Chapter 14
On Deformations of Toric Fano Varieties

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Abstract In this note we collect some results on the deformation theory of toric Fano varieties.

Keywords Deformations · Fano varieties · Toric varieties · Reflexive polytopes

14.1 Introduction

A *Fano variety* is a normal projective variety X over $\mathbb{C}$ such that its anticanonical divisor $-K_X$ is $\mathbb{Q}$-Cartier and ample. Fano varieties constitute the basic building blocks of algebraic varieties, according to the Minimal Model Program. The geometry of Fano varieties is a well studied area. In particular, moduli (and consequently deformations) of Fano varieties constitute a very interesting and important topic in algebraic geometry, e.g. [21, 62, 69].

Here we will concentrate on deformations and smoothings of *toric* Fano varieties. These varieties occupy a prominent role in Mirror Symmetry, a large part of which is based on the phenomenon of toric degeneration as in [17, 18, 30, 43].

Toric Fano varieties correspond to certain polytopes which are called *Fano polytopes*. The goal of this note is to present some combinatorial criteria on Fano polytopes which can detect whether the corresponding toric Fano variety is smoothable, i.e. can be deformed to a smooth (Fano) variety.

Special attention is given to toric Fano threefolds with Gorenstein singularities. These varieties correspond to the 4319 reflexive polytopes of dimension 3, which were classified by Kreuzer and Skarke [66]. In this case, thanks to the use of the software MAGMA [22], we were able to produce a lot of examples for the combinatorial criteria discussed in this note.

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14.1.1 Outline

In Sect. 14.2.1 the very classical theory of infinitesimal deformations of algebraic varieties is recalled. In Sect. 14.2.2 we survey some properties of smoothings of algebraic varieties. In Sect. 14.2.3 two well-studied deformation invariants for Fano varieties are introduced.

In Sect. 14.3.1 we recall some results on the deformation theory of affine toric varieties. We provide an example in Sect. 14.3.2.

The core of this note is Sect. 14.4. We recall the definition of Fano polytopes in Sect. 14.4.1. In Sect. 14.4.2 we present a couple of sufficient conditions that ensure that a toric Fano variety is non-smoothable. The rigidity of toric Fano varieties is examined in Sect. 14.4.3. In Sects. 14.4.4 and 14.4.5 we study the smoothability of toric Fano surfaces and toric Fano threefolds with isolated singularities; an example is presented in Sect. 14.4.6. In Sect. 14.4.7 we present another sufficient condition that ensures that a toric Fano threefold is non-smoothable. In Sect. 14.4.8 we include more results on deformations of toric Fano varieties.

In Sect. 14.5 we write down the lists of the reflexive polytopes of dimension 3 which satisfy the several combinatorial conditions considered in Sect. 14.4.

14.1.2 Notation and Conventions

We work over $\mathbb{C}$, but everything will hold over a field of characteristic zero with appropriate modifications.

In Sects. 14.3 and 14.4 we assume that the reader is familiar with the basic notions of toric geometry, which can be found in [34, 41]. All toric varieties considered here are normal. A lattice is a finitely generated free abelian group. The letters N, $\overline{N}$, $\tilde{N}$ stand for lattices and M, $\overline{M}$, $\tilde{M}$ for their duals, e.g. $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$; the duality pairing $M \times N \to \mathbb{Z}$ and its extension $M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$ are denoted by $\langle \cdot, \cdot \rangle$.

In a real vector space of finite dimension a polytope is the convex hull of finitely many points, or equivalently a compact subset which is the intersection of finitely many closed halfspaces. We refer the reader to the book [99] for the geometry of polytopes.

14.2 Deformations

14.2.1 Infinitesimal Deformations

Let (Comp) be the category of noetherian complete local $\mathbb{C}$-algebras with residue field $\mathbb{C}$. For every $R \in$ (Comp) we denote by $\mathfrak{m}_R$ the maximal ideal of R. Let (Art) be the subcategory of (Comp) whose objects are artinian, i.e. local finite $\mathbb{C}$-algebras.

A functor of Artin rings is a functor F from the category (Art) to the category of sets such that $F(\mathbb{C})$ is the set with one element. We will only consider functor of Artin rings which satisfy some additional properties: Schlessinger's axioms (H1) and (H2) [91] and Fantechi–Manetti condition (L) [37, (2.9)]. We will not specify these conditions here, but we refer the reader to [37, Sect. 2] for a quick introduction. Precise formulations and additional details about the notions we introduce below can be found in any reference about deformation theory, e.g. [13, 36, 50, 70, 91, 92, 95, 97].

A natural transformation (or briefly map) of functors $\phi\colon F \to G$ is called *smooth* if the lifting property in Grothendieck's definition of formally smooth morphisms holds, i.e. for every local surjection $A' \twoheadrightarrow A$ in (Art) the natural map $F(A') \to F(A) \times_{G(A)} G(A')$ is surjective; in particular, if ϕ is smooth then $\phi(A)\colon F(A) \to G(A)$ is surjective for all $A \in$ (Art). A functor F is called smooth if the map from F to the trivial functor is smooth.

For a functor F, the set $F(\mathbb{C}[t]/(t^2))$ has a natural structure of a $\mathbb{C}$-vector space, denoted by $\mathrm{T}F$ and called the *tangent space* of F. One can prove that F is the trivial functor if and only if $\mathrm{T}F = 0$. If $\phi\colon F \to G$ is a map, then the function $\phi(\mathbb{C}[t]/(t^2))$ is linear and denoted by $\mathrm{T}\phi\colon \mathrm{T}F \to \mathrm{T}G$.

If $R \in$ (Comp) one can consider the functor $h_R = \mathrm{Hom}(\cdot, R)$ prorepresented by R. A map $h_R \to F$ is equivalent to a pro-object of F on $R = \varprojlim R/\mathfrak{m}_R^{n+1}$, i.e. an element of the set $\varprojlim F(R/\mathfrak{m}_R^{n+1})$. A *hull* for a functor F is a ring $R \in$ (Comp) together with a smooth morphism $\phi\colon h_R \to F$ such that $\mathrm{T}\phi$ is bijective. A hull exists if and only if $\mathrm{T}F$ has finite dimension. If a hull exists, it is unique. Provided that $\mathrm{T}F$ has finite dimension r, then F is smooth if and only if the hull of F is isomorphic to $\mathbb{C}[\![t_1, \dots, t_r]\!]$.

For a functor F, consider the set $\mathcal{E}$ made up of pairs (π, ξ), where $\pi : A' \to A$ is a surjection in (Art) such that $\mathfrak{m}_{A'} \cdot (\ker \pi) = 0$ and $\xi \in F(A)$. A $\mathbb{C}$-vector space V is called an *obstruction space* for F if there exists a function $\omega\colon \mathcal{E} \to \coprod_{(\pi,\xi)\in\mathcal{E}} \ker \pi \otimes_{\mathbb{C}} V$ such that the two following conditions are satisfied:

1. for every $(\pi, \xi) \in \mathcal{E}$, $\omega(\pi, \xi) \in \ker \pi \otimes_{\mathbb{C}} V$;
2. for every $(\pi, \xi) \in \mathcal{E}$, we have that $\omega(\pi, \xi) = 0$ if and only if there exists $\xi' \in F(A')$ which maps to ξ.

There are infinitely many obstruction spaces for a functor F because any vector space containing an obstruction space is an obstruction space. A functor F is smooth if and only if 0 is an obstruction space for F; in this case we also say that F is unobstructed. There is a notion of compatible obstruction spaces for a map $\phi\colon F \to G$: this will be a linear map $\mathrm{o}\phi$ from an obstruction space of F to an obstruction space of G with some compatibility properties with respect to ϕ.

The following is an important smoothness criterion. Assume that $\phi\colon F \to G$ is a map with compatible obstruction map $\mathrm{o}\phi$ from an obstruction space of F to an obstruction space of G. If $\mathrm{T}\phi$ is surjective and $\mathrm{o}\phi$ is injective, then ϕ is smooth.

Let X be a scheme of finite type over $\mathbb{C}$. We denote by Def_X the functor of (infinitesimal) deformations of X. If $R \in$ (Comp), a pro-object of Def_X on R is called a *formal deformation* of X over R. If R is a hull for Def_X, then the corresponding formal deformation of X over R is called the *miniversal deformation* of X. We say that X is *rigid* if all deformations of X are trivial. If X is reduced, then the tangent space of Def_X is $\mathrm{Ext}^1(\Omega_X, \mathcal{O}_X)$; in this case X is rigid if and only if $\mathrm{Ext}^1(\Omega_X, \mathcal{O}_X) = 0$. If X is either normal or reduced and local complete intersection (l.c.i. for short), then $\mathrm{Ext}^2(\Omega_X, \mathcal{O}_X)$ is an obstruction space for Def_X. If X is smooth, then $\mathrm{H}^i(X, T_X) = \mathrm{Ext}^i(\Omega_X, \mathcal{O}_X)$ for all $i \geq 0$. In particular, if X is smooth and affine then it is rigid.

Proposition 1 *If X is a smooth Fano variety, then $\mathrm{H}^i(X, T_X) = 0$ for each $i \geq 2$. In particular, the infinitesimal deformations of X are unobstructed, i.e. Def_X is smooth.*

Proof Let n be the dimension of X. Since the anticanonical line bundle $\omega_X^\vee$ is ample, by Kodaira–Nakano vanishing we have $\mathrm{H}^i(X, \Omega_X^{n-1} \otimes \omega_X^\vee) = 0$ whenever $i + n - 1 > n$, i.e. $i \geq 2$. We conclude because the tangent sheaf T_X is isomorphic to $\Omega_X^{n-1} \otimes \omega_X^\vee$. □

Let X be a scheme of finite type over $\mathbb{C}$ and let Def_X^{lt} be the subfunctor of Def_X made up of the locally trivial deformations of X. The tangent space of Def_X^{lt} is $\mathrm{H}^1(X, T_X)$ and $\mathrm{H}^2(X, T_X)$ is an obstruction space for Def_X^{lt}.

Proposition 2 *Let X be a reduced scheme of finite type over $\mathbb{C}$ such that X is either l.c.i. or normal. If $\mathrm{H}^0(X, \mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)) = 0$, then all deformations of X are locally trivial, i.e. $Def_X^{\mathrm{lt}} = Def_X$.*

Proof The local-to-global spectral sequence for Ext gives the following exact sequence.

$$0 \to \mathrm{H}^1(T_X) \to \mathrm{Ext}^1(\Omega_X, \mathcal{O}_X) \to \mathrm{H}^0(\mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)) \to \mathrm{H}^2(T_X) \to \mathrm{Ext}^2(\Omega_X, \mathcal{O}_X)$$

The vanishing of $\mathrm{H}^0(\mathcal{E}xt^1(\Omega_X, \mathcal{O}_X))$ implies that the inclusion $\phi \colon Def_X^{\mathrm{lt}} \hookrightarrow Def_X$ induces an isomorphism on tangent spaces and an injection on obstruction spaces. Therefore ϕ is smooth, and consequently surjective. □

In particular, all deformations of a smooth scheme are locally trivial.

Let X be a reduced scheme of finite type over $\mathbb{C}$ with isolated singularities. For each singular point $x \in X$, let U_x be an affine open neighbourhood of x such that $U_x \setminus \{x\}$ is smooth. Then define

$$Def_X^{\mathrm{loc}} := \prod_{x \in \mathrm{Sing}(X)} Def_{U_x}.$$

The tangent space of Def_X^{loc} is $\mathrm{H}^0(X, \mathcal{E}xt^1(\Omega_X, \mathcal{O}_X))$. If X is either l.c.i. or normal, then $\mathrm{H}^0(X, \mathcal{E}xt^2(\Omega_X, \mathcal{O}_X))$ is an obstruction space for Def_X^{loc}. There is an obvious map $Def_X \to Def_X^{\mathrm{loc}}$ which restricts a deformation of X to a deformation of U_x for each x.

