\alpha(y), \quad y \in \mathbf{X}, \tag{46}$$

$$\mathfrak{T}h_\alpha(z) = C^i(z,\widehat{\varphi}_\alpha(z)) + Qh_\alpha(z,\widehat{\varphi}_\alpha(z)), \quad z \in \boldsymbol{\Xi}. \tag{47}$$

According to Proposition 1 (i), we can find a sequence $\{x_n\} \in \mathbf{X}$ such that $x_n \to x$ and $\underline{\lim}_{n\to\infty} h_{\alpha_n}(x_n) = h_*(x)$. In what follows set, for notational simplicity, $x_n(t) = \phi(x_n,t)$, $x(t) = \phi(x,t)$, $a_n(t) = \widehat{\varphi}_{\alpha_n}(x_n(t))$, $t_n^* = t^*(x_n)$. From continuity of t^* and ϕ (see Assumption (B4)) we have that, as $n \to \infty$, $x_n(t) \to x(t)$, and, whenever $t^*(x) < \infty$, $x_n(t_n^*) \to \phi(x,t^*(x))$. From the fact that $\mathfrak{R}h_\alpha$ is bounded from below and ρ_α is bounded, we can apply the Fatou's lemma in (45) to obtain that

$$\begin{aligned} h_*(x) = \underline{\lim}_{n\to\infty} h_{\alpha_n}(x_n) \geq \int_{]0,+\infty[} \underline{\lim}_{n\to\infty} \Big(I_{[0,t_n^*)}(t)e^{-(K+\alpha_n)t}\left[\mathfrak{R}h_{\alpha_n}(x_n(t)) - \rho_{\alpha_n}\right]\Big)dt \\ + \underline{\lim}_{n\to\infty} e^{-(K+\alpha_n)t_n^*}\mathfrak{T}h_{\alpha_n}(x_n(t_n^*)). \end{aligned} \tag{48}$$

The convergence of ρ_{α_n} to ρ together with Assumption (B5) implies that, *a.s.* on $[0,\infty)$,

$$\begin{aligned} &\underline{\lim}_{n\to\infty} I_{[0,t_n^*)}(t)e^{-(K+\alpha_n)t}\Big\{\mathfrak{R}h_{\alpha_n}(x_n(t)) - \rho_{\alpha_n}\Big\} \\ &\qquad = I_{[0,t^*(x))}(t)e^{-Kt}\Big\{\underline{\lim}_{n\to\infty} \mathfrak{R}h_{\alpha_n}(x_n(t)) - \rho\Big\}, \end{aligned} \tag{49}$$

and $\underline{\lim}_{n\to\infty} e^{-(K+\alpha_n)t_n^*}\mathfrak{T}h_{\alpha_n}(x_n(t_n^*)) = e^{-Kt^*(x)}\,\underline{\lim}_{n\to\infty}\mathfrak{T}h_{\alpha_n}(x_n(t_n^*))$. The goal now is to show that

$$\underline{\lim}_{n\to\infty} \Re h_{\alpha_n}(x_n(t))) \geq \Re h_*(x(t)), \tag{50}$$

and that

$$\underline{\lim}_{n\to\infty} \Im h_{\alpha_n}(x_n(t_n^*)) \geq \Im h_\alpha(x(t^*(x))). \tag{51}$$

Let us first show (50). For a fixed $t \in (0, t^*(x))$, there is no loss of generality in assuming that $t < t_n^*$ for any $n \in \mathbb{N}$ and thus $x_n(t) \in \mathbf{X}$. Consider a subsequence $\{n_j\}$ of $\{n\}$ such that

$$\underline{\lim}_{n\to\infty} \Re h_{\alpha_n}(x_n(t)) = \lim_{j\to\infty} \Re h_{\alpha_{n_j}}(x_{n_j}(t)).$$

From Assumptions (B1) and (C1) the multifunction Ψ^g is compact valued and upper semi-continuous so that, from the fact that $x_{n_j}(t) \to x(t)$, we can find a subsequence of $\{a_{n_j}(t)\} \in \mathbf{A}(x_{n_j}(t))$, still denoted by $\{a_{n_j}(t)\}$ such that $a_{n_j}(t) \to a \in \mathbf{A}(x(t))$ (see Theorem 17.16 in [1]) as $j \to \infty$. From (46) we have that

$$\begin{aligned}\underline{\lim}_{n\to\infty} \Re h_{\alpha_n}(x_n(t)) = \lim_{j\to\infty} &\Big(C^g(x_{n_j}(t), a_{n_j}(t)) + q h_{\alpha_{n_j}}(x_{n_i}(t), x_{n_j}(t))\Big)\\ &+ \lim_{j\to\infty} \Big(K h_{\alpha_{n_j}}(x_{n_j}(t))\Big),\end{aligned}$$

and therefore

$$\begin{aligned}\underline{\lim}_{n\to\infty} \Re h_{\alpha_n}(x_n(t)) \geq\ &\underline{\lim}_{j\to\infty} C^g(x_{n_j}(t), a_{n_j}(t))\\ &+ \underline{\lim}_{j\to\infty} \Big(q h_{\alpha_{n_j}}(x_{n_i}(t), x_{n_j}(t)) + K h_{\alpha_{n_j}}(x_{n_j}(t))\Big).\end{aligned} \tag{52}$$

