Chapter 2
On the Fine Interior of Three-Dimensional Canonical Fano Polytopes

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Abstract The Fine interior Δ^{FI} of a d-dimensional lattice polytope Δ is a rational subpolytope of Δ which is important for constructing minimal birational models of non-degenerate hypersurfaces defined by Laurent polynomials with Newton polytope Δ. This paper presents some computational results on the Fine interior of all 674,688 three-dimensional canonical Fano polytopes.

Keywords Lattice polytope · Fine interior · Hypersurface

2.1 Introduction

Let $M \cong \mathbb{Z}^d$ be a free abelian group of rank d. We set $M_{\mathbb{Q}} := M \otimes \mathbb{Q}$ and denote by N the dual group $\mathrm{Hom}(M, \mathbb{Z})$ in the dual $\mathbb{Q}$-linear vector space $N_{\mathbb{Q}} := \mathrm{Hom}(M, \mathbb{Q})$. Let $\langle \cdot, \cdot \rangle : M_{\mathbb{Q}} \times N_{\mathbb{Q}} \to \mathbb{Q}$ be the natural pairing.

A convex compact d-dimensional polytope $\Delta \subseteq M_{\mathbb{Q}}$ is called *lattice d-tope* if all vertices of Δ belong to the lattice $M \subseteq M_{\mathbb{Q}}$, i.e., Δ equals the convex hull $\mathrm{conv}(\Delta \cap M)$ of all lattice points in Δ. The usual interior Δ° of Δ is the complement $\Delta \setminus \partial\Delta$, where $\partial\Delta$ is the boundary of Δ. Another interior of a lattice polytope Δ was introduced by Fine [3, 13, 15, 20]:

Definition 1 Let $\Delta \subseteq M_{\mathbb{Q}}$ be a lattice d-tope. Denote by ord_{Δ} the piecewise linear function $N_{\mathbb{Q}} \to \mathbb{Q}$ with

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$$\text{ord}_\Delta(y) := \min_{x \in \Delta} \langle x, y \rangle \quad (y \in N_\mathbb{Q}).$$

Then the convex subset

$$\Delta^{\text{FI}} := \bigcap_{n \in N \setminus \{0\}} \{x \in M_\mathbb{Q} \mid \langle x, n \rangle \geq \text{ord}_\Delta(n) + 1\}$$

is called the *Fine interior* of Δ.

One can show that only finitely many linear inequalities $\langle x, n \rangle \geq \text{ord}_\Delta(n) + 1$ are necessary to define Δ^{FI}. Therefore, Δ^{FI} is a convex hull of finitely many rational points $p \in M_\mathbb{Q}$. Moreover, any lattice point $p \in \Delta^\circ \cap M$ in the usual interior of Δ is contained in Δ^{FI}. Therefore, Δ^{FI} contains the convex hull of $\Delta \cap M$, i.e., we get the inclusion $\text{conv}(\Delta^\circ \cap M) \subseteq \Delta^{\text{FI}}$. In particular, Δ^{FI} is non-empty if $\Delta^\circ \cap M$ is non-empty. Moreover, for any lattice polytope Δ of dimension $d \leq 2$ one has the equality $\text{conv}(\Delta^\circ \cap M) = \Delta^{\text{FI}}$ [3]. The Fine interior Δ^{FI} of a lattice polytope Δ of dimension $d \geq 3$ may happen to be strictly larger than the convex hull $\text{conv}(\Delta^\circ \cap M)$. The simplest famous example of such a situation is due to M. Reid. Other similar examples based on hollow 3-topes can be found in Sect. 2.7:

Example 2 ([20, Example 4.15]) Let $M \subseteq \mathbb{Q}^4$ be 3-dimensional affine lattice defined by

$$M := \left\{ (m_1, m_2, m_3, m_4) \in \mathbb{Z}^4 \,\middle|\, \sum_{i=1}^{4} m_i = 5, \ \sum_{i=1}^{4} i m_i \equiv 0 \pmod 5 \right\}.$$

Consider the M-lattice 3-tope $\Delta \subseteq M_\mathbb{Q}$ defined as the convex hull of 4 lattice points

$$(5, 0, 0, 0), \ (0, 5, 0, 0), \ (0, 0, 5, 0), \ \text{and } (0, 0, 0, 5) \in M.$$

Then $\text{conv}(\Delta^\circ \cap M) = \emptyset$, but Δ^{FI} is the 3-dimensional M-rational simplex

$$\text{conv}\{(2, 1, 1, 1), (1, 2, 1, 1), (1, 1, 2, 1), (1, 1, 1, 2)\}$$

and $\Delta^{\text{FI}} \cap M$ is empty.

In this paper, we are interested in lattice d-topes $\Delta \subseteq M_\mathbb{Q}$ obtained as Newton polytopes of Laurent polynomials f_Δ in d variables $x_1, \ldots, x_d$, i.e.,

$$f_\Delta(\mathbf{x}) = \sum_{m \in \Delta \cap M} a_m \mathbf{x}^m,$$

where $a_m \in \mathbb{C}$ are sufficiently general complex numbers. The importance of the Fine interior is explained by the following theorem [3, 15, 20]:

Theorem 3 *Let $\mathcal{Z}_\Delta \subseteq \mathbb{T}^d$ be a non-degenerate affine hypersurface in the d-dimensional algebraic torus $\mathbb{T}^d$ defined by a Laurent polynomial f_Δ with Newton d-tope Δ. Then the following conditions are equivalent:*

1. *a smooth projective compactification $\mathcal{V}_\Delta$ of $\mathcal{Z}_\Delta$ has non-negative Kodaira dimension, i.e., $\kappa \geq 0$;*
2. *$\mathcal{Z}_\Delta$ is birational to a minimal model $\mathcal{S}_\Delta$ with abundance;*
3. *the Fine interior Δ^{FI} of Δ is non-empty.*

Remark 4 By well known results of Khovanskii [14], one has vanishing of the cohomology groups

$$h^i(\mathcal{O}_{\mathcal{V}_\Delta}) = 0 \ \ (1 \leq i \leq d-2)$$

and the equation $h^{d-1}(\mathcal{O}_{\mathcal{V}_\Delta}) = |\Delta^\circ \cap M|$. The numbers $h^i(\mathcal{O}_{\mathcal{V}_\Delta})$ are birational invariants of $\mathcal{Z}_\Delta$; they do not depend on a smooth projective compactification $\mathcal{V}_\Delta$ of $\mathcal{Z}_\Delta$. In particular, the number $|\Delta^\circ \cap M|$ is the geometric genus p_g of the affine hypersurface $\mathcal{Z}_\Delta \subseteq \mathbb{T}^d$.

Smooth projective compactifications of non-degenerate hypersurfaces in $\mathbb{T}^d$ can be obtained using the theory of toric varieties [14].

Let $\Delta \subseteq M_\mathbb{Q}$ be a lattice d-tope. We consider the *normal fan* Σ^Δ of Δ in the dual space $N_\mathbb{Q}$, i.e., $\Sigma^\Delta := \{\sigma^\theta \,|\, \theta \preceq \Delta\}$, where σ^θ is the cone generated by all inward-pointing facet normals of facets containing the face $\theta \preceq \Delta$ of Δ. One has $\dim(\sigma^\theta) + \dim(\theta) = d$ for any face $\theta \preceq \Delta$. We denote by X_Δ the normal projective toric variety constructed via the normal fan Σ^Δ. In particular, the above function $\mathrm{ord}_\Delta : N_\mathbb{Q} \to \mathbb{Q}$ is a piecewise linear function with respect to this fan defining an ample Cartier divisor on X_Δ. In particular, the cones $\sigma^\theta \in \Sigma^\Delta$ are defined as

$$\sigma^\theta = \left\{ y \in N_\mathbb{Q} \,\middle|\, \mathrm{ord}_\Delta(y) = \langle x, y \rangle \ \text{ for all } x \in \theta \right\}.$$

Remark 5 Using the normal fan Σ^Δ, one can compute the fundamental group $\pi_1(\mathcal{V}_\Delta)$ of a smooth projective birational model $\mathcal{V}_\Delta$ of a non-degenerate affine hypersurface $\mathcal{Z}_\Delta$ (given as in Theorem 3). The fundamental group $\pi_1(\mathcal{V}_\Delta)$ does not depend on the choice of the smooth birational model and it is isomorphic to the quotient of the lattice N modulo the sublattice N' generated by all lattice points in $(d-1)$-dimensional cones σ^θ of the normal fan Σ^Δ [4].

Example 6 The minimal model $\mathcal{S}_\Delta$ of a non-degenerate affine surface $\mathcal{Z}_\Delta$ defined by a Laurent polynomial with the Newton polytope Δ from Example 2 is a *Godeaux surface*. It is a surface of general type with $p_g = q = 0$, $K^2 = 1$, and $\pi_1(\mathcal{S}_\Delta) \cong \mathbb{Z}/5\mathbb{Z}$.

Definition 7 A lattice d-tope Δ is called *canonical Fano d-tope* if $|\Delta^\circ \cap M| = 1$. Up to a shift by a lattice vector, we will assume without loss of generality that $0 \in M$ is the single lattice point in the interior Δ° of the canonical Fano d-tope Δ, i.e., $\Delta^\circ \cap M = \{0\}$.

All canonical Fano 3-topes have been classified [16]. There exists exactly 674,688 canonical Fano 3-topes Δ. The aim of this paper is to present computational results of their Fine interiors Δ^{FI} and some related combinatorial invariants. These data are important for computing minimal smooth projective surfaces $\mathcal{S}_\Delta$ with $p_g = 1$ and $q = 0$ which are birational to affine non-degenerate hypersurfaces $\mathcal{Z}_\Delta \subseteq \mathbb{T}^3 \cong (\mathbb{C}^\times)^3$.

The simplest description of the minimal surface $\mathcal{S}_\Delta$ has been obtained when Δ is a reflexive 3-tope [5].

Definition 8 A lattice d-tope $\Delta \subseteq M_{\mathbb{Q}}$ containing the origin $0 \in M$ in its interior is called *reflexive* if the dual polytope

$$\Delta^* := \{y \in N \mid \langle x, y\rangle \geq -1 \ \text{ for all } x \in \Delta\} \subseteq N_{\mathbb{Q}}$$

is a lattice polytope.

There exist 4,319 reflexive 3-topes, classified by Kreuzer and Skarke [17]. They form a small subset in the list of all 674,688 canonical Fano 3-topes [16]. Reflexive 4-topes are also classified by Kreuzer and Skarke [18]. There exist 473,800,776 reflexive 4-topes, but the complete list of all canonical Fano 4-topes is unknown and expected to be much bigger.

If Δ is a reflexive d-tope, then X_Δ is a Gorenstein toric Fano d-fold and the Zariski closure $\overline{Z}_\Delta$ in X_Δ is a Gorenstein Calabi-Yau $(d-1)$-fold. If $d = 3$, then $\overline{Z}_\Delta$ is a $K3$-surface with at worst finitely many Du Val singularities of type A_k. The minimal surface $\mathcal{S}_\Delta$ is a smooth $K3$-surface which is obtained as the minimal (crepant) desingularization of $\overline{Z}_\Delta$ [5].

One motivation for the present paper is due to Corti and Golyshev, who have found 9 interesting examples of canonical Fano 3-simplices Δ such that the affine surfaces $\mathcal{Z}_\Delta$ are birational to elliptic surfaces of Kodaira dimension $\kappa = 1$ [11].

The computation of the Fine interior Δ^{FI} for all canonical Fano 3-topes $\Delta \subseteq M_{\mathbb{Q}}$ has shown that the dimension of the Fine interior Δ^{FI} has only three values: 0, 1, and 3. It is rather surprising that there are no canonical Fano 3-topes Δ with $\dim(\Delta^{\mathrm{FI}}) = 2$.

The condition $\dim(\Delta^{\mathrm{FI}}) = 0$ holds if and only if Δ^{FI} equals the lattice point $0 \in M$. There exist exactly 665,599 canonical Fano 3-topes with $\Delta^{\mathrm{FI}} = \{0\}$, where $0 \in M$ is the only interior lattice point of Δ. These polytopes are characterized in [3, Proposition 3.4] by the condition that $0 \in N$ is an interior lattice point of the lattice 3-tope

$$[\Delta^*] := \mathrm{conv}(\Delta^* \cap N).$$

Remark 9 If Δ is a canonical Fano 3-tope, then $\Delta^{\mathrm{FI}} = \{0\}$ if and only if the non-degenerate affine surface $\mathcal{Z}_\Delta$ is birational to a $K3$-surface [3, Theorem 2.26].

The case $\dim(\Delta^{\mathrm{FI}}) = 1$ splits in two subcases. There exists exactly 20 canonical Fano 3-topes Δ such that $0 \in M$ is the midpoint of the Fine interior Δ^{FI}. Therefore, we call this Fine interior *symmetric*. Canonical Fano 3-topes with 1-dimensional symmetric Fine interior are characterized by the condition that $[\Delta^*]$ is a reflexive 2-tope.

The Fine interior of the remaining 9,020 canonical Fano 3-topes with $\dim(\Delta^{\mathrm{FI}}) = 1$ contains $0 \in M$ as a vertex. Therefore, we call this Fine interior *asymmetric*. Canonical Fano 3-topes with 1-dimensional asymmetric Fine interior are combinatorially characterized by the condition that $0 \in N$ is contained in the relative interior of a facet $\Theta \preceq [\Delta^*]$ of the lattice 3-tope $[\Delta^*]$. The minimal surfaces $\mathcal{S}_\Delta$ corresponding to canonical Fano 3-topes with 1-dimensional Fine interior (symmetric and asymmetric) are elliptic surfaces of Kodaira dimension $\kappa = 1$.

There exist exactly 49 canonical Fano 3-topes with $\dim(\Delta^{\mathrm{FI}}) = 3$. These polytopes are characterized by the condition that $0 \in N$ is a vertex of the lattice 3-tope $[\Delta^*]$. The surfaces $\mathcal{S}_\Delta$ corresponding to canonical Fano 3-topes Δ with 3-dimensional Fine interior Δ^{FI} are of general type (i.e., $\mathcal{S}_\Delta$ has maximal Kodaira dimension $\kappa = \dim(\mathcal{S}_\Delta) = 2$).

Remark 10 The Fine interior computations were done using

$$\Delta^{\mathrm{FI}} = \bigcap_{\theta \preceq \Delta} \bigcap_{n \in \mathcal{H}(\sigma^\theta)} \left\{ x \in M_{\mathbb{Q}} \,\middle|\, \langle x, n \rangle \geq \mathrm{ord}_\Delta(n) + 1 \right\},$$

where $\mathcal{H}(\sigma^\theta)$ denotes the set of all irreducible elements in the monoid $\sigma^\theta \cap N$. It is the minimal generating set of the monoid $\sigma^\theta \cap N$ and is called *Hilbert basis* of $\sigma^\theta \cap N$.

In the next sections we consider examples and discuss additional properties of canonical Fano 3-topes Δ in dependence of their Fine interiors Δ^{FI}. All computations were done using the Graded Ring Database [8], including the data of all 674,688 canonical Fano 3-topes, and MAGMA [7]. Therefore, all statements have been checked by computer calculations. The canonical Fano 3-topes used as examples in this paper appear with an ID that is the example's ID in the Graded Ring Database.[1]

2.2 Almost Reflexive Polytopes of Dimension 3 and 4

Definition 11 A canonical Fano d-tope $\Delta \subseteq M_{\mathbb{Q}}$ is called *almost reflexive* if the convex hull of all N-lattice points in the dual polytope Δ^* is reflexive.

It is easy to show the following statement:

Proposition 12 *If a canonical Fano d-tope Δ is almost reflexive, then*

$$\Delta^{FI} = \{0\}.$$

Proof If $[\Delta^*]$ is reflexive, then $\Delta = (\Delta^*)^*$ is contained in the dual reflexive polytope $[\Delta^*]^*$. Therefore, the Fine interior of Δ is contained in the Fine interior of the reflexive polytope $[\Delta^*]^*$ and $([\Delta^*]^*)^{\mathrm{FI}} = \{0\}$. Thus, $\Delta^{\mathrm{FI}} = \{0\}$. □

[1] http://www.grdb.co.uk/forms/toricf3c.

The converse statement is not true in general for $d \geq 5$, but there exist many equivalent characterizations of reflexive and almost reflexive d-topes among canonical Fano d-topes if $d = 3$ or $d = 4$.

Let us recall some combinatorial invariants of arbitrary lattice d-topes.

Definition 13 The *Ehrhart power series* of an arbitrary lattice d-tope $\Delta \subseteq M_{\mathbb{Q}}$ is defined as

$$P_\Delta(t) := \sum_{k \geq 0} |k\Delta \cap M|\, t^k,$$

where $|k\Delta \cap M|$ denotes the number of lattice points in the k-th dilate $k\Delta$ of Δ.