Proposition 3 *Let X be a reduced scheme of finite type over $\mathbb{C}$ with isolated singularities. Assume that X is either l.c.i. or normal. If $\mathrm{H}^2(X, T_X) = 0$ then there are no local-to-global obstructions for the infinitesimal deformations of X, i.e. the map $Def_X \to Def_X^{\mathrm{loc}}$ is smooth.*

Proof We consider the local-to-global spectral sequence for $\mathrm{Ext}^\bullet(\Omega_X, \mathcal{O}_X)$. The second page is given by $E_2^{p,q} = \mathrm{H}^p(\mathcal{E}xt^q(\Omega_X, \mathcal{O}_X))$. Since X has isolated singularities, the sheaves $\mathcal{E}xt^q(\Omega_X, \mathcal{O}_X)$ are supported on isolated points for $q \geq 1$; in particular they do not have higher cohomology. This means that $E_2^{p,q}$ is supported on the lines $p = 0$ and $q = 0$. Therefore, in E_2 the only non-zero differential is

$$d_2 \colon \mathrm{H}^0(\mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)) \longrightarrow \mathrm{H}^2(T_X).$$

We obtain that the bottom left corner of the third page E_3 is the following.

$$\begin{array}{llll}
\mathrm{H}^3(T_X) & 0 & 0 & 0 \\
\mathrm{coker} d_2 & 0 & 0 & 0 \\
\mathrm{H}^1(T_X) & 0 & 0 & 0 \\
\mathrm{H}^0(T_X) & \ker d_2 & \mathrm{H}^0(\mathcal{E}xt^2(\Omega_X, \mathcal{O}_X)) & \mathrm{H}^0(\mathcal{E}xt^3(\Omega_X, \mathcal{O}_X))
\end{array}$$

In E_3 the only non-zero differential is

$$d_3 \colon \mathrm{H}^0(\mathcal{E}xt^2(\Omega_X, \mathcal{O}_X)) \longrightarrow \mathrm{H}^3(T_X).$$

The bottom left corner of the fourth page E_4 is the following.

$$\begin{array}{llll}
\mathrm{coker} d_3 & 0 & 0 & 0 \\
\mathrm{coker} d_2 & 0 & 0 & 0 \\
\mathrm{H}^1(T_X) & 0 & 0 & 0 \\
\mathrm{H}^0(T_X) & \ker d_2 & \ker d_3 & \mathrm{H}^0(\mathcal{E}xt^3(\Omega_X, \mathcal{O}_X))
\end{array}$$

From the fourth page on, the pieces of total degree ≤ 3 do not change any more. Therefore we have two short exact sequences:

$$0 \longrightarrow \mathrm{H}^1(T_X) \longrightarrow \mathrm{Ext}^1(\Omega_X, \mathcal{O}_X) \longrightarrow \ker d_2 \longrightarrow 0,$$
$$0 \longrightarrow \mathrm{coker} d_2 \longrightarrow \mathrm{Ext}^2(\Omega_X, \mathcal{O}_X) \longrightarrow \ker d_3 \longrightarrow 0.$$

These can be joined to construct the following long exact sequence.

$$\begin{array}{lllllll}
0 \longrightarrow & \mathrm{H}^1(T_X) & \longrightarrow & \mathrm{Ext}^1(\Omega_X, \mathcal{O}_X) & \longrightarrow & \mathrm{H}^0(\mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)) & \xrightarrow{d_2} \\
\xrightarrow{d_2} & \mathrm{H}^2(T_X) & \longrightarrow & \mathrm{Ext}^2(\Omega_X, \mathcal{O}_X) & \longrightarrow & \mathrm{H}^0(\mathcal{E}xt^2(\Omega_X, \mathcal{O}_X)) &
\end{array}$$

So far we did not use the assumption $\mathrm{H}^2(T_X) = 0$. From this vanishing, via the long exact sequence above we deduce that the map $Def_X \to Def_X^{\mathrm{loc}}$ induces a surjection on tangent spaces and an injection on obstruction spaces. □

14.2.2 Smoothings

Here we discuss smoothability conditions for schemes of finite type over $\mathbb{C}$. We will only consider the case of equidimensional schemes and we will refer the reader to [50, Sect. 29] for a more general treatment, which uses the Lichtenbaum–Schlessinger functors.

If X is a proper scheme over $\mathbb{C}$, a *smoothing* of X is a proper flat morphism $\mathcal{X} \to B$ such that B is an integral scheme of finite type over $\mathbb{C}$ of positive dimension and there exists a closed point $b_0 \in B$ such that the fibre over b_0 is X and all the other fibres are smooth. By restricting to a curve in B and normalising it, we may require that the base B is a smooth affine curve and that the maximal ideal corresponding to b_0 is principal. We say that X is *smoothable* if it admits a smoothing.

For every $n \geq 0$, set $S_n := \operatorname{Spec} \mathbb{C}[t]/(t^{n+1})$. If X is a scheme of finite type over $\mathbb{C}$ with pure dimension d, then a *formal smoothing* of X is a formal deformation $\{X_n \to S_n\}_n$ of X over $\mathbb{C}[\![t]\!]$ such that there exists m such that t^m is in the dth Fitting ideal of Ω_{X_m/S_m}. We refer the reader to [35, Sect. 20.2] for the definition and the properties of Fitting ideals. We say that X is *formally smoothable* if it admits a formal smoothing. It is clear that if X is formally smoothable, then every open subscheme of X is formally smoothable.

Remark 4 If $\{X_n \to S_n\}_n$ is a formal deformation of X over $\mathbb{C}[\![t]\!]$ and t^m is in the dth Fitting ideal of Ω_{X_m/S_m}, then for all $n \geq m$ we have that t^n is in the dth Fitting ideal of Ω_{X_n/S_n}.

The proof of this fact is as follows. We have $\mathcal{O}_{X_n} = \mathcal{O}_{X_{n+1}}/t^{n+1}\mathcal{O}_{X_{n+1}}$. Since the formation of Fitting ideals commutes with base change, we have the equality

$$\mathrm{Fitt}_d(\Omega_{X_n/S_n}) = (\mathrm{Fitt}_d(\Omega_{X_{n+1}/S_{n+1}}) + t^{n+1}\mathcal{O}_{X_{n+1}})/t^{n+1}\mathcal{O}_{X_{n+1}}.$$

Therefore if $t^n \in \mathrm{Fitt}_d(\Omega_{X_n/S_n})$ then $t^n \in \mathrm{Fitt}_d(\Omega_{X_{n+1}/S_{n+1}}) + t^{n+1}\mathcal{O}_{X_{n+1}}$, hence $t^{n+1} \in t\mathrm{Fitt}_d(\Omega_{X_{n+1}/S_{n+1}}) \subseteq \mathrm{Fitt}_d(\Omega_{X_{n+1}/S_{n+1}})$ as $t^{n+2} = 0$ in $\mathcal{O}_{X_{n+1}}$.

Lemma 5 *Let X be a Cohen–Macaulay proper scheme over $\mathbb{C}$ of pure dimension d. Let B be a smooth curve over $\mathbb{C}$, $b_0 \in B$ be a closed point, and $\pi : \mathcal{X} \to B$ be a proper flat morphism such that the fibre over b_0 is X. Let ξ be the formal $\mathfrak{m}_{b_0}$-adic completion of π at b_0, i.e. $\xi = \{\mathcal{X} \times_B \operatorname{Spec} \mathcal{O}_{B,b_0}/\mathfrak{m}_{b_0}^{n+1} \to \operatorname{Spec} \mathcal{O}_{B,b_0}/\mathfrak{m}_{b_0}^{n+1}\}_n$. Then:*

1. *if π is a smoothing of X, then ξ is a formal smoothing of X;*
2. *if ξ is a formal smoothing of X, then there exists an open neighbourhood B' of b_0 in B such that $\mathcal{X} \times_B B' \to B'$ is a smoothing of X.*

Proof This proof comes from [14, Sect. 0E7S].

Notice that ξ does not change if we restrict π to an open neighbourhood of b_0 in B. Therefore, in order to prove the statements (1) and (2) we can arbitrarily restrict to an open neighbourhood of b_0 in B. Hence we may assume that B is affine and the maximal ideal corresponding to the point b_0 is principal, generated by $t \in \mathcal{O}_B$.

We consider the set $W \subseteq \mathcal{X}$ made up of the points $x \in \mathcal{X}$ such that the local ring of the fibre $\mathcal{X}_{\pi(x)}$ at x is Cohen–Macaulay. By [44, 12.1.7], W is open in $\mathcal{X}$. As π is closed, $B \setminus \pi(\mathcal{X} \setminus W)$ is an open neighbourhood of b_0 in B. Therefore, if we restrict B to an open neighbourhood of b_0 in B, we may assume that all fibres of π are Cohen–Macaulay. By [14, Lemma 02NM], we may assume that π has relative dimension d.

Let $I \subseteq \mathcal{O}_\mathcal{X}$ be the dth Fitting ideal of $\Omega_{\mathcal{X}/B}$. For each n, set

$$S_n = \operatorname{Spec} \mathcal{O}_{B,b_0}/\mathfrak{m}_{b_0}^{n+1} = \operatorname{Spec} \mathcal{O}_B/t^{n+1}\mathcal{O}_B$$

and $X_n = \mathcal{X} \times_B S_n$; let $I_n \subseteq \mathcal{O}_{X_n}$ be the dth Fitting ideal of Ω_{X_n/S_n}. Since Fitting ideals commute with base change, we have $\mathcal{O}_{X_n} = \mathcal{O}_\mathcal{X}/t^{n+1}\mathcal{O}_\mathcal{X}$ and $I_n = I\mathcal{O}_{X_n} = (I + t^{n+1}\mathcal{O}_\mathcal{X})/t^{n+1}\mathcal{O}_\mathcal{X}$.

Since π is flat of relative dimension d, the zero locus of I is the singular locus of π. Moreover, the fibre over b_0 is the closed subset $\mathrm{V}(t)$. Therefore, the fibre of b_0 is the unique singular fibre if and only if $t \in \sqrt{I}$.

(1) If π is a smoothing, then there exists m such that $t^m \in I$. Since $I_m = (I + t^{m+1}\mathcal{O}_\mathcal{X})/t^{m+1}\mathcal{O}_\mathcal{X}$, this implies that $t^m \in I_m$. So ξ is a formal smoothing.

(2) If ξ is a formal smoothing, then $t^m \in I_m = (I + t^{m+1}\mathcal{O}_\mathcal{X})/t^{m+1}\mathcal{O}_\mathcal{X}$ for some m. So in $\mathcal{O}_\mathcal{X}$ we have the equality $t^m = p + t^{m+1}q$, for some $p \in I$ and $q \in \mathcal{O}_\mathcal{X}$. Writing $t^m(1 - tq) = p$ and noticing that the function $1 - tq$ does not vanish at the points of $X = \mathrm{V}(t)$, we deduce that t^m belongs to the stalk I_x of I at all points $x \in X$. This implies that t^m lies in I in an open neighbourhood U of X in $\mathcal{X}$. Since π is closed, by restricting B to $B \setminus \pi(\mathcal{X} \setminus U)$ we have $t^m \in I$. Therefore π is a smoothing. □

Proposition 6 *Let X be a Cohen–Macaulay scheme proper over $\mathbb{C}$.*

1. *If X is smoothable, then every open subscheme of X is formally smoothable.*
2. *Assume that X is projective and $\mathrm{H}^2(X, \mathcal{O}_X) = 0$; if X is formally smoothable, then X is smoothable.*

Proof We may assume that X is connected. Therefore X has pure dimension, say d.

(1) This follows immediately from Lemma 5 and from the fact that if X is formally smoothable then every open subscheme of X is formally smoothable.

(2) Set $d := \dim X$. Let $\xi = \{X_n \to S_n\}_n$ be a formal smoothing of X, where S_n is $\operatorname{Spec} \mathbb{C}[t]/(t^{n+1})$ as usual. Let m be such that t^m is in the dth Fitting ideal of Ω_{X_m/S_m}.

As X is proper over $\mathbb{C}$, the tangent space of Def_X has finite dimension, therefore Def_X has a hull $R \in$ (Comp). Let $\eta = \{\eta_n \colon Y_n \to \operatorname{Spec} R/\mathfrak{m}_R^{n+1}\}_n$ be the miniversal deformation of X. By [95, Proposition 6.51] or [92, Theorem 2.5.13], from $\mathrm{H}^2(\mathcal{O}_X) = 0$ we deduce that η is effective, i.e. there exists a projective flat morphism $\mathscr{X} \to \operatorname{Spec} R$ whose $\mathfrak{m}_R$-adic completion is η.

