Chapter 15
Polygons of Finite Mutation Type

Thomas Prince

Abstract We classify Fano polygons with finite mutation class. This classification exploits a correspondence between Fano polygons and cluster algebras, refining the notion of singularity content due to Akhtar and Kasprzyk. We also introduce examples of cluster algebras associated to Fano polytopes in dimensions greater than two.

Keywords Polytopes · Toric Varieties · Mirror Symmetry

15.1 Introduction

The notion of combinatorial, or polytope, mutation was introduced by Akhtar–Coates–Galkin–Kasprzyk [3] to describe mirror partners to Fano manifolds. Following Givental [17–19], Kontsevich [28], and Hori–Vafa [25], the mirror partner to a Fano manifold consists of a complex manifold together with a holomorphic function, the *superpotential*. If this mirror manifold contains a complex torus we can write down a collection of volume preserving birational maps of this complex torus which preserve the regularity of the superpotential. We call these rational maps (algebraic) mutations, following [3] and work of Galkin–Usnich [16]. Combinatorial mutation is the operation induced on the Newton polyhedra of the restriction of the superpotential to such tori.

All the polytopes we consider are *Fano*, that is, polytopes which contain the origin in the interior and such that its vertices are primitive lattice vectors. In joint work [27] with Kasprzyk and Nill we showed that, in dimension two, the notion of polytope mutation is compatible with the construction of a quiver and cluster algebras one can associate to each Fano polygon.

The idea of associating a polygon with a quiver—or toric diagram—has a reasonably long history, particularly in the physics literature. In that setting the polygon describes a toric Calabi–Yau singularity and the quiver is used to describe the matter

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content of a gauge theory arising on a stack of D3-branes probing the toric Calabi–Yau singularity, (see for example [1, 5, 10, 15, 21, 22, 29] for a selection of the literature on this subject). The construction of a quiver (and cluster algebra) from a polygon has also been used by Gross–Hacking–Keel [20] in the study of associated log Calabi–Yau varieties, and to study the derived category of the toric variety, or the associated local toric Calabi–Yau as pursued, for example, in [7, 23, 24, 31, 32]. In each setting the basic construction is the same, and we recall the version relevant to our applications in Sect. 15.3.

Our main result, Theorem 30, is a classification of the mutation classes of polygons which contain only finitely many polygons. This parallels a finite type result of Mandel [30], for rank two cluster varieties. In particular we see that finite mutation classes of polygons fall into four types A_1^n, for $n \in \mathbb{Z}_{\geq 0}$, A_2, A_3, and D_4.

There is a close connection between mutation classes of Fano polygons and $\mathbb{Q}$-Gorenstein deformations of the corresponding toric varieties which is described in detail in [2]. Following these ideas we predict the existence of a finite type parameter space for these deformations, together with a boundary stratification such that each 0-stratum corresponds to a polygon in the given mutation class, and the 1-strata corresponds to the mutation families constructed by Ilten [26].

While our main result applies in dimension two, we note that polytope mutation is defined in all dimensions, and the construction of a quiver and cluster algebra we provide applies to 'compatible collection' of mutations in any dimension, see Definition 16. This definition is, unfortunately, less well behaved in dimensions greater than two, but we provide an example indicating that polytope mutation can detect known examples of cluster structures appearing on linear sections of Grassmannians of planes. We expect this to extend to a wide variety of other cluster structures found in Fano manifolds and their mirror manifolds.

15.2 Quivers and Cluster Algebras

We devote this section to fixing the various conventions and notation, as well as recalling the basic definitions. We recall the definition of cluster algebra, and in order to address both geometric and combinatorial applications we shall adapt our treatment from the work of Fomin–Zelevinsky [13], and the work of Fock–Goncharov [11] and Gross–Hacking–Keel [20]. We first fix the following data:

1. N, a fixed lattice with skew-symmetric form $\{-,-\}\colon N \times N \to \mathbb{Z}$;
2. a saturated sublattice $N_{uf} \subseteq N$, the *unfrozen* sublattice;
3. an index set I, $|I| = \mathrm{rk}(N)$ together with a subset $I_{uf} \subseteq I$ such that $|I_{uf}| = \mathrm{rk}(N_{uf})$. For later convenience we set $n := |I_{uf}|$.

Remark 1 The requirement that the form is integral is not necessary, but is sufficiently general for our applications and simplifies the exposition considerably.

Definition 2 A *(labelled) seed* is a pair $\mathbf{s} = (\mathcal{E}, C)$, where:

1. $\mathcal{E}$ is a basis of N indexed by I, such that the subset indexed by I_{uf} is a basis of N_{uf};

2. C is a transcendence basis of $\mathcal{F}$, the field of rational functions in n independent variables over $\mathbb{Q}(x_i : i \in I \backslash I_{uf})$, referred to as a *cluster*. Elements of C are also referred to as *cluster variables*.

Remark 3 The basis $\mathcal{E}$ is referred to as *seed data* in [11, 20]. Since we have fixed the lattice N and skew-symmetric form $\{-,-\}$ the elements of C can be identified with coordinate functions on the *seed torus* T_N.

Definition 4 Fix a seed $\mathbf{s} = (\mathcal{E}, C)$, where $\mathcal{E} = \{e_i : i \in I\}$ and $C = \{x_i : i \in I_{uf}\}$. Fixing an element $j \in I_{uf}$, the *jth mutation* of $(\mathcal{E}, C)$ is the seed $(\mathcal{E}', C')$, where $\mathcal{E}' = \{e'_i : i \in I\}$ is defined by setting

$$e'_k = \begin{cases} -e_j & \text{if } k = j, \\ e_k + \max\big(b_{kj}, 0\big)e_j & \text{otherwise,} \end{cases}$$

where $b_{kl} = \{e_k, e_l\}$. While the cluster $C' = \{x'_i : i \in I_{uf}\}$ is defined by setting,

$$x'_k = x_k, \text{ if } k \neq j, \qquad \text{and} \qquad x_j x'_j = \prod_{\substack{k \text{ such that} \\ b_{jk} > 0}} x_k^{b_{jk}} + \prod_{\substack{l \text{ such that} \\ b_{jl} < 0}} x_l^{b_{lj}}. \tag{15.1}$$

Recall that the matrix $B := (b_{kl})_{k,l \in I_{uf}}$ is typically referred to as the *exchange matrix* of the seed.