This Ehrhart series is a rational function of the form

$$P_\Delta(t) = \frac{\psi_d(\Delta)t^d + \cdots + \psi_1(\Delta)t + \psi_0(\Delta)}{(1-t)^{d+1}},$$

where $\psi_i(\Delta)$ are non-negative integers for all $0 \leq i \leq d$ [22] such that $\psi_0(\Delta) = 1$ and $\psi_1(\Delta) = |\Delta \cap M| - d - 1$. Moreover, $\sum_{i=0}^{d} \psi_i(\Delta) = v(\Delta)$, where $v(\Delta) := d! \cdot \mathrm{vol}(\Delta)$ denotes the *normalized volume* of Δ.

One has the following characterization of reflexive d-topes:

Proposition 14 ([6, Theorem 4.6]) *A canonical Fano d-tope Δ is reflexive if and only if*

$$\psi_i(\Delta) = \psi_{d-i}(\Delta) \quad (0 \leq i \leq d).$$

The Ehrhart reciprocity implies that the power series

$$Q_\Delta(t) := \sum_{k \geq 1} |(k\Delta)^\circ \cap M|\, t^k$$

is a rational function

$$Q_\Delta(t) = \frac{\varphi_{d+1}(\Delta)t^{d+1} + \cdots + \varphi_2(\Delta)t + \varphi_1(\Delta)t + \varphi_0(\Delta)}{(1-t)^{d+1}},$$

where $\varphi_0(\Delta) = 0$ and $\varphi_1(\Delta) = |\Delta^\circ \cap M|$. Using Serre duality, one obtains [12, Sect. 4, 5.11]

$$\varphi_i(\Delta) = \psi_{d+1-i}(\Delta) \quad (1 \leq i \leq d+1),$$

i.e., in particular

$$\psi_d(\Delta) = \varphi_1(\Delta) = |\Delta^\circ \cap M|$$

and

$$\psi_{d-1}(\Delta) = \varphi_2(\Delta) = |2\Delta^\circ \cap M| - (d+1)|\Delta^\circ \cap M|.$$

Therefore, the lattice d-tope Δ is a canonical Fano d-tope if and only if $\psi_d(\Delta) = 1$. Moreover,

$$\psi_{d-1}(\Delta) = |(2\Delta)^\circ \cap M| - (d+1)$$

if Δ is a canonical Fano d-tope.

Applying the above equations, one immediately obtains the following criterion for reflexivity of canonical Fano d-topes in the case $d = 3, 4$:

Proposition 15 *Let* $\Delta \subseteq M_{\mathbb{Q}}$ *be a canonical Fano* d*-tope with* $d \in \{3, 4\}$. *Then for* $d = 3$, *one has*

$$P_\Delta(t) = \frac{t^3 + (|(2\Delta)^\circ \cap M| - 4)t^2 + (|\Delta \cap M| - 4)t + 1}{(1-t)^4}$$

and for $d = 4$, *one obtains*

$$P_\Delta(t) = \frac{t^4 + (|(2\Delta)^\circ \cap M| - 5)t^3 + \psi_2(\Delta)t^2 + (|\Delta \cap M| - 5)t + 1}{(1-t)^5}.$$

In particular, Δ *is reflexive if and only if*

$$|\Delta \cap M| = |(2\Delta)^\circ \cap M|.$$

Proposition 16 *Let* $\Delta \subseteq M_{\mathbb{Q}}$ *be a canonical Fano* d*-tope with* $d \in \{3, 4\}$ *such that* $0 \in N$ *is an interior lattice point of* $[\Delta^*]$. *Then* $[\Delta^*]$ *is reflexive, i.e.,* Δ *is almost reflexive.*

Proof Let $n \in N$ be an interior lattice point of $[\Delta^*]$. Then $\langle x, n\rangle \geq 0$ for all $x \in \Delta \cap M$ because

$$\Delta^* = \{y \in N_{\mathbb{Q}} \mid \langle x, y\rangle \geq -1 \quad \text{for all } x \in \Delta\}$$

and $\langle x, n\rangle$ is an integer. Since $0 \in \Delta^\circ \cap M$, $M_{\mathbb{Q}}$ is the set of all non-negative $\mathbb{Q}$-linear combinations of all lattice points in $\Delta \cap M$. This implies $\langle x', n\rangle \geq 0$ for all $x' \in M_{\mathbb{Q}}$, i.e., $n = 0$. Therefore, $[\Delta^*]$ has only one interior lattice point $0 \in N$, i.e., $[\Delta^*]$ is a canonical Fano d-tope.

It is clear that $[\Delta^*]$ is contained in the interior of $2[\Delta^*]$. Therefore, we have $[\Delta^*] \cap N \subseteq (2[\Delta^*])^\circ \cap N$. On the other hand, for any lattice point $n \in (2[\Delta^*])^\circ$, $\langle x, n\rangle > -2$ for all $x \in \Delta \cap M$. Since $\langle x, n\rangle$ is an integer, $n \in \Delta^* \cap N$, i.e.,

$$[\Delta^*] \cap N = (2[\Delta^*])^\circ \cap N.$$

Using Proposition 15, $[\Delta^*]$ is reflexive. □

Corollary 17 *Let* $\Delta \subseteq M_{\mathbb{Q}}$ *be a canonical Fano* d*-tope with* $d \in \{3, 4\}$ *such that* $0 \in N$ *is an interior lattice point of* $[\Delta^*]$. *Then* $[\Delta^*]^*$ *is the smallest* (*referring to inclusion*) *reflexive polytope containing* Δ.

Proof Let $\Delta' \subseteq M_{\mathbb{Q}}$ be a reflexive d-tope such that $\Delta \subseteq \Delta'$. Then $(\Delta')^* \subseteq \Delta^*$. Since $(\Delta')^*$ is a lattice polytope, it is contained in $[\Delta^*]$. Thus, $[\Delta^*]^*$ is contained in $((\Delta')^*)^* = \Delta'$. □

Remark 18 If Δ is a reflexive d-tope, then $[2\Delta^\circ] = \Delta$. If Δ is a canonical Fano d-tope with $d \in \{3, 4\}$ such that $\Delta^{\mathrm{FI}} = \{0\}$ and Δ is contained in a reflexive d-tope Δ', then $[2\Delta^\circ]$ is contained in $[(2\Delta')^\circ] = \Delta'$. Therefore, $[2\Delta^\circ]$ is contained in the smallest reflexive polytope $[\Delta^*]^*$ containing Δ, i.e.,

$$[2\Delta^\circ] \subseteq [\Delta^*]^*.$$

Computations showed that among all 665,599 canonical Fano 3-topes Δ with $\Delta^{\mathrm{FI}} = \{0\}$ there exist exactly 211,941 canonical Fano 3-tops such that $[2\Delta^\circ]$ is reflexive. For the remaining canonical Fano 3-topes Δ the lattice 3-topes $[2\Delta^\circ]$ are larger than Δ, but are not equal to the reflexive hull $[\Delta^*]^*$.

Remark 19 Let Δ be an almost reflexive 3-tope. We denote by $\tau(\Delta)$ the lattice d-tope $[2\Delta^\circ]$. If $\tau(\Delta)$ is not reflexive, then it is almost reflexive and we can consider the larger lattice d-tope $\tau^2(\Delta) := \tau(\tau(\Delta)) \subseteq [\Delta^*]^*$. After at most five steps, $\tau^k(\Delta)$ is equal to the reflexive hull $[\Delta^*]^*$ of Δ.

In dimension 4, the situation is comparable:

Example 20 Let $\Delta \subseteq \mathbb{R}^4$ be the almost reflexive 4-tope defined by the inequalities $x_i \geq -1$ $(1 \leq i \leq 4)$, $x_1 \leq 2$, and $x_1 + x_2 + x_3 + x_4 \leq 1$. Then $\Delta^{FI} = \{0\}$ and the smallest reflexive 4-tope containing Δ is the 4-simplex $[\Delta^*]^*$ defined by the inequalities $x_i \geq -1$ $(1 \leq i \leq 4)$ and $x_1 + x_2 + x_3 + x_4 \leq 1$. It is easy to see that $\tau(\Delta)$ is not the reflexive 4-tope $[\Delta^*]^*$ because the vertex $(4, -1, -1, -1) \in \mathrm{vert}([\Delta^*]^*)$ is not in $2\Delta^\circ$. However, $\tau^2(\Delta) = [\Delta^*]^*$.

2.3 Canonical Fano 3-Topes with $\Delta^{\mathrm{FI}} = \{0\}$

We note that the set of all reflexive 3-topes forms a rather small part of the set of all canonical Fano 3-topes. The majority of the canonical Fano 3-topes belong to the subset of almost reflexive 3-topes. The proof of the following statement is based on the result of Skarke [21] and the explanations in the previous section.

Proposition 21 *A canonical Fano 3-tope Δ is almost reflexive if one of the following equivalent conditions is satisfied:*

1. $\Delta^{\text{FI}} = \{0\}$;
2. $0 \in N$ *is an interior lattice point of* $[\Delta^*]$;
3. Δ *is contained in some reflexive* 3-*tope;*
4. $\tau^k(\Delta)$ *is the reflexive* 3-*tope* $[\Delta^*]^*$ *for some sufficiently large* k $(1 \leq k \leq 5)$;
5. *the lattice* 3-*tope* $[2\Delta^\circ]$ *has exactly one interior lattice point;*
6. *the non-degenerate affine hypersurface* $\mathcal{Z}_\Delta$ *defined by a Laurent polynomial with Newton polytope* Δ *is birational to a smooth* $K3$-*surface.*

Computations show that there exist exactly 665,599 almost reflexive canonical Fano 3-topes. The set of almost reflexive 3-topes includes all 4,319 reflexive 3-topes. We have shown that for any almost reflexive 3-tope Δ, the reflexive polytope $\Delta^{\text{ref}} := [\Delta^*]^*$ is the smallest reflexive 3-tope containing Δ. We call Δ^{ref} the *reflexive hull* of Δ. Thus we obtain a natural surjective map $\Delta \mapsto \Delta^{\text{ref}}$ from the set of almost reflexive 3-topes to the set of reflexive 3-topes, which is the identity on the set of reflexive 3-topes. The minimal surface $\mathcal{S}_\Delta$ is a $K3$-surface if and only if Δ is an almost reflexive 3-tope. If Δ is an almost reflexive 3-tope, but not reflexive, then the minimal surface $\mathcal{S}_\Delta$ is a crepant desingularization of the Zariski closure of $\mathcal{Z}_\Delta$ in the Gorenstein toric Fano threefold $X_{\Delta^{\text{ref}}}$ defined by the reflexive hull of Δ.

A generalization of the reflexive hull of almost reflexive 3-topes for arbitrary lattice d-topes with non-empty Fine interior can be obtained using the notion of the support of the Fine interior Δ^{FI}.

Definition 22 Let $\Delta \subseteq M_{\mathbb{Q}}$ be a lattice d-tope with $\Delta^{\text{FI}} \neq \emptyset$. Then the set

$$\text{supp}(\Delta^{\text{FI}}) := \{n \in N \mid \text{ there exists } x \in \Delta^{\text{FI}} \text{ with } \langle x, n\rangle = \text{ord}_\Delta(n) + 1\}$$

is called *support of the Fine interior* of Δ.

Example 23 If Δ is a reflexive d-tope, then the support of the Fine interior of Δ is the set of all non-zero lattice points in $\Delta^* \cap N$.

Remark 24 It is easy to show that one always has

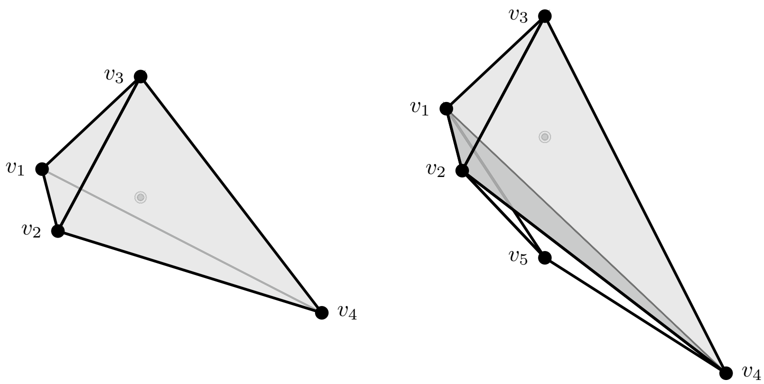
$$\Delta^{\text{FI}} = \bigcap_{n \in \text{supp}(\Delta^{\text{FI}})} \{x \in M_{\mathbb{Q}} \mid \langle x, n\rangle \geq \text{ord}_\Delta(n) + 1\}.$$

Definition 25 Let $\Delta \subseteq M_{\mathbb{Q}}$ be a lattice d-tope with $\Delta^{\text{FI}} \neq \emptyset$. Then the rational polytope

$$\Delta^{\text{can}} := \bigcap_{n \in \text{supp}(\Delta^{\text{FI}})} \{x \in M_{\mathbb{Q}} \mid \langle x, n\rangle \geq \text{ord}_\Delta(n)\}$$

contains Δ and is called the *canonical hull* of Δ.

Example 26 If Δ is an almost reflexive 3-tope, then $\text{supp}(\Delta^{\text{FI}})$ is the set $(\Delta^* \cap N) \setminus \{0\}$ of boundary lattice points in the reflexive 3-tope $[\Delta^*]$ and the canonical



(a) ID 547386. (b) ID 547385.

Fig. 2.1 Canonical Fano 3-topes Δ with Δ^{FI}=$\{0\}$. Shaded faces are occluded and the Fine interior $\{0\}$ is shown in grey with a double border. The whole polytope is the canonical hull Δ^{can} as well as the reflexive hull Δ^{ref} and the grey coloured polytope is Δ. **a** Reflexive polytope $\Delta = \mathrm{conv}\{v_1, v_2, v_3, v_4\}$ with $v_1 = (1, 0, 0), v_2 = (0, 1, 0), v_3 = (0, 0, 1), v_4 = (-1, -1, -1)$, and $\Delta^{\mathrm{ref}} = \Delta^{\mathrm{can}} = \Delta$. All facets of Δ have lattice distance 1 to the origin. **b** Almost reflexive polytope $\Delta = \mathrm{conv}\{v_1, v_2, v_3, v_4\}$ with $v_1 = (1, 0, 0), v_2 = (0, 1, 0), v_3 = (0, 0, 1), v_4 = (-1, -1, -2)$, and $\Delta^{\mathrm{ref}} = \Delta^{\mathrm{can}} = \mathrm{conv}\{v_1, v_2, v_3, v_4, v_5\}$ with $v_5 = (0, 0, -1)$ reflexive. The dark grey coloured facet of Δ has lattice distance 2 and all other facets have lattice distance 1 to the origin

hull Δ^{can} equals the reflexive hull Δ^{ref} of the polytope Δ, i.e., $\Delta^{\mathrm{can}} = \Delta^{\mathrm{ref}} = [\Delta^*]^*$. In particular, in this case Δ^{can} is always a lattice 3-tope.

There exists a smooth projective toric variety X_Σ defined by a fan Σ whose 1-dimensional cones are generated by all lattice vectors from the finite set $\mathrm{supp}(\Delta^{\mathrm{FI}})$. Then the minimal surface $\mathcal{S}_\Delta$ is a $K3$-surface which is the Zariski closure of $\mathcal{Z}_\Delta$ in X_Σ [3].

Example 27 Let us consider the (almost) reflexive canonical Fano 3-tope $\Delta = \mathrm{conv}\{v_1, v_2, v_3, v_4\} \subseteq M_{\mathbb{Q}}$ (ID 547386, Fig. 2.1a) with vertices

$$v_1 := (1, 0, 0),\ v_2 := (0, 1, 0),\ v_3 := (0, 0, 1),\ \text{and } v_4 := (-1, -1, -1)$$

and $\Delta^{\mathrm{FI}} = \{0\}$. Moreover,

$$\Delta^{\mathrm{ref}} = \mathrm{conv}(2\Delta^\circ \cap M) = \mathrm{conv}(\Delta \cap M) = \Delta$$

and

$$\Delta^{\mathrm{can}} = [\Delta^*]^* = (\Delta^*)^* = \Delta$$

because Δ is reflexive, i.e., $\Delta^{\mathrm{ref}} = \Delta^{\mathrm{can}} = \Delta$ reflexive (Fig. 2.1a).

