



Control in Hilbert Space and First-Order Mean Field Type Problem

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Abstract We extend the work [9] by two of the coauthors, which dealt with a deterministic control problem for which the Hilbert space could be generic and investigated a novel form of the ‘lifting’ technique proposed by P. L. Lions. In [9], we only showed the local existence and uniqueness of solutions to the FBODEs in the Hilbert space which were associated to the control problems with drift function consisting of the control only. In this article, we establish the global existence and uniqueness of the solutions to the FBODEs in Hilbert space corresponding to control problems with separable drift function which is nonlinear in state and linear in control. We shall also prove the sufficiency of the Pontryagin Maximum Principle and derive the corresponding Bellman equation. Finally, by using the ‘lifting’ idea as in [6, 7], we shall apply the result to solve the linear quadratic mean field type control problems, and to show the global existence of the corresponding Bellman equations.

Dedicated in memory of Mark Davis for his outstanding contribution in control theory.

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1 Introduction

In recent years, Mean Field Game (MFG) and Mean Field Type Control Theory (MFTCT) are burgeoning. Carmona and Delarue [14] proved the existence of the general forward-backward systems of equations of McKean-Vlasov type using the probabilistic approach, and therefore obtained the classical solution to the master equation arisen from MFG. Their assumptions restricted their application to LQ models only. Cardaliaguet et al. [12] proved the existence of the classical solution to the master equation arisen from MFG by PDE techniques and the method of characteristics. To do so, they required the state space to be compact, and the Hamiltonian to be smooth, globally Lipschitz continuous and to satisfy a certain coercivity condition. Buckdahn et al. [11] adopted a similar approach to study forward flows, proving that the semigroup of a standard McKean-Vlasov stochastic differential equation with smooth coefficients is the classical solution of a particular type of master equation. A crucial assumption was made therein on the smoothness of the coefficients, which restricted the scope of applications. Gangbo and Mészáros [19] constructed global solutions to the master equation in potential Mean Field Games, where displacement convexity was used as a substitution for the monotonicity condition. Besides the notion of classical solutions, Mou and Zhang in [26] gave a notion of weak solution of the master equation arisen from mean field games, using their results of mollifiers on the infinite dimensional space. More results can be found in the papers of Cosso and Pham [16], Pham and Wei [29] and Djete et al. [18], which concern the Bellman and Master equations of Mean Field Games and Mean Field Type Control Theory.

By Pontryagin Maximum Principle, MFG and MFTCT are deeply connected to mean field forward backward stochastic differential equations. Pardoux and Tang [27], Antonelli [2] and Hu and Yong [21] showed the existence and uniqueness of FBSDEs under small time intervals by a fixed point argument. For Markovian FBSDEs, to get rid of the small time issue, Ma et al. [24] employed the Four Step Scheme. They constructed decoupling functions by the use of the classical solutions of quasi-linear PDEs, hence non-degeneracy of the diffusion coefficient and the strong regularity condition on the coefficients were required. Another way to remove time constraints in Markovian FBSDEs was by Delarue [17]. Local solutions were patched together by the use of decoupling functions. PDE methods were used to bound the coefficients of the terminal function relative to the initial data in order for the problem to be well-posed. It was later extended to the case of non-Markovian FBSDEs by Zhang in [32]. Moreover, to deal with non-Markovian FBSDEs with arbitrary time length, there was the pure probabilistic method – method of continuation. It required monotonicity conditions on the coefficients. For seminal works one may consult [20, 28, 30, 31]. With the help of decoupling functions as in [17], but using a BSDE to control the terminal coefficient instead of PDEs, Ma et al. [25] covered most of the above cases, but in the case of codomain being \mathbb{R} . For mean field type FBSDE. A rather general existence result but with a restrictive assumption (boundedness of the coefficients with respect to the state variable) was first done in [13] by Carmona and Delarue. Taking advantage of the convexity of the underlying Hamiltonian and ap-

plying the continuation method, Carmona and Delarue extended their results in [14]. Bensoussan et al. [10] exploited the condition in [14] and gave weaker conditions for which the results in [14] still hold. By the method of continuation, Ahuja et al. [1] extended the above result to the FBSDEs which allow coefficients to be functionals of the processes. More details can be found in the monographs [15, 3] and [4, 5, 8].

We establish the global existence and uniqueness of the solutions to the FBODEs in Hilbert space corresponding to control problems with separable drift function which is nonlinear in state and linear in control. The result can be applied to solve linear quadratic mean field type control problems. We exploit the ‘lifting to Hilbert space’ approach suggested by P. L. Lions in [22, 23], but lift to another Hilbert space instead of the space of random variables. After lifting, the problems are akin to standard control problems, but the drawback is that they are in the infinite dimensional space. By the Pontryagin Maximum Principle, the control problems are reduced to FBODEs in the Hilbert space. In order to accommodate nonlinear settings, we make use of the idea of decoupling. By a Banach fixed point argument, we are able to locally find a decoupling function for the FBODEs. We then derive *a priori* estimates of the decoupling function and extend the solution from local to global as in Delarue [17] by the *a priori* estimates. Finally we apply our result to solve linear quadratic mean field type control problems and obtain their corresponding Bellman equations.

The rest of this article is organized as follows. In Section 2, we introduce the model in the Hilbert space. In Section 3, we express the related FBODE and define the decoupling function. *A priori* estimates of the decoupling function are derived in Section 4. In Section 5, we prove the local existence and uniqueness of the FBODE by using a Banach fixed point argument on the function space containing the decoupling function. In Section 6.1, we construct the global solution by our *a priori* estimates. We show the sufficiency of the Maximum Principle in Section 6.2 and write the corresponding Bellman function in Section 6.3. In Section 7, we apply our result in the Hilbert space as in [7], to solve the optimal control problem, and show the global existence to the corresponding Bellman equation.

2 The Model

2.1 Assumptions

Let \mathcal{H} be a Hilbert space, with scalar product denoted by (\cdot, \cdot) . We consider a non-linear operator A , $x \in \mathcal{H} \mapsto A(x) \in \mathcal{H}$, such that

$$A(0) = 0. \tag{1}$$

We assume that $x \mapsto A(x)$ is C^1 and that $DA(x)(= D_x A(x)) \in \mathcal{L}(\mathcal{H}; \mathcal{H})$, that means

$$D_x A(x)(\cdot) : y \in \mathcal{H} \mapsto D_x A(x)(y) = \lim_{\epsilon \rightarrow 0} \frac{A(x + \epsilon y) - A(x)}{\epsilon},$$

and it satisfies:

$$\text{The operator norm } \|DA(x)\| \leq \gamma. \quad (2)$$

By definition, we have the result:

$$(DA(x)y, z) = (D_x(A(x), z), y); \quad (3)$$

indeed, we can see this by noting that

$$\begin{aligned} (DA(x)y, z) &= \lim_{\epsilon \rightarrow 0} \left(\frac{1}{\epsilon} (A(x + \epsilon y) - A(x)), z \right) \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} [(A(x + \epsilon y), z) - (A(x), z)] \\ &= (D_x(A(x), z), y), \end{aligned}$$

where the last step follows by differentiating the functional $(A(\cdot), z) : \mathcal{H} \rightarrow \mathbb{R}$. We also assume that $DA(x)$ is differentiable with a second derivative $D^2A(x) \in \mathcal{L}(\mathcal{H}; \mathcal{L}(\mathcal{H}; \mathcal{H}))$, similar to (3), such that

$$\begin{cases} (D_x(D_x(A(x), z), y), w) = (D_{xx}^2 A(x)(y)w, z), \\ D_x(D_x(A(x), z), y) = (D_{xx}^2 A(x)(y), z). \end{cases} \quad (4)$$

We assume the Lipschitz property:

$$\|DA(x_1) - DA(x_2)\| \leq \frac{b|x_1 - x_2|}{1 + \max(|x_1|, |x_2|)}, \quad (5)$$

which implies

$$\|D^2A(x)\| \leq \frac{b}{1 + |x|}. \quad (6)$$

In the sequel, we shall make restrictions on the size of b .

We next consider $x \mapsto F(x)$ and $x \mapsto F_T(x)$, functionals on \mathcal{H} , which are C^2 , with the properties:

$$\begin{cases} F(0) = 0, D_x F(0) = 0, \\ \nu |\xi|^2 \leq (D_{xx}^2 F(x)\xi, \xi) \leq M |\xi|^2; \end{cases} \quad (7)$$

$$\begin{cases} F_T(0) = 0, D_x F_T(0) = 0, \\ \nu_T |\xi|^2 \leq (D_{xx}^2 F_T(x)\xi, \xi) \leq M_T |\xi|^2, \end{cases} \quad (8)$$

and $\nu, \nu_T > 0$. \mathcal{H} is the state space. In addition, there is a control space \mathcal{V} , also a Hilbert space and a linear bounded operator $B \in \mathcal{L}(\mathcal{V}; \mathcal{H})$, an invertible-self adjoint operator on \mathcal{V} , denoted by N . We assume that

$$(BN^{-1}B^*\xi, \xi) \geq m|\xi|^2, \quad m > 0. \quad (9)$$

Remark 1 The assumption (1) and the first line assumptions (7), (8) are of course not necessary. It is just to simplify the calculations.

2.2 The Problem

We consider the following control problem. The state evolution is governed by the differential equation in \mathcal{H} :

$$\begin{cases} \frac{dx}{ds} = A(x) + Bv(s), \\ x(t) = x, \end{cases} \quad (10)$$

in which $v(\cdot)$ is in $L^2(t, T; \mathcal{V})$. It is easy to check that the state $x(\cdot)$ is uniquely defined and belongs to $H^1(t, T; \mathcal{H})$. We define the payoff functional:

$$J_{xt}(v(\cdot)) := \int_t^T F(x(s))ds + F_T(x(T)) + \frac{1}{2} \int_t^T (v(s), Nv(s))ds. \quad (11)$$

This functional is continuous and coercive. If \mathcal{H} were \mathbb{R}^n , it would be classical that it has a minimum and thus we could write the necessary conditions of optimality. But the proof does not carry over to general Hilbert spaces. Moreover, since A is not linear, we do not have the convexity property, which would guarantee the existence and uniqueness of a minimum, and thus a solution of the necessary conditions of optimality. We shall then write the necessary conditions of optimality and prove directly the existence and uniqueness of a solution.