By a theorem of Artin [11, Theorem 1.6] (see also [50, Theorem 21.3]), the morphism $\mathscr{X} \to \operatorname{Spec} R$ is algebraizable in the following sense: there exist a scheme Z of finite type over $\mathbb{C}$, a closed point $z_0 \in Z$, and a proper flat morphism $\mathcal{X} \to Z$, with fibre X over z_0, such that R is the completion $\widehat{\mathcal{O}}_{Z,z_0}$ of the local ring of Z at z_0 and $\mathscr{X}$ is isomorphic, as R-schemes, to $\mathcal{X} \times_Z \operatorname{Spec} R$. In particular, the miniversal deformation η is the collection $\{\mathcal{X} \times_Z \operatorname{Spec} \mathcal{O}_{Z,z_0}/\mathfrak{m}_{z_0}^{n+1} \to \operatorname{Spec} \mathcal{O}_{Z,z_0}/\mathfrak{m}_{z_0}^{n+1}\}_n$. The situation is summarised in the following cartesian squares, for all n.

$$\begin{array}{ccccc} Y_n & \hookrightarrow & \mathscr{X} & \longrightarrow & \mathcal{X} \\ \downarrow{\scriptstyle \eta_n} & & \downarrow & & \downarrow \\ \operatorname{Spec} R/\mathfrak{m}_R^{n+1} & \hookrightarrow & \operatorname{Spec} R = \operatorname{Spec} \widehat{\mathcal{O}}_{Z,z_0} & \longrightarrow & Z \end{array}$$

As η is miniversal, there exists a local $\mathbb{C}$-algebra homomorphism

$$\varphi \colon \widehat{\mathcal{O}}_{Z,z_0} = R \longrightarrow \mathbb{C}[\![t]\!]$$

such that ξ is induced by η via φ, i.e. X_n is isomorphic to $Y_n \times_{\operatorname{Spec} R/\mathfrak{m}_R^{n+1}} S_n$ as S_n-schemes for every n. By another theorem of Artin [10, Corollary 2.5], the map φ has an algebraic approximation up to order m in the following sense: there exist a smooth affine curve B over $\mathbb{C}$ with a closed point $b_0 \in B$ and a $\mathbb{C}$-morphism $f \colon B \to Z$ such that $f(b_0) = z_0$ and the completion

$$\varphi' \colon \widehat{\mathcal{O}}_{Z,z_0} = R \longrightarrow \widehat{\mathcal{O}}_{B,b_0} = \mathbb{C}[\![t]\!]$$

of $f^{\#}_{b_0} \colon \mathcal{O}_{Z,z_0} \to \mathcal{O}_{B,b_0}$ satisfies the following property:

$$\varphi \equiv \varphi' \text{ modulo } t^{m+1}. \tag{14.1}$$

Let π be the base change $\mathcal{X} \times_Z B \to B$ along $f \colon B \to Z$. Let ξ' be the formal $\mathfrak{m}_{b_0}$-adic completion of π, i.e. $\xi' = \{\mathcal{X} \times_Z \operatorname{Spec} \mathcal{O}_{B,b_0}/\mathfrak{m}_{b_0}^{n+1} \to \operatorname{Spec} \mathcal{O}_{B,b_0}/\mathfrak{m}_{b_0}^{n+1}\}_n$. The two formal deformations ξ and ξ' of X over $\mathbb{C}[\![t]\!]$ are in general different, but they coincide up to order m because of (14.1). This implies that t^m is in the dth Fitting ideal of the sheaf of Kähler differentials of $\mathcal{X} \times_Z \operatorname{Spec} \mathcal{O}_{B,b_0}/\mathfrak{m}_{b_0}^{m+1} \to \operatorname{Spec} \mathcal{O}_{B,b_0}/\mathfrak{m}_{b_0}^{m+1}$. Therefore, ξ' is a formal smoothing. By Lemma 5, up to restrict B to an open neighbourhood of b_0 in B, we have that $\pi : \mathcal{X} \times_Z B \to B$ is a smoothing. □

The following theorem ensures that a projective scheme with formally smoothable isolated singularities is smoothable, provided that some local and cohomological conditions hold.

Theorem 7 *Let X be a projective scheme over $\mathbb{C}$ such that:*

1. *X is reduced and Cohen–Macaulay;*
2. *X is either l.c.i. or normal;*

3. $\mathrm{H}^2(X, T_X) = 0$ *and* $\mathrm{H}^2(X, \mathcal{O}_X) = 0$*;*
4. *X has isolated singularities and for each singular point $x \in X$ there exists an open affine neighbourhood of x which is formally smoothable.*

Then X is smoothable.

Proof Set $d = \dim X$. Let $x_1, \ldots, x_r$ be the singular points of X. Let U_i be an affine open neighbourhood of x_i in X which is formally smoothable and such that $U_i \setminus \{x_i\}$ is smooth. Let $\xi_i = \{U_{i,n} \to S_n\}_n$ be a formal smoothing of U_i, where S_n is $\operatorname{Spec} \mathbb{C}[t]/(t^{n+1})$ as usual.

By Proposition 3, from $\mathrm{H}^2(T_X) = 0$ we deduce that the map $Def_X \to Def_X^{\mathrm{loc}} = \prod_{i=1}^{r} Def_{U_i}$ is smooth. Therefore there exists a formal deformation $\xi = \{X_n \to S_n\}_n$ of X over $\mathbb{C}[\![t]\!]$ such that for each i the restriction of ξ to U_i is ξ_i, i.e. for all n the restriction of X_n to U_i is $U_{i,n}$. By Remark 4 we have that ξ is a formal smoothing of X. We conclude by Proposition 6. □

We now see some conditions that imply that a scheme is not smoothable.

Proposition 8 *Let X be a singular scheme of finite type over $\mathbb{C}$ of pure dimension. Assume that at least one of the following conditions holds:*

1. *every infinitesimal deformation of X is locally trivial;*
2. *the functor Def_X has an artinian hull.*

Then X is not formally smoothable.

Proof Set $d = \dim X$.

(1) Let U be a singular affine open subscheme of X. Let $\{X_n \to S_n\}_n$ be a formal deformation of X over $\mathbb{C}[\![t]\!]$. Let U_n be the restriction of X_n to U. By (1) we get that U_n is isomorphic, as S_n-scheme, to the trivial deformation $U \times_{\operatorname{Spec} \mathbb{C}} S_n$. Therefore $\operatorname{Fitt}_d(\Omega_{U_n/S_n}) = \operatorname{Fitt}_d(\Omega_{U/\mathbb{C}})\mathcal{O}_{U_n}$. As U is singular, $\operatorname{Fitt}_d(\Omega_{U/\mathbb{C}}) \subsetneq \mathcal{O}_U$. This implies that $t^n \notin \operatorname{Fitt}_d(\Omega_{U_n/S_n})$.

(2) Let R be the hull of Def_X. Every formal deformation of X over $\mathbb{C}[\![t]\!]$ is induced by the miniversal one via a local $\mathbb{C}$-algebra homomorphism $f \colon R \to \mathbb{C}[\![t]\!]$. As every element in $\mathfrak{m}_R$ is nilpotent and $\mathbb{C}[\![t]\!]$ is a domain, the homomorphism f factors as $R \twoheadrightarrow R/\mathfrak{m}_R = \mathbb{C} \hookrightarrow \mathbb{C}[\![t]\!]$. This implies that every formal deformation of X over $\mathbb{C}[\![t]\!]$ is trivial. Using a similar argument as in (1), we can prove that X cannot have a formal smoothing. □

The following corollary, which is a direct consequence of Propositions 2, 6 and 8, gives some obstructions to the smoothability of a Cohen–Macaulay proper scheme.

Corollary 9 *Let X be a Cohen–Macaulay scheme proper over $\mathbb{C}$. Let $U \subseteq X$ be an open subscheme of X such that U is singular, reduced, and either l.c.i. or normal. If $\mathrm{H}^0(U, \mathcal{E}xt^1(\Omega_U, \mathcal{O}_U)) = 0$ or Def_U has an artinian hull (e.g. if $\operatorname{Ext}^1(\Omega_U, \mathcal{O}_U) = 0$), then X is not smoothable.*

14.2.3 Invariants

Here we introduce a couple of invariants for Fano varieties.

The *Hilbert series* of a Fano variety X is the power series defined by its anti-plurigenera:

$$\mathrm{Hilb}(X, -K_X) := \sum_{m \geq 0} h^0(X, -mK_X)t^m \in \mathbb{Z}[\![t]\!].$$

The *(anticanonical) degree* of a Fano variety X is the positive rational number $(-K_X)^n$, where $n = \dim X$. If X is Gorenstein, i.e. K_X is Cartier, then the degree is an integer. The degree can be recovered from the Hilbert series because, up to a constant which depends on the dimension of X, it is the leading term of the Hilbert polynomial of $-K_X$.

The following proposition shows that the Hilbert series and the anticanonical degree are deformation invariants for Fano varieties with Gorenstein log terminal singularities.

Proposition 10 *Let S be a noetherian scheme over $\mathbb{Q}$ and let $\pi : X \to S$ be a proper flat morphism whose geometric fibres are Fano varieties with Gorenstein log terminal singularities. Then the Hilbert series and the degree of the fibres are locally constant on S.*

Proof The morphism π is a relatively Gorenstein. Therefore, by [47, V.9.7], the dualising sheaf ω_π is a line bundle on X and its restriction to each fibre X_s is $\mathcal{O}_{X_s}(K_{X_s})$.

By Serre duality and Kawamata–Viehweg vanishing [63, Theorem 2.70], we get $\mathrm{H}^1(X_s, \mathcal{O}_{X_s}(-mK_{X_s})) = 0$ for all $m \geq 0$ and $s \in S$. By cohomology and base change [48, Theorem III.12.11], for all $m \geq 0$, we get that the sheaf $\pi_*\omega_\pi^{\otimes -m}$ is locally free and has rank $h^0(X_s, \mathcal{O}_{X_s}(-mK_{X_s}))$ at the point $s \in S$. This implies that the Hilbert series of the fibres is locally constant on S. □

14.3 Deformations of Affine Toric Varieties

14.3.1 Toric Singularities

In this section we will consider deformations of toric singularities, that is affine toric varieties. We refer the reader to [34, 41] for an introduction to toric geometry.

If X is an affine toric variety of dimension 2, then X is a cyclic quotient surface singularity. There is extensive literature about deformations of this kind of singularities, e.g. [19, 27, 64, 88, 93, 94]. In particular, it is known that every affine toric variety of dimension 2 is smoothable [12].

The study of the deformation theory of affine toric varieties of dimension at least 3 has been initiated by Altmann [4–8]. For example, he computed the tangent space of the deformation functor of an affine toric variety. We will not write down the explicit description of $\mathrm{Ext}^1(\Omega_X, \mathcal{O}_X)$ when X is an affine toric variety, but we will mention a consequence.

Proposition 11 (Altmann [5, Corollary 6.5.1]) *If X is a $\mathbb{Q}$-Gorenstein affine toric variety which is smooth in codimension 2 and $\mathbb{Q}$-factorial in codimension 3, then X is rigid.*

Corollary 12 *Every isolated $\mathbb{Q}$-Gorenstein toric singularity of dimension ≥ 4 is rigid.*

Now we need to do a brief detour on Minkowski sums. If $F_0, F_1, \ldots, F_r$ are polytopes in a real vector space, their *Minkowski sum* is the polytope

$$F_0 + F_1 + \cdots + F_r := \{v_0 + v_1 + \cdots + v_r \mid v_0 \in F_0, v_1 \in F_1, \ldots, v_r \in F_r\}.$$

When we have $F = F_0 + F_1 + \cdots + F_r$, we say that we have a *Minkowski decomposition* of the polytope F. We consider Minkowski decompositions up to translation: for instance, we consider the Minkowski decomposition $F = (v + F_0) + (-v + F_1)$ to be equivalent to $F = F_0 + F_1$ for every vector v. Moreover, in what follows we require that the summands F_j are lattice polytopes, i.e. their vertices belong to a fixed lattice.

