Chapter 9
The Ring of Conditions for Horospherical Homogeneous Spaces

Johannes Hofscheier

Abstract These are notes of a five talks lecture series during the "Graduate Summer School in Algebraic Group Actions", at McMaster University, June 11th–15th, 2018. The aim of this lecture series is to introduce the ring of conditions of a spherical homogeneous space with a special emphasis on the horospherical case, i.e., homogeneous spaces with respect to a connected complex reductive group which are torus bundles over a flag variety. In these notes, we start with an example from enumerative geometry which naturally yields first instances of spherical varieties. We continue with a recollection of the necessary background on reductive groups needed throughout the rest of the manuscript. After that we introduce spherical varieties: we discuss the Luna–Vust theory of spherical embeddings and explain the complete combinatorial description of horospherical varieties (an important subfamily of spherical varieties). We conclude with the definition of the ring of conditions of spherical homogeneous spaces and give an explicit description for the horospherical case.

Keywords Spherical variety · Linear algebraic group · Enumerative geometry · Ring of conditions

9.1 Motivation

In the following, some elementary knowledge of algebraic geometry is expected from the reader. Introductory texts which cover the required topics are, e.g., [9, 23, 24, 27]. Parts of this manuscript follow the lecture notes [11] by Kiritchenko.

First examples of spherical varieties emerged from enumerative geometry such as Grassmannians. It turns out that many enumerative problems reduce to intersection theoretic questions on algebraic varieties equipped with a "good" transitive (or almost transitive) action of an algebraic group. Here is a classical example:

J. Hofscheier (✉)
School of Mathematical Sciences, University of Nottingham, Nottingham NG7 2RD, UK
e-mail: johannes.hofscheier@nottingham.ac.uk

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Problem 1 How many lines in 3-space $\mathbb{C}^3$ intersect 4 given lines in general position?

Recall the general trick how to rephrase an affine geometric question into a linear one: Suppose X is an affine geometric object in $\mathbb{C}^n$. Introduce one further dimension and consider the linear span of X regarded as a subset of the affine hyperplane $\{x_{n+1} = 1\}$ where x_{n+1} denotes the additional coordinate.

Hence, the above question reduces to a problem in the Grassmannian $\mathrm{Gr}(2, 4)$ (2-planes in $\mathbb{C}^4$). This algebraic variety admits a transitive action by GL_4 (the general linear group of invertible 4×4-matrices with complex entries). Indeed, let $e_1, \ldots, e_4$ be the standard basis in $\mathbb{C}^4$ and consider the natural action of GL_4 on $\mathbb{C}^4$. Clearly GL_4 acts transitively on planes in $\mathbb{C}^4$ and the stabilizer P of the coordinate plane $\mathrm{span}\{e_1, e_2\}$ is given by

$$P = \left\{ \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} : A, B \in \mathrm{GL}_2, C \in \mathrm{Mat}(2 \times 2) \right\}.$$

Hence $\mathrm{Gr}(2, 4) \cong \mathrm{GL}_4 / P$ is a homogeneous space under GL_4 and P is an example of a parabolic subgroup (see definition below).

Note that by the transition to $\mathrm{Gr}(2, 4)$, we implicitly consider the lines as sitting in the projective 3-space $\mathbb{P}^3$ and intersections are taken in the projective sense. Indeed, two parallel lines do not intersect in the affine 3-space, but their corresponding 2-planes do. This corresponds to the fact that two parallel lines intersect at infinity in $\mathbb{P}^3$.

Let us recall the crucial ideas of Schubert's solution to Problem 1. To solve it, he developed the calculus of "conditions" (see [22]), which has since become known as *Schubert Calculus*. Examples of conditions are:

1. for a given point a, denote by σ_a the condition that a line contains a;
2. for a given line ℓ, denote by σ_ℓ the condition that a line intersects ℓ; or
3. for a plane Π, denote by σ_Π the condition that a line is contained in Π.

Schubert's brilliant idea was that conditions can be added and multiplied and this corresponds to logical "or" and "and" operations on the conditions, e.g., $\sigma_{\ell_1} + \sigma_{\ell_2}$ is the condition that a line intersects line ℓ_1 or line ℓ_2 while $\sigma_{\ell_1} \cdot \sigma_{\ell_2}$ is the condition that a line intersects both lines ℓ_1 and ℓ_2. So, for instance, Problem 1 can be reformulated to: What is $\sigma_{\ell_1} \cdots \sigma_{\ell_4}$ where ℓ_i are four lines in general position? In particular, we can reformulate the problem in an algebraic equation and obtain

$$\sigma_{\ell_1} \cdot \sigma_{\ell_2} \cdot \sigma_{\ell_3} \cdot \sigma_{\ell_4} = (\sigma_{\ell_1} \cdot \sigma_{\ell_2}) \cdot (\sigma_{\ell_3} \cdot \sigma_{\ell_4}) = ?.$$

So we have to understand the conditions $\sigma_{\ell_1} \cdot \sigma_{\ell_2}$ and $\sigma_{\ell_3} \cdot \sigma_{\ell_4}$. By using some heuristics, Schubert came to the conclusion that "perturbations" of the condition $\sigma_{\ell_1} \cdots \sigma_{\ell_4}$ do not change the answer (the conservation of number principle), i.e., we are allowed to move the lines ℓ_i. In particular, we may assume that ℓ_1, ℓ_2 lie on a plane, and so do ℓ_3 and ℓ_4 (see Fig. 9.1). From that it straightforwardly follows that a line intersects both ℓ_1 and ℓ_2 if and only if it is either contained in the

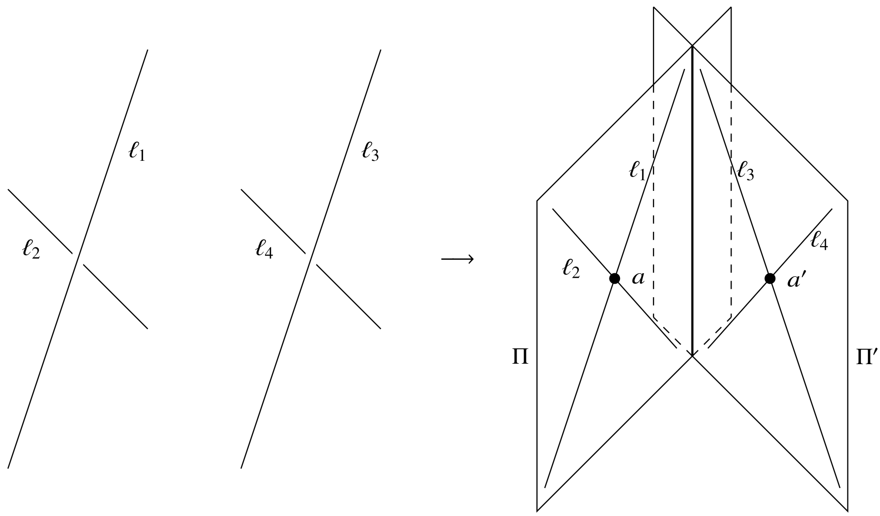

Fig. 9.1 The conservation of number principle implies that we are allowed to move the lines ℓ_i

plane Π spanned by ℓ_1 and ℓ_2 (recall that we take intersections in $\mathbb{P}^3$) or it contains the intersection point of ℓ_1 and ℓ_2. Using Schubert calculus this means

$$\sigma_{\ell_1} \cdot \sigma_{\ell_2} = \sigma_a + \sigma_\Pi \qquad \text{and} \qquad \sigma_{\ell_3} \cdot \sigma_{\ell_4} = \sigma_{a'} + \sigma_{\Pi'}$$

where a is the intersection point of ℓ_1 and ℓ_2 and Π is the plane spanned by ℓ_1 and ℓ_2 and similarly for ℓ_3, ℓ_4.

Thus, we get

$$\sigma_{\ell_1} \cdot \sigma_{\ell_2} \cdot \sigma_{\ell_3} \cdot \sigma_{\ell_4} = (\sigma_a + \sigma_\Pi) \cdot (\sigma_{a'} + \sigma_{\Pi'}) = \sigma_a \cdot \sigma_{a'} + \sigma_a \cdot \sigma_{\Pi'} + \sigma_{a'} \cdot \sigma_\Pi + \sigma_\Pi \cdot \sigma_{\Pi'}.$$

Clearly there is exactly one line passing through both a and a' and there is exactly one line contained in both Π and Π'. On the other hand, as a is not contained in Π', the condition $\sigma_a \cdot \sigma_{\Pi'}$ is not satisfied by any line, and similarly for $\sigma_{a'} \cdot \sigma_\Pi$.