3 Necessary Conditions of Optimality

3.1 The System

It is standard to check the following system of forward-backward equations in \mathcal{H} :

$$\begin{cases} \frac{dy}{ds} = A(y) - BN^{-1}B^*z(s), \quad t < s < T, \\ -\frac{dz}{ds} = (DA(y(s)))^*z(s) + DF(y(s)), \\ y(t) = x, \quad z(T) = DF_T(y(T)). \end{cases} \quad (12)$$

The optimal state is $y(\cdot)$, and $z(\cdot)$ is the adjoint state. The optimal control is then:

$$u(s) = -N^{-1}B^*z(s). \quad (13)$$

The system (12) expresses the Pontryagin Maximum Principle. The objective is to study the system of Equations (12).

3.2 Decoupling

We set

$$z(t) = \Gamma(x, t). \quad (14)$$

It is standard to check that $z(s) = \Gamma(y(s), s)$. So $y(s)$ is the solution of the differential equation in \mathcal{H} :

$$\begin{cases} \frac{dy}{ds} = A(y) - BN^{-1}B^*\Gamma(y(s), s), \\ y(t) = x, \end{cases} \quad (15)$$

and $\Gamma(x, s)$ is the solution of the nonlinear partial differential equation:

$$\begin{cases} -\frac{\partial \Gamma}{\partial s} = D_x \Gamma(x)A(x) + (D_x A(x))^* \Gamma(x) - D_x \Gamma(x)BN^{-1}B^* \Gamma(x, s) + D_x F(x), \\ \Gamma(x, T) = D_x F_T(x). \end{cases} \quad (16)$$

If $A(x) = Ax$, $F(x) = \frac{1}{2}(x, Mx)$ and $F_T(x) = \frac{1}{2}(x, M_T x)$, then $\Gamma(x, s) = P(s)x$, and $P(s)$ is solution of the Riccati equation:

$$\begin{cases} -\frac{dP}{ds} = P(s)A + A^*P(s) - P(s)BN^{-1}B^*P(s) + M, \\ P(T) = M_T. \end{cases} \quad (17)$$

4 A Priori Estimates

4.1 First Estimate

We state the first result:

Proposition 1 *We assume (1), (2), (5), (7), (8), (9) and*

$$\frac{b^2}{16} < (m-k)(v-k), \quad 0 < k < \min(m, v), \quad (18)$$

then we have the a priori estimate:

$$|\Gamma(x, t)| \leq |x| \left(\frac{M_T^2}{\nu_T} + \frac{\gamma^2 + M^2}{k} (T-t) \right). \quad (19)$$

Proof From the system (12), we obtain:

$$\frac{d}{ds}(y(s), z(s)) = (A(y(s)) - BN^{-1}B^*z(s), z(s)) - ((DA(y(s)))^*z(s) + DF(y(s)), y(s)).$$

Integration yields:

$$\begin{aligned} & (D_x F_T(y(T)), y(T)) + \int_t^T (BN^{-1}B^*z(s), z(s)) ds + \int_t^T (D_x F(y(s)), y(s)) ds \\ &= (x, z(t)) + \int_t^T (A(y(s)) - DA(y(s))y(s), z(s)) ds. \end{aligned} \quad (20)$$

We note that

$$|A(x) - DA(x)x| \leq \frac{b}{2}|x|; \quad (21)$$

indeed, $A(x) - DA(x)x = \int_0^1 (DA(\theta x) - DA(x))x d\theta$, and from the assumption (5), we get:

$$|A(x) - DA(x)x| \leq \int_0^1 \frac{b|x|^2(1-\theta)}{1+|x|} d\theta,$$

which implies (21). Therefore, from (20), we obtain, using assumptions:

$$(x, z(t)) \geq \nu_T |y(T)|^2 + m \int_t^T |z(s)|^2 ds + \nu \int_t^T |y(s)|^2 ds - \frac{b}{2} \int_t^T |y(s)||z(s)| ds.$$

Using (18), we can state:

$$(x, z(t)) \geq \nu_T |y(T)|^2 + k \int_t^T (|y(s)|^2 + z(s)|^2) ds. \quad (22)$$

On the other hand, from the second equation (12), we write $z(t) = z(T) + \int_t^T ((DA(y(s)))^*z(s) + DF(y(s))) ds$, hence

$$\begin{aligned} (x, z(t)) &= (x, DF_T(y(T)) + \int_t^T (DA(y(s))x, z(s)) ds + \int_t^T (x, DF(y(s))) ds, \\ (x, z(t)) &\leq |x||z(t)| \leq |x|(M_T|y(T)| + \int_t^T \gamma|z(s)| ds + \int_t^T |y(t)| dt) \\ &\leq \frac{1}{2} \left(\nu_T |y(T)|^2 + k \int_t^T (|y(s)|^2 + z(s)|^2) ds \right) \\ &\quad + \frac{|x|^2}{2} \left(\frac{M_T^2}{\nu_T} + \frac{\gamma^2 + M^2}{k} (T-t) \right). \end{aligned}$$

From this relation and (22), we get:

$$v_T |y(T)|^2 + k \int_t^T (|y(s)|^2 + |z(s)|^2) ds \leq |x|^2 \left(\frac{M_T^2}{v_T} + \frac{\gamma^2 + M^2}{k} (T-t) \right).$$

Therefore,

$$|x| |z(t)| \leq |x|^2 \left(\frac{M_T^2}{v_T} + \frac{\gamma^2 + M^2}{k} (T-t) \right),$$

and the result follows. We write

$$\alpha_t = \frac{M_T^2}{v_T} + \frac{\gamma^2 + M^2}{k} (T-t). \quad (23)$$

Note that in the system (12), we can write

$$|z(s)| \leq \alpha_s |y(s)|. \quad (24)$$

4.2 Second Estimate

The second estimate concerns the gradient $D_x \Gamma(x, t)$. We have the following result:

Proposition 2 *We make the assumptions of Proposition 1 and*

$$v - b\alpha_0 > 0, \quad (25)$$

then we have the a priori estimate:

$$\|D_x \Gamma(x, t)\| \leq \frac{M_T^2}{v_T} + \frac{\gamma^2}{m} (T-t) + \int_t^T \frac{(M + b\alpha_s)^2}{v - b\alpha_s} ds. \quad (26)$$

Proof We differentiate the system (12) with respect to x . We denote

$$\mathcal{Y}(s) = D_x y(s), \quad \mathcal{Z}(s) = D_x z(s). \quad (27)$$

Differentiating (12), we can write, by recalling notation (4):

$$\begin{aligned} \frac{d}{ds} \mathcal{Y}(s) \xi &= D_x A(y(s)) \mathcal{Y}(s) \xi - BN^{-1} B^* \mathcal{Z}(s) \xi, \\ -\frac{d}{ds} \mathcal{Z}(s) \xi &= (D_{xx}^2 A(y(s)) \mathcal{Y}(s)(\xi), z(s)) + (DA(y(s)))^* \mathcal{Z}(s) \xi + D_{xx}^2 F(y(s)) \mathcal{Y}(s) \xi, \end{aligned} \quad (28)$$

$$\mathcal{Y}(t) \xi = \xi, \quad \mathcal{Z}(T) \xi = D_{xx}^2 F_T(y(T)) \mathcal{Y}(T) \xi. \quad (29)$$

We compute $\frac{d}{ds} (\mathcal{Y}(s) \xi, \mathcal{Z}(s) \xi)$ and then integrate. We obtain that

$$\begin{aligned}
(\mathcal{Z}(t)\xi, \xi) &= (D_{xx}^2 F_T(y(T))\mathcal{Y}(T)\xi, \mathcal{Y}(T)\xi) + \int_t^T (BN^{-1}B^*\mathcal{Z}(s)\xi, \mathcal{Z}(s)\xi)ds \\
&\quad + \int_t^T (D_{xx}^2 F(y(s))\mathcal{Y}(s)\xi, \mathcal{Y}(s)\xi)ds \\
&\quad + \int_t^T (D_{xx}^2 A(y(s))\mathcal{Y}(s)(\xi)\mathcal{Y}(s)\xi, z(s)) \\
&\geq \nu_T |\mathcal{Y}(T)\xi|^2 + m \int_t^T |\mathcal{Z}(s)\xi|^2 ds + \int_t^T (\nu - b\alpha_s) |\mathcal{Y}(s)\xi|^2 ds.
\end{aligned} \tag{30}$$

Also, from the second line of (28),

$$|\mathcal{Z}(t)\xi| \leq M_T |\mathcal{Y}(T)\xi| + \int_t^T (M + b\alpha_s) |\mathcal{Y}(s)\xi| ds + \gamma \int_t^T |\mathcal{Z}(s)\xi| ds. \tag{31}$$

Combining (30) and (31) as in Proposition 1, we conclude that

$$|\mathcal{Z}(t)\xi| \leq |\xi| \left(\frac{M_T^2}{\nu_T} + \frac{\gamma^2}{m} (T-t) + \int_t^T \frac{(M + b\alpha_s)^2}{\nu - b\alpha_s} ds \right).$$

Since $\mathcal{Z}(t)\xi = D_x \Gamma(x, t)$, the result (26) follows immediately. The proof is complete. \square

We shall call

$$\beta_t = \frac{M_T^2}{\nu_T} + \frac{\gamma^2}{m} (T-t) + \int_t^T \frac{(M + b\alpha_s)^2}{\nu - b\alpha_s} ds. \tag{32}$$

Since

$$\Gamma(x, t) = \int_0^1 D_x \Gamma(\theta x, t) x d\theta,$$

we also have:

$$|\Gamma(x, t)| \leq \beta_t |x|, \tag{33}$$

so in fact,

$$\begin{cases} |\Gamma(x, t)| \leq \min(\alpha_t, \beta_t) |x|, \\ \|D_x \Gamma(x, t)\| \leq \beta_t. \end{cases} \tag{34}$$

5 Local Time Solution

5.1 Fixed Point Approach

We want to solve (12) by a fixed point approach. Suppose we have a function $\lambda(x, t)$ with values in \mathcal{H} such that:

$$\begin{cases} |\lambda(x, t)| \leq \mu_t |x|, \\ \|D_x \lambda(x, t)\| \leq \rho_t, \end{cases} \quad (35)$$

where μ_t and ρ_t are bounded functions on $[T-h, T]$, for some convenient h . These functions will be chosen conveniently in the sequel, with $\mu_t < \rho_t$. We then solve

$$\begin{cases} \frac{d}{ds} y(s) = A(y(s)) - BN^{-1}B^* \lambda(y(s), s), \\ y(t) = x. \end{cases} \quad (36)$$

This differential equation defines uniquely $y(s)$, thanks to the assumptions (35). We then define

$$\Lambda(x, t) := D_x F_T(y(T)) + \int_t^T (DA(y(s)))^* \lambda(y(s), s) ds + \int_t^T D_x F(y(s)) ds. \quad (37)$$

We want to show that μ_t and ρ_t can be chosen such that

$$|\Lambda(x, t)| \leq \mu_t |x|, \quad \|D_x \Lambda(x, t)\| \leq \rho_t, \quad (38)$$

and that the map $\lambda \mapsto \Lambda$ has a fixed point. This will be only possible when t remains close to T , namely $T-h < t < T$, with h small.