Example 28 Consider the almost reflexive canonical Fano 3-tope $\Delta \subseteq M_{\mathbb{Q}}$ (ID 547385, Fig. 2.1b) with vertices

$$v_1 := (1, 0, 0),\ v_2 := (0, 1, 0),\ v_3 := (0, 0, 1),\ \text{and } v_4 := (-1, -1, -2)$$

and $\Delta^{\mathrm{FI}} = \{0\}$. Moreover,

$$\Delta^{\mathrm{ref}} = \mathrm{conv}((\Delta \cap M) \cup \{v_5\}) = \mathrm{conv}\{v_1, v_2, v_3, v_4, v_5\}$$

and

$$\Delta^{\mathrm{can}} = [\Delta^*]^* = \mathrm{conv}\{v_1, v_2, v_3, v_4, v_5\}$$

with $v_5 := (0, 0, -1)$ because Δ is almost reflexive, i.e., $\Delta^{\mathrm{ref}} = \Delta^{\mathrm{can}} = \Delta$ reflexive (Fig. 2.1b).

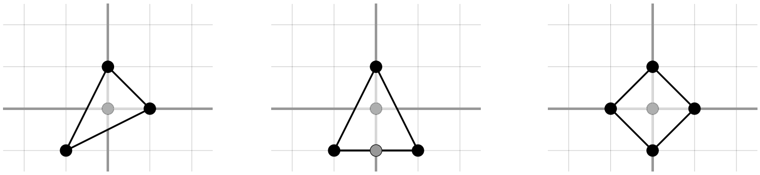
2.4 Asymmetric Fine Interior of Dimension 1

There exist exactly 9,020 canonical Fano 3-topes Δ with 1-dimensional Fine interior such that $0 \in N$ belongs to a facet $\Theta \preceq [\Delta^*]$ of the lattice 3-tope $[\Delta^*]$. This class of canonical Fano 3-topes is characterized by the property that the lattice 3-tope $[2\Delta^\circ]$ has exactly 2 interior lattice points.

The corresponding minimal surfaces $\mathcal{S}_\Delta$ are *simply connected* (i.e., have trivial fundamental group $\pi_1(\mathcal{S}_\Delta)$) elliptic surfaces of Kodaira dimension $\kappa = 1$. We observed that the facet $\Theta \preceq [\Delta^*]$ is a reflexive 2-tope corresponding to one of the three types pictured in Fig. 2.2. All N-lattice points on the boundary of Θ belong to $\mathrm{supp}(\Delta^{\mathrm{FI}})$. It was checked that for all these 3-topes Δ the canonical hull Δ^{can} is again a lattice 3-tope. Moreover, the Fine interior Δ^{FI} is contained in the ray generated by the primitive lattice vector $v_\Delta \in M$ which is the primitive inward-pointing facet normal of Θ, i.e., $\langle x, y\rangle = 0$ for all $x \in \Delta^{\mathrm{FI}}$, $y \in \Theta$. The lattice point $0 \in M$ is a vertex of Δ^{FI}. More precisely, one has

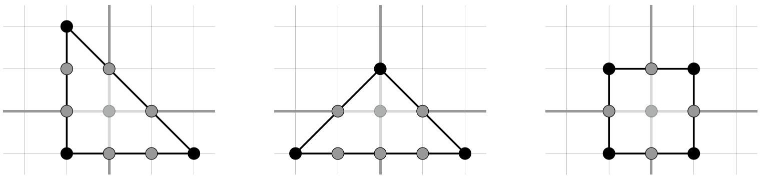
$$\Delta^{\mathrm{FI}} = \mathrm{conv}\{0, \lambda v_\Delta\},$$

where $\lambda \in \{1/2, 2/3\}$. The primitive lattice vector v_Δ is the unique interior lattice point on a reflexive facet $\theta_+ \preceq \Delta$ of Δ of one of the three possible types pictured in Fig. 2.2. These three reflexive polygons θ_+ are characterized by the condition that the dual reflexive polygons θ_+^* are obtained from θ_+ (Fig. 2.3) by enlarging the lattice $\mathbb{Z}^2$ in the following ways: $\mathbb{Z}^2 + \mathbb{Z}(1/3, 2/3)$ (Fig. 2.3a), $\mathbb{Z}^2 + \mathbb{Z}(1/2, 0)$ (Fig. 2.3b), and $\mathbb{Z}^2 + \mathbb{Z}(1/2, 1/2)$ (Fig. 2.3c). Moreover, the reflexive facet θ_+ of Δ is isomorphic to the facet Θ of $[\Delta^*]$. The projection $M \to M/\mathbb{Z}v_\Delta$ of Δ or of θ_+ along v_Δ is a reflexive polygon of one of the three types pictured in Fig. 2.3, which is dual to θ_+ and Θ. The lattice vector v_Δ defines a character of the 3-dimensional torus $\chi : \mathbb{T}^3 \to \mathbb{C}^\times$. For almost all $\alpha \in \mathbb{C}^\times$, the fiber $\chi^{-1}(\alpha)$ is an affine elliptic curve defined by a Laurent



(a) (b) (c)

Fig. 2.2 Reflexive Facets of Δ Containing $\pm v_\Delta$. Three types of reflexive facets $\theta_\pm \preceq \Delta$ of Δ containing $\pm v_\Delta$ for all 9,020 + 20 canonical Fano 3-topes Δ with $\dim(\Delta^{\mathrm{FI}}) = 1$. Vertices are coloured black, boundary points that are not vertices grey, and the origin light grey. **a** $\mathrm{conv}\{(1, 0), (0, 1), (-1, -1)\}$. **b** $\mathrm{conv}\{(1, 0), (-1, 1), (-1, -1)\}$. **c** $\mathrm{conv}\{(\pm 1, 0), (0, \pm 1)\}$



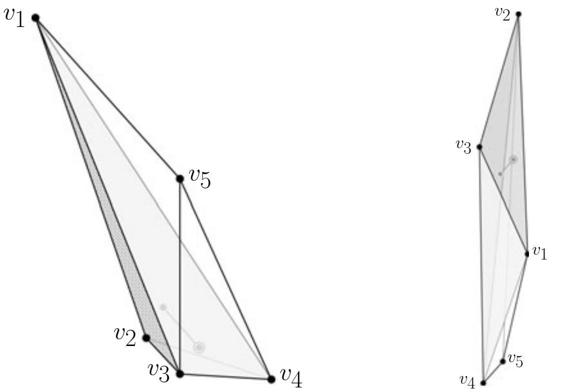
(a) (b) (c)

Fig. 2.3 Reflexive Projection Polytopes. Three types of reflexive polytopes obtained via a projection of Δ along $\pm v_\Delta$ for all 9,020 + 20 canonical Fano 3-topes Δ with $\dim(\Delta^{\mathrm{FI}}) = 1$. Vertices are coloured black, boundary points that are not vertices grey, and the origin light grey. **a** $\mathrm{conv}\{(-1, 2), (-1, -1), (2, -1)\}$. **b** $\mathrm{conv}\{(-2, -1), (0, 1), (2, -1)\}$. **c** $\mathrm{conv}\{(\pm 1, \pm 1)\}$

polynomial with the reflexive Newton polytope $\Theta^* \cong \theta_+^*$ of one of the three types pictured in Fig. 2.3 with the distribution shown in Table 2.1. So χ defines birationally an elliptic fibration.

Table 2.1 Distribution of the Reflexive Facets of Δ Containing $\pm \mathbf{v}_\Delta$. Table contains: Type of the reflexive facet $\theta_\pm$ containing $\pm v_\Delta$, type of the dual reflexive facet $\theta_\pm^*$, the enlarged lattice used to obtain $\theta_\pm^*$ from $\theta_\pm$, the number of canonical Fano 3-topes $\Delta_{\mathrm{asym}} := \{\Delta \mid$ 1-dim. asym. $\Delta^{\mathrm{FI}}\}$, and the number of canonical Fano 3-topes $\Delta_{\mathrm{sym}} := \{\Delta \mid$ 1-dim. sym. $\Delta^{\mathrm{FI}}\}$ with respect to the facet type of $\theta_\pm$ pictured in Fig. 2.2

$\theta_\pm$	$\theta_\pm^*$	Enlarged lattice	$\#\Delta_{\mathrm{asym}}$	$\#\Delta_{\mathrm{sym}}$
Figure 2.2a	Figure 2.3a	$\mathbb{Z}^2 + \mathbb{Z}(1/3, 2/3)$	3,038	7
Figure 2.2b	Figure 2.3b	$\mathbb{Z}^2 + \mathbb{Z}(1/2, 0)$	4,663	9
Figure 2.2c	Figure 2.3c	$\mathbb{Z}^2 + \mathbb{Z}(1/2, 1/2)$	1,319	4



(a) ID 547324 (b) ID 547323

Fig. 2.4 Canonical Fano 3-topes with Asymmetric Fine Interior of Dimension 1 Shaded faces are occluded. The Fine interior and the origin are shown in grey, with a double border around the origin. The facet θ_+ is grey dotted. **a** The whole polytope is $\Delta = \text{conv}\{v_1, v_2, v_3, v_4\}$ with $v_1 = (2, 3, 8)$, $v_2 = (1, 0, 0)$, $v_3 = (0, 1, 0)$, and $v_4 = (-1, -1, -1)$. Moreover, $\Delta^{\text{FI}} = \text{conv}\{(0, 0, 0), (1/2, 1/2, 1)\}, \theta_+ = \text{conv}\{v_1, v_2, v_3\}$, and $\Delta^{\text{can}} = \text{conv}\{v_1, v_2, v_3, v_4, v_5\}$ with $v_5 = (0, 1, 4)$. **b** The whole polytope is $\Delta = \text{conv}\{v_1, v_2, v_3, v_4\}$ with $v_1 = (-1, 1, -2)$, $v_2 = (1, -2, 3)$, $v_3 = (1, 0, 0)$, and $v_4 = (-2, 5, -3)$. Moreover, $\Delta^{\text{FI}} = \text{conv}\{(0, 0, 0), (0, 2/3, 0)\}$ and $\theta_+ = \text{conv}\{v_2, v_3, v_4\}$, and $\Delta^{\text{can}} = \text{conv}\{v_1, v_2, v_3, v_4, v_5\}$ with $v_5 = (-2, 4, -3)$

Example 29 Let $\Delta \subseteq M_{\mathbb{Q}}$ be a canonical Fano 3-tope given as the convex hull of

$$v_1 := (2, 3, 8),\ v_2 := (1, 0, 0),\ v_3 := (0, 1, 0),\ \text{and } v_4 := (-1, -1, -1)$$

(ID 547324, Fig. 2.4a, Tables 2.2 and 2.4). Then

$$\Delta^{\text{FI}} = \text{conv}\{(0, 0, 0), (1/2, 1/2, 1)\} = \text{conv}\{0, 1/2 \cdot v_\Delta\},$$

where $v_\Delta = (1, 1, 2)$. One has $v_1 + 2v_2 + v_3 = 4v_\Delta$. Therefore, v_Δ is the interior lattice point of the reflexive facet θ_+ of Δ with vertices v_1, v_2, v_3 and the images $\overline{v}_1, \overline{v}_2, \overline{v}_3$ of v_1, v_2, v_3 in $M/\mathbb{Z}v_\Delta$ are vertices of the dual reflexive triangle θ_+^* (Fig. 2.3b) satisfying the relation

$$\overline{v}_1 + 2\overline{v}_2 + \overline{v}_3 = 0.$$

To compute the canonical hull Δ^{can} of Δ, we obtain $\text{supp}(\Delta^{\text{FI}}) = \{s_i \mid 1 \leq i \leq 18\}$ with $s_1 := (-1, -1, 1)$, $s_2 := (-1, -1, 2)$, $s_3 := (-1, -1, 3)$, $s_4 := (-1, 0, 1)$, $s_5 := (-1, 0, 2)$, $s_6 := (-1, 1, 0)$, $s_7 := (-1, 1, 1)$, $s_8 := (-1, 2, 0)$, $s_9 := (-1, 3, -1)$, $s_{10} := (0, -1, 1), \ldots, s_{18} := (-2, -2, 1)$, which leads to

$$\Delta^{\text{can}} = \text{conv}\{v_1, v_2, v_3, v_4, v_5\}$$

with $v_5 := (0, 1, 4)$ (Fig. 2.4a).

Table 2.2 9 Canonical Fano 3-topes with Asymmetric Fine Interior of Dimension 1. Table contains: vertices vert(Δ) of Δ, vertices vert(Δ^{FI}) of the Fine interior Δ^{FI}, unique primitive lattice point $v_\Delta \in \theta_+$ in the reflexive facet $\theta_+ \preceq \Delta$, and weights $(w_i)_{0\le i\le 3}$ of the weighted projective 3-space $\mathbb{P}(w_0, \dots, w_3)$ appearing in [11]

ID	vert(Δ)	vert(Δ^{FI})	v_Δ	$(w_i)_{0\le i\le 3}$
547324	(2, 3, 8), (1, 0, 0), (0, 1, 0), (−1, −1, −1)	$0, 1/2 \cdot v_\Delta$	(1, 1, 2)	(1, 5, 6, 8)
547323	(−1, 1, −2), (1, −2, 3), (1, 0, 0), (−2, 5, −3)	$0, 2/3 \cdot v_\Delta$	(0, 1, 0)	(1, 4, 7, 9)
547311	(−1, 4, 2), (−1, −1, 0), (0, 0, −1), (2, 0, 1)	$0, 2/3 \cdot v_\Delta$	(0, 1, 1)	(2, 5, 8, 9)
547490	(1, 2, 4), (1, 0, 0), (1, −2, 3), (−1, 1, −2)	$0, 1/2 \cdot v_\Delta$	(0, 1, 0)	(1, 5, 8, 14)
547321	(1, −2, 3), (0, 1, 0), (1, 0, 0), (−6, 3, −8)	$0, 1/2 \cdot v_\Delta$	(−1, 1, −2)	(3, 7, 8, 10)
547305	(0, 1, 0), (1, 0, 0), (1, 2, 4), (−4, −6, −7)	$0, 2/3 \cdot v_\Delta$	(−1, −1, −1)	(4, 7, 9, 10)
547526	(1, 0, 0), (0, 1, 0), (−2, 1, 5), (2, −4, −9)	$0, 2/3 \cdot v_\Delta$	(1, −1, −3)	(5, 9, 8, 11)
547454	(2, 1, 7), (1, 0, 0), (0, 1, 0), (−2, −3, −3)	$0, 1/2 \cdot v_\Delta$	(0, 0, 1)	(3, 7, 8, 18)
547446	(0, 1, 1), (−6, 7, −15), (1, −2, 3), (1, 0, 0)	$0, 1/2 \cdot v_\Delta$	(−1, 1, −2)	(5, 8, 9, 22)

Table 2.3 9 Canonical Fano 3-topes with Asymmetric Fine Interior of Dimension 1. Table contains: primitive inward-pointing facet normals $(n_i)_{1\le i\le 4}$ of Δ, vertices vert(θ_+) of the reflexive facet $\theta_+ \preceq \Delta$, and primitive inward-pointing facet normal n_{θ_+} of the reflexive facet $\theta_+ \preceq \Delta$

ID	$(n_i)_{1\le i\le 4}$	vert(θ_+)	n_{θ_+}
547324	(−2, −2, 1), (−1, −1, 3), (−1, 3, −1), (7, −3, −1)	(2, 3, 8), (1, 0, 0), (0, 1, 0)	(−2, −2, 1)
547323	(−3, −3, −2), (−1, 0, 1), (−1, 6, 4), (17, 3, −5)	(1, −2, 3), (1, 0, 0), (−2, 5, −3)	(−3, −3, −2)
547311	(−1, −1, 1), (−1, 2, 1), (1, 2, −5), (7, −2, 5)	(−1, 4, 2), (−1, −1, 0), (2, 0, 1)	(1, 2, −5)
547490	(−2, −2, 1), (−1, 0, 0), (−1, 6, 4), (23, 2, −8)	(1, 2, 4), (1, 0, 0), (−1, 1, −2)	(−2, −2, 1)
547321	(−3, −3, −2), (−2, −2, 1), (−1, 3, 2), (9, −5, −8)	(0, 1, 0), (1, 0, 0), (−6, 3, −8)	(−2, −2, 1)
547305	(−7, −7, 11), (−2, −2, 1), (−1, 2, −1), (7, −3, −1)	(0, 1, 0), (1, 2, 4), (−4, −6, −7)	(7, −3, −1)
547526	(−5, −5, −2), (−3, −3, 1), (−1, 2, −1), (25, −8, 10)	(1, 0, 0), (0, 1, 0), (2, −4, −9)	(−3, −3, 1)
547454	(−7, −7, 2), (−1, −1, 2), (−1, 1, 0), (7, −2, −2)	(2, 1, 7), (0, 1, 0), (−2, −3, −3)	(7, −2, −2)
547446	(−9, 21, 14), (−5, −3, −2), (−1, −1, 0), (9, 1, −3)	(0, 1, 1), (−6, 7, −15), (1, −2, 3)	(9, 1, −3)

