Maximally Distributed Random Fields under Sublinear Expectation

Xinpeng Li and Shige Peng

Abstract This paper focuses on the maximal distribution on sublinear expectation space and introduces a new type of random fields with the maximally distributed finite-dimensional distribution. The corresponding spatial maximally distributed white noise is constructed, which includes the temporal-spatial situation as a special case due to the symmetrical independence property of maximal distribution. In addition, the stochastic integrals with respect to the spatial or temporal-spatial maximally distributed white noises are established in a quite direct way without the usual assumption of adaptability for integrand.

1 Introduction

In mathematics and physics, a random field is a type of parameterized family of random variables. When the parameter is time $t \in \mathbb{R}^+$, we call it a stochastic process, or a temporal random field. Quite often the parameter is space $x \in \mathbb{R}^d$, or time-space $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$. In this case, we call it a spatial or temporal-spatial random field. A typical example is the electromagnetic wave dynamically spread everywhere in our $\mathbb{R}^3$-space or more exactly, in $\mathbb{R}^+ \times \mathbb{R}^3$-time-space. In principle, it is impossible to know the exact state of the electromagnetic wave of our real world , namely, it is a nontrivial random field parameterized by the time-space $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3$.

Classically, a random field is defined on a given probability space $(\Omega, \mathcal{F}, P)$. But for the above problem, can we really get to know the probability P? This involves the so called problem of uncertainty of probabilities.

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Over the past few decades, non-additive probabilities or nonlinear expectations have become active domains for studying uncertainties, and received more and more attention in many research fields, such as mathematical economics, mathematical finance, statistics, quantum mechanics. A typical example of nonlinear expectation is sublinear one, which is used to model the uncertainty phenomenon characterized by a family of probability measures $\{P_\theta\}_{\theta\in\Theta}$ in which the true measure is unknown, and such sublinear expectation is usually defined by

$$\mathbb{E}[X] := \sup_{\theta\in\Theta} E_{P_\theta}[X].$$

This notion is also known as the upper expectation in robust statistics (see Huber [9]), or the upper prevision in the theory of imprecise probabilities (see Walley [20]), and has the closed relation with coherent risk measures (see Artzner et al. [1], Delbaen [4], Föllmer and Schied [6]). A first dynamical nonlinear expectation, called g-expectation was initiated by Peng [12].

The foundation of sublinear expectation theory with a new type of G-Brownian motion and the corresponding Itô's stochastic calculus was laid in Peng [13], which keeps the rich and elegant properties of classical probability theory except linearity of expectation. Peng [15] initially defined the notion of independence and identical distribution (i.i.d.) based on the notion of nonlinear expectation instead of the capacity. Based on the notion of new notions, the most important distribution called G-normal distribution introduced, which can be characterized by the so-called G-heat equation. The notions of G-expectation and G-Brownian motion can be regarded as a nonlinear generalization of Wiener measure and classical Brownian motion. The corresponding limit theorems as well as stochastic calculus of Itô's type under G-expectation are systematically developed in Peng [18]. Besides that, there is also another important distribution, called maximal distribution. The distribution of maximally distributed random variable X can be calculated simply by

$$\mathbb{E}[\varphi(X)] = \max_{v\in[-\mathbb{E}[-X],\mathbb{E}[X]]} \varphi(v), \quad \varphi \in C_b(\mathbb{R}).$$

The law of large numbers under sublinear expectation (see Peng [18]) shows that if $\{X_i\}_{i=1}^\infty$ is a sequence of independent and identical distributed random variables with $\lim_{c\to\infty} \mathbb{E}[(|X_1| - c)^+] = 0$, then the sample average converges to maximal distribution in law, i.e.,

$$\lim_{n\to\infty} \mathbb{E}[\varphi(\frac{X_1 + \cdots + X_n}{n})] = \max_{v\in[-\mathbb{E}[-X_1],\mathbb{E}[X_1]]} \varphi(v), \quad \forall\varphi \in C_b(\mathbb{R}).$$

We note that the finite-dimensional distribution for quadratic variation process of G-Brownian motion is also maximal distributed.

Recently, Ji and Peng [10] introduced a new G-Gaussian random fields, which contains a type of spatial white noise as a special case. Such white noise is a natural generalization of the classical Gaussian white noise (for example, see Walsh [21], Dalang [2] and Da Prato and Zabczyk [3]). As pointed in [10], the space-indexed increments do not satisfy the property of independence. Once the sublinear G-expectation degenerates to linear case, the property of independence for the space-indexed part turns out to be true as in the classical probability theory.

In this paper, we introduce a very special but also typical random field, called maximally distributed random field, in which the finite-dimensional distribution is maximally distributed. The corresponding space-indexed white noise is also constructed. It is worth mentioning that the space-indexed increments of maximal white noise is independent, which is essentially different from the case of G-Gaussian white noise. Thanks to the symmetrical independence of maximally distributed white noise, it is natural to view the temporal-spatial maximally distributed white noise as a special case of the space-indexed maximally distributed white noise. The stochastic integrals with respect to spatial and temporal-spatial maximally distributed white noises can be constructed in a quite simple way, which generalize the stochastic integral with respect to quadratic variation process of G-Brownian motion introduced in Peng [18]. Furthermore, due to the boundedness of maximally distributed random field, the usual assumption of adaptability for integrand can be dropped. We emphasize that the structure of maximally distributed white noise is quite simple, it can be determined by only two parameters $\underline{\mu}$ and $\overline{\mu}$, and the calculation of the corresponding finite-dimensional distribution is taking the maximum of continuous function on the domain determined by $\underline{\mu}$ and $\overline{\mu}$. The use of maximally distributed random fields for modelling purposes in applications can be explained mainly by the simplicity of their construction and analytic tractability combined with the maximal distributions of marginal which describe many real phenomena due to the law of large numbers with uncertainty.

This paper is organized as follows. In Section 2, we review basic notions and results of nonlinear expectation theory and the notion and properties of maximal distribution. In Section 3, we first recall the general setting of random fields under nonlinear expectations, and then introduce the maximally distributed random fields. In Section 4, we construct the spatial maximally distributed white noise and study the corresponding properties. The properties of spatial as well as temporal-spatial maximally distributed white noise and the related stochastic integrals are established in Section 5.

2 Preliminaries

In this section, we recall some basic notions and properties in the nonlinear expectation theory. More details can be found in Denis et al. [5], Hu and Peng [8] and Peng [13, 14, 15, 16, 18, 19].

Let Ω be a given nonempty set and $\mathcal{H}$ be a linear space of real-valued functions on Ω such that if $X \in \mathcal{H}$, then $|X| \in \mathcal{H}$. $\mathcal{H}$ can be regarded as the space of random variables. In this paper, we consider a more convenient assumption: if random variables $X_1,\cdots,X_d \in \mathcal{H}$, then $\varphi(X_1, X_2, \cdots, X_d) \in \mathcal{H}$ for each $\varphi \in C_{b.Lip}(\mathbb{R}^d)$. Here $C_{b.Lip}(\mathbb{R}^d)$ is the space of all bounded and Lipschitz functions on $\mathbb{R}^d$.