5.2 Choice of Functions μ_t and ρ_t

From (36), we obtain:

$$\frac{d}{ds} |y(s)| \leq \left| \frac{d}{ds} y(s) \right| \leq (\gamma + \|BN^{-1}B^*\| \mu_s) |y(s)|,$$

which implies

$$|y(s)| \leq |x| \exp\left(\int_t^s (\gamma + \|BN^{-1}B^*\| \mu_\tau) d\tau\right), \quad (39)$$

and thus from (37) it follows that

$$|\Lambda(x, t)| \leq M_T |y(T)| + \int_t^T (M + \gamma \mu_s) |y(s)| ds.$$

Using (39), we obtain:

$$|\Lambda(x, t)| \leq |x| \left(M_T \exp \left(\int_t^T (\gamma + \|BN^{-1}B^*\|\mu_\tau) d\tau \right) + \int_t^T (M + \gamma\mu_s) \exp \left(\int_t^s (\gamma + \|BN^{-1}B^*\|\mu_\tau) d\tau \right) ds \right).$$

To obtain the first inequality (38), we must choose the function μ_t such that

$$\mu_t = M_T \exp \left(\int_t^T (\gamma + \|BN^{-1}B^*\|\mu_\tau) d\tau \right) + \int_t^T (M + \gamma\mu_s) \exp \left(\int_t^s (\gamma + \|BN^{-1}B^*\|\mu_\tau) d\tau \right) ds. \quad (40)$$

So μ_t must be solution of the differential equation of Riccati type:

$$\begin{cases} \frac{d}{dt} \mu_t = -\|BN^{-1}B^*\|\mu_t^2 - 2\gamma\mu_t - M, \\ \mu_T = M_T. \end{cases} \quad (41)$$

To proceed, we need to assume that

$$\gamma^2 < M\|BN^{-1}B^*\|, \quad (42)$$

and we define μ_t by the formula:

$$\begin{aligned} & \arctan \frac{\mu_t\|BN^{-1}B^*\| + \gamma}{\sqrt{M\|BN^{-1}B^*\| - \gamma^2}} \\ &= \arctan \frac{M_T\|BN^{-1}B^*\| + \gamma}{\sqrt{M\|BN^{-1}B^*\| - \gamma^2}} + \left(\sqrt{M\|BN^{-1}B^*\| - \gamma^2} \right) (T - t). \end{aligned} \quad (43)$$

For $h > 0$, define θ_h with

$$\begin{aligned} & \arctan \frac{\theta_h\|BN^{-1}B^*\| + \gamma}{\sqrt{M\|BN^{-1}B^*\| - \gamma^2}} \\ &= \arctan \frac{M_T\|BN^{-1}B^*\| + \gamma}{\sqrt{M\|BN^{-1}B^*\| - \gamma^2}} + \left(\sqrt{M\|BN^{-1}B^*\| - \gamma^2} \right) h. \end{aligned} \quad (44)$$

The number h must be small enough to guarantee that

$$\arctan \frac{M_T\|BN^{-1}B^*\| + \gamma}{\sqrt{M\|BN^{-1}B^*\| - \gamma^2}} + \left(\sqrt{M\|BN^{-1}B^*\| - \gamma^2} \right) h < \frac{\pi}{2}. \quad (45)$$

Formula (43) defines uniquely μ_t for $T - h < t < T$. It is decreasing in t , with $M_T < \mu_t < \theta_h$.

Therefore, for $T - h < t < T$, we have defined by (37) a function $\Lambda(x, t)$ which satisfies the first condition (38), with μ_t defined by equation (43). We turn now to

the definition of ρ_t . Define $\mathcal{Y}(s) = D_x y(s)$, see (36). We have:

$$\begin{cases} \frac{d}{ds} \mathcal{Y}(s) = (DA(y(s)) - BN^{-1}B^*D_x \lambda(y(s), s)) \mathcal{Y}(s), \\ \mathcal{Y}(t) = I. \end{cases} \quad (46)$$

We obtain, by techniques already used:

$$\|\mathcal{Y}(s)\| \leq \exp\left(\int_t^s (\gamma + \|BN^{-1}B^*\|\rho_\tau) d\tau\right). \quad (47)$$

We then differentiate $\Lambda(x, t)$ in x , see (37). We get:

$$\begin{aligned} D_x \Lambda(x, t) &= D_{xx}^2 F_T(y(T)) \mathcal{Y}(T) + \int_t^T (D_{xx}^2 A(y(s)) \mathcal{Y}(s), \lambda(y(s), s)) ds, \\ &+ \int_t^T (D_x A(y(s)))^* D_x \lambda(y(s), s) \mathcal{Y}(s) ds + \int_t^T D_{xx}^2 F(y(s)) \mathcal{Y}(s) ds, \end{aligned}$$

and we obtain:

$$\|D_x \Lambda(x, t)\| \leq M_T \|\mathcal{Y}(T)\| + \int_t^T (M + b\mu_s + \gamma\rho_s) \|\mathcal{Y}(s)\| ds.$$

Since $T - h < t < T$, we can majorize, using also (47), to obtain:

$$\begin{aligned} \|D_x \Lambda(x, t)\| &\leq M_T \exp\left(\int_t^T (\gamma + \|BN^{-1}B^*\|\rho_s) ds\right) \\ &+ \int_t^T (M + b\theta_h + \gamma\rho_s) \exp\left(\int_t^s (\gamma + \|BN^{-1}B^*\|\rho_\tau) d\tau\right) ds. \end{aligned} \quad (48)$$

We are thus led to looking for ρ_t solution of

$$\begin{aligned} \rho_t &= M_T \exp\left(\int_t^T (\gamma + \|BN^{-1}B^*\|\rho_s) ds\right) \\ &+ \int_t^T (M + b\theta_h + \gamma\rho_s) \exp\left(\int_t^s (\gamma + \|BN^{-1}B^*\|\rho_\tau) d\tau\right) ds. \end{aligned} \quad (49)$$

This equation is similar to the one defining μ_t , see (40), with the change of M into $M + b\theta_h$. Hence, by analogy with (43), we can assert that:

$$\begin{aligned} \arctan \frac{\rho_t \|BN^{-1}B^*\| + \gamma}{\sqrt{(M + b\theta_h) \|BN^{-1}B^*\| - \gamma^2}} &= \arctan \frac{M_T \|BN^{-1}B^*\| + \gamma}{\sqrt{(M + b\theta_h) \|BN^{-1}B^*\| - \gamma^2}} \\ &+ \left(\sqrt{(M + b\theta_h) \|BN^{-1}B^*\| - \gamma^2}\right) (T - t). \end{aligned} \quad (50)$$

In order to get a bounded solution for ρ_t , we need that the right hand side of (50) be smaller than $\frac{\pi}{2}$. We need to restrict h more than with (45), namely:

$$\arctan \frac{M_T \|BN^{-1}B^*\| + \gamma}{\sqrt{M \|BN^{-1}B^*\| - \gamma^2}} + \left(\sqrt{(M + b\theta_h) \|BN^{-1}B^*\| - \gamma^2} \right) h < \frac{\pi}{2}. \quad (51)$$

Then the function ρ_t is well defined on $(T - h, T]$, by formula (50) and the function $\Lambda(x, t)$ defined by (37), for $t \in (T - h, T]$ satisfies (38) if $\lambda(x, t)$ satisfies (35). We also claim that

$$\rho_t > \mu_t. \quad (52)$$

Indeed, ρ_t satisfies the Riccati equation:

$$\begin{cases} \frac{d}{dt} \rho_t = -\|BN^{-1}B^*\| \rho_t^2 - 2\gamma \rho_t - (M + b\theta_h), \\ \rho_T = M_T, \end{cases} \quad (53)$$

and comparing (41) and (53), it is standard to show the property (52).

5.3 Contraction Mapping

We define the space of functions $(x, t) \in \mathcal{H} \times (T - h, T) \mapsto \lambda(x, t) \in \mathcal{H} \times (T - h, T)$, equipped with the norm:

$$\|\lambda\|_h = \sup_{x \in \mathcal{H}, t \in (T-h, T)} \frac{|\lambda(x, t)|}{|x|}. \quad (54)$$

This space is a Banach space, denoted by \mathcal{B}_h . We next consider the convex closed subset of \mathcal{B}_h of functions such that:

$$|\lambda(x, t)| \leq \mu_t |x|, \|D_x \lambda(x, t)\| \leq \rho_t, \forall t \in (T - h, T], \quad (55)$$

where μ_t and ρ_t are defined by (43) and (50), respectively. The subset (55) is denoted by C_h . The map $\lambda \mapsto \Lambda$, defined by (36) and (37), is defined from C_h to C_h . We want to show that it leads to a contraction.