Altmann [5] has noticed that certain Minkowski decompositions induce deformations of affine toric varieties. In Sect. 14.3.2 we will see an example of this fact. For the proof we refer the reader to the original reference [5] and to [71, 81].

Now let us concentrate on Gorenstein toric singularities. They are associated to lattice polytopes of dimension one less than the dimension of the singularity. More precisely, let F be a lattice polytope of dimension $n-1$ in a lattice $\overline{N}$ of rank $n-1$ and let U_F be the affine toric variety associated to the cone $\sigma_F = \mathbb{R}_{\geq 0}(F \times \{1\})$ in the lattice $N := \overline{N} \oplus \mathbb{Z}$, i.e. $U_F = \operatorname{Spec} \mathbb{C}[\sigma_F^\vee \cap M]$, where $M = \overline{M} \oplus \mathbb{Z}$ is the dual of N and $\sigma_F^\vee$ is the dual cone of σ_F. We have that U_F has dimension n and is Gorenstein. All Gorenstein affine toric varieties without torus factors arise in this way from a lattice polytope. The isomorphism class of U_F does not change if we change F via an affine transformation in $\overline{N} \rtimes \mathrm{GL}(\overline{N}, \mathbb{Z})$.

As usual in toric geometry, the geometric properties of U_F can be deduced from the combinatorial properties of F. For instance:

1. U_F is smooth in codimension k if and only if all faces of F with dimension $< k$ are standard simplices;
2. U_F is $\mathbb{Q}$-factorial in codimension k if and only if all faces of F with dimension $< k$ are simplices.

It is always the case that U_F is smooth in codimension 1 and $\mathbb{Q}$-factorial in codimension 2.

If F is a segment of lattice length $m+1$, then U_F is the A_m surface singularity $\operatorname{Spec} \mathbb{C}[x, y, z]/(xy - z^{m+1})$. This is an isolated hypersurface singularity, therefore it is very easy to write down the miniversal deformation: $xy = z^{m+1} + t_m z^{m-1} + \cdots + t_1$ over $\mathbb{C}[\![t_1, \ldots, t_m]\!]$. It is clear that this singularity is smoothable.

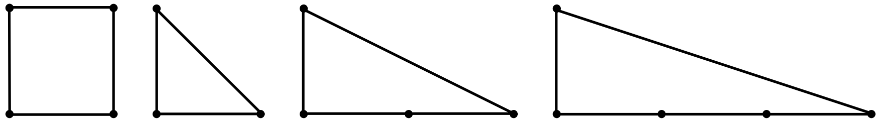

Fig. 14.1 A standard square, a standard triangle, an A_1-triangle and an A_2-triangle

If F is a lattice polygon, then the affine toric threefold U_F has the following properties:

1. U_F has, at most, an isolated singularity if and only if the edges of F are unitary, i.e. have lattice length 1;
2. U_F is $\mathbb{Q}$-factorial if and only if F is a triangle.

Now we provide some examples of lattice polygons and their corresponding toric Gorenstein affine threefolds.

Example 13 A lattice polygon F is called a *standard square* (Fig. 14.1) if it is a quadrilateral such that all its lattice points are vertices, or equivalently if it is $\mathbb{Z}^2 \rtimes \mathrm{GL}_2(\mathbb{Z})$-equivalent to $\mathrm{conv}\{(0,0),(1,0),(1,1),(0,1)\} \subseteq \mathbb{R}^2$. If F is a standard square, then U_F is the ordinary double point (i.e. node) $\mathrm{Spec}\,\mathbb{C}[x,y,z,w]/(xy - zw)$. This singularity is clearly smoothable as it is a hypersurface singularity. Its miniversal deformation is given by $xy - zw = t$ over $\mathbb{C}[\![t]\!]$.

A lattice polygon F is called a *standard triangle* if it is a triangle such that all its lattice points are vertices, or equivalently if it is $\mathbb{Z}^2 \rtimes \mathrm{GL}_2(\mathbb{Z})$-equivalent to $\mathrm{conv}\{(0,0),(1,0),(0,1)\} \subseteq \mathbb{R}^2$. F is a standard triangle if and only if U_F is isomorphic to $\mathbb{A}^3$.

If $m \geq 1$, then a lattice polygon F is called an *A_m-triangle* if it is a triangle such that there are no interior lattice points and the edges have lattice lengths $1, 1, m+1$, respectively. Equivalently, a polygon is an A_m-triangle if and only if it is $\mathbb{Z}^2 \rtimes \mathrm{GL}_2(\mathbb{Z})$-equivalent to $\mathrm{conv}\{(0,0),(m+1,0),(0,1)\} \subseteq \mathbb{R}^2$. If F is an A_m-triangle, then U_F is the cA_m-singularity $\mathrm{Spec}\,\mathbb{C}[x,y,z,w]/(xy - z^{m+1})$. This singularity is clearly smoothable as it is a hypersurface singularity.

Altmann [7] explicitly constructed the miniversal deformation of an isolated Gorenstein toric singularity of dimension 3. (By Corollary 12 it is trivial to construct the miniversal deformation of an isolated Gorenstein toric singularity of dimension ≥ 4.) A consequence of his construction is the following description of the irreducible components of the base of the miniversal deformation.

Theorem 14 (Altmann [7]) *Let F be a lattice polygon with unitary edges and let U_F be the corresponding isolated Gorenstein toric singularity of dimension* 3. *Let R be the hull of Def_{U_F}. Then there exists a one-to-one correspondence between minimal primes of R and maximal Minkowski decompositions of F. Moreover, if a minimal prime $\mathfrak{p} \subset R$ corresponds to the maximal Minkowski decomposition $F = F_0 + F_1 + \cdots + F_r$, then* $r = \dim R/\mathfrak{p}$.

Corollary 15 *Let F be a lattice polygon with unitary edges and let U_F be the associated isolated Gorenstein toric singularity of dimension* 3*. Then Def_{U_F} has an artinian hull if and only if F is Minkowski indecomposable.*

14.3.2 The Affine Cone over the Del Pezzo Surface of Degree 7

Here we study an explicit example of what has been considered in Sect. 14.3.1. In the lattice $\overline{N} = \mathbb{Z}^2$ consider the pentagon

$$F = \operatorname{conv}\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix} \right\} \subseteq \overline{N}_{\mathbb{R}}, \qquad (14.2)$$

which is depicted on the left of Fig. 14.2. The toric variety associated to the face fan of F is the smooth del Pezzo surface of degree 7, which is denoted by dP_7 and is the blow up of $\mathbb{P}^2$ in 2 distinct points. The anticanonical map of dP_7 is a closed embedding into $\mathbb{P}^7$.

Now we put the pentagon F at height 1 in the lattice $N = \overline{N} \oplus \mathbb{Z}$ and we consider the cone over it:

$$\sigma_F = \operatorname{cone}\left\{ \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\-1\\1\end{pmatrix} \right\} \subseteq \overline{N}_{\mathbb{R}} \oplus \mathbb{R}.$$

The affine toric variety $U_F = \operatorname{Spec} \mathbb{C}[\sigma_F^\vee \cap (\overline{M} \oplus \mathbb{Z})]$ is the affine cone over the anticanonical embedding of dP_7 and has an isolated Gorenstein canonical non-terminal singularity at the vertex of the cone.

Altmann [7, (9.1)] shows that the hull of Def_{U_F} is $\mathbb{C}[[t_1, t_2]]/(t_1^2, t_1 t_2)$, which is a line with an embedded point. The reduction of the miniversal deformation, i.e. the base change to the reduction of the hull, is induced by the unique maximal Minkowski decomposition of the pentagon F in the following way.

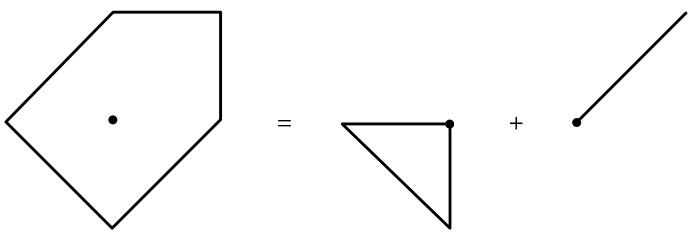

Fig. 14.2 The Minkowski decomposition (14.3) of the pentagon F in (14.2)

In the lattice $\overline{N}$ we have the Minkowski decomposition

$$F = \operatorname{conv}\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix}\right\} + \operatorname{conv}\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}, \tag{14.3}$$

which is illustrated in Fig. 14.2. Following [5, (3.4)], in the lattice $\tilde{N} = \overline{N} \oplus \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$ we construct the cone

$$\tilde{\sigma} = \operatorname{cone}\left\{\begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\\1\end{pmatrix}\right\} \subseteq \tilde{N}_{\mathbb{R}}.$$

Notice that the the first three rays of $\tilde{\sigma}$ come from the vertices of the first summand of F in (14.3), whereas the last two rays of $\tilde{\sigma}$ come from the vertices of the second summand of F in (14.3). Let $\tilde{U} = \operatorname{Spec} \mathbb{C}[\tilde{\sigma}^\vee \cap \tilde{M}]$ be the affine toric variety associated to the cone $\tilde{\sigma}$, where $\tilde{M}$ denotes the dual of $\tilde{N}$. One can prove that $\tilde{U}$ has only an isolated terminal Gorenstein singularity. Let f_1 and f_2 be the regular functions on $\tilde{U}$ associated to the characters $(0, 0, 1, 0) \in \tilde{M}$ and $(0, 0, 0, 1) \in \tilde{M}$, respectively. The variety U_F is the zero locus of the function $f_1 - f_2$, i.e. we have a cartesian diagram

$$\begin{array}{ccc} U_F & \longrightarrow & \tilde{U} \\ \big\downarrow & & \big\downarrow \pi \\ \operatorname{Spec} \mathbb{C} & \longrightarrow & \mathbb{A}^1_{\mathbb{C}} \end{array} \tag{14.4}$$

where π is given by the function $f_1 - f_2$ and the bottom morphism is given by the origin of $\mathbb{A}^1_{\mathbb{C}}$. Since $f_1 - f_2$ is not constant and $\mathbb{A}^1_{\mathbb{C}}$ is regular of dimension 1, the morphism π is flat. The reduction of the miniversal deformation of U_F is the formal deformation of U_F over $\mathbb{C}[\![t]\!]$ obtained from the square (14.4) by base change via $\operatorname{Spec} \mathbb{C}[t]/(t^{n+1}) \hookrightarrow \operatorname{Spec} \mathbb{C}[t] = \mathbb{A}^1_{\mathbb{C}}$ for all n. The following proposition shows that this is a formal smoothing.

Proposition 16 *Let F be the pentagon defined in (14.2) and let U_F be the corresponding Gorenstein toric threefold singularity. Then the collection of the base change of π in (14.4) via* $\operatorname{Spec} \mathbb{C}[t]/(t^{n+1}) \to \operatorname{Spec} \mathbb{C}[t] = \mathbb{A}^1_{\mathbb{C}}$ *for all n is a formal smoothing of U_F. In particular, U_F is formally smoothable.*

Proof We want to study the closed fibres of π. The fibre over the origin of $\mathbb{A}^1_{\mathbb{C}}$ is U_F. Let us fix $\lambda \in \mathbb{C} \setminus \{0\}$ and we consider the fibre $\pi^{-1}(\lambda)$ of π over the closed point $(t - \lambda)$ of $\mathbb{A}^1_{\mathbb{C}}$ corresponding to λ. We consider the subcone τ_1 (resp. τ_2) of $\tilde{\sigma}$ that is generated by the first three (resp. last two) rays of $\tilde{\sigma}$. We consider the affine toric variety $W_j = \operatorname{Spec} \mathbb{C}[\tau_j^\vee \cap \tilde{M}]$, for $j = 1, 2$. We have that W_1 and W_2 are open subschemes of $\tilde{U}$.

