Pairs Trading under Geometric Brownian Motion Models

Phong Luu, Jingzhi Tie, and Qing Zhang

Abstract This survey paper is concerned with pairs trading strategies under geometric Brownian motion models. Pairs trading is about trading simultaneously a pair of securities, typically stocks. The idea is to monitor the spread of their price movements over time. A pairs trade is triggered by their price divergence (e.g., one stock moves up a significant amount relative to the other) and consists of a short position in the strong stock and a long position in the weak one. Such a strategy bets on the reversal of their price strengths and the eventual convergence of the price spread. Pairs trading is popular among trading institutions because its risk neutral nature. In practice, the trader needs to decide when to initiate a pairs position (how much divergence is enough) and when to close the position (how to take profits or cut losses). It is the main goals of this paper to address these issues and theoretical findings along with related practical considerations.

Key words: pairs trading, optimal trading strategy, geometric Brownian motions

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1 Introduction

This paper is about strategies for simultaneously trading a pair of stocks. The idea is to track the price movements of these two securities over some period of time and compare their relative price strength. A pairs trade is triggered when their prices diverge, e.g., one stock moves up substantially relative to the other. A pairs trade is entered and consists of a short position in the stronger stock and a long position in the weaker one. Such a strategy bets on the reversal of their price strength and eventual convergence of their price spread.

Pairs trading was introduced by Garry Bamberger and followed by Nunzio Tartaglia's quantitative group at Morgan Stanley in the 1980s. Tartaglia's group used advanced statistical tools and developed high tech trading systems by incorporating trader's intuition and disciplined filtering rules. They were able to identify pairs of stocks and trade them with a great success. See Gatev et al. [7] for related background details. In addition, there are studies addressing why pairs trading works. For related in-depth discussions in connection with the cause of the price divergence and subsequent convergence, we refer the reader to the books by Vidyamurthy [21] and Whistler [22].

Empirical studies and related considerations can be found in papers by Do and Faff [4, 5], Gatev et al. [7], and books by Vidyamurthy [21] and Whistler [22]. Issues involved in these works include statistical characterization of the spread process, performance of pairs trading with various trading thresholds, and the impact of trading costs in connection with pairs trading.

A major advantage of pairs trading is its 'market neutral' nature in the sense that it helps to hedge market risks. For example, if the market crashes and takes both stocks with it, the trade would result in a gain on the short side and a loss on the long side of the position. The gain and loss cancel out each other and to some extent, reduce the market risk.

In pairs trading, a crucial step is to determine when to initiate a pairs trade (i.e., how much spread divergence is sufficient to trigger a trade) and when to close the position (when to lock in profits). Following empirical developments documented in Gatev et al. [7], increasing efforts were made addressing theoretical aspects of pairs trading. The main focus was devoted to development of mathematical models that capture the spread movements, filtering techniques, optimal entry and exit timings, money management and risk control. For example, in Elliott et al. [6], the price spread is assumed to be a mean reversion process with additive noise. Several filtering techniques were explored to identify entry points. One exit rule with a fixed holding period was discussed in detail. In Deshpande and Barmish [3], a general (mean-reversion based) framework was developed. Using a 'spread' function, they were able to determine the numbers of shares of each stock every moment and how to adjust them over time. They showed that such an algorithm leads to positive expected returns.

Some recent efforts on pairs trading have been devoted to in-depth analysis based on mean reversion models. For example, Kuo et al. [11] considered an optimal selling rule. The objective is to determine the time of closing an existing pairs position in order to maximize an expected return or to cut losses short. In particular, given a fixed cut-loss level, the optimal target level can be determined under a mean reversion model. Further results on mean reversion models can be found in Song and Zhang [18]. They have developed a complete system with both entry and exit signals. They have shown that the optimal trading rule can be determined by threshold levels. The calculation of these levels only involves algebraic equations.

We would like to point out that almost all literature on pairs trading is mean reversion based one way or the other. On the one hand, this makes the trading more intuitive. On the other, such constraint adds a severe limitation on its potential applications. In order to meet the mean-reversion requirement, tradable pairs are typically selected among stocks from the same industrial sector. From a practical viewpoint, it is highly desirable to have a broad range of stock selections for pairs trading. Mathematically speaking, this amounts to the possibility of treating pairs trading under models other than mean reversion. In Tie et al. [19], they have developed a new method to treat the pairs-trading problem under general geometric Brownian motions. In particular, under a two-dimensional geometric Brownian motion model, they were able to fully characterize the optimal policy in terms of two threshold lines obtained by solving the associated variational inequalities. The principal idea of pairs trading is that one builds the position of a pair when the cost is low and closes the position when the pairs' value is high. These two threshold switching lines quantify exactly how low is low and how high is high. These policies are easy to compute and implement. The most striking feature of these results is the simplicity of the solution: Clean-cut assumptions and closed-form trading policies.

One important consideration in trading has yet received deserved attention: How to trade with cutting losses. There are many scenarios when cutting losses may arise. A typical one is a margin call. This is often proceeded with heavy losses leading to an enforced closure of part or the entire pairs position. Often in practice, a pairs trader chooses a pre-determined stop-loss level due to a money management consideration. From a modeling point of view, the prices of the pairs may cease to behave as the model prescribes due to undesirable events such as acquisition (or bankruptcy) of one stock in the pairs position. It is necessary to modify the trading rule accordingly in order to accommodate a pre-determined stop loss level. On the other hand, from a control theoretical viewpoint, forcing a stop loss amounts to imposing a hard state constraint. This often poses substantial challenges when solving the problem. Such issues were addressed in Liu et al. [13] recently. They were able to establish regions in terms of threshold lines to characterize trading rules. They also obtained sufficient conditions that guarantee the optimality of these trading rules.

In this paper, we mainly involve stocks. Nevertheless, the idea of pairs trading is not limited to stock trading. For example, the optimal timing of investments in irreversible projects can also be considered as a pairs-trading problem. Back in 1986, McDonald and Siegel [15] considered optimal timing of investment in an irreversible project. Two factors are included in their model: The value of the project and its cost. Greater project value growth potential and lesser future project cost will postpone the transaction. See also Hu and Øksendal [9] for more rigorous mathematical treatment. In terms of pairs trading, their results are about a pairs trading selling rule. Extension along this line can be found in Tie and Zhang [20]. They treated the pairs selling rule under a regime-switching model. They were also able to show threshold type selling policies.

The problem under consideration is closely related to traditional portfolio selection problems. Following Merton's work in the late 60's, much progress along this direction has been made. A thorough treatment of the problem can be found in Davis and Norman [2] in which they studied Merton's investment/consumption problem with the transaction costs and established wedge-shaped regions for the pair of bank and stock holdings. To some extent, pairs trading resembles portfolio selection. Rather than balancing between bank and stock holdings, pairs trading involves positions consisting of two stocks. In portfolio selection, risk control is achieved through adjusting proportion of stock holdings; while, in pairs trading, the risk is limited by focusing on highly correlated stocks that are traded in opposite directions. Early theoretical development along portfolio selection with transaction costs using viscosity solutions can be found in Zariphopoulou [23]. Further in-depth studies and a complete solution to investment and consumption problem with transaction costs can be found in Shreve and Soner [17].

Mathematical trading rules have been studied for many years. In addition to the work by Hu and Øksendal [9] and Song and Zhang [18], Zhang [25] considered a selling rule determined by two threshold levels, a target price and a stop-loss limit. In [25], such optimal threshold levels are obtained by solving a set of two-point boundary value problems. Guo and Zhang [8] studied the optimal selling rule under a model with switching Geometric Brownian motion. Using a smooth-fit technique, they obtained the optimal threshold levels by solving a set of algebraic equations. These papers are concerned with the selling side of trading in which the underlying price models are of GBM type. Some subsequent efforts were devoted to strategies on complete trading systems including buying and selling decision making. For example, Dai et al. [1] developed a trend-following rule based on a conditional probability indicator. They showed that the optimal trading rule can be determined by two threshold curves which can be obtained by solving the associated Hamilton-Jacobi-Bellman (HJB) equations. A similar idea was developed following a confidence interval approach by Iwarere and Barmish [10]. In addition, Merhi and Zervos [16] studied an investment capacity expansion/reduction problem following a dynamic programming approach under a geometric Brownian motion market model. In connection with mean reversion trading, Zhang and Zhang [24] obtained a buy-low and sell-high policy by characterizing the 'low' and 'high' levels in terms of the mean reversion parameters.

In this paper, we focus on the mathematical aspects of pairs trading. We present key ideas used in derivation of solutions to the associated HJB equations and summarize the main results. In §2, we consider pairs trading under geometric Brownian motions. It can be seen that pairs trading ideas are more general and they do not have to be cast under a mean reversion framework. In §3, we address pairs trading with a stop-loss constraint. We establish threshold type trading policies and provide sufficient conditions that guarantee the optimality of these policies. In §4, we consider a two-dimensional geometric Brownian model with regime-switching. We focus on related optimal pairs selling rules. Proofs of these results are omitted and can be found in [13, 19, 20]. Finally, some concluding remarks are given in §5.

