Chapter 3
Lattice Distances in 3-Dimensional Quantum Jumps

Mónica Blanco

Abstract For Q a lattice polytope and $x \notin Q$ a lattice point, we say that (Q, x) is a quantum jump if $\mathrm{conv}(Q \cup \{x\})$ contains exactly one more lattice point than Q. Usually this can only happen when the lattice distance between x and Q is somehow bounded. In this paper I collect several results and information on the bound for that distance in 3-dimensional quantum jumps, and the consequences on the distance between the boundary of a polytope and its interior lattice points.

Keywords Lattice polytope · Lattice distance · Quantum jump · Interior points

3.1 Introduction

Throughout my research on classifying lattice 3-polytopes by their number of lattice points [2–4] there has been a recurrent situation: *suppose there is a lattice polytope Q, and a lattice point $x \notin Q$, what can I say about x with respect to Q so that* $\mathrm{conv}(Q \cup \{x\})$ *does not contain more lattice points other than those of Q and x*? Usually the answer had to do with the distance from x to Q being bounded.

In order to explain things more formally we need to introduce notation and some basic definitions. A *lattice* point is an element of $\mathbb{Z}^d$, and a *lattice polytope* is the convex hull of finitely many lattice points. We write lattice d-polytope if the polytope is d-dimensional. Two polytopes P and Q are *equivalent*, or *unimodularly equivalent*, if there exists a *unimodular transformation* that maps one to the other. That is, if there exists an affine map $t : \mathbb{R}^d \to \mathbb{R}^d$ such that $t(\mathbb{Z}^d) = \mathbb{Z}^d$ and $t(P) = Q$. The *size* of a lattice polytope $P \subset \mathbb{R}^d$ is the number of lattice points in it. An affine functional $f : \mathbb{R}^d \to \mathbb{R}$ is *integer* if $f(\mathbb{Z}^d) \subseteq \mathbb{Z}$ and it is *primitive* if $f(\mathbb{Z}^d) = \mathbb{Z}$. The *(lattice) width* of a lattice d-polytope $P \subset \mathbb{R}^d$ *with respect to* an integer functional f is the length of $f(P) \subset \mathbb{R}$, and the *width* of P is the minimum among those, for f non-constant.

M. Blanco (✉)
Department of Mathematics, Statistics and Computation, University of Cantabria, Avda. Los Castros s/n, 39005 Santander, Cantabria, Spain
e-mail: m.blanco.math@gmail.com

A. M. Kasprzyk and B. Nill (eds.), *Interactions with Lattice Polytopes*,
Springer Proceedings in Mathematics & Statistics 386,
https://doi.org/10.1007/978-3-030-98327-7_3

Finally, for $S \subset \mathbb{R}^d$, we denote by $\text{conv}(S)$ and $\text{aff}(S)$ the *convex* and *affine hulls* of S. In particular, an affine subspace $S \subset \mathbb{R}^d$ is *lattice* if $\text{aff}(S \cap \mathbb{Z}^d) = S$.

Let us now introduce the main two definitions in this paper:

Definition 1 The *lattice distance* between a lattice hyperplane $H \subset \mathbb{R}^d$ and a lattice point $x \in \mathbb{Z}^d$ is $\text{dist}(x, H) := |f(x)|$, where f is a primitive functional with $f(H) = 0$.

Definition 2 Let $Q \subset \mathbb{R}^d$ be a lattice polytope, not necessarily full-dimensional, and let $x \in \mathbb{Z}^d \setminus Q$. We say that the pair (Q, x) is a *quantum jump* if

$$\text{conv}\left(Q \cup \{x\}\right) \cap \mathbb{Z}^d = \left(Q \cap \mathbb{Z}^d\right) \cup \{x\}.$$

More generally, let $Q, R \subset \mathbb{R}^d$ be lattice polytopes, not necessarily full-dimensional, with $Q \cap R = \emptyset$. We say that the pair (Q, R) is a *quantum union* if

$$\text{conv}\left(Q \cup R\right) \cap \mathbb{Z}^d = \left(Q \cap \mathbb{Z}^d\right) \cup \left(R \cap \mathbb{Z}^d\right).$$

That is, if the lattice points of $\text{conv}(Q \cup R)$ are either in Q or in R.

The name of *quantum jump* was first used by Bruns, Gubeladze, and Michałek [5]. Notice that they restrict the concept of quantum jump (Q, x) for when both Q and $\text{conv}(Q \cup \{x\})$ are full-dimensional and *normal*. Remember that a lattice d-polytope Q is *normal* if, for all $k \in \mathbb{N}$, every lattice point in kQ can be written as the sum of k lattice points in Q.

Now, if we want to take a look at the distance of quantum jumps, we first need to define the distance between a point and a polytope. Following Definition 1, they are well and naturally defined the following distances:

Definition 3 1. Let $Q \subset \mathbb{R}^d$ be a lattice $(d-1)$-polytope, and let $x \in \mathbb{Z}^d \setminus \text{aff}(Q)$, then

$$\text{dist}(x, Q) := \text{dist}(x, \text{aff}(Q)).$$

2. Let $H_1, H_2 \subset \mathbb{R}^d$ be parallel lattice hyperplanes ($H_1 \cap H_2 = \emptyset$), then

$$\text{dist}(H_1, H_2) := \text{dist}(x, H_2), \qquad \text{for any } x \in H_1.$$

3. Let $\ell_1, \ell_2 \subset \mathbb{R}^3$ be lattice lines such that $\text{aff}(\ell_1 \cup \ell_2) = \mathbb{R}^3$, then

$$\text{dist}(\ell_1, \ell_2) := \text{dist}(H_1, H_2),$$

where H_1, H_2 are the unique pair of parallel lattice hyperplanes such that $\ell_i \subset H_i$.

4. Let $s_1, s_2 \subset \mathbb{R}^3$ be lattice segments such that $\text{aff}(s_1 \cup s_2) = \mathbb{R}^3$, then

$$\text{dist}(s_1, s_2) := \text{dist}\left(\text{aff}(s_1), \text{aff}(s_2)\right).$$

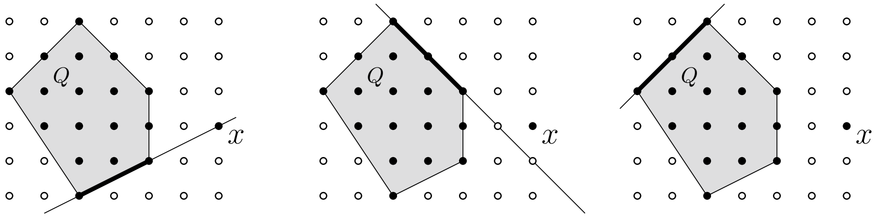

Fig. 3.1 The three figures show different facets of a lattice polygon Q, the hyperplane they are contained in, and the lattice point x. Only the facet in the middle figure is visible from x

Notice that the width and the distance are heavily related. In broad terms, the distance between two lower-dimensional objects $R_1, R_2 \subset \mathbb{R}^d$ with $\text{aff}(R_1 \cup R_2) = \mathbb{R}^d$ is the width of $\text{conv}(R_1 \cup R_2)$ with respect to a specific functional that is determined by the relative position between R_1 and R_2. In general, if $R := \text{conv}(R_1 \cup R_2) \subset \mathbb{R}^d$ is not full-dimensional, the distance between R_1 and R_2 is measured in the lattice $\text{aff}(R) \cap \mathbb{Z}^d \cong \mathbb{Z}^{\dim(R)}$. In the case of lattice segments, we call (lattice) *length* of a segment the distance between its two endpoints (vertices). Notice that a lattice segment of length k has exactly $k + 1$ lattice points. We say that a segment is *primitive* if it has length one.

Now, the distance that is not necessarily well-defined is the distance from a point to a full-dimensional polytope. This notion will be written in terms of the distance to the *visible* facets (see Fig. 3.1):

Definition 4 Let $Q \subset \mathbb{R}^d$ be a lattice d-polytope, $F \subset Q$ a facet of Q and $x \in \mathbb{Z}^d \setminus Q$. F is *visible* from x if $\text{aff}(F)$ strictly separates x from Q.

Definition 5 Let $Q \subset \mathbb{R}^d$ be a lattice d-polytope and let $x \in \mathbb{Z}^d \setminus Q$.

1. The *minimum distance* between x and Q is

$$d_x(Q) := \min \{\text{dist}(x, \text{aff}(F)) \mid F \text{ facet visible from } x\}.$$

2. The *maximum distance* between x and Q is

$$D_x(Q) := \max \{\text{dist}(x, \text{aff}(F)) \mid F \text{ facet visible from } x\}.$$

See Fig. 3.2 for a 2-dimensional example of the maximum and minimum distances between a point and a polygon. Notice that $D_x(Q)$ is the *height of x over Q* as in [5, Definition 4.1].

Let us see what we know about the distance of quantum jumps in each dimension. For this, we can also think of a quantum jump as follows: any d-dimensional quantum jump is of the form (P^v, v), for $P \subset \mathbb{R}^d$ a lattice d-polytope, $v \in \text{vert}(P)$ a vertex of P and $P^v := \text{conv}(P \cap \mathbb{Z}^d \setminus \{v\})$. Notice that the dimension of P^v can be d or $d - 1$.