Let $\lambda^1(x, t)$, $\lambda^2(x, t)$ in C_h and the corresponding functions $\Lambda^1(x, t)$, $\Lambda^2(x, t)$, which also belong to C_h . Let $y^1(s)$, $y^2(s)$ be the solutions of (36) corresponding to λ^1 , λ^2 . We call $\tilde{y}(s) = y^1(s) - y^2(s)$. We have:

$$\begin{cases} \frac{d}{ds} \tilde{y}(s) = A(y^1(s)) - A(y^2(s)) - BN^{-1}B^*(\lambda^1(y^1(s)) - \lambda^2(y^2(s))), \\ \tilde{y}(t) = 0, \end{cases}$$

hence

$$\frac{d}{ds}|\tilde{y}(s)| \leq \gamma|\tilde{y}(s)| + \|BN^{-1}B^*\| |\lambda^1(y^1(s)) - \lambda^2(y^2(s))|.$$

Next,

$$\begin{aligned} & |\lambda^1(y^1(s)) - \lambda^2(y^2(s))| \\ & \leq |\lambda^1(y^1(s)) - \lambda^1(y^2(s))| + |\lambda^1(y^2(s)) - \lambda^2(y^2(s))| \\ & \leq \rho_s|\tilde{y}(s)| + \|\lambda^1 - \lambda^2\|_h |x| \exp\left(\int_t^s (\gamma + \|BN^{-1}B^*\|\mu_\tau) d\tau\right). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{d}{ds}|\tilde{y}(s)| & \leq (\gamma + \|BN^{-1}B^*\|\rho_s)|\tilde{y}(s)| \\ & \quad + \|BN^{-1}B^*\| |x| \|\lambda^1 - \lambda^2\|_h \exp\left(\int_t^s (\gamma + \|BN^{-1}B^*\|\mu_\tau) d\tau\right). \end{aligned}$$

We obtain that

$$\begin{aligned} & |\tilde{y}(s)| \exp\left(-\int_t^s (\gamma + \|BN^{-1}B^*\|\rho_\tau) d\tau\right) \\ & \leq \|BN^{-1}B^*\| |x| \|\lambda^1 - \lambda^2\|_h \int_t^s \exp\left(-\|BN^{-1}B^*\| \int_t^\tau (\rho_\theta - \mu_\theta) d\theta\right) d\tau, \end{aligned}$$

which implies:

$$|\tilde{y}(s)| \leq h\|BN^{-1}B^*\| |x| \|\lambda^1 - \lambda^2\|_h \exp\left(\int_t^s (\gamma + \|BN^{-1}B^*\|\rho_\tau) d\tau\right). \quad (56)$$

We next have from the definition of the map $\Lambda(x, t)$ that:

$$\begin{aligned} \Lambda^1(x, t) - \Lambda^2(x, t) & = DF_T(y^1(T)) - DF_T(y^2(T)) \\ & \quad + \int_t^T (DA^*(y^1(s))\lambda^1(y^1(s)) - DA^*(y^2(s))\lambda^2(y^2(s))) ds \\ & \quad + \int_t^T (DF(y^1(s)) - DF(y^2(s))) ds. \end{aligned} \quad (57)$$

We have:

$$\begin{aligned} & |DA^*(y^1(s))\lambda^1(y^1(s)) - DA^*(y^2(s))\lambda^2(y^2(s))| \\ & \leq (b\theta_h + \gamma\rho_s)|\tilde{y}(s)| + \gamma|x| \|\lambda^1 - \lambda^2\|_h \exp\left(\int_t^s (\gamma + \|BN^{-1}B^*\|\mu_\tau) d\tau\right). \end{aligned}$$

So, from (57), we obtain:

$$\begin{aligned}
|\Lambda^1(x,t) - \Lambda^2(x,t)| &\leq M_T |\widetilde{y}(T)| + \int_t^T (M + b\theta_h + \gamma\rho_s) |\widetilde{y}(s)| ds \\
&\quad + \gamma|x| \|\lambda^1 - \lambda^2\|_h \int_t^T \exp\left(\int_t^s (\gamma + \|BN^{-1}B^*\|_{\mu_\tau}) d\tau\right) ds,
\end{aligned} \tag{58}$$

and from (56):

$$\begin{aligned}
&|\Lambda^1(x,t) - \Lambda^2(x,t)| \\
&\leq |x| \|\lambda^1 - \lambda^2\|_h h \times \left[\|BN^{-1}B^*\| \left(M_T \exp\left(\int_t^T (\gamma + \|BN^{-1}B^*\|_{\rho_\tau}) d\tau\right) \right. \right. \\
&\quad \left. \left. + \int_t^T (M + b\theta_h + \gamma\rho_s) \left(\int_t^s (\gamma + \|BN^{-1}B^*\|_{\rho_\tau}) d\tau \right) ds \right) \right] \\
&\quad + \gamma|x| \|\lambda^1 - \lambda^2\|_h \int_t^T \exp\left(\int_t^s (\gamma + \|BN^{-1}B^*\|_{\mu_\tau}) d\tau\right) ds,
\end{aligned}$$

then from the definition of ρ_t (see (49)), we obtain:

$$\begin{aligned}
&|\Lambda^1(x,t) - \Lambda^2(x,t)| \\
&\leq |x| \|\lambda^1 - \lambda^2\|_h h \left(\rho_t \|BN^{-1}B^*\| + \gamma \exp\left(\int_{T-h}^T (\gamma + \|BN^{-1}B^*\|_{\mu_\tau}) d\tau\right) \right). \tag{59}
\end{aligned}$$

Similarly to the definition of θ_h (see (44)), we define the quantity σ_h by the formula:

$$\begin{aligned}
\arctan \frac{\sigma_h \|BN^{-1}B^*\| + \gamma}{\sqrt{(M + b\theta_h) \|BN^{-1}B^*\| - \gamma^2}} &= \arctan \frac{M_T \|BN^{-1}B^*\| + \gamma}{\sqrt{(M + b\theta_h) \|BN^{-1}B^*\| - \gamma^2}} \\
&\quad + \left(\sqrt{(M + b\theta_h) \|BN^{-1}B^*\| - \gamma^2} \right) h. \tag{60}
\end{aligned}$$

From (50), we see that $M_T < \rho_t < \sigma_h$. Therefore from (59),

$$\|\Lambda^1 - \Lambda^2\|_h \leq \|\lambda^1 - \lambda^2\|_h h \left(\sigma_h \|BN^{-1}B^*\| + \gamma \exp\left(h(\gamma + \|BN^{-1}B^*\|_{\theta_h})\right) \right). \tag{61}$$

Using the fact that $\theta_h \rightarrow M_T$ as $h \rightarrow 0$, equation (60) shows that $\sigma_h \rightarrow M_T$ as $h \rightarrow 0$. We deduce that:

$$h \left(\sigma_h \|BN^{-1}B^*\| + \gamma \exp\left(h(\gamma + \|BN^{-1}B^*\|_{\theta_h})\right) \right) \rightarrow 0, \text{ as } h \rightarrow 0. \tag{62}$$

We can restrict h such that

$$h \left(\sigma_h \|BN^{-1}B^*\| + \gamma \exp\left(h(\gamma + \|BN^{-1}B^*\|_{\theta_h})\right) \right) < 1, \tag{63}$$

and thus for h sufficiently small, the map $\lambda \mapsto \Lambda$ is paradoxical and leads to a contradiction. We can summarize the results in the following theorem:

Theorem 1 *We assume (42). We choose h small enough to satisfy conditions (45), (51), (63). For $T - h < t < T$, there exists one and only one solution of the system of forward-backward equations (12). We have also one and only one solution of equation (16) on the same interval.*

6 Global Solution

6.1 Statement of Results

We have proven in Theorem 1 the existence and uniqueness of a local solution of the system (12). We want to state that this solution is global, under the assumptions of Proposition 2.

Theorem 2 *We make the assumptions of Proposition 2 and (42). The local solution defined in Theorem 1 can be extended. Thus there exists one and only one solution of the system (12) on any finite interval $[0, T]$, and there exists one and only one solution of equation (16) on any finite interval $[0, T]$.*

Proof Defining by $\Gamma(x, t)$ the fixed point obtained in Theorem 1, it is the unique solution of the parabolic equation:

$$\begin{cases} -\frac{\partial \Gamma}{\partial t} = D_x \Gamma(x) A(x) + (D_x A(x))^* \Gamma(x) - D_x \Gamma(x) B N^{-1} B^* \Gamma(x, s) \\ \quad + D_x F(x), \quad T - h < t < T, \\ \Gamma(x, T) = D_x F_T(x), \end{cases} \quad (64)$$

with h restricted as stated in Theorem 1. We also have the estimates:

$$\begin{cases} |\Gamma(x, t)| \leq \min(\alpha_t, \beta_t) |x|, \\ \|D_x \Gamma(x, t)\| \leq \beta_t, \end{cases} \quad (65)$$

with

$$\begin{cases} \alpha_t = \frac{M_T^2}{\nu_T} + \frac{\gamma^2 + M^2}{k} (T - t), \\ \beta_t = \frac{M_T^2}{\nu_T} + \frac{\gamma^2}{m} (T - t) + \int_t^T \frac{(M + b\alpha_s)^2}{\nu - b\alpha_s} ds. \end{cases} \quad (66)$$

These estimates follow from the *a priori* estimates stated in Proposition 1 and 2. They do not depend on h . Now we want to extend (64) for $t < T - h$. To avoid confusion, we define

$$U_{T-h}(x) := \Gamma(x, T - h). \quad (67)$$

We set $M_{T-h} = \beta_0$. We can then state:

$$\begin{cases} |U_{T-h}(x)| \leq M_{T-h}|x|, \\ \|D_x U_{T-h}(x)\| \leq M_{T-h}, \end{cases} \quad (68)$$

and we consider the parabolic equation:

$$\begin{cases} -\frac{\partial \Gamma}{\partial t} = D_x \Gamma(x)A(x) + (D_x A(x))^* \Gamma(x) - D_x \Gamma(x)BN^{-1}B^* \Gamma(x, s) \\ \quad + D_x F(x), \quad t < T-h, \\ \Gamma(x, T-h) = U_{T-h}(x). \end{cases} \quad (69)$$

We associate to this equation the system:

$$\begin{cases} \frac{dy}{ds} = A(y) - BN^{-1}B^*z(s), \quad t < s < T-h, \\ -\frac{dz}{ds} = (DA(y(s)))^*z(s) + DF(y(s)), \\ y(t) = x, \quad z(T-h) = U_{T-h}(y(T-h)). \end{cases} \quad (70)$$

Proceeding like in Theorem 1, we can solve this system on an interval $[T-h-l, T-h]$, for a sufficiently small $l \neq h$. The difference is due to the fact that $M_{T-h} \neq M_T$. So in (64), we can replace $T-h$ by $T-h-l$. This time the estimates on $\Gamma(x, T-h-l)$ and $D_x \Gamma(x, T-h-l)$ are identical to those of $\Gamma(x, T-h)$ and $D_x \Gamma(x, T-h)$, thanks to the *a priori* estimates. So the intervals we can extend further will have the same length. Clearly, this implies that we can extend (64) up to $t = 0$. So, we obtain the global existence and uniqueness of equation (16) on $[0, T]$. The proof is complete. \square

6.2 Optimal Control

In Theorem 2, we have obtained the existence and uniqueness of the solution of the pair $(y(s), z(s))$ of the system (12), for any $t \in [0, T]$. We want now to check that the control $u(s)$ defined by (13) is solution of the control problem (10), (11), and that the optimal control is unique.