Definition 5 A *cluster algebra* is the subalgebra of $\mathcal{F}$ generated by the union of all clusters obtained by mutation from a given seed.

Any skew-symmetric $n \times n$ matrix B determines a skew-symmetric form on a (based) lattice $\mathbb{Z}^n$. Set $N = \mathbb{Z}^n$, $I = I_{uf} = \{1, \dots, n\}$, $\mathcal{E}$ to be the standard basis on $\mathbb{Z}^n$, and let $C = \{x_1, \dots, x_n\}$. We let $\mathcal{A}(B)$ denote the cluster algebra associated to the seed $(\mathcal{E}, C)$.

Definition 5 is really a special case of the definition of a cluster algebra, a class referred to as the *skew-symmetric cluster algebras of geometric type*. In the general case the form $\{-,-\}$ need only be *skew-symmetrizable*. One consequence of the skew-symmetry of the form $\{-,-\}$ is the identification of each exchange matrix with an (unfrozen) *quiver*. One may assign this quiver in the obvious way, assigning a vertex v_i to each element $i \in I_{uf}$, and b_{ij} arrows $v_i \to v_j$, oriented according to the sign of b_{ij}. We may also add 'frozen' vertices v_i for each element of $i \in I \backslash I_{uf}$, with arrows introduced between frozen and unfrozen vertices similarly. Equivalently we may consider the quiver associated to the *extended exchange matrix*, but we do not make further use of this terminology. There is a well-known notion of quiver mutation, going back to Fomin–Zelevinsky [13], generalising the reflection functors of Bernstein–Gelfand–Ponomarev [6]. Mutating a seed in a skew-symmetric cluster algebra induces a corresponding mutation of the associated quiver.

Definition 6 Given a quiver Q and an element $i \in I_{uf}$, the *mutation of Q at v_i* is the quiver $\mathrm{mut}(Q, v)$ obtained from Q by:

1. adding, for each subquiver $v_1 \to v \to v_2$, an arrow from v_1 to v_2;
2. deleting a maximal set of disjoint two-cycles;
3. reversing all arrows incident to v.

The resulting quiver is well-defined up to isomorphism, regardless of the choice of two-cycles in (2).

Since we shall refer to quivers frequently we shall make the following conventions. Given a quiver Q, we define:

1. Q_0 to be the set of vertices of Q;
2. $\mathrm{Arr}(v_i, v_j)$ to be the set of arrows from $v_i \in Q_0$ to $v_j \in Q_0$;
3. b_{ij} to be the cardinality of $\mathrm{Arr}(v_i, v_j)$, with sign indicating orientation.

We shall always assume Q has no vertex-loops or 2-cycles.

Given a seed $\mathbf{s}$ we shall also fix notation for the dual basis $\mathcal{E}^\star$ of $M := \hom(N, \mathbb{Z})$ and for each $i \in I$, set $v_i := \{e_i, -\} \in M$. We now define the $\mathcal{A}$ and $\mathcal{X}$ cluster *varieties* defined by Fock–Goncharov [11]. Toward this, observe to a seed $\mathbf{s}$ we can associate a pair of tori:

$$\mathcal{X}_{\mathbf{s}} = T_M \qquad \mathcal{A}_{\mathbf{s}} = T_N.$$

The dual pair of bases for the respective lattices define identifications of these tori with split tori,

$$\mathcal{X}_{\mathbf{s}} \longrightarrow \mathbb{G}_m^{|I|}, \qquad \mathcal{A}_{\mathbf{s}} \longrightarrow \mathbb{G}_m^{|I|}.$$

Letting $\mathbf{s}'$ denote the kth mutation of $\mathbf{s}$, we associate birational maps $\mu_k \colon \mathcal{X}_{\mathbf{s}} \dashrightarrow \mathcal{X}_{\mathbf{s}'}$ and $\mu_k \colon \mathcal{A}_{\mathbf{s}} \dashrightarrow \mathcal{A}_{\mathbf{s}'}$ to each seed, defined by setting

$$\mu_k^\star z^n = z^n(1+z^{e_k})^{-\{n,e_k\}} \qquad \mu_k^\star z^m = z^m(1+z^{v_k})^{\langle e_k,m\rangle},$$

where $n \in N$, $m \in M$, z^m (resp. z^n) denotes the function on $T_N := \mathrm{Spec}(\mathbb{C}[M])$ corresponding to m (resp. the function on T_M corresponding to n), and $\langle -, - \rangle$ denotes the canonical pairing between M and N. Note that the tori $\mathcal{X}_{\mathbf{s}}$ and $\mathcal{X}_{\mathbf{s}'}$ are canonically identified with T_N, but the maps μ_k between them are not isomorphisms.

Pulling these birational maps back along the identifications with the split torus given by the seed, the birational map $\mu_k \colon \mathcal{A}_{\mathbf{s}} \dashrightarrow \mathcal{A}_{\mathbf{s}'}$ is given by the exchange relation (15.1). That is, this birational map is the coordinate-free expression of the exchange relation once we identify the standard coordinates on T_N with the cluster variables $x_i \in C$ (including the frozen variables $x_{n+1} \dots, x_m$). We obtain schemes $\mathcal{X}$ and $\mathcal{A}$ by gluing the seed tori $\mathcal{A}_s$ and $\mathcal{X}_s$ along the birational maps defined by the mutations μ_k. For more details and related results we refer to [11, 20].

We conclude this section by recalling the classifications of cluster algebras of *finite type* and *finite mutation type*.

Definition 7 A cluster algebra is said to be of *finite type* if it contains finitely many clusters.

Given an undirected graph G we say that a quiver Q is an *orientation* of G if it has the same set of vertices and for each edge of G there is precisely one arrow between the respective vertices. Given a simply-laced Dynkin diagram D we say that Q is of *type* D if it is an orientation of the underlying graph of D.