2 Pairs Trading under a GBM

In this section, we consider pairs trading under a two-dimensional geometric Brownian motion model. A share of pairs position $\mathbf{Z}$ consists of one share long position in stocks $\mathbf{X}^1$ and one share short position in $\mathbf{X}^2$. Let (X_t^1, X_t^2) denote their prices at t satisfying the following stochastic differential equation:

$$d\begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix} = \begin{pmatrix} X_t^1 & \\ & X_t^2 \end{pmatrix} \left[\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} dt + \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} d \begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix} \right], \tag{1}$$

where μ_i, $i = 1,2$, are the return rates, σ_{ij}, $i,j = 1,2$, the volatility constants, and (W_t^1, W_t^2) a 2-dimensional standard Brownian motion.

We consider the case that the net position at any time can be either long (with one share of $\mathbf{Z}$) or flat (no stock position of either $\mathbf{X}^1$ or $\mathbf{X}^2$). Let $i = 0,1$ denote the initial net position and let $\tau_0 \le \tau_1 \le \tau_2 \le \cdots$ denote a sequence of buying and selling (stopping) times. If initially the net position is long ($i = 1$), then one should sell $\mathbf{Z}$ before acquiring any shares in the future. That is, to first sell the pair at τ_0, then buy at τ_1, sell at τ_2, buy at τ_3, and so on. The corresponding trading sequence is denoted by $\Lambda_1 = (\tau_0, \tau_1, \tau_2, \ldots)$. Likewise, if initially the net position is flat ($i = 0$), then one should start to buy a share of $\mathbf{Z}$. That is, to first buy at τ_1, sell at τ_2, then buy at τ_3, an so forth. The corresponding sequence of stopping times is denoted by $\Lambda_0 = (\tau_1, \tau_2, \ldots)$.

Let K denote the fixed percentage of transaction costs associated with buying or selling of stocks $\mathbf{X}^i$, $i = 1,2$. For example, the cost to establish the pairs position $\mathbf{Z}$ at $t = t_1$ is $(1+K)X_{t_1}^1 - (1-K)X_{t_1}^2$ and the proceeds to close it at a later time $t = t_2$ is $(1-K)X_{t_2}^1 - (1+K)X_{t_2}^2$. For ease of notation, let $\beta_{\mathrm{b}} = 1+K$ and $\beta_{\mathrm{s}} = 1-K$.

Given the initial state (x_1, x_2), net position $i = 0,1$, and the decision sequences Λ_0 and Λ_1, the corresponding reward functions

$$\begin{aligned}
J_0(x_1,x_2,\Lambda_0) = E\Big\{&[e^{-\rho\tau_2}(\beta_s X^1_{\tau_2} - \beta_b X^2_{\tau_2}) I_{\{\tau_2<\infty\}} - e^{-\rho\tau_1}(\beta_b X^1_{\tau_1} - \beta_s X^2_{\tau_1}) I_{\{\tau_1<\infty\}}] \\
&+[e^{-\rho\tau_4}(\beta_s X^1_{\tau_4} - \beta_b X^2_{\tau_4}) I_{\{\tau_4<\infty\}} - e^{-\rho\tau_3}(\beta_b X^1_{\tau_3} - \beta_s X^2_{\tau_3}) I_{\{\tau_3<\infty\}}] + \cdots\Big\}, \\
J_1(x_1,x_2,\Lambda_1) = E\Big\{&e^{-\rho\tau_0}(\beta_s X^1_{\tau_0} - \beta_b X^2_{\tau_0}) I_{\{\tau_0<\infty\}} \\
&+[e^{-\rho\tau_2}(\beta_s X^1_{\tau_2} - \beta_b X^2_{\tau_2}) I_{\{\tau_2<\infty\}} - e^{-\rho\tau_1}(\beta_b X^1_{\tau_1} - \beta_s X^2_{\tau_1}) I_{\{\tau_1<\infty\}}] \\
&+[e^{-\rho\tau_4}(\beta_s X^1_{\tau_4} - \beta_b X^2_{\tau_4}) I_{\{\tau_4<\infty\}} - e^{-\rho\tau_3}(\beta_b X^1_{\tau_3} - \beta_s X^2_{\tau_3}) I_{\{\tau_3<\infty\}}] + \cdots\Big\},
\end{aligned} \tag{2}$$

where $\rho > 0$ is a given discount factor and I_A is the indicator function of an event A.

For $i = 0, 1$, let $V_i(x_1,x_2)$ denote the value functions with $(X^1_0, X^2_0) = (x_1,x_2)$ and initial net positions $i = 0, 1$. That is, $V_i(x_1,x_2) = \sup_{\Lambda_i} J_i(x_1,x_2,\Lambda_i)$, $i = 0, 1$.

Remark. Note that the 'one-share' assumption can be easily relaxed. For example, one can consider any pairs $\mathbf{Z}$ consisting of n_1 shares of long position in $\mathbf{X}^1$ and n_2 shares of short position in $\mathbf{X}^2$. This case can be treated by changing of the state variables $(X^1_t, X^2_t) \to (n_1 X^1_t, n_2 X^2_t)$. Due to the nature of GBMs, the corresponding system equation in (1) will stay the same. The new allocations will only affect the reward function in (2) implicitly. In addition, we only focus on the 'long' side of pairs trading and note that the 'short' side of trading can also be treated by simply switching the roles of the two stocks $\mathbf{X}^1$ and $\mathbf{X}^2$. □

Example. In this example, we consider stock prices of Target Corp. (TGT) and Wal-Mart Stores Inc. (WMT). In Figure 1, daily closing prices of both stocks from 1985 to 2014 are plotted. The data is divided into two parts. The first part (1985-1999) will be used to calibrate the model and the second part (2000-2014) to backtest the performance of our results. Using the prices (1985-1999) and following the traditional least squares method, we obtain $\mu_1 = 0.2059$, $\mu_2 = 0.2459$, $\sigma_{11} = 0.3112$, $\sigma_{12} = 0.0729$, $\sigma_{21} = 0.0729$, $\sigma_{22} = 0.2943$.

We assume **(A1):** $\rho > \mu_1$ and $\rho > \mu_2$. Under these conditions, we can show that, for all $x_1, x_2 > 0$,

$$0 \le V_0(x_1,x_2) \le x_2, \quad \text{and} \quad \beta_s x_1 - \beta_b x_2 \le V_1(x_1,x_2) \le \beta_b x_1 + K x_2. \tag{3}$$

Formally, the associated HJB equations have the form: For $x_1, x_2 > 0$,

$$\begin{aligned}
&\min\Big\{\rho v_0(x_1,x_2) - \mathcal{A} v_0(x_1,x_2),\ v_0(x_1,x_2) - v_1(x_1,x_2) + \beta_b x_1 - \beta_s x_2\Big\} = 0, \\
&\min\Big\{\rho v_1(x_1,x_2) - \mathcal{A} v_1(x_1,x_2),\ v_1(x_1,x_2) - v_0(x_1,x_2) - \beta_s x_1 + \beta_b x_2\Big\} = 0,
\end{aligned} \tag{4}$$

where

$$\mathcal{A} = \frac{1}{2}\left\{a_{11} x_1^2 \frac{\partial^2}{\partial x_1^2} + 2 a_{12} x_1 x_2 \frac{\partial^2}{\partial x_1 \partial x_2} + a_{22} x_2^2 \frac{\partial^2}{\partial x_2^2}\right\} + \mu_1 x_1 \frac{\partial}{\partial x_1} + \mu_2 x_2 \frac{\partial}{\partial x_2},$$

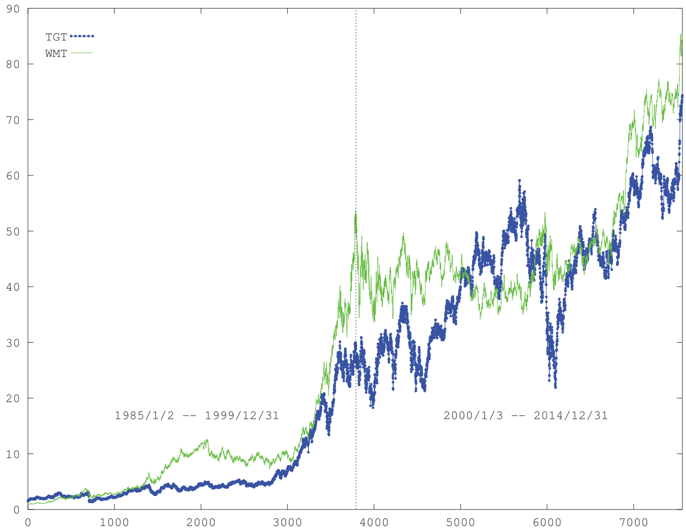

Fig. 1 Daily Closing Prices of TGT and WMT from 1985 to 2014.

and $a_{11} = \sigma_{11}^2 + \sigma_{12}^2$, $a_{12} = \sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}$, and $a_{22} = \sigma_{21}^2 + \sigma_{22}^2$.