For example, the lattice distance in quantum jumps of dimension ≤ 2 is always one:

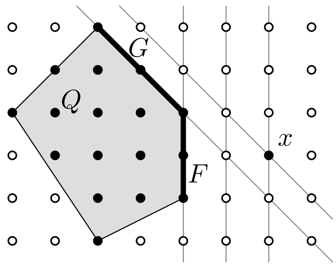

Fig. 3.2 A lattice polygon Q, a lattice point x, and the two facets of Q visible from x, F and G. In this case, $\text{dist}(x, F) = 2$ and $\text{dist}(x, G) = 1$. Hence $d_x(Q) = 1$ and $D_x(Q) = 2$

Lemma 6 *Let P be a lattice polytope of dimension $d \in \{1, 2\}$, and let v be a vertex of P.*

If P^v is of dimension $d - 1$, then $\text{dist}(v, P^v) = 1$ *and, if P^v is of dimension d, we have $d_v(P^v) = D_v(P^v) = 1$.*

To understand the idea in general: let P^v be d-dimensional. For any $(d-1)$-dimensional face of P^v that is visible from v, chose S an empty $(d-1)$-dimensional simplex in it. Since (P^v, v) is a quantum jump, so is (S, v), which implies that the convex hull of S and v is an empty d-simplex. Remember that an *empty simplex of dimension d* is a lattice d-polytope with $d+1$ vertices and such that those vertices are its only lattice points.

In the cases of $d = 1, 2$, any empty simplex has to be unimodular, hence the *vertex-facet distance* (lattice distance between a vertex and the only facet that does not contain it) is always 1. In dimension 3 things get more complicated since we have empty tetrahedra of arbitrarily high volume, and hence arbitrarily high vertex-facet distance (e.g. Reeve tetrahedra [8]). That is, quantum jumps between a unimodular triangle and a lattice point that is at arbitrarily high lattice distance from it.

In Sect. 3.2 of this paper I put together some information on the lattice distance of 3-dimensional quantum jumps (Q, x) that derives partially from previous research [2–4]. We distinguish when Q is 2 or 3-dimensional:

1. If Q is 2-dimensional (Sect. 3.2.1) it so happens that the classifications of lattice 3-polytopes of size 5 and 6 [2, 3], together with a suitable classification of lattice polygons, give all the information there is to know about the distance from Q to x. It can be summarized as follows:

Theorem 7 (see Corollary 14) *Let $Q \subset \mathbb{R}^3$ be a lattice polygon, and let $x \in \mathbb{Z}^3 \setminus$* $\text{aff}(Q)$ *such that (Q, x) is a quantum jump. Then, the lattice distance from x to Q is at most* 3 *unless Q is a lattice triangle of width one, in which case the distance is unbounded.*

2. As a direct consequence of the results of the previous section, in Sect. 3.2.2 we have the following result on the distance of a quantum union of lattice segments:

Theorem 8 (see Corollary 16) *Let $s, t \subset \mathbb{R}^3$ be lattice segments with* $\text{aff}(s \cup t) = \mathbb{R}^3$ *such that (s, t) is a quantum union. Then, the lattice distance from s to t is one, unless both s and t are primitive, in which case it is unbounded.*

3. For the case of Q being 3-dimensional (Sect. 3.2.3), Example 18 shows that there exist quantum jumps of this type at arbitrarily high distance. We also look at all the lattice 3-polytopes P of size 11 and width > 1 (database of [4]), and for each vertex v of P such that P^v is 3-dimensional we compute the minimum and maximum distances from v to P^v. Looking at the numbers one can easily see that there is no hope in trying to bound these distances, having very high numbers for both the minimum and the maximum distances.

Finally, in Sect. 3.3 I use all the information gathered in the previous section to study the distance between the boundary of a polytope and its interior. More precisely, for $P \subset \mathbb{R}^3$ a lattice 3-polytope we look at the distance between a lattice point or segment in ∂P (the boundary of P) and the *inner lattice polytope of* P, which is $I_P := \text{conv}\{\text{int}(P) \cap \mathbb{Z}^3\}$. Notice that I_P together with a point (or segment) of the boundary is always a quantum jump (or union). The definition of inner polytope also applies to rational polytopes.

We only look at inner polytopes I_P of size ≥ 3 (see Remark 20), and we separate cases according to its dimension:

1. I_P has dimension 1, that is, it is a lattice segment of length ≥ 2. In this case we look at the distance between I_P and a segment in the boundary. By Corollary 16, this distance must be one, leading to:

Theorem 9 (see Theorem 21) *The projection of P in the direction of the segment I_P is a* reflexive *polygon (polygon with a unique interior lattice point).*

2. I_P has dimension 2. In Sect. 3.3.2 we prove a specific property that a polygon has to satisfy in order to appear as the inner polygon of a 3-dimensional lattice polytope (see Theorem 23). Together with the results of Corollary 14 we obtain:

Corollary 10 (see Corollary 24) *For I_P of dimension* 2 *and size* ≥ 12*, the distance from any boundary point of P to I_P is at most* 1.

3. I_P has dimension 3. Again we look at the classification of lattice 3-polytopes of size ≤ 11 and width > 1 [4], take the polytopes with 3-dimensional inner polytope, and look at the minimum and maximum distances from any vertex to the inner polytope. In Sect. 3.3.3 we simply collect some information on the numbers obtained, without exploring it further. This time the values look more promising, since the largest value that appears is a maximum distance of 6, and in very high proportion the maximum and minimum distances are 1.

For future work one could try and complete the results on distances between a lattice point of the boundary of a polytope, and its 2 or 3-dimensional inner polytope. For the inner polytope of dimension 2, it is left to explore the cases when I_P has up to 11 lattice points. This seems perfectly doable with the help of the classification of polygons of Proposition 11, together with the results of Sect. 3.2.1. On the other hand, for I_P of dimension 3, one would have to identify in the used database all the polytopes that yield maximum and minimum distances equal to 1 and try to derive the properties they have as opposed to those that yield larger distances. One would have then to try and extend this to lattice 3-polytopes of size larger than 11.

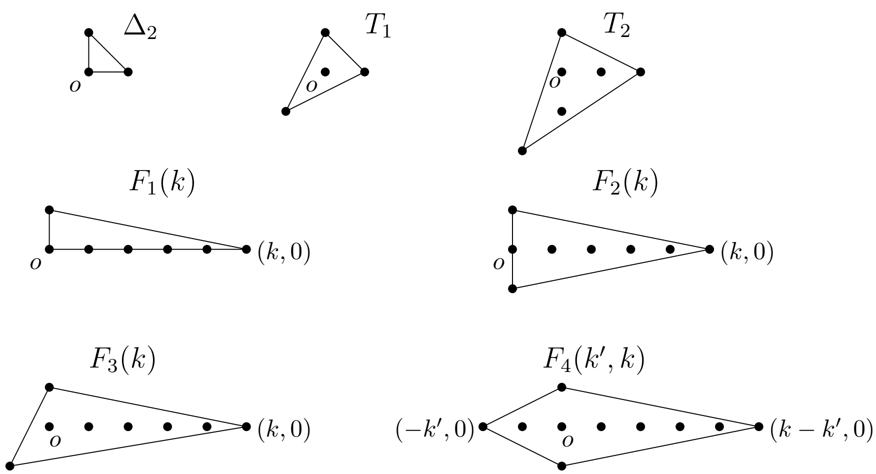

Fig. 3.3 The complete classification of lattice polygons that do not contain a unit square

3.2 Distances in 3-Dimensional Quantum Jumps or Unions

Let $Q \subset \mathbb{Z}^3$ be a lattice polytope and let $x \in \mathbb{Z}^3 \setminus Q$. We study the distance between x and Q, provided that (Q, x) is a quantum jump.

3.2.1 Quantum Jumps (Q, x) with Q of Dimension 2

We first see at what distance can a lattice point be from a lattice polygon, so that they form a quantum jump. For this, we first classify lattice polygons in a way that is suitable distance-wise. In the following lemma, we call *unit square* any lattice polygon unimodularly equivalent to $[0, 1]^2$.

Proposition 11 *Let $Q \subset \mathbb{R}^2$ be a lattice polygon. Then Q either contains a unit square or is equivalent to one of the following configurations:*

1. Δ_2*, the unimodular triangle;*
2. $T_1 := \mathrm{conv}\{(1, 0), (0, 1), (-1, -1)\}$*, the unique terminal triangle;*
3. $T_2 := \mathrm{conv}\{(2, 0), (0, 1), (-1, -2)\}$*, a clean triangle with three non-collinear interior lattice points;*
4. $F_1(k) := \mathrm{conv}\{(0, 0), (0, 1), (k, 0)\}$*, for* $k \geq 2$*;*
5. $F_2(k) := \mathrm{conv}\{(0, 1), (0, -1), (k, 0)\}$*, for* $k \geq 2$*;*
6. $F_3(k) := \mathrm{conv}\{(-1, -1), (0, 1), (k, 0)\}$*, for* $k \geq 2$*; or*
7. $F_4(k', k) := \mathrm{conv}\{(0, 1), (0, -1), (-k', 0), (k - k', 0)\}$*, for* $0 < k' < k$*.*

See Fig. 3.3 for a depiction of the polygons of Proposition 11. For its proof, let us first establish the following notation.