We obtain

$$\sigma_{\ell_1} \cdot \sigma_{\ell_2} \cdot \sigma_{\ell_3} \cdot \sigma_{\ell_4} = \underbrace{\sigma_a \cdot \sigma_{a'}}_{=1} + \underbrace{\sigma_a \cdot \sigma_{\Pi'}}_{=0} + \underbrace{\sigma_{a'} \cdot \sigma_\Pi}_{=0} + \underbrace{\sigma_\Pi \cdot \sigma_{\Pi'}}_{=1} = 2.$$

Of course, we haven't given a precise explanation yet and in his fifteenth problem Hilbert asked for a rigorous foundation of Schubert Calculus. Our goal will be to understand De Concini's and Procesi's solution to Hilbert's problem. For that, we also need to understand spherical geometry, a topic which is exciting in its own right.

9.2 Linear Algebraic Groups: A Crash Course

The classical books by Borel, Humphreys and Springer [2, 10, 25] are excellent references for what follows. A more modern and accessible book is [18]. For convenience, we work over the field of complex numbers $\mathbb{C}$.

An algebraic variety G is called an *algebraic group* if G is a group and the maps $G \times G \to G$, $(g, h) \mapsto gh$ and $G \to G$, $g \mapsto g^{-1}$ are morphisms of algebraic varieties. The *Lie algebra* of G, usually denoted by $\mathfrak{g}$, is the tangent space $T_e G$ at the identity element $e \in G$ equipped with a binary operation $[\cdot, \cdot]$ called the *Lie bracket*. Important examples of algebraic groups are GL_n (=the general linear group of invertible $n \times n$-matrices with complex entries), SL_n (=the special linear group of $n \times n$-matrices with complex entries and determinant 1), abelian varieties (=complete connected algebraic groups) or elliptic curves (=1-dimensional abelian varieties). We will work with *linear algebraic groups*, i.e., Zariski closed subgroups of GL_n. If $G \subseteq \mathrm{GL}_n$ is a linear algebraic group, then $T_e G = \mathfrak{g} \subseteq \mathfrak{gl}_n = T_e \,\mathrm{GL}_n = \{(n \times n) - \text{matrices}\}$, and the Lie bracket can be defined as the commutator of matrices

$$[A, B] := AB - BA.$$

Remark 2 If one replaces "algebraic varieties" and "morphisms of algebraic varieties" by "smooth manifolds" and "smooth maps", one obtains the definition of a Lie group.

Exercise 3 Let G be an algebraic group.

1. Show that only one irreducible component of G can pass through e. This component is called the *identity component* of G, usually denoted by G°.
2. Show that G° is a normal subgroup of finite index in G, whose cosets are the connected as well as irreducible components of G.

Exercise 4 Which of the following algebraic groups are linear?

1. $(\mathbb{C}^n, +)$,
2. An elliptic curve,
3. PGL_n.

From now on all algebraic groups are assumed to be linear, unless stated otherwise.

Definition 5 An element $g \in G \subseteq \mathrm{GL}_n$ is called *semisimple* if the matrix g is diagonalizable. It is called *unipotent* if all eigenvalues of g are equal to 1. (This definition is independent of the choice of the embedding $G \subseteq \mathrm{GL}_n$.)

Exercise 6 Let $\pi : G \to \mathrm{GL}_n$ be a (rational) representation of an algebraic group G, i.e., π is a morphism of algebraic groups. Show that:

1. If $G = (\mathbb{C}^*)^n$, then any matrix in $\pi(G)$ is diagonalizable.
2. If $G = \mathbb{C}^n$, then any matrix in $\pi(G)$ is unipotent.

Exercise 7 (*Jordan decomposition*) Show that every element $g \in G$ has a unique decomposition $g = g_s g_u$, where $g_s \in G$ is semisimple, $g_u \in G$ is unipotent, and g_s and g_u commute.

The *radical*, denoted by $R(G)$, of an algebraic group G is the identity component of its maximal normal solvable subgroup. The *unipotent radical*, denoted by $R_u(G)$, is the set of unipotent elements in the radical of G.

Definition 8 An algebraic group G is *reductive* if $R_u(G) = \{e\}$. It is *semisimple* if $R(G) = \{e\}$.

Theorem 9 (Characterization of reductive groups) *Let G be an algebraic group. The following conditions are equivalent:*

1. *G is reductive;*
2. *$R(G)$ is a torus;*
3. *$G^\circ = T \cdot S$, where T is a torus and S is a connected semisimple subgroup;*
4. *any finite-dimensional rational representation of G is completely reducible (recall: a* rational representation *of G in a vector space V is a homomorphism $G \to \mathrm{GL}(V)$ of algebraic groups);*
5. *G admits a faithful finite-dimensional completely reducible rational representation;*
6. *the Lie algebra of G admits a direct sum decomposition $\mathfrak{g} = \mathfrak{h} \oplus i\mathfrak{h}$ where $\mathfrak{h}$ is the Lie algebra of a maximal compact real Lie subgroup of G.*

Exercise 10 Which of the following groups are reductive?

1. $\mathbb{C}^n$,
2. GL_n,
3. An elliptic curve.

Exercise 11 Show that an algebraic group G is reductive if and only if G does not contain a normal subgroup isomorphic to $\mathbb{C}^n$.

A *character* of an algebraic group G is a homomorphism of algebraic groups $\chi : G \to \mathbb{C}^*$ and the set of all characters gives the *character group* of G, i.e., $\mathfrak{X}(G) := \{\chi : G \to \mathbb{C}^* \text{ character}\}$.

An *algebraic torus* is an algebraic group T that is isomorphic to

$$(\mathbb{C}^*)^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : z_i \neq 0\}.$$

If G is an algebraic group, then a maximal element of the set

$$\{H \subseteq G \text{ closed subgroup, } H \text{ an algebraic torus}\}$$

(which is ordered by inclusion) is called a *maximal torus* of G.

Theorem 12 *In an algebraic group, any two maximal tori are conjugated.*

The dimension of T is called the *rank* of G. The character lattice $\mathfrak{X}(T)$ of T is also called the *weight lattice* of G, and its elements are called *weights* of G.

Exercise 13 Find a maximal torus of the following groups:

1. GL_n,
2. SL_n,
3. SO_n.

The set $\{H \subseteq G$ closed connected solvable subgroup$\}$ is partially ordered by inclusion. A maximal element of this set is called a *Borel subgroup*.

Exercise 14 Show that the upper-triangular invertible matrices form a Borel subgroup in GL_n and that any two Borel subgroups are conjugated. (Hint: Use the Lie–Kolchin theorem [10, Theorem 17.6].)

Theorem 15 *In an algebraic group, any two Borel subgroups are conjugated.*

Definition 16 A (Zariski) closed subgroup $P \subseteq G$ is called *parabolic* if P contains a Borel subgroup of G.

Exercise 17 Let G be a linear algebraic group, $B \subseteq G$ a Borel subgroup and $T \subseteq G$ a maximal torus.

1. Show that up to conjugation $T \subseteq B$.
2. Show that restricting characters from B to T induces an isomorphism of character lattices $\mathfrak{X}(B) \cong \mathfrak{X}(T)$.

The *Weyl group* W of G is defined as $N_G(T)/C_G(T)$, where $N_G(T)$ and $C_G(T)$ denote the normalizer and centralizer, respectively, of a maximal torus $T \subseteq G$. The Weyl group acts on T by conjugation: if $w = nC_G(T)$ for $n \in N_G(T)$, then $w(t) := (ntn^{-1})$ for $t \in T$.

Theorem 18 *If G is a connected reductive group, then $C_G(T) = T$ for any maximal torus $T \subseteq G$. In particular, the Weyl group is given by $W = N_G(T)/T$.*

Exercise 19 If G is a connected reductive group, $T \subseteq G$ a maximal torus, and $B \subseteq G$ a Borel subgroup with $T \subseteq B$, show that for any $w \in W$ the double coset $B\dot{w}B$ is independent of the choice of a representative $\dot{w} \in N_G(T)$. Thus, by abuse of notation, we will denote this double coset by BwB.

Theorem 20 (Bruhat decomposition) *If G is a connected reductive group, $T \subseteq G$ a maximal torus, and $B \subseteq G$ a Borel subgroup with $T \subseteq B$, then there is a disjoint union, i.e., $BwB = Bw'B$ if and only if $w = w'$ in W,*

$$G = \bigsqcup_{w \in W} BwB$$

In particular,

$$G/B = \bigsqcup_{w \in W} BwB/B.$$

Exercise 21 You may want to start with $n = 3$ or $n = 4$ in the following problems.