We convert these HJB equations into single variable equations. Let $y = x_2/x_1$ and $v_i(x_1, x_2) = x_1 w_i(x_2/x_1)$, for some function $w_i(y)$ and $i = 0, 1$. By direct calculation, we have

$$\frac{\partial v_i}{\partial x_1} = w_i(y) - y w_i'(y), \; \frac{\partial v_i}{\partial x_2} = w_i'(y),$$
$$\frac{\partial^2 v_i}{\partial x_1^2} = \frac{y^2 w_i''(y)}{x_1}, \; \frac{\partial^2 v_i}{\partial x_2^2} = \frac{w_i''(y)}{x_1}, \text{ and } \frac{\partial^2 v_1}{\partial x_1 \partial x_2} = -\frac{y w_i''(y)}{x_1}.$$

We can write $\mathcal{A}v_i$ in terms of w_i and obtain

$$\mathcal{A}v_i = x_1 \left\{ \frac{1}{2}[a_{11} - 2a_{12} + a_{22}] y^2 w_i''(y) + (\mu_2 - \mu_1) y w_i'(y) + \mu_1 w_i(y) \right\}.$$

Let $\mathcal{L}[w_i(y)] = \lambda y^2 w_i''(y) + (\mu_2 - \mu_1) y w_i'(y) + \mu_1 w_i(y)$ with $\lambda = (a_{11} - 2a_{12} + a_{22})/2$. Then, the above HJB equations can be given as follows:

$$\begin{aligned} &\min \left\{ \rho w_0(y) - \mathcal{L} w_0(y), \; w_0(y) - w_1(y) + \beta_b - \beta_s y \right\} = 0, \\ &\min \left\{ \rho w_1(y) - \mathcal{L} w_1(y), \; w_1(y) - w_0(y) - \beta_s + \beta_b y \right\} = 0. \end{aligned} \tag{5}$$

In this paper, we only consider the case when $\lambda \neq 0$. If $\lambda = 0$, the problem reduces to a first order case and can be similarly treated. To solve these equations, we first focus on $(\rho - \mathcal{L})w_i(y) = 0$, $i = 0, 1$. These are the Euler equations and their solutions are of the form y^δ, for some δ. We substitute this into the equation $(\rho - \mathcal{L})w_i = 0$ and obtain the corresponding characteristic equation $\delta^2 - (1 + (\mu_1 - \mu_2)/\lambda)\delta - (\rho - \mu_1)/\lambda = 0$. There are two real roots

$$\begin{aligned} \delta_1 &= \frac{1}{2}\left(1 + \frac{\mu_1 - \mu_2}{\lambda} + \sqrt{\left(1 + \frac{\mu_1 - \mu_2}{\lambda}\right)^2 + \frac{4\rho - 4\mu_1}{\lambda}}\right) > 1, \\ \delta_2 &= \frac{1}{2}\left(1 + \frac{\mu_1 - \mu_2}{\lambda} - \sqrt{\left(1 + \frac{\mu_1 - \mu_2}{\lambda}\right)^2 + \frac{4\rho - 4\mu_1}{\lambda}}\right) < 0. \end{aligned} \tag{6}$$

The general solution of $(\rho - \mathcal{L})w_i(y) = 0$ should be of the form: $w_i(y) = c_{i1}y^{\delta_1} + c_{i2}y^{\delta_2}$, for some c_{i1} and c_{i2}, $i = 1, 2$.

Intuitively, if X_t^1 is small and X_t^2 is large, then one should buy $\mathbf{X}^1$ and sell (short) $\mathbf{X}^2$. I.e., to open a pairs position $\mathbf{Z}$. If, on the other hand, X_t^1 is large and X_t^2 is small, then one should close the pairs position $\mathbf{Z}$ by selling $\mathbf{X}^1$ and buying back $\mathbf{X}^2$. In view of this, the first quadrant $P = \{(x_1, x_2) : x_1 > 0 \text{ and } x_2 > 0\}$ into three regions Γ_1, Γ_2, and Γ_3 where $\Gamma_1 = \{(x_1, x_2) \in P : x_2 \le k_1x_1\}$, $\Gamma_2 = \{(x_1, x_2) \in P : k_1x_1 < x_2 < k_2x_1\}$, and $\Gamma_3 = \{(x_1, x_2) \in P : x_2 \ge k_2x_1\}$. This is illustrated in Figure 2.

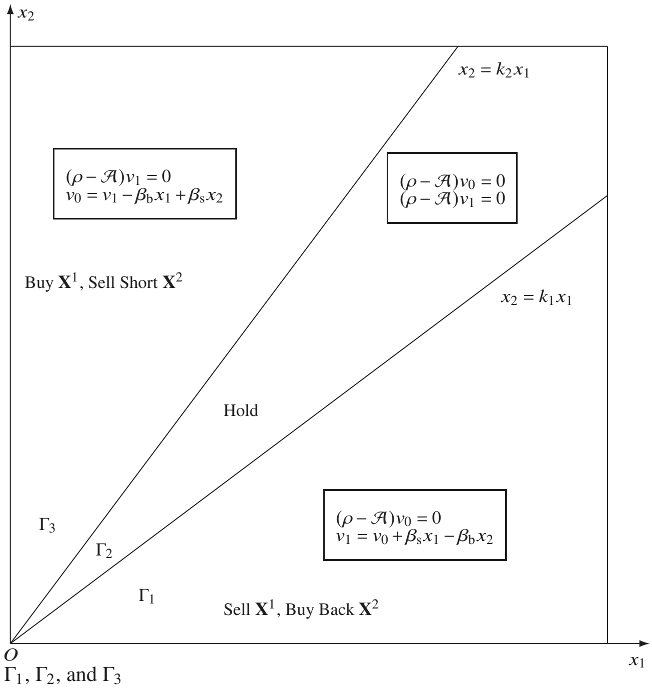

Fig. 2 Regions Γ_1, Γ_2, and Γ_3

With a little bit abuse of notation, we can write the corresponding Γ_i, $i = 1,2,3$, in terms of $y(= x_2/x_1)$: $\Gamma_1 = \{y:\ 0 < y \le k_1\}$, $\Gamma_2 = \{y:\ k_1 < y < k_2\}$, and $\Gamma_3 = \{y : y \ge k_2\}$. Here $0 < k_1 < k_2$ are slopes (thresholds) to be determined so that on

$$\begin{aligned} &\Gamma_1 : (\rho - \mathcal{L})w_0 = 0, && w_1 = w_0 + \beta_s - \beta_b y;\\ &\Gamma_2 : (\rho - \mathcal{L})w_0 = 0, && (\rho - \mathcal{L})w_1 = 0;\\ &\Gamma_3 : w_0 = w_1 - \beta_b + \beta_s y, && (\rho - \mathcal{L})w_1 = 0. \end{aligned} \tag{7}$$

Recall the boundedness of the value function in (3) and $\delta_2 < 0$. The coefficient of the term y^{δ_2} in w_0 on Γ_1 has to be zero. Thus, $w_0 = C_0 y^{\delta_1}$ for some C_0 on Γ_1. Likewise, on Γ_3, the coefficient of y^{δ_1} must be zero because $\delta_1 > 1$. The solution is $w_1 = C_1 y^{\delta_2}$ for some C_1 on Γ_3. Finally, these functions are extended to Γ_2 and are given by $w_0 = C_0 y^{\delta_1}$ and $w_1 = C_1 y^{\delta_2}$. The solutions on each region should have the form:

$$\begin{aligned} &\Gamma_1 : w_0 = C_0 y^{\delta_1}, && w_1 = C_0 y^{\delta_1} + \beta_s - \beta_b y;\\ &\Gamma_2 : w_0 = C_0 y^{\delta_1}, && w_1 = C_1 y^{\delta_2};\\ &\Gamma_3 : w_0 = C_1 y^{\delta_2} - \beta_b + \beta_s y, && w_1 = C_1 y^{\delta_2}. \end{aligned}$$

Next we use smooth-fit conditions to determine the values for parameters: k_1, k_2, C_0, and C_1. Necessarily, the continuity of w_1 and its first order derivative at $y = k_1$ imply $C_1 k_1^{\delta_2} = C_0 k_1^{\delta_1} + \beta_s - \beta_b k_1$ and $C_1 \delta_2 k_1^{\delta_2 - 1} = C_0 \delta_1 k_1^{\delta_1 - 1} - \beta_b$. These equations can be written in matrix form:

$$\begin{pmatrix} k_1^{\delta_1} & -k_1^{\delta_2} \\ \delta_1 k_1^{\delta_1 - 1} & -\delta_2 k_1^{\delta_2 - 1} \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \end{pmatrix} = \begin{pmatrix} \beta_b k_1 - \beta_s \\ \beta_b \end{pmatrix}. \tag{8}$$

Similarly, the smooth-fit conditions for w_0 at $y = k_2$ lead to equations:

$$\begin{pmatrix} k_2^{\delta_1} & -k_2^{\delta_2} \\ \delta_1 k_2^{\delta_1 - 1} & -\delta_2 k_2^{\delta_2 - 1} \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \end{pmatrix} = \begin{pmatrix} \beta_s k_2 - \beta_b \\ \beta_s \end{pmatrix}. \tag{9}$$