Theorem 8 ([14]) *There is a canonical bijection between the Cartan matrices of finite type and cluster algebras (without frozen variables) of finite type. Under this bijection, a Cartan matrix A of finite type corresponds to the cluster algebra $\mathcal{A}(B)$, where B is an arbitrary skew-symmetrizable matrix with Cartan companion equal to A.*

Theorem 8 describes skew-symmetric cluster algebras with finitely many clusters. We can ask instead for the weaker condition that only finitely many *quivers* appear associated to seeds of the cluster algebra. This is the notion of *finite mutation type* cluster algebra, for which a classification is also known.

Theorem 9 ([9, Theorem 6.1]) *Given a quiver Q with finite mutation class, its adjacency matrix b_{ij} is the adjacency matrix of a triangulation of a bordered surface or is mutation equivalent to one of eleven exceptional types.*

The class of quivers coming from triangulations of surfaces is well-studied and we make use of a combinatorial characterisation of this class of quivers via *block decomposition*. A quiver Q is said to admit a *block decomposition* if it may be assembled from the six *blocks* shown in Fig. 15.5 by identifying the vertices of quivers shown with unfilled circles, the *outlets*. More precisely, we choose an injection from a subset of the combined set of outlets O into O such that no outlet is mapped to a vertex of the same block, including itself. We form Q by gluing the quiver along these vertices and cancelling any two cycles formed by this process. See [12, Definition 13.1] for further discussion and examples of this definition.

Theorem 10 ([12, Theorem 13.3]) *A quiver Q given by the adjacency matrix of a triangulation of a surface is mutation equivalent to a quiver which admits a* block decomposition.

15.3 Mutations of Polytopes

In two dimensions all combinatorial mutations are 'tropicalisations' of cluster mutations. While this ceases to be true in higher dimensions there is a natural class of combinatorial mutations, the *edge mutations* which do appear in this way. In terms of the definition of combinatorial mutation given in [3], edge mutations are those which have one-dimensional *factor*. In particular each edge mutation is obtained by studying the effect of the following birational maps—an *algebraic mutation* [3]—on the Newton polyhedra of certain Laurent polynomials. Throughout this section N denotes an n-dimensional lattice (not necessarily related to the definition of a cluster algebra). We recall that, working over $\mathbb{C}$, if M is the lattice dual to N, the torus T_M is defined to be $\mathrm{Spec}(\mathbb{C}[N])$.

Definition 11 Given an element $w \in M$, the *weight vector*, and $f \in \mathrm{Ann}(w)$, the *factor*, define a birational map $\phi_{w,f} \colon T_M \dashrightarrow T_M$ sending

$$z^n \mapsto z^n(1 + z^f)^{\langle w,n \rangle}.$$

Given a Laurent polynomial $W \in \mathbb{C}[N]$ such that $\phi^\star_{w,f}(W) \in \mathbb{C}[N]$ say that W is mutable with weight vector w and factor f.

Definition 12 (*Cf.* [3, pg. 12]) Fix a Fano polytope $P \subset N_\mathbb{Q}$ and its dual $P^\star \subset M_\mathbb{Q}$, a weight vector $w \in M$, and factor $f \in \mathrm{Ann}(w)$. Define a piecewise linear map $T_{w,f} \colon M_\mathbb{Q} \to M_\mathbb{Q}$ by setting

$$T_{w,f} \colon m \mapsto m + \max(0, \langle m, f \rangle)w.$$

If $T_{w,f}(P^\star)$ is a convex polytope then we say P admits the mutation (w, f) and that P mutates to $(T_{w,f}(P^\star))^\star$.

Remark 13 This definition of mutation is really a 'dual characterisation' of [3, Definition 5], which encodes how the Newton polytope of a Laurent polynomial changes under algebraic mutation.

Remark 14 In [3] the authors show that the result of applying a mutation to a Fano polytope produces another Fano polytope, so the last dualization in Definition 12 is well-defined.

Proposition 15 *Given* $w \in M$, $f \in \mathrm{Ann}(w)$ *and a mutable Laurent polynomial* $W \in \mathbb{C}[N]$ *we have the following identity;*

$$\mathrm{Newt}\left(\phi^\star_{w,f} W\right)^\star = T_{w,f}\left(\mathrm{Newt}(W)^\star\right).$$

Proof The notion of combinatorial mutation is compatible with the mutation W by construction. The interpretation of a combinatorial mutation as a piecewise linear map is made in the proof of Proposition 4 in [3]. □

Definition 16 We define *mutation data* to be elements $(w, f) \in M \oplus N$ such that w and f are primitive, and $f \in \mathrm{Ann}(w)$. A set of mutation data $\{(w_i, f_i) \in M \oplus N : i \in I\}$, for a finite index set I, is called *compatible* if

$$\langle w_i, f_j \rangle = -\langle w_j, f_i \rangle \qquad \text{for all } i, j \in I.$$

Remark 17 If $\dim N = 2$ mutation data is automatically compatible; indeed $\langle w_i, f_j \rangle$ can be identified with $w_i \wedge w_j$ for a suitable orientation of M.

Definition 18 To a compatible collection of mutation data $\mathcal{E}$ we define a quiver $Q_\mathcal{E}$ as follows:

1. the vertex set of $Q_{\mathcal{E}}$ is $\mathcal{E}$;
2. between two vertices (w_i, f_i) and (w_j, f_j) there are $\langle w_i, f_j\rangle$ arrows, with sign indicating orientation.

Observe that, as $\langle w_i, f_j\rangle$ is skew-symmetric, the quiver $Q_{\mathcal{E}}$ contains no loops or two cycles. Note that we can use this definition to assign a *cluster algebra* to a compatible collection of mutation data. We define a rule governing how compatible collections of mutations themselves mutate.

Definition 19 Given a compatible collection of mutation data $\mathcal{E}$, let L be the sublattice of $M \oplus N$ generated by the elements of $\mathcal{E}$, and let $\{(w_i, f_i), (w_j, f_j)\} := \langle w_i, f_j\rangle$ define a skew-symmetric form on L. Fixing a pair $E_k = (w_k, f_k) \in \mathcal{E}$ we *mutate* $\mathcal{E}$ to a new collection $\mathcal{E}_k$ as follows:

1. $E_k \mapsto -E_k$;
2. $E_i \mapsto E_i - \max(\{E_i, E_k\}, 0)E_k$, if $i \neq k$.