Example 30 Let $\Delta \subseteq M_{\mathbb{Q}}$ be a canonical Fano 3-tope given as the convex hull of

$$v_1 := (-1, 1, -2),\ v_2 := (1, -2, 3),\ v_3 := (1, 0, 0),\ \text{and } v_4 := (-2, 5, -3)$$

(ID 547323, Fig. 2.4b, Tables 2.2 and 2.4). Then (Table 2.3)

$$\Delta^{\mathrm{FI}} = \mathrm{conv}\{(0, 0, 0), (0, 2/3, 0)\} = \mathrm{conv}\{0, 2/3 \cdot v_\Delta\},$$

where $v_\Delta = (0, 1, 0)$. One has $v_2 + v_3 + v_4 = 3v_\Delta$. Therefore, v_Δ is the interior lattice point of the reflexive facet θ_+ of Δ with vertices v_2, v_3, v_4 and the images $\overline{v}_2, \overline{v}_3, \overline{v}_4$ of v_2, v_3, v_4 in $M/\mathbb{Z}v_\Delta$ are vertices of the dual reflexive triangle θ_+^* (Fig. 2.3a) satisfying the relation

$$\overline{v}_2 + \overline{v}_3 + \overline{v}_4 = 0.$$

Table 2.4 9 Canonical Fano 3-topes with Asymmetric Fine Interior of Dimension 1. Table contains: vertices $\mathrm{vert}(\Theta)$ of the reflexive facet $\Theta \preceq [\Delta^*]$, support $\mathrm{supp}(\Delta^{FI})$ of the Fine interior Δ^{FI}, and vertices $\mathrm{vert}(\Delta^{can})$ of the canonical hull Δ^{can}

ID	$\mathrm{vert}(\Theta)$	$\mathrm{supp}(\Delta^{FI})$	$\mathrm{vert}(\Delta^{can})$
547324	$(-1,3,-1), (-1,-1,1), (1,-1,0)$	$(-2,-2,1), (-1,-1,1), S_1$	$\mathrm{vert}(\Delta), (0,1,4)$
547323	$(-1,0,1), (-1,0,0), (2,0,-1)$	$(-3,-3,-2), (-1,0,0), S_2$	$\mathrm{vert}(\Delta), (-2,4,-3)$
547311	$(-1,-1,1), (0,1,-1), (1,0,0)$	$(-1,-1,1), (-1,0,1), S_3$	$\mathrm{vert}(\Delta), (-1,2,0)$
547490	$(-1,0,0), (-1,0,1), (3,0,-1)$	$(-2,-2,1), (-1,0,0), S_4$	$\mathrm{vert}(\Delta), (1,-1,4)$
547321	$(-1,-1,0), (-1,3,2), (1,-1,-1)$	$(-2,-2,1), (-1,-1,0), (-1,0,0), (-1,1,1), (-1,3,2),$ $(0,-1,-1), (1,-1,-1)$	$\mathrm{vert}(\Delta), (1,0,1), (0,-3,4)$
547305	$(-1,2,-1), (1,-1,0), (0,-1,1)$	$(-1,-1,1), (-1,0,0), (-1,2,-1), (0,-1,1), (1,-1,0),$ $(7,-3,-1)$	$\mathrm{vert}(\Delta), (0,-2,-3), (1,2,2)$
547526	$(-1,-1,0), (-1,2,-1), (2,-1,1)$	$(-3,-3,1), (-1,-1,0), (-1,2,-1), (0,-1,0), (2,-1,1)$	$\mathrm{vert}(\Delta)\setminus\{(-2,1,5)\}, (0,1,3), (-3,1,6)$
547454	$(-1,1,0), (0,-1,0), (2,-1,0)$	$(-1,-1,1), (-1,-1,2), S_5$	$\mathrm{vert}(\Delta), (2,1,2)$
547446	$(-1,-1,0), (0,2,1), (2,0,-1)$	$(-1,-1,0), (-1,0,0), (0,2,1), (1,1,0), (2,0,-1), (9,1,-3)$	$\mathrm{vert}(\Delta), (1,0,-1), (1,0,3)$

where $S_1 := (-1,-1,2), (-1,-1,3), (-1,0,1), (-1,0,2), (-1,1,0), (-1,1,1), (-1,2,0), (-1,3,-1), (0,-1,1), (0,-1,2), (0,0,1), (0,1,0), (1,-1,0), (1,-1,1), (1,0,0), (2,-1,0)$

$S_2 := (-1,0,1), (-1,1,1), (-1,2,2), (-1,3,2), (-1,4,3), (-1,6,4), (0,1,1), (0,2,1), (0,3,2), (0,5,3), (1,1,0), (1,2,1), (1,4,2), (2,0,-1), (2,1,0), (2,3,1), (3,2,0), (4,1,-1)$

$S_3 := (-1,1,1), (-1,2,1), (0,0,1), (0,1,-1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,2,-5)$

$S_4 := (-1,0,1), (-1,1,1), (-1,2,2), (-1,3,2), (-1,4,3), (-1,6,4), (0,1,1), (0,2,1), (0,3,2), (0,5,3), (1,0,0), (1,1,0), (1,2,1), (1,4,2), (2,1,0), (2,3,1), (3,0,-1), (3,2,0), (4,1,-1)$

$S_5 := (-1,0,1), (-1,1,0), (0,-1,0), (0,-1,1), (1,-1,0), (2,-1,0), (7,-2,-2)$

To compute the canonical hull Δ^{can} of Δ, we obtain $\text{supp}(\Delta^{\text{FI}}) = \{s_i \mid 1 \le i \le 20\}$ with $s_1 := (-3, -3, -2)$, $s_2 := (-1, 0, 0)$, $s_3 := (-1, 0, 1)$, $s_4 := (-1, 1, 1)$, $s_5 := (-1, 2, 2)$, $s_6 := (-1, 3, 2)$, $s_7 := (-1, 4, 3)$, $s_8 := (-1, 6, 4)$, $s_9 := (0, 1, 1)$, $s_{10} := (0, 2, 1)$, ..., $s_{20} := (4, 1, -1)$, which leads to

$$\Delta^{\text{can}} = \text{conv}\{v_1, v_2, v_3, v_4, v_5\}$$

with $v_5 := (-2, 4, -3)$ (Fig. 2.4b).

Remark 31 The detailed information about a small selection of the 9,020 canonical Fano 3-topes with $\dim(\Delta^{\text{FI}}) = 1$ and $0 \in \text{vert}(\Delta^{\text{FI}})$ can be found in Tables 2.2, 2.3, and 2.4.

2.5 Symmetric Fine Interior of Dimension 1

There exist exactly 20 canonical Fano 3-topes Δ such that 0 is the center of 1-dimensional Fine interior Δ^{FI}. In this case, $\mathcal{S}_\Delta$ is an elliptic surface of Kodaira dimension $\kappa = 1$ with non-trivial fundamental group $\pi_1(\mathcal{S}_\Delta)$ of order 2 or 3. Computations show that one always has $\Delta = \Delta^{\text{can}}$ and

$$\Delta^{\text{FI}} = \text{conv}\{-\lambda v_\Delta, \lambda v_\Delta\}$$

with $\lambda = \frac{1}{2}$ if and only if $|\pi_1(\mathcal{S}_\Delta)| = 2$ and

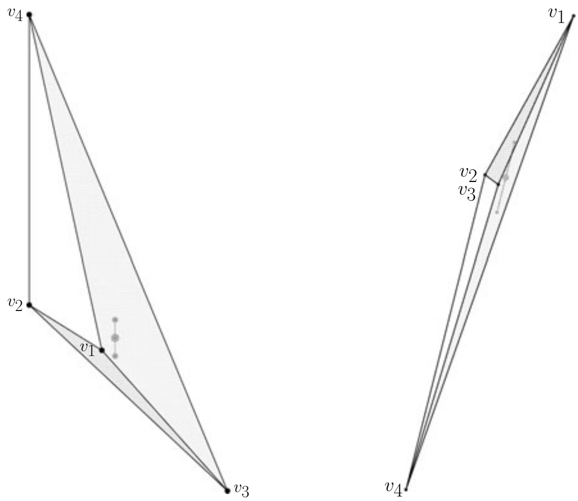
$$\Delta^{\text{FI}} = \text{conv}\{-\mu v_\Delta, \mu v_\Delta\}$$

with $\mu = \frac{2}{3}$ if and only if $|\pi_1(\mathcal{S}_\Delta)| = 3$. The primitive lattice vectors $\pm v_\Delta$ are the two unique interior lattice points in two reflexive facets $\theta_\pm \preceq \Delta$ of one of the three possible types pictured in Fig. 2.2. The reflexive facets $\theta_\pm$ of Δ are isomorphic to the facet Θ of $[\Delta^*]$. The projections $M \to M/\mathbb{Z}(\pm v_\Delta)$ of Δ or of $\theta_\pm$ along $\pm v_\Delta$ reveal a reflexive polygon of one of the three types pictured in Fig. 2.3, which is dual to $\theta_\pm$ and Θ. The lattice vector v_Δ defines a character of the 3-dimensional torus $\chi : \mathbb{T}^3 \to \mathbb{C}^\times$. For almost all $\alpha \in \mathbb{C}^\times$, the fiber $\chi^{-1}(\alpha)$ is an affine elliptic curve defined by a Laurent polynomial with the reflexive Newton polytope $\Theta^* \cong \theta_\pm^*$ of one of the three types pictured in Fig. 2.3 with the distribution shown in Table 2.1. So χ defines birationally an elliptic fibration. The vertex sets of Δ and these reflexive facets are related via $\text{vert}(\Delta) = \text{vert}(\theta_+) \cup \text{vert}(\theta_-)$. Moreover, every edge of Δ is either an edge of θ_+ or θ_- of these two facets or it is parallel to v_Δ.

Example 32 Let $\Delta \subseteq M_\mathbb{Q}$ be a canonical Fano 3-tope given as the convex hull

$$v_1 := (0, 1, 0),\ v_2 := (2, 1, 1),\ v_3 := (-2, -3, -5),\ \text{and}\ v_4 := (2, 1, 9)$$

(ID 547393, Fig. 2.5b, Tables 2.5 and 2.6). Then



(a) ID 547393 (b) ID 547409

Fig. 2.5 Canonical Fano 3-topes with Symmetric Fine Interior of Dimension 1. Shaded faces are occluded. The Fine interior and the origin are shown in grey with a double border around the origin. The facets $\theta_\pm$ are grey dotted. **a** The whole polytope is $\Delta = \text{conv}\{v_1, v_2, v_3, v_4\}$ with $v_1 = (1, 0, 0)$, $v_2 = (2, 1, 1)$, $v_3 = (-2, -3, -5)$, and $v_4 = (2, 1, 9)$. Moreover, $\Delta^{\text{FI}} = \text{conv}\{(0, 0, -1/2), (0, 0, 1/2)\}$, $\theta_+ = \text{conv}\{v_1, v_3, v_4\}$, $\theta_- = \text{conv}\{v_1, v_2, v_3\}$, and $\Delta^{\text{can}} = \Delta$. **b** The whole polytope is $\Delta = \text{conv}\{v_1, v_2, v_3, v_4\}$ with $v_1 = (-4, 2, 9)$, $v_2 = (1, 0, 0)$, $v_3 = (0, 1, 0)$, and $v_4 = (7, -6, -18)$. Moreover, $\Delta^{\text{FI}} = \text{conv}\{(-2/3, 2/3, 2), (2/3, -2/3, -2)\}$, $\theta_+ = \text{conv}\{v_1, v_2, v_3\}$, $\theta_- = \text{conv}\{v_1, v_3, v_4\}$, and $\Delta^{\text{can}} = \Delta$

$$\Delta^{\text{FI}} = \text{conv}\{(0, 0, -1/2), (0, 0, 1/2)\} = \text{conv}\{-\lambda v_\Delta, \lambda v_\Delta\}$$

with $\lambda = \frac{1}{2}$, where $v_\Delta = (0, 0, 1)$. One has $2v_1 + v_3 + v_4 = 4v_\Delta$ and $2v_1 + v_2 + v_3 = 4(-v_\Delta)$. Therefore, v_Δ is the interior lattice point of the reflexive facet $\theta_+ = \theta_{134}$ of Δ and $-v_\Delta$ is the interior lattice point of the reflexive facet $\theta_- = \theta_{123}$ of Δ (Fig. 2.2b). The images $\overline{v}_1, \overline{v}_3, \overline{v}_4$ of v_1, v_3, v_4 in $M/\mathbb{Z}v_\Delta$ and the images $\overline{v}_1, \overline{v}_2, \overline{v}_3$ of v_1, v_2, v_3 in $M/\mathbb{Z}(-v_\Delta)$ are vertices of the dual reflexive triangle $\theta_\pm^*$ (Fig. 2.3b) satisfying the relation

$$2\overline{v}_1 + \overline{v}_3 + \overline{v}_4 = 0$$

and

$$2\overline{v}_1 + \overline{v}_2 + \overline{v}_3 = 0,$$

respectively.

To compute the canonical hull Δ^{can} of Δ, we obtain $\text{supp}(\Delta^{\text{FI}}) = \{s_i \mid 1 \leq i \leq 6\}$ with $s_1 := (-1, -2, 2)$, $s_2 := (-1, 1, 0)$, $s_3 := (0, -1, 0)$, $s_4 := (1, -1, 0)$, $s_5 := (2, -1, 0)$, and $s_6 := (9, -2, -2)$, which leads to $\Delta^{\text{can}} = \Delta$.

Table 2.5 20 Canonical Fano 3-topes with Symmetric Fine Interior of Dimension 1. Table contains: vertices vert(Δ) of Δ

ID	vert(Δ)
547393	(0, 1, 0), (2, 1, 1), (−2, −3, −5), (2, 1, 9)
547409	(−4, 2, 9), (1, 0, 0), (0, 1, 0), (7, −6, −18)
547461	(0, 1, 0), (2, 1, 1), (−2, −3, −5), (0, 1, 4)
544442	(1, 0, 0), (0, 1, 0), (3, −6, 8), (1, −4, 4), (−5, 6, −12)
544443	(−1, −2, 0), (3, −6, 8), (0, 1, 0), (1, 0, 0), (−3, 4, −8)
544651	(−4, 1, −3), (4, −2, 3), (0, 1, 0), (1, −2, 3), (−1, 1, −3)
544696	(5, −4, −15), (1, 0, 0), (0, 1, 0), (−4, 2, 9), (−3, 1, 6)
544700	(−2, −3, −3), (0, 1, 0), (1, 0, 0), (−1, −4, −6), (2, 5, 9)
544749	(−6, −5, −8), (0, 1, 0), (1, 0, 0), (−2, −1, 0), (3, 2, 4)
520925	(0, 1, 0), (0, 0, 1), (−2, −1, 0), (−2, 0, −1), (8, 2, 3), (−2, −3, −2)
520935	(3, 4, 6), (2, 1, 2), (−3, −2, −2), (1, 0, 0), (0, 1, 0), (−6, −5, −8)
522056	(−1, −1, 0), (0, 1, 0), (1, 0, 0), (−1, −1, −3), (−5, −3, −6), (6, 4, 9)
522059	(2, 5, 6), (−2, −3, −3), (0, 1, 0), (1, 0, 0), (−1, −4, −6), (0, 1, 3)
522087	(1, 0, −3), (1, 0, 0), (0, 1, 0), (−4, 2, 9), (−3, 1, 6), (5, −4, −12)
522682	(2, 1, 4), (−3, −2, −4), (−2, −3, −4), (1, 2, 4), (1, 0, 0), (0, 1, 0)
522684	(−2, −1, −4), (3, 2, 4), (−2, −1, 0), (1, 0, 0), (0, 1, 0), (−4, −3, −4)
526886	(−3, 4, −6), (1, 0, 0), (0, 1, 0), (3, −6, 8), (0, 1, −2), (2, −5, 6)
439403	(1, 2, 2), (−1, 0, 0), (−1, 1, −1), (1, 0, 0), (−1, −2, −2), (1, 1, 3), (1, −3, −1)
275525	(4, 1, 2), (0, 1, 0), (−2, −1, 0), (1, 1, 2), (−3, −1, −2), (−2, −1, −2), (1, 1, 0), (1, −1, 0)
275528	(−1, 0, −1), (−3, −2, 1), (−2, −1, 2), (0, −1, 0), (0, 1, 0), (1, 0, 1), (2, 1, −2), (3, 2, −1)

Example 33 Let $\Delta \subseteq M_{\mathbb{Q}}$ be a canonical Fano 3-tope given as the convex hull

$$v_1 := (-4, 2, 9),\ v_2 := (1, 0, 0),\ v_3 := (0, 1, 0),\ \text{and } v_4 := (7, -6, -18)$$

(ID 547409, Fig. 2.5b, Tables 2.5 and 2.6). Then

$$\Delta^{\mathrm{FI}} = \mathrm{conv}\{(-2/3, 2/3, 2), (2/3, -2/3, -2)\} = \mathrm{conv}\{-\mu v_\Delta, \mu v_\Delta\}$$

with $\mu = \frac{2}{3}$, where $v_\Delta = (1, -1, -3)$. One has $v_1 + v_2 + v_3 = -3v_\Delta$ and $v_1 + v_3 + v_4 = -3(-v_\Delta)$. Therefore, v_Δ is the interior lattice point of the reflexive facet $\theta_+ = \theta_{123}$ of Δ and $-v_\Delta$ is the interior lattice point of the reflexive facet $\theta_- = \theta_{134}$ of Δ (Fig. 2.2b). The images $\overline{v}_1, \overline{v}_2, \overline{v}_3$ of v_1, v_2, v_3 in $M/\mathbb{Z}v_\Delta$ and the images $\overline{v}_1, \overline{v}_3, \overline{v}_4$ of v_1, v_3, v_4 in $M/\mathbb{Z}(-v_\Delta)$ are vertices of the dual reflexive triangle $\theta_\pm^*$ (Fig. 2.3b) satisfying the relation

$$\overline{v}_1 + \overline{v}_2 + \overline{v}_3 = 0,$$

and

$$\overline{v}_1 + \overline{v}_3 + \overline{v}_4 = 0,$$

respectively.