Solve for C_0 an C_1 and express the corresponding inverse matrices in terms of k_1 and k_2 to obtain

$$\begin{pmatrix} C_0 \\ C_1 \end{pmatrix} = \frac{1}{\delta_1 - \delta_2} \begin{pmatrix} \beta_{\rm b}(1-\delta_2)k_1^{1-\delta_1} + \beta_{\rm s}\delta_2 k_1^{-\delta_1} \\ \beta_{\rm b}(1-\delta_1)k_1^{1-\delta_2} + \beta_{\rm s}\delta_1 k_1^{-\delta_2} \end{pmatrix} = \frac{1}{\delta_1 - \delta_2} \begin{pmatrix} \beta_{\rm s}(1-\delta_2)k_2^{1-\delta_1} + \beta_{\rm b}\delta_2 k_2^{-\delta_1} \\ \beta_{\rm s}(1-\delta_1)k_2^{1-\delta_2} + \beta_{\rm b}\delta_1 k_2^{-\delta_2} \end{pmatrix}. \tag{10}$$

The second equality yields two equations of k_1 and k_2. We can simplify them and write

$$(1-\delta_2)(\beta_{\rm b}k_1^{1-\delta_1} - \beta_{\rm s}k_2^{1-\delta_1}) = \delta_2(\beta_{\rm b}k_2^{-\delta_1} - \beta_{\rm s}k_1^{-\delta_1}),$$
$$(1-\delta_1)(\beta_{\rm b}k_1^{1-\delta_2} - \beta_{\rm s}k_2^{1-\delta_2}) = \delta_1(\beta_{\rm b}k_2^{-\delta_2} - \beta_{\rm s}k_1^{-\delta_2}).$$

To solve these equations, let $r = k_2/k_1$ and replace k_2 by rk_1 to obtain

$$(1-\delta_2)(\beta_{\rm b} - \beta_{\rm s}r^{1-\delta_1})k_1 = \delta_2(\beta_{\rm b}r^{-\delta_1} - \beta_{\rm s})$$

and

$$(1-\delta_1)(\beta_{\rm b} - \beta_{\rm s}r^{1-\delta_2})k_1 = \delta_1(\beta_{\rm b}r^{-\delta_2} - \beta_{\rm s}).$$

We have

$$k_1 = \frac{\delta_2(\beta_{\rm b}r^{-\delta_1} - \beta_{\rm s})}{[(1-\delta_2)(\beta_{\rm b} - \beta_{\rm s}r^{1-\delta_1})]} = \frac{\delta_1(\beta_{\rm b}r^{-\delta_2} - \beta_{\rm s})}{[(1-\delta_1)(\beta_{\rm b} - \beta_{\rm s}r^{1-\delta_2})]}.$$

Using the second equality and write the difference of both sides, we have

$$f(r) := \delta_1(1-\delta_2)(\beta_{\rm b}r^{-\delta_2} - \beta_{\rm s})(\beta_{\rm b} - \beta_{\rm s}r^{1-\delta_1})$$
$$-\delta_2(1-\delta_1)(\beta_{\rm b}r^{-\delta_1} - \beta_{\rm s})(\beta_{\rm b} - \beta_{\rm s}r^{1-\delta_2}) = 0.$$

where $\beta = \beta_b/\beta_s (> 1)$. Then we can show $f(\beta^2) > 0$ and $f(r) \to -\infty$, as $r \to \infty$. Therefore, there exists $r_0 > \beta^2$ so that $f(r_0) = 0$. Using this r_0, we write k_1 and k_2:

$$k_1 = \frac{\delta_2(\beta_{\rm b}r_0^{-\delta_1} - \beta_{\rm s})}{(1-\delta_2)(\beta_{\rm b} - \beta_{\rm s}r_0^{1-\delta_1})} = \frac{\delta_1(\beta_{\rm b}r_0^{-\delta_2} - \beta_{\rm s})}{(1-\delta_1)(\beta_{\rm b} - \beta_{\rm s}r_0^{1-\delta_2})},$$
$$k_2 = \frac{\delta_2(\beta_{\rm b}r_0^{1-\delta_1} - \beta_{\rm s}r_0)}{(1-\delta_2)(\beta_{\rm b} - \beta_{\rm s}r_0^{1-\delta_1})} = \frac{\delta_1(\beta_{\rm b}r_0^{1-\delta_2} - \beta_{\rm s}r_0)}{(1-\delta_1)(\beta_{\rm b} - \beta_{\rm s}r_0^{1-\delta_2})}. \tag{11}$$

Finally, we can use these k_1 and k_2 to express C_0 and C_1 given in (10).

Theorem. Assume (A1). Then the solutions of the HJB equations (4) can be given as $v_0(x_1, x_2) = x_1 w_0(x_2/x_1)$ and $v_1(x_1, x_2) = x_1 w_1(x_2/x_1)$ where

$$w_0(y) = \begin{cases} \left(\dfrac{\beta_b(1-\delta_2)k_1^{1-\delta_1} + \beta_s\delta_2 k_1^{-\delta_1}}{\delta_1-\delta_2}\right) y^{\delta_1}, & \text{if } 0 < y < k_2, \\ \left(\dfrac{\beta_b(1-\delta_1)k_1^{1-\delta_2} + \beta_s\delta_1 k_1^{-\delta_2}}{\delta_1-\delta_2}\right) y^{\delta_2} + \beta_s y - \beta_b, & \text{if } y \ge k_2, \end{cases}$$

$$w_1(y) = \begin{cases} \left(\dfrac{\beta_b(1-\delta_2)k_1^{1-\delta_1} + \beta_s\delta_2 k_1^{-\delta_1}}{\delta_1-\delta_2}\right) y^{\delta_1} + \beta_s - \beta_b y, & \text{if } 0 < y \le k_1, \\ \left(\dfrac{\beta_b(1-\delta_1)k_1^{1-\delta_2} + \beta_s\delta_1 k_1^{-\delta_2}}{\delta_1-\delta_2}\right) y^{\delta_2}, & \text{if } y > k_1. \end{cases}$$

The optimal trading rule can be determined by two threshold lines ($x_2 = k_1 x_1$ and $x_2 = k_2 x_1$) as follows:

Theorem. Assume (A1). Then, $v_i(x_1, x_2) = x_1 w_i(x_2/x_1) = V_i(x_1, x_2)$, $i = 0, 1$. Moreover, if initially $i = 0$, let $\Lambda_0^* = (\tau_1^*, \tau_2^*, \tau_3^*, \ldots)$ such that $\tau_1^* = \inf\{t \ge 0 : (X_t^1, X_t^2) \in \Gamma_3\}$, $\tau_2^* = \inf\{t \ge \tau_1^* : (X_t^1, X_t^2) \in \Gamma_1\}$, $\tau_3^* = \inf\{t \ge \tau_2^* : (X_t^1, X_t^2) \in \Gamma_3\}$, and so on. Similarly, if initially $i = 1$, let $\Lambda_1^* = (\tau_0^*, \tau_1^*, \tau_2^*, \ldots)$ such that $\tau_0^* = \inf\{t \ge 0 : (X_t^1, X_t^2) \in \Gamma_1\}$, $\tau_1^* = \inf\{t \ge \tau_0^* : (X_t^1, X_t^2) \in \Gamma_3\}$, $\tau_2^* = \inf\{t \ge \tau_1^* : (X_t^1, X_t^2) \in \Gamma_1\}$, and so on. Then Λ_0^* and Λ_1^* are optimal. □

Example 1 (cont.) We backtest our pairs trading rule using the stock prices of TGT and WMT from 2000 to 2014. Using the parameters mentioned earlier, based on the historical prices from 1985 to 1999, we obtain the pair $(k_1, k_2) = (1.03905, 1.28219)$. A pairs trading (long $\mathbf{X}^1$ and short $\mathbf{X}^2$) is triggered when (X_t^1, X_t^2) enters Γ_3. The position is closed when (X_t^1, X_t^2) enters Γ_1. Initially, we allocate trading the capital \$100K. When the first long signal is triggered, buy \$50K TGT stocks and short the same amount of WMT. Such half-and-half capital allocation between long and short applies to all trades. In addition, each pairs transaction is charged \$5 commission. In Figure 3, the corresponding ratio X_t^2/X_t^1, the threshold levels k_1 and k_2, and the corresponding equity curve are plotted. There are total 3 trades and the end balance is \$155.914K.

We can also switch the roles of $\mathbf{X}^1$ and $\mathbf{X}^2$, i.e., to long WMT and short TGT by taking $\mathbf{X}^1$=WMT and $\mathbf{X}^2$=TGT. In this case, the new $(\tilde{k}_1, \tilde{k}_2) = (1/k_2, 1/k_1) = (1/1.28219, 1/1.03905)$. These levels and the corresponding equity curve is given in Figure 4. Such trade leads to the end balance \$132.340K. Note that both types of trades have no overlap, i.e., they do not compete for the same capital. The grand total profit is \$88254 which is a 88.25% gain.

Note also that there are only 5 trades in the fifteen year period leaving the capital in cash most of the time. This is desirable because the cash sitting in the account can be used for other types of shorter term trading in between, at least drawing interest over time.