Theorem 3 *Under the assumptions of Theorem 2, the control $u(\cdot)$ defined by (13) is the unique optimal control for the problem (10), (11).*

Proof Let $v(\cdot)$ be another control. We shall prove that

$$J(u(\cdot) + v(\cdot)) \geq J(u(\cdot)), \quad (71)$$

which will prove the optimality of $u(\cdot)$. We define by $y_v(\cdot)$ the state corresponding to the control $u(\cdot) + v(\cdot)$. It is the solution of

$$\begin{cases} \frac{d}{ds} y_v(s) = A(y_v(s)) + B(u(s) + v(s)), \\ y_v(t) = x, \end{cases} \quad (72)$$

and we have:

$$J(u(\cdot) + v(\cdot)) = \int_t^T F(y_v(s)) ds + F_T(y_v(T)) + \frac{1}{2} \int_t^T (u(s) + v(s), N(u(s) + v(s))) ds,$$

and

$$\begin{aligned} & J(u(\cdot) + v(\cdot)) - J(u(\cdot)) \\ &= \int_t^T (F(y_v(s)) - F(y(s))) ds + F_T(y_v(T)) - F_T(y(T)) \\ &\quad + \frac{1}{2} \int_t^T (v(s), Nv(s)) ds + \int_t^T (Nu(s), v(s)) ds. \end{aligned}$$

We denote $\tilde{y}_v(s) := y_v(s) - y(s)$. It satisfies:

$$\begin{cases} \frac{d}{ds} \tilde{y}_v(s) = A(y_v(s)) - A(y(s)) + Bv(s), \\ \tilde{y}_v(t) = 0. \end{cases} \quad (73)$$

Then,

$$\begin{aligned} & J(u(\cdot) + v(\cdot)) - J(u(\cdot)) \\ &= \int_t^T (D_x F(y(s)), \tilde{y}_v(s)) ds \\ &\quad + \int_t^T \int_0^1 \int_0^1 \theta (D_{xx}^2 F(y(s) + \lambda\theta\tilde{y}_v(s)) \tilde{y}_v(s), \tilde{y}_v(s)) ds d\lambda d\theta \\ &\quad + (D_x F_T(y(T)), \tilde{y}_v(T)) + \int_0^1 \int_0^1 \theta (D_{xx}^2 F_T(y(T) + \lambda\theta\tilde{y}_v(T)) \tilde{y}_v(T), \tilde{y}_v(T)) d\lambda d\theta \\ &\quad + \frac{1}{2} \int_t^T (v(s), Nv(s)) ds - \int_t^T (z(s), Bv(s)) ds. \end{aligned}$$

From the assumptions (7), we can write:

$$\begin{aligned} J(u(\cdot) + v(\cdot)) - J(u(\cdot)) &\geq \int_t^T \left(-\frac{d}{ds} z(s) - DA^*(y(s))z(s), \tilde{y}_v(s) \right) ds \\ &\quad + \frac{\nu}{2} \int_t^T |\tilde{y}_v(s)|^2 ds + (z(T), \tilde{y}_v(T)) \\ &\quad + \frac{\nu_T}{2} |\tilde{y}_v(T)|^2 + \frac{1}{2} \int_t^T (v(s), Nv(s)) ds \\ &\quad - \int_t^T \left(z(s), \frac{d}{ds} \tilde{y}_v(s) - (A(y_v(s)) - A(y(s))) \right) ds, \end{aligned}$$

which reduces to:

$$\begin{aligned}
J(u(\cdot) + v(\cdot)) - J(u(\cdot)) &\geq \frac{\nu}{2} \int_t^T |\tilde{y}_v(s)|^2 ds + \frac{\nu_T}{2} |\tilde{y}_v(T)|^2 + \frac{1}{2} \int_t^T (v(s), Nv(s)) ds \\
&\quad + \int_t^T (z(s), A(y_v(s)) - A(y(s)) - DA(y(s))\tilde{y}_v(s)) ds.
\end{aligned} \tag{74}$$

Note that

$$|(z(s), A(y_v(s)) - A(y(s)) - DA(y(s))\tilde{y}_v(s))| \leq \frac{b|z(s)||\tilde{y}_v(s)|^2}{2(1+|y(s)|)} \leq \frac{b\alpha_s}{2} |\tilde{y}_v(s)|^2.$$

Finally, we can state that

$$J(u(\cdot) + v(\cdot)) - J(u(\cdot)) \geq \frac{1}{2} \int_t^T (v - b\alpha_s) |\tilde{y}_v(s)|^2 ds + \frac{\nu_T}{2} |\tilde{y}_v(T)|^2 - \frac{1}{2} \int_t^T (v(s), Nv(s)) ds. \tag{75}$$

Thanks to the assumption (25), the right hand side of (75) is positive, which proves (71) and completes the proof of the result. \square

6.3 Bellman Equation

We have proven, under the assumptions of Theorem 2, that the control problem (10), (11) has a unique solution $u(\cdot)$. Defining the value function

$$V(x, t) := \inf_{v(\cdot)} J_{xt}(v(\cdot)) = J_{xt}(u(\cdot)), \tag{76}$$

we can state that:

$$V(x, t) = \int_t^T F(y(s)) ds + F_T(y(T)) + \frac{1}{2} \int_t^T (BN^{-1}B^*\Gamma(y(s), s), \Gamma(y(s), s)) ds, \tag{77}$$

with

$$\begin{cases} \frac{d}{ds}y(s) = A(y(s)) - BN^{-1}B^*\Gamma(y(s), s), \\ y(t) = x. \end{cases} \tag{78}$$

We first have:

Proposition 3 *We have the following property:*

$$\Gamma(x, t) = D_x V(x, t). \tag{79}$$

Proof Since the minimum of $J_{xt}(v(\cdot))$ is attained in the unique value $u(\cdot)$, we can rely on the envelope theorem to claim that:

$$(D_x V(x, t), \xi) = \int_t^T (D_x F(y(s)), \mathcal{X}(s)\xi) ds + (D_x F_T(y(T)), \mathcal{X}(T)\xi), \tag{80}$$

in which $\mathcal{X}(s)$ is the solution of

$$\begin{cases} \frac{d}{ds}\mathcal{X}(s) = D_x A(y(s))\mathcal{X}(s), \\ \mathcal{X}(t) = I. \end{cases}$$

Recalling the equation (12) for $z(s)$ and performing integration by parts in (80), the result $(D_x V(x, t), \xi) = (\Gamma(x, t), \xi)$ is easily obtained. This proves the result (79). \square

We can then obtain the Bellman equation for the value function $V(x, t)$.

Theorem 4 *We make the assumptions of Theorem 2. The function $V(x, t)$ is the unique solution of*

$$\begin{cases} -\frac{\partial V}{\partial t} - (D_x V, A(x)) + \frac{1}{2}(D_x V, BN^{-1}B^*D_x V) = F(x), \\ V(x, T) = F_T(x). \end{cases} \quad (81)$$

Proof We know that $V(x, t)$ is Gâteaux differentiable in x , with the derivative $\Gamma(x, t)$. From (12), $\Gamma(x, t)$ is continuous in t . From equation (77), we can write:

$$\begin{aligned} V(x, t) - V(x, t + \epsilon) &= \int_t^{t+\epsilon} F(y(s))ds + \frac{1}{2} \int_t^{t+\epsilon} (BN^{-1}B^*\Gamma(y(s), s), \Gamma(y(s), s))ds \\ &\quad + V(y(\epsilon), t + \epsilon) - V(x, t + \epsilon). \end{aligned} \quad (82)$$

We then have:

$$\begin{aligned} &V(y(\epsilon), t + \epsilon) - V(x, t + \epsilon) \\ &= V\left(x + \int_t^{t+\epsilon} A(y(s))ds - \int_t^{t+\epsilon} BN^{-1}B^*\Gamma(y(s), s)ds, t + \epsilon\right) - V(x, t + \epsilon) \\ &= \left(\Gamma(x, t + \epsilon), \int_t^{t+\epsilon} A(y(s))ds - \int_t^{t+\epsilon} BN^{-1}B^*\Gamma(y(s), s)ds\right) \\ &\quad + \int_0^1 \left(\Gamma\left(x + \theta \int_t^{t+\epsilon} (A(y(s)) - BN^{-1}B^*\Gamma(y(s)))ds, t + \epsilon\right) - \Gamma(x, t + \epsilon), \right. \\ &\quad \left. \int_t^{t+\epsilon} (A(y(s)) - BN^{-1}B^*\Gamma(y(s)))ds\right) d\theta. \end{aligned} \quad (83)$$

Using the fact that $\Gamma(x, t)$ is uniformly Lipschitz in x and continuous in t , we obtain easily from (83) that:

$$\frac{V(y(\epsilon), t + \epsilon) - V(x, t + \epsilon)}{\epsilon} \rightarrow (\Gamma(x, t), A(x) - BN^{-1}B^*\Gamma(x, t)).$$

Then, dividing (82) by ϵ and letting ϵ tend to 0, we obtain the PDE (81), recalling (79). The initial condition in (81) is trivial. If we take the gradient in x of (81), we recognize equation (16). Since this equation has a unique solution, the solution of (81) is also unique (easy checking). This completes the proof. \square

7 Application to Mean Field Type Control Theory

7.1 Wasserstein Space

Denote by $\mathcal{P}_2(\mathbb{R}^n)$ the Wasserstein space of Borel probability measures m on \mathbb{R}^n such that $\int_{\mathbb{R}^n} |x|^2 dm(x) < \infty$, with the metric

$$W_2(\mu, \nu) = \sqrt{\inf \left\{ \int |x - y|^2 d\pi(x, y) : \pi \in \Pi(\mu, \nu) \right\}}, \quad (84)$$

where $\Pi(\mu, \nu)$ is the space of all Borel probability measures on $\mathbb{R}^n \times \mathbb{R}^n$ whose first and second marginals are μ and ν respectively.

7.2 Functional Derivatives

Let F be a functional on $\mathcal{P}_2(\mathbb{R}^n)$. We recall the idea of the functional derivative here.

Definition 1 F is said to have a functional derivative if there exists a continuous function $\frac{dF}{dm} : \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^n \rightarrow \mathbb{R}$, such that for some $c : \mathcal{P}_2(\mathbb{R}^n) \rightarrow [0, \infty)$ which is bounded on bounded subsets, we have

$$\left| \frac{dF}{dm}(m, x) \right| \leq c(m)(1 + |x|^2) \quad (85)$$

and

$$F(m') - F(m) = \int_0^1 \int_{\mathbb{R}^n} \frac{dF}{dm}(m + \theta(m' - m))(x) d(m' - m)(x) d\theta. \quad (86)$$

We require also $\int_{\mathbb{R}^n} \frac{dF}{dm}(m, x) dm(x) = 0$ as it is unique up to a constant by definition.