Lower semicontinuity of C^g on $\mathbf{K}^g$ yields to

$$\underline{\lim}_{j\to\infty} C^g(x_{n_j}(t), a_{n_j}(t)) \geq C^g(x(t), a). \tag{53}$$

From Proposition 1 (i) and (iii), the fact that Q is weakly continuous on $\mathbf{K}^g$ (Assumption (B2)), and the continuity of λ (Assumption (B3)), we get that

$$\underline{\lim}_{j\to\infty} \lambda(x_{n_j}(t), a_{n_j}(t)) Q h_{\alpha_{n_j}}(x_{n_j}(t), a_{n_j}(t)) \geq \lambda(x(t), a) Q h_*(x(t), a) \tag{54}$$

and, recalling that $K - \lambda(x_{n_j}(t), a_{n_j}(t)) \geq 0$ from Assumption (A1), we get that

$$\underline{\lim}_{j\to\infty} [K - \lambda(x_{n_j}(t), a_{n_j}(t))] h_{\alpha_{n_j}}(x_{n_j}(t), a_{n_j}(t)) \geq [K - \lambda(x(t), a)] h_*(x(t), a). \tag{55}$$

Combining (46), (52), (53), (54), (55), we conclude that

$$\underline{\lim}_{n\to\infty} \Re h_{\alpha_n}(x_n(t))) \geq C^g(x(t), a) + q h_*(x(t), a) + K h_*(x(t)) \geq \Re h_*(x(t)),$$

showing (50).

Let us now show (51) for $t^*(x) < \infty$. From the fact that Ψ^i is compact valued and upper semi-continuous and $x_n(t_n^*) \to x(t^*(x))$, and using similar arguments as before (in particular equation (47)), we can find a subsequence $\{a_{n_j}(t_{n_j}^*)\} \in \mathbf{A}(x_{n_j}(t_{n_j}^*))$ such that $a_{n_j}(t_{n_j}^*) \to b \in \mathbf{A}(x(t^*(x)))$ (see again Theorem 17.16 in [1]), and that

$$\underline{\lim}_{n\to\infty} \mathfrak{T}h_{\alpha_n}(x_n(t_n^*)) \geq C^i(x(t^*(x)),b) + Qh_*(x(t^*(x)),b) \geq \mathfrak{T}h_\alpha(x(t^*(x)))$$

showing (51).

Combining (48), (49), (50) and (51) we get that (27) holds, showing Lemma 5. □

Proof of the Lemma 6: From Theorem 3 we get that for any $x \in \mathbf{X}$

$$\rho \geq \mathcal{X}H(\phi(x,s)) + C^{g,\widehat{\varphi}}(\phi(x,s)) + q^{\widehat{\varphi}}H((\phi(x,s))), \tag{56}$$

and for the case $t^*(x) < \infty$,

$$H(\phi(x,t^*(x))) \geq C^{i,\widehat{\varphi}}(\phi(x,t^*(x))) + Q^{\widehat{\varphi}}H(\phi(x,t^*(x))). \tag{57}$$

Multiplying (56) by $e^{-\Lambda^{\widehat{\varphi}}(x,s)}$ and taking the integral from 0 to t we obtain that

$$\begin{aligned}\rho\int_{]0,t[} e^{-\Lambda^{\widehat{\varphi}}(x,s)}ds &\geq \int_{]0,t[} e^{-\Lambda^{\widehat{\varphi}}(x,s)}(\mathcal{X}H(\phi(x,s)) - \lambda^{\widehat{\varphi}}(\phi(x,s))H(\phi(x,s)))ds \\ &+ \int_{]0,t[} e^{-\Lambda^{\widehat{\varphi}}(x,s)}C^{g,\widehat{\varphi}}(\phi(x,s))ds + \int_{]0,t[} e^{-\Lambda^{\widehat{\varphi}}(x,s)}\lambda^{\widehat{\varphi}}Q^{\widehat{\varphi}}H(\phi(x,s))ds. \end{aligned} \tag{58}$$

Replacing

$$\begin{aligned}&\int_{]0,t[} e^{-\Lambda^{\widehat{\varphi}}(x,s)}(\mathcal{X}H(\phi(x,s)) - \lambda^{\widehat{\varphi}}(\phi(x,s))H(\phi(x,s)))ds \\ &\qquad = e^{-\Lambda^{\widehat{\varphi}}(x,t)}H(\phi(x,t)) - H(x)\end{aligned}$$

into (58) yields to

$$\begin{aligned}H(x) \geq &-\rho\int_{]0,t[} e^{-\Lambda^{\widehat{\varphi}}(x,s)}ds + \int_{]0,t[} e^{-\Lambda^{\widehat{\varphi}}(x,s)}\lambda^{\widehat{\varphi}}Q^{\widehat{\varphi}}H(\phi(x,s))ds \\ &+ e^{-\Lambda^{\widehat{\varphi}}(x,t)}H(\phi(x,t)) + \int_{]0,t[} e^{-\Lambda^{\widehat{\varphi}}(x,s)}C^{g,\widehat{\varphi}}(\phi(x,s))ds.\end{aligned}$$

Taking the limit as $t \to t^*(x)$ and using (57) for the case $t^*(x) < \infty$ we obtain (37). From (37) and Proposition 3.4 in [8] we obtain (38).

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