1. Explicitly compute the Bruhat decomposition of GL_n (take T to be the maximal torus of diagonal matrices and B the Borel subgroup of upper triangular matrices).
2. Classify all parabolic subgroups in GL_n (up to conjugation). (Hint: There is a relationship between parabolic subgroups in GL_n and flags in $\mathbb{C}^n$. Recall that a *flag* is an increasing sequence of subspaces of $\mathbb{C}^n$, i.e., $\{0\} = V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_k = \mathbb{C}^n$. The dimensions $d_i := \dim V_i$ yield an increasing sequence of integers $0 = d_0 < d_1 < \cdots < d_k = n$, called the *signature* of the flag.)

Exercise 22 Let G be a connected reductive group, $T \subseteq G$ a maximal torus, $B \subseteq G$ a Borel subgroup with $T \subseteq B$ and $P \subseteq G$ a parabolic subgroup with $B \subseteq P$. Show that

$$G/P = \bigsqcup_{w \in W/W_P} BwP/P$$

where $W_P = N_P(T)/T$ is the Weyl group of P.

The closure of B-orbits in G/P are the *Schubert varieties* (denoted by $X(w)$). They play an important role in the study of G/P. The dimension of $X(w)$ equals the *length* $l(w)$ of w (i.e., the minimal number of factors needed to write w as a product of simple reflections). In particular, there exists a unique element w_0 of maximal length in W/W_P.

Example 23 Let $G = \mathrm{SL}_n$ and T be the maximal torus of diagonal matrices contained in the Borel subgroup B of upper-triangular matrices. The Lie algebra $\mathfrak{g} = \mathfrak{sl}_n$ (i.e., the tangent space $T_e \mathrm{SL}_n$ equipped with the Lie bracket $[\cdot, \cdot]$) is the set of traceless matrices in $\mathrm{Mat}(n \times n)$ equipped with the commutator bracket $[A, B] = AB - BA$. Furthermore, the Lie algebra $\mathfrak{t}$ of T coincides with the subspace of diagonal matrices in $\mathfrak{sl}_n$. Observe that the Lie bracket induces a map $\mathrm{ad}\colon \mathfrak{t} \to \mathrm{End}(\mathfrak{g})$; $A \mapsto [A, \cdot]$ which is a *representation of Lie algebras*, i.e., $\mathrm{ad}([A, B]) = \mathrm{ad}(A)\,\mathrm{ad}(B) - \mathrm{ad}(B)\,\mathrm{ad}(A)$ for any $A, B \in \mathfrak{t}$ (check this!). Let $\varepsilon_1, \ldots, \varepsilon_n$ be the linear forms in $\mathfrak{t}^*$ induced by the diagonal entries, i.e., $\varepsilon_i(\mathrm{diag}(t_1, \ldots, t_n)) = t_i$ and set $\varepsilon_{ij} = \varepsilon_i - \varepsilon_j$. It is straightforward to show that the Lie algebra decomposes into eigenspaces as follows

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\varepsilon_{ij} \in R} \mathfrak{g}_{\varepsilon_{ij}}, \text{ where } \mathfrak{g}_{\varepsilon_{ij}} = \mathbb{C} \begin{pmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 0 \end{pmatrix} \begin{matrix} \\ i\text{th} \\ \\ \end{matrix} \quad (1 \text{ in the } j\text{th column}),$$

and $R = \{\varepsilon_{ij} \colon 1 \leq i, j \leq n, i \neq j\}$. If $\mathfrak{b}$ is the Lie algebra of B, then

$$\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha$$

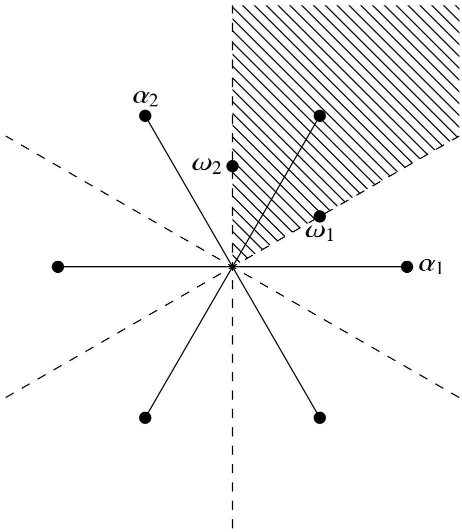

Fig. 9.2 Illustration of the root system A_2. We identify the hyperplane $\{x + y + z = 0\} \subseteq \mathbb{R}^3$ with $\mathbb{R}^2$ via the basis obtained by applying the Gram–Schmidt algorithm to the basis (α_1, α_2) where $\mathbb{R}^3$ and $\mathbb{R}^2$ are equipped with the usual euclidian scalar products

where $R^+ = \{\varepsilon_{ij} : i < j\}$ and this set is called the set of *positive roots*. The set of *simple roots* $S = \{\alpha_i := \varepsilon_{i,i+1} : i = 1, \ldots, n-1\}$ (cf. Fig. 9.2) forms a basis of $\mathfrak{t}^*$ (check this!) and induces an isomorphism $\mathfrak{t}^* \cong \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 + \ldots + x_n = 0\} \subseteq \mathbb{R}^n$. To any simple root α_i one associates a reflection s_i, namely the linear transformation on $\mathbb{R}^n$ which swaps the coordinates with index i and $(i+1)$. We identify s_i with an element in $W = N_G(T)/T$:

$$\begin{pmatrix} I_{i-1} & & & \\ & 0 & 1 & \\ & -1 & 0 & \\ & & & I_{n-i-1} \end{pmatrix} T \in W.$$

Then W is generated by the s_i, i.e., $W = \langle s_i : i = 1, \ldots, n-1 \rangle$ (check this!). It straightforwardly follows that W is isomorphic to the group S_n of permutations on the coordinates of $\mathbb{R}^n$ via $s_i \mapsto (i, i+1)$ (transposition swapping i with $i+1$).

In general, the Lie bracket induces a natural representation $\mathrm{ad} \colon \mathfrak{t} \to \mathrm{End}(\mathfrak{g}); x \mapsto [x, \cdot]$. There is a set of linear forms, called *roots*, $R \subseteq \mathfrak{t}^*$ such that

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha$ denotes the linear subspace of eigenvectors of weight α, i.e., the set of vectors $x \in \mathfrak{g}$ such that $[h, x] = \alpha(h)x$ for all $h \in \mathfrak{t}$. The Lie algebra $\mathfrak{b}$ of B can be written as

$$\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha$$

for some subset $R^+ \subseteq R$, called the set of *positive roots*. There exists a unique basis S contained in R^+ such that all positive roots are linear combinations of elements in S with nonnegative integer coefficients. The elements of S are called *simple roots*.

The following fundamental theorem on parabolic subgroups can be found in any introductory text on algebraic groups. Recall the definition of the Weyl group: $W = N_G(T)/T$. Let R be the set of roots and let S be the set of simple roots induced by the choice of the Borel subgroup B.

Theorem 24 *The assignment $I \mapsto P_I = \bigsqcup_{w \in W_I} BwB$ induces a bijection between subsets of the set of simple roots S and parabolic subgroups of G which contain B (here W_I denotes the group generated by the simple reflections s_α for $\alpha \in I$).*

9.3 Spherical Varieties

Recall the definition of a toric variety:

Definition 25 Let T be an algebraic torus. A normal irreducible T-variety is called *toric variety* if it contains an open dense T-orbit.

Spherical varieties can be thought of as a generalization of toric varieties where one allows also non-abelian group actions. Unfortunately, the straightforward generalization does not work:

Definition 26 (*Incorrect definition*) Let G be a connected linear algebraic group. A normal irreducible G-variety is called *spherical* if it contains an open dense G-orbit.

Exercise 27 Show that the "incorrect definition" of spherical varieties does not imply finiteness of the number of orbits, a property one would expect from a generalization of toric varieties. (Hint: Consider the action of GL_n on the space of $(n \times n)$-matrices by left multiplication. Show that the GL_n-orbits are classified by matrices in reduced row echelon form. If $n \geq 2$, deduce that, although there is an open GL_n orbit, the number of GL_n-orbits is infinite.)

So the definition of spherical varieties is more subtle: Let G be a connected reductive complex linear algebraic group (this assumption has several implications which make this choice important: finite generation properties, good representation theory, cf. Theorem 9).

Definition 28 A normal irreducible G-variety is said to be *spherical* if it contains an open orbit under the action of a Borel subgroup of G. (In particular, it contains an open G-orbit.)

Example 29 Examples of spherical varieties are toric varieties (a Borel subgroup of $(\mathbb{C}^*)^n$ is $(\mathbb{C}^*)^n$ itself).

Another point of view on spherical varieties, important to the theory, is as follows: First consider the open G-orbit which is a homogeneous space G/H for some subgroup H of G. Then consider the embedding of G/H in X. So we make the following definitions:

Definition 30 A closed subgroup $H \subseteq G$ is called *spherical* if G/H has a dense open orbit for a Borel subgroup of G. In this case, G/H is called a *spherical homogeneous space*.