This formula is identical to the mutation of seed data given in [11]; a connection we now make precise. Fix a compatible collection of mutations $\mathcal{E}$ and define a skew-symmetric form $[-,-]$ on $\mathbb{Z}^{|\mathcal{E}|}$ defined by setting $[e_i, e_j] := \{\theta(e_i), \theta(e_j)\}$, where $\theta\colon \mathbb{Z}^{|\mathcal{E}|} \to M \oplus N$ is defined by sending $e_i \mapsto (w_i, f_i)$. The following Lemma follows immediately by comparison of the formulae for mutating seed data in a cluster algebra with Definition 19.

Lemma 20 *The operations of mutation given in Definition 19, and of mutation of the seeds defined above, are intertwined by θ.*

Example 21 In dimensions higher than two a compatible collection of mutation data which defines a set of combinatorial mutations of a given polytope can transform by mutation to a compatible collection of mutation data which does not define a set of combinatorial mutations of the transformed polytope. In particular the piecewise linear maps may fail to preserve convexity. This appears to be a important obstruction to generalising the two-dimensional theory of mutations to higher dimensional polytopes. For example, consider the polytope

$$P := \operatorname{conv}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\-1\end{pmatrix}, \begin{pmatrix}-1\\-1\\0\end{pmatrix}, \begin{pmatrix}-1\\-1\\-1\end{pmatrix}\right).$$

Consider mutation data $(w_1, f_1) := (e_1^\star, e_3)$ and $(w_2, f_2) = (e_2^\star, e_3)$. Since $f_1 = f_2$, we have that $\langle w_i, f_j\rangle = 0$ for all $i, j \in \{1, 2\}$; hence these mutations are compatible. However, while P admits both these mutations, the composition of these two (in either order) is the mutation corresponding to the pair $(w, f) := (e_1^\star + e_2^\star, e_3)$, which does *not* define a mutation of P.

Proposition 22 *Given seed data $\mathcal{E}$ such that $Q_{\mathcal{E}}$ is a directed simply-laced Dynkin diagram the number of polytopes obtained by successive edge mutation is bounded by the numbers of seeds in the cluster algebra determined by $Q_{\mathcal{E}}$. If $Q_{\mathcal{E}}$ is of type A_n this bound is the Catalan number C_{n+1}.*

In fact, compatible collections of mutations appear whenever we have a cluster algebra with skew-symmetric exchange matrix.

Proposition 23 *Every compatible collection of mutations determines and is determined by a skew-symmetric cluster algebra without frozen variables, together with a subspace V of the kernel of the skew-symmetric form $\{-,-\}$ defined by the exchange matrix.*

Proof Fix a skew-symmetric cluster algebra without frozen variables and a nominated subspace $V \subset \ker\{-,-\}$. Recall that a seed defines a basis e_i of a lattice, which we denote $\widetilde{N}$. Define $M := \widetilde{N}/V$ and let $p\colon \widetilde{N} \to M$ be the canonical projection. The map $\theta\colon \widetilde{N} \to M \oplus \hom(M,\mathbb{Z})$ defined by $\theta\colon n \mapsto (p(n), \{n,-\})$ defines a compatible collection of mutation data with weight vectors in the lattice M. □

Note that N and M play dual roles to those in [11], and we insist throughout that $P \subset N_{\mathbb{Q}}$. This exchange of roles explains the odd definition of M in the proof of Proposition 23. To compare the birational maps associated to the two notions of mutations let $\mathbf{s}$ be a seed of the cluster algebra determined by a compatible collection of mutation data, and let $\mathcal{E}$ be the compatible collection corresponding to $\mathbf{s}$. Fix an element $E_k = (w_k, f_k) \in \mathcal{E}$ and consider the following diagram,

$$\begin{array}{ccc} \mathcal{A}_{\mathbf{s}} & \overset{\mu_k}{\dashrightarrow} & \mathcal{A}_{\mu_k(\mathbf{s})} \\ \downarrow p & & \downarrow p \\ T_M & \underset{\phi_{(w_k,f_k)}}{\dashrightarrow} & T_M, \end{array} \tag{15.2}$$

Proposition 24 *The diagram shown in* (15.2) *commutes.*

Proof This is an exercise in writing out the definitions of the respective mutations: see [27, Sect. 3]. □

Example 25 The del Pezzo surface of degree 5 admits a toric degeneration to a toric surface Z with a pair of A_1 singularities. Given a three-dimensional linear section X of the Grassmannian $\mathrm{Gr}(2,5)$ X admits a toric degeneration to the projective cone over Z. The fan determined by this toric threefold is formed by taking the cones over the faces of the reflexive polytope with PALP id 245.

In Fig. 15.1 we show a pentagon of polytopes obtained by successively mutating the polytope shown in the top-right with respect to the mutation data

$$\mathcal{E} := \{(w_1, f_1), (w_2, f_2)\}$$

where,

$$w_1 := (-1,0,0), \qquad f_1 := (0,1,1)^T,$$
$$w_2 := (0,0,-1), \qquad f_2 := (-1,0,0)^T.$$

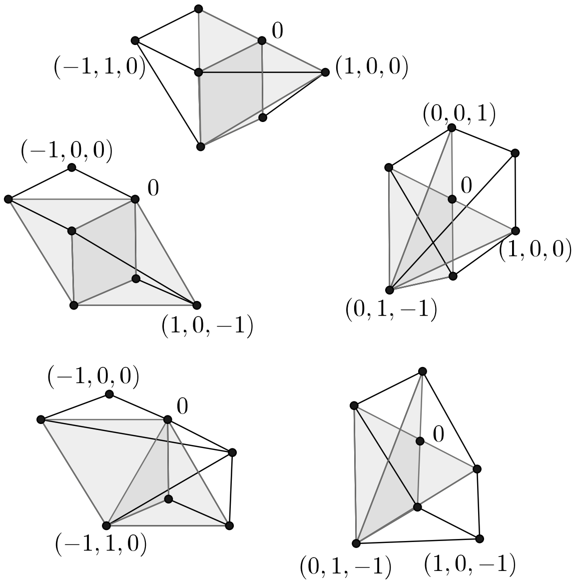


Fig. 15.1 Pentagon of edge mutations among toric degenerations of B_5

We recall that there is an A_2 cluster structure on the co-ordinate ring of the Grassmannian, and a toric degeneration of $\mathrm{Gr}(2, 5)$ for each cluster chart in the dual Grassmannian [33]. We expect that cluster structures in the mirror to a Fano variety to be detected by such compatible collections of mutations.