Table 2.6 20 Canonical Fano 3-topes with Symmetric Fine Interior of Dimension 1. Table contains: vertices $\mathrm{vert}(\Delta^{\mathrm{FI}})$ of the Fine interior Δ^{FI}, unique primitive lattice points $\pm v_\Delta \in \theta_\pm$ in the reflexive facets $\theta_\pm \preceq \Delta$, vertices $\mathrm{vert}(\theta_\pm)$ of the reflexive facets $\theta_\pm \preceq \Delta$, and support $\mathrm{supp}(\Delta^{\mathrm{FI}})$ of the Fine interior Δ^{FI} (here: $\Delta^{\mathrm{can}} = \Delta$)

ID	$\mathrm{vert}(\Delta^{\mathrm{FI}})$	$\pm v_\Delta$	$\mathrm{vert}(\theta_\pm)$	$\mathrm{supp}(\Delta^{\mathrm{FI}})$
547393	$\pm 1/2 \cdot v_\Delta$	$\pm(0, 0, 1)$	(0, 1, 0), (2, 1, 1), (−2, −3, −5) (0, 1, 0), (−2, −3, −5), (2, 1, 9)	(−1, −2, 2), (−1, 1, 0), (0, −1, 0), (1, −1, 0), (2, −1, 0), (9, −2, −2)
547409	$\pm 2/3 \cdot v_\Delta$	$\pm(1, -1, -3)$	(−4, 2, 9), (1, 0, 0), (0, 1, 0) (−4, 2, 9), (0, 1, 0), (7, −6, −18)	(−3, −3, −1), (−1, −1, 0), (−1, 2, −1), (2, −1, 1), (15, −3, 7)
547461	$\pm 1/2 \cdot v_\Delta$	$\pm(0, 0, 1)$	(0, 1, 0), (2, 1, 1), (−2, −3, −5) (2, 1, 1), (−2, −3, −5), (0, 1, 4)	(−3, 6, −2), (−1, −2, 2), (−1, 1, 0), (0, −1, 0), (1, −1, 0), (2, −1, 0)
544442	$\pm 1/2 \cdot v_\Delta$	$\pm(1, -1, 2)$	(0, 1, 0), (1, −4, 4), (−5, 6, −12) (1, 0, 0), (0, 1, 0), (3, −6, 8)	(−2, −2, −1), (−1, −1, 0), (−1, 1, 1), (1, −1, −1), (3, −1, −2), (10, −2, −5)
544443	$\pm 1/2 \cdot v_\Delta$	$\pm(1, -1, 2)$	(−1, −2, 0), (0, 1, 0), (−3, 4, −8) (3, −6, 8), (0, 1, 0), (1, 0, 0)	(−2, −2, −1), (−1, −1, 0), (−1, 1, 1), (1, −1, −1), (3, −1, −2), (6, −2, −3)
544651	$\pm 2/3 \cdot v_\Delta$	$\pm(1, 0, 0)$	(−4, 1, −3), (0, 1, 0), (1, −2, 3) (4, −2, 3), (0, 1, 0), (−1, 1, −3)	(−3, −3, 1), (0, −1, −1), (0, −1, 0), (0, 2, 1), (3, −3, −4)
544696	$\pm 2/3 \cdot v_\Delta$	$\pm(1, -1, -3)$	(1, 0, 0), (0, 1, 0), (−4, 2, 9) (5, −4, −15), (1, 0, 0), (−3, 1, 6)	(−3, −3, −1), (−3, 12, −4), (−1, −1, 0), (−1, 2, −1), (2, −1, 1)
544700	$\pm 2/3 \cdot v_\Delta$	$\pm(1, 2, 3)$	(−2, −3, −3), (0, 1, 0), (−1, −4, −6) (0, 1, 0), (1, 0, 0), (2, 5, 9)	(−3, −3, 2), (−1, −1, 1), (−1, 2, −1), (2, −1, 0), (3, −3, 2)
544749	$\pm 1/2 \cdot v_\Delta$	$\pm(1, 1, 2)$	(−6, −5, −8), (0, 1, 0), (1, 0, 0) (0, 1, 0), (−2, −1, 0), (3, 2, 4)	(−2, −2, 3), (−1, −1, 1), (−1, 1, 0), (−1, 3, −1), (1, −1, 0), (2, −2, −1)

(continued)

Table 2.6 (continued)

ID	vert(Δ^{FI})	$\pm v_\Delta$	vert($\theta_\pm$)	supp(Δ^{FI})
520925	$\pm 1/2 \cdot v_\Delta$	$\pm(2, 1, 1)$	$(-2, -1, 0), (-2, 0, -1), (-2, -3, -2)$ $(0, 1, 0), (0, 0, 1), (8, 2, 3)$	$(-1, -1, 3), (0, -1, 1), (0, 1, -1), (1, -2, -2), (1, -1, -1), (1, 0, 0)$
520935	$\pm 1/2 \cdot v_\Delta$	$\pm(1, 1, 2)$	$(1, 0, 0), (0, 1, 0), (-6, -5, -8)$ $(3, 4, 6), (2, 1, 2), (-3, -2, -2)$	$(-2, -2, 3), (-1, -1, 1), (-1, 1, 0), (-1, 3, -1), (0, 4, -3), (1, -1, 0)$
522056	$\pm 2/3 \cdot v_\Delta$	$\pm(2, 1, 3)$	$(0, 1, 0), (-1, -1, -3), (-5, -3, -6)$ $(-1, -1, 0), (1, 0, 0), (6, 4, 9)$	$(-3, 6, -1), (-1, -1, 1), (-1, 2, 0), (0, -3, 2), (2, -1, -1)$
522059	$\pm 2/3 \cdot v_\Delta$	$\pm(1, 2, 3)$	$(-2, -3, -3), (0, 1, 0), (-1, -4, -6)$ $(2, 5, 6), (1, 0, 0), (0, 1, 3)$	$(-3, 3, -2), (-1, -1, 1), (-1, 2, -1), (2, -1, 0), (3, -3, 2)$
522087	$\pm 2/3 \cdot v_\Delta$	$\pm(1, -1, -3)$	$(1, 0, 0), (0, 1, 0), (-4, 2, 9)$ $(1, 0, -3), (-3, 1, 6), (5, -4, -12)$	$(-3, -3, -1), (-1, -1, 0), (-1, 2, -1), (2, -1, 1), (9, 0, 4)$
522682	$\pm 1/2 \cdot v_\Delta$	$\pm(1, 1, 2)$	$(-3, -2, -4), (-2, -3, -4), (1, 0, 0), (0, 1, 0)$ $(2, 1, 4), (1, 2, 4), (1, 0, 0), (0, 1, 0)$	$(-2, -2, 1), (-2, -2, 3), (-1, -1, 1), (-1, 1, 0), (1, -1, 0), (1, 1, -1)$
522684	$\pm 1/2 \cdot v_\Delta$	$\pm(1, 1, 2)$	$(-2, -1, -4), (1, 0, 0), (-4, -3, -4)$ $(3, 2, 4), (-2, -1, 0), (0, 1, 0)$	$(-2, 2, 1), (-1, -1, 1), (-1, 1, 0), (-1, 3, -1), (1, -1, 0), (2, -2, -1)$
526886	$\pm 1/2 \cdot v_\Delta$	$\pm(1, -1, 2)$	$(-3, 4, -6), (0, 1, -2), (2, -5, 6)$ $(1, 0, 0), (0, 1, 0), (3, -6, 8)$	$(-2, -2, -1), (-1, -1, 0), (-1, 1, 1), (0, 4, 3), (1, -1, -1), (3, -1, -2)$
439403	$\pm 1/2 \cdot v_\Delta$	$\pm(0, 1, 1)$	$(-1, 1, -1), (1, 0, 0), (-1, -2, -2), (1, -3, -1)$ $(1, 2, 2), (-1, 0, 0), (-1, 1, -1), (1, 1, 3)$	$(-2, -1, 3), (-1, -1, 1), (-1, 0, 0), (1, 0, 0), (1, 1, -1), (2, -1, -1)$
275525	$\pm 1/2 \cdot v_\Delta$	$\pm(1, 0, 0)$	$(0, 1, 0), (-2, -1, 0), (1, 1, 2), (-3, -1, -2)$ $(4, 1, 2), (-2, -1, -2), (1, 1, 0), (1, -1, 0)$	$(-2, 0, 3), (0, -1, 0), (0, -1, 1), \pm(0, 1, -1), (0, 1, 0), (2, -2, -1)$
275528	$\pm 1/2 \cdot v_\Delta$	$\pm(1, 1, -1)$	$(-3, -2, 1), (-2, -1, 2), (0, -1, 0), (1, 0, 1)$ $(-1, 0, -1), (0, 1, 0), (2, 1, -2), (3, 2, -1)$	$(-1, 1, 0), (-1, 2, -1), (0, -1, -1), (0, 1, 1), (1, -2, 1), (1, -1, 0)$

To compute the canonical hull Δ^{can} of Δ, we obtain $\text{supp}(\Delta^{\text{FI}}) = \{s_i \mid 1 \le i \le 5\}$ with $s_1 := (-3, -3, -1), s_2 := (-1, -1, 0), s_3 := (-1, 2, -1), s_4 := (2, -1, 1)$, and $s_5 := (15, -3, 7)$, which leads to $\Delta^{\text{can}} = \Delta$.

Remark 34 Information about all 20 canonical Fano 3-topes with $\dim(\Delta^{\text{FI}}) = 1$ and $0 \in (\Delta^{\text{FI}})^\circ$ can be found in Tables 2.5 and 2.6.

2.6 Fine Interior of Dimension 3

There exist 49 canonical Fano 3-topes Δ such that $\dim(\Delta^{\text{FI}}) = 3$. Exactly 3 of these polytopes Δ define minimal surface $\mathcal{S}_\Delta$ with non-trivial fundamental group of order 2 and $K^2 = 2$. For these 3 polytopes one has $\Delta = \Delta^{\text{can}}$. The surfaces $\mathcal{S}_\Delta$ were investigated by Todorov [24] as well as Catanese and Debarre [10].

The remaining 46 canonical Fano 3-topes Δ define simply connected minimal surfaces $\mathcal{S}_\Delta$ with $K^2 = 1$. These surfaces were investigated by Kanev [19], Catanese [9], and Todorov [23]. Among these 46 canonical Fano 3-topes there exist exactly 26 polytopes Δ such that $\Delta = \Delta^{\text{can}}$.

Example 35 ([19]) Let $M \subseteq \mathbb{Q}^4$ be the 3-dimensional affine lattice defined by

$$M := \{(m_1, m_2, m_3, m_4) \in \mathbb{Z}^4 \mid m_1 + m_2 + m_3 + 2m_4 = 6,\ m_2 + 2m_3 \equiv 0 \pmod 3\}$$

and $\Delta' \subseteq M_{\mathbb{Q}}$ be the convex hull of 4 lattice points

$$(6, 0, 0, 0),\ (0, 6, 0, 0),\ (0, 0, 6, 0),\ \text{and } (0, 0, 0, 3) \in M.$$

Then $(\Delta')^{\text{FI}}$ is the 3-dimensional rational simplex

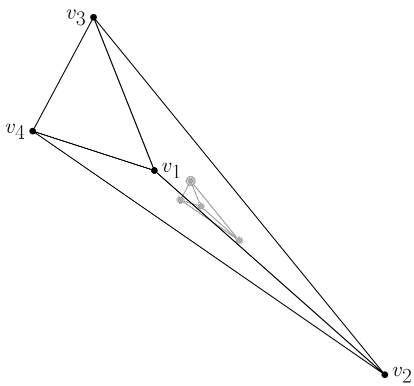
$$\text{conv}\{(2, 1, 1, 1), (1, 2, 1, 1), (1, 1, 2, 1), (1, 1, 1, 3/2)\}$$

and $(\Delta')^{\text{FI}} \cap M = \{(2, 1, 1, 1)\}$.

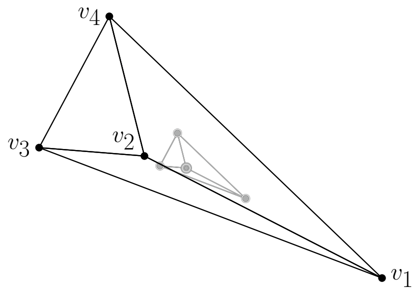
The canonical Fano 3-tope Δ' is the Newton polytope of the μ_3-cyclic quotient $\overline{Z}_{\Delta'}$ of the projective surface of degree 6 defined by the polynomial $z_1^6 + z_2^6 + z_3^6 + z_4^3 = 0$ in the weighted projective space $\mathbb{P}(1, 1, 1, 2)$, where the cyclic group μ_3 acts via $(z_1 : z_2 : z_3 : z_4) \mapsto (z_1 : \varepsilon z_2 : \varepsilon^2 z_3 : z_4)$. The single interior lattice point in Δ' corresponds to the monomial $z_1^2 z_2 z_3 z_4$. The surface $\overline{Z}_{\Delta'}$ has 3 cyclic quotient singularities of type A_2. The minimal desingularization $\mathcal{S}_{\Delta'}$ of $\overline{Z}_{\Delta'}$ is a simply connected surface of general type with $K^2 = 1$.

One can identify Δ' with the canonical Fano 3-simplex Δ given as the convex hull of

$$v_1 := (1, 0, 0),\ v_2 := (-2, -4, -5),\ v_3 := (1, 2, 4),\ \text{and } v_4 := (1, 4, 2)$$



(a) ID 547444



(b) ID 547465

Fig. 2.6 Canonical Fano 3-topes with Fine Interior of Dimension 3. Shaded faces are occluded. The Fine interior and the origin are shown in grey with a double border around the origin. **a** The whole polytope is $\Delta = \mathrm{conv}\{v_1, v_2, v_3, v_4\}$ with $v_1 = (1, 0, 0)$, $v_2 = (-2, -4, -5)$, $v_3 = (1, 2, 4)$, and $v_4 = (1, 4, 2)$. Moreover, $\Delta^{\mathrm{FI}} = \mathrm{conv}\{(0, 0, 0), (-1/2, -1, -3/2), (0, -1/3, -2/3), (0, 1/3, -1/3)\}$ and $\Delta^{\mathrm{can}} = \Delta$. **b** The whole polytope is $\Delta = \mathrm{conv}\{v_1, v_2, v_3, v_4\}$ with $v_1 = (-3, -2, -2)$, $v_2 = (1, 0, 0)$, $v_3 = (1, 3, 1)$, and $v_4 = (1, 1, 3)$. Moreover, $\Delta^{\mathrm{FI}} = \mathrm{conv}\{(0, 0, 0), (-1, -1/2, -1/2), (0, 3/4, 1/4), (0, 1/4, 3/4)\}$ and $\Delta^{\mathrm{can}} = \Delta$

(ID 547444, Fig. 2.6a, Tables 2.7, 2.8, and 2.9). The primitive inward-pointing facet normals of the facets $\theta_{124}, \theta_{234}, \theta_{123}$, and $\theta_{134} \preceq \Delta$ of this simplex Δ are

$$n_1 := (-2, -1, 2),\ n_2 := (5, -1, -1),\ n_3 := (-1, 2, -1),\ \text{and}\ n_4 := (-1, 0, 0),$$

respectively. They satisfy the relation

$$n_1 + n_2 + n_3 + 2n_4 = 0.$$

To compute the canonical hull Δ^{can} of Δ, we obtain $\mathrm{supp}(\Delta^{\mathrm{FI}}) = \{s_i \mid 1 \le i \le 6\}$ with $s_1 := (-2, -1, 2)$, $s_2 := (-1, 0, 0)$, $s_3 := (-1, 2, -1)$, $s_4 := (1, 1, -1)$, $s_5 := (3, 0, -1)$, and $s_6 := (5, -1, -1)$, which leads to $\Delta^{\mathrm{can}} = \Delta$.