Definition 2 F is said to have a second order functional derivative if there exists a continuous function $\frac{d^2 F}{dm^2} : \mathcal{P}_2 \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that, for some $c : \mathcal{P}_2(\mathbb{R}^n) \rightarrow [0, \infty)$ which is bounded on bounded subsets, we have

$$\left| \frac{d^2 F}{dm^2}(m, x, \tilde{x}) \right| \leq c(m)(1 + |x|^2 + |\tilde{x}|^2) \quad (87)$$

and

$$\begin{aligned}
& F(m') - F(m) \\
&= \int_{\mathbb{R}^n} \frac{dF}{dm}(m)(x) d(m' - m)(x) \\
&+ \int_0^1 \int_0^1 \theta \frac{d^2F}{dm^2}(m + \lambda\theta(m' - m))(x, \tilde{x}) d(m' - m)(x) d(m' - m)(\tilde{x}) d\lambda d\theta.
\end{aligned} \tag{88}$$

Again, we require that $\int_{\mathbb{R}^n} \frac{d^2F}{dm}(m, x, \tilde{x}) dm(\tilde{x}) = 0$, for all $x \in \mathbb{R}^n$, and $\int_{\mathbb{R}^n} \frac{d^2F}{dm}(m, x, \tilde{x}) dm(x) = 0$, for all $\tilde{x} \in \mathbb{R}^n$, as it is unique up to a constant. Note also that

$$\frac{d^2F}{dm^2}(m)(x, \tilde{x}) = \frac{d^2F}{dm^2}(m)(\tilde{x}, x). \tag{89}$$

We write $D \frac{dF}{dm}(m)(x)$ to mean differentiating with respect to x , and $D_1 \frac{d^2F}{dm^2}(m)(x_1, x_2)$ and $D_2 \frac{d^2F}{dm^2}(m)(x_1, x_2)$ to denote partial differentiation with respect to x_1 and x_2 , respectively.

7.3 Mean Field Type Control Problems

We introduce the setting of a mean-field type control problem. Consider real-valued functions $f(x, m)$ and $h(x, m)$ defined on $\mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n)$. We define

$$\begin{aligned}
F(m) &:= \int_{\mathbb{R}^n} f(x, m) dm(x), \\
F_T(m) &:= \int_{\mathbb{R}^n} h(x, m) dm(x).
\end{aligned}$$

Fix a $m \in \mathcal{P}_2(\mathbb{R}^n)$. Let $A, B : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be matrices, and $N : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a self-adjoint invertible matrix. We make the following assumptions on f, h, B, N, A . We assume that

(A1) $\forall x \in \mathbb{R}^n$,

$$BN^{-1}B^*x \cdot x \geq m|x|^2, m > 0. \tag{90}$$

(A2) f is regular enough such that the following is justifiable. $\forall y \in \mathbb{R}^n$,

$$v|y|^2 \leq \frac{\partial^2 f}{\partial x^2}(x, m)y \cdot y \leq M|y|^2, \quad (91)$$

$$v|y|^2 \leq D_\xi^2 \frac{\partial f}{\partial m}(x, m)(\xi)y \cdot y \leq M|y|^2, \quad (92)$$

$$D_\xi \frac{\partial^2 f}{\partial x \partial m}(x, m)(\xi) = 0, \quad (93)$$

$$D_{\xi_1} D_{\xi_2} \frac{\partial^2 f}{\partial m^2}(x, m)(\xi_1, \xi_2) = 0. \quad (94)$$

(A3) h is regular enough such that the following is justifiable. $\forall y \in \mathbb{R}^n$,

$$v_T|y|^2 \leq \frac{\partial^2 h}{\partial x^2}(x, m)y \cdot y \leq M_T|y|^2, \quad (95)$$

$$v_T|y|^2 \leq D_\xi^2 \frac{\partial h}{\partial m}(x, m)(\xi)y \cdot y \leq M_T|y|^2, \quad (96)$$

$$D_\xi \frac{\partial^2 h}{\partial x \partial m}(x, m)(\xi) = 0, \quad (97)$$

$$D_{\xi_1} D_{\xi_2} \frac{\partial^2 h}{\partial m^2}(x, m)(\xi_1, \xi_2) = 0. \quad (98)$$

(A4) For the matrices, we have

$$|A| < M|BN^{-1}B^*|, \text{ with } |\cdot| \text{ the matrix 2-norm.} \quad (99)$$

The set of our feasible control is $L^2(t, T; L_m^2(\mathbb{R}^n; \mathbb{R}^n))$, i.e.,

$$v_{\cdot, m, t}(\cdot) \in L^2(t, T; L_m^2(\mathbb{R}^n; \mathbb{R}^n))$$

if and only if

$$\int_t^T \int_{\mathbb{R}^n} |v_{x, m, t}(s)|^2 dm(x) ds < \infty.$$

To each $v_{\cdot, m, t}(\cdot) \in L^2(t, T; L_m^2(\mathbb{R}^n; \mathbb{R}^n))$ and $x \in \mathbb{R}^n$ we associate the state

$$x_{x, m, t}(s; v) := x + \int_t^s [Ax_{x, m, t}(\tau; v) + Bv_{x, m, t}(\tau)] d\tau. \quad (100)$$

Note that $x_{\cdot, m, t}(\cdot) \in L^2(t, T; L_m^2(\mathbb{R}^n; \mathbb{R}^n))$. We define the objective functional on $L^2(t, T; L_m^2(\mathbb{R}^n; \mathbb{R}^n))$ by

$$\begin{aligned} J_{m, t}(v) := & \int_t^T F(x_{\cdot, m, t}(s; v) \# m) ds + F_T(x_{\cdot, m, T}(s; v) \# m) \\ & + \frac{1}{2} \int_t^T \int_{\mathbb{R}^n} v_{x, m, t}^*(\tau) N v_{x, m, t}(\tau) dm(x) d\tau. \end{aligned} \quad (101)$$

Thus the value function is

$$V(m, t) := \inf_{v \in L^2(t, T; L_m^2(\mathbb{R}^n; \mathbb{R}^n))} J_{m, t}(v). \quad (102)$$

7.4 The Hilbert Space \mathcal{H}_m and the Push-Forward Map

We proceed as our previous works [6, 7].

7.4.1 Settings

Fix $m \in \mathcal{P}_2(\mathbb{R}^n)$, we define $\mathcal{H}_m := L_m^2(\mathbb{R}^n; \mathbb{R}^n)$, the set of all measurable vector field Φ such that $\int_{\mathbb{R}^n} |\Phi(x)|^2 dm(x) < \infty$. We equip \mathcal{H}_m with the inner product

$$\langle X, Y \rangle_{\mathcal{H}_m} := \int_{\mathbb{R}^n} X(x) \cdot Y(x) dm(x). \quad (103)$$

Write the corresponding norm as $\|X\|_{\mathcal{H}_m} = \sqrt{\langle X, X \rangle_{\mathcal{H}_m}}$.

Definition 3 For $m \in \mathcal{P}_2$, $X \in \mathcal{H}_m$, define $X \otimes m \in \mathcal{P}_2$ as follow: for all $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $x \mapsto \frac{|\phi(x)|}{1 + |x|^2}$ is bounded, define

$$\int \phi(x) d(X \otimes m)(x) := \int \phi(X(x)) dm(x). \quad (104)$$

Remark 2 This actually is the push-forward map as we are working on the deterministic case. We write as $X \otimes m$ to align with our treatment of the stochastic case in [7].

We recall several useful properties from [7].

Proposition 4 *We have the following properties:*

1. Let $X, Y \in \mathcal{H}_m$, and suppose $X \circ Y \in \mathcal{H}_m$. Then $(X \circ Y) \otimes m = X \otimes (Y \otimes m)$.
2. If $X(x) = x$ is the identity map, then $X \otimes m = m$.
3. Let $X \in \mathcal{H}_m$, denote the space $L_X^2(t, T; \mathcal{H}_m)$ to be the set of all processes in $L^2(t, T; \mathcal{H}_m)$ that is adapted to $\sigma(X)$. There exists a natural linear isometry between $L_X^2(t, T; \mathcal{H}_m)$ and $L^2(t, T; \mathcal{H}_{X \otimes m})$.

Proof Please refer to [7] Section 2 and Section 3. □

7.4.2 Extending the Domain of Functions to \mathcal{H}_m

The proofs in this section is standard, we therefore omit unless specified. Readers may refer to [7] Section 2. Let $F : \mathcal{P}_2(\mathbb{R}^n) \rightarrow \mathbb{R}$, we extend F to be a function on \mathcal{H}_m

by $X \mapsto F(X \otimes m)$, $\forall X \in \mathcal{H}_m$. When the domain is \mathcal{H}_m , we can talk about Gâteaux derivative. We actually have the following relation between the Gâteaux derivative on \mathcal{H}_m and its functional derivative:

Proposition 5 *Let $F : \mathcal{P}_2(\mathbb{R}^n) \mapsto \mathbb{R}$ have a functional derivative $\frac{dF}{dm}$, and $x \mapsto \frac{dF}{dm}(m, x)$ is continuously differentiable in \mathbb{R}^n . Assume that $D\frac{dF}{dm}(m, x)$ is continuous in both m and x , and*

$$\left| D\frac{dF}{dm}(m)(x) \right| \leq c(m)(1 + |x|) \quad (105)$$

for some constant $c(m)$ depending only on m . Denote the Gâteaux derivative as $D_X F(X \otimes m)$, we have

$$D_X F(X \otimes m) = D\frac{dF}{dm}(X \otimes m)(X(\cdot)). \quad (106)$$

We now look at the second order Gâteaux derivative, denoted as $D_X^2 F(X \otimes m)$, note that $D_X^2 F(X \otimes m)$ is a bounded linear operator from \mathcal{H}_m to \mathcal{H}_m .

Proposition 6 *In addition to the assumptions in Proposition 5, let F has a second order functional derivative $\frac{d^2 F}{dm^2}(m)(x_1, x_2)$, assume also $D^2\frac{dF}{dm}(m)(x)$, $D_1\frac{d^2 F}{dm^2}(m)(x_1, x_2)$, $D_2\frac{d^2 F}{dm^2}(m)(x_1, x_2)$ and $D_1 D_2\frac{d^2 F}{dm^2}(m)(x_1, x_2)$ exist and are continuous, such that*

$$\left| D^2\frac{dF}{dm}(m)(x) \right| \leq d(m), \quad (107)$$

$$\left| D_1 D_2\frac{d^2 F}{dm^2}(m)(x_1, x_2) \right| \leq d'(m), \quad (108)$$

where d, d' are constants depending on m only, and $|\cdot|$ is the matrix 2-norm. Then we have:

$$\begin{aligned} D_X^2 F(X \otimes m)Y(x) &= D^2\frac{dF}{dm}(X \otimes m)(X(x))Y(x) \\ &+ \int_{\mathbb{R}^n} D_1 D_2\frac{d^2 F}{dm^2}(X \otimes m)(X(x), X(x'))Y(x')dm(x'). \end{aligned} \quad (109)$$

Besides, we can view $F(X \otimes m)$ as $m \mapsto F(X \otimes m)$, in this case, we can talk about differentiation with respect to m , denote it as $\frac{\partial F}{\partial m}$. The following relation between $\frac{\partial F}{\partial m}$ and $\frac{dF}{dm}$ holds.