Note that the polytopes we show in Fig. 15.1 are not dual to Fano polytopes. However, recalling that B_5 has Fano index 2, we can obtain a reflexive polytope by dilating each of the polytopes shown in Fig. 15.1 by a factor of two, and translating.

In the two dimensional case, we can canonically define a maximal set of compatible mutations, making use of the notion of singularity content [4].

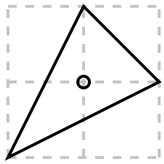
Definition 26 (*Cf.* [27, Sect. 1.2]) Given a Fano polygon $P \subset N_{\mathbb{Q}}$ with singularity content $(n, \mathcal{B})$ and $m := |\mathcal{B}| + n$, we define:

1. an index set I of size m containing a subset I_{uf} of size n, together with functions

$$\phi_{uf} \colon I_{uf} \to \{\text{edges of } P\} \qquad \phi_f \colon I \backslash I_{uf} \to \mathcal{B}$$

   such that fibres $\phi_{uf}^{-1}(E)$ contain $m_E := \lfloor \ell(E)/r_E \rfloor$ elements, where $\ell(E)$ is the lattice length of the edge E, and r_E is the Gorenstein (or *local*) index of the cone over E, while the map ϕ_f is a bijection;
2. a lattice map $\rho \colon \mathbb{Z}^m \to M$ sending each basis element to the primitive, inward-pointing normal to the edge of P defined by the cone given by the specified functions ϕ_{uf} and ϕ;
3. a form $\{e_i, e_j\} := \rho(e_i) \wedge \rho(e_j)$. Note that this is an integral skew-symmetric form.

Fig. 15.2 The Fano polygon for $\mathbb{P}^2$

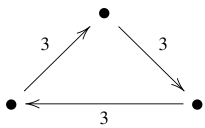


The value m_E appears in the definition of the *singularity content* of a two dimensional cone; and is equal to the maximal number of T-cones of Gorenstein index r_E which fit inside the cone on E.

By [27, Proposition 3.17] the construction of a quiver from mutation data provided by Definition 26 intertwines polygon and quiver mutations. We let $(\mathcal{E}_P, C_P)$ denote the seed associated to a Fano polygon, where $\mathcal{E}_P$ is the standard basis e_i of $\mathbb{Z}^m$, and C_P is the standard transcendence basis of the field of rational functions in n variables over $\mathbb{Q}(x_i : i \in I \backslash I_{uf})$. We let Q_P denote the *unfrozen* quiver associated to $(\mathcal{E}_P, C_P)$. We say a Fano polygon is of *finite mutation type* if it is mutation equivalent to only finitely many Fano polygons.

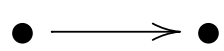
Conjecture 27 The cluster algebra C_P associated to a Fano polygon P, together with a bijection between the set of frozen variables and $\mathcal{B}$, is a complete mutation invariant of the Fano polygon P.

Example 28 Consider the Fano polygon P for $\mathbb{P}^2$ (this is depicted in Fig. 15.2). Computing the determinant of the inward-pointing normals we obtain the quiver Q_P



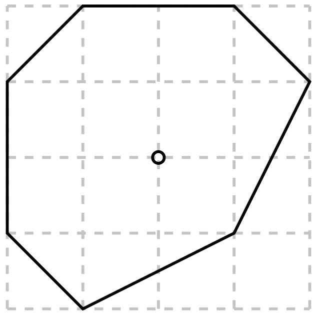
The mutations of this quiver are well-known, and the triple $(3a, 3b, 3c)$ of non-zero entries of the exchange matrix satisfy the Markov equation $a^2 + b^2 + c^2 = 3abc$. Indeed, as the polygon P is mutated the corresponding toric surfaces are $\mathbb{P}(a^2, b^2, c^2)$ for the same triples (a, b, c). We see that in this case the mutations of the quivers exactly capture the mutations of the polygon.

Example 29 Consider the toric surface (using the notation for these surfaces appearing in [8]), $X_{5,5/3}$ associated with the Fano polygon shown in Fig. 15.3. The unfrozen quiver associated to this surface is simply the A_2 quiver:



This example is important, both in this section, because it is an example of a *finite type* polygon, and since a smoothing of this surface is given by 5 Pfaffian equations, see [8, Sect. 3.3], a fact closely connected to the A_2 quiver we construct here.

Fig. 15.3 The Fano polygon for $X_{5,5/3}$



15.4 Finite Type Classification

We now make use of the classification of finite type and finite mutation type cluster algebras to establish the following result.

Theorem 30 *P is of finite mutation type if and only if Q_P is mutation equivalent to a quiver of type $(A_1)^n$, A_2, A_3, or D_4.*

Remark 31 The types referred to in Theorem 30 may also be referred to as type I_n, II, III, and IV respectively; in analogy with Kodaira's monodromy matrices. The relationship between these matrices, log Calabi–Yau manifolds, and monodromy in certain integral affine manifolds is explored by Mandel in [30].

Remark 32 We remark that all the cases which appear in Theorem 30 do occur as (unfrozen) quivers associated to polygons. Several examples can be found in [8, p. 42] and are tabulated below.