We have that W_1 is the open subset of $\tilde{U}$ where the function f_2 does not vanish, i.e. $W_1 = \{f_2 \neq 0\} \subseteq \tilde{U}$, and analogously $W_2 = \{f_1 \neq 0\} \subseteq \tilde{U}$. It is clear that there is an isomorphism $W_1 \simeq \mathbb{A}^3 \times \mathbb{G}_\mathrm{m}$ with respect to which the function $f_1|_{W_1}$ becomes a projection onto a $\mathbb{A}^1$-factor in $\mathbb{A}^3$ and the function $f_2|_{W_1}$ becomes the projection onto the $\mathbb{G}_\mathrm{m}$-factor. There is also an isomorphism $W_2 \simeq \mathbb{A}^2 \times \mathbb{G}_\mathrm{m}^2$ with respect to which the function $f_1|_{W_2}$ becomes a projection onto a $\mathbb{G}_\mathrm{m}$-factor and the function $f_2|_{W_2}$ becomes a projection onto an $\mathbb{A}^1$-factor.

Now $\pi^{-1}(\lambda) \cap W_1 = \{f_1 = f_2 + \lambda\} \cap W_1$ is isomorphic to $\mathbb{A}^2 \times \mathbb{G}_\mathrm{m}$ and $\pi^{-1}(\lambda) \cap W_2 = \{f_2 = f_1 - \lambda\} \cap W_2$ is isomorphic to $\mathbb{A}^1 \times \mathbb{G}_\mathrm{m}^2$. Since $\lambda \neq 0$, it is clear that $\pi^{-1}(\lambda) \subseteq W_1 \cup W_2$. Therefore we have proved that $\pi^{-1}(\lambda)$ is smooth.

One can also show that the generic fibre of π is smooth over the generic point of $\mathbb{A}^1_\mathbb{C}$. In order to prove this, it is enough to base change to the spectrum of the field $\mathbb{C}(t)$ of rational function of $\mathbb{A}^1_\mathbb{C}$ and pursue a similar argument, which deals with toric varieties over the field $\mathbb{C}(t)$.

In particular, π is flat of relative dimension 3 and has Cohen–Macaulay fibres. As in the proof of Lemma 5, from the fact that all non-special fibres of π are smooth we can deduce that π induces a formal smoothing of U_F. □

14.4 Deformations of Toric Fano Varieties

14.4.1 Fano Polytopes

Fano polytopes are the combinatorial-polyhedral avatars of toric Fano varieties.

Definition 17 A polytope P in a lattice N of rank n is called *Fano* if:

1. P has dimension n;
2. the origin 0 lies in the interior of P;
3. the vertices of P are primitive lattice elements of N.

If P is a Fano polytope, we denote by X_P the complete toric variety associated to the *spanning fan* (also called the *face fan*) of P.

If P is a Fano polytope, then X_P is a Fano variety. All toric Fano varieties arise in this way from a Fano polytope [34, Sect. 8.3]. The variety X_P is Gorenstein, i.e. its (anti)canonical divisor is Cartier, if and only if P is reflexive, i.e. the facets of P lie on hyperplanes with height 1 with respect to the origin. The maximal toric affine charts of X_P (or equivalently the torus-fixed points of X_P) are in one-to-one correspondence with the facets of P. If n is the dimension of P, for every $0 \leq k \leq n$ there is a one-to-one correspondence between the k-dimensional torus-orbits of X_P and the $(n-k-1)$-dimensional faces of P.

Fano polytopes of small dimension with specific properties have been classified [15, 16, 57–59, 65–67, 77, 78, 89, 90, 98]. We refer the reader to [60] for a survey on the classification of Fano polytopes.

14.4.2 Two Sufficient Conditions for Non-smoothability

It is an open problem to understand whether an arbitrary toric Fano variety is smoothable. Here we state a couple of conditions that forbid the smoothability. Both conditions on a toric Fano variety X are based on the existence of an open affine singular subscheme U such that U is not formally smoothable.

Theorem 18 *Let N be a lattice, let P be a Fano polytope in N, and let X be the toric Fano variety associated to the spanning fan of P. Assume that there exists a face F of P which satisfies the following conditions:*

1. *for each 1-face F' of F, there exists a basis of N which contains the two vertices of F';*
2. *each 2-face of F is a triangle;*
3. *there exists no basis of N which contains all the vertices of F.*

Then X is not smoothable.

Proof Let U be the affine toric open subscheme of X associated to the cone spanned by the face F. The condition (1) means that U is smooth in codimension 2. The condition (2) means that U is $\mathbb{Q}$-factorial in codimension 3. Therefore U is rigid by Proposition 11. The condition (3) implies that U is singular. Therefore, by Corollary 9, X is not smoothable. □

If P is a reflexive polytope of dimension 3, then the theorem above applies if there exists a triangular facet F with unitary edges and such that it is not a standard triangle. Below we see that we can relax the condition of F being triangular to F being Minkowski-indecomposable.

Proposition 19 *Let P be a reflexive polytope of dimension 3 and let X be the toric Fano threefold associated to the spanning fan of P. Assume that there exists a facet F of P such that:*

1. *F has unitary edges;*
2. *F is Minkowski-indecomposable;*
3. *F is not a standard triangle (i.e. the vertices of F do not form a basis of the lattice).*

Then X is not smoothable.

Proof The proof is very similar to the proof of Theorem 18. Let U be the affine toric open subscheme of X associated to the cone spanned by F. The conditions (1) and (3) means that U has an isolated singularity. Since P is reflexive, U is Gorenstein. By Corollary 15, from (2) we deduce that Def_U has an artinian hull. Therefore, by Proposition 8, U is not formally smoothable. By Corollary 9, X is not smoothable. □

14.4.3 Rigidity

Here we will see that if a toric Fano variety has very mild singularities then it is rigid.

Lemma 20 *Let X be a toric Fano variety. Then* $\mathrm{H}^i(X, T_X) = 0$ *for each* $i \geq 1$. *In particular, all locally trivial deformations of X are trivial.*

Proof Set $n = \dim X$. Consider the smooth locus $j : U \hookrightarrow X$. Let D be the toric boundary of X. The sheaves T_X and $(j_* \Omega_U^{n-1} \otimes \mathcal{O}_X(D))^{\vee\vee}$ are reflexive on X and their restrictions to U coincide, because U is smooth and T_U is isomorphic to $\Omega_U^{n-1} \otimes \omega_U^\vee$. Therefore, since the complement of U has codimension at least 2, by [49, Proposition 1.6] we have that T_X is isomorphic to $(j_* \Omega_U^{n-1} \otimes \mathcal{O}_X(D))^{\vee\vee}$. Since D is ample, we conclude by Bott–Steenbrink–Danilov vanishing [34, Theorem 9.3.1] (see also [24, 40, 75]). □

An immediate consequence of the lemma above is the following result.

Proposition 21 *Every smooth toric Fano variety is rigid.*

This result was originally proved by Bien and Brion [20]. Later de Fernex and Hacon [38] proved the rigidity of $\mathbb{Q}$-factorial terminal toric Fano varieties. The following theorem, due to Totaro, is the most general rigidity theorem for toric Fano varieties of which we are aware.

Theorem 22 (Totaro [96, Theorem 5.1]) *A Fano toric variety which is smooth in codimension* 2 *and* $\mathbb{Q}$*-factorial in codimension* 3 *is rigid.*

Proof By Lemma 20, $\mathrm{H}^1(T_X) = 0$. By Proposition 11, the sheaf $\mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)$ is zero. From the five term exact sequence of Ext, which is written in the proof of Proposition 2, we deduce that $\mathrm{Ext}^1(\Omega_X, \mathcal{O}_X)$ is zero. □

If P is a Fano polytope, then X_P satisfies the hypotheses of this theorem if and only if all 2-faces of P are triangles and each edge, i.e. 1-face, of P has lattice length 1 and is contained in some hyperplane which has height 1 with respect to the origin.

Corollary 23 *Let X be a toric Fano variety of dimension* ≥ 4. *If X has isolated singularities, then X is rigid.*

In Sects. 14.4.4 and 14.4.5 we will study deformations of toric Fanos with isolated singularities and of dimension 2 or 3.

14.4.4 Toric del Pezzo Surfaces

A *del Pezzo surface* is a Fano variety of dimension 2. A toric del Pezzo surface is associated to a Fano polygon, which is a Fano polytope of dimension 2.

Theorem 24 *Every toric del Pezzo surface is smoothable.*

Proof Let X be an arbitrary toric del Pezzo surface. It is well known that X is a normal Cohen–Macaulay projective variety. By Demazure vanishing [34, Theorem 2.9.3], $\mathrm{H}^2(\mathcal{O}_X) = 0$. By Lemma 20, $\mathrm{H}^2(T_X) = 0$. Since X is normal and of dimension 2, X has isolated singularities. By Theorem 7 it is enough to check that the singularities of X are formally smoothable.

The singularities of X are cyclic quotient surface singularities. This kind of singularities is always smoothable; indeed, it is enough to pick the Artin component of the base of the miniversal deformation [12]. □

Remark 25 When the canonical divisor of a normal variety X is not Cartier, flat deformations of X are too wild for hoping to study moduli of varieties. For a normal $\mathbb{Q}$-Gorenstein non-Gorenstein variety X one should consider a subfunctor of Def_X which is made up of the deformations of X in which the canonical divisor deforms well. This is the theory of $\mathbb{Q}$-Gorenstein deformations, developed by Kollár–Shepherd-Barron [64] (see also [1, 9, 45, 68]).

In the context of $\mathbb{Q}$-Gorenstein deformations the analogous statement of Theorem 24 is false: there exist non-Gorenstein toric del Pezzo surfaces which cannot be deformed via $\mathbb{Q}$-Gorenstein deformations to a smooth del Pezzo surface, e.g. the weighted projective space $\mathbb{P}(1, 1, 3)$. Nonetheless, it is true that for $\mathbb{Q}$-Gorenstein deformations of del Pezzo surfaces there are no local-to-global obstructions [2, Lemma 6]. Therefore, a del Pezzo surface is $\mathbb{Q}$-Gorenstein smoothable if and only if its singularities are $\mathbb{Q}$-Gorenstein smoothable.

Since the main focus of this note is the study of deformations of Gorenstein toric Fano threefolds, we will omit to discuss the theory of $\mathbb{Q}$-Gorenstein deformations. We refer the reader to [46, 82] for the study of toric del Pezzo surfaces which have $\mathbb{Q}$-Gorenstein smoothings.

14.4.5 Toric Fano Threefolds with Isolated Singularities

Theorem 26 *Let X be a toric Fano variety of dimension* 3 *with isolated singularities. Then X is smoothable if and only if its singularities are formally smoothable.*

Proof By Proposition 6, if X is smoothable then its singularities are formally smoothable. Conversely, suppose that the singularities of X are formally smoothable. Then we argue as in the proof of Theorem 24: X is a normal Cohen–Macaulay projective variety with $\mathrm{H}^2(\mathcal{O}_X) = 0$, by [34, Theorem 2.9.3], and $\mathrm{H}^2(T_X) = 0$, by Lemma 20. By Theorem 7, X is smoothable. □

Corollary 27 *Let P be a reflexive polytope of dimension 3 and let X be the toric Fano threefold associated to the spanning fan of P. If each facet of P is either a standard triangle or a standard square (see the definition in Example 13), then X is smoothable.*

Proof By Example 13 we have that the singularities of X are at most ordinary double points (i.e. nodes). These singularities are formally smoothable. By Theorem 26 we conclude. □

The proof of this corollary is essentially a specific case of [39, Sect. 4.a]. The corollary could have been deduced also from a more general result by Namikawa according to which every Fano threefold with Gorenstein terminal singularities is smoothable [76]. The smooth Fano threefolds which are the smoothings of the toric Fano threefold appearing in Corollary 27 have been studied by Galkin [42].

For $d \in \{6, 7\}$, let dP_d be the smooth del Pezzo surface of degree d; it is toric. The complete anticanonical linear system on dP_d induces a closed embedding $\mathrm{dP}_d \hookrightarrow \mathbb{P}^d$. We consider the projective cone $C(\mathrm{dP}_d) \subseteq \mathbb{P}^{d+1}$ over this embedding; we have that $C(\mathrm{dP}_d)$ is a toric Fano threefold with a Gorenstein canonical non-terminal isolated singularity. In Sect. 14.4.6 we will see that $C(\mathrm{dP}_7)$ is smoothable. In [80] it is shown that $C(\mathrm{dP}_6)$ has two smoothings (see also [56, Example 3.3]).