Recall that in Exercise 27, we have seen that an open G-orbit does not guarantee the finiteness of G-orbits, a property one would expect from a generalization of toric varieties. It is interesting that one can use this property as a characterization of spherical homogeneous spaces:

Theorem 31 ([1]) *A homogeneous G-space O is spherical if and only if any G-variety X with an open orbit equivariantly isomorphic to O has finitely many G-orbits.*

A morphism $\varphi\colon X \to Y$ of G-varieties is called *equivariant* if $\varphi(g \cdot x) = g \cdot \varphi(x)$ for any $g \in G$ and all $x \in X$.

Definition 32 Suppose G/H is a spherical homogeneous space. An equivariant open embedding of G/H into a normal irreducible G-variety X is called a *spherical embedding*, and X is called a *spherical variety*.

In particular, the description of spherical varieties splits into two parts:

1. Classify all spherical homogeneous spaces G/H.
2. For a fixed spherical homogeneous space, classify all G-equivariant open embeddings $G/H \hookrightarrow X$ into normal irreducible G-varieties.

Historically, the second problem has been answered first by the work of Luna and Vust [17]. Only recently, the first problem has been answered by work of several researchers (see [3, 6, 15, 16]).

Exercise 33 Show the following statements:

1. A closed subgroup $H \subseteq \mathrm{SL}_2$ is spherical if and only if $\dim H > 0$.
2. Table 9.1 shows a list of all spherical subgroups of SL_2 (up to conjugation). (Although spherical varieties with an SL_2-action seem to be special, they actually play a crucial role in the development of spherical varieties. See, for example, [17] or [8, 14].) Hints: If this is too hard, then verify explicitly that the subgroups given in Table 9.1 are spherical:

   a. $\mathrm{SL}_2 / B \cong \mathbb{P}^1$ where SL_2 naturally acts on $\mathbb{P}^1$,
   b. it is enough to show that U_1 is spherical (why?) and to do that consider the natural action of SL_2 on $\mathbb{A}^2$,
   c. consider the natural action of SL_2 on $\mathbb{P}^1 \times \mathbb{P}^1$ to show that T is a spherical subgroup of SL_2,
   d. consider the natural action of SL_2 on the symmetric (2×2)-matrices to show that N is a spherical subgroup of SL_2.

Exercise 34 A closed subgroup $H \subseteq G$ is called *horospherical* if it contains a maximal unipotent subgroup U of G. Show that horospherical subgroups are spherical. In particular, flag varieties are spherical. (Hint: Use the Bruhat decomposition.)

Table 9.1 Classification of spherical subgroups of SL_2 (up to conjugation)

H	Details
SL_2	
B	Borel subgroup
$U_k = \left\{ \begin{pmatrix} \xi & * \\ 0 & \xi^{-1} \end{pmatrix} : \xi \in \mu_k \right\}$	$k \in \mathbb{N}$, μ_k group of kth roots of unity
T	Maximal torus
N	Normalizer of a maximal torus

9.3.1 The Luna–Vust Theory of Spherical Embeddings

Recall that for a fixed algebraic torus T, all toric embeddings $T \hookrightarrow X$ into a normal irreducible T-variety can be combinatorially described by polyhedral fans in the vector space $\mathrm{Hom}(\mathfrak{X}(T), \mathbb{Q})$. A similar description exists for spherical embeddings which we now explain. This is called the *Luna–Vust theory of spherical embeddings*. Good references for this theory are [13, 17]:

Let G be a connected reductive complex algebraic group and fix a spherical subgroup $H \subseteq G$. Let B be a Borel subgroup of G and T a maximal torus of G contained in B. We now explain how all spherical embeddings $G/H \hookrightarrow X$ can be described combinatorially.

Definition 35 The combinatorial objects needed in the Luna–Vust theory are listed in Table 9.2. The rank of $\mathcal{M}$ is also called the *rank of the spherical homogeneous space* G/H, i.e., $\mathrm{rk}(G/H) = \mathrm{rk}(\mathcal{M})$. Let $\mathcal{N} := \mathrm{Hom}(\mathcal{M}, \mathbb{Z})$ be the dual lattice of $\mathcal{M}$ and note that we have a dual pairing $\langle \cdot, \cdot \rangle \colon \mathcal{N} \times \mathcal{M} \to \mathbb{Z}$. Furthermore, set $\mathcal{M}_{\mathbb{Q}} := \mathcal{M} \otimes \mathbb{Q}$ and $\mathcal{N}_{\mathbb{Q}} = \mathrm{Hom}(\mathcal{M}, \mathbb{Q})$. Recall, that in our context a *valuation* is a map $\nu \colon \mathbb{C}(G/H)^* = \mathbb{C}(G/H) \setminus \{0\} \to \mathbb{Q}$ which satisfies:

1. $\nu(f_1 + f_2) \geq \min\{\nu(f_1), \nu(f_2)\}$ whenever $f_1, f_2, f_1 + f_2 \in \mathbb{C}(G/H)^*$;
2. $\nu(f_1 f_2) = \nu(f_1) + \nu(f_2)$ for all $f_1, f_2 \in \mathbb{C}(G/H)^*$; and
3. $\nu(\mathbb{C}^*) = 0$.

A valuation ν is called G-invariant if $\nu(g \cdot f) = \nu(f)$ for all $g \in G$ and $f \in \mathbb{C}(G/H)^*$.

As the set of B-semi-invariant rational functions on G/H will appear frequently below, we introduce the notation $\mathbb{C}(G/H)^{(B)}$ for it.

Lemma 36 *As G/H has an open B-orbit, a B-semi-invariant rational function f is determined by its weight χ_f up to a scalar multiple. Said in other words: For any $\chi \in \mathcal{M}$, there is $f_\chi \in \mathbb{C}(G/H)^{(B)}$ (unique up to a scalar multiple) such that $b \cdot f_\chi = \chi(b) f_\chi$.*

Table 9.2 The *Luna–Vust data*

Object	Definition
Weight lattice	$\mathcal{M} = \{\chi \in \mathfrak{X}(B)\colon$ $\exists f \in \mathbb{C}(G/H)^*$, B-semi-invariant with $b \cdot f = \chi(b) f$ for $b \in B\}$
Colors	$\mathcal{D} = \{B$-invariant prime divisors in $G/H\}$
Valuation cone	$\mathcal{V} = \{\nu\colon \mathbb{C}(G/H)^* \to \mathbb{Q},\ G$-invariant valuation$\}$

Table 9.3 The "Luna–Vust data" for the toric case

Object	Toric case
Weight lattice	$M = \mathfrak{X}(T)$
Colors	$\mathcal{D} = \varnothing$
Valuation cone	$\mathcal{V} = \mathcal{N}_{\mathbb{Q}}$

The following interpretation of a valuation $\nu\colon \mathbb{C}(G/H) \to \mathbb{Q}$ (invariant or not) will be useful:

$$\rho\colon \{\nu\colon \mathbb{C}(G/H) \to \mathbb{Q} \text{ valuation}\} \to \mathcal{N}_{\mathbb{Q}};\ \nu \mapsto [\chi \mapsto \nu(f_\chi)].$$

Theorem 37 ([4]) *The map $\rho|_{\mathcal{V}}\colon \mathcal{V} \hookrightarrow \mathcal{N}_{\mathbb{Q}}$ is injective and its image is a polyhedral cone whose dual cone is simplicial and not necessarily full-dimensional.*

Any color $D \in \mathcal{D}$ induces a valuation ν_D and by abuse of notation, we will write $\rho(D) := \rho(\nu_D)$. In general, the map $\rho|_{\mathcal{D}}\colon \mathcal{D} \to \mathcal{N}_{\mathbb{Q}}$ is not that well-behaved (see Exercise 38).

Exercise 38 Find the "Luna–Vust data" for the spherical homogeneous spaces from Exercise 33. In particular, you should see the following phenomena:

1. SL_2/T: the map $\rho\colon \mathcal{D} \to \mathcal{N}$ might not be injective;
2. SL_2/N: the image of a color $\rho(D)$ might not be primitive in $\mathcal{N}$;
3. SL_2/B: the image of a color might even be zero, i.e., $\rho(D) = 0$.

Example 39 The "Luna–Vust data" of the toric case is listed in Table 9.3.