We call $X = (X_1, \cdots, X_n)$, $X_i \in \mathcal{H}$, $1 \leq i \leq n$, an n-dimensional random vector, denoted by $X \in \mathcal{H}^n$.

Definition 1 A nonlinear expectation $\hat{E}$ on $\mathcal{H}$ is a functional $\hat{E} : \mathcal{H} \to \mathbb{R}$ satisfying the following properties: for each $X, Y \in \mathcal{H}$,

(i) Monotonicity: $\hat{E}[X] \geq \hat{E}[Y]$ if $X \geq Y$;
(ii) Constant preserving: $\hat{E}[c] = c$ for $c \in \mathbb{R}$;

The triplet $(\Omega, \mathcal{H}, \hat{E})$ is called a nonlinear expectation space. If we further assume that

(iii) Sub-additivity: $\hat{E}[X + Y] \leq \hat{E}[X] + \hat{E}[Y]$;
(iv) Positive homogeneity: $\hat{E}[\lambda X] = \lambda \hat{E}[X]$ for $\lambda \geq 0$.

Then $\hat{E}$ is called a sublinear expectation, and the corresponding triplet $(\Omega, \mathcal{H}, \hat{E})$ is called a sublinear expectation space.

Let $(\Omega, \mathcal{H}, \hat{E})$ be a nonlinear (resp., sublinear) expectation space. For each given n-dimensional random vector X, we define a functional on $C_{b.Lip}(\mathbb{R}^n)$ by

$$\mathbb{F}_X[\varphi] := \hat{E}[\varphi(X)], \text{ for each } \varphi \in C_{b.Lip}(\mathbb{R}^n).$$

$\mathbb{F}_X$ is called the distribution of X. It is easily seen that $(\mathbb{R}^n, C_{b.Lip}(\mathbb{R}^n), \mathbb{F}_X)$ forms a nonlinear (resp., sublinear) expectation space. If $\mathbb{F}_X$ is not a linear functional on $C_{b.Lip}(\mathbb{R}^n)$, we say X has distributional uncertainty.

Definition 2 Two n-dimensional random vectors X_1 and X_2 defined on nonlinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{E}_2)$ respectively, are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$, if $\mathbb{F}_{X_1} = \mathbb{F}_{X_2}$, i.e.,

$$\hat{E}_1[\varphi(X_1)] = \hat{E}_2[\varphi(X_2)], \ \forall \varphi \in C_{b.Lip}(\mathbb{R}^n).$$

Definition 3 Let $(\Omega, \mathcal{H}, \hat{E})$ be a nonlinear expectation space. An n-dimensional random vector Y is said to be independent from another m-dimensional random vector X under the expectation $\hat{E}$ if, for each test function $\varphi \in C_{b.Lip}(\mathbb{R}^{m+n})$, we have

$$\hat{E}[\varphi(X, Y)] = \hat{E}[\hat{E}[\varphi(x, Y)]_{x=X}].$$

Remark 1 Peng [15] (see also Peng [18]) introduced the notions of the distribution and the independence of random variables under a nonlinear expectation, which play a crucially important role in the nonlinear expectation theory.

For simplicity, the sequence $\{X_i\}_{i=1}^n$ is called independence if X_{i+1} is independent from $(X_1, \cdots, X_i)$ for $i = 1, 2, \cdots, n-1$. Let $\bar{X}$ and X be two n-dimensional random vectors on $(\Omega, \mathcal{H}, \hat{E})$. $\bar{X}$ is called an independent copy of X, if $\bar{X} \overset{d}{=} X$ and $\bar{X}$ is independent from X.

Remark 2 It is important to note that "Y is independent from X" does not imply that "X is independent from Y" (see Peng [18]).

In this paper, we focus on an important distribution on sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$, called maximal distribution.

Definition 4 An n-dimensional random vector $X = (X_1, \cdots, X_n)$ on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ is said to be maximally distributed, if there exists a bounded and closed convex subset $\Lambda \subset \mathbb{R}^n$ such that, for every continuous function $\varphi \in C(\mathbb{R}^n)$,

$$\hat{E}[\varphi(X)] = \max_{x \in \Lambda} \varphi(x).$$

Remark 3 Here Λ characterizes the uncertainty of X. It is easy to check that this maximally distributed random vector X satisfies

$$X + \bar{X} \overset{d}{=} 2X,$$

where $\bar{X}$ is an independent copy of X. Conversely, suppose a random variable X satisfying $X + \bar{X} \overset{d}{=} 2X$, if we further assume the uniform convergence condition $\lim_{c\to\infty} \hat{E}[(|X| - c)^+] = 0$ holds, then we can deduce that X is maximally distributed by the law of large numbers (see Peng [18]). An interesting problem is that is X still maximally distributed without such uniform convergence condition? We emphasize that the law of large numbers does not hold in this case, a counterexample can be found in Li and Zong [11].

Proposition 1 *Let $g(p) = \max_{v\in\Lambda} v \cdot p$ be given. Then an n-dimensional random variable is maximally distributed if and only if for each $\varphi \in C(\mathbb{R}^n)$, the following function*

$$u(t, x) := \hat{E}[\varphi(x + tX)] = \max_{v \in \Lambda} \varphi(x + tv), \ (t, x) \in [0, \infty) \times \mathbb{R}^n \tag{1}$$

is the unique viscosity solution of the the following nonlinear partial differential equation

$$\partial_t u - g(D_x u) = 0, \ \ u|_{t=0} = \varphi(x). \tag{2}$$

This property implies that, each sublinear function g on $\mathbb{R}^n$ determines uniquely a maximal distribution. The following property is easy to check.

Proposition 2 *Let X be an n-dimensional maximally distributed random vector characterized by its generating function*

$$g(p) := \hat{E}[X \cdot p], \ \ p \in \mathbb{R}^n.$$

Then, for any function $\psi \in C(\mathbb{R}^n)$, $Y = \psi(X)$ is also an $\mathbb{R}$-valued maximally distributed random variable:

$$\mathbb{E}[\varphi(Y)] = \max_{v \in [\underline{\rho}, \overline{\rho}]} \varphi(v), \qquad \overline{\rho} = \max_{\gamma \in \Lambda} \psi(\gamma), \quad \underline{\rho} = \min_{\gamma \in \Lambda} \psi(\gamma).$$

Proposition 3 *Let $X = (X_1, \cdots, X_n)$ be an n-dimensional maximal distribution on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$. If the corresponding generating function satisfies, for all $p = (p_1, \cdots, p_n) \in \mathbb{R}^n$,*

$$g(p) = \hat{E}[X_1 p_1 + \cdots + X_n p_n] = \hat{E}[X_1 p_1] + \cdots + \hat{E}[X_n p_n],$$

then $\{X_i\}_{i=1}^n$ is a sequence of independent maximally distributed random variables.

Moreover, for any permutation π of $\{1, 2, \cdots, n\}$, the sequence $\{X_{\pi(i)}\}_{i=1}^n$ is also independent.