14.4.6 The Projective Cone over the Del Pezzo Surface of Degree 7

Here we study the deformations of an explicit toric Fano threefold with an isolated Gorenstein non-terminal singularity.

Fix the lattice $\overline{N} = \mathbb{Z}^2$. Consider the pentagon $F \subseteq \overline{N}_{\mathbb{R}}$ defined in (14.2), imagine to put it into the plane $\overline{N}_{\mathbb{R}} \times \{1\}$ in $\overline{N}_{\mathbb{R}} \oplus \mathbb{R} \simeq \mathbb{R}^3$, and create the pyramid over it with apex at the point $(0, 0, -1)$: this is the polytope

$$P = \operatorname{conv}\left\{ \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\-1\\1\end{pmatrix}, \begin{pmatrix}0\\0\\-1\end{pmatrix} \right\} \tag{14.5}$$

in the lattice $\overline{N} \oplus \mathbb{Z}$ and is depicted in Fig. 14.3. It is clear that P is a Fano polytope.

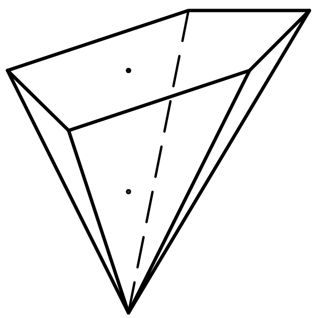

Fig. 14.3 The 3-dimensional lattice polytope P defined in (14.5) and associated to the projective cone over the del Pezzo surface of degree 7

Let X be the toric variety associated to the spanning fan of P. Then X is the projective cone over the anticanonical embedding of the smooth del Pezzo surface of degree 7. The affine toric variety U_F considered in Sect. 14.3.2 is the affine open toric subscheme of X associated to the pentagonal facet F of P. We have that X is a Fano threefold with an isolated non-terminal canonical Gorenstein singularity at the vertex of the cone.

Proposition 28 *Let X be the toric Fano threefold associated to the polytope P in (14.5), i.e. X is the projective cone over the anticanonical embedding of the smooth del Pezzo surface of degree 7. Then X is smoothable and can be deformed to the smooth Fano threefold $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$.*

Proof By Proposition 16, X has an isolated singularity which is formally smoothable. By Theorem 26 we know that X is smoothable. We need to know to which smooth Fano threefold X can be deformed.

From toric geometry [34, Theorem 13.4.3], we have that the anticanonical degree $(-K_X)^3$ is the normalised volume of the polar polytope of P, which is 56 in this case. Since X has Gorenstein canonical singularities, by Proposition 10 we have that the anticanonical degree is preserved in the smoothing. By inspecting the list of smooth Fano threefolds (see [53, 54, 72–74] or [55, Sect. 12]), there is a unique smooth Fano threefold of anticanonical degree 56, namely $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$. □

14.4.7 Another Sufficient Condition for Non-smoothability

In addition to the result of Proposition 19, here we present another obstruction for the smoothability of a toric Fano threefold with Gorenstein singularities.

Theorem 29 ([79]) *Let N be a lattice of rank 3, let $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, let $\langle \cdot, \cdot \rangle \colon M \times N \to \mathbb{Z}$ be the duality pairing, let P be a reflexive polytope in N, and let X be the toric Fano threefold associated to the spanning fan of P. Assume that there are two adjacent facets F_0 and F_1 of P such that:*

1. *both F_0 and F_1 are A_n-triangles for some integer $n \geq 1$ (see the definition in Example 13);*
2. *$F_0 \cap F_1$ is a segment with $n + 2$ lattice points;*
3. *$\langle w_1, v_0 \rangle = 0$, where $w_1 \in M$ is such that $F_1 \subseteq \{v \in N_{\mathbb{R}} \mid \langle w_1, v \rangle = 1\}$ and $v_0 \in N$ is the vertex of F_0 which does not lie on the segment $F_0 \cap F_1$.*

Then X is not smoothable.

With the terminology of [79], the two triangles F_0 and F_1 are called "two adjacent almost-flat A_n-triangles".

Proof (Sketch of the proof of Theorem 29) We refer the reader to [79] for all the details missing here. Let U_i be the toric open affine subscheme of X associated to the facet F_i, for each $i = 0, 1$. Set $U = U_0 \cup U_1$.

One can show that U admits an A_n-bundle structure over $\mathbb{P}^1$. More precisely, one can construct a toric morphism $\pi : U \to \mathbb{P}^1$ such that, for each $i = 0, 1$, if V_i denotes the ith standard affine chart of $\mathbb{P}^1$ then $\pi^{-1}(V_i) = U_i$ and the restriction $\pi|_{U_i} : U_i \to V_i$ is the projection $\operatorname{Spec} \mathbb{C}[x, y, z, w]/(xy - z^{n+1}) \to \operatorname{Spec} \mathbb{C}[w]$. This A_n-bundle may be non trivial, depending on the relative position of the two triangles F_0 and F_1. Set $d = \langle w_1, v_0 \rangle$. By [79, Proposition 3.5] there exists an isomorphism of coherent sheaves on $\mathbb{P}^1$:

$$\pi_* \mathcal{E}xt^1_{\mathcal{O}_U}(\Omega_U, \mathcal{O}_U) \simeq \bigoplus_{2 \le j \le n+1} \mathcal{O}_{\mathbb{P}^1}(-jd - j).$$

Since $d = 0$, the sheaf on the right is a direct sum of negative line bundles on $\mathbb{P}^1$, hence we have $\mathrm{H}^0(U, \mathcal{E}xt^1_{\mathcal{O}_U}(\Omega_U, \mathcal{O}_U)) = 0$. By Corollary 9, X is not smoothable. □

With the same technique of the theorem above one can also construct some rigid toric Fano threefolds with only cA_1-singularities (see [79, Theorem 1.2]). This refutes a conjecture of Prokhorov [85] according to which every Fano threefold with compound Du Val singularities is smoothable.

14.4.8 Other Methods

Here we briefly collect some other results on deformations and smoothings of toric Fano varieties. Most of these results have been motivated by Mirror Symmetry for Fano varieties (see [2, 3, 26, 30, 31, 61, 86, 87]).

By analysing cluster transformations of tori, Akhtar–Coates–Galkin–Kasprzyk [3] have introduced the notion of *mutation* of Fano polytopes. A mutation is a combinatorial procedure that, under certain conditions, transforms a Fano polytope P into another Fano polytope P'. Ilten [51] has proved that mutations of Fano polytopes induce deformations of the corresponding toric Fano varieties; more precisely, if P and P' are related via a mutation, then he has constructed a flat family over $\mathbb{P}^1$ such that the fibre over 0 is X_P and the fibre over ∞ is $X_{P'}$.

Ilten, Lewis and Przyjalkowski [52] have constructed toric degenerations of smooth Fano threefolds with Picard rank 1.

Christophersen and Ilten [29] have constructed degenerations of smooth Fano threefolds of low degree to certain unobstructed Fano Stanley–Reisner schemes. Since these unobstructed Fano Stanley–Reisner schemes are also degenerations of singular toric Fano varieties, this implies the following result.

Theorem 30 (Christophersen–Ilten [28, Proposition 4.2, Theorems 5.1, 7.1]) *Let X be a toric Fano threefold with Gorenstein singularities. If $(-K_X)^3 \in \{4, 6, 8, 10, 12\}$, then X is smoothable.*

Coates–Kasprzyk–Prince [32] have introduced a combinatorial gadget, called *scaffolding*, on a Fano polytope P which induces a closed embedding of the toric Fano variety X_P into a bigger toric variety Y. Often X_P is a complete intersection in the Cox coordinates of Y, therefore it is easy to construct embedded deformations of X_P in Y. In many cases this produces smoothings of X_P. For instance, Cavey and Prince [25] have successfully applied the scaffolding method to construct deformations of toric del Pezzo surfaces to del Pezzo surfaces with a single $\frac{1}{k}(1, 1)$ singularity.

Moreover, Prince [83] has found necessary and sufficient conditions in order to have that the ambient toric variety Y is smooth: this is the notion of *cracked polytope*. He has also found a sufficient condition for a smoothing of X_P to exist inside Y. Via the scaffolding method and cracked polytopes, in [84] he constructs a degeneration of each smooth Fano threefold with very ample anticanonical bundle and Picard rank ≥ 2 to a Gorenstein toric Fano threefold.

14.5 Lists of Reflexive Polytopes of Dimension 3

Below we write lists of reflexive polytopes of dimension 3 which satisfy specific properties. There are exactly 4319 reflexive polytopes of dimension 3: the classification is due to Kreuzer and Skarke [66]. The IDs we use are numbers between 1 and 4319 and come from the Graded Ring Database [23]. All polytopes we consider below are reflexive of dimension 3. They correspond to toric Fano threefolds with Gorenstein singularities. We denote by X_P the toric Fano threefold associated to the spanning fan of P.

Let $\mathcal{S}_{\text{smoothable}}$ be the set of polytopes P such that the corresponding toric Fano threefold X_P is smoothable. It is an open question to explicitly compute $\mathcal{S}_{\text{smoothable}}$.

Let $\mathcal{S}_{\text{smooth}}$ be the set of polytopes which have only standard triangles as facets. These 18 polytopes correspond to the smooth toric Fano threefolds.

Let $\mathcal{S}_{\text{isol}}$ be the set of polytopes with unitary edges such that at least one facet is not a standard triangle. These 137 polytopes correspond to the singular toric Fano threefolds with isolated Gorenstein singularities.

Let $\mathcal{S}_{\text{nodes}}$ be the set of polytopes such that all facets are either standard triangles or standard squares and there is at least a square facet. These 82 polytopes correspond to the singular toric Fano threefolds with at most ordinary double points, or equivalently to the singular toric Fano threefolds with Gorenstein terminal singularities. By Corollary 27 these varieties are smoothable.

Let $\mathcal{S}_{\text{low}}$ be the set of polytopes P such that the normalised volume of the polar P^* of P belongs to $\{4, 6, 8, 10, 12\}$. These 220 polytopes correspond to the toric Gorenstein Fano threefolds X such that $(-K_X)^3 \in \{4, 6, 8, 10, 12\}$.

Let $\mathcal{S}_{\text{indec}}$ be the set of polytopes which contain a facet F which has unitary edges, is Minkowski indecomposable and is not a standard triangle. By Proposition 19 the corresponding toric Fano threefolds are not smoothable.

Let $\mathcal{S}_{\text{aft}}$ be the set of polytopes which contain a pair of adjacent almost-flat A_n-triangles, for some $n \geq 1$. In other words, the set $\mathcal{S}_{\text{aft}}$ contains exactly all polytopes P to which Theorem 29 applies. Therefore, the corresponding toric Fano threefolds are not smoothable.

Let $\mathcal{S}$ denote the set of all reflexive polytopes of dimension 3, i.e. the set of positive integers not greater than 4319. We have:

$$\mathcal{S}_{\text{nodes}} \subseteq \mathcal{S}_{\text{isol}} \subseteq \mathcal{S} \setminus \mathcal{S}_{\text{smooth}},$$
$$\mathcal{S}_{\text{smooth}} \cup \mathcal{S}_{\text{nodes}} \cup \mathcal{S}_{\text{low}} \subseteq \mathcal{S}_{\text{smoothable}},$$
$$\mathcal{S}_{\text{indec}} \cup \mathcal{S}_{\text{aft}} \subseteq \mathcal{S} \setminus \mathcal{S}_{\text{smoothable}}.$$

Below we write down the elements of most of the sets mentioned above.