Quiver Q_P	Polygon	Surface
$\varnothing = A_1^0$	9	$X_{6,2}$
A_1^3	11	$X_{4,7/3}$
A_1^6	12	$B_{2,8/3}$
A_2	7	$X_{5,5/3}$
A_3	17	$X_{3,4}$
D_4	5	$X_{4,4/3}$

Examples of polygons P with $Q_P = A_1^{2k+2}$ and A_1^{2k+1} for $k \geq 5$ are given by the quadrilaterals

$$\operatorname{conv}\left(\begin{pmatrix}-1\\-3\end{pmatrix}, \begin{pmatrix}1\\-3\end{pmatrix}, \begin{pmatrix}-1\\k-2\end{pmatrix}, \begin{pmatrix}1\\k-2\end{pmatrix}\right), \text{ and}$$
$$\operatorname{conv}\left(\begin{pmatrix}-1\\-3\end{pmatrix}, \begin{pmatrix}1\\-2\end{pmatrix}, \begin{pmatrix}-1\\k-2\end{pmatrix}, \begin{pmatrix}1\\k-2\end{pmatrix}\right).$$

We first make two straightforward observations. First we note that the cluster algebra C_P induces a sequence of surjections:

$$\begin{array}{c}\{\text{Clusters of } C_P\} \\ \downarrow \\ \{\text{Polygons mutation equivalent to } P\} \\ \downarrow \\ \{\text{Quivers mutation equivalent to } Q_P\}.\end{array} \tag{15.3}$$

The first vertical arrow follows from the fact that algebraic mutations determine combinatorial mutations, the second from Lemma 20. For example, using this tower of surjections in the case of a type A_2 cluster algebra, we can immediately state the following result.

Proposition 33 *If a Fano polygon P has singularity content $(2, \mathcal{B})$ and the primitive inward-pointing normal vectors of the two edges corresponding to the unfrozen variables of C_P form a basis of the lattice M, then the mutation-equivalence class of P has at most five members.*

Proof The quiver associated to P is precisely an orientation of the A_2 quiver. The cluster algebra C_P is well-known and its cluster exchange graph forms a pentagon. Note however that the *quiver* mutation graph is trivial, as the A_2 quiver mutates only to itself. □

Proposition 24 implies that the mutation class of P has at most five elements. Note that we do not have a non-trivial lower bound: there is only one polygon in mutation equivalent to the polygon described in Example 29 up to $\mathrm{GL}(2, \mathbb{Z})$ equivalence. Next observe that the sequence of surjections shown in (15.3) immediately implies that

$$C_P \text{ finite type} \Rightarrow P \text{ finite mutation type} \Rightarrow C_P \text{ finite mutation type.}$$

Lemma 34 *Given a Fano polygon P of finite mutation type, Q_P does not contain a Kronecker subquiver*

$$Q_k := \{\ v_1 \xrightarrow{k} v_2\ \},$$

where $k > 1$ is the number of arrows from v_1 to v_2.

Remark 35 This result is expected from results on the corresponding cluster algebra. The Kronecker quiver defines a rank 2 cluster algebra which is known not to be of finite type when $k > 1$. Given that P is the Newton polygon of a superpotential which is itself a combination of cluster monomials, we expect the polygon P to grow as we mutate.

Proof (***Proof of Lemma*** 34) Assume there is a Q_k subquiver of Q_P, with vertices v_1, v_2 corresponding to edges E_1 E_2 of P. We define $\rho \colon \mathbb{Z}^2 \to M$ by mapping the standard basis to the primitive inward normal vectors w_i to E_i for $i \in \{1, 2\}$.

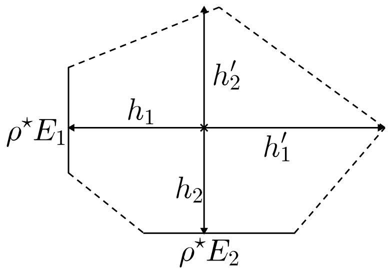


Fig. 15.4 Schematic diagram of a polygon in standard form

Let $P' \subset \mathbb{Q}^2$ be the image of P under $\rho^\star$. The resulting polygon in $\mathbb{Q}^2$ is shown schematically in Fig. 15.4.

The *local index* of each cone in P is the integral height of the edge from the origin. Let h_i denote the local indices of E_i for $i \in \{1, 2\}$. Note that, as $h_i = \langle e_i, \rho^\star u\rangle$ for any $u \in E_i$, h_i is also the local index of $\rho^\star(E_i)$ in P'. Mutating at v_1 and v_2 we denote the new local indices,

$$(h_1, h_2') \longleftarrow (h_1, h_2) \longrightarrow (h_1', h_2).$$

We first show that $\rho^\star$ increases the lattice lengths of E_i by a factor of $k := |w_1 \wedge w_2|$ for each $i \in \{1, 2\}$. Let u_1^i and u_2^i denote the vertices of E_i, and $f_i := (u_1^i - u_2^i)/\ell(E_i)$; where $\ell(E_i)$ denotes the lattice length of E_i. Note that $\langle w_1, f_2\rangle = \langle w_2, f_1\rangle = w_1 \wedge w_2$ for a suitable choice of orientation of M. Moreover, since – for each $i \in \{1, 2\}$ – $\langle \rho^\star(u), e_i\rangle = h_i$ is constant as u varies in E_i, the edge $\rho^\star(E_i)$ has direction vector $e^\star_{3-i}$; where $\{e_1^\star, e_2^\star\}$ is the dual basis to $\{e_1, e_2\}$. In other words,

$$\rho^\star(u_1^i) - \rho^\star(u_2^i) = \ell(\rho^\star(E_i))e^\star_{3-i},$$

and hence we have that

$$\begin{aligned}
\ell(\rho^\star(E_i)) &= \langle \ell(\rho^\star(E_i))e^\star_{3-i}, e_{3-i}\rangle \\
&= \langle \rho^\star(u_1^i) - \rho^\star(u_2^i), e_{3-i}\rangle \\
&= \langle u_1^i - u_2^i, \rho(e_{3-i})\rangle \\
&= \ell(E_i)\langle f_i, w_{3-i}\rangle \\
&= \pm\ell(E_i)(w_1 \wedge w_2),
\end{aligned}$$

where signs and orientations are chosen such that $\ell(E)$ is always positive. Studying Fig. 15.4 note that $h_1 + h_1' \geq \ell(\rho^\star(E_2))$, however—by the calculation above—$\ell(\rho^\star(E_2)) = k\ell(E_2)$. Moreover, we have that $\ell(E_2) \geq h_2$, since the Fano polygon P admits a mutation along this edge. Hence we observe that

$$h_1' \geq kh_2 - h_1 \qquad h_2' \geq kh_1 - h_2.$$

Consider the case $k \geq 3$, and assume without loss of generality that $h_2 \geq h_1$. We have that $h'_1 \geq 3h_2 - h_1 \geq 2h_2 \geq 2h_1$. Thus in this case the values in the pair (h_1, h_2) grow (at least) exponentially with mutation, and in particular take infinitely many values.