Proof For $i = 1, \cdots, n$, we denote $\overline{\mu}_i = \hat{E}[X_i]$ and $\underline{\mu}_i = -\hat{E}[-X_i]$. Since

$$\begin{aligned} g(p) &= \hat{E}[X_1 \cdot p_1 + \cdots + X_n \cdot p_n] = \hat{E}[X_1 \cdot p_1] + \hat{E}[X_2 \cdot p_2] + \cdots + \hat{E}[X_n \cdot p_n] \\ &= \sum_{i=1}^n \max_{v_i \in [\underline{\mu}_i, \overline{\mu}_i]} p_i v_i = \max_{(v_1, \cdots, v_n) \in \otimes_{i=1}^n [\underline{\mu}_i, \overline{\mu}_i]} (p_1 v_1 + \cdots + p_n v_n), \end{aligned}$$

it follows Proposition 1 that $(X_1, \cdots, X_n)$ is an n-dimensional maximally distributed random vector such that, $\forall \varphi \in C(\mathbb{R}^n)$,

$$\hat{E}[\varphi(X_1, \cdots, X_n)] = \max_{(v_1, \cdots, v_n) \in \otimes_{i=1}^n [\underline{\mu}_i, \overline{\mu}_i]} \varphi(v_1, \cdots, v_n).$$

It is easy to check that $\{X_i\}_{i=1}^n$ is independent, and so does the permuted sequence $\{X_{\pi(i)}\}_{i=1}^n$. □

Remark 4 The independence of maximally distributed random variables is symmetrical. But, as discussed in Remark 2, under a sublinear expectation, X is independent from Y does not automatically imply that Y is also independent from X. In fact, Hu and Li [7] proved that, if X is independent from Y, and Y is also independent from X, and both of X and Y have distributional uncertainty, then (X, Y) must be maximally distributed.

3 Maximally distributed random fields

In this section, we first recall the general setting of random fields defined on a nonlinear expectation space introduced by Ji and Peng [10].

Definition 5 Under a given nonlinear expectation space $(\Omega, \mathcal{H}, \hat{E})$, a collection of m-dimensional random vectors $W = (W_\gamma)_{\gamma \in \Gamma}$ is called an m-dimensional random field indexed by Γ, if for each $\gamma \in \Gamma$, $W_\gamma \in \mathcal{H}^m$.

In order to introduce the notion of finite-dimensional distribution of a random field W, we denote the family of all sets of finite indices by

$$\mathcal{J}_\Gamma := \{\underline{\gamma} = (\gamma_1, \cdots, \gamma_n) : \ \forall n \in \mathbb{N}, \ \gamma_1, \cdots, \gamma_n \in \Gamma, \ \gamma_i \neq \gamma_j \text{ if } i \neq j\}.$$

Definition 6 Let $(W_\gamma)_{\gamma\in\Gamma}$ be an m-dimensional random field defined on a nonlinear expectation space $(\Omega, \mathcal{H}, \hat{E})$. For each $\underline{\gamma} = (\gamma_1, \cdots, \gamma_n) \in \mathcal{J}_\Gamma$ and the corresponding random vector $W_{\underline{\gamma}} = (W_{\gamma_1}, \cdots, W_{\gamma_n})$, we define a functional on $C_{b.Lip}(\mathbb{R}^{n\times m})$ by

$$\mathbb{F}^W_{\underline{\gamma}}[\varphi] = \hat{E}[\varphi(W_{\underline{\gamma}})]$$

The collection $(\mathbb{F}^W_{\underline{\gamma}}[\varphi])_{\underline{\gamma}\in\mathcal{J}_\Gamma}$ is called the family of finite-dimensional distributions of $(W_\gamma)_{\gamma\in\Gamma}$.

It is clear that, for each $\underline{\gamma} \in \mathcal{J}_\Gamma$, the triple $(\mathbb{R}^{n\times m}, C_{b.Lip}(\mathbb{R}^{n\times m}), \mathbb{F}^W_{\underline{\gamma}})$ constitutes a nonlinear expectation space.

Let $(W^{(1)}_\gamma)_{\gamma\in\Gamma}$ and $(W^{(2)}_\gamma)_{\gamma\in\Gamma}$ be two m-dimensional random fields defined on nonlinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{E}_2)$ respectively. They are said to be identically distributed, denoted by $(W^{(1)}_\gamma)_{\gamma\in\Gamma} \stackrel{d}{=} (W^{(2)}_\gamma)_{\gamma\in\Gamma}$, or simply $W^{(1)} \stackrel{d}{=} W^{(2)}$, if for each $\underline{\gamma} = (\gamma_1, \cdots, \gamma_n) \in \mathcal{J}_\Gamma$,

$$\hat{E}_1[\varphi(W^{(1)}_{\underline{\gamma}})] = \hat{E}_2[\varphi(W^{(2)}_{\underline{\gamma}})], \ \ \forall \varphi \in C_{b.Lip}(\mathbb{R}^{n\times m}).$$

For any given m-dimensional random field $W = (W_\gamma)_{\gamma\in\Gamma}$, the family of its finite-dimensional distributions satisfies the following properties of consistency:

(1) Compatibility: For each $(\gamma_1, \cdots, \gamma_n, \gamma_{n+1}) \in \mathcal{J}_\Gamma$ and $\varphi \in C_{b.Lip}(\mathbb{R}^{n\times m})$,

$$\mathbb{F}^W_{\gamma_1,\cdots,\gamma_n}[\varphi] = \mathbb{F}^W_{\gamma_1,\cdots,\gamma_n,\gamma_{n+1}}[\widetilde{\varphi}], \tag{3}$$

where the function $\widetilde{\varphi}$ is a function on $\mathbb{R}^{(n+1)\times m}$ defined for any $y_1, \cdots, y_n, y_{n+1} \in \mathbb{R}^m$,

$$\widetilde{\varphi}(y_1, \cdots, y_n, y_{n+1}) = \varphi(y_1, \cdots, y_n);$$

(2) Symmetry: For each $(\gamma_1, \cdots, \gamma_n) \in \mathcal{J}_\Gamma$, $\varphi \in C_{b.Lip}(\mathbb{R}^{n\times m})$ and each permutation π of $\{1, \cdots, n\}$,

$$\mathbb{F}^W_{\gamma_{\pi(1)},\cdots,\gamma_{\pi(n)}}[\varphi] = \mathbb{F}^W_{\gamma_1,\cdots,\gamma_n}[\varphi_\pi] \tag{4}$$

where we denote $\varphi_\pi(y_1, \cdots, y_n) = \varphi(y_{\pi(1)}, \cdots, y_{\pi(n)})$, for $y_1, \cdots, y_n \in \mathbb{R}^m$.

The following theorem generalizes the classical Kolmogorov's existence theorem to the situation of sublinear expectation space, which is a variant of Theorem 3.8 in Peng [17]. The proof can be founded in Ji and Peng [10].