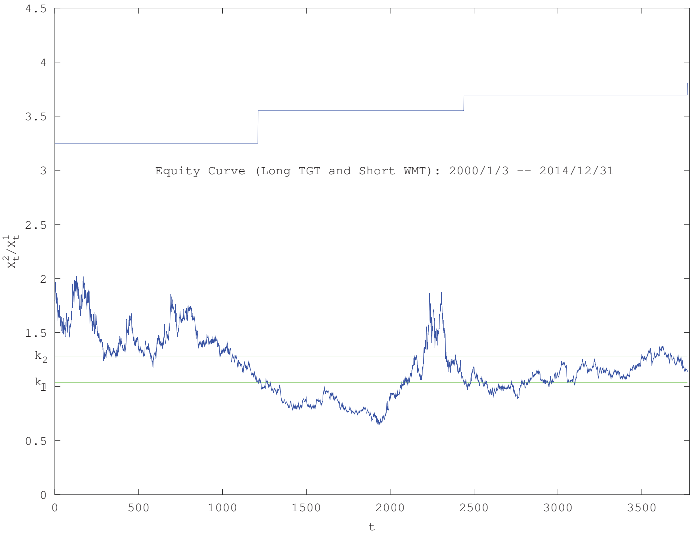

Fig. 3 $\mathbf{X}^1$=TGT, $\mathbf{X}^2$=WMT: The threshold levels k_1, k_2 and the corresponding equity curve

Fig. 4 $\mathbf{X}^1$=WMT, $\mathbf{X}^2$=TGT: The threshold levels k_1, k_2 and the corresponding equity curve

3 Pairs Trading with Cutting Losses

In this section, we consider our above-mentioned pairs trading rule with cutting losses. Recall that a pairs position consists of a long position in stock $\mathbf{X}^1$ and a short position in $\mathbf{X}^2$. The objective is to open (buy) and close (sell) the pairs positions sequentially to maximize the discounted reward function J_0 and J_1 in (2). In practice, unexpected events could cause substantial losses. This normally occurs when the long side X_t^1 shrinks while the short side X_t^2 rises. To limit the downside risk of the pairs position, we impose a hard cut loss level and require $X_t^2/X_t^1 \le M$. Here M is a constant representing a stop-loss level to account for market reaction to undesirable events. The introduction of such stop-loss level amounts to imposing a hard state constraint which makes the corresponding optimal control problem much more difficult.

Let τ_M denote the corresponding exit time, i.e., $\tau_M = \{t : X_t^2/X_t^1 \ge M\}$. Then, $\tau_n \le \tau_M$, for all n.

Our goal is to find Λ_0 and Λ_1 so as to maximize the reward functions $J_0(x_1,x_2,\Lambda_0)$ and $J_1(x_1,x_2,\Lambda_1)$ under such state constraints. For $i = 0,1$, let $V_i(x_1,x_2)$ denote the corresponding value functions with the initial state $(X_0^1, X_0^2) = (x_1,x_2)$ and net positions $i = 0,1$.

Example The main purpose of imposing a hard stop-loss level M is to limit losses to an acceptable level to account for undesirable market moves to unforeseeable events. The stock prices of Ford Motor (F) and General Motors (GM) are highly correlated historically. They make good candidates for pairs trading. In Figure 5, the daily closing price ratio (F/GM) from 1977 to 2009 is plotted. It can be seen that the ratio remains 'normal' for most of the time during this period of time. The ratio starts to rise when approaching the subprime crisis. This would normally trigger a pairs position longing GM and shorting F. Finally, it spikes prior to GM's chapter 11 filing on June 1, 2009 causing heavy losses to any F/GM pair positions. Such hypothetical losses can be limited if one had a hard limit M in place to begin with to force close the position before prices getting out of control.

The choice of M depends on the investor's risk preference. Smaller M (tighter stop-loss control) will cause frequent stop outs and limit profit potential. Larger M (loose stop-loss), on the other hand, will leave more room for the position to run with higher risks. □

Let H denote the feasible region under the hard state constraint $x_2/x_1 \le M$. Then, $H = \{(x_1,x_2) : 0 < x_1, 0 < x_2 \le Mx_1\}$. We can show the same inequalities in (3) hold on H. The associated HJB equations on H can be given as follows:

$$\begin{cases} \min\{\rho v_0(x_1,x_2) - \mathcal{A} v_0(x_1,x_2),\ v_0(x_1,x_2) - v_1(x_1,x_2) + \beta_{\mathrm{b}} x_1 - \beta_{\mathrm{s}} x_2\} = 0, \\ \min\{\rho v_1(x_1,x_2) - \mathcal{A} v_1(x_1,x_2),\ v_1(x_1,x_2) - v_0(x_1,x_2) - \beta_{\mathrm{s}} x_1 + \beta_{\mathrm{b}} x_2\} = 0, \end{cases} \tag{12}$$

with the boundary conditions $v_0(x_1, Mx_1) = 0$ and $v_1(x_1, Mx_1) = \beta_{\mathrm{s}} x_1 - \beta_{\mathrm{b}} M x_1$.

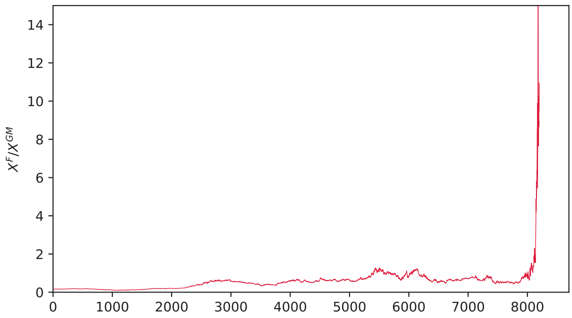

Fig. 5 Daily closing ratio of F/GM from 1977 to 2009

Following similar approach as in the previous section, we divide the feasible region H into four regions $\Gamma_1 = \{(x_1,x_2) \in H : 0 < x_2 \le k_1x_1\}$, $\Gamma_2 = \{(x_1,x_2) \in H : k_1x_1 < x_2 \le k_2x_1\}$, $\Gamma_3 = \{(x_1,x_2) \in H : k_2x_1 < x_2 \le k_3x_1\}$, and $\Gamma_4 = \{(x_1,x_2) \in H : k_3x_1 < x_2 \le Mx_1\}$, where $0 < k_1 < k_2 < k_3 < M$ are threshold slopes to be determined. The control actions on Γ_1, Γ_2, and Γ_3 are similar as before. Γ_4 is the hold and see region due to possible cut-loss at $x_2 = Mx_1$. This is illustrated in Figure 6.

Using the smooth-fit approach, we can show that the k_1 and k_2 are identical as the ones given in (11) with δ_1 and δ_2 in (6).

To determine k_3, let

$$f_1(x) = \frac{M^{\delta_1}\beta_s(x(1-\delta_2)+\beta\delta_2)}{x^{\delta_1}} + \frac{M^{\delta_2}\beta_s(x(\delta_1-1)-\beta\delta_1)}{x^{\delta_2}} + \beta_s(1-M\beta)(\delta_1-\delta_2).$$

We assume **(A2)**: There is a k_3 in (k_2, M) such that $f_1(k_3) = 0$.

A sufficient condition for this can be given as **(A2')** $f_1(k_3) > 0$.

On each of the regions Γ_i, $i = 1,2,3,4$, we can write the solutions of the HJB equations in terms of δ_i, $i = 1,2$, with coefficients C_j, $j = 0,1,...,4$. Then using smooth-fit conditions, we can specify these constants as follows:

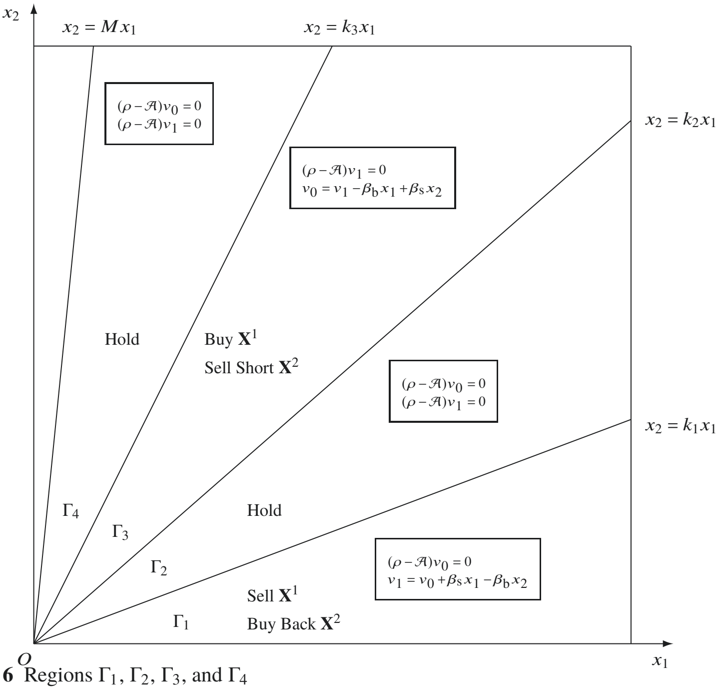

Fig. 6 Regions Γ_1, Γ_2, Γ_3, and Γ_4

$$
\begin{cases}
C_0 = \dfrac{1}{M^{\delta_1}}\Big[(M^{\delta_1}, M^{\delta_2})K^0(k_1)\begin{pmatrix}\beta_{\rm s}k_1-\beta_{\rm b}\\ \beta_{\rm b}\end{pmatrix}+\beta_{\rm s}-M\beta_{\rm b}\Big],\\
C_1 = C_0 + \dfrac{(\delta_2-1)\beta_{\rm b}k_1^{1-\delta_1}-\beta_{\rm s}\delta_2 k_1^{-\delta_1}}{\delta_1-\delta_2},\\
C_2 = \dfrac{(1-\delta_1)\beta_{\rm b}k_1^{1-\delta_2}+\beta_{\rm s}\delta_1 k_1^{-\delta_2}}{\delta_1-\delta_2},\\
C_3 = C_1 + \dfrac{(1-\delta_2)\beta_{\rm s}k_3^{1-\delta_1}+\beta_{\rm b}\delta_2 k_3^{-\delta_1}}{\delta_1-\delta_2},\\
C_4 = C_2 + \dfrac{(\delta_1-1)\beta_{\rm s}k_3^{1-\delta_2}-\beta_{\rm b}\delta_1 k_3^{-\delta_2}}{\delta_1-\delta_2},
\end{cases}
\tag{13}
$$