Example 36 ([24]) Let $M \subseteq \mathbb{Q}^4$ be the 3-dimensional affine lattice defined by

$$M := \{(m_1, m_2, m_3, m_4) \in \mathbb{Z}^4 \mid m_1 + m_2 + 2m_3 + 2m_4 = 8,\ 3m_2 + m_3 + 3m_4 \equiv 0 \pmod 4\}$$

and $\Delta' \subseteq M_{\mathbb{Q}}$ be the convex hull of 4 lattice points

$$(8, 0, 0, 0),\ (0, 8, 0, 0),\ (0, 0, 4, 0),\ \text{and}\ (0, 0, 0, 4) \in M.$$

Table 2.7 49 Canonical Fano 3-topes with Fine Interior of Dimension 3 Table contains: vertices vert(Δ) of Δ

ID	vert(Δ)
547444	(1, 0, 0), (−2, −4, −5), (1, 2, 4), (1, 4, 2)
547465	(−3, −2, −2), (1, 0, 0), (1, 3, 1), (1, 1, 3)
547524	(0, 2, 1), (−2, −3, −5), (2, 1, 1), (0, 0, 1)
547525	(0, 0, 1), (0, 1, 0), (2, 1, 1), (−2, −5, −7)
545317	(−3, 4, −6), (0, 1, 0), (1, 0, 0), (1, −2, 4), (3, −5, 6)
545932	(0, −1, −1), (1, −1, −3), (−2, 1, 5), (1, 0, 0), (1, 2, −2)
546013	(3, −5, 6), (1, −2, 4), (1, 0, 0), (−1, 1, −2), (−1, 3, −2)
546062	(0, 1, 3), (−2, 1, −1), (0, 1, 0), (1, 0, 0), (−1, −2, −2)
546070	(0, −2, −3), (0, 2, 1), (−2, −3, −5), (2, 1, 1), (0, 0, 1)
546205	(1, 2, −2), (−1, 0, 2), (1, 0, 0), (−2, 1, 5), (1, −1, −3)
546219	(1, 1, 1), (−3, −2, −2), (1, 0, 0), (1, 3, 1), (−1, −1, 1)
546663	(2, −3, −1), (1, 0, 0), (0, 1, 0), (0, 0, 1), (−2, −3, −3)
546862	(1, 0, 0), (0, 1, 0), (−2, 1, 5), (1, −1, −3), (1, 2, −2)
546863	(−1, −1, 1), (1, 3, 1), (0, 0, 1), (1, 0, 0), (−3, −2, −2)
547240	(−1, 1, −2), (0, 1, 0), (1, 0, 0), (1, −2, 4), (3, −5, 6)
547246	(0, −2, −3), (−2, −3, −5), (2, 1, 1), (0, 1, 0), (0, 0, 1)
532384	(1, −1, −3), (−2, 1, 5), (1, 0, 0), (1, −1, −2), (0, −1, −1), (1, 2, −2)
532606	(0, −1, 2), (−1, −1, 0), (0, 1, 0), (1, 0, 0), (2, 2, −3), (−2, 0, −3)
533513	(−1, 1, 2), (1, 0, 0), (0, 1, 0), (1, 1, 2), (−1, −2, −4), (−2, −3, −4)
534667	(1, 0, 3), (−1, −1, −1), (0, 1, 0), (1, 0, 0), (−1, −1, 0), (5, 2, 3)
534669	(1, 3, 0), (5, 3, 2), (−1, −1, −1), (0, 0, 1), (1, 0, 0), (−1, −1, 0)
534866	(−1, −1, −3), (1, 0, 0), (0, 1, 0), (1, 1, 1), (−1, −1, 0), (−3, −5, −3)
535952	(3, −5, 6), (1, −2, 4), (1, 0, 0), (0, 1, 0), (−1, 1, −2), (−1, 2, −2)
536013	(0, 1, 1), (0, 0, 1), (0, 1, 0), (2, 1, 1), (−2, −3, −5), (0, −2, −3)
536498	(1, 2, −2), (1, −1, −2), (1, 0, 0), (0, 1, 0), (−2, 1, 5), (1, −1, −3)
537834	(0, 0, 1), (1, 0, 0), (0, 1, 0), (−2, 1, 5), (1, −1, −3), (1, 2, −2)
538356	(−2, −3, −3), (−1, −3, −1), (0, 0, 1), (0, 1, 0), (1, 0, 0), (−1, −1, −3)
539063	(−1, 1, −1), (1, 1, 3), (−3, −2, −2), (1, 0, 0), (0, 1, 0), (1, 1, 2)
539304	(1, 0, 1), (−3, −1, −2), (1, 1, 2), (−2, −1, 0), (1, 0, 0), (1, 2, 0)
539313	(1, −1, −2), (1, 1, −1), (−1, 2, 2), (1, −1, −3), (−2, 1, 5), (1, 0, 0)
540602	(0, 0, 1), (1, 0, 0), (−2, 1, 5), (1, −1, −3), (−1, 2, 2), (1, 1, −1)
540663	(1, 0, 0), (0, 1, 0), (1, 1, 2), (−3, −1, −2), (1, 1, 1), (−3, −2, 0)
474457	(−1, 2, −3), (1, 0, 2), (0, 0, 1), (0, 1, 0), (1, 0, 0), (−1, −1, 0), (−3, −2, −3)
481575	(3, 2, 4), (−1, −1, −2), (−3, −1, −2), (−2, −1, 0), (0, 1, 0), (1, 0, 0), (0, 0, −1)
483109	(3, 0, 2), (1, −2, −2), (0, 0, −1), (−1, −1, 0), (1, 1, 1), (0, 1, 0), (−1, 0, 0)
490478	(1, −1, −2), (1, 1, −1), (−1, 2, 2), (1, −1, −3), (−2, 1, 5), (1, 0, 0), (−1, 0, 2)
490481	(−3, −2, 0), (−5, −3, −2), (1, 0, 0), (0, 1, 0), (1, 1, 2), (−1, −1, −1), (2, 1, 1)
490485	(−1, −1, 0), (1, 2, 0), (1, 0, 0), (−2, −1, 0), (1, 1, 2), (−3, −1, −2), (1, 0, 1)
490511	(1, 0, 0), (0, 1, 0), (−2, −1, 0), (1, 1, 2), (2, 1, 1), (1, 0, 1), (−5, −2, −4)
495687	(0, 0, −1), (1, 1, −1), (−1, 2, 2), (1, −1, −3), (−2, 1, 5), (1, 0, 0), (0, 0, 1)
499287	(1, 1, 1), (−1, −1, −3), (1, 0, 0), (0, 1, 0), (0, 0, 1), (−1, −3, −1), (−2, −3, −3)
499291	(−1, −1, −1), (−1, −1, −3), (1, 0, 0), (0, 1, 0), (0, 0, 1), (−1, −3, −1), (−2, −3, −3)
499470	(1, 0, 0), (0, 1, 0), (−2, −1, 0), (1, 1, 2), (0, 0, 1), (−5, −2, −4), (2, 1, 1)
501298	(3, −6, 8), (−1, 1, −2), (1, −2, 3), (0, 1, 0), (1, 0, 0), (0, 1, −1), (3, −5, 6)
501330	(1, 0, 0), (0, 1, 0), (−2, −1, 0), (1, 1, 2), (1, 1, 1), (0, 0, 1), (−5, −2, −4)
354912	(3, 1, 2), (1, 0, 0), (0, 1, 0), (−2, −1, 0), (1, 1, 2), (2, 1, 1), (1, 0, 1), (−5, −2, −4)
372528	(2, 1, 1), (−1, −1, −1), (1, 1, 2), (0, 1, 0), (1, 0, 0), (−5, −3, −2), (−3, −2, 0), (1, 1, 0)
372973	(−5, −2, −4), (1, 0, 1), (2, 1, 1), (1, 1, 2), (−2, −1, 0), (0, 1, 0), (1, 0, 0), (2, 1, 2)
388701	(1, 1, 1), (−2, −3, −3), (−1, −3, −1), (0, 0, 1), (0, 1, 0), (1, 0, 0), (−1, −1, −3), (−1, −1, −1)

Table 2.8 49 Canonical Fano 3-topes with Fine Interior of Dimension 3. Table contains: vertices vert(Δ^{FI}) of the Fine interior Δ^{FI}

ID	vert(Δ^{FI})
547444	0, (−1/2, −1, −3/2), (0, −1/3, −2/3), (0, 1/3, −1/3)
547465	0, (−1, −1/2, −1/2), (0, 3/4, 1/4), (0, 1/4, 3/4)
547524	0, (0, 1/2, 0), (1/3, 1/3, 0), (−1/3, −1/3, −1)
547525	0, (0, 0, −1/2), (1/3, 0, −1/3), (−1/3, −1, −5/3)
545317	0, (1, −3/2, 2), (2/3, −2/3, 1), (1/2, −1/2, 1), (2/3, −1, 5/3)
545932	0, (−1/2, 1/2, 3/2), (0, 1/3, 2/3), (0, 2/3, 1/3)
546013	0, (1, −3/2, 2), (0, 1/2, 0), (1/2, −1/4, 1/2), (1/2, −3/4, 3/2)
546062	0, (−1/2, −1/2, −1/2), (−2/3, 0, −1/3), (−1/3, 0, 1/3)
546070	0, (0, 1/2, 0), (1/2, 1/4, 0), (0, −1/2, −1), (−1/2, −3/4, −3/2)
546205	0, (−1/2, 1/2, 3/2), (0, 1/3, 2/3), (0, 2/3, 1/3)
546219	0, (−1, −1/2, −1/2), (−1/3, 1/3, 0), (−2/3, −1/3, 0)
546663	0, (0, −1/2, 0), (1/3, −1, −1/3), (−1/3, −1, −2/3)
546862	0, (−1/2, 1/2, 3/2), (0, 1/3, 2/3), (0, 2/3, 1/3)
546863	0, (−1, −1/2, −1/2), (−1/3, 1/3, 0), (−2/3, −1/3, 0)
547240	0, (1, −3/2, 2), (2/3, −2/3, 1), (1/2, −1/2, 1), (2/3, −1, 5/3)
547246	0, (0, 0, −1/2), (1/3, 0, −1/3), (0, −1/2, −1), (−1/3, −2/3, −4/3)
532384	0, (−1/2, 1/2, 3/2), (0, 1/3, 2/3), (0, 2/3, 1/3)
532606	0, (0, 1/2, −1/2), (1/3, 2/3, −1), (−1/3, 1/3, −1)
533513	0, (−1/2, −1/2, −1), (−1/2, 0, 0), (−1/3, 0, −1/3), (−2/3, −2/3, −1)
534667	0, (1/2, 1/2, 1/2), (4/3, 2/3, 1), (2/3, 1/3, 1)
534669	0, (1/2, 1/2, 1/2), (4/3, 1, 2/3), (2/3, 1, 1/3)
534866	0, (0, −1/2, −1/2), (−1/3, −2/3, −1), (−2/3, −4/3, −1)
535952	0, (1, −3/2, 2), (2/3, −2/3, 1), (1/2, −1/2, 1), (2/3, −1, 5/3)
536013	0, (0, 0, −1/2), (1/3, 0, −1/3), (0, −1/2, −1), (−1/3, −2/3, −4/3)
536498	0, (−1/2, 1/2, 3/2), (0, 1/3, 2/3), (0, 2/3, 1/3)
537834	0, (−1/2, 1/2, 3/2), (0, 1/3, 2/3), (0, 2/3, 1/3)
538356	0, (0, −1/2, −1/2), (−1/3, −2/3, −1), (−1/3, −1, −2/3), (−1/2, −1, −1)
539063	0, (−1, −1/2, −1/2), (−2/3, 0, −1/3), (−1/3, 0, 1/3)
539304	0, (0, 1/2, 0), (−1/2, 0, 0), (0, 1/3, 1/3), (−2/3, 0, −1/3)
539313	0, (−1/2, 1/2, 3/2), (0, 1/3, 2/3), (0, 1/2, 1/2), (−1/3, 2/3, 1)
540602	0, (−1/2, 1/2, 3/2), (0, 1/3, 2/3), (0, 1/2, 1/2), (−1/3, 2/3, 1)
540663	0, (−1/2, 0, 0), (−1, −1/2, 0), (−1/3, 0, 1/3), (−1, −1/3, −1/3)
474457	0, (0, 0, −1/2), (−1/3, 1/3, −1), (−2/3, −1/3, −1)
481575	0, (−1/2, 0, 0), (1/2, 1/2, 1), (0, 1/3, 1/3), (−1/3, 0, 1/3)
483109	0, (0, −1/2, 0), (2/3, −1/3, 1/3), (1/3, −2/3, −1/3)
490478	0, (−1/2, 1/2, 3/2), (0, 1/3, 2/3), (0, 1/2, 1/2), (−1/3, 2/3, 1)
490481	0, (−1/2, 0, 0), (−1, −1/2, 0), (−1/3, 0, 1/3), (−4/3, −2/3, −1/3)
490485	0, (0, 1/2, 0), (−1/2, 0, 0), (0, 1/3, 1/3), (−2/3, 0, −1/3)
490511	0, (−3/2, −1/2, −1), (−1/2, 0, 0), (−2/3, 0, −1/3), (−1, −1/3, −1/3)
495687	0, (−1/2, 1/2, 3/2), (0, 1/3, 2/3), (0, 1/2, 1/2), (−1/3, 2/3, 1)
499287	0, (0, −1/2, −1/2), (−1/3, −2/3, −1), (−1/3, −1, −2/3), (−1/2, −1, −1)
499291	0, (0, −1/2, −1/2), (−1/3, −2/3, −1), (−1/3, −1, −2/3), (−1/2, −1, −1)
499470	0, (−3/2, −1/2, −1), (−1/2, 0, 0), (−2/3, 0, −1/3), (−1, −1/3, −1/3)
501298	0, (1/2, −1/2, 1), (2/3, −2/3, 1), (1, −3/2, 2), (1, −5/3, 7/3)
501330	0, (−3/2, −1/2, −1), (−1/2, 0, 0), (−2/3, 0, −1/3), (−1, −1/3, −1/3)
354912	0, (−3/2, −1/2, −1), (−1/2, 0, 0), (−2/3, 0, −1/3), (−1, −1/3, −1/3)
372528	0, (−1/2, 0, 0), (−1, −1/2, 0), (−1/3, 0, 1/3), (−4/3, −2/3, −1/3)
372973	0, (−3/2, −1/2, −1), (−1/2, 0, 0), (−2/3, 0, −1/3), (−1, −1/3, −1/3)
388701	0, (0, −1/2, −1/2), (−1/3, −2/3, −1), (−1/3, −1, −2/3), (−1/2, −1, −1)

Table 2.9 49 Canonical Fano 3-topes with Fine Interior of Dimension 3. Table contains: support supp(Δ^{FI}) of the Fine interior Δ^{FI}, vertices vert(Δ^{can}) of the canonical hull Δ^{can}, and order of fundamental group $|\pi_1(\mathcal{S}_\Delta)|$ of the minimal model $\mathcal{S}_\Delta$