Theorem 1 *Let $\{\mathbb{F}_{\underline{\gamma}}, \underline{\gamma} \in \mathcal{J}_\Gamma\}$ be a family of finite-dimensional distributions satisfying the compatibility condition* (3) *and the symmetry condition* (4)*. Then there exists an m-dimensional random field $W = (W_\gamma)_{\gamma\in\Gamma}$ defined on a nonlinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ whose family of finite-dimensional distributions coincides with $\{\mathbb{F}_{\underline{\gamma}}, \underline{\gamma} \in \mathcal{J}_\Gamma\}$. Moreover, if we assume that each $\mathbb{F}_{\underline{\gamma}}$ in $\{\mathbb{F}_{\underline{\gamma}}, \underline{\gamma} \in \mathcal{J}_\Gamma\}$ is sublinear, then the corresponding expectation $\hat{E}$ on the space of random variables $(\Omega, \mathcal{H})$ is also sublinear.*

Now we consider a new random fields under a sublinear expectation space.

Definition 7 Let $(W_\gamma)_{\gamma\in\Gamma}$ be an m-dimensional random field, indexed by Γ, defined on a sublinear expectation space $(\Omega,\mathcal{H},\hat{E})$. $(W_\gamma)_{\gamma\in\Gamma}$ is called a maximally distributed random field if for each $\underline{\gamma}=(\gamma_1,\cdots,\gamma_n)\in\mathcal{J}_\Gamma$, the following $(n\times m)$-dimensional random vector

$$\begin{aligned}W_{\underline{\gamma}}=&(W_{\gamma_1},\cdots,W_{\gamma_n})\\=&(W_{\gamma_1}^{(1)},\cdots W_{\gamma_1}^{(m)},\cdots,W_{\gamma_n}^{(1)},\cdots,W_{\gamma_n}^{(m)}),\ \ W_{\gamma_i}^{(j)}\in\mathcal{H},\end{aligned}$$

is maximally distributed.

For each $\underline{\gamma}=(\gamma_1,\cdots,\gamma_n)\in\mathcal{J}_\Gamma$, we define

$$g_{\underline{\gamma}}^W(p)=\hat{E}[W_{\underline{\gamma}}\cdot p],\quad p\in\mathbb{R}^{n\times m},$$

Then $(g_{\underline{\gamma}}^W)_{\underline{\gamma}\in\mathcal{J}_\Gamma}$ constitutes a family of sublinear functions:

$$g_{\underline{\gamma}}^W:\mathbb{R}^{n\times m}\mapsto\mathbb{R},\quad \underline{\gamma}=(\gamma_1,\cdots,\gamma_n),\ \gamma_i\in\Gamma,\ 1\le i\le n,\ n\in\mathbb{N},$$

which satisfies the properties of consistency in the following sense:

(1) Compatibility: For any $(\gamma_1,\cdots,\gamma_n,\gamma_{n+1})\in\mathcal{J}_\Gamma$ and $p=(p_i)_{i=1}^{n\times m}\in\mathbb{R}^{n\times m}$,

$$g_{\gamma_1,\cdots,\gamma_n,\gamma_{n+1}}^W(\bar{p})=g_{\gamma_1,\cdots,W_{\gamma_n}}^W(p),\tag{5}$$

where $\bar{p}=\begin{pmatrix}p\\0\end{pmatrix}\in\mathbb{R}^{(n+1)\times m}$;

(2) Symmetry: For any permutation π of $\{1,\cdots,n\}$,

$$g_{\gamma_{\pi(1)},\cdots,\gamma_{\pi(n)}}^W(p)=g_{\gamma_1,\cdots,\gamma_n}^W(\pi^{-1}(p)),\tag{6}$$

where $\pi^{-1}(p)=(p^{(1)},\ldots,p^{(n)})$,

$$p^{(i)}=(p_{(\pi^{-1}(i)-1)m+1},\ldots,p_{(\pi^{-1}(i)-1)m+m}),\ 1\le i\le n.$$

If the above type of family of sublinear functions $(g_{\underline{\gamma}})_{\underline{\gamma}\in\mathcal{J}_\Gamma}$ is given, following the construction procedure in the proof of Theorem 3.5 in Ji and Peng [10], we can construct a maximally distributed random field on sublinear expectation space.

Theorem 2 *Let $(g_{\underline{\gamma}})_{\underline{\gamma}\in\mathcal{J}_\Gamma}$ be a family of real-valued functions such that, for each $\underline{\gamma}=(\gamma_1,\cdots,\gamma_n)\in\mathcal{J}_\Gamma$, the real function $g_{\underline{\gamma}}$ is defined on $\mathbb{R}^{n\times m}\mapsto\mathbb{R}$ and satisfies the sub-linearity. Moreover, this family $(g_{\underline{\gamma}})_{\underline{\gamma}\in\mathcal{J}_\Gamma}$ satisfies the compatibility condition* (5) *and symmetry condition* (6)*. Then there exists an m-dimensional maximally distributed random field $(W_\gamma)_{\gamma\in\Gamma}$ on a sublinear expectation space $(\Omega,\mathcal{H},\hat{E})$ such that for each $\underline{\gamma}=(\gamma_1,\cdots,\gamma_n)\in\mathcal{J}_\Gamma$, $W_{\underline{\gamma}}=(W_{\gamma_1},\cdots,W_{\gamma_n})$ is maximally distributed with generating function*

$$g^W_{\underline{\gamma}}(p) = \hat{E}[W_{\underline{\gamma}} \cdot p] = g_{\underline{\gamma}}(p), \quad \textit{for any } p \in \mathbb{R}^{n\times m}.$$

Furthermore, if there exists another maximally distributed random field $(\bar{W}_\gamma)_{\gamma\in\Gamma}$*, with the same index set* Γ*, defined on a sublinear expectation space* $(\bar{\Omega}, \bar{\mathcal{H}}, \bar{E})$ *such that for each* $\underline{\gamma} = (\gamma_1, \cdots, \gamma_n) \in \mathcal{J}_\Gamma$*,* $\bar{W}_{\underline{\gamma}}$ *is maximally distributed with the same generating function* $g_{\underline{\gamma}}$*, namely,*

$$\bar{E}[\bar{W}_{\underline{\gamma}} \cdot p] = g_{\underline{\gamma}}(p) \quad \textit{for any } p \in \mathbb{R}^{n\times m},$$

then we have $W \overset{d}{=} \bar{W}$*.*

4 Maximally distributed white noise

In this section, we formulate a new type of maximally distributed white noise on $\mathbb{R}^d$.

Given sublinear expectation space $\Omega, \mathcal{H}, \hat{E}$, let $\mathbb{L}^p(\Omega)$ be the completion of $\mathcal{H}$ under the Banach norm $\|X\| := \hat{E}[|X|^p]^{\frac{1}{p}}$. For any $X, Y \in \mathbb{L}^1(\Omega)$, we say that $X = Y$ if $\hat{E}[|X - Y|] = 0$. As shown in Chapter 1 of Peng [18], $\hat{E}$ can be continuously extended to the mapping from $\mathbb{L}^1(\Omega)$ to $\mathbb{R}$ and properties (i)-(iv) of Definition 1 still hold. Moreover, $(\Omega, \mathbb{L}^1(\Omega), \hat{E})$ also forms a sublinear expectation space, which is called the complete sublinear expectation space.