where

$$
K^0(x) = \frac{1}{\delta_1-\delta_2}\begin{pmatrix}-\delta_2 x^{-\delta_1} & x^{1-\delta_1}\\ \delta_1 x^{-\delta_2} & -x^{1-\delta_2}\end{pmatrix}.
$$

Finally, we need an additional condition to guarantee all inequalities in the HJB equations to hold. We assume (**A3**): Either $f_2'(M) < 0$, or $f_2''(M) < 0$, where,

$$f_2(y) = (C_1 y^{\delta_1} + C_2 y^{\delta_2}) - (C_3 y^{\delta_1} + C_4 y^{\delta_2}) + \beta_b y - \beta_s.$$

A sufficient condition fo (A3) can be given as **(A3')**: $\mu_1 \geq \mu_2$. Under these conditions, we have the following theorems.

Theorem Assume (A1), (A2), and (A3). Then the following functions $v_i(x_1, x_2) = x_1 w_i(x_2/x_1)$, $i = 0, 1$, satisfy the HJB equations (12) where

$$w_0(y) = \begin{cases} C_0 y^{\delta_1}, & 0 < y < k_2, \\ C_1 y^{\delta_1} + C_2 y^{\delta_2} + \beta_s y - \beta_b, & k_2 \leqslant y \leqslant k_3, \\ C_3 y^{\delta_1} + C_4 y^{\delta_2}, & k_3 < y \leqslant M; \end{cases}$$
$$w_1(y) = \begin{cases} C_0 y^{\delta_1} + \beta_s - \beta_b y, & 0 < y < k_1, \\ C_1 y^{\delta_1} + C_2 y^{\delta_2}, & k_1 \leqslant y \leqslant M. \end{cases}$$

Theorem Assume (A1), (A2), and (A3) and $v_0(x_1, x_2) \geq 0$. Then, $v_i(x_1, x_2) = x_1 w_i(x_2/x_1) = V_i(x_1, x_2)$, $i = 0, 1$. Moreover, if $i = 0$, let $\Lambda_0^* = (\tau_1^*, \tau_2^*, \tau_3^*, \ldots) = (\tau_1^0, \tau_2^0, \tau_3^0, \ldots) \wedge \tau_M$ where $\tau_1^0 = \inf\{t \geq 0 : (X_t^1, X_t^2) \in \Gamma_3\}$, $\tau_2^0 = \inf\{t \geq \tau_1^0 : (X_t^1, X_t^2) \in \Gamma_1\}$, $\tau_3^0 = \inf\{t \geq \tau_2^0 : (X_t^1, X_t^2) \in \Gamma_3\}, \ldots$.

Similarly, if $i = 1$, let $\Lambda_1^* = (\tau_0^*, \tau_1^*, \tau_2^*, \ldots) = (\tau_0^0, \tau_1^0, \tau_2^0, \ldots) \wedge \tau_M$ where $\tau_0^0 = \inf\{t \geq 0 : (X_t^1, X_t^2) \in \Gamma_1\}$, $\tau_1^0 = \inf\{t \geq \tau_0^0 : (X_t^1, X_t^2) \in \Gamma_3\}$, $\tau_2^0 = \inf\{t \geq \tau_1^0 : (X_t^1, X_t^2) \in \Gamma_1\}$,.... Then Λ_0^* and Λ_1^* are optimal. □

Next, we consider the daily closing prices of Target Corp. (TGT) and Wal-Mart Stores Inc. (WMT) from 1985 to 2019. The data are divided into two parts. The first part (1985-1999) is used to calibrate the model and the second part (2000-2014) to backtest the performance of our results. Let $\mathbf{X}^1$=WMT and $\mathbf{X}^2$=TGT. Using the traditional least squares method, we have

$$\mu_1 = 0.2459, \mu_2 = 0.2059, \sigma_{11} = 0.2943, \sigma_{12} = 0.0729, \sigma_{21} = 0.0729, \sigma_{22} = 0.3112. \tag{14}$$

And also, we take $K = 0.001$ and $\rho = 0.5$. Using these parameters, we obtain $(k_1, k_2, k_3) = (0.780, 0.963, 1.913)$.

Backtesting 1: (WMT-TGT): We backtest our pairs-trading rule using the daily closing prices of WMT and TGT from 2000/1/2 to 2019/3/15. Use $(k_1, k_2, k_3) = (0.780, 0.963, 1.913)$. Assume initial capital \$100K. We keep the 50:50 allocations in longs and shorts. In Figure 7, the ratio of $\mathbf{X}_t^{TGT}/\mathbf{X}_t^{WMT}$, the threshold levels (k_1, k_2, k_3), and the equity curve are plotted with the x-axis representing the number of trading days. Also, when there is no pairs position, we factor in a 3% interest for the cash position. The overall end balance is \$195.46K. For comparison purpose, a money market return with 3% interest rate is also plotted in Figure 7. In this example, the stop loss with $M = 2$ was not triggered and there was no forced stops.

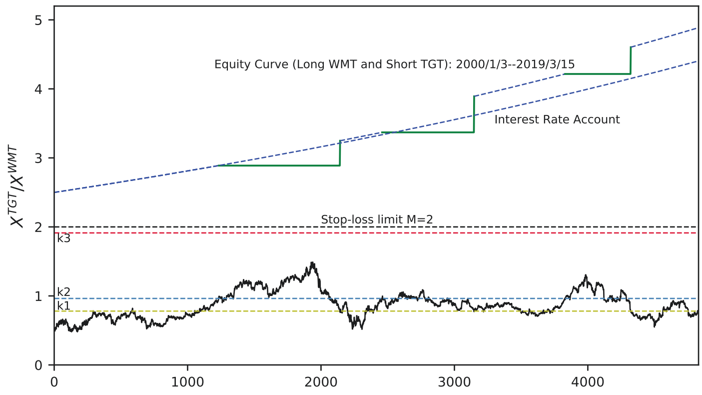

Fig. 7 The threshold levels k_1, k_2, k_3 and the equity curve

Backtesting 2: (GM-F). Next, we backtest using the daily closing prices of GM and F from 1998/1/2 to 2009/6/30. We take $M = 2$ and follow similar calculation (with 2:1 ratio of F/GM) to obtain $(k_1, k_2, k_3) = (0.760, 0.892, 1.946)$. Also assume the initial capital \$100K. We keep the 50:50 distribution in dollar amount between longs and shorts. In Figure 8, the ratio $2\mathbf{X}_t^F/\mathbf{X}_t^{GM}$, the threshold levels (k_1, k_2, k_3), and the equity curve are plotted. Similarly as in the previous example, when there is no pairs position, a 3% interest was factored in for the cash position. The overall end balance is \$149.52K after hitting stop-loss limit $M = 2$ on 2009/3/6.

On the other hand, without cutting losses, the initial \$100K will end up with \$86.38K in debt when the last pairs closed on GM's bankruptcy (2009/6/1). A pure money market return with 3% interest rate is also provided in Figure 8.

4 A Pairs Selling Rule with Regime Switching

Market models with regime-switching are important in market analysis. In this paper, we consider a geometric Brownian motion with regime-switching. The market mode is represented by a two-state Markov chain. We focus on the selling part of pairs trading and generalize the results of Hu and Øksendal [9] by incorporating models with regime switching. We show that the optimal selling rule can be determined by two threshold curves and establish a set of sufficient conditions that guarantee the optimality of the policy. We also include several numerical examples under a different set of parameter values.