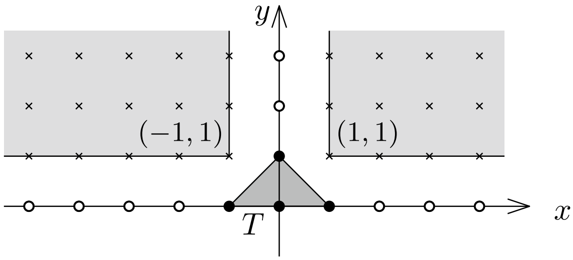

Fig. 3.4 The situation in the proof of Proposition 11 when Q contains three collinear lattice points in a facet

Remark 12 Let $P \subset \mathbb{R}^d$ be a polytope, and let $R \subsetneq P$. If a point $x \in \mathbb{R}^d$ is not a point of P, then $P \cap C_x(R) = \emptyset$, where $C_x(R) := x - \mathbb{R}_{\geq 0}\big(R - x\big)$. This fact follows trivially from convexity of polytopes. We use it whenever we want to determine a polytope P, and we know R a subset of P and x a point not in P.

Proof (Proof of Proposition 11) For Q with 3 lattice points, we have $Q \cong \Delta_2$. If Q has 4 lattice points, then $Q \cong T_1$, $Q \cong F_1(2)$ or Q is equivalent to the unit square.

So assume for the rest of the proof that Q has size at least 5. In particular we know that Q has 3 collinear lattice points and we can assume, without loss of generality, that these are $(-1, 0)$, $(0, 0)$ and $(1, 0)$.

If Q contains a unit square, we have finished. Assume for the rest of the proof that Q does not contain a unit square. That is, we have Q a lattice polygon of size ≥ 5, containing the lattice points $(-1, 0)$, $(0, 0)$ and $(1, 0)$, and not containing a unit square. Since Q is 2-dimensional, it has some lattice point outside of the line $\ell := \{y = 0\}$ and, by Lemma 6, we can choose one in either $\ell_+ := \{y = 1\}$ or $\ell_- := \{y = -1\}$. Without loss of generality, we assume that the point $(0, 1) \in \ell_+$ is in Q. That is, the triangle $T := \operatorname{conv}\{(-1, 0), (1, 0), (0, 1)\} \subset Q$.

Let us now distinguish the cases according to whether Q has three collinear lattice points in a facet or not.

1. Suppose the three collinear points $(-1, 0)$, $(0, 0)$ and $(1, 0)$ are in a facet of Q. Then $Q \subset \{y \geq 0\}$. Since Q does not contain a unit square, the points $(-1, 1)$ and $(1, 1)$ cannot be in Q. Moreover, by Remark 12 this implies that no point in the affine cones $C_{(-1,1)}(T)$ and $C_{(1,1)}(T)$ is in Q. See Fig. 3.4. That means that the only lattice points that can lie in $Q \setminus T$ are in the following sets:

$$A := \big\{(i, 0), i \in \mathbb{Z} \setminus [-1, 1]\big\}, \quad B := \big\{(0, j), j \in \mathbb{Z}, j \geq 2\big\}$$

   Q can contain points of A or points of B, but in order for $(1, 1)$ and $(-1, 1)$ not to be in Q, it cannot contain points of A and B at the same time. Adding points of A to T gives rise to polygons of the type $F_1(k)$, and adding points of B gives rise to $F_2(k)$.
2. If no facet of Q contains three collinear points, then the origin, which is in the relative interior of the segment $\operatorname{conv}\{(-1, 0), (1, 0)\} \subset Q$, must be an interior point of Q. So Q must contain some lattice point in $\{y < 0\}$ and, by Lemma 6,

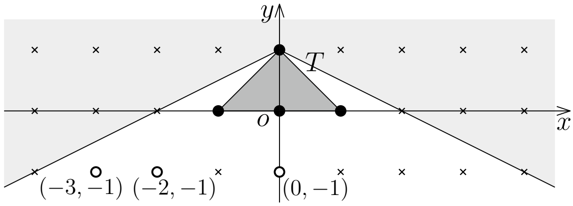

Fig. 3.5 The situation in the proof of Proposition 11 when Q does not contains three collinear lattice points in a facet

it contains some point of ℓ_-. Let us denote this point by p_-. We can assume without loss of generality that $(-2, 0), (2, 0) \notin \text{conv}(T \cup \{p_-\})$, or else we can simply choose a different triple of collinear lattice points in $\{y = 0\}$. That is, $p_- \in \{\pm 3, \pm 2, \pm 1, 0\} \times \{-1\}$. By symmetry of the already established points with respect to the line $\{x = 0\}$, we can assume that $p_- \in \{x \leq 0\}$. Also we know that $p_- \neq (-1, -1)$ since Q does not contain a unit square. The three remaining possibilities $p_- \in \{(-3, -1), (-2, -1), (0, -1)\}$ are depicted in Fig. 3.5. Let $T' := \text{conv}(T \cup \{p_-\}) \subseteq Q$, and let us study the three options for p_-.

a. $p_- = (-3, -1)$. We can apply the unimodular transformation $(x, y) \mapsto (x - y + 1, y)$ so that T' is mapped to $F_3(2)$ and assume now that $F_3(2) \subseteq Q$. See the left-most picture in Fig. 3.6. The lattice points $(-1, 0)$, $(0, -1)$ and $(1, 1)$ cannot lie in Q, or else it would contain a unit square. This already implies that the cones $C_{(-1,0)}(F_3(2))$, $C_{(0,-1)}(F_3(2))$ and $C_{(1,1)}(F_3(2))$ do not intersect Q. With that, the only lattice points that can be in Q are the points $(r, 0)$, with $r > 2$. We can take as many as wanted and this gives rise to configurations $F_3(k)$.
b. $p_- = (-2, -1)$. Again we apply the same unimodular transformation so that T' is, in this case, mapped to $F_2(2) \subseteq Q$. In order for Q not to have unit squares, no point in $\{-1, 1\}^2$ can lie in Q which, after removing the corresponding lattice cones, leaves the following possibilities for further points of Q:

$$A := \{(-1, 2)\}, \quad B := \{(-1, -2)\},$$

$$C := \{(r, 0), r \in \mathbb{Z}, r \geq 3\}, \quad D := \{(s, 0), s \in \mathbb{Z}, s \leq -1\}$$

The points in A and B cannot be in Q at the same time, and each gives rise to a configuration equivalent to T_2. The points in C or D cannot be in Q at the same time as the points in A or B. If Q has points of D, we have configurations $F_4(k', k)$ and, if Q only has points of C we get configurations $F_2(k)$.
c. $p_- = (0, -1)$. In this case, $T' = F_4(1, 2)$. After excluding the points in the cones with apex in $\{-1, 1\}^2$, Q can have other lattice points in:

$$A := \{(r, 0), r \in \mathbb{Z} \setminus [-1, 1]\}, \quad B := \{(0, s), s \in \mathbb{Z} \setminus [-1, 1]\}$$

Q cannot have points of A and B at the same time, and adding to T' points of either A or B gives rise to configurations equivalent to $F_4(k', k)$. □

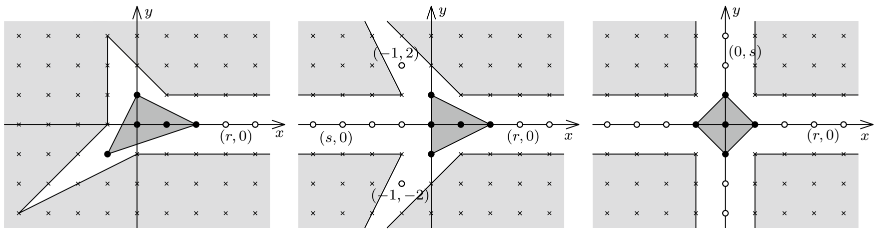

Fig. 3.6 The three possible polygons T' in the proof of Proposition 11. In each figure, the dark gray area is the polygon T' (or equivalent). The light gray area is the union of the cones that do not intersect Q. Black dots are lattice points of T', crosses are lattice points that cannot be in Q, and white dots are the possible lattice points of $Q \setminus T'$

Let us now take that classification and see what are the conditions on the coordinates of a lattice point $x \in \mathbb{Z}^3$ so that a polygon Q and $x \notin \mathrm{aff}(Q)$ form a quantum jump.

Lemma 13 *Let $Q \subset \mathbb{R}^2 \times \{0\}$ be a lattice polygon and let $x = (a, b, c) \in \mathbb{Z}^3$ be a lattice point with $c \neq 0$ and such that (Q, x) is a quantum jump. Then:*

1. *if $Q = \Delta_2$, then at least one of the following happens:*
   - *i.* $a \equiv 1 \pmod{c}$ *and* $\gcd(b, c) = 1$;
   - *ii.* $b \equiv 1 \pmod{c}$ *and* $\gcd(a, c) = 1$;
   - *iii.* $a + b \equiv 0 \pmod{c}$ *and* $\gcd(a, c) = 1$;
2. *if Q contains a unit square, then $c = \pm 1$;*
3. *if $Q = T_1$, then $c = \pm 1$, or $c = \pm 3$ and $a \equiv -b \equiv \pm 1 \mod 3$;*
4. *if $Q = F_1(k)$, for $k \geq 2$, then $b \equiv 1 \mod c$ and* $\gcd(a, c) = 1$;
5. *if $Q = F_3(k)$, for $k \geq 2$, then $c = \pm 1$; or*
6. *if $Q = T_2$, $F_2(k)$, $F_4(k', k)$, for $k > k' > 0$, then $c = \pm 1$, or $c = \pm 2$ and* $a \equiv b \equiv 1 \mod 2$.