Proposition 7 *Let $F : \mathcal{P}_2(\mathbb{R}^n) \mapsto \mathbb{R}^n$ have a functional derivative and fix $X \in \mathcal{H}_m$. We have*

$$\frac{\partial F}{\partial m}(X \otimes m)(x) = \frac{dF}{dm}(X \otimes m)(X(x)). \quad (110)$$

Now let $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$, we extend it as $\forall X \in \mathcal{H}_m, X \mapsto A(X) \in \mathcal{H}_m$,

$$A(X)(x) = A(X(x)). \quad (111)$$

It is trivial to see that if A^{-1} exists in \mathbb{R}^n , then $A^{-1}(X)(x) = A^{-1}(X(x))$ is the inverse of A in \mathcal{H}_m . So is the transpose of A , if A is a matrix. Again, we can talk about its Gâteaux derivative.

Proposition 8 *Let A to be continuously differentiable. Denote its derivative to be dA . Assume that there exists k such that $|dA(x)| \leq k$ for all $x \in \mathbb{R}^n$, where $|\cdot|$ is the matrix 2-norm. Then for all $X, Y \in \mathcal{H}_m$, we have*

$$D_X A(X)Y(x) = dA(X(x))Y(x). \quad (112)$$

Proof Let $X, Y, H \in \mathcal{H}_m$, then

$$\begin{aligned} & \frac{1}{\epsilon} \left\langle A(X + \epsilon Y) - A(X), H \right\rangle_{\mathcal{H}_m} \\ &= \frac{1}{\epsilon} \int_{\mathbb{R}^n} \left[A(X(\xi) + \epsilon Y(\xi)) - A(X(\xi)) \right] \cdot H(\xi) dm(\xi) \\ &= \int_{\mathbb{R}^n} \int_0^1 dA(X(\xi) + \theta \epsilon Y(\xi)) Y(\xi) \cdot H(\xi) d\theta dm(\xi) \\ &\rightarrow \int_{\mathbb{R}^n} dA(X(\xi)) Y(\xi) \cdot H(\xi) dm(\xi) = \left\langle dA(X(\cdot)) Y(\cdot), H \right\rangle_{\mathcal{H}_m}. \end{aligned}$$

Proposition 9 *Let A be twice continuously differentiable. Denote its second derivative to be $d^2 A$. Note that $d^2 A(x)(a, b) \in \mathbb{R}^n$, and $d^2 A(x)(a, b) = d^2 A(x)(b, a)$. Assume that there exists $k(x)$ such that $\forall a, b \in \mathbb{R}^n, |d^2 A(x)(a, b)| \leq k(x)$, then we have*

$$d^2 A(X)(Y, W)(x) = d^2 A(X(x))(Y(x), W(x)). \quad (113)$$

Proof Let $X, Y, W, H \in \mathcal{H}_m$, then

$$\begin{aligned} & \frac{1}{\epsilon} \left\langle D_X A(X + W)Y - D_X A(X)Y, H \right\rangle_{\mathcal{H}_m} \\ &= \frac{1}{\epsilon} \int_{\mathbb{R}^n} \left[dA(X(\xi) + \epsilon W(\xi))Y(\xi) - dA(X(\xi))Y(\xi) \right] \cdot H(\xi) dm(\xi) \\ &= \int_{\mathbb{R}^n} \int_0^1 d^2 A(X(\xi) + \theta \epsilon W(\xi))(Y(\xi), W(\xi)) \cdot H(\xi) d\theta dm(\xi) \\ &\rightarrow \int_{\mathbb{R}^n} d^2 A(X(\xi))(Y(\xi), W(\xi)) \cdot H(\xi) dm(\xi). \end{aligned}$$

7.5 Control Problem in the Hilbert Space \mathcal{H}_m

Recall the definitions of A , B , N , F , F_T in Section 7.3. Extend the functions as in Section 7.4.2. We assume (A1), (A2), (A3) and (A4). It is not hard to derive (9), (7) and (42) from the assumptions. Note that in our case, $b = 0$.

Now fix $X \in \mathcal{H}_m$ as our initial data. For given $v_{Xt} \in L^2_X(t, T; \mathcal{H}_m)$ (subscript X and t to address the measurability and starting time), consider the dynamics:

$$X(s) = X + \int_t^s [AX(\tau) + Bv_{Xt}(\tau)] d\tau. \quad (114)$$

Denote the process as $X_{Xt}(s) = X_{Xt}(s; v_{Xt})$. Define the cost functional:

$$J_{Xt}(v_{Xt}) := \int_t^T F(X_{Xt}(s) \otimes m) ds + F_T(X_{Xt}(T) \otimes m) + \frac{1}{2} \int_t^T \langle v_{Xt}(\tau), Nv_{Xt}(\tau) \rangle_{\mathcal{H}_m} d\tau, \quad (115)$$

and the value function is

$$V(X, t) := \inf_{v_{Xt} \in L^2_X(t, T; \mathcal{H}_m)} J_{Xt}(v_{Xt}). \quad (116)$$

This is in the form of our concerned model in Section 2, with the Hilbert space being \mathcal{H}_m .

While (114) is infinite dimensional, there is a finite dimensional view point of it. For $v_{Xt} \in L^2_X(t, T; \mathcal{H}_m)$, by Proposition 4, let $\tilde{v} \in L^2(t, T; \mathcal{H}_{X \otimes m})$ be the representative of v_{Xt} . Consider

$$x(s) = x + \int_t^s [Ax(\tau) + B\tilde{v}(\tau, x)] d\tau. \quad (117)$$

Denote the solution to be $x(s; x, \tilde{v}(\cdot, x))$. Then we have

$$X_{Xt}(s; v_{Xt})(x) = x(s; X(x), \tilde{v}(\cdot, X(x))).$$

We introduce the notation $X_{xt}(\cdot)$ with a lowercase letter for x to mean $x(\cdot; x, \tilde{v}(\cdot, x))$, and $v_t(s)$ to mean $\tilde{v}(s, \cdot)$. From above we can conclude that the law of $X_{Xt}(s; v_{Xt}(\cdot))$ is $x(s; \cdot, \tilde{v}(\cdot, \cdot)) \otimes (X \otimes m)$. Hence the cost functional (115) can be written as

$$\begin{aligned}
& J_{X_t}(v_{X_t}) \\
&= \int_t^T F(X_{X_t}(s) \otimes m) ds + F_T(X_{X_t}(T) \otimes m) + \frac{1}{2} \int_t^T \langle v_{X_t}(\tau), N v_{X_t}(\tau) \rangle_{\mathcal{H}_m} d\tau \\
&= \int_t^T F(x(s; \cdot, \tilde{v}(\cdot, \cdot)) \otimes (X \otimes m)) ds + F_T(x(T; \cdot, \tilde{v}(\cdot, \cdot)) \otimes (X \otimes m)) \\
&\quad + \frac{1}{2} \int_t^T \langle v_{X_t}(\tau), N v_{X_t}(\tau) \rangle_{\mathcal{H}_m} d\tau \\
&=: J_{X \otimes m, t},
\end{aligned} \tag{118}$$

that means J depends on X only through $X \otimes m$. Respectively,

$$V(X, t) = \inf_{v_{X_t} \in L_X^2(t, T; \mathcal{H}_m)} J_{X_t}(v_{X_t}) = \inf_{v_{X_t} \in L_X^2(t, T; \mathcal{H}_m)} J_{X \otimes m, t}(v_{X_t}) =: V(X \otimes m, t). \tag{119}$$

7.6 Necessary and Sufficient Condition for Optimality

Assume (A1), (A2), (A3) and (A4), we conclude from Theorem 2 that there exists unique optimal control $\hat{v}_{X_t}(s) = -N^{-1}B^*Z_{X_t}(s)$, where $Z_{X_t}(s)$ together with $Y_{X_t}(s)$ are the unique solution of the system

$$Y_{X_t}(s) = X + \int_t^s [AY_{X_t}(\tau) - BN^{-1}B^*Z_{X_t}(\tau)] d\tau, \tag{120}$$

$$Z_{X_t}(s) = \int_s^T [(AY_{X_t}(\tau))^* Z_{X_t}(\tau) + D_X F(Y_{X_t}(\tau) \otimes m)] + D_X F_T(Y_{X_t}(T) \otimes m). \tag{121}$$

Again, because $L_X^2(t, T; \mathcal{H}_m)$ is isometric to $L^2(t, T; \mathcal{H}_{X \otimes m})$, there exists $Y_{\xi_t}(s)$, $Z_{\xi_t}(s)$ such that $Y_{X_t} = Y_{\xi_t}|_{\xi=X}$ and $Z_{X_t} = Z_{\xi_t}|_{\xi=X}$, (Y_{ξ_t}, Z_{ξ_t}) solving

$$Y_{\xi_t}(s) = \xi + \int_t^s [AY_{\xi_t}(\tau) - BN^{-1}B^*Z_{\xi_t}(\tau)] d\tau, \tag{122}$$

$$\begin{aligned}
Z_{\xi_t}(s) &= \int_s^T \left[(AY_{\xi_t}(\tau))^* Z_{\xi_t}(\tau) + D \frac{dF}{dm}(Y_t(\tau) \otimes (X \otimes m))(Y_{\xi_t}(\tau)) \right] \\
&\quad + D \frac{dF_T}{dm}(Y_t(T) \otimes (X \otimes m))(Y_{\xi_t}(T)).
\end{aligned} \tag{123}$$