Example 40 Consider the natural action of SL_2 on $\mathbb{C}^2$. Let B be the Borel subgroup of upper triangular matrices, T the maximal torus of diagonal matrices and U the unipotent radical of B. Denote the standard basis of $\mathbb{C}^2$ by e_1, e_2. Then $\mathrm{SL}_2/U \cong \mathrm{SL}_2 \cdot e_1 = \mathbb{C}^2 \setminus \{0\}$ and $B \cdot e_2 = \mathbb{C} \times \mathbb{C}^*$ is the open B-orbit. The rational functions on SL_2/U are given by $\mathbb{C}(x, y) = \mathbb{C}(\mathbb{A}^2)$. It is straightforward to verify that

$$\mathbb{C}(\mathrm{SL}_2/U)^{(B)} \cong \{cy^k\colon c \in \mathbb{C}, k \in \mathbb{Z}\}.$$

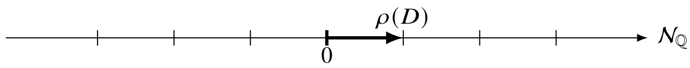

Fig. 9.3 Illustrating the "Luna–Vust data" of SL_2/U

Hence, $\mathcal{M} = \mathbb{Z}\omega$ where ω is the fundamental weight of SL_2 induced by the diagonal elements of T. Clearly $D = \mathrm{div}(y)$ is the only B-stable prime divisor in SL_2/U, i.e., we only have one color $\mathcal{D} = \{D\}$ and $\rho(D) = \check{\alpha}|_{\mathcal{M}}$ where $\check{\alpha}$ denotes the coroot associated to the simple root α of SL_2 (with respect to our choice of Borel). If we consider the blowup of $\mathbb{A}^2$ at the origin, we obtain an exceptional SL_2-invariant divisor E which induces an SL_2-invariant valuation ν_E such that $\rho(\nu_E) = \check{\alpha}|_{\mathcal{M}}$. In particular, the ray $\mathbb{Q}_{\geq 0}\check{\alpha}$ is contained in the valuation cone $\mathcal{V}$. If we consider the spherical embedding $\mathrm{SL}_2/U \hookrightarrow \mathbb{P}^2$, we see that the line at infinity induces a G-invariant valuation ν with $\rho(\nu) = -\check{\alpha}|_{\mathcal{M}}$, and thus $\mathcal{V} = \mathcal{N}_{\mathbb{Q}}$.

One usually illustrates the combinatorial data in a picture (see Fig. 9.3).

Definition 41 We introduce the following "colored" extensions of the notions of polyhedral cone, face and fan from the toric case.

1. A *colored cone* is a pair $(C, \mathcal{F})$ with
   a. $\mathcal{F} \subseteq \mathcal{D}$,
   b. $C \subseteq \mathcal{N}_{\mathbb{Q}}$ convex cone generated by $\rho(\mathcal{F})$ and finitely many elements of $\mathcal{V} \cap \mathcal{N}$,
   c. the relative interior of C intersects $\mathcal{V}$,
   d. C contains no lines and $0 \notin \rho(\mathcal{F})$.

   Such colored cones are usually called strictly convex colored cones, but as we are only interested in strictly convex cones, we will omit the specifier and just speak of colored cones.
2. A *colored face* of a colored cone $(C, \mathcal{F})$ is a pair $(C', \mathcal{F}')$ such that
   a. C' is a face of C (in the usual sense),
   b. the relative interior of C' intersects $\mathcal{V}$,
   c. $\mathcal{F}' = \{D \in \mathcal{F} \colon \rho(D) \in C'\}$.
3. A *colored fan* is a finite set Σ of colored cones with the following properties:
   a. every colored face of a colored cone of Σ is in Σ,
   b. for all $\nu \in \mathcal{V}$, there exists at most one colored cone $(C, \mathcal{F}) \in \Sigma$ such that ν is in the relative interior of C.
4. The *support* of a color fan Σ is the set $\bigcup_{(C,\mathcal{F})}(C \cap \mathcal{V}) \subseteq \mathcal{V}$ where $(C, \mathcal{F})$ runs through all elements in Σ.

Let us explain how to associate a colored fan Σ_X to a spherical embedding $G/H \hookrightarrow X$.

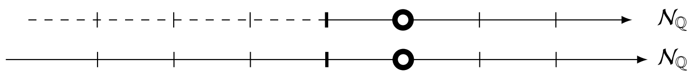

Fig. 9.4 The colored fans of $\mathrm{SL}_2/U \hookrightarrow \mathbb{A}^2$ and $\mathrm{SL}_2/U \hookrightarrow \mathbb{P}^2$

Theorem 42 *X is covered by finitely many G-invariant open subvarieties of X containing a unique closed G-orbit (such varieties are called* simple embeddings*).*

Let $X' \subseteq X$ be an open G-invariant subvariety which is a simple embedding and denote the G-invariant divisors of X' by $X_1, \dots, X_{m'}$. Let $\mathcal{F}'$ be the set of colors $D \in \mathcal{D}$ whose closure in X contain the closed orbit of X'. We define C' to be the cone in $\mathcal{N}_{\mathbb{Q}}$ generated by $\rho(D)$ for $D \in \mathcal{F}'$ and $\rho(X_i) := \rho(\nu_{X_i})$ for $i = 1, \dots, m'$. Then $(C', \mathcal{F}')$ is a colored cone in $\mathcal{N}_{\mathbb{Q}}$. Moreover the set of colored cones constructed this way (together with their colored faces) forms a colored fan, which we denote by Σ_X.

Theorem 43 (Luna–Vust) *The map $X \mapsto \Sigma_X$ is a bijection from the isomorphism classes of spherical G/H-embeddings and the set of colored fans.*

Example 44 (*Example* 40 *continued*) Clearly $\mathrm{SL}_2/U \hookrightarrow \mathbb{A}^2$ is a simple spherical embedding (the origin is the only closed SL_2-orbit). On the other hand the spherical embedding $\mathrm{SL}_2/U \hookrightarrow \mathbb{P}^2$ is not simple (indeed we can cover it with an affine chart $\mathbb{A}^2$ and the complement of the unique SL_2 fixed point). The corresponding colored fans are illustrated in Fig. 9.4 (understand how to get them and which colored fan corresponds to which spherical embedding). Note that the circle means that the cone $\mathbb{Q}_{\geq 0}$ is "colored" by the unique color of SL_2/U, i.e., $(\mathbb{Q}_{\geq 0}, \{D\})$.

Exercise 45 Use the Luna–Vust theory to classify all spherical embeddings of SL_2/T and SL_2/N. Draw the corresponding colored fans. Hint: You should find 2 in both cases.

Many results about spherical varieties are known. Unfortunately, due to lack of time, we won't be able to dig any deeper.

Theorem 46 *A list of selected results:*

1. *The number of B-orbits is finite;*
2. *X is complete if and only if any $v \in \mathcal{V}$ is contained in some colored cone of Σ_X;*
3. *there is a bijective correspondence between G-orbits in X and colored cones in Σ_X;*
4. *a combinatorial smoothness criterion;*
5. *combinatorial descriptions of the Picard group and the divisor class group;*
6. *ampleness criterion for divisors;*

and many more ...

To learn more about the features of spherical varieties, the interested reader is encouraged to consult [26] for further reading.

9.3.2 The Classification of Spherical Homogeneous Spaces

The classification of spherical homogeneous spaces G/H turns out to be quite hard. Luna's brilliant insight in spherical varieties made it possible to find such a description from Wasserman's list of certain spherical varieties of rank 2 [28]. Inspired by it, Luna [16] formulated a conjectural description and proved it for spherical varieties of type A. Only recently Luna's Conjecture was proven in general with the combined efforts of several researchers [3, 6, 15]. Unfortunately, time does not permit to give more details on this exciting topic, instead we will see a complete answer for an interesting special case.

9.3.3 The Complete Picture in the Horospherical Case

Recall from Exercise 34 that a closed subgroup $H \subseteq G$ is called *horospherical* if it contains a maximal unipotent subgroup of G. An exceedingly well written introduction to horospherical varieties can be found in [19, 20] by Pasquier.

Fix a maximal unipotent subgroup $U \subseteq G$, a Borel subgroup $U \subseteq B$ of G and a maximal torus $T \subseteq B$.

Let us list some fundamental properties of horospherical subgroups. We refer to [19] for further details and references.

Proposition 47 *For a horospherical subgroup $H \subseteq G$ with $U \subseteq H$, the following statements hold:*

1. *the normalizer $P := N_G(H)$ is a parabolic subgroup containing B. Let $I \subseteq S$ be the unique set of simple roots such that $P = P_I$ (see Theorem 24);*
2. $\mathcal{M} = \{\chi \in \mathfrak{X}(P) \colon \chi|_H = 1\} \subseteq \{\chi \in \mathfrak{X}(T) \colon \langle\check{\alpha}, \chi\rangle = 0$ *for all* $\alpha \in I\}$*;*
3. $H = \bigcap_{\chi\in\mathcal{M}} \ker(\chi)$*;*
4. $\mathcal{D} = \{D_\alpha := \overline{Bw_0 s_\alpha P/H} \colon \alpha \in S \setminus I\}$ *where $w_0 \in W$ is the longest element in the Weyl group $W = N_G(T)/T$ and s_α denotes the simple reflection associated to the simple root α;*
5. $P_{-w_0(I)}$ *coincides with the stabilizer of the open B-orbit in G/H;*
6. $\mathcal{V} = \mathcal{N}_{\mathbb{Q}}$.