ID	supp(Δ^{FI})	vert(Δ^{can})	$\lvert\pi_1(\mathcal{S}_\Delta)\rvert$
547444	(−2, −1, 2), (−1, 0, 0), (−1, 2, −1), (1, 1, −1), (3, 0, −1), (5, −1, −1)	vert(Δ)	1
547465	(−1, −1, 3), (−1, 0, 0), (−1, 0, 1), (−1, 0, 2), (−1, 1, 0), (−1, 1, 1), (−1, 2, 0), (−1, 3, −1), (2, −1, −1)	vert(Δ)	2
547524	(−1, −2, 2), (−1, 1, 0), (−1, 2, −1), (0, 0, −1), (0, 1, −1), (0, 2, −1), (1, 0, −1), (1, 1, −1), (2, 0, −1), (3, 0, −1)	vert(Δ), (0, −1, −1)	1
547525	(−1, −2, 2), (−1, 2, −1), (0, −1, 0), (0, 0, −1), (0, 1, −1), (1, −1, −1), (1, −1, 0), (1, 0, −1),	vert(Δ), (1, 1, 1), (−1, −2, −3)	1
	(1, 1, −1), (2, −1, −1), (2, −1, 0), (2, 0, −1), (3, −1, −1), (3, −1, 0), (3, 0, −1), (4, −1, −1), (4, 0, −1), (5, −1, −1), (6, −1, −1)		
545317	(−2, −2, −1), (−1, −1, 0), (−1, 2, 2), (1, −1, −1), (1, 2, 1), (3, 2, 0)	vert(Δ)	1
545932	(−2, −1, −1), (−1, 0, 0), (−1, 2, −1), (0, 1, 0), (1, 1, 0), (2, 0, 1), (3, 0, 1), (5, −1, 2)	vert(Δ), (1, −1, −2), (1, 0, −3)	1
546013	(−2, −2, −1), (−1, 0, 1), (−1, 2, 2), (0, 1, 1), (1, 0, 0), (1, 2, 1), (2, 1, 0), (3, 0, −1), (3, 2, 0)	vert(Δ)	2
546062	(−1, −1, 0), (−1, −1, 1), (−1, −1, 2), (−1, 0, 0), (−1, 0, 1), (−1, 1, 0), (−1, 2, −1), (0, −1, 0), (2, 1, −1)	vert(Δ)	1

(continued)

Table 2.9 (continued)

ID	supp(Δ^{FI})	vert(Δ^{can})	$\lvert\pi_1(\mathcal{S}_\Delta)\rvert$
546070	(−1, −2, 2), (−1, 2, −1), (0, 0, −1), (0, 1, −1), (0, 2, −1), (1, 0, −1), (1, 1, −1), (2, 0, −1), (3, 0, −1)	vert(Δ)	2
546205	(−2, −1, −1), (−1, 0, 0), (−1, 2, −1), (−1, 3, −1), (0, 1, 0), (1, 1, 0), (1, 2, 0), (2, 0, 1), (3, 0, 1), (3, 1, 1), (5, −1, 2), (5, 0, 2), (7, −1, 3)	vert(Δ)	1
546219	(−1, −1, 3), (−1, 0, 0), (−1, 0, 1), (−1, 0, 2), (−1, 1, −1), (−1, 1, 0), (−1, 1, 1), (−1, 2, 0), (0, 0, −1), (2, −1, −1)	vert(Δ)	1
546663	(−1, −1, −1), (−1, −1, 0), (−1, −1, 1), (−1, −1, 2), (−1, 0, −1), (0, −1, −1), (0, −1, 0), (0, −1, 1), (0, 0, −1), (1, −1, −1), (1, −1, 0), (1, 0, −1), (1, 2, −2), (2, −1, −1), (2, −1, 0), (2, 0, −1), (3, −1, −1)	vert(Δ), (−1, −1, −1)	1
546862	(−2, −1, −1), (−1, 0, 0), (−1, 2, −1), (−1, 3, −1), (0, 1, 0), (0, 4, −1), (1, 1, 0), (1, 2, 0), (2, 0, 1), (2, 3, 0), (3, 0, 1), (3, 1, 1), (4, 2, 1), (5, −1, 2), (5, 0, 2), (6, 1, 2), (7, −1, 3), (8, 0, 3), (10, −1, 4)	vert(Δ), (0, 0, 1)	1
546863	(−1, −1, 3), (−1, 0, 0), (−1, 0, 1), (−1, 0, 2), (−1, 1, −1), (−1, 1, 0), (−1, 1, 1), (−1, 2, 0), (0, 0, −1), (2, −1, −1)	vert(Δ)\{(0, 0, 1)}, (1, 1, 1)	1
547240	(−2, −2, −1), (−1, −1, 0), (−1, 0, 1), (−1, 2, 2), (0, −1, 0), (0, 1, 1), (1, −1, −1), (1, 0, 0), (1, 2, 1), (2, −1, −1), (2, 1, 0), (3, 0, −1), (3, 2, 0)	vert(Δ), (0, 1, −1), (0, 0, 1)	1
547246	(−1, −2, 2), (−1, 2, −1), (0, −1, 0), (0, 0, −1), (0, 1, −1), (0, 2, −1), (1, −1, −1), (1, −1, 0), (1, 0, −1), (1, 1, −1), (2, −1, −1), (2, −1, 0), (2, 0, −1), (3, −1, −1), (3, 0, −1), (4, −1, −1)	vert(Δ), (1, 1, 1), (−1, −1, −2)	1

(continued)

Table 2.9 (continued)

ID	$\mathrm{supp}(\Delta^{\mathrm{FI}})$	$\mathrm{vert}(\Delta^{\mathrm{can}})$	$\lvert\pi_1(\mathcal{S}_\Delta)\rvert$
532384	(−2, −1, −1), (−1, 0, 0), (−1, 2, −1), (0, 1, 0), (1, 1, 0), (2, 0, 1), (3, 0, 1), (5, −1, 2)	vert(Δ), (1, 0, −3)	1
532606	(−1, −1, −1), (−1, 1, 0), (−1, 2, 1), (0, −1, −1), (0, 1, 0), (1, −2, 0), (1, −1, −1), (2, −1, −1)	vert(Δ), (0, −1, 1)	1
533513	(−1, −1, 1), (−1, 0, 0), (−1, 1, 0), (−1, 2, −1), (0, −1, 0), (0, 1, −1), (0, 3, −2), (2, −2, 1)	vert(Δ), (1, 1, 1), (0, −1, −1)	1
534667	(−1, −1, 2), (−1, −1, 3), (−1, 0, 2), (−1, 1, 1), (−1, 2, 0), (0, −1, 1), (0, −1, 2), (0, 0, 1), (0, 1, 0), (1, −2, −1), (1, −1, 0), (1, −1, 1), (1, 0, 0), (2, −1, −1), (2, −1, 0)	vert(Δ)	1
534669	(−1, 0, 2), (−1, 1, 1), (−1, 2, −1), (−1, 2, 0), (0, 0, 1), (0, 1, −1), (0, 1, 0), (1, −1, −2), (1, 0, −1), (1, 0, 0), (2, −1, −1), (2, −1, 0)	vert(Δ)	1
534866	(−2, 1, 1), (−1, −1, 1), (−1, 0, 0), (−1, 1, −1), (−1, 2, −2), (0, −1, 0), (0, 0, −1), (0, 1, −2), (1, −1, −1), (1, −1, 0), (1, 0, −2), (1, 0, −1), (2, −1, −2), (2, −1, −1), (2, −1, 0)	vert(Δ)	1
535952	(−2, −2, −1), (−1, −1, 0), (−1, 0, 1), (−1, 2, 2), (0, 1, 1), (1, −1, −1), (1, 0, 0), (1, 2, 1), (2, 1, 0), (3, 0, −1), (3, 2, 0)	vert(Δ)	1
536013	(−1, −2, 2), (−1, 2, −1), (0, −1, 0), (0, 0, −1), (0, 1, −1), (0, 2, −1), (1, −1, 0), (1, 0, −1), (1, 1, −1), (2, −1, 0), (2, 0, −1), (3, 0, −1)	vert(Δ)	1
536498	(−2, −1, −1), (−1, 0, 0), (−1, 2, −1), (0, 1, 0), (1, 1, 0), (1, 2, 0), (2, 0, 1), (2, 3, 0), (3, 0, 1), (3, 1, 1), (4, 2, 1), (5, −1, 2), (5, 0, 2), (6, 1, 2), (7, −1, 3), (8, 0, 3), (10, −1, 4)	vert(Δ)	1
537834	(−2, −1, −1), (−1, 0, 0), (−1, 2, −1), (−1, 3, −1), (0, 1, 0), (0, 4, −1), (1, 1, 0), (1, 2, 0), (2, 0, 1), (2, 3, 0), (3, 0, 1), (3, 1, 1), (4, 2, 1), (5, −1, 2), (5, 0, 2), (6, 1, 2), (7, −1, 3), (8, 0, 3), (10, −1, 4)	vert(Δ)	1

(continued)

Table 2.9 (continued)

ID	supp(Δ^{FI})	vert(Δ^{can})	$\|\pi_1(\mathcal{S}_\Delta)\|$
538356	(−2, 1, 1), (−1, −1, −1), (−1, −1, 0), (−1, −1, 1), (−1, 0, −1), (−1, 0, 0), (−1, 1, −1), (0, −1, −1), (0, −1, 0), (0, 0, −1), (1, −1, −1), (1, −1, 0), (1, 0, −1), (2, −1, −1), (2, −1, 0), (2, 0, −1), (3, −1, −1)	vert(Δ), (−1, −1, −1)	1
539063	(−1, −1, 1), (−1, 0, 0), (−1, 0, 1), (−1, 0, 2), (−1, 1, 0), (−1, 1, 1), (−1, 2, 0), (−1, 3, −1), (0, −1, 0), (2, −1, −1)	vert(Δ) \ {(0, 1, 0), (1, 1, 2)}, (1, 1, 1)	1
539304	(−1, 0, 0), (−1, 0, 1), (−1, 0, 2), (−1, 1, 0), (−1, 1, 1), (−1, 2, 0), (−1, 2, 1), (−1, 3, 0), (0, 1, −1), (0, 1, 0), (2, −2, −1)	vert(Δ), (−2, −1, −1)	1
539313	(−2, −1, −1), (−1, 0, 0), (−1, 2, −1), (0, 1, 0), (1, −1, 1), (1, 1, 0), (1, 2, 0), (2, 0, 1), (2, 3, 0), (3, 0, 1), (3, 1, 1), (4, 2, 1), (5, 0, 2), (6, 1, 2)	vert(Δ), (−1, 1, 2)	1
540602	(−2, −1, −1), (−1, 0, 0), (−1, 2, −1), (−1, 3, −1), (0, 1, 0), (0, 4, −1), (1, −1, 1), (1, 1, 0), (1, 2, 0), (2, 0, 1), (2, 3, 0), (3, 0, 1), (3, 1, 1), (4, 2, 1), (5, 0, 2), (6, 1, 2)	vert(Δ), (−1, 1, 2)	1
540663	(−1, −1, 1), (−1, −1, 2), (−1, 0, 0), (−1, 0, 1), (−1, 0, 2), (−1, 1, 0), (−1, 1, 1), (−1, 2, −1), (−1, 2, 0), (−1, 2, 1), (0, −1, 0), (0, −1, 1), (2, −2, −1)	vert(Δ), (−1, 0, −1)	1
474457	(−2, 1, 2), (−1, −1, 0), (−1, 0, 0), (−1, 1, 0), (−1, 2, 0), (1, −1, −1), (1, 0, −1), (2, −1, −1)	vert(Δ)	1
481575	(−1, −1, 1), (−1, 0, 1), (−1, 1, 0), (−1, 2, 0), (−1, 3, −1), (0, −1, 1), (0, 1, 0), (2, −2, −1)	vert(Δ), (−1, −1, −1)	1
483109	(−1, −1, 1), (0, −1, 0), (0, −1, 1), (0, 2, −1), (1, −1, −1), (1, −1, 0), (1, −1, 1), (1, 0, −2), (1, 0, −1), (1, 0, 0), (1, 0, 1)	vert(Δ)	1

(continued)

Table 2.9 (continued)

ID	supp(Δ^{FI})	vert(Δ^{can})	$\lvert\pi_1(\mathcal{S}_\Delta)\rvert$
490478	(−2, −1, −1), (−1, 0, 0), (−1, 2, −1), (0, 1, 0), (1, −1, 1), (1, 1, 0), (1, 2, 0), (2, 0, 1), (3, 0, 1), (3, 1, 1), (5, 0, 2)	vert(Δ)	1
490481	(−1, −1, 2), (−1, −1, 3), (−1, 0, 1), (−1, 0, 2), (−1, 1, 0), (−1, 1, 1), (−1, 2, −1), (−1, 2, 0), (0, −1, 0), (0, −1, 1), (0, −1, 2), (2, −2, −1)	vert(Δ)	1
490485	(−1, 0, 0), (−1, 0, 1), (−1, 0, 2), (−1, 1, 0), (−1, 1, 1), (−1, 2, 0), (−1, 2, 1), (0, 1, −1), (0, 1, 0), (2, −2, −1)	vert(Δ)	1
490511	(−1, −1, 2), (−1, 0, 1), (−1, 1, 0), (−1, 1, 1), (−1, 2, 0), (−1, 3, 0), (0, −1, 0), (0, 1, −1), (2, −2, −1)	vert(Δ), (2, 1, 2)	1
495687	(−2, −1, −1), (−1, 0, 0), (−1, 2, −1), (−1, 3, −1), (0, 1, 0), (0, 4, −1), (1, −1, 1), (1, 1, 0), (1, 2, 0), (2, 0, 1), (2, 3, 0), (3, 0, 1), (3, 1, 1), (4, 2, 1)	vert(Δ)	1
499287	(−2, 1, 1), (−1, −1, 1), (−1, 0, 0), (−1, 1, −1), (0, −1, 0), (0, 0, −1), (1, −1, −1), (1, −1, 0), (1, 0, −1), (2, −1, −1), (2, −1, 0), (2, 0, −1), (3, −1, −1)	vert(Δ), (−1, −1, −1)	1
499291	(−2, 1, 1), (−1, −1, −1), (−1, −1, 0), (−1, −1, 1), (−1, 0, −1), (−1, 0, 0), (−1, 1, −1), (0, −1, −1), (0, −1, 0), (0, 0, −1), (1, −1, −1), (1, −1, 0), (1, 0, −1), (2, −1, −1), (2, −1, 0), (2, 0, −1), (3, −1, −1)	vert(Δ)	1
499470	(−1, −1, 2), (−1, 0, 1), (−1, 1, 0), (−1, 1, 1), (−1, 2, −1), (−1, 2, 0), (−1, 3, −1), (−1, 3, 0), (0, −1, 0), (0, 1, −1), (2, −2, −1)	vert(Δ)	1

(continued)

Table 2.9 (continued)

ID	supp(Δ^{FI})	vert(Δ^{can})	$\lvert\pi_1(\mathcal{S}_\Delta)\rvert$
501298	(−2, −2, −1), (−1, −1, 0), (−1, 0, 1), (−1, 2, 2), (0, −1, 0), (0, 1, 1), (1, −1, −1), (1, 0, 0), (1, 2, 1), (2, −1, −1), (2, 1, 0), (3, −1, −2), (3, 0, −1), (4, −1, −2), (5, 0, −2), (6, −1, −3)	vert(Δ)	1
501330	(−1, −1, 1), (−1, −1, 2), (−1, 0, 0), (−1, 0, 1), (−1, 1, 0), (−1, 1, 1), (−1, 2, −1), (−1, 2, 0), (−1, 3, −1), (−1, 3, 0), (0, −1, 0), (0, 1, −1), (2, −2, −1)	vert(Δ)	1
354912	(−1, −1, 2), (−1, 0, 1), (−1, 1, 1), (−1, 2, 0), (−1, 3, 0), (0, −1, 0), (0, 1, −1), (2, −2, −1)	vert(Δ)	1
372528	(−1, 0, 1), (−1, 0, 2), (−1, 1, 0), (−1, 1, 1), (−1, 2, −1), (−1, 2, 0), (0, −1, 0), (0, −1, 1), (0, −1, 2), (2, −2, −1)	vert(Δ)	1
372973	(−1, −1, 2), (−1, 0, 1), (−1, 1, 0), (−1, 1, 1), (−1, 2, 0), (−1, 3, 0), (0, −1, 0), (0, 1, −1), (2, −2, −1)	vert(Δ)	1
388701	(−2, 1, 1), (−1, −1, 1), (−1, 0, 0), (−1, 1, −1), (0, −1, 0), (0, 0, −1), (1, −1, −1), (1, −1, 0), (1, 0, −1), (2, −1, −1), (2, −1, 0), (2, 0, −1), (3, −1, −1)	vert(Δ)	1

Then $(\Delta')^{\mathrm{FI}}$ is the 3-dimensional rational simplex

$$\mathrm{conv}\{(3,1,1,1),(1,3,1,1),(1,1,2,1),(1,1,1,2)\}$$

and $(\Delta')^{\mathrm{FI}} \cap M = \{(1,1,2,1)\}$.

The canonical Fano 3-tope Δ' is the Newton polytope of the μ_4-cyclic quotient $\overline{Z}_{\Delta'}$ of the projective surface of degree 8 defined by the polynomial $z_1^8 + z_2^8 + z_3^4 + z_4^4 = 0$ in the weighted projective space $\mathbb{P}(1,1,2,2)$, where the cyclic group μ_4 acts via $(z_1 : z_2 : z_3 : z_4) \mapsto (z_1 : i^3 z_2 : i z_3 : i^3 z_4)$. The single interior lattice point in this lattice simplex Δ' corresponds to the monomial $z_1 z_2 z_3^2 z_4$. The projective surface $\overline{Z}_{\Delta'}$ has two Gorenstein cyclic quotient singularities of type A_3. The minimal desingularization $\mathcal{S}_{\Delta'}$ of $\overline{Z}_{\Delta'}$ is a surface of general type with $K^2 = 2$ and fundamental group $\pi_1(\mathcal{S}_\Delta)$ of order 2.