As (Y_{ξ_t}, Z_{ξ_t}) depends on m through $X \otimes m$, we write $(Y_{\xi, X \otimes m, t}, Z_{\xi, X \otimes m, t})$. We can write the value function as

$$\begin{aligned}
V(X, t) &= \int_t^T F(Y_{X_t}(s) \otimes m) ds + F_T(Y_{X_t}(T) \otimes m) \\
&\quad + \frac{1}{2} \int_t^T \langle N^{-1} B^* Z_{X_t}(\tau), B^* Z_{X_t}(\tau) \rangle_{\mathcal{H}_m} d\tau \\
&= \int_t^T F(Y_{\cdot, X \otimes m, t}(s) \otimes (X \otimes m)) ds + F_T(Y_{\cdot, X \otimes m, t}(T) \otimes (X \otimes m)) \\
&\quad + \frac{1}{2} \int_t^T \int_{\mathbb{R}^n} N^{-1} B^* Z_{\xi, X \otimes m, t}(\tau) \cdot B^* Z_{\xi, X \otimes m, t}(\tau) d(X \otimes m)(\xi) d\tau \\
&= V(X \otimes m, t).
\end{aligned} \tag{124}$$

In particular, if we choose X to be the identity function, i.e., $X(x) = x$, recall that $X \otimes m = m$, there exists $(Y_{x, m, t}, Z_{x, m, t})$ solving

$$Y_{x, m, t}(s) = x + \int_t^s [AY_{x, m, t}(\tau) - BN^{-1}B^*Z_{x, m, t}(\tau)] d\tau, \tag{125}$$

$$\begin{aligned}
Z_{x, m, t}(s) &= \int_s^T \left[(AY_{x, m, t}(\tau))^* Z_{x, m, t}(\tau) + D \frac{dF}{dm}(Y_{\cdot, m, t}(\tau) \otimes m)(Y_{x, m, t}(\tau)) \right] \\
&\quad + D \frac{dF_T}{dm}(Y_{\cdot, m, t}(T) \otimes (X \otimes m))(Y_{x, m, t}(T)),
\end{aligned} \tag{126}$$

which is the system of optimality condition for our mean field type control problem in Section 7.3. For the value function, we have

$$\begin{aligned}
V(m, t) &= \int_t^T F(Y_{\cdot, m, t}(s) \otimes m) ds + F_T(Y_{\cdot, m, t}(T) \otimes m) \\
&\quad + \frac{1}{2} \int_t^T \int_{\mathbb{R}^n} N^{-1} B^* Z_{x, m, t}(\tau) \cdot B^* Z_{x, m, t}(\tau) dm(x) d\tau.
\end{aligned} \tag{127}$$

7.7 Properties of the Value Function

We give the functional derivative of the value function V , and the relation between the solution of the FBSDE and V . As the proofs are standard, we omit here and readers may refer to Section 4 of [7].

Proposition 10 *Assume (A1), (A2), (A3), (A4). We have the following properties for the value function:*

1. *By Proposition 3, we have*

$$D_X V(X \otimes m, t) = Z_{X_t}(t). \tag{128}$$

2. *We have*

$$\begin{aligned}
\frac{dV}{dm}(m,t)(x) &= \int_t^T \frac{dF}{dm}(Y_{\cdot,m,t}(s) \otimes m)(Y_{x,m,s}(s)) ds \\
&+ \frac{dF_T}{dm}(Y_{\cdot,m,t}(T) \otimes m)(Y_{x,m,s}(T)) \\
&+ \frac{1}{2} \int_t^T N^{-1} B^* Z_{x,m,t}(\tau) \cdot B^* Z_{x,m,t}(\tau) d\tau.
\end{aligned} \tag{129}$$

3. We have

$$D \frac{d}{dm} V(m,t)(x) = Z_{x,m,t}(t), \tag{130}$$

$$D_X V(X \otimes m,t) = D \frac{d}{dm} V(X \otimes m,t)(X) \tag{131}$$

4. Also, the feedback nature of Z in Y , i.e., for any $x \in \mathbb{R}^n$, $\forall s \in [t, T]$, we have

$$Z_{x,m,t}(s) = D \frac{d}{dm} V(Y_{\cdot,m,t} \otimes m, s)(Y_{x,m,t}(s)), \tag{132}$$

and for any $X \in \mathcal{H}_m$, $\forall s \in [t, T]$,

$$Z_{X,t}(s) = D_X V(Y_{X,t}(s) \otimes m, s). \tag{133}$$

7.8 Bellman Equation

Assume (A1), (A2), (A3), (A4). By Theorem 4, we deduce that for any $T > 0$, $V(X \otimes m, t)$ is the unique solution to the following Bellman equation:

$$\begin{cases}
-\frac{\partial V}{\partial t}(X \otimes m, t) - \left\langle D_X V(X \otimes m, t), AX \right\rangle_{\mathcal{H}_m} \\
+ \frac{1}{2} \left\langle D_X V(X \otimes m, t), BN^{-1} B^* D_X V(X \otimes m, t) \right\rangle_{\mathcal{H}_m} = F(X \otimes m), \\
V(X \otimes m) = F_T(X \otimes m).
\end{cases} \tag{134}$$

As before, let X be the identity function, together with Proposition 10, we conclude that for any $T > 0$, $V(m, t)$ solves the following PDE on the space of probability measures:

$$\begin{cases}
-\frac{\partial V}{\partial t}(m, t) - \int_{\mathbb{R}^n} D \frac{dV}{dm}(m, t)(x) \cdot Ax dm(x) \\
+ \frac{1}{2} \int_{\mathbb{R}^n} D \frac{dV}{dm}(m, t)(x) \cdot BN^{-1} B^* D \frac{dV}{dm}(m, t)(x) dm(x) = F(m), \\
V(m, T) = F_T(m).
\end{cases} \tag{135}$$

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References

1. S. Ahuja, W. Ren, and T. Yang. Forward-backward stochastic differential equations with monotone functionals and mean field games with common noise. *Stochastic Processes and their Applications*, 129(10):3859–3892, 2019.
2. F. Antonelli. Backward-forward stochastic differential equations. *Ann. Appl. Probab.*, 3:777–793, 1993.
3. A. Bensoussan, J. Frehse, and P. Yam. *Mean Field Games and Mean Field Type Control Theory*. Springer, 2013.
4. A. Bensoussan, J. Frehse, and S. C. P. Yam. The master equation in mean field theory. *Journal de Mathématiques Pures et Appliquées*, 2014.
5. A. Bensoussan, J. Frehse, and S. C. P. Yam. On the interpretation of the master equation. *Stochastic Processes and their Applications*, 127(7):2093–2137, 2017.
6. A. Bensoussan, P. J. Graber, and S. C. P. Yam. Stochastic control on space of random variables. *arXiv:1903.12602*, 2019.
7. A. Bensoussan, P. J. Graber, and S. C. P. Yam. Control on Hilbert spaces and application to mean field type control theory. *arXiv:2005.10770*, 2020.
8. A. Bensoussan, K. C. J. Sung, S.C.P. Yam, and S.P. Yung. Linear quadratic mean field games. *Journal of Optimization Theory and Applications*, 169(2):469–529, 2016.
9. A. Bensoussan and S. C. P. Yam. Control problem on space of random variables and master equation. *ESAIM: Control, Optimisation and Calculus of Variations*, 25:10, 2019.
10. A. Bensoussana, S. C. P. Yam, and Z. Zhang. Well-posedness of mean-field type forward-backward stochastic differential equations. *Stochastic Processes and their Applications*, 125(9):3327–3354, 2015.
11. R. Buckdahn, J. Li, S. Peng, and C. Rainer. Mean-field stochastic differential equation and associated PDEs. *The Annals of Probability*, 45:824–878, 2017.
12. P. Cardaliaguet, F. Delarue, J.M. Lasry, and P.L. Lions. *The Master Equation and the Convergence Problem in Mean Field Games: (AMS-201)*, volume 201. Princeton University Press, 2019.

13. R. Carmona and F. Delarue. Mean field forward-backward stochastic differential equations. *Electron. Commun. Probab.*, 18, 2013.
14. R. Carmona and F. Delarue. Forward-backward stochastic differential equations and controlled McKean-Vlasov dynamics. *Ann. Probab.*, 43(5):2647–2700, 2015.
15. R. Carmona and F. Delarue. *Probabilistic Theory of Mean Field Games with Applications I-II*. Springer, 2018.
16. A. Cosso and H. Pham. Zero-sum stochastic differential games of generalized McKean-Vlasov type. *Journal de Mathématiques Pures et Appliquées*, 129:180–212, 2019.
17. F. Delarue. On the existence and uniqueness of solutions to FBSDEs in a non-degenerate case. *Stochastic Processes and their Applications*, 99:209–286, 2002.
18. M.F. Djete, D. Possamai, and X. Tan. McKean-Vlasov optimal control: The dynamic programming principle. *arXiv:1907.08860*, 2019.
19. W. Gangbo and A.R. Mészáros. Global well-posedness of master equations for deterministic displacement convex potential mean field games. *arXiv:2004.01660*, 2020.
20. Y. Hu and S. Peng. Solution of forward-backward stochastic differential equations. *Probab. Theory and Related Fields*, 103:273–283, 1995.
21. Y. Hu and J. Yong. Forward-backward stochastic differential equations with nonsmooth coefficients. *Stochastic Processes and their Applications*, 87(1):93–106, 2000.
22. P. L. Lions. Lectures at college de france. <http://www.college-de-france.fr>.
23. P. L. Lions. Seminar at college de france. 2014.
24. J. Ma, P. Protter, and J. Yong. Solving forward-backward stochastic differential equations explicitly—a four step scheme. *Probab. Theory Related Fields*, 98:339–359, 1994.
25. J. Ma, Z. Wu, D. Zhang, and J. Zhang. On well-posedness of Forward-Backward SDEs — a unified approach. *Annals of Applied Probability*, 25(4):2168–2214, 2015.
26. C. Mou and J. Zhang. Weak solutions of mean field game master equations. *arXiv:1903.09907*, 2019.
27. E. Pardoux and S. Tang. Forward-backward stochastic differential equations and quasilinear parabolic PDEs. *Probab. Theory and Related Fields*, 114(2):123–150, 1999.
28. S. Peng and Z. Wu. Fully coupled forward-backward stochastic differential equations and applications to optimal control. *SIAM J. Control Optim.*, 37(3):825–843, 1999.
29. H. Pham and X. Wei. Dynamic programming for optimal control of stochastic McKean-Vlasov dynamics. *SIAM Journal on Control and Optimization*, 55(2):1069–1101, 2017.
30. J. Yong. Finding adapted solutions of forward-backward stochastic differential equations: Method of continuation. *Probab. Theory and Related Fields*, 107(4):537–572, 1997.
31. J. Yong. Forward-backward stochastic differential equations with mixed initial-terminal conditions. *Trans. Amer. Math. Soc.*, 362(2):1047–1096, 2010.
32. J. Zhang. The wellposedness of FBSDEs. *Discrete & continuous dynamical systems - B*, 6(4):927–940, 2006.