Definition 8 Let $(\Omega, \mathbb{L}^1(\Omega), \hat{E})$ be a complete sublinear expectation space and $\Gamma = \mathcal{B}_0(\mathbb{R}^d) := \{A \in \mathcal{B}(\mathbb{R}^d), \lambda_A < \infty\}$, where λ_A denotes the Lebesgue measure of $A \in \mathcal{B}(\mathbb{R}^d)$. Let $g : \mathbb{R} \mapsto \mathbb{R}$ be a given sublinear function, i.e.,

$$g(p) = \overline{\mu}p^+ - \underline{\mu}p^-, \quad -\infty < \underline{\mu} \le \overline{\mu} < +\infty.$$

A random field $W = \{W_A\}_{A\in\Gamma}$ is called a one-dimensional maximally distributed white noise if

(i) For each $A_1, \cdots, A_n \in \Gamma$, $(W_{A_1}, \cdots, W_{A_n})$ is a $\mathbb{R}^n$-maximally distributed random vector under $\hat{E}$, and for each $A \in \Gamma$,

$$\hat{E}[W_A \cdot p] = g(p)\lambda_A, \quad p \in \mathbb{R}. \tag{7}$$

(ii) Let $A_1, A_2, \cdots, A_n$ be in Γ and mutually disjoint, then $\{W_{A_i}\}_{i=1}^n$ are independent sequence, and

$$W_{A_1\cup A_2\cup\cdots\cup A_n} = W_{A_1} + W_{A_2} + \cdots + W_{A_n}. \tag{8}$$

Remark 5 For each $A \in \Gamma$, we can restrict that W_A takes values in $[\lambda_A\underline{\mu}, \lambda_A\overline{\mu}]$. Indeed, let

$$d_A(x) := \min_{y\in[\lambda_A\underline{\mu}, \lambda_A\overline{\mu}]} \{|x - y|\},$$

by the definition of maximal distribution,

$$\hat{E}[d_A(W_A)] = \max_{v\in[\lambda_A\underline{\mu},\lambda_A\overline{\mu}]} \min_{y\in[\lambda_A\underline{\mu},\lambda_A\overline{\mu}]} \{|v-y|\} = 0,$$

which implies that $d_A(W_A) = 0$.

We can construct a spatial maximal white noise satisfying Definition 8 in the following way.

For each $\underline{\gamma} = (A_1, \cdots, A_n) \in \mathcal{J}_\Gamma$, $\Gamma = \mathcal{B}_0(\mathbb{R}^d)$, consider the mapping $g_{\underline{\gamma}}(\cdot) : \mathbb{R}^n \to \mathbb{R}$ defined as follows:

$$g_{\underline{\gamma}}(p) := \sum_{k\in\{0,1\}^n} g(k\cdot p)\lambda_{B(k)}, \quad p \in \mathbb{R}^n, \tag{9}$$

where $k = (k_1, \cdots, k_n) \in \{0, 1\}^n$, and $B(k) = \cap_{j=1}^n B_j$, with

$$B_j = \begin{cases} A_j & \text{if } k_j = 1, \\ A_j^c & \text{if } k_j = 0. \end{cases}$$

For example, given $A_1, A_2, A_3 \in \Gamma$ and $p = (p_1, p_2, p_3) \in \mathbb{R}^3$,

$$\begin{aligned} &g_{A_1,A_2,A_3}(p) = g(p_1+p_2+p_3)\lambda_{A_1\cap A_2\cap A_3} \\ &+ g(p_1+p_2)\lambda_{A_1\cap A_2\cap A_3^c} + g(p_2+p_3)\lambda_{A_1^c\cap A_2\cap A_3} + g(p_1+p_3)\lambda_{A_1\cap A_2^c\cap A_3} \\ &+ g(p_1)\lambda_{A_1\cap A_2^c\cap A_3^c} + g(p_2)\lambda_{A_1^c\cap A_2\cap A_3^c} + g(p_3)\lambda_{A_1^c\cap A_2^c\cap A_3}. \end{aligned}$$

Obviously, for each $\underline{\gamma} = (A_1, \cdots, A_n) \subset \Gamma$, $g_{\underline{\gamma}}(\cdot)$ defined by (9) is a sublinear function defined on $\mathbb{R}^n$ due to the sub-linearity of function $g(\cdot)$. The following property shows that the consistency conditions (5) and (6) also hold for $\{g_{\underline{\gamma}}\}_{\gamma\in\mathcal{J}_\Gamma}$.

Proposition 4 *The family $\{g_{\underline{\gamma}}\}_{\gamma\in\mathcal{J}_\Gamma}$ defined by (9) satisfies the consistency conditions (5) and (6).*

Proof For compatibility (5), given $A_1, \cdots, A_n, A_{n+1} \in \Gamma$ and $\bar{p}^T = (p^T, 0) \in \mathbb{R}^{n+1}$, we have

$$\begin{aligned} g_{A_1,\cdots,A_{n+1}}(\bar{p}) &= \sum_{k\in\{0,1\}^{n+1}} g(k\cdot\bar{p})\lambda_{B(k)} \\ &= \sum_{k'\in\{0,1\}^n} g(k'\cdot p)(\lambda_{B(k')\cap A_{n+1}} + \lambda_{B(k')\cap A_{n+1}^c}) \\ &= \sum_{k'\in\{0,1\}^n} g(k'\cdot p)\lambda_{B(k')} = g_{A_1,\cdots,A_n}(p). \end{aligned}$$

The symmetry (6) can be easily verified since the operators $k\cdot p$ and $B(k) = \cap_{j=1}^n B_j$ are also symmetry. □

Now we present the existence of the maximally distributed white noises under the sublinear expectation.

Theorem 3 *For each given sublinear function*

$$g(p) = \max_{\mu\in[\underline{\mu},\overline{\mu}]}(\mu\cdot p) = \overline{\mu}p^+ - \underline{\mu}p^-,\ p\in\mathbb{R},$$

there exists a one-dimensional maximally distributed random field $(W_\gamma)_{\gamma\in\Gamma}$ on a sublinear expectation space $(\Omega, \mathbb{L}^1(\Omega), \hat{E})$ such that, for each $\underline{\gamma} = (A_1, \cdots, A_n) \in \mathcal{J}_\Gamma$, $W_{\underline{\gamma}} = (W_{A_1}, \cdots, W_{A_n})$ is maximally distributed.

Furthermore, $(W_\gamma)_{\gamma\in\Gamma}$ is a spatial maximally distributed white noise under $(\Omega, \mathbb{L}^1(\Omega), \hat{E})$, namely, conditions (i) and (ii) of Definition 8 are satisfied.