For the purpose of simplifying notation in Lemma 13 and its proof, let us denote by Q (resp. R) both the lattice polygon in $\mathbb{R}^2$ and its embedding $Q \times \{0\}$ (resp. $R \times \{0\}$) in $\mathbb{R}^3$.

Proof Part 1 of the statement follows from the classification of empty tetrahedra [9], which states that a lattice tetrahedron is empty if one of the three pairs of opposite edges are at lattice distance one. It is also required that these opposite edges are primitive segments (gcd condition in the statement).

In each of the cases 2–6, we choose a subpolygon R of Q of size 4 or 5:

2. R is the unit square in Q;
3. $R := Q$ of size 4;
4. $R := F_1(2) \subseteq Q$ of size 4;
5. $R := F_3(2) \subseteq Q$, of size 5;

6. $R := F_2(2) \subseteq Q$ or $R := F_4(1, 2) \subseteq Q$, of size 5.

Since (Q, x) is a quantum jump, and $R \subseteq Q$, so is (R, x). That is, the polytope $P := \operatorname{conv}(R \cup \{x\})$ is of size 5 or 6. Let us find the possible equivalences of P in the classification of lattice 3-polytopes of size 5 or 6 [2, 3]. Notice that in all the cases, P is a pyramid over a known polygon with apex x, so it suffices to find these in the mentioned classification:

2. $P \cong \operatorname{conv}\{(0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 0), (0, 0, 1)\}$ (the unique configuration of signature (2, 2) in [2]);
3. $P \cong \operatorname{conv}\{(0, 0, 0), (1, 0, 0), (0, 1, 0), (-1, -1, 0), (0, 0, 1)\}$ or $P \cong \operatorname{conv}\{(0, 0, 0), (1, 0, 0), (0, 1, 0), (-1, -1, 0), (1, 2, 3)\}$ (the two configurations of signature (3, 1) in [2]);
4. $P \cong \operatorname{conv}\{(0, 0, 0), (1, 0, 0), (-1, 0, 0), (0, 1, 0), (p, q, 1)\}$, with $\gcd(p, q) = 1$ (the configurations of signature (2, 1) in [2]);
5. $P \cong \operatorname{conv}\{(-1, -1, 0), (1, 0, 0), (0, 0, 0), (0, 1, 0), (0, 2, 0), (0, 0, 1)\}$ (a pyramid of width one in [3]);
6. $P \cong \operatorname{conv}\{(-1, 0, 0), (1, 0, 0), (0, 0, 0), (0, 1, 0), (0, 2, 0), (0, 0, 1)\}$, $P \cong \operatorname{conv}\{(-1, 0, 0), (1, 0, 0), (0, 0, 0), (0, 1, 0), (0, 2, 0), (1, 1, 2)\}$, $P \cong \operatorname{conv}\{(-1, 1, 0), (1, 1, 0), (0, 0, 0), (0, 1, 0), (0, 2, 0), (0, 0, 1)\}$, or $P \cong \operatorname{conv}\{(-1, 1, 0), (1, 1, 0), (0, 0, 0), (0, 1, 0), (0, 2, 0), (1, 0, 2)\}$ (two pyramids of width one, and configurations $A.1$ and $A.2$ in [3]).

It is left to the reader to see that adding all the lattice points in $Q \setminus R$ does not put any further restrictions on the coordinates of x. That is, for each value of x so that (R, x) is a quantum jump, we also have that (Q, x) is a quantum jump. Notice that the different possibilities for the values of a, b and c that appear in the statement in each of the cases, appear by applying to P all the unimodular transformations in $\mathbb{R}^3$ that are automorphisms of Q. □

In terms of the distance from x to Q, which in Lemma 13 is the value $|c|$, we have the following result.

Corollary 14 *Let $Q \subset \mathbb{R}^3$ be a lattice polygon and let $x \in \mathbb{Z}^3 \setminus \operatorname{aff}(Q)$ be a lattice point such that (Q, x) is a quantum jump. Then exactly one of the following happens:*

1. *Q contains a unit square or $Q \cong F_3(k)$, and $\operatorname{dist}(x, Q) = 1$;*
2. *$Q \cong T_2$, $F_2(k)$ or $F_4(k', k)$, and $\operatorname{dist}(x, Q) = 1$ or 2;*
3. *$Q \cong T_1$ and $\operatorname{dist}(x, Q) = 1$ or 3; or*
4. *$Q \cong \Delta_2$ or $F_1(k)$, and the distance from x to Q is unbounded.*

The polygons Δ_2, T_i and $F_i(k)$ are as in Proposition 11, for $0 < k' < k$.

Notice that the only cases when the distance is unbounded are $Q \cong \Delta_2$ or $Q \cong F_1(k)$, that is, when *Q is a triangle of width one*.

3.2.2 Quantum Unions of Lattice Segments

The two cases where the distance is unbounded in the previous section also have in common that *all the lattice points are along two lattice segments*. Let us think about them as *quantum unions of lattice segments*.

Remark 15 1. In the case of $Q = \Delta_2$, we have that (Q, x) is a quantum jump if $P := \text{conv}(Q \cup \{x\})$ is an empty tetrahedron. We can also write it as $P = \text{conv}(s_1 \cup s_2)$, with (s_1, s_2) a quantum union of primitive segments, where s_1 is an edge from x and one of the three vertices of Δ_2, and s_2 is the opposite edge. Notice that in this case there are three possible choices for the pair of primitive segments.
2. In the case $Q = F_1(k)$, we have that (Q, x) is a quantum jump if $(s_1(k), s_2)$ is a quantum union between the lattice segment $s_1(k) := \text{conv}\{(0, 0, 0), (k, 0, 0)\}$ and the primitive segment $s_2 := \text{conv}\{(0, 1, 0), x\}$.

Cases 1 and 4 of Lemma 13, reformulated in terms of the distance between segments that form a quantum union, are as follows:

Corollary 16 *Let $s, t \subset \mathbb{R}^3$ be lattice segments such that* $\text{aff}(s \cup t) = \mathbb{R}^3$ *and such that (s, t) is a quantum union. Then:*

1. *if one of s or t is not primitive, then* $\text{dist}(s, t) = 1$*;*
2. *if both s and t are primitive, the distance* $\text{dist}(s, t)$ *can be arbitrarily high, but one of the following distances must be one:*

$$\text{dist}(s, t), \quad \text{dist}\big(\text{conv}\{s_1, t_1\}, \text{conv}\{s_2, t_2\}\big), \quad \text{dist}\big(\text{conv}\{s_1, t_2\}, \text{conv}\{s_2, t_1\}\big)$$

where $s_i, t_i \in \mathbb{Z}^3$ are the end-points of s and t, respectively.

3.2.3 Quantum Jumps (Q, x) with Q of Dimension 3

For the case when $Q \subset \mathbb{R}^3$ is a lattice 3-polytope, remember that we defined the distance from a point $x \in \mathbb{Z}^3 \setminus Q$ to Q in terms of the distance to the facets of Q that are visible from x (Definition 5). In particular, one can study the distance from x to Q, for (Q, x) a quantum jump, by combining the results of the previous section on the facets of Q that are visible from x.

Remark 17 Let $Q \subset \mathbb{R}^3$ be a lattice 3-polytope. For each facet F of Q, let H_F^- be the open halfspace from which the facet F is visible, and denote by $H_F^+ = \mathbb{R}^3 \setminus H_F^-$ the closed halfspace with $Q \subset H_F^+$. Then subdivide $\mathbb{R}^3 \setminus Q$ into the regions

$$\mathcal{R}_\mathcal{I} := \bigcap_{F \in \mathcal{I}} H_F^- \cap \bigcap_{F \notin \mathcal{I}} H_F^+, \qquad \mathcal{I} \neq \emptyset,$$

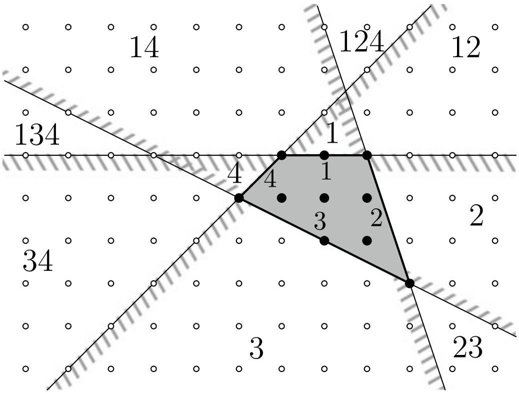

Fig. 3.7 The subdivision of the space of Remark 17 for a lattice polygon. The facets of the polygon are labeled 1 to 4 and each semi-open region of the space is labeled by the facets that are visible from it. Each region can *see* one, two or three facets

so that, for all $x \in \mathcal{R}_{\mathcal{I}}$, the facets of Q that are visible from x are exactly those of $\mathcal{I}$. Notice that the closures of the regions $\mathcal{R}_{\mathcal{I}}$ are rational polytopes or polyhedra. See Fig. 3.7 for a 2-dimensional example of this subdivision of the space.

Let $\mathcal{R}_{\mathcal{I}}$ be one of those regions and suppose that $x \in \mathcal{R}_{\mathcal{I}} \cap \mathbb{Z}^3$ is such that (Q, x) is a quantum jump. We can have three different types of situations.