Theorem 48 ([13, Theorem 6.1]) *If $H \subseteq G$ is a spherical subgroup, then $N_G(H)/H$ is diagonalizable. In particular, if H contains U, then P/H is a torus where $P := N_G(H)$.*

Now, we come to an important geometric characterization of horospherical homogeneous spaces.

Recall that a continuous surjective map $p\colon E \to X$ of topological spaces is called a *fiber bundle* with fiber F (another topological space) if X can be covered with open subsets U such that there are homeomorphisms $\varphi\colon p^{-1}(U) \to U \times F$ in such a way that p agrees with the projection onto the first factor (see Fig. 9.5). It is said to be

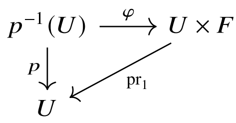

Fig. 9.5 Local trivialization of fiber bundles

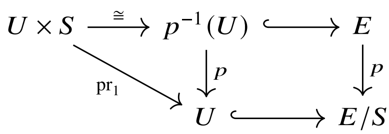

Fig. 9.6 Quotient by free torus action is locally trivial in Zariski topology

a *principal* Γ-*bundle*, for Γ a topological group, if in addition E is equipped with a continuous Γ-action $\Gamma \times E \to E$ preserving the fibers of p, i.e., if $y \in p^{-1}(x)$ for some $x \in X$, then $\gamma \cdot y \in p^{-1}(x)$ for any $\gamma \in \Gamma$, and acts freely and transitively on them.

Here is a crucial observation on principal torus bundles in algebraic geometry.

Lemma 49 *Let the torus S act freely on the normal irreducible variety E with good geometric quotient $p\colon E \to E/S$. Then for each $y \in E/S$ there exists an affine open neighbourhood $U \subseteq E/S$ of x such that the diagram in Fig. 9.6 commutes and the upper left isomorphism is S-equivariant.*

Exercise 50 Show that if an algebraic torus S acts freely on a normal irreducible variety E with good geometric quotient $p\colon E \to E/S$, then p admits Zariski open trivializations (i.e., prove Lemma 49). What if we replace S by a connected reductive G? Hint: For the second part of the question, you may want to consider the morphism $\phi\colon X \to Y$; $(A, B) \mapsto (\det(A), \operatorname{tr}(AB), \det(B))$ where $X = \{(A, B) \in \operatorname{Mat}(2 \times 2, \mathbb{C})^2 \colon \det(AB - BA) \neq 0, \operatorname{tr}(A) = \operatorname{tr}(B) = 0\}$ and $Y = \{y \in \mathbb{C}^3 \colon 4y_1y_3 - y_2^2 \neq 0\}$.

Let $p\colon E \to X$ be a Γ-principal bundle. Suppose that E and X are G-spaces for another topological group G. Then $p\colon E \to X$ is an *equivariant principal* Γ-*bundle* if p is equivariant (i.e., $p(g \cdot y) = g \cdot p(y)$ for any $g \in G$ and $y \in E$) and the two actions by Γ and G commute (usually one assumes that Γ acts from the right while G acts from the left).

Proposition 51 *If $H \subseteq G$ is a closed subgroup, then the following statements are equivalent:*

1. *H is horospherical, i.e., contains the unipotent radical of a Borel subgroup;*
2. *G/H is an (algebraic) equivariant principal torus bundle over a flag variety G/P where G naturally acts on G/H resp. G/P by left translations (the dimension of the torus fiber coincides with the* rank *of G/H);*
3. *$H = \bigcap_{\chi \in M} \ker \chi$ for some parabolic subgroup P of G and some sublattice $M \subseteq \mathfrak{X}(P)$.*

Furthermore, $P = N_G(H)$ *and* $P = TH = BH$ *for all maximal tori* T *of* B *contained in* P *and all Borel subgroups* B *of* G *contained in* P.

Proof (1) ⇒ (2): By Theorem 48, $S := P/H = N_G(H)/H$ is an algebraic torus. It acts on G/H by right-translations, i.e.,

$$S \times G/H \to G/H; (pH, xH) \mapsto xp^{-1}H.$$

It is straightforward to check that S acts freely on G/H, and thus the result follows by Lemma 49.

(2) ⇒ (1): Suppose that the fibers of the torus bundle $p\colon G/H \to G/P$ are isomorphic to the algebraic torus T. As the two actions by G and T commute, the morphisms $\varphi_t\colon G/H \to G/H; xH \mapsto t \cdot xH$ for $t \in T$ are G-equivariant automorphisms. It follows by [26, Proposition 1.8] that we may consider T as a subgroup of $N_G(H)/H$. Set $N := N_G(H)$. Let $\widetilde{T}$ be the preimage of T under the natural projection map $N \to N/H$. Note that $N \to N/H$ is a morphism of algebraic groups and that $\widetilde{T}$ is a closed subgroup of G. Since $p^{-1}(xP) \cong T$ for any $xP \in G/P$, it follows that a conjugate of P is contained in $\widetilde{T}$, and thus it contains a maximal unipotent subgroup U. As the natural projection morphism of algebraic groups $\widetilde{T} \to T$ maps unipotent elements on unipotent elements, it follows that U is in its kernel which implies that $U \subseteq H$.

(1) ⇔ (3) straightforwardly follows from Proposition 47 (3). □

Exercise 52 Show that SL_2/U_k is indeed a torus bundle over SL_2/B.

Exercise 53 Use the Luna–Vust theory to classify all spherical embeddings of SL_2/U where $U \subseteq \mathrm{SL}_2$ is a maximal unipotent subgroup. Draw the corresponding colored fans. Hint: You should find 6.

Proposition 54 ([20, Proposition 1.6]) *The assignment which associates to a horospherical subgroup* $H \subseteq G$ *the pair* $(\mathcal{M}, I)$ *(see Proposition 47) induces a bijection between horospherical subgroups of* G *and pairs* (M, J) *where* $J \subseteq S$ *and* $M \subseteq \mathfrak{X}(T)$ *is a sublattice such that* $\langle \check{\alpha}, \chi \rangle = 0$ *for any* $\alpha \in J$ *and all* $\chi \in M$.

The horospherical subgroup associated to a pair (M, J) as in Proposition 54 is given by $H = \bigcap_{\chi \in M} \ker \chi$ where $M \subseteq \mathfrak{X}(P_J)$.

Exercise 55 Use the combinatorial description of horospherical subgroups to classify those contained in SL_2.

A colored fan Σ is called *toroidal* if $\mathcal{F} = \varnothing$ for any $(\mathcal{C}, \mathcal{F}) \in \Sigma$. Observe that in the horospherical case toroidal fans coincide with fans in the toric sense. In this special case, we have the following explicit construction of horospherical toroidal varieties:

Proposition 56 ([20, Examples 1.13 (2)]) *If* $H \subseteq G$ *is a horospherical subgroup containing* U *and* Σ *is a toroidal fan, then the corresponding spherical embedding is* G*-equivariantly isomorphic to* $G \times_P X_\Sigma$ *where* X_Σ *denotes the toric variety corresponding to the fan* Σ *(with acting torus* P/H *where* $P = N_G(H)$*).*

In the situation of Proposition 56, recall that P acts on $G \times X_\Sigma$ by $p \cdot (g, x) := (gp^{-1}, pH \cdot x)$ with good geometric quotient $G \times_P X_\Sigma = (G \times X_\Sigma)/P$.

9.4 The Ring of Conditions of a Horospherical Variety

A good reference for the ring of conditions is the classical paper by De Concini and Procesi [7].

Let G be a connected complex algebraic group and $H \subseteq G$ a closed subgroup (not necessarily spherical). Consider the homogeneous space G/H.

Recall that two subvarieties $X, Y \subseteq G/H$ are said to *intersect properly* if either $X \cap Y = \varnothing$ or each irreducible component of the intersection $X \cap Y$ has dimension $\dim(X) + \dim(Y) - \dim(G/H)$. They are said to *intersect transversally* if the intersection $X \cap Y$ is smooth and has pure dimension $\dim(X) + \dim(Y) - \dim(G/H)$.

Theorem 57 (Kleiman's transversality theorem [12, Corollary 4]) *Let $X, Y \subseteq G/H$ be two irreducible subvarieties. The left translate of X by $g \in G$ we denote by gX.*

1. *There exists a dense open subset $U \subseteq G$ such that gX and Y intersect properly for each $g \in U$.*
2. *If X, Y are smooth, then there exists a dense open subset $U \subseteq G$ such that gX and Y intersect transversally for any $g \in U$.*

In particular, if X and Y have complementary dimensions (but are not necessarily smooth), the intersection $gX \cap Y$ consists of finitely many points and this number is constant for generic $g \in G$.