$\mathcal{S}_{\text{smooth}}$ = {1, 5, 6, 7, 8, 25, 26, 27, 28, 29, 30, 31, 82, 83, 84, 85, 219, 220}

$\mathcal{S}_{\text{isol}}$ = {3, 4, 11, 12, 17, 21, 22, 23, 24, 42, 48, 49, 50, 51, 54, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 155, 156, 158, 159, 160, 167, 168, 170, 177, 187, 188, 198, 199, 200, 201, 202, 203, 204, 205, 206, 207, 208, 209, 210, 211, 212, 213, 214, 215, 216, 217, 218, 360, 363, 364, 365, 366, 376, 377, 378, 380, 385, 403, 410, 411, 412, 413, 414, 415, 416, 417, 418, 419, 420, 421, 422, 423, 424, 425, 426, 427, 686, 688, 689, 692, 693, 694, 695, 696, 707, 710, 725, 729, 730, 731, 732, 733, 734, 735, 736, 737, 738, 739, 740, 741, 1085, 1086, 1087, 1091, 1092, 1093, 1109, 1110, 1111, 1112, 1113, 1114, 1517, 1518, 1519, 1524, 1528, 1529, 1530, 1941, 1943, 2355, 2356}

$\mathcal{S}_{\text{nodes}}$ = {4, 21, 22, 23, 24, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 198, 199, 200, 201, 202, 203, 204, 205, 206, 207, 208, 209, 210, 211, 212, 213, 214, 215, 216, 217, 218, 410, 411, 412, 413, 414, 415, 416, 417, 418, 419, 420, 421, 422, 423, 424, 425, 426, 427, 729, 730, 731, 732, 733, 734, 735, 736, 737, 738, 739, 740, 741, 1109, 1110, 1111, 1112, 1113, 1114, 1528, 1529, 1530, 1943, 2356}

$\mathcal{S}_{\text{low}}$ = {1946, 2711, 2756, 2817, 3043, 3051, 3053, 3079, 3314, 3319, 3329, 3331, 3349, 3350, 3390, 3393, 3406, 3416, 3447, 3452, 3453, 3505, 3573, 3620, 3625, 3626, 3667, 3683, 3702, 3727, 3728, 3731, 3733, 3735, 3736, 3738, 3739, 3740, 3756, 3760, 3762, 3777, 3790, 3791, 3792, 3795, 3796, 3844, 3845, 3846, 3848, 3853, 3857, 3868, 3869, 3874, 3875, 3879, 3901, 3903, 3922, 3923, 3927, 3928, 3933, 3936, 3937, 3938, 3946, 3962, 3964, 3965, 3966, 3967, 3981, 3983, 3984, 3985, 3991, 3995, 4003, 4004, 4005, 4006, 4007, 4022, 4023, 4024, 4027, 4031, 4032, 4041, 4042, 4043, 4044, 4056, 4058, 4059, 4060, 4070, 4074, 4075, 4076, 4080, 4088, 4092, 4094, 4095, 4102, 4104, 4117, 4118, 4119, 4122, 4124, 4131, 4132, 4133, 4134, 4135, 4143, 4144, 4145, 4149, 4159, 4160, 4161, 4167, 4168, 4169, 4170, 4179, 4180, 4181, 4182, 4183, 4184, 4186, 4190, 4191, 4194, 4200, 4202, 4203, 4205, 4206, 4214, 4215, 4216, 4217, 4218, 4219, 4220, 4225, 4228, 4229, 4231, 4232, 4233, 4235, 4236, 4238, 4239, 4241, 4244, 4245, 4246, 4247, 4249, 4250, 4251, 4252, 4254, 4255, 4256, 4258, 4260, 4261, 4263, 4267, 4268, 4269, 4270, 4272, 4273, 4275, 4278, 4280, 4281, 4282, 4284, 4285, 4286, 4287, 4288, 4290, 4291, 4292, 4293, 4294, 4295, 4297, 4298, 4299, 4300, 4301, 4303, 4304, 4307, 4308, 4309, 4310, 4311, 4312, 4313, 4314, 4315, 4317, 4318, 4319}

$\mathcal{S}_{\text{indec}}$ = {3, 12, 17, 32, 38, 48, 49, 51, 54, 88, 91, 94, 98, 99, 100, 101, 102, 103, 105, 115, 119, 121, 134, 137, 138, 141, 142, 155, 158, 159, 170, 188, 228, 235, 239, 242, 243, 247, 248, 252, 254, 256, 260, 262, 265, 271, 278, 293, 294, 298, 299, 301, 317, 318, 330, 351, 353, 360, 378, 380, 438, 439, 440, 443, 445, 455, 468, 480, 491, 492, 493, 497, 501, 502, 515, 525, 526, 529, 530, 532, 539, 541, 543, 546, 550, 553, 562, 570, 575, 604, 608, 609, 614, 620, 645, 650, 660, 663, 688, 744, 752, 753, 754, 756, 760, 774, 775, 776, 780, 784, 790, 791, 792, 800, 834, 841, 844, 845, 852, 856, 859, 864, 866, 887, 900, 908, 912, 914, 923, 935, 963, 979, 990, 991, 1012, 1019, 1020, 1130, 1151, 1154, 1183, 1199, 1204, 1205, 1208, 1215, 1218, 1220, 1261, 1275, 1277, 1283, 1299, 1302, 1309, 1311, 1352, 1370, 1384, 1397, 1547, 1585, 1598, 1631, 1636, 1638, 1679, 1683, 1687, 1693, 1728, 1750, 1751, 1777, 1791, 1992, 2014, 2046, 2047, 2050, 2051, 2080, 2081, 2084, 2096, 2124, 2129, 2379, 2404, 2425, 2427, 2455, 2456, 2716, 2750, 2751, 2755}

$\mathcal{S}_{\text{aft}}$ = {15, 16, 36, 41, 45, 53, 58, 59, 61, 65, 66, 102, 105, 110, 111, 112, 113, 116, 117, 124, 125, 128, 135, 141, 142, 144, 146, 147, 148, 149, 152, 162, 172, 179, 183, 189, 192, 193, 197, 230, 236, 244, 248, 261, 268, 271, 272, 277, 278, 279, 280, 281, 282, 286, 288, 290, 292, 302, 310, 324, 325, 327, 331, 332, 333, 334, 335, 337, 340, 343, 347, 349, 351, 355, 356, 358, 361, 362, 386, 399, 400, 407, 443, 445, 448, 452, 453, 456, 457, 463, 467, 487, 490, 496, 497, 499, 501, 502, 505, 507, 508, 509, 511, 512, 516, 523, 540, 545, 550, 563, 569, 577, 579, 581, 582, 583, 594, 599, 600, 601, 605, 606, 617, 629, 633, 658, 670, 671, 672, 674, 679, 682, 687, 705, 760, 764, 770, 771, 780, 781, 786, 787, 792, 797, 799, 809, 811, 812, 815, 816, 824, 859, 865, 868, 873, 875, 878, 883, 884, 889, 891, 892, 893, 894, 895, 902, 905, 929, 956, 960, 965, 987, 1003, 1004, 1006, 1011, 1021, 1038, 1045, 1051, 1156, 1160, 1168, 1175, 1177, 1199, 1203, 1209, 1216, 1217, 1225, 1232, 1234, 1251, 1252, 1253, 1255, 1256, 1260, 1262, 1265, 1275, 1286, 1287, 1293, 1300, 1305, 1308, 1324, 1327, 1351, 1371, 1383, 1398, 1533, 1545, 1550, 1551, 1554, 1561, 1579, 1589, 1613, 1614, 1615, 1620, 1637, 1638, 1656, 1665, 1666, 1671, 1686, 1690, 1693, 1697, 1711, 1747, 1748, 1760, 1763, 1989, 2000, 2001, 2027, 2045, 2051, 2052, 2068, 2071, 2072, 2076, 2084, 2096, 2098, 2102, 2379, 2380, 2385, 2403, 2405, 2423, 2424, 2425, 2427, 2738, 2777, 2778, 2792, 3047, 3057, 3063, 3064}

We have $|\mathcal{S}_{\text{indec}} \cup \mathcal{S}_{\text{aft}}| = 442$. Therefore there exist at least 442 non-smoothable toric Fano threefolds with Gorenstein singularities.

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References

1. Abramovich, D., Hassett, B.: Stable varieties with a twist. In: Classification of algebraic varieties, EMS Series of Congress Reports, pp. 1–38. European Mathematical Society, Zürich (2011)
2. Akhtar, M., Coates, T., Corti, A., Heuberger, L., Kasprzyk, A.M., Oneto, A., Petracci, A., Prince, T., Tveiten, K.: Mirror symmetry and the classification of orbifold del Pezzo surfaces. Proc. Amer. Math. Soc. **144**(2), 513–527 (2016)

3. Akhtar, M., Coates, T., Galkin, S., Kasprzyk, A.M.: Minkowski polynomials and mutations. SIGMA Symmetry Integr. Geom. Methods Appl. **8**, Paper 094, 17 (2012)
4. Altmann, K.: Computation of the vector space T^1 for affine toric varieties. J. Pure Appl. Algebra **95**(3), 239–259 (1994)
5. Altmann, K.: Minkowski sums and homogeneous deformations of toric varieties. Tohoku Math. J. (2) **47**(2), 151–184 (1995)
6. Altmann, K.: Infinitesimal deformations and obstructions for toric singularities. J. Pure Appl. Algebra **119**(3), 211–235 (1997)
7. Altmann, K.: The versal deformation of an isolated toric Gorenstein singularity. Invent. Math. **128**(3), 443–479 (1997)
8. Altmann, K.: One parameter families containing three-dimensional toric-Gorenstein singularities. In: Explicit birational geometry of 3-folds. London Mathematical Society Lecture Note series, vol. 281, pp. 21–50. Cambridge University Press, Cambridge (2000)
9. Altmann, K., Kollár, J.: The dualizing sheaf on first-order deformations of toric surface singularities. J. Reine Angew. Math. **753**, 137–158 (2019)
10. Artin, M.: Algebraic approximation of structures over complete local rings. Inst. Hautes Études Sci. Publ. Math. **36**, 23–58 (1969)
11. Artin, M.: Algebraization of formal moduli. I. In: Global Analysis (Papers in Honor of K. Kodaira), pp. 21–71. University Tokyo Press, Tokyo (1969)
12. Artin, M.: Algebraic construction of Brieskorn's resolutions. J. Algebra **29**, 330–348 (1974)
13. Artin, M.: Lectures on Deformations of Singularities. Tata Institute of Fundamental Research, Bombay (1976)
14. The Stacks project (2018). https://stacks.math.columbia.edu
15. Batyrev, V.V.: Toric Fano threefolds. Izv. Akad. Nauk SSSR Ser. Mat. **45**(4), 704–717, 927 (1981)
16. Batyrev, V.V.: On the classification of toric Fano 4-folds. pp. 1021–1050 (1999). Algebraic geometry, 9
17. Batyrev, V.V.: Toric degenerations of Fano varieties and constructing mirror manifolds. In: The Fano Conference, pp. 109–122. University Torino, Turin (2004)
18. Batyrev, V.V., Ciocan-Fontanine, I., Kim, B., van Straten, D.: Mirror symmetry and toric degenerations of partial flag manifolds. Acta Math. **184**(1), 1–39 (2000)
19. Behnke, K., Riemenschneider, O.: Quotient surface singularities and their deformations. In: Singularity Theory (Trieste, 1991), pp. 1–54. World Scientific Publishing, River Edge (1995)
20. Bien, F., Brion, M.: Automorphisms and local rigidity of regular varieties. Compositio Math. **104**(1), 1–26 (1996)
21. Birkar, C.: Singularities of linear systems and boundedness of Fano varieties. Ann. of Math. (2) **193**(2), 347–405 (2021)
22. Bosma, W., Cannon, J., Playoust, C.: The Magma algebra system. I. The user language. pp. 235–265 (1997). Computational algebra and number theory (London, 1993)
23. Brown, G., Kasprzyk, A.M.: Graded Ring Database. http://www.grdb.co.uk
24. Buch, A., Thomsen, J.F., Lauritzen, N., Mehta, V.: The Frobenius morphism on a toric variety. Tohoku Math. J. (2) **49**(3), 355–366 (1997)
25. Cavey, D., Prince, T.: Del Pezzo surfaces with a single $1/k(1, 1)$ singularity. J. Math. Soc. Japan **72**(2), 465–505 (2020)
26. Cheltsov, I., Katzarkov, L., Przyjalkowski, V.: Birational geometry via moduli spaces. In: Birational Geometry, Rational Curves, and Arithmetic, Simons Symposia, pp. 93–132. Springer, Cham (2013)
27. Christophersen, J.A.: On the components and discriminant of the versal base space of cyclic quotient singularities. In: Singularity theory and its applications, Part I (Coventry, 1988/1989). Lecture Notes in Math., vol. 1462, pp. 81–92. Springer, Berlin (1991)
28. Christophersen, J.A., Ilten, N.: Hilbert schemes and toric degenerations for low degree Fano threefolds. J. Reine Angew. Math. **717**, 77–100 (2016)
29. Christophersen, J.A., Ilten, N.O.: Degenerations to unobstructed Fano Stanley-Reisner schemes. Math. Z. **278**(1–2), 131–148 (2014)