One can identify Δ' with the canonical Fano 3-simplex Δ given as the convex hull of

$$v_1 := (-3,-2,-2),\ v_2 := (1,0,0),\ v_3 := (1,3,1),\ \text{and}\ v_4 := (1,1,3)$$

(ID 547465, Fig. 2.6b, Tables 2.7, 2.8, and 2.9). The primitive inward-pointing facet normals of the facets $\theta_{123}, \theta_{124}, \theta_{234}, \theta_{134} \preceq \Delta$ of this simplex Δ are

$$n_1 := (-1,-1,3),\ n_2 := (-1,3,-1),\ n_3 := (-1,0,0),\ \text{and}\ n_4 := (2,-1,-1),$$

respectively. They satisfy the relation

$$n_1 + n_2 + 2n_3 + 2n_4 = 0.$$

To compute the canonical hull Δ^{can} of Δ, we obtain $\mathrm{supp}(\Delta^{\mathrm{FI}}) = \{s_i \mid 1 \le i \le 9\}$ with $s_1 := (-1,-1,3)$, $s_2 := (-1,0,0)$, $s_3 := (-1,0,1)$, $s_4 := (-1,0,2)$, $s_5 := (-1,1,0)$, $s_6 := (-1,1,1)$, $s_7 := (-1,2,0)$, $s_8 := (-1,3,-1)$, and $s_9 := (2,-1,-1)$, which leads to $\Delta^{\mathrm{can}} = \Delta$.

Remark 37 Information about all 49 canonical Fano 3-topes with $\dim(\Delta^{\mathrm{FI}}) = 3$ can be found in Tables 2.7, 2.8, and 2.9.

2.7 Hollow 3-Topes with Non-empty Fine Interior

A lattice polytope $\Delta \subseteq M_{\mathbb{Q}}$ is called *hollow* if it has no interior lattice points in its relative interior, i.e., $\Delta^\circ \cap M = \emptyset$. By [25, Theorem 1.3], any 3-dimensional hollow lattice polytope can be projected to the unimodular 1-simplex, to the double unimodular 2-simplex, or is an exceptional hollow 3-tope, whereas up to unimodular transformation there exist only a finite number of these. This theorem implies that a hollow 3-tope with non-empty Fine interior has to be exceptional because the unimodular 1-simplex and the double unimodular 2-simplex have empty Fine interior. Treutlein has found 9 maximal exceptional hollow polytopes, which was not an complete list. Averkov et al. [1, 2] have found the complete list consisting of 12 maximal exceptional hollow 3-topes Δ_i $(1 \leq i \leq 12)$ (Table 2.10, Fig. 2.7). Computations show that exactly 9 of 12 maximal exceptional hollow 3-topes Δ_i have non-empty Fine interior Δ_i^{FI} (Table 2.10). Moreover, no one of these 9 polytopes contains a proper lattice 3-subpolytope with non-empty Fine interior. Thus, there exist exactly 9 hollow 3-topes Δ_i with non-empty Fine interior Δ_i^{FI}.

It is remarkable that all minimal surfaces $\mathcal{S}_{\Delta_i}$ corresponding to these 9 hollow 3-topes Δ_i have non-trivial fundamental group $\pi_1(\mathcal{S}_\Delta)$ of order 2, 3, or 5 (Table 2.10). There exist exactly 5 hollow 3-topes Δ_i with 0-dimensional Fine interior $\Delta_i^{\mathrm{FI}} = \{R\}$, where $R \in \frac{1}{2}M \setminus M$ is a rational point (Table 2.10). The normal fans Σ^{Δ_i} of these 5 hollow polytopes Δ_i define 5 toric Fano threefolds $X_{\Sigma^{\Delta_i}}$ with at worst canonical singularities (Table 2.11). These Fano threefolds can be obtained as quotients of Gorenstein toric Fano threefolds $X_{\Sigma_{\Delta''}}$ in the following 5 ways:

1. $\mathbb{P}(1, 1, 2, 4)$ with a μ_2-action given by

$$(x_0, x_1, x_2, x_3) \mapsto (x_0, -x_1, -x_2, -x_3);$$

2. $\mathbb{P}^3$ with a μ_4-action given by

$$(x_0, x_1, x_2, x_3) \mapsto (x_0, ix_1, -x_2, -ix_3);$$

3. $\{x_1x_2 - x_3x_4 = 0\} \subseteq \mathbb{P}(2, 1, 1, 1, 1)$ with a μ_2-action given by

$$(x_0, x_1, x_2, x_3, x_4) \mapsto (-x_0, -x_1, -x_2, x_3, x_4);$$

4. $\mathbb{P}^1 \times \mathbb{P}(1, 1, 2)$ with a μ_2-action given by

$$(x_0, x_1, y_0, y_1, y_2) \mapsto (x_0, -x_1, y_0, -y_1, -y_2);$$

5. $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with a μ_2-action given by

$$(x_0, x_1, y_0, y_1, z_0, z_1) \mapsto (x_0, -x_1, y_0, -y_1, z_0, -z_1).$$

In addition, Table 2.12 contains the support $\mathrm{supp}(\Delta_i^{\mathrm{FI}})$ of the Fine interior Δ_i^{FI} and the vertices of the canonical hull Δ_i^{can} for all 9 hollow polytopes Δ_i with non-empty Fine interior Δ_i^{FI}.

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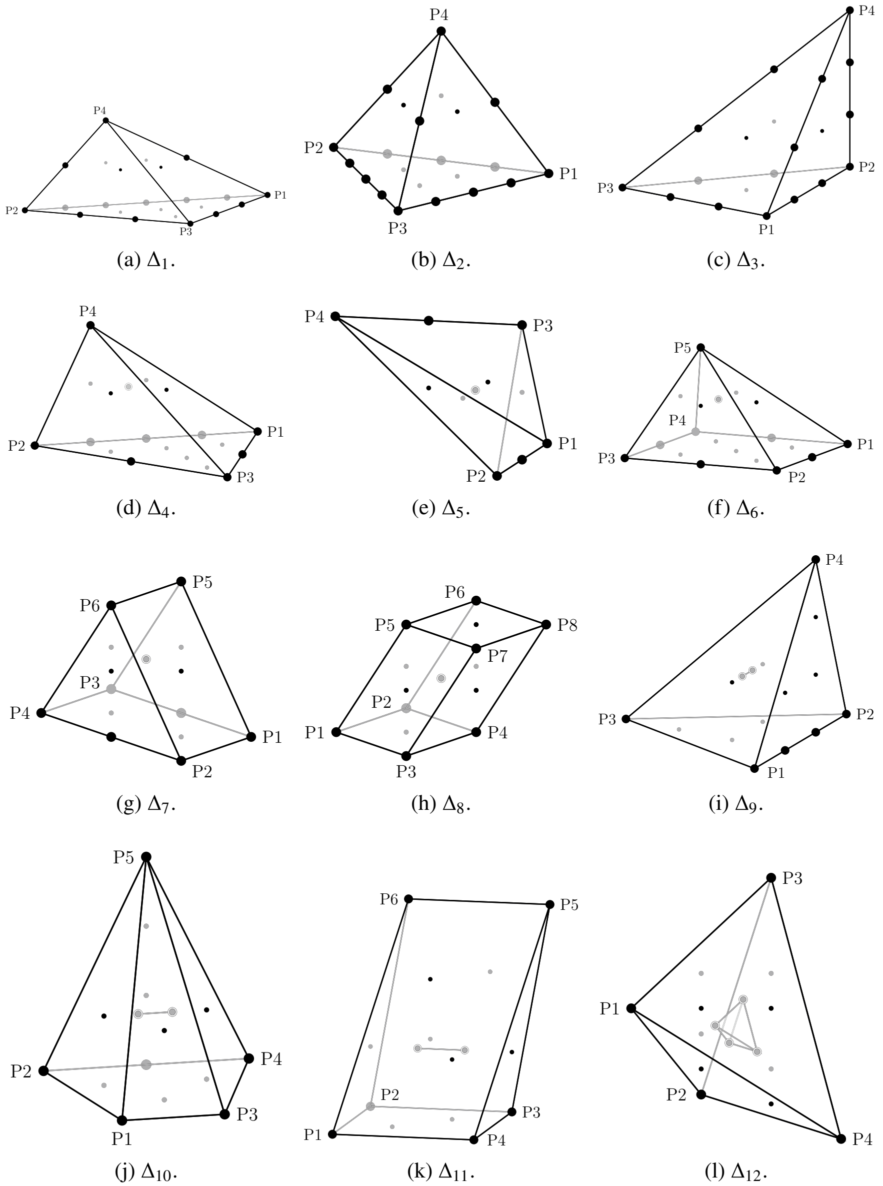
Table 2.10 12 Maximal Hollow 3-topes. Table contains: index i of the maximal hollow 3-tope Δ_i, vertices $\mathrm{vert}(\Delta_i)$ of Δ_i, lattice width $\mathrm{w}(\Delta_i)$ of Δ_i, dimension $\dim(\Delta_i^{\mathrm{FI}})$ of Fine interior Δ_i^{FI}, vertices $\mathrm{vert}(\Delta_i^{\mathrm{FI}})$ of Δ_i^{FI}, and order of fundamental group $|\pi_1(\mathcal{S}_\Delta)|$ of the minimal model $\mathcal{S}_{\Delta_i}$

i	$\mathrm{vert}(\Delta_i)$	$\mathrm{w}(\Delta_i)$	$\dim(\Delta_i^{\mathrm{FI}})$	$\mathrm{vert}(\Delta_i^{\mathrm{FI}})$	$\lvert\pi_1(\mathcal{S}_\Delta)\rvert$
1	(0, 0, 0), (6, 0, 0), (3, 3, 0), (4, 0, 2)	2	−1	∅	1
2	(0, 0, 0), (4, 0, 0), (0, 4, 0), (2, 0, 2)	2	−1	∅	1
3	(0, 0, 0), (3, 0, 0), (0, 3, 0), (3, 0, 3)	3	−1	∅	1
4	(0, 0, 0), (4, 0, 0), (2, 4, 0), (3, 0, 2)	2	0	1/2 · (5, 1, 2)	2
5	(0, 0, 0), (2, 2, 0), (1, 1, 2), (3, −1, 2)	2	0	1/2 · (3, 1, 2)	2
6	(0, 0, 0), (2, 2, 0), (4, 0, 0), (2, −2, 0), (3, 1, 2)	2	0	1/2 · (5, 1, 2)	2
7	(0, 0, 0), (1, 1, 0), (2, −2, 0), (3, −1, 0), (1, −1, 2), (2, 0, 2)	2	0	1/2 · (3, −1, 2)	2
8	(0, 0, 0), (1, 1, 0), (1, −1, 0), (2, 0, 0), (1, −1, 2), (2, 0, 2), (2, −2, 2), (3, −1, 2)	2	0	1/2 · (3, −1, 2)	2
9	(0, 0, 0), (3, 0, 0), (1, 3, 0), (2, 0, 3)	3	1	(4/3, 1, 1), (5/3, 1, 1)	3
10	(0, 0, 0), (1, 2, 0), (1, −1, 0), (3, 0, 0), (2, 1, 3)	3	1	(4/3, 2/3, 1), (5/3, 1/3, 1)	3
11	(0, 0, 0), (1, 1, 0), (3, 0, 0), (2, −1, 0), (4, 1, 3), (2, 2, 3)	3	1	(5/3, 2/3, 1), (7/3, 1/3, 1)	3
12	(−1, 0, 0), (0, 1, −2), (1, 2, 1), (2, −2, −1)	3	3	(1/5, 1/5, −2/5), (2/5, 2/5, −4/5), (3/5, 3/5, −1/5), (4/5, −1/5, −3/5)	5



(a) Δ_1. (b) Δ_2. (c) Δ_3.

(d) Δ_4. (e) Δ_5. (f) Δ_6.

(g) Δ_7. (h) Δ_8. (i) Δ_9.

(j) Δ_{10}. (k) Δ_{11}. (l) Δ_{12}.

Fig. 2.7 12 Maximal Hollow 3-topes. Shaded faces are occluded. The Fine interior is shown in grey with double borders around its vertices

Table 2.11 5 Hollow 3-topes with 0-dimensional Fine Interior. Table contains: index i of the maximal hollow 3-tope Δ_i, rays of the normal fan Σ^{Δ_i} corresponding to Δ_i, ID of the canonical Fano 3-tope Δ' such that $\Sigma^{\Delta_i} \cong \Sigma_{\Delta'}$, rays of the spanning fan $\Sigma_{\Delta'}$, ID of the reflexive canonical Fano 3-tope Δ'' used to construct the Gorenstein toric Fano threefold $X_{\Sigma_{\Delta''}}$ to obtain the toric Fano threefold $X_{\Sigma_{\Delta'}}$ with at worst canonical singularities as a μ_2 quotient, and reference to the corresponding Gorenstein toric Fano threefold $X_{\Sigma_{\Delta''}}$ including the precise μ_2-action on Sect. 2.7

i	$\Sigma^{\Delta_i}(1)$	ID(Δ')	$\Sigma_{\Delta'}(1)$	ID(Δ'')	$X_{\Sigma_{\Delta''}}$
4	(2, −1, −3), (0, 0, 1), (0, 1, 0), (−2, −1, −1)	547354	(−2, −3, −5), (2, 1, 1), (0, 1, 0), (0, 0, 1)	547363	(i)
5	(1, −1, 0), (1, 1, −1), (−1, 1, 2), (−1, −1, −1)	547364	(0, 0, 1), (0, 2, 1), (2, 1, 0), (−2, −3, −2)	547367	(ii)
6	(0, 0, 1), (1, 1, −2), (1, −1, −1), (−1, −1, 0), (−1, 1, −1)	544353	(1, 0, 0), (0, 1, 0), (−2, −1, 0), (1, 1, 2), (−3, −1, −2)	544357	(iii)
7	(0, 0, 1), (−1, 1, −1), (1, −1, −1), (1, 1, 0), (−1, −1, 0)	544310	(−1, −1, −2), (1, 1, 2), (−2, −1, 0), (0, 1, 0), (1, 0, 0)	544342	(iv)
8	(−1, 1, 1), (0, 0, −1), (−1, −1, 0), (1, 1, 0), (1, −1, −1), (0, 0, 1)	520134	(1, 0, 0), (0, 1, 0), (1, 1, 2), (−1, 0, 0), (0, −1, 0),	520140	(v)
			(−1, −1, −2)		

Table 2.12 9 Hollow 3-topes with Non-empty Fine Interior. Table contains: index i of the maximal hollow 3-tope Δ_i, support $\mathrm{supp}(\Delta_i^{\mathrm{FI}})$ of Δ_i^{FI}, and vertices $\mathrm{vert}(\Delta_i^{\mathrm{can}})$ of the canonical hull Δ_i^{can}

i	$\mathrm{supp}(\Delta_i^{\mathrm{FI}})$	$\mathrm{vert}(\Delta_i^{\mathrm{can}})$
4	$(-2,-1,-1), (0,-1,-2), (2,-1,-3), (0,0,1), (0,0,-1), (0,1,0)$	$\mathrm{vert}(\Delta_i)$
5	$(1,-1,0), (1,1,-1), (0,0,1), (0,0,-1), (-1,-1,-1), (-1,1,2)$	$\mathrm{vert}(\Delta_i)$
6	$(1,1,-2), (1,-1,-1), (-1,-1,0), (-1,1,-1), (0,0,1), (0,0,-1)$	$\mathrm{vert}(\Delta_i)$
7	$(1,1,0), (1,-1,-1), (-1,-1,0), (-1,1,-1), (0,0,1), (0,0,-1)$	$\mathrm{vert}(\Delta_i)$
8	$(1,1,0), (1,-1,-1), (-1,-1,0), (0,0,1), (0,0,-1), (-1,1,1)$	$\mathrm{vert}(\Delta_i)$
9	$(0,-1,-1), (0,0,1), (3,-1,-2), (0,1,0), (-3,-2,-1)$	$\mathrm{vert}(\Delta_i)$
10	$(-1,2,-1), (1,1,-1), (-1,-1,0), (2,-1,-1), (0,0,1)$	$\mathrm{vert}(\Delta_i)$
11	$(1,-1,0), (0,0,1), (-1,-2,1), (-1,1,0), (1,2,-2)$	$\mathrm{vert}(\Delta_i)$
12	$(1,1,1), (1,-1,0), (-2,-1,1), (0,1,-2)$	$\mathrm{vert}(\Delta_i)$

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