We consider two stocks $\mathbf{X}^1$ and $\mathbf{X}^2$. Let $\{X_t^1, t \geq 0\}$ denote the prices of stock $\mathbf{X}^1$ and $\{X_t^2, t \geq 0\}$ that of stock $\mathbf{X}^2$. Let also α_t be a two-state Markov chain representing

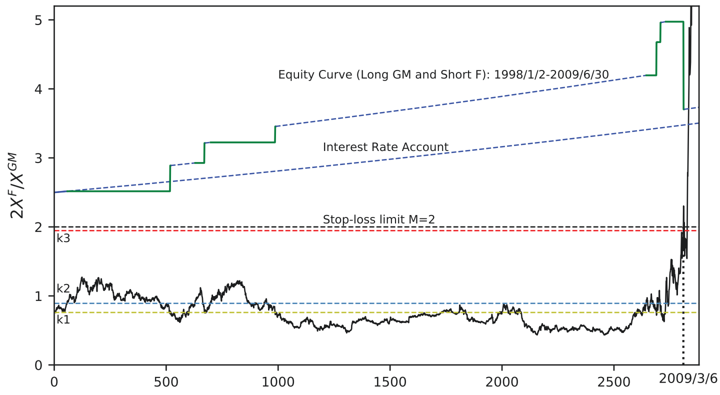

Fig. 8 The threshold levels k_1, k_2, k_3 and the equity curve

regime mode. They satisfy the following stochastic differential equation:

$$d\begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix} = \begin{pmatrix} X_t^1 & \\ & X_t^2 \end{pmatrix}\left[\begin{pmatrix} \mu_1(\alpha_t) \\ \mu_2(\alpha_t) \end{pmatrix} dt + \begin{pmatrix} \sigma_{11}(\alpha_t) & \sigma_{12}(\alpha_t) \\ \sigma_{21}(\alpha_t) & \sigma_{22}(\alpha_t) \end{pmatrix} d\begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix}\right], \tag{15}$$

where μ_i, $i = 1,2$, are the return rates, σ_{ij}, $i,j = 1,2$, the volatility constants, and (W_t^1, W_t^2) a 2-dimensional standard Brownian motion.

Let $\mathcal{M} = \{1,2\}$ denote the state space for α_t and let $Q = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix}$, with $\lambda_1 > 0$ and $\lambda_2 > 0$, be its generator. We assume α_t and (W_t^1, W_t^2) are independent.

In this section, we consider a pairs selling rule under the regime switching model. Again, we assume the corresponding pairs position consists of a one-share long position in stock $\mathbf{X}^1$ and a one-share short position in stock $\mathbf{X}^2$. The problem is to determine an optimal stopping time τ to close the pairs position by selling $\mathbf{X}^1$ and buying back $\mathbf{X}^2$.

Given the initial state $(X_0^1, X_0^2) = (x_1, x_2)$, $\alpha_0 = i = 1,2$, and the selling time τ, the corresponding reward function

$$J(x_1, x_2, i, \tau) = E\left[e^{-\rho\tau}(\beta_s X_\tau^1 - \beta_b X_\tau^2)\right], \tag{16}$$

where $\rho > 0$ is a given discount factor, $\beta_b = 1 + K$, $\beta_s = 1 - K$, and K is the transaction cost in percentage.

The problem is to find an $\{\mathcal{F}_t\} = \sigma\{(X_r^1, X_r^2, \alpha_r) : r \le t\}$ stopping time τ to maximize J. Let $V(x_1, x_2, i)$ denote the corresponding value functions:

$$V(x_1,x_2,i) = \sup_{\tau} J(x_1,x_2,i,\tau). \tag{17}$$

As in the previous sections, we impose the following conditions: **(B1)** For $i = 1,2$, $\rho > \mu_1(i)$ and $\rho > \mu_2(i)$.

Under these conditions, we can obtain

$$\beta_s x_1 - \beta_b x_2 \le V(x_1,x_2,i) \le \beta_s x_1. \tag{18}$$

To consider the associated HJB equations, for $i = 1,2$, let

$$\begin{aligned}\mathcal{A}_i = \frac{1}{2}\left[a_{11}(i)x_1^2\frac{\partial^2}{\partial x_1^2} + 2a_{12}(i)x_1x_2\frac{\partial^2}{\partial x_1\partial x_2} + a_{22}(i)x_2^2\frac{\partial^2}{\partial x_2^2}\right]\\ +\mu_1(i)x_1\frac{\partial}{\partial x_1} + \mu_2(i)x_2\frac{\partial}{\partial x_2}\end{aligned} \tag{19}$$

where

$$\begin{cases} a_{11}(i) = \sigma_{11}^2(i) + \sigma_{12}^2(i),\\ a_{12}(i) = \sigma_{11}(i)\sigma_{21}(i) + \sigma_{12}(i)\sigma_{22}(i),\\ a_{22}(i) = \sigma_{21}^2(i) + \sigma_{22}^2(i). \end{cases}$$

Using these generators, the associated HJB equations have the form:

$$\begin{cases} \min\{(\rho - \mathcal{A}_1)v(x_1,x_2,1) - \lambda_1(v(x_1,x_2,2) - v(x_1,x_2,1)),\\ \qquad\qquad v(x_1,x_2,1) - \beta_s x_1 + \beta_b x_2\} = 0,\\ \min\{(\rho - \mathcal{A}_2)v(x_1,x_2,2) - \lambda_2(v(x_1,x_2,1) - v(x_1,x_2,2)),\\ \qquad\qquad v(x_1,x_2,2) - \beta_s x_1 + \beta_b x_2\} = 0. \end{cases} \tag{20}$$

To solve the HJB equations (20), we can introduce change of variables: $y = x_2/x_1$ and $v(x_1,x_2,i) = x_1 w_i(x_2/x_1)$, for some functions $w_i(y)$ and $i = 1,2$.

Consider characteristic equations for $(\rho - \mathcal{A}_1)v_1 - \lambda_1(v_2 - v_1) = 0$ and $(\rho - \mathcal{A}_2)v_2 - \lambda_2(v_1 - v_2) = 0$:

$$[\rho + \lambda_1 - \theta_1(\delta)][\rho + \lambda_2 - \theta_2(\delta)] - \lambda_1\lambda_2 = 0, \tag{21}$$

where, for $i = 1,2$. $\theta_i(\delta) = \sigma_i\delta(\delta - 1) + [(\mu_2(i) - \mu_1(i)]\delta + \mu_1(i)$ and $\sigma_i = [a_{11}(i) - 2a_{12}(i) + a_{22}(i)]/2$.

It can be seen the above equation has four zeros: $\delta_1 \ge \delta_2 > 1 > 0 > \delta_3 \ge \delta_4$.

Heuristically, one should close the pairs position when X_t^1 is large and X_t^2 is small. In view of this, we introduce $H_1 = \{(x_1,x_2) : \ x_2 \le k_1x_1\}$ and $H_2 = \{(x_1,x_2) : \ x_2 \le k_2x_1\}$, for some k_1 and k_2 (to be determined) so that one should sell when (X_t^1, X_t^2)

enters H_i provided $\alpha_t = i$, $i = 1, 2$. In this paper, we only consider the case: $k_1 < k_2$. Other cases can be treated similarly.

To represent the solutions to the HJB equations on each of these regimes, we apply the smooth-fit approach and obtain:

$$\begin{cases} C_1 = \dfrac{-\delta_4\beta_{\rm s} + (\delta_4 - 1)\beta_{\rm b}k_2}{\eta_3(\delta_3-\delta_4)k_2^{\delta_3}}, \\ C_2 = \dfrac{\delta_3\beta_{\rm s} + (1-\delta_3)\beta_{\rm b}k_2}{\eta_4(\delta_3-\delta_4)k_2^{\delta_4}}, \\ C_3 = \dfrac{\gamma_2(\beta_{\rm s}-a_1) + (1-\gamma_2)(\beta_{\rm b}+a_2)k_1}{(\gamma_2-\gamma_1)k_1^{\gamma_1}}, \\ C_4 = \dfrac{-\gamma_1(\beta_{\rm s}-a_1) + (\gamma_1-1)(\beta_{\rm b}+a_2)k_1}{(\gamma_2-\gamma_1)k_1^{\gamma_2}}, \end{cases}$$

where $\eta_j = (\rho + \lambda_1 - \theta_1(\delta_j))/\lambda_1$, for $j = 1, 2, 3, 4$, and

$$\begin{cases} \gamma_1 = \dfrac{1}{2} + \dfrac{\mu_1(1)-\mu_2(1)}{2\sigma_1} + \sqrt{\left(\dfrac{1}{2} + \dfrac{\mu_1(1)-\mu_2(1)}{2\sigma_1}\right)^2 + \dfrac{\rho+\lambda_1-\mu_1(1)}{\sigma_1}}, \\ \gamma_2 = \dfrac{1}{2} + \dfrac{\mu_1(1)-\mu_2(1)}{2\sigma_1} - \sqrt{\left(\dfrac{1}{2} + \dfrac{\mu_1(1)-\mu_2(1)}{2\sigma_1}\right)^2 + \dfrac{\rho+\lambda_1-\mu_1(1)}{\sigma_1}}. \end{cases} \tag{22}$$