If $(\bar{W}_\gamma)_{\gamma\in\Gamma}$ is another maximally distributed white noise with the same sublinear function g in (9)*, then $\bar{W} \overset{d}{=} W$.*

Proof Thanks to Proposition 4 and Theorem 2, the existence and uniqueness of the maximally distributed random field W in a sublinear expectation space $(\Omega, \mathbb{L}^1(\Omega), \hat{E})$ with the family of generating functions defined by (9) hold. We only need to verify that the maximally distributed random field W satisfies conditions (i) and (ii) of Definition 8.

For each $A \in \Gamma$, $\hat{E}[W_A \cdot p] = g(p)\lambda_A$ by Theorem 2 and (9), thus (i) of Definition 8 holds.

We note that if $\{A_i\}_{i=1}^n$ are mutually disjoint, then for $p = (p_1, \cdots, p_n) \in \mathbb{R}^n$, by (9), we have

$$\hat{E}[p_1 W_{A_1} + \cdots + p_n W_{A_n}] = g(p_1)\lambda_{A_1} + \cdots + g(p_n)\lambda_{A_n},$$

thus the independence of $\{W_{A_i}\}_{i=1}^n$ can be implied by Proposition 3.

In order to prove (8), we only consider the case of two disjoint sets. Suppose that

$$A_1 \cap A_2 = \emptyset, \quad A_3 = A_1 \cup A_2,$$

an easy computation of (9) shows that

$$\begin{aligned} g_{A_1,A_2,A_3}(p) =& g(p_1 + p_3)\lambda_{A_1} + g(p_2 + p_3)\lambda_{A_2} \\ =& \max_{v_1\in[\underline{\mu}\lambda_{A_1},\overline{\mu}\lambda_{A_1}]}\ \max_{v_2\in[\underline{\mu}\lambda_{A_2},\overline{\mu}\lambda_{A_2}]}\ \max_{v_3=v_1+v_2}(p_1\cdot v_1 + p_2\cdot v_2 + p_3\cdot v_3). \end{aligned}$$

Thus, for each $\varphi \in C(\mathbb{R}^3)$,

$$\hat{E}[\varphi(W_{A_1}, W_{A_2}, W_{A_3})] = \max_{v_1\in[\underline{\mu}\lambda_{A_1},\overline{\mu}\lambda_{A_1}]}\ \max_{v_2\in[\underline{\mu}\lambda_{A_2},\overline{\mu}\lambda_{A_2}]}\ \max_{v_3=v_1+v_2}\varphi(v_1, v_2, v_3).$$

In particular, we set $\varphi(v_1, v_2, v_3) = |v_1 + v_2 - v_3|$, it follows that

$$\hat{E}[|W_{A_1} + W_{A_2} - W_{A_1\cup A_2}|] = 0.$$

which implies that

$$W_{A_1 \cup A_2} = W_{A_1} + W_{A_2}.$$

Finally, (ii) of Definition 8 holds. □

Remark 6 The finite-dimensional distribution of maximally distributed whiten noise can be uniquely determined by two parameters $\overline{\mu}$ and $\underline{\mu}$, which can be simply calculated by taking the maximum of the continuous function over the domain determined by $\overline{\mu}$ and $\underline{\mu}$.

Similar to the invariant property of G-Gaussian white noise introduced in Ji and Peng [10], it also holds for maximally distributed white noise due to the well-known invariance of the Lebesgue measure under rotation and translation.

Proposition 5 *For each $p \in \mathbb{R}^d$ and $O \in \mathbb{O}(d) := \{O \in \mathbb{R}^{d\times d} : O^T = O^{-1}\}$, we set*

$$T_{p,O}(A) = O \cdot A + p, \quad A \in \Gamma.$$

Then, for each $A_1, \cdots, A_n \in \Gamma$,

$$(W_{A_1}, \cdots, W_{A_n}) \overset{d}{=} (W_{T_{p,O}(A_1)}, \cdots, W_{T_{p,O}(A_n)}).$$

5 Spatial and temporal maximally distributed white noise and related stochastic integral

In Ji and Peng [10], we see that a spatial G-white noise is essentially different from the temporal case or the temporal-spatial case, since there is no independence property for the spatial G-white noise. But for the maximally distributed white noise, spatial or temporal-spatial maximally distributed white noise has the independence property due to the symmetrical independence for maximal distribution.

Combining symmetrical independence and boundedness properties of maximal distribution, the integrand random fields can be largely extended when we consider the stochastic integral with respect to spatial maximally distributed white noise. For stochastic integral with respect to temporal-spatial case, the integrand random fields can even contain the "non-adapted" situation.

5.1 Stochastic integral with respect to the spatial maximally distributed white noise

We firstly define the stochastic integral with respect to the spatial maximally distributed white noise in a quite direct way.

Let $\{W_\gamma\}_{\gamma\in\Gamma}$, $\Gamma = \mathcal{B}_0(\mathbb{R}^d)$, be a one-dimensional maximally distributed white noise defined on a complete sublinear expectation space $(\Omega, \mathbb{L}^1(\Omega), \hat{E})$, with $g(p) =$

$\overline{\mu}p^+ - \underline{\mu}p^-$, $-\infty < \underline{\mu} \leq \overline{\mu} < \infty$. We introduce the following type of random fields, called simple random fields.

Given $p \geq 1$, set

$$M_g^{p,0}(\Omega) = \{\eta(x,\omega) = \sum_{i=1}^{n} \xi_i(\omega) 1_{A_i}(x),\ A_1, \cdots, A_n \in \Gamma \text{ are mutually disjoint}$$
$$i = 1, 2, \ldots, n,\ \ \xi_1, \cdots, \xi_n \in \mathbb{L}^p(\Omega), \quad n = 1, 2, \cdots, \}.$$

For each simple random fields $\eta \in M_g^{p,0}(\Omega)$ of the form

$$\eta(x,\omega) = \sum_{i=1}^{n} \xi_i(\omega) \mathbf{1}_{A_i}(x), \tag{10}$$

the related Bohner's integral for η with respect to the Lebesgue measure λ is

$$I_B(\eta) = \int_{\mathbb{R}^d} \eta(x,\omega)\lambda(dx) := \sum_{i=1}^{n} \xi_i(\omega)\lambda_{A_i}.$$

It is immediate that $I_B(\eta) : M_g^{p,0}(\Omega) \mapsto \mathbb{L}^p(\Omega)$ is a linear and continuous mapping under the norm for η, defined by,

$$\|\eta\|_{M^p} = \hat{E}[\int_{\mathbb{R}^d} |\eta(x,\omega)|^p \lambda(dx)]^{\frac{1}{p}}.$$

The completion of $M_g^{p,0}(\Omega)$ under this norm is denoted by $M_g^p(\Omega)$ which is a Banach space. The unique extension of the mapping I_B is denoted by

$$\int_{\mathbb{R}^d} \eta(x,\omega)\lambda(dx) := I_B(\eta),\ \ \eta \in M_g^p(\Omega).$$

Now for a simple random field $\eta \in M_g^{p,0}(\Omega)$ of form (10), we define its stochastic integral with respect to W as

$$I_W(\eta) := \int_{\mathbb{R}^d} \eta(x,\omega)W(dx) = \sum_{i=1}^{n} \xi_i(\omega)W_{A_i}.$$

With this formulation, we have the following estimation.