1. If $\mathcal{R}_{\mathcal{I}}$ is bounded (or if $\mathcal{R}_{\mathcal{I}} \cap \mathbb{Z}^3$ is finite), the distance of x to Q is automatically bounded.
2. If $\mathcal{R}_{\mathcal{I}} \cap \mathbb{Z}^3$ has infinitely many points (in particular $\mathcal{R}_{\mathcal{I}}$ is unbounded), and some facet of $\mathcal{I}$ is not a triangle of width one, then the distance of x to Q is bounded by the results of the previous section (Corollary 14).
3. Finally, if $\mathcal{R}_{\mathcal{I}} \cap \mathbb{Z}^3$ has infinitely many points and all the facets in $\mathcal{I}$ are triangles of width one, the distance from x to Q *may not be bounded*. Notice that, even in this last case the distance from x to Q could still be bounded by combining the restrictions given for the coordinates of x as in parts 1 and 4 of Lemma 13 for all the different facets of $\mathcal{I}$.

For instance, we can find arbitrarily high distance in these types of quantum jumps.

Example 18 Let $h \in \mathbb{Z}$, $h > 0$. Let:

1. $x = (0, 0, 0)$;
2. $F := \text{conv}\{(1, 0, h), (0, 1, h), (1, 1, h)\}$, a unimodular triangle in $\{z = h\}$;
3. $Q \subset \mathbb{R}^3$ be a lattice 3-polytope such that $Q \subset C$, for $C := \mathbb{R}_{\geq 0}(F) = \{xh \leq z\} \cap \{yh \leq z\} \cap \{(x + y)h \geq z\}$ (the triangular cone of F with apex at the origin) and such that $x \notin Q$, $F \subset Q$.

Then F is a facet of Q, it is the only facet that is visible from x, and Q and x are such that (Q, x) is a quantum jump with $D_x(Q) = d_x(Q) = \text{dist}(x, F) = h$. If moreover Q is contained in $C_x := C \cap \{x \leq 1\}$ or $C_y := C \cap \{y \leq 1\}$, then Q and $\text{conv}(Q \cup \{x\})$ are polytopes of width one.

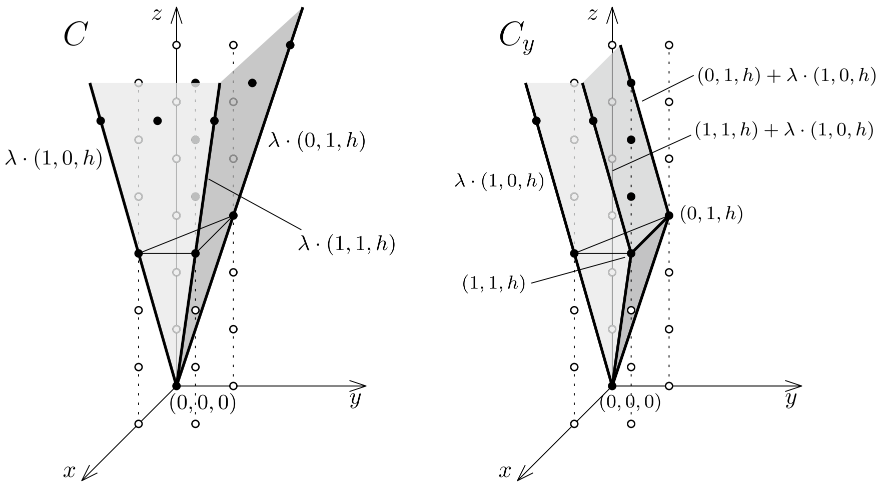

Fig. 3.8 The regions C and C_y of Example 18

We can also choose Q of size n, for any $n \geq 4$, since the regions C, C_x or C_y have infinitely many lattice points. See Fig. 3.8 for a depiction of the polyhedral regions C and C_y.

Even though it is clear that arbitrarily bad examples can occur, how often does this happen? Are they rare or does the general picture look bad? For this, we look at our database of lattice 3-polytopes of size ≤ 11 and width > 1 [4]. For each of those polytopes $P \subset \mathbb{R}^3$ and for each $v \in \text{vert}(P)$ such that P^v is full dimensional, we look at the distance in the quantum jump (P^v, v). Notice that our database contains all the information on the types of quantum jumps when P^v is of size ≤ 10 and extends to a polytope P of width > 1. That is, of size ≤ 10, we do not have the information on polytopes of width one that extend to polytopes of width one, which are infinitely many for each size (and no enumeration exists).

For each quantum jump (P^v, v) we compute the values $d_v(P^v)$ and $D_v(P^v)$ and store the following vectors:

1. $\overline{d}_P := \big(d_v(P^v)\big)_{v \in \text{vert}(P),\, P^v \text{ full-dimensional}}$
2. $\overline{D}_P := \big(D_v(P^v)\big)_{v \in \text{vert}(P),\, P^v \text{ full-dimensional}}$

We separate the 216, 453 polytopes of our database in three different groups. Notice that the entries of each vector $\overline{d}_P$ and $\overline{D}_P$ are positive integers.

1. $\overline{d}_P = (1, 1, \ldots, 1) = \overline{D}_P$. This is the best case scenario we can find, since *every* vertex v of P, with P^v full-dimensional, is at distance one from *all* the facets of P^v that are visible from v. However, only 5,796 polytopes (about 2.7%) fall into this category.
2. $\overline{d}_P = (1, 1, \ldots, 1)$, $\overline{D}_P \neq (1, 1, \ldots, 1)$. In this case, things are not as nice, but we still have that *every* vertex v of P, with P^v full-dimensional, is at distance one

from *at least one* facet of P^v that is visible from v. In this category we have 77,443 polytopes (~35.8%).

3. $\overline{d}_P, \overline{D}_P \neq (1, 1, \ldots, 1)$. This is the worst case we can have, in which some vertex v of P, with P^v full-dimensional, is at distance *larger than one* from *all* the facets of P^v that are visible form v. This is the case for most of the polytopes in our database: 133,214 polytopes, or ~61.5% of the total.

In terms of the magnitudes of the entries, we have that the largest entries in the vectors $\overline{d}_P$ and $\overline{D}_P$, for each n the size of P, are:

n	5	6	7	8	9	10	11
$\max d_v(P^v)$	5	7	13	19	25	31	37
$\max D_v(P^v)$	7	13	19	25	31	37	43

Notice that the maximum values for $d_v(P^v)$ and $D_v(P^v)$, for P of size n, are $6(n-5)+1$ and $6(n-4)+1$, respectively (for $n \neq 5$ in the first case). This has to do with the fact that, as h grows (see Example 18) we need more lattice points to construct a polytope of width > 1 that yields a vertex at distance h.

The average values of the $d_v(I_P)$ and $D_v(I_P)$ are, respectively, 1.42 and 3.35.

Remark 19 If we were to follow the lines of Sect. 3.2.1, we would want to have, in this section, an irredundant list of lattice 3-polytopes Q, and the maximum and minimum distances a point x can be from Q, for (Q, x) a quantum jump.

However, we need to consider that we have 216,453 polytopes and that, for each of those polytopes P and each vertex v of P we have a different polytope P^v. Organizing the information on the distances with no redundancies among the P^v does not seem to be worth undertaking, in light of the distances that appear and the arguments made.

3.3 Distance from the Boundary to the Inner Polytope

Let $P \subset \mathbb{R}^3$ be a lattice 3-polytope with $I_P \neq \emptyset$.

Remark 20 For I_P of size 1 or 2, the classification of lattice 3-polytopes with 1 and 2 interior lattice points was completed, respectively, by Kasprzyk [7] and by Balletti and Kasprzyk [1].

1. The 3-*dimensional* distances that can be measured in the case of I_P consisting of one lattice point are the distances between this point and the facets of P.
2. In the case of I_P having two lattice points, we would have to look at the distance between I_P and a non-coplanar lattice segment in the boundary.

In these two situations the distance is a priori unbounded if we look at it locally: we can have a quantum jump between a unimodular triangle and a point in the first case, and a quantum union of primitive segments in the second, at arbitrarily high distance (see Corollaries 14 and 16). There will be a bound following from the fact

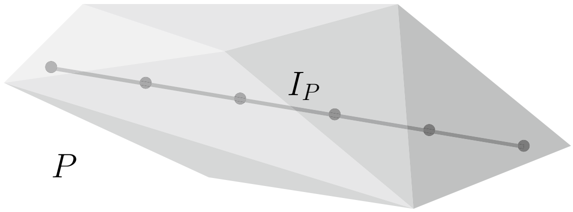

Fig. 3.9 A polytope P having inner polytope I_P of dimension 1

that there are only finitely many lattice 3-polytopes with 1 and 2 interior lattice points, but the author does not believe it is worth exploring the more than 23 million such polytopes.

So let P be such that I_P has size at least three. Let $S \subset \partial P$ be a lattice point or primitive segment in the boundary of P, we look at the distance between S and I_P, relying on the fact that (I_P, S) is a quantum jump (or union). This happens because $\mathrm{conv}(I_P \cup S) \setminus S \subset \mathrm{int}(P)$, and the only interior lattice points of P are those of I_P.