Let

$$g(r) = \frac{A_1 - \gamma_2(\beta_{\rm s}-a_1)r^{\gamma_1}}{(1-\gamma_2)(\beta_{\rm b}+a_2)r^{\gamma_1} - B_1 r} - \frac{A_2 + \gamma_1(\beta_{\rm s}-a_1)r^{\gamma_2}}{(\gamma_1-1)(\beta_{\rm b}+a_2)r^{\gamma_2} - B_2 r}, \tag{23}$$

where

$$\begin{cases} A_1 = \dfrac{-\delta_4\beta_{\rm s}(\gamma_2-\delta_3)}{\eta_3(\delta_3-\delta_4)} + \dfrac{\delta_3\beta_{\rm s}(\gamma_2-\delta_4)}{\eta_4(\delta_3-\delta_4)} - \gamma_2 a_1, \\ A_2 = \dfrac{-\delta_4\beta_{\rm s}(\delta_3-\gamma_1)}{\eta_3(\delta_3-\delta_4)} + \dfrac{\delta_3\beta_{\rm s}(\delta_4-\gamma_1)}{\eta_4(\delta_3-\delta_4)} + \gamma_1 a_1, \\ B_1 = \dfrac{(\delta_4-1)(\gamma_2-\delta_3)\beta_{\rm b}}{\eta_3(\delta_3-\delta_4)} + \dfrac{(1-\delta_3)\beta_{\rm b}(\gamma_2-\delta_4)}{\eta_4(\delta_3-\delta_4)} - (\gamma_2-1)a_2, \\ B_2 = \dfrac{(\delta_4-1)(\delta_3-\gamma_1)\beta_{\rm b}}{\eta_3(\delta_3-\delta_4)} + \dfrac{(1-\delta_3)\beta_{\rm b}(\delta_4-\gamma_1)}{\eta_4(\delta_3-\delta_4)} - (1-\gamma_1)a_2, \end{cases}$$

with $a_1 = \lambda_1\beta_{\rm s}/(\rho + \lambda_1 - \mu_1(1))$ and $a_2 = -\lambda_1\beta_{\rm b}/(\rho + \lambda_1 - \mu_2(1))$.

We assume **(B2)**: $g(r)$ has a zero $r_0 > 1$.

Using this r_0, we can obtain

$$\begin{cases} k_1 = \dfrac{A_1 - \gamma_2(\beta_s - a_1)r_0^{\gamma_1}}{(1-\gamma_2)(\beta_b + a_2)r_0^{\gamma_1} - B_1 r_0}, \\ k_2 = r_0 k_1 = \dfrac{A_1 r_0 - \gamma_2(\beta_s - a_1)r_0^{\gamma_1+1}}{(1-\gamma_2)(\beta_b + a_2)r_0^{\gamma_1} - B_1 r_0}. \end{cases} \tag{24}$$

We can express C_1, C_2, C_3, and C_4 in terms of k_1 and k_2. The solutions to the HJB equations have the form $v(x_1, x_2, \alpha) = x_1 w_\alpha(x_2/x_1)$, $\alpha = 1, 2$, with

$$w_1(y) = \begin{cases} \beta_s - \beta_b y & \text{for } y \in \Gamma_1, \\ C_3 y^{\gamma_1} + C_4 y^{\gamma_2} + a_1 + a_2 y & \text{for } y \in \Gamma_2, \\ C_1 y^{\delta_3} + C_2 y^{\delta_4} & \text{for } y \in \Gamma_3; \end{cases}$$

$$w_2(y) = \begin{cases} \beta_s - \beta_b y & \text{for } y \in \Gamma_1 \cup \Gamma_2, \\ C_1 \eta_3 y^{\delta_3} + C_2 \eta_4 y^{\delta_4} & \text{for } y \in \Gamma_3, \end{cases}$$

where

$$\Gamma_1 = (0, k_1], \quad \Gamma_2 = (k_1, k_2), \quad \text{and} \quad \Gamma_3 = [k_2, \infty).$$

To guarantee the variational inequalities in the HJB equations, we need the following conditions:

$$k_1 \le \min\left\{ \frac{(\rho - \mu_1(1))\beta_s}{(\rho - \mu_2(1))\beta_b}, \frac{(\rho - \mu_1(2))\beta_s}{(\rho - \mu_2(2))\beta_b} \right\}; \tag{25}$$

$$w_1(y) \le \beta_s - \beta_b y + \frac{1}{\lambda_2}[(\rho - \mu_1(2))\beta_s - (\rho - \mu_2(2))\beta_b y] \text{ on } \Gamma_2. \tag{26}$$

In addition, let $\phi(y) = w_1(y) - \beta_s + \beta_b y$. Then

$$\begin{cases} \phi''(k_1) = C_3\gamma_1(\gamma_1 - 1)k_1^{\gamma_1 - 2} + C_4\gamma_2(\gamma_2 - 1)k_1^{\gamma_2 - 2}, \text{ and} \\ \phi(k_2) = C_3 k_2^{\gamma_1} + C_4 k_2^{\gamma_2} + a_1 + a_2 k_2 - \beta_s + \beta_b y. \end{cases}$$

We need conditions

$$\phi''(k_1) \ge 0 \text{ and } \phi(k_2) \ge 0. \tag{27}$$

Finally, on Γ_3, let $\psi(y) = w_2(y) - \beta_s + \beta_b y$. Then,

$$\psi''(k_2) = C_1\eta_3\delta_3(\delta_3 - 1)k_2^{\delta_3 - 2} + C_2\eta_4\delta_4(\delta_4 - 1)k_2^{\delta_4 - 2}.$$

We need

$$\psi''(k_2) \ge 0 \text{ and } C_1 y^{\delta_3} + C_2 y^{\delta_4} \ge \beta_s - \beta_b y \text{ on } \Gamma_3. \tag{28}$$

Theorem. Assume (B1) and (B2). Assume also (25), (26), (27), and (28) hold. Then, $v(x_1,x_2,\alpha) = x_1 w_\alpha(x_2/x_1) = V(x_1,x_2,\alpha)$, $\alpha = 1,2$. Let $D = \{(x_1,x_2,1) : x_2 > k_1 x_1\} \cup \{(x_1,x_2,2) : x_2 > k_2 x_1\}$. Let $\tau^* = \inf\{t : (X_t^1, X_t^2, \alpha_t) \notin D\}$. Then τ^* is optimal. □

Finally, we give an example to illustrate the results.

Example In this example, we take

$$\mu_1(1) = 0.20, \quad \mu_2(1) = 0.25, \quad \mu_1(2) = -0.30, \; \mu_2(2) = -0.35,$$

$$\sigma_{11}(1) = 0.30, \; \sigma_{12}(1) = 0.10, \; \sigma_{21}(1) = 0.10, \; \sigma_{22}(1) = 0.35,$$

$$\sigma_{11}(2) = 0.40, \; \sigma_{12}(2) = 0.20, \; \sigma_{21}(2) = 0.20, \; \sigma_{22}(2) = 0.45,$$

$$\lambda_1 = 6.0, \qquad \lambda_2 = 10.0, \qquad K = 0.001, \qquad \rho = 0.50.$$

Then, we use the function $g(r)$ in (23) and find the unique zero $r_0 = 1.020254 > 1$. Using this r_0 and (24), we obtain $k_1 = 0.723270$ and $k_2 = 0.737920$. Then, we calculate and get $C_1 = 0.11442$, $C_2 = -0.00001$, $C_3 = 0.29121$, $C_4 = 0.00029$, $\eta_3 = 0.985919$, and $\eta_4 = -1.541271$. With these numbers, we verify all variational inequalities required in (B2). The graphs of the value functions are given in Figure 9.
□

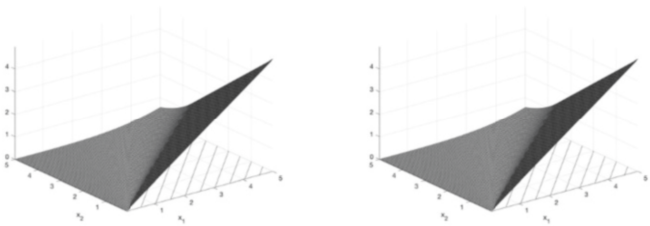

Fig. 9 Value Functions $V(x_1, x_2, 1)$ and $V(x_1, x_2, 2)$

5 Conclusions

In this paper, we have surveyed pairs trading under geometric Brownian motion models. We were able to obtain closed-form solutions. The trading rules are given in terms of threshold levels and are simple and easy to implement. The major advantage of pairs trading is its risk-neutral nature, i.e., it can be profitable regardless of the general market directions. Pairs trading is a natural extension to McDonald and Siegel's [15] irreversible project investment decision making. We were able to obtain similar results under suitable conditions.

Some initial efforts in connection with numerical computations and implementation have been done in Luu [14]. In particular, stochastic approximation techniques (see Kushner and Yin [12]) can be used to effectively estimate these threshold levels directly. Finally, it would be interesting to examine how these methods work through backtests for a larger selection of stocks.

It would be interesting to extend the results to incorporate more involved models (e.g., models with incomplete observation in market mode α_t). In this case, nonlinear filtering methods such as the Wonham filter can be used for calculation of the conditional probabilities of $\alpha = i$ given the stock prices up to time t. Some ideas along this line have been used in Dai et al. [1] in connection with the trend-following trading.

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