Remark 58 There is a slight strengthening of Kleiman's transversality theorem in [7, Sect. 6.1].

Recall that the free abelian group $\mathcal{Z}^k(G/H) = \bigoplus_{X \subseteq G/H} \mathbb{Z}X$, where the sum is over closed irreducible subvarieties of codimension k, is said to be the group of algebraic cycles of codimension k. Theorem 57 makes it possible to introduce an intersection pairing between groups of algebraic cycles of complementary codimensions

$$\mathcal{Z}^k(G/H) \qquad \text{and} \qquad \mathcal{Z}^{\dim(G/H)-k}(G/H).$$

It is enough to define it for irreducible cycles and then extend bilinearly:

$$\mathcal{Z}^k(G/H) \times \mathcal{Z}^{\dim(G/H)-k}(G/H) \to \mathbb{Z};$$
$$(X, Y) \mapsto (X \cdot Y) := \#(gX \cap Y) \quad \text{(for generic } g \in G).$$

Here $X, Y \subseteq G/H$ are assumed to be irreducible subvarieties.

Definition 59 Two algebraic cycles $X, Y \in \mathcal{Z}^k(G/H)$ are said to be equivalent, i.e., $X \sim Y$, if for any algebraic cycle of complementary codimension $Z \in \mathcal{Z}^{\dim(G/H)-k}(G/H)$ the intersection products are the same $(X \cdot Z) = (Y \cdot Z)$. We denote the group of equivalence classes by $C^k(G/H) := \mathcal{Z}^k(G/H)/\sim$ and consider it as the "group of conditions of dimension $\dim(G/H) - k$".

Clearly the intersection pairing factors through the equivalence relation, so that we obtain an intersection pairing on the groups of conditions: $C^k(G/H) \times C^{\dim(G/H)-k}(G/H) \to \mathbb{Z}$. So far $C^*(G/H) := \bigoplus_{k=0}^{\dim(G/H)} C^k(G/H)$ is only a group, but we want to introduce a product on it, so that it becomes a ring. Again, it is enough to define a product structure for classes of irreducible cycles $X \in \mathcal{Z}^k(G/H)$ and $Y \in \mathcal{Z}^l(G/H)$ and then extend bilinearly. Here is a naive approach:

Definition 60 Define the intersection product of $[X]$ and $[Y]$ where $X, Y \subseteq G/H$ are two irreducible subvarieties by $[X] \cdot [Y] := [gX \cap Y]$ for generic $g \in G$.

Unfortunately this definition of an intersection product may not be well-defined in general (see Exercise 61).

Exercise 61 Show that the naive definition of an intersection product of two irreducible subvarieties $X, Y \subseteq G/H$ is not well-defined in general. (Hint: Consider $G = (\mathbb{C}^3, +)$ acting on $\mathbb{A}^3$ by translations. Let $H = \{0\}$ and compute the intersection product of $X = \{y = 0\}$, $Y = \{x = yz\} \subseteq \mathbb{C}^3$.)

Proposition 62 ([7]) *For a flag variety G/P, the intersection product in Definition 60 is well-defined and the ring $C^*(G/P)$ can be identified with the Chow ring $A^*(G/P)$ or with the cohomology ring $H^*(G/P, \mathbb{Z})$.*

Led by this observation, De Concini and Procesi showed the remarkable fact that the intersection product of Definition 60 is well-defined on spherical homogeneous spaces. Let C be the set of smooth (or complete) spherical embeddings $G/H \hookrightarrow X$. This set C admits the partial ordering defined such that a spherical embedding $G/H \hookrightarrow X_1$ is greater than $G/H \hookrightarrow X_2$ if there exists an equivariant morphism $X_1 \to X_2$. De Concini's and Procesi's idea is to show that for any $X \in C$ and any algebraic cycle $Y \subseteq G/H$, there is an $X' \in C$ with $X \leq X'$ such that the closure $\overline{Y}$ of Y in X' intersects the boundary of the open G-orbit in X' properly. The existence of such a "good compactification" ensures that if one considers the embedding X' then we may always assume (up to generic translations by G) that the intersection with $\overline{Y}$ takes place in the open G-orbit G/H. To get an isomorphism of rings, we have to consider "good compactifications" of all algebraic cycles at once.

Theorem 63 ([7, Sect. 6.3]) *The intersection product from Definition 60 is well-defined on a spherical homogeneous space G/H and there is a canonical isomorphism of graded rings*

$$C^*(G/H) = \varinjlim_{X'} A^*(X') = \varinjlim_{X'} H^*(X', \mathbb{Z})$$

where the limit is taken over complete (or equivalently smooth) spherical embeddings $G/H \hookrightarrow X'$.

Remark 64 Any complete spherical embedding is dominated by a smooth projective toroidal one, and thus they form a cofinal set.

Exercise 65 Explicitly compute the ring of conditions for some spherical homogeneous spaces SL_2/H where $H \subseteq \mathrm{SL}_2$ is a spherical subgroup. (Hint: In this case, the rank of the spherical homogeneous space is bounded by 1, and thus there are only finitely many spherical embeddings, so that we can straightforwardly compute the direct limit of cohomology rings.)

Exercise 66 Use the ring of conditions of $\mathrm{Gr}(2, 4)$ to solve the "4-lines problem".

9.4.1 The Horospherical Case

From now on let $H \subseteq G$ be a horospherical subgroup containing the unipotent radical U of a Borel subgroup B. Set $P := N_G(H)$ which is a parabolic subgroup containing B.

Any character $\alpha \in \mathfrak{X}(P)$ induces an action of P on the affine line $\mathbb{C}_\alpha$ by $p \cdot x = \alpha(p)x$. We obtain an action of P on $G \times \mathbb{C}_\alpha$ by $p \cdot (g, x) = (gp^{-1}, \alpha(p)x)$. The geometric quotient by this action exists and is denoted by $G \times_P \mathbb{C}_\alpha$, i.e., $G \times_P \mathbb{C}_\alpha = (G \times \mathbb{C}_\alpha)/P$. For the equivalence classes in $G \times_P \mathbb{C}_\alpha$ we write $g \star x$. The projection map $G \times_P \mathbb{C}_\alpha \to G/P$; $g \star x \mapsto gP$ yields an equivariant line bundle on G/P where G acts on the left, i.e., $g' \cdot (g \star x) = (g'g) \star x$. We write $\delta(\alpha) := G \times_P \mathbb{C}_\alpha$ and note that these bundles are usually called homogeneous fiber bundles (see [26, Section 2.1]). If we compose the map $\mathfrak{X}(P) \to \mathrm{Pic}(G/P)$ with the inclusion $\mathcal{M} \subseteq \mathfrak{X}(P)$, we obtain a map $\delta \colon \mathcal{M} \to \mathrm{Pic}(G/P)$.

The following statement combinatorially describes the cohomology ring of smooth projective toroidal horospherical varieties. It is a special case of a more general result.

Theorem 67 ([21, Theorem 1.2]) *Let X_Σ be a smooth projective toroidal horospherical variety defined by an (uncolored) fan Σ with rays $\rho_1, \ldots, \rho_n$. Let $v_1, \ldots, v_n \in \mathcal{N}$ be the primitive vectors along the rays ρ_i. Then the cohomology ring $H^*(X_\Sigma, \mathbb{Q})$ is isomorphic as an $H^*(G/P, \mathbb{Q})$-algebra to the quotient of $H^*(G/P, \mathbb{Q})[x_1, \ldots, x_n]$ by the sum of ideals*

$$\left\langle x_{j_1} \cdots x_{j_k} \colon \rho_{j_1}, \ldots, \rho_{j_k} \text{ do not span a cone of } \Sigma \right\rangle + \left\langle c_1(\delta(m)) - \sum_{i=1}^{n} \langle v_i, m \rangle x_i \colon m \in \mathcal{M} \right\rangle,$$

where $c_1(\delta(m)) \in H^2(G/P, \mathbb{Z})$ denotes the first Chern class of the line bundle $\delta(m)$.

Note that the first ideal in the sum of ideals in Theorem 67 corresponds to the Stanley-Reisner ideal of the toric variety. The challenge is to find a good description of the ring in Theorem 67 as we want to take the direct limit over all smooth projective toroidal fans Σ.