3.3.1 Inner Polytope of Dimension 1

If I_P is a lattice segment (see Fig. 3.9), and since I_P has size at least 3, by Corollary 16, the distance from I_P to any lattice segment in the boundary must be one. A consequence of this is the following result[1]:

Theorem 21 (Averkov–Balletti–Blanco–Nill–Soprunov) *Let $P \subset \mathbb{R}^3$ be a lattice* 3-*polytope with I_P a lattice segment of lattice length k ($k+1$ collinear lattice points), for $k \geq 2$. If $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ is the lattice projection that maps I_P to the origin then $\pi(P)$ is a reflexive polygon.*

Proof Since $k > 0$, the projection π is well defined and unique, up to unimodular transformation. Because the $k+1$ collinear lattice points are in the interior of P, their projection, i. e. the origin, is an interior point of $\pi(P)$. Let e be an edge of $\pi(P)$, then there exists a lattice segment e' in the boundary of P such that $\pi(e') = e$. Take the following polytope $R_e := \mathrm{conv}(I_P \cup e') \subset P$. Since $e' \subset \partial P$ and $I_P \subset \mathrm{int}(P)$, then R_e cannot contain any extra lattice points. That is, it is the quantum union of two lattice segments. By Corollary 16, and since I_P is not primitive, the distance between I_P and e' must be one. In the projection, this directly implies that the distance from the edge e and the origin (the respective projections of the segments) is one. Hence $\pi(P)$ is reflexive. □

[1] Discussed in the Oberwolfach mini-workshop *Lattice polytopes: Methods, Advances and Applications*, September 2017.

This result can help, for example, in the full classification of lattice 3-polytopes P with I_P a lattice segment. The projection has 16 possibilities: the 16 reflexive polygons. For one such Q fixed, all the lattice points in P must be in $\pi^{-1}(Q)$.

3.3.2 Inner Polytope of Dimension 2

For P having inner polytope I_P of dimension 2 (see Fig. 3.10), our main result resides in proving a specific property that a polygon must have so that it can actually appear as the inner polytope of a lattice 3-polytope. For this, let us introduce the concept of *front*:

Definition 22 Let $Q \subset \mathbb{Z}^2$ be a lattice polygon and let v be a vertex of Q. A *front* of Q from v is a facet of the polygon $Q^v := \mathrm{conv}(Q \setminus \{v\} \cap \mathbb{Z}^2)$ that is visible from v. See Fig. 3.11 for an example of the fronts of a polygon.

Theorem 23 *If P is a lattice 3-polytope with I_P of dimension 2, then the fronts of I_P have length ≤ 8.*

Proof Let $F \subset I_P$ be the longest front of I_P, of length $\ell > 0$. We can assume without loss of generality that $I_P \subset \{z = 0\}$, $\ell \geq 3$, that $v := (0, 1, 0)$ is a vertex of I_P

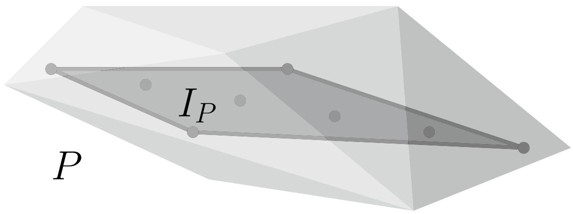

Fig. 3.10 A polytope P having inner polytope I_P of dimension 2

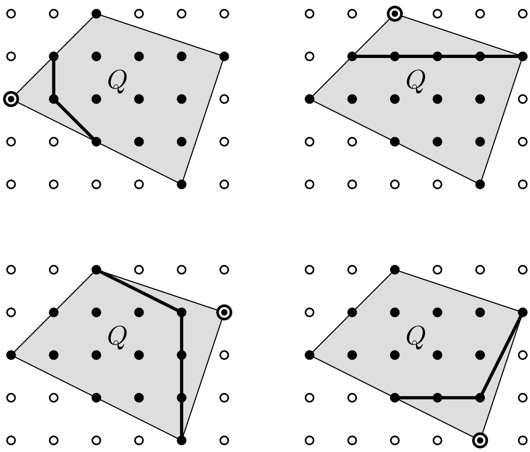

Fig. 3.11 Four copies of a polygon Q, each showing the fronts of Q from a specific vertex

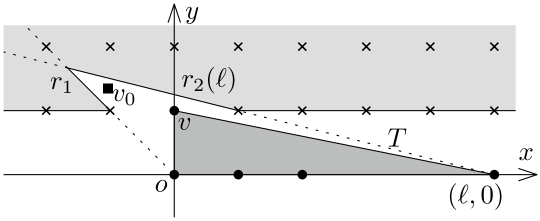

Fig. 3.12 The setting of the proof of Theorem 23

and $F := \text{conv}\{(0,0,0), (\ell,0,0)\}$ (that is, F is a front of I_P from v). In particular, we have that $T := F_1(\ell) \times \{0\} = \text{conv}(F \cup \{v\}) \subseteq I_P$ (for $F_1(\ell)$ as in Proposition 11) and that the only lattice points of I_P in $\{y \geq 0\}$ are those of F and v. We need to prove that $\ell \leq 8$.

The intersection of P with the plane $\{z = 0\}$ is the rational polygon $P_0 := P \cap \{z = 0\}$. We have that $I_P = \text{conv}(\text{relint}(P_0) \cap \mathbb{Z}^3)$. That is, the inner polytope of P coincides with the *relative* inner polygon of P_0. For now let us identify $\mathbb{R}^2 \times \{0\}$ and $\mathbb{R}^2$ in the trivial way, so from now on we simply say *interior of* P_0 for the relative interior of it embedded in the space $\mathbb{R}^3$.

The vertex $v = (0,1)$ of I_P is an interior point of P_0, so for any line passing through v there must be a vertex of P_0 in each of the open halfspaces determined by this line. In particular, there must be a vertex of P_0 in the open halfspace $\{y > 1\}$. Let us denote this vertex by v_0 and consider the rational polygon $T' := \text{conv}(T \cup \{v_0\})$. Since $T \subseteq I_P \subset \text{int}(P_0)$ and $v_0 \in \partial P_0$, then $T' \setminus \{v_0\} \subset \text{int}(P_0)$. That is, the only lattice points of $T' \setminus \{v_0\}$ are those of T. In particular, $(-1,1), (1,1) \notin T' \setminus \{v_0\}$, which implies that $v_0 \notin C_{(-1,1)}(T) \cup C_{(1,1)}(T)$ (see Remark 12).

This in turn implies that v_0 must lie in the open rational triangle R_ℓ determined by the hyperplanes $\{y = 1\}$, $r_1 := \text{aff}\{(0,0), (-1,1)\} = \{x + y = 0\}$ and $r_2(\ell) := \text{aff}\{(\ell,0), (1,1)\} = \{x + (\ell - 1)y = \ell\}$ (see Fig. 3.12).

That is,

$$v_0 \in R_\ell = \text{int}\left(\text{conv}\left\{(-1,1), (1,1), \left(\frac{-\ell}{\ell-2}, \frac{\ell}{\ell-2}\right)\right\}\right),$$

which is well defined for $\ell \geq 3$.

Observe that $R_\ell \subset R_3$ for all $\ell > 3$, and that $R_3 \cap \mathbb{Z}^2 = \emptyset$. That is, there is no lattice point in R_ℓ. In particular, $v_0 \notin \mathbb{Z}^3$, and the only possibility is that v_0 is the intersection of a primitive segment $uw := \text{conv}\{u, w\} \subset P$ with the plane $\{z = 0\}$, with neither u nor w in this plane (and one in each of the halfspaces). This segment uw is contained in an edge of P, although it is not necessarily equal to it.

In order to find out more about the coordinates of v_0, we need to know the distances $d_u := \text{dist}(u, H)$ and $d_w := \text{dist}(w, H)$ in the full-dimensional polytope P, for $H := \text{aff}\{P_0\} = \{z = 0\}$.

For this, let us look at a 3-dimensional proper subpolytope of P:

$$K := \text{conv}\{(0,0,0), (1,0,0), (2,0,0), (0,1,0), u, w\} \subset P$$

So far, the information we have is that $K = \text{conv}(K_0 \cup \{u, w\})$, for $K_0 := K \cap \{z = 0\} = \text{conv}\{o, (2,0,0), (0,1,0), v_0\}$ and u and w lying one in $\{z > 0\}$ and the other in $\{z < 0\}$, the edge uw being primitive and cutting the plane $\{z = 0\}$ at the rational point $v_0 \in R_\ell$. Let us prove the following properties of K:

1. *K has size* 6. On one hand, $K \cap \{z = 0\} = K_0$ does not contain more lattice points other than $(0,0,0)$, $(1,0,0)$, $(2,0,0)$ and $(0,1,0)$, since $K_0 \subset T'$ and $T' \cap \mathbb{Z}^3 = T \cap \mathbb{Z}^3$. On the other hand, if K contains an extra lattice point other than those four and u or w, this lattice point would have to lie outside of uw (which is a primitive edge) and outside of $\{z = 0\}$. Since $K \setminus uw \subset \text{int}(P)$, this would be an interior lattice point of P outside of the plane $\{z = 0\}$, which is impossible by hypothesis.
2. *K has width* > 1. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a linear primitive functional. If f is not constant in the line $\{y = 0 = z\}$, then the width of K with respect to f is > 1 since f will take three different values in the points $(0,0,0), (1,0,0), (2,0,0) \in K$. Take now f to be constant in that line. Then the width of K with respect to f is the width of $\pi(K)$ with respect to f', for π the lattice projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2$, $\pi(x, y, z) = (y, z)$, and f' the primitive functional $f' : \mathbb{R}^2 \to \mathbb{R}$, $f'(y, z) = f(\pi^{-1}(y, z))$. Notice that f' is well defined because f is constant in the fibers of the projection. But under this projection, the point $(0, 1)$ is an interior point of $\pi(K)$, hence the width of $\pi(K)$ with respect to any functional is > 1. See the picture on the right in Fig. 3.13.