Lemma 1 *For each $\eta \in M_g^{1,0}(\Omega)$ of form (10), we have*

$$\hat{E}\left[\left|\int_{\mathbb{R}^d} \eta(x,\omega)W(dx)\right|\right] \leq \kappa \hat{E}\left[\int_{\mathbb{R}^d} |\eta(x,\omega)|\lambda(dx)\right] \tag{11}$$

where $\kappa = \max\{|\underline{\mu}|, |\overline{\mu}|\}$.

Proof We have

$$\hat{E}[|\int_{\mathbb{R}^d}\eta(x,\omega)W(dx)|] = \hat{E}[|\sum_{i=1}^{N}\xi_i(\omega)W_{A_i}|] \le \hat{E}[\sum_{i=1}^{N}|\xi_i(\omega)|\cdot|W_{A_i}|]$$
$$\le \kappa\hat{E}[\sum_{i=1}^{N}|\xi_i(\omega)|\cdot\lambda_{A_i}] = \kappa\hat{E}[\|\eta\|_{M_g^1(\Omega)}].$$

The last inequality is due to the boundedness of maximal distribution (see Remark 5). □

This lemma shows that $I_W : M_g^{1,0}(\Omega) \mapsto \mathbb{L}^1(\Omega)$ is a linear continuous mapping. Consequently, I_W can be uniquely extended to the whole domain $M_g^1(\Omega)$. We still denote this extended mapping by

$$\int_{\mathbb{R}^d}\eta W(dx) := I_W(\eta).$$

Remark 7 Different from the stochastic integrals with respect to G-white noise in Ji and Peng [10] which is only defined for the deterministic integrand, here the integrand can be a random field.

5.2 Maximally distributed random fields of temporal-spatial types and related stochastic integral

It is well-known that the framework of the classical white noise defined in a probability space $(\Omega, \mathcal{F}, P)$ with 1-dimensional temporal and d-dimensional spatial parameters is in fact a $\mathbb{R}^{1+d}$-indexed space type white noise. But Peng [17] and then Ji and Peng [10] observed a new phenomenon: Unlike the classical Gaussian white noise, the d-dimensional space-indexed G-white noise cannot have the property of incremental independence, thus spatial G-white noise is essentially different from temporal-spatial or temporal one. Things will become much direct for the case of maximally distributed white noise due to the incremental independence property of maximal distributions. This means that a time-space maximally distributed $(1+d)$-white noise is essentially a $(1+d)$-spatial white noise. The corresponding stochastic integral is also the same. But in order to make clear the dynamic properties, we still provide the description of the temporal-spatial white-noise on the time-space framework:

$$\mathbb{R}^+ \times \mathbb{R}^d = \{(t, x_1, \ldots, x_d) \in \mathbb{R}^+ \times \mathbb{R}^d\},$$

where the index $t \in [0, \infty)$ is specially preserved to be the index for time.

Let $\Gamma = \{A \in \mathcal{B}(\mathbb{R}^+ \times \mathbb{R}^d), \lambda_A < \infty\}$, the maximally distributed white noise $\{W_A\}_{A\in\Gamma}$ is just like in the spatial case with dimension $1 + d$.

More precisely, let

$$\Omega = \{\omega \in \mathbb{R}^\Gamma : \omega(A \cup B) = \omega(A) + \omega(B),\\ \forall A, B \in \Gamma, \quad A \cup B = \emptyset\},$$

and $W = (W_\gamma(\omega) = \omega_\gamma)_{\gamma \in \Gamma}$ the canonical random field.

For $T > 0$, denote the temporal-spatial sets before time T by

$$\Gamma_T := \{A \in \Gamma : (s, x) \in A \Rightarrow 0 \leq s < T\}.$$

Set $\mathcal{F}_T = \sigma\{W_A, A \in \Gamma_T\}$, $\mathcal{F} = \bigvee_{T \geq 0} \mathcal{F}_T$, and

$$L_{ip}(\Omega_T) = \{\varphi(W_{A_1}, \ldots, W_{A_n}), \ \forall n \in \mathbb{N},\\ A_i \in \Gamma_T, i = 1, \ldots, n, \varphi \in C_{b.Lip}(\mathbb{R}^n)\}.$$

We denote

$$L_{ip}(\Omega) = \bigcup_{n=1}^{\infty} L_{ip}(\Omega_n).$$

For each $X \in L_{ip}(\Omega)$, without loss of generality, we assume X has the form

$$X = \varphi(W_{A_{11}}, \cdots, W_{A_{1m}}, \cdots, W_{A_{n1}}, \cdots, W_{A_{nm}}),$$

where $A_{ij} = [t_{i-1}, t_i) \times A_j$, $1 \leq i \leq n, 1 \leq j \leq m$, $0 = t_0 < t_1 < \cdots < t_n < \infty$, $\{A_1, \cdots, A_m\} \subset \mathcal{B}_0(\mathbb{R}^d)$ are mutually disjoint and $\varphi \in C_{b.Lip}(\mathbb{R}^{n \times m})$. Then the corresponding sublinear expectation for X can be defined by

$$\begin{aligned}\hat{E}[X] &= \hat{E}[\varphi(W_{A_{11}}, \cdots, W_{A_{1m}}, \cdots, W_{A_{n1}}, \cdots, W_{A_{nm}})\\ &= \max_{v_{ij} \in [\underline{\mu}, \overline{\mu}]} \varphi(\lambda_{A_{11}} v_{11}, \cdots, \lambda_{A_{1m}} v_{1m}, \cdots, \lambda_{A_{n1}} v_{v_{n1}}, \cdots, \lambda_{A_{nm}} v_{nm}),\\ &\qquad\qquad 1 \leq i \leq m, 1 \leq j \leq n\end{aligned}$$

and the related conditional expectation of X under $\mathcal{F}_t$, where $t_j \leq t < t_{j+1}$, denoted by $\hat{E}[X|\mathcal{F}_t]$, is defined by

$$\hat{E}[\varphi(W_{A_{11}}, \cdots, W_{A_{1m}}, \cdots, W_{A_{n1}}, \cdots, W_{A_{nm}})|\mathcal{F}_t]\\ = \psi(W_{A_{11}}, \cdots, W_{A_{1m}}, \cdots, W_{A_{j1}}, \cdots, W_{A_{jm}}),$$

where

$$\psi(x_{11}, \cdots, x_{1m}, \cdots, x_{j1}, \cdots, x_{jm}) = \hat{E}[\varphi(x_{11}, \cdots, x_{1m}, \cdots, x_{j1}, \cdots, x_{jm}, \tilde{W})].$$

Here

$$\tilde{W} = (W_{A_{(j+1)1}}, \cdots, W_{A_{(j+1)m}}, \cdots, W_{A_{n1}}, \cdots, W_{A_{nm}}).$$

It is easy to verify that $\hat{E}[\cdot]$ defines a sublinear expectation on $L_{ip}(\Omega)$ and the canonical process $(W_\gamma)_{\gamma\in\Gamma}$ is a one-dimensional temporal-spatial maximally distributed white noise on $(\Omega, L_{ip}(\Omega), \hat{E})$.