Next consider the case $k = 2$. The inequalities above become,

$$h'_1 \geq 2h_2 - h_1 \qquad h'_2 \geq 2h_1 - h_2,$$

and we are again free to assume that $h_2 \geq h_1$. Thus $h'_1 \geq 2h_2 - h_1 \geq h_1$, and if $h_2 > h_1$, $h'_1 \geq 2h_2 - h_1 > h_2$. Thus, assuming $h_1 \neq h_2$, one can generate an infinite increasing sequence of local indices. The only remaining case is if $h := h_1 = h_2 = h'_1 = h'_2$. To eliminate this possibility observe that, since $k = 2$, the edges $\rho^\star(E_1)$, $\rho^\star(E_2)$ must meet in a vertex with coordinates $(-h, -h)$ (indeed, assuming this does not hold, a mutation returns us to the previous case and one of the above inequalities is strict). Note that the sublattice $\rho^\star(N)$ is determined by the fact that $\rho^\star$ doubles the edge lengths of E_1 and E_2. The lattice vectors (a, a) are in this sublattice for all $a \in \mathbb{Z}$. Thus, by primitivity of the vertices in P, $h = 1$. Since the origin is in the interior of P, mutating in one of v_1 or v_2 returns us to the previous case. □

Remark 36 Proposition 34 implies all the quivers that we consider from now on are directed graphs. Hence we refer to vertices as *adjacent* if they are adjacent in the underlying graph.

As well as the non-existence of Kronecker quivers in Q_P for finite mutation type polygons P, we use heavy use of a connectedness result for quivers Q_P which follows immediately from the definition of Q_P via determinants in the plane; or equivalently from the fact the exchange matrix has rank 2 (Fig. 15.5).

Lemma 37 *Given a Fano polygon P and vertices v_1, v_2, v_3 of Q_P such that v_i and v_{i+1} are not adjacent for $i = 1, 2$, then v_1 and v_3 are not adjacent.*

Proof (*Proof of Theorem* 30) By Lemma 37, if Q_P is not connected, $Q_P \cong A_1^n$ for some n. Similarly, if Q_P is of type A or D, then it must be one of A_2, A_3 or D_4. Thus we only need to show that there is no Fano polygon P of finite mutation type such that C_P is not of finite-type. However C_P is of finite mutation type, and we use the classification described in Theorems 9 and 10, following [9, 12]. In fact, using Lemma 37, none of the eleven exceptional types can occur as Q_P for a Fano polygon P. Hence we can restrict to quivers which admit a *block decomposition* and work case-by-case.

We claim that every quiver Q_P associated to a Fano polygon P which admits a block decomposition is either mutation equivalent to an orientation of a simply-laced Dynkin diagram or to a quiver which contains a subquiver Q_k for $k > 1$. We assume for contradiction that Q_P is the quiver associated to a Fano polygon P of finite-type which is not mutation equivalent to a simply laced Dynkin diagram.

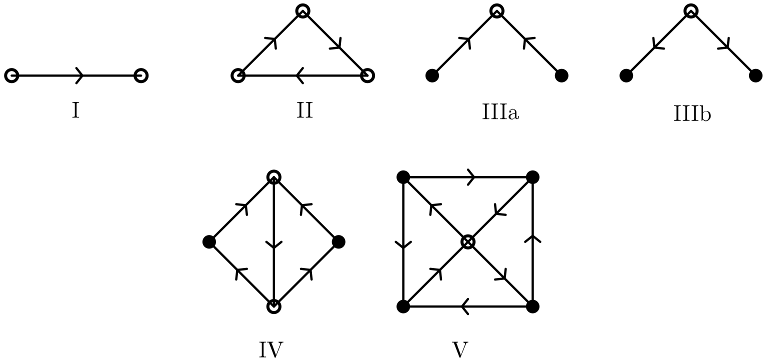


Fig. 15.5 The blocks of a *block decomposition*

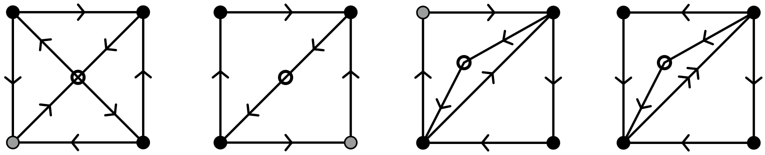
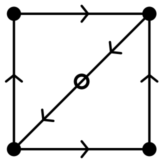
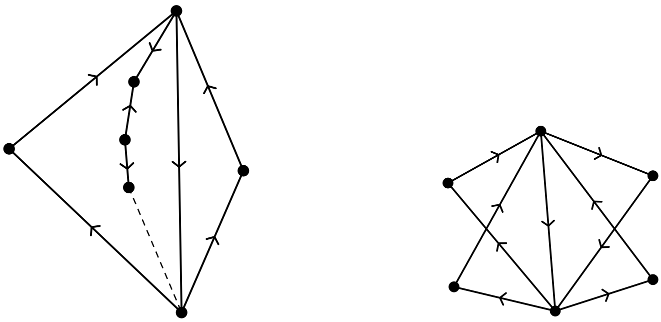