That is, K is a lattice 3-polytope of size 6 and width > 1, and the classification of such polytopes appears in [3]. However, not all of these polytopes are a possible candidate for K. To narrow the possibilities, we can figure out the oriented matroid of our configuration K, since the classification in [3] is also organized according to this combinatorial information. Remember that the *oriented matroid* of a finite set of points is the information recording the affine dependencies, in particular coplanarities and collinearities between the points (see [6] for information on oriented matroids).

For this, one extra thing that we can notice is that v cannot be a vertex of K. Suppose otherwise, then the polytope $K' := K^{(0,1,0)} = \text{conv}\{o, (1,0,0), (2,0,0), u, w\}$ is a polytope of size 5 with three collinear lattice points. That is, K' is the convex hull

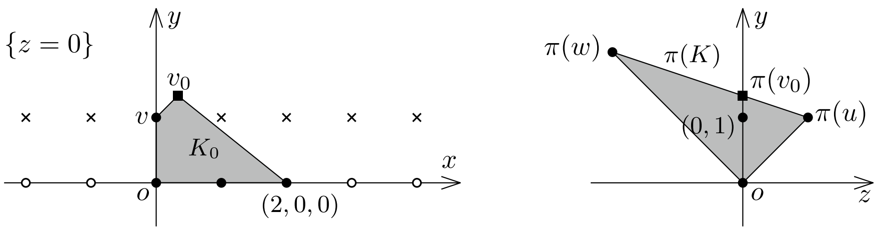

Fig. 3.13 The intersection K_0 of K with the plane $\{z = 0\}$, and the projection of K under π

of a lattice segment of length 2 and a primitive lattice segment uw. By Corollary 16, the lattice distance between these two lattice segments must be one. Taking again projection π, this is equivalent to the segment $\pi(uw)$ being at distance one from the origin. Which is impossible since (0, 1) is a lattice point strictly in between $\pi(uw)$ and the origin (see the picture in the right of Fig. 3.13). In particular, v is not a vertex of K_0 and the picture on the left of Fig. 3.13 is not accurate.

That is, the oriented matroid of the six lattice points of K can be described as follows:

1. four of them are vertices (o, $(2, 0, 0)$, u and w);
2. one non-vertex point is in an edge ($(1, 0, 0) = \frac{1}{2}((0, 0, 0) + (2, 0, 0))$);
3. the hyperplane containing the three collinear points and the other non-vertex point ($\{z = 0\}$) leaves the remaining two vertices (u and w) strictly in opposite sides of it.

Notice that there are four different types (or orbits) of points: the endpoints of the collinearity (o and $(2, 0, 0)$), the middle point of the collinearity ($(1, 0, 0)$), the other two vertices (u and w), and the remaining non-vertex point (v). This sixth point v has three different possibilities, in terms of the oriented matroid. The three possibilities for v are: (I) it is in the relative interior of one of the facets $\operatorname{conv}\{o, u, w\}$ or $\operatorname{conv}\{(2, 0, 0), u, w\}$; (II) it is in the relative interior of the triangle $\operatorname{conv}\{(1, 0, 0), u, w\}$; or (III) it lies in the interior of one of the tetrahedra $\operatorname{conv}\{o, (1, 0, 0), u, w\}$ or $\operatorname{conv}\{(1, 0, 0), (2, 0, 0), u, w\}$. These three options are shown in Fig. 3.14. For each of these three cases, the oriented matroid is fully described. Without going into details of how the oriented matroids are represented and classified in [3], one can derive that the oriented matroid of the three options (I), (II) and (III) are, respectively, oriented matroids 3.6, 3.8 and 4.11 as encoded in [3, Fig. 1].

In [3, Tables 8 and 9] we can see that the only lattice 3-polytopes of size 6, width > 1 and with one of the three specified oriented matroids are B.7 (oriented matroid 3.8), C.1 (oriented matroid 3.6), and F.13 to F.17 (oriented matroid 4.11). The following 3×6 matrices have, as columns, the six lattice points of each of those seven polytopes:

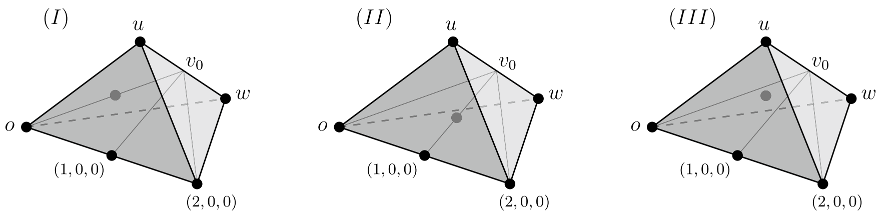

Fig. 3.14 The three possibilities (I), (II), and (III) in the proof of Theorem 23

$$B.7\begin{pmatrix}0&1&0&-1&0&0\\0&0&1&-1&0&2\\0&0&0&0&1&-1\end{pmatrix}\quad F.13\begin{pmatrix}0&1&0&-2&1&4\\0&1&0&-1&0&1\\0&1&1&-2&0&2\end{pmatrix}\quad F.16\begin{pmatrix}0&1&-1&0&1&2\\0&0&-2&0&3&6\\0&0&-1&1&1&1\end{pmatrix}$$

$$F.14\begin{pmatrix}0&0&-1&1&1&1\\0&0&-1&2&0&-2\\0&1&-1&1&0&-1\end{pmatrix}$$

$$C.1\begin{pmatrix}0&1&0&-1&0&-1\\0&0&1&-1&0&0\\0&0&0&0&1&2\end{pmatrix}\quad F.15\begin{pmatrix}0&0&1&1&-1&-3\\0&0&0&2&-1&-4\\0&1&0&1&-1&-3\end{pmatrix}\quad F.17\begin{pmatrix}0&1&1&0&-1&-2\\0&0&3&0&-2&-4\\0&0&1&1&-1&-3\end{pmatrix}$$

That is, our polytope K must be equivalent to one of them, say $\tilde{K}$, and let $t : \mathbb{R}^3 \to \mathbb{R}^3$ be any unimodular transformation that maps K to $\tilde{K}$. Then t will send the edge $\operatorname{conv}\{(0, 0, 0), (2, 0, 0)\}$ to the unique collinearity of three lattice points in $\tilde{K}$, and $v = (0, 1, 0)$ to the only non-vertex of the remaining lattice points.

Since unimodular transformations preserve distances, we have that $\{d_u, d_w\} = \{d'_u, d'_w\}$, for $d'_u := \operatorname{dist}(t(u), t(H))$ and $d'_w := \operatorname{dist}(t(w), t(H))$. Moreover, we can assume without loss of generality that $d_u \leq d_w$. Then:

$$(d_u, d_w) = \begin{cases} (1, 1) & \text{if } K \cong \text{B.7, C.1, F.13, F.15} \\ (1, 2) & \text{if } K \cong \text{F.14, F.17} \\ (1, 3) & \text{if } K \cong \text{F.16} \end{cases}$$

The distance in our original coordinates of K, since $H = \{z = 0\}$, is measured on the z-coordinate of the points. That is, let z_u and z_w be the respective z-coordinates of u and w, we have that $d_u = |z_u|$ and $d_w = |z_w|$. Without loss of generality $z_u > 0 > z_w$:

$$(z_u, z_w) = \begin{cases} (1, -1) & \text{if } K \cong \text{B.7, C.1, F.13, F.15} \\ (1, -2) & \text{if } K \cong \text{F.14, F.17} \\ (1, -3) & \text{if } K \cong \text{F.16} \end{cases}$$

Let us now see that the denominator of the rational coordinates of v_0 can only be 2, 3 or 4:

$$v_0 = (1 - \lambda)u + \lambda w, \text{ for some } \lambda \in [0, 1]$$

where λ is such that the z-coordinate of v_0 is 0:

$$0 = (1 - \lambda)z_u + \lambda z_w \Longrightarrow \lambda = \frac{z_u}{z_u - z_w} = \frac{d_u}{d_u + d_w}$$

That is, $\lambda \in \frac{1}{2}\mathbb{Z}$, $\frac{1}{3}\mathbb{Z}$ or $\frac{1}{4}\mathbb{Z}$, hence

$$v_0 = (a, b, 0), \text{ for } (a, b) \in \left(\frac{1}{2}\mathbb{Z}\right)^2 \cup \left(\frac{1}{3}\mathbb{Z}\right)^2 \cup \left(\frac{1}{4}\mathbb{Z}\right)^2.$$