The following approach is inspired by [5]. To keep notation simple, set $\mathcal{M}_{\mathbb{Q}} = \mathcal{M} \otimes \mathbb{Q}$ and $\mathcal{N}_{\mathbb{Q}} = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{M}, \mathbb{Q})$. Let Σ be a smooth projective toroidal fan in $\mathcal{N}_{\mathbb{Q}}$. A map $f : \mathcal{N}_{\mathbb{Q}} \to \mathbb{Q}$ is *piecewise polynomial* if for any $\sigma \in \Sigma$, the map $f|_{\sigma} : \sigma \to \mathbb{Q}$ extends to a polynomial function on the linear space $\mathrm{span}_{\mathbb{Q}}\{\sigma\}$, i.e., a piecewise polynomial function f on Σ is a collection of compatible polynomial functions $f_{\sigma} : \sigma \to \mathbb{Q}$. In particular, such a function is continuous. We denote by $\mathcal{R}_{\Sigma}$ the set of all piecewise polynomial functions on Σ which is a ring under pointwise addition and multiplication. Let $S^*(\mathcal{M}_{\mathbb{Q}})$ be the symmetric algebra of the $\mathbb{Q}$-vector space $\mathcal{M}_{\mathbb{Q}}$. Recall that $S^*(\mathcal{M}_{\mathbb{Q}})$ can be naturally identified with the polynomial functions on $\mathcal{N}_{\mathbb{Q}}$. Note that $\mathcal{R}_{\Sigma}$ is a positively graded $\mathbb{Q}$-algebra with graded subalgebra $S^*(\mathcal{M}_{\mathbb{Q}})$. Indeed, any piecewise polynomial function $f = (f_{\sigma})_{\sigma \in \Sigma}$ uniquely decomposes into a sum of homogeneous piecewise polynomial functions.

Exercise 68 Let Σ be a smooth projective toroidal fan in $\mathcal{N}_{\mathbb{Q}}$. Show that for any ray ρ there is a piecewise linear function φ_{ρ} on Σ which vanishes on all the other rays and satisfies $\varphi(u_{\rho}) = 1$ where u_{ρ} is the primitive ray generator in $\mathcal{N}$ of the ray ρ.

Let us write $\Sigma(1)$ for the set of rays of a fan Σ and u_{ρ} for the primitive generator in $\mathcal{N}$ of the ray $\rho \in \Sigma(1)$.

Lemma 69 *If Σ is a smooth projective toroidal fan, then $\{\varphi_{\rho} : \rho \in \Sigma(1)\}$ (where φ_{ρ} is defined in Exercise 68) forms a basis of $\mathcal{R}^1_{\Sigma}$ the space of piecewise linear functions on Σ.*

Exercise 70 Let Σ be a smooth projective toroidal fan. Show that $\mathcal{R}_{\Sigma}$ is isomorphic to the *Stanley-Reisner algebra* R_{Σ}, i.e., the quotient ring of $\mathbb{Q}[T_{\rho} : \rho \in \Sigma(1)]$ by the relations $\prod_{i=1}^{k} T_{\rho_i} = 0$ whenever $\rho_1, \ldots, \rho_k$ are distinct rays which do not generate a cone of Σ. (Hint: Clearly $\prod_{i=1}^{k} \varphi_{\rho_i} = 0$ whenever $\rho_1, \ldots, \rho_k$ do not generate a cone of Σ. Therefore, there is a unique algebra homomorphism from R_{Σ} to $\mathcal{R}_{\Sigma}$, which sends T_{ρ} to φ_{ρ}. Show that this map is an isomorphism.)

We can now reformulate Theorem 67:

Proposition 71 *Let X_{Σ} be a smooth projective toroidal horospherical variety defined by an (uncolored) fan Σ. Then the cohomology ring $H^*(X_{\Sigma}, \mathbb{Q})$ is isomorphic as an $H^*(G/P, \mathbb{Q})$-algebra to the quotient of $H^*(G/P, \mathbb{Q}) \otimes \mathcal{R}_{\Sigma}$ by the ideal*

$$\left\langle c_1\left(\delta(m)\right) \otimes 1 - \sum_{\rho \in \Sigma(1)} \langle u_{\rho}, m \rangle 1 \otimes \varphi_{\rho} : m \in \mathcal{M} \right\rangle = \left\langle c_1(\delta(m)) \otimes 1 - 1 \otimes \langle \cdot, m \rangle : m \in \mathcal{M} \right\rangle,$$

where $\langle \cdot, m \rangle \in S^*(\mathcal{M}_{\mathbb{Q}})$ is a (piecewise) linear function on Σ.

Proof The statement is a reformulation of Theorem 67 except the equality of the two ideals which remains to be shown. Recall from Lemma 69 that the set of piecewise linear functions $\{\varphi_\rho : \rho \in \Sigma(1)\}$ (where φ_ρ are defined in Exercise 68) forms a basis of $\mathcal{R}^1_\Sigma$. Then the (piecewise) linear function $\langle \cdot, m\rangle$ for $m \in \mathcal{M}$ can be expressed as a linear combination in this basis, namely $\langle \cdot, m\rangle = \sum_{\rho\in\Sigma(1)} \langle u_\rho, m\rangle \varphi_\rho$. □

We denote by $\mathcal{R}$ the set of all piecewise polynomial functions on smooth projective toroidal fans in $\mathcal{N}_\mathbb{Q}$, i.e., $\mathcal{R} = \bigcup_\Sigma \mathcal{R}_\Sigma$ where the union is taken over all smooth projective toroidal fans Σ.

Theorem 72 *We have that*

$$C^*(G/H) \otimes \mathbb{Q} \cong \big(H^*(G/P, \mathbb{Q}) \otimes \mathcal{R}\big) \,/\, \langle c_1(\delta(m)) \otimes 1 - 1 \otimes \langle \cdot, m\rangle : m \in \mathcal{M}\rangle \,,$$

where $\langle \cdot, m\rangle \in S^*(\mathcal{M}_\mathbb{Q})$ *is a piecewise linear function on any smooth projective toroidal fan.*

Proof For convenience, let us write $A := H^*(G/P, \mathbb{Q})$.

By Theorem 63, we have

$$C^*(G/H) \otimes \mathbb{Q} = \left(\varinjlim_{X'} H^*(X', \mathbb{Z})\right) \otimes \mathbb{Q} = \varinjlim_{X'} H^*(X', \mathbb{Q})$$

where the limit is taken over all smooth projective toroidal embeddings of G/H which is a directed set. Indeed, for any two smooth projective toroidal embeddings with corresponding fans Σ_1, Σ_2, we can find a third smooth projective toroidal fan Σ which refines both fans Σ_1 and Σ_2. We introduce the relation $\Sigma \preceq \Sigma'$ whenever Σ' refines Σ. Suppose $\Sigma \preceq \Sigma'$, so that we obtain an equivariant map $X_{\Sigma'} \to X_\Sigma$. Our goal is to understand how the representation of cohomology rings given in Proposition 71 behaves under this map. By Proposition 71, the cohomology rings (as A-algebras) are generated by classes of divisors, so that the map corresponding to $X_{\Sigma'} \to X_\Sigma$ is given by pulling back divisors which in turn induces the natural inclusion $\mathcal{R}_\Sigma \subseteq \mathcal{R}_{\Sigma'}$. Let $I_\Sigma := \langle c_1(\delta(m)) \otimes 1 - 1 \otimes \langle \cdot, m\rangle : m \in \mathcal{M}\rangle \subseteq A \otimes \mathcal{R}_\Sigma$. Similarly define $I_{\Sigma'}$ in $A \otimes \mathcal{R}_{\Sigma'}$. As $I_\Sigma \subseteq I_{\Sigma'}$, we obtain the natural map $\mu_{\Sigma,\Sigma'} : (A \otimes \mathcal{R}_\Sigma)/I_\Sigma \to (A \otimes \mathcal{R}_{\Sigma'})/I_{\Sigma'}$. Then $((A \otimes \mathcal{R}_\Sigma)/I_\Sigma, \mu_{\Sigma,\Sigma'})$ is the direct system yielding the direct limit $\varinjlim H^*(X', \mathbb{Q})$. Moreover, we obtain two more direct systems, namely $(I_\Sigma, I_\Sigma \subseteq I_{\Sigma'})$ and $(A \otimes \mathcal{R}_\Sigma, A \otimes \mathcal{R}_\Sigma \subseteq A \otimes \mathcal{R}_{\Sigma'})$ (for $\Sigma \preceq \Sigma'$). Indeed, we obtain a direct system of exact sequences:

$$0 \to I_\Sigma \to A \otimes \mathcal{R}_\Sigma \to (A \otimes \mathcal{R}_\Sigma)/I_\Sigma \to 0.$$

The statement follows by the fact that taking direct limits in the category of modules is an exact functor, $\varinjlim A \otimes \mathcal{R}_\Sigma = A \otimes \mathcal{R}$, and $\varinjlim I_\Sigma = I$, where I denotes the ideal in the statement. □

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