Fig. 15.6 Mutations of block V

Fig. 15.7 Quiver V′



<u>Block V:</u> First observe that, since only one vertex of the block V is an outlet, the V block quiver is a subquiver of any quiver which contains the V block in its decomposition. However this quiver mutates to a quiver with a Q_2 subquiver as shown in Fig. 15.6. Therefore block V never appears in a decomposition of a quiver Q_P. For later use we shall fix the following intermediate quiver, V′, shown in Fig. 15.7.
<u>Blocks IIIa and IIIb:</u> Assume there is a type III block (a or b) connected to a quiver Q' at a vertex v. If there is a vertex v' of Q' such that v and v' are not adjacent, the quiver violates Lemma 37. In particular the vertex set of Q' must be the vertex set of a single block. In particular, using the previous part, Q' has at most four vertices. Case by case study shows that only the A_3 and D_4 types appear.
<u>Block IV:</u> Consider the case of a decomposition only using type IV blocks. Note that the type IV block is itself of type D4. Consider attaching two type IV blocks. If the blocks are attached at a single outlet the resulting quiver contradicts Lemma 37. In fact it is easy to see that it is impossible to add additional type IV blocks to meet this condition. If both pairs of outlets are matched there are two possible quivers depending on the relative orientations of the arrow between the outlets, one orienta-



(a) Attaching I blocks to a IV block (b) Attaching II blocks to a IV block

Fig. 15.8 A type IV block

tion produces a Q_2 subquiver automatically, the other produces a quiver containing the quiver V' as a subquiver. Thus, for a type IV block to appear in a decomposition of Q_P it must include a type I or II block.

Now consider decompositions using type I and II blocks as well as type IV blocks. First note there must be exactly one IV block (assuming there is at least one). Indeed, if type IV blocks are not connected using both vertices, a non-outlet vertex of a IV block is not adjacent to some outlet, and some non-outlet vertex of a (different) IV block. However outlets and non-outlets of a type IV block are always adjacent, violating Lemma 37.

Thus we must attach I and II blocks to a single type IV block. By Lemma 37 the vertex set of the final quiver must be equal to the vertex set obtained by attaching a single block to each outlet of the IV block. Considering these cases in turn, we note first that attaching a type I block to cancel the arrow between the two outlets produces a quiver mutation equivalent to D_4 and therefore eliminated. For chains type I blocks of length two, if a 3-cycle is produced, a mutation in the vertex between the type I blocks produces the V' quiver. If not, the same mutation produces a Q_2 subquiver.

Attaching a type II block along two outlets of the type IV block recovers the V' or Q_2 subquiver cases we have already seen. Attaching type II blocks to a single outlet each we observe that every new vertex must be adjacent to both outlets of the IV block. Hence the only case without a Q_2 subquiver is shown on the right of Fig. 15.8, however this quiver mutates to one with a Q_2 subquiver. Attaching further type II blocks any quiver we obtain must contradict Lemma 37.

Blocks I and II: From what we have shown above, the block decomposition of Q_P consists only of type I and type II blocks. Any connected quiver with a block decomposition into type I blocks is a path (with possibly changing orientations), which possibly closes up into a cycle. The only cases not violating Lemma 37 are mutation equivalent to orientations of simply laced Dynkin diagrams.

For decompositions of Q_P with type I and II blocks we divide the proof into cases indexed by the number of type II blocks. For a single type II block, we can attach a type I block to two outlets and in this way reduce to the type III case. Attaching each type I block to a type II block in at most one outlet, we use the fact that every new vertex must be adjacent to at least two of the vertices of the type II block. Thus we can obtain only two undirected graphs—the underlying graph of a type IV block or an orientation of a tetrahedron, these cases can easily be eliminated. For example, there is no orientation of the tetrahedron making every cycle oriented; hence after a single mutation we obtain a quiver violating Lemma 34.

Consider the case of a pair of type II blocks. If these have disjoint vertex sets, each outlet of a type II block cannot be adjacent to *two* of the outlets of the other type II block. Thus we must cancel the arrow between these two outlets with a type I block. However this creates a pair of 1-valent non-outlet vertices which can be eliminated similarly to the type III case. At the other extreme, if we attach along all three outlets, we produce two easy cases. Attaching along a pair of outlets we generate either a Q_2 subquiver or a 4-cycle. Considering the 4-cycle with two outlets v_1 and v_2 (on non-adjacent corners) to meet the conditions of Lemma 37 any vertex adjacent to one of v_1 or v_2 must be adjacent to the other. Moreover, if the resulting quiver contains an arrow between v_1 and v_2, a mutation at one of the non-outlet vertices gives a Q_2 subquiver. Given a vertex v adjacent to v_1 and v_2, if this defines a path between them, mutating at this node and a non-outlet in the four cycle produces a Q_2 subquiver. If v does not lie on a path between v_1 and v_2 then mutating at both outlets produces a Q_2 subquiver.

Attaching the type II blocks at a single outlet, the four arrows incident to this vertex are now fixed, so any new vertex must be adjacent to each of the remaining four outlets by Lemma 37. However this cannot be achieved with type I blocks.

Attaching more than two type II blocks together, we can eliminate the case where two are connected to form a 4-cycle as above. Since we can easily eliminate the case that two type II blocks meet in three outlets, we assume that each type II block meets every other in at most one outlet. Some pair of type II blocks must be attached in an outlet (otherwise we can argue as in the case of type II block separated by type I blocks). Thus, since every new vertex must be adjacent to all four outlets formed by attaching two type II blocks, all possible quivers can be represented as an octahedron with some orientation, see Fig. 15.9.

Considering an orientation of the octahedron; if any triangular face does not form a cycle we can mutate to form a Q_2 subquiver. Assuming every triangle is a cycle, and possibly mutating, the vertices adjacent to the 'top' of the octahedron form a type V block subquiver. Following the same reasoning as for the type V block case (although note that the type V block is not part of a block decomposition here) these cases can be eliminated. □

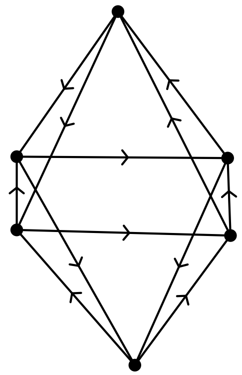


Fig. 15.9 Octahedron of type II blocks

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