30. Coates, T., Corti, A., Galkin, S., Golyshev, V., Kasprzyk, A.M.: Mirror symmetry and Fano manifolds. In: European Congress of Mathematics, pp. 285–300. European Mathematical Society, Zürich (2013)
31. Coates, T., Corti, A., Galkin, S., Kasprzyk, A.M.: Quantum periods for 3-dimensional Fano manifolds. Geom. Topol. **20**(1), 103–256 (2016)
32. Coates, T., Kasprzyk, A., Prince, T.: Laurent inversion. Pure Appl. Math. Q. **15**(4), 1135–1179 (2019)
33. Coates, T., Kasprzyk, A.M.: Code repository (2019). https://bitbucket.org/fanosearch/magma-core
34. Cox, D.A., Little, J.B., Schenck, H.K.: Toric varieties. Graduate Studies in Mathematics, vol. 124. American Mathematical Society, Providence (2011)
35. Eisenbud, D.: Commutative algebra. Graduate Texts in Mathematics, vol. 150. Springer, New York (1995). With a view toward algebraic geometry
36. Fantechi, B., Göttsche, L.: Local properties and Hilbert schemes of points. In: Fundamental Algebraic Geometry. Mathematical Surveys and Monographs, vol. 123, pp. 139–178. American Mathematical Society, Providence (2005)
37. Fantechi, B., Manetti, M.: Obstruction calculus for functors of Artin rings. I. J. Algebra **202**(2), 541–576 (1998)
38. de Fernex, T., Hacon, C.D.: Deformations of canonical pairs and Fano varieties. J. Reine Angew. Math. **651**, 97–126 (2011)
39. Friedman, R.: Simultaneous resolution of threefold double points. Math. Ann. **274**(4), 671–689 (1986)
40. Fujino, O.: Multiplication maps and vanishing theorems for toric varieties. Math. Z. **257**(3), 631–641 (2007)
41. Fulton, W.: Introduction to toric varieties. Annals of Mathematics Studies, vol. 131. Princeton University Press, Princeton (1993). The William H. Roever Lectures in Geometry
42. Galkin, S.: Small toric degenerations of Fano threefolds (2018). arXiv:1809.02705 [math.AG]
43. Gross, M., Siebert, B.: From real affine geometry to complex geometry. Ann. of Math. (2) **174**(3), 1301–1428 (2011)
44. Grothendieck, A.: Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III. Inst. Hautes Études Sci. Publ. Math. (28), 255 (1966)
45. Hacking, P.: Compact moduli of plane curves. Duke Math. J. **124**(2), 213–257 (2004)
46. Hacking, P., Prokhorov, Y.: Smoothable del Pezzo surfaces with quotient singularities. Compos. Math. **146**(1), 169–192 (2010)
47. Hartshorne, R.: Residues and duality. Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20. Springer, Berlin (1966)
48. Hartshorne, R.: Algebraic Geometry. Springer, New York (1977). Graduate Texts in Mathematics, No. 52
49. Hartshorne, R.: Stable reflexive sheaves. Math. Ann. **254**(2), 121–176 (1980)
50. Hartshorne, R.: Deformation theory. Graduate Texts in Mathematics, vol. 257. Springer, New York (2010)
51. Ilten, N.O.: Mutations of Laurent polynomials and flat families with toric fibers. SIGMA Symmetry Integr. Geom. Methods Appl. **8**, Paper 047, 7 (2012)
52. Ilten, N.O., Lewis, J., Przyjalkowski, V.: Toric degenerations of Fano threefolds giving weak Landau-Ginzburg models. J. Algebra **374**, 104–121 (2013)
53. Iskovskih, V.A.: Fano threefolds. I. Izv. Akad. Nauk SSSR Ser. Mat. **41**(3), 516–562, 717 (1977)
54. Iskovskih, V.A.: Fano threefolds. II. Izv. Akad. Nauk SSSR Ser. Mat. **42**(3), 506–549 (1978)
55. Iskovskikh, V.A., Prokhorov, Y.G.: Fano varieties. In: Algebraic Geometry, V, Encylopaedia of Mathematical Sciences, vol. 47, pp. 1–247. Springer, Berlin (1999)
56. Jahnke, P., Radloff, I.: Terminal Fano threefolds and their smoothings. Math. Z. **269**(3–4), 1129–1136 (2011)

57. Kasprzyk, A.M.: Toric Fano three-folds with terminal singularities. Tohoku Math. J. (2) **58**(1), 101–121 (2006)
58. Kasprzyk, A.M.: Canonical toric Fano threefolds. Canad. J. Math. **62**(6), 1293–1309 (2010)
59. Kasprzyk, A.M., Kreuzer, M., Nill, B.: On the combinatorial classification of toric log del Pezzo surfaces. LMS J. Comput. Math. **13**, 33–46 (2010)
60. Kasprzyk, A.M., Nill, B.: Fano polytopes. In: Strings. Gauge Fields, and the Geometry Behind, pp. 349–364. World Scientific Publishing, Hackensack (2013)
61. Katzarkov, L., Przyjalkowski, V.: Landau-Ginzburg models—old and new. In: Proceedings of the Gökova Geometry-Topology Conference 2011, pp. 97–124. International Press, Somerville, MA (2012)
62. Kollár, J., Miyaoka, Y., Mori, S.: Rational connectedness and boundedness of Fano manifolds. J. Differ. Geom. **36**(3), 765–779 (1992)
63. Kollár, J., Mori, S.: Birational geometry of algebraic varieties. Cambridge Tracts in Mathematics, vol. 134. Cambridge University Press, Cambridge (1998). With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original
64. Kollár, J., Shepherd-Barron, N.I.: Threefolds and deformations of surface singularities. Invent. Math. **91**(2), 299–338 (1988)
65. Kreuzer, M., Nill, B.: Classification of toric Fano 5-folds. Adv. Geom. **9**(1), 85–97 (2009)
66. Kreuzer, M., Skarke, H.: Classification of reflexive polyhedra in three dimensions. Adv. Theor. Math. Phys. **2**(4), 853–871 (1998)
67. Kreuzer, M., Skarke, H.: Complete classification of reflexive polyhedra in four dimensions. Adv. Theor. Math. Phys. **4**(6), 1209–1230 (2000)
68. Lee, Y., Nakayama, N.: Grothendieck duality and $\mathbb{Q}$-Gorenstein morphisms. Publ. Res. Inst. Math. Sci. **54**(3), 517–648 (2018)
69. Li, C., Wang, X., Xu, C.: On the proper moduli spaces of smoothable Kähler-Einstein Fano varieties. Duke Math. J. **168**(8), 1387–1459 (2019)
70. Manetti, M.: Differential graded Lie algebras and formal deformation theory. In: Algebraic Geometry—Seattle 2005. Part 2. Proceedings of Symposia in Pure Mathematics, vol. 80, pp. 785–810. American Mathematical Society, Providence (2009)
71. Mavlyutov, A.R.: Deformations of toric varieties via Minkowski sum decompositions of polyhedral complexes (2009). arXiv:0902.0967 [math.AG]
72. Mori, S., Mukai, S.: Classification of Fano 3-folds with $B_2 \geq 2$. Manuscripta Math. **36**(2), 147–162 (1981/82)
73. Mori, S., Mukai, S.: On Fano 3-folds with $B_2 \geq 2$. In: Algebraic Varieties and Analytic Varieties (Tokyo, 1981). Advanced Studies in Pure Mathematics, vol. 1, pp. 101–129. North-Holland, Amsterdam (1983)
74. Mori, S., Mukai, S.: Erratum: "Classification of Fano 3-folds with $B_2 \geq 2$" [Manuscripta Math. **36**(2), 147–162 (1981/82); MR0641971 (83f:14032)]. Manuscripta Math. **110**(3), 407 (2003)
75. Mustaţă, M.: Vanishing theorems on toric varieties. Tohoku Math. J. (2) **54**(3), 451–470 (2002)
76. Namikawa, Y.: Smoothing Fano 3-folds. J. Algebraic Geom. **6**(2), 307–324 (1997)
77. Nill, B., Øbro, M.: $\mathbb{Q}$-factorial Gorenstein toric Fano varieties with large Picard number. Tohoku Math. J. (2) **62**(1), 1–15 (2010)
78. Øbro, M.: An algorithm for the classification of smooth Fano polytopes (2007). arXiv:0704.0049 [math.CO]
79. Petracci, A.: Some examples of non-smoothable Gorenstein Fano toric threefolds. Math. Z. **295**(1–2), 751–760 (2020)
80. Petracci, A.: An example of mirror symmetry for Fano threefolds. In: Birational geometry and moduli spaces. Springer INdAM Serirs, vol. 39, pp. 173–188. Springer, Cham ([2020] © 2020)
81. Petracci, A.: Homogeneous deformations of toric pairs. Manuscripta Math. **166**(1–2), 37–72 (2021)
82. Prince, T.: Smoothing toric Fano surfaces using the Gross-Siebert algorithm. Proc. Lond. Math. Soc. (3) **117**(3), 617–660 (2018)
83. Prince, T.: Cracked polytopes and Fano toric complete intersections. Manuscripta Math. **163**(1–2), 165–183 (2020)

84. Prince, T.: From cracked polytopes to Fano threefolds. Manuscripta Math. **164**(1–2), 267–320 (2021)
85. Prokhorov, Y.G.: The degree of Fano threefolds with canonical Gorenstein singularities. Mat. Sb. **196**(1), 81–122 (2005)
86. Przhiyalkovskiĭ, V.V.: Weak Landau-Ginzburg models of smooth Fano threefolds. Izv. Ross. Akad. Nauk Ser. Mat. **77**(4), 135–160 (2013)
87. Przyjalkowski, V.: On Landau-Ginzburg models for Fano varieties. Commun. Number Theory Phys. **1**(4), 713–728 (2007)
88. Riemenschneider, O.: Deformationen von Quotientensingularitäten (nach zyklischen Gruppen). Math. Ann. **209**, 211–248 (1974)
89. Sato, H.: Toward the classification of higher-dimensional toric Fano varieties. Tohoku Math. J. (2) **52**(3), 383–413 (2000)
90. Sato, H.: Smooth toric Fano five-folds of index two. Proc. Japan Acad. Ser. A Math. Sci. **82**(7), 106–110 (2006)
91. Schlessinger, M.: Functors of Artin rings. Trans. Amer. Math. Soc. **130**, 208–222 (1968)
92. Sernesi, E.: Deformations of algebraic schemes, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 334. Springer, Berlin (2006)
93. Stevens, J.: On the versal deformation of cyclic quotient singularities. In: Singularity Theory and Its Applications, Part I (Coventry, 1988/1989). Lecture Notes in Mathematics, vol. 1462, pp. 302–319. Springer, Berlin (1991)
94. Stevens, J.: The versal deformation of cyclic quotient singularities. In: Deformations of Surface Singularities. Bolyai Society Mathematical Studies, vol. 23, pp. 163–201. János Bolyai Mathematical Society, Budapest (2013)
95. Talpo, M., Vistoli, A.: Deformation theory from the point of view of fibered categories. In: Handbook of Moduli. Vol. III, Advanced Lectures in Mathematics (ALM), vol. 26, pp. 281–397. International Press, Somerville (2013)
96. Totaro, B.: Jumping of the nef cone for Fano varieties. J. Algebraic Geom. **21**(2), 375–396 (2012)
97. Vistoli, A.: The deformation theory of local complete intersections (1997). arXiv:alg-geom/9703008
98. Watanabe, K., Watanabe, M.: The classification of Fano 3-folds with torus embeddings. Tokyo J. Math. **5**(1), 37–48 (1982)
99. Ziegler, G.M.: Lectures on Polytopes. Graduate Texts in Mathematics, vol. 152. Springer, New York (1995)