For each $p \geq 1$, $T \geq 0$, we denote by $L_g^p(\Omega_T)$(resp., $L_g^p(\Omega)$) the completion of $L_{ip}(\Omega_T)$(resp., $L_{ip}(\Omega)$) under the norm $\|X\|_p := (\hat{E}[|X|^p])^{1/p}$. The conditional expectation $\hat{E}[\cdot|\mathcal{F}_t] : L_{ip}(\Omega) \to L_{ip}(\Omega_t)$ is a continuous mapping under $\|\cdot\|_p$ and can be extended continuously to the mapping $L_g^p(\Omega) \to L_g^p(\Omega_t)$ by

$$|\hat{E}[X|\mathcal{F}_t] - \hat{E}[Y|\mathcal{F}_t]| \leq \hat{E}[|X-Y||\mathcal{F}_t] \quad \text{for } X, Y \in L_{ip}(\Omega).$$

It is easy to verify that the conditional expectation $\hat{E}[\cdot|\mathcal{F}_t]$ satisfies the following properties, and the proof is very similar to the corresponding one of Proposition 5.3 in Ji and Peng [10].

Proposition 6 *For each $t \geq 0$, the conditional expectation $\hat{E}[\cdot|\mathcal{F}_t] : L_g^p(\Omega) \to L_g^p(\Omega_t)$ satisfies the following properties: for any $X, Y \in L_g^p(\Omega)$, $\eta \in L_g^p(\Omega_t)$,*

(i) $\hat{E}[X|\mathcal{F}_t] \geq \hat{E}[Y|\mathcal{F}_t]$ *for* $X \geq Y$.
(ii) $\hat{E}[\eta|\mathcal{F}_t] = \eta$.
(iii) $\hat{E}[X+Y|\mathcal{F}_t] \leq \hat{E}[X|\mathcal{F}_t] + \hat{E}[Y|\mathcal{F}_t]$.
(iv) $\hat{E}[\eta X|\mathcal{F}_t] = \eta^+\hat{E}[X|\mathcal{F}_t] + \eta^-\hat{E}[-X|\mathcal{F}_t]$ *if* η *is bounded.*
(v) $\hat{E}[\hat{E}[X|\mathcal{F}_t]|\mathcal{F}_s] = \hat{E}[X|\mathcal{F}_{t\wedge s}]$ *for* $s \geq 0$.

Now we define the stochastic integral with respect to the spatial-temporal maximally distributed white noise W, which is similar to the spatial situation.

For each given $p \geq 1$, let $M^{p,0}(\Omega_T)$ be the collection of simple processes with the form:

$$f(s,x;\omega) = \sum_{i=0}^{n-1}\sum_{j=1}^{m} X_{ij}(\omega)\mathbf{1}_{A_j}(x)\mathbf{1}_{[t_i,t_{i+1})}(s), \tag{12}$$

where $X_{ij} \in L_g^p(\Omega_T)$, $i = 0, \cdots, n-1$, $j = 1, \cdots, m$, $0 = t_0 < t_1 < \cdots < t_n = T$, and $\{A_j\}_{j=1}^m \subset \Gamma$ is mutually disjoint.

Remark 8 Since we only require $X_{ij} \in L_g^p(\Omega_T)$, the integrand may "non-adapted". This issue is essentially different from the requirement of adaptability in the definition of stochastic integral with respect to temporal-spatial G-white noise in Ji and Peng [10].

The completion of $M^{p,0}(\Omega_T)$ under the norm $\|\cdot\|_{M^p}$, denoted by $M_g^p(\Omega_T)$, is a Banach space, where the Banach norm $\|\cdot\|_{M^p}$ is defined by

$$\begin{aligned}\|f\|_{M^p} &:= \left(\hat{E}\left[\int_0^T \int_{\mathbb{R}^d} |f(s,x)|^p ds\lambda(dx)\right]\right)^{\frac{1}{p}} \\ &= \left\{\hat{E}\left[\sum_{i=0}^{n-1}\sum_{j=1}^{m} |X_{ij}|^p(t_{i+1}-t_i)\lambda_{A_j}\right]\right\}^{\frac{1}{p}}.\end{aligned}$$

For $f \in M^{p,0}(\Omega_T)$ with the form as (12), the related stochastic integral with respect to the temporal-spatial maximally distributed white noise W can be defined as follows:

$$I_W(f) = \int_0^T \int_{\mathbb{R}^d} f(s,x)W(ds,dx) := \sum_{i=0}^{n-1}\sum_{j=1}^{m} X_{ij} W([t_j, t_{j+1}) \times A_j). \tag{13}$$

Similar to Lemma 1, we have

Lemma 2 *For each* $f \in M^{1,0}([0,T] \times \mathbb{R}^d)$,

$$\hat{E}\left[\left|\int_0^T \int_{\mathbb{R}^d} f(s,x)W(ds,dx)\right|\right] \leq \kappa \hat{E}\left[\int_0^T \int_{\mathbb{R}^d} |f(s,x)| ds dx\right], \tag{14}$$

where $\kappa = \max\{|\overline{\mu}|, |\underline{\mu}|\}$.

Thus $I_W : M^{1,0}(\Omega_T) \mapsto L_g^1(\Omega_T)$ is a continuous linear mapping. Consequently, I_W can be uniquely extend to the domain $M_g^1(\Omega_T)$. We still denote this mapping by

$$\int_0^T \int_{\mathbb{R}^d} f(s,x)W(ds,dx) := I_W(f) \ \text{ for } f \in M_g^1(\Omega_T).$$

Remark 9 Thanks to the boundedness of maximally distributed white noise, the domain of integrand $M_g^1(\Omega_T)$ is much larger since the usual requirement of adaptability for integrand can be dropped.

It is easy to check that the stochastic integral has the following properties.

Proposition 7 *For each* $f, g \in M_g^1(\Omega_T)$, $0 \leq s \leq r \leq t \leq T$,
(i) $\int_s^t \int_{\mathbb{R}^d} f(u,x)W(du,dx) = \int_s^r \int_{\mathbb{R}^d} f(u,x)W(du,dx) + \int_r^t \int_{\mathbb{R}^d} f(u,x)W(du,dx)$.
(ii) $\int_s^t \int_{\mathbb{R}^d} (\alpha f(u,x) + g(u,x))W(du,dx)$
$= \alpha \int_s^t \int_{\mathbb{R}^d} f(u,x)W(du,dx) + \int_s^t \int_{\mathbb{R}^d} g(u,x)W(du,dx)$, *where* $\alpha \in L_g^1(\Omega_T)$ *is bounded.*

Remark 10 In particular, if we only consider temporal maximally distributed white noise and further assume that $\underline{\mu} \geq 0$. In this case, the index set $\Gamma = \{[s,t) : 0 \leq s < t < \infty\}$. The canonical process $W([0,t))$ is the quadratic variation process of G-Brownian motion, more details about the quadratic variation process can be found in Peng [18].

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