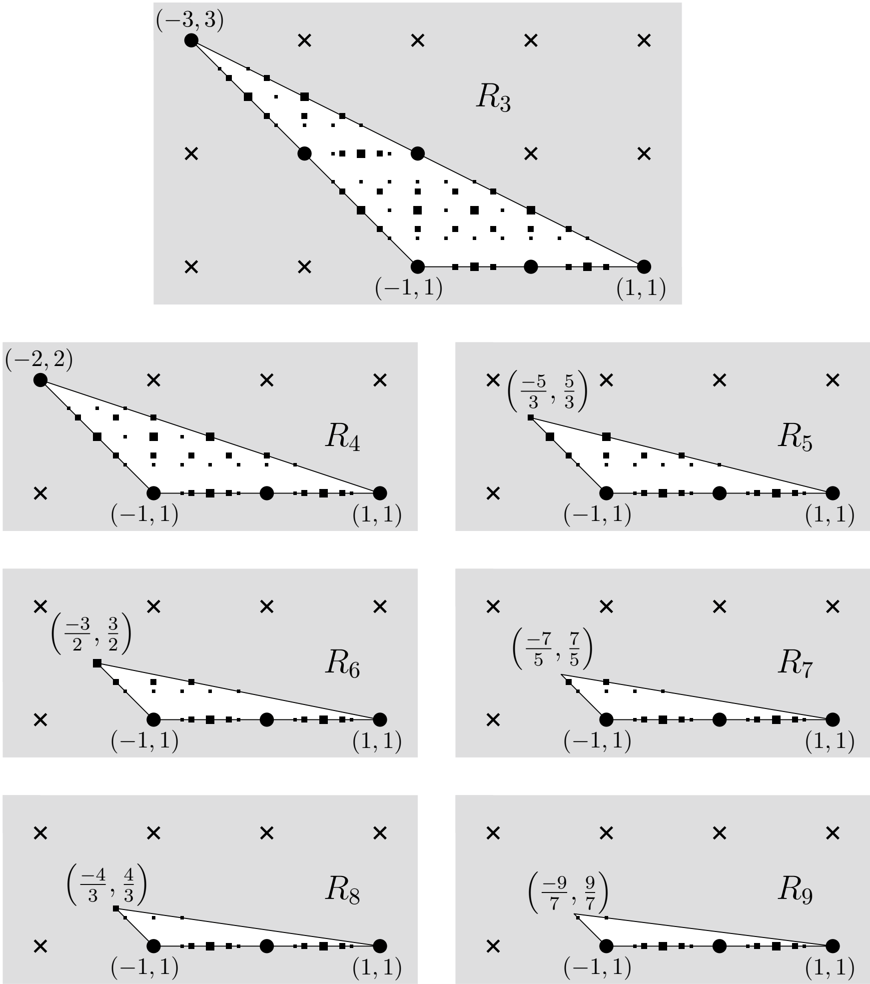

Fig. 3.15 The regions R_3 to R_9, with the points of the lattices L_2, L_3 and L_4 contained in them. Large squares are points of $L_2 \setminus \mathbb{Z}^2$, medium squares are the points of $L_3 \setminus \mathbb{Z}^2$, and small squares the points of $L_4 \setminus L_2$. Black dots and crosses represent points of $\mathbb{Z}^2$ in ∂R_ℓ and $\mathbb{R}^2 \setminus \overline{R_\ell}$, respectively

Remember also that v_0 must lie in the open triangle R_ℓ. To prove the statement of the theorem it remains to see that the intersection of R_ℓ with any of the lattices L_2, L_3 or L_4, for $L_i := \left(\frac{1}{i}\mathbb{Z}\right)^2 \times \{0\}$ is empty for $\ell \geq 9$. This is true since R_9 does not contain any point of those lattices, and since $R_\ell \subseteq R_9$ for $\ell \geq 9$. To help the reader visualize this we have drawn in Fig. 3.15 all the regions R_3 to R_9 with the possible positions for the point v_0. □

As a consequence of the theorem, we find that polygons that do not contain a unit square can only be inner polytopes of lattice 3-polytopes if they have few lattice points:

Corollary 24 *Let $P \subset \mathbb{R}^3$ be a lattice 3-polytope with I_P of dimension 2. Then exactly one of the following happens:*

1. *I_P contains a unit square and all its fronts are of length ≤ 8;*
2. *$I_P \cong T_1$;*
3. *$I_P \cong F_3(k)$, for $2 \leq k \leq 8$;*
4. *$I_P \cong T_2$, $F_2(k)$ or $F_4(k', k)$, for $0 < k' < k \leq 8$; or*
5. *$I_P \cong \Delta_2$ or $I_P \cong F_1(k)$, for $2 \leq k \leq 8$.*

In particular, in cases 1–4 any lattice point in the boundary of P is at distance at most 1, 3, 1 and 2, respectively, from I_P.

Proof The first part of the statement follows from Proposition 11 and Theorem 23, considering that the longest fronts in $F_1(k)$, $F_2(k)$, $F_3(k)$ and $F_4(k', k)$ have length k. The second part follows from Corollary 14. □

Remark 25 In case 5, the distance of any boundary lattice point of P to I_P will also be bounded since there are only finitely many lattice 3-polytopes with those particular polygons as inner polytopes. However, this bound can only be found globally, and not locally, since the distance from a single lattice point to I_P is a priori unbounded (see Corollary 14).

3.3.3 Inner Polytope of Dimension 3

Now there is only left the case where I_P is 3-dimensional. In particular, these are quantum jumps of the type considered in Sect. 3.2.3, and we can apply the results of Sect. 3.2.1 as explained in Remark 17. However, notice that we cannot use the results of Sect. 3.3.2, since they heavily rely on I_P being 2-dimensional.

We do the same as we did in Sect. 3.2.3: we check our database of lattice 3-polytopes of size ≤ 11, width > 1 and 3-dimensional inner polytope, of which there are 15,763 polytopes [4]. In this case, since a polytope with interior lattice points cannot have width one, we are not losing cases by having only polytopes of width > 1, but we only have polytopes with at most 11 lattice points in total. Since I_P is 3-dimensional, it has at least size 4, and since $\text{vert}(P) \subset P \setminus I_P$, then I_P has at most 7 lattice points. That is, our database contains the information on I_P of size $k \in \{4, 5, 6, 7\}$, with P of size $n \in \{k + 4, \ldots, 11\}$ and with $n - k \leq 7$ lattice points in the boundary. In particular, both P and I_P are very *clean* (few points in the boundary) among the polytopes being checked.

We consider the following vectors:

1. $d_P := \left(d_v(I_P)\right)_{v \in \mathrm{vert}(P)}$
2. $D_P := \left(D_v(I_P)\right)_{v \in \mathrm{vert}(P)}$

In this case, the results we get are much more hopeful, since most of the polytopes P have the distance from vertices to I_P all ones:

1. 8,786 polytopes (∼55.74%) have $d_P = (1, 1, \ldots, 1) = D_P$. That is, *every* vertex v of P is at distance one from *all* the facets of I_P that are visible from v.
2. 5,804 polytopes (∼36.82%) have $d_P = (1, 1, \ldots, 1)$, $D_P \neq (1, 1, \ldots, 1)$. In this case, *every* vertex v of P is at distance one from *at least one* facet of I_P that is visible from v.
3. 1,173 polytopes (∼7.44%) have $d_P, D_P \neq (1, 1, \ldots, 1)$. That is, there exists a vertex v of P that is at distance *larger than one* from *all* the facets of I_P that are visible form v.

Moreover, the values of the distances are much smaller. The maximum and minimum values for each size n are as follows:

n	8	9	10	11
max $D_v(I_P)$	3	4	5	6
max $d_v(I_P)$	3	3	3	4

and the average values of the $D_v(I_P)$ and $d_v(I_P)$ are, respectively, 1.12 and 1.02.

Remark 26 Following the reasonings of Remark 19, in this case we would want to have a list of polytopes Q and the maximum distance we can have a point x so that (Q, x) is a quantum jump and there exists a polytope P such that $Q = I_P$ and $x \in \partial P$. From our database we are only considering 15,763 polytopes and, for each of those polytopes P, we have exactly one polytope I_P.

Putting together all the equivalent inner polytopes, we find out that there are only 39 equivalence classes of inner polytopes. Moreover, around 9,000 polytopes in the database (more than half) have the unimodular tetrahedron as its inner polytope.

The maximum and minimum distances for I_P of size k are as follows:

k	4	5	6	7
max $D_v(I_P)$	4	4	5	6
max $d_v(I_P)$	4	3	2	2

Altogether, it seems that we could find manageable bounds for the distance in quantum jumps (I_P, v), although further work is required.

Acknowledgements I would like to thank the referee for the useful comments and suggestions.

References

1. Balletti, G., Kasprzyk, A.M.: Three-dimensional lattice polytopes with two interior lattice points (2016). arXiv:1612.08918 [math.CO]
2. Blanco, M., Santos, F.: Lattice 3-polytopes with few lattice points. SIAM J. Discrete Math. **30**(2), 669–686 (2016)
3. Blanco, M., Santos, F.: Lattice 3-polytopes with six lattice points. SIAM J. Discrete Math. **30**(2), 687–717 (2016)
4. Blanco, M., Santos, F.: Enumeration of lattice 3-polytopes by their number of lattice points. Discrete Comput. Geom. **60**(3), 756–800 (2018)
5. Bruns, W., Gubeladze, J., Michałek, M.: Quantum jumps of normal polytopes. Discrete Comput. Geom. **56**(1), 181–215 (2016)
6. De Loera, J.A., Rambau, J., Santos, F.: Triangulations. Algorithms and Computation in Mathematics, vol. 25. Springer, Berlin (2010). Structures for algorithms and applications
7. Kasprzyk, A.M.: Canonical toric Fano threefolds. Canad. J. Math. **62**(6), 1293–1309 (2010)
8. Reeve, J.E.: On the volume of lattice polyhedra. Proc. Lond. Math. Soc. **3**(7), 378–395 (1957)
9. White, G.K.: Lattice tetrahedra. Can. J. Math. **16**, 389–396 (1964)