Chapter 13
Toric Degenerations in Symplectic Geometry

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Abstract Toric degeneration in algebraic geometry is a process of degenerating a given projective variety into a toric one. Then one can obtain information about the original variety via analyzing the toric one, which is a much easier object to study. Harada and Kaveh described how one incorporates a symplectic structure into this process, providing a very useful tool for solving certain problems in symplectic geometry. Below we present two applications of this method: questions about the Gromov width, and cohomological rigidity problems.

Keywords Symplectic toric manifold · Bott manifold · Toric degeneration · Gromov width

13.1 Introduction

Manifolds and algebraic varieties equipped with a group action are usually better understood as a presence of an action is a sign of certain symmetries. In particular, *toric varieties* form a very well understood class of varieties. These are varieties which contain an algebraic torus $T^n_{\mathbb{C}} := (\mathbb{C}^*)^n$ as a dense open subset and are equipped with an action of $T^n_{\mathbb{C}}$ which extends the usual action of $T^n_{\mathbb{C}}$ on itself. For more about toric varieties see, for example, [5, 12]. To understand a given projective variety X one can try to "degenerate" it to a toric one, i.e., form a family of varieties with generic member X and one special member some toric variety X_0. The varieties X and X_0 are closely related and thus one can obtain information about X by studying X_0. Moreover, such a degeneration gives a map from X to X_0 which, in certain situations, preserves some special structures X and X_0 might be equipped with (for example: a symplectic structure).

One can use the method of toric degenerations to solve problems in symplectic geometry. In this work we discuss the following two applications:

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1. calculating lower bounds for the Gromov width, i.e. trying to find the largest ball which can be symplectically embedded into a given symplectic manifold;
2. constructing symplectomorphisms needed for a cohomological rigidity problem for symplectic toric manifolds. This problem is about checking whether any two symplectic toric manifolds with isomorphic integral cohomology rings (via an isomorphism preserving the class of symplectic form) are symplectomorphic.

Recall that an $2n$-dimensional symplectic manifold (M, ω) equipped with an effective Hamiltonian action of an n-dimensional torus $T = (S^1)^n$ is called a *symplectic toric manifold*. The action being Hamiltonian means that there exists a moment map[1] $\mu\colon M \to \mathbb{R}^n$. Such a manifold can be given a complex structure interacting well with the symplectic one so that one calls ω a Kähler form and (M, ω) a Kähler manifold. In particular, symplectic toric manifolds are toric varieties in the sense of algebraic geometry. A theorem of Delzant states that we have a bijection[2]

{$2n$-dim compact symplectic toric manifolds} up to equivariant symplectomorphisms	$\Longleftrightarrow$	{rational and smooth polytopes in $\mathbb{R}^n$} up to translations and $\mathrm{GL}(n, \mathbb{Z})$ transformations.

In this bijection, a manifold corresponds to an image of its moment map, therefore the associated polytope is often called a moment polytope or a moment image. Not much is known about a classification of symplectic toric manifolds up to symplectomorphism. The cohomological rigidity mentioned in the second bullet above asks if such classification might be given by the integral cohomology rings.

In Sects. 13.3 and 13.4 respectively we describe the above problems in detail and explain how one can use toric degenerations to solve problems of this type. In particular we prove (rather, outline the proofs of) the following two results, obtained in projects joint with I. Halacheva, X. Fang, P. Littelmann, and S. Tolman. As to apply a toric degeneration to (M, ω) one needs ω to be an integral symplectic form, in both theorems we demand that the symplectic form is integral up to scaling, i.e. that $a[\omega] \in \mathrm{H}^*(M; \mathbb{Z})$ for some $a \in \mathbb{R}$. To simplify the exposition we slightly abuse the notation and given a map F defined on $\mathrm{H}^*(M; \mathbb{Z})$ we use F to also denote the map induced by F on $\mathrm{H}^*(M; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$.

Theorem 1 ([11, 14]) *Let K be a compact connected simple Lie group. The Gromov width of a coadjoint orbit $\mathcal{O}_\lambda$ through a point λ, integral up to scaling, equipped with the Kostant–Kirillov–Souriau symplectic form, is at least*

$$\min\{ \left|\langle \lambda, \alpha^\vee \rangle\right| ;\ \alpha^\vee \text{ a coroot and } \langle \lambda, \alpha^\vee \rangle \neq 0\}. \tag{13.1}$$

[1] A moment map is a T-invariant map $\mu\colon M \to \mathrm{Lie}(T)^* \cong \mathbb{R}^n$ such that for every $X \in \mathrm{Lie}(T)$ it holds that $\iota_{X^\sharp}\omega = d\mu^X$ where $X^\sharp$ denotes the vector field on M induced by X and $\mu^X : M \to \mathbb{R}$ is defined by $\mu^X(p) = \langle \mu(p), X \rangle$..

[2] Recall that a polytope in $\mathbb{R}^n$ is called rational if the directions of its edges are in $\mathbb{Z}^n$. It is called smooth if for every vertex the primitive vectors in the directions of edges meeting at that vertex form a $\mathbb{Z}$-basis for $\mathbb{Z}^n$.

Theorem 2 ([27]) *Let* (M, ω_M) *and* (N, ω_N) *be Bott manifolds with symplectic forms integral up to scaling. Moreover, assume that* $\mathrm{H}^*(M; \mathbb{Q})$ *and* $\mathrm{H}^*(N; \mathbb{Q})$ *are isomorphic to* $\mathbb{Q}[x_1, \ldots, x_n]/\langle x_1^2, \ldots, x_n^2\rangle$. *For any ring isomorphism* $F\colon \mathrm{H}^*(M; \mathbb{Z}) \to \mathrm{H}^*(N; \mathbb{Z})$ *with* $F([\omega_M]) = [\omega_N]$, *there exists a symplectomorphism* $f\colon (N, \omega_N) \to (M, \omega_M)$ *such that the map* $\mathrm{H}^*(f)\colon \mathrm{H}^*(M; \mathbb{Z}) \to \mathrm{H}^*(N; \mathbb{Z})$ *induced by* f *on integral cohomology rings is exactly* F.

There are other applications of toric degenerations in symplectic geometry. For example, one can obtain information about Ginzburg–Landau potential function on X from that of X_0 and thus detect some non-displaceable Lagrangians of X, see [25].

13.2 Toric Degenerations

A **toric degeneration** of a projective variety X is a flat family $\pi\colon \mathfrak{X} \to \mathbb{C}$ with generic fiber X and one special fiber $X_0 = \pi^{-1}(0)$, a (not necessarily normal) toric variety. A construction of such a degeneration of a projective variety X, equipped with a very ample line bundle satisfying certain conditions, can be found in Anderson [1, Theorem 2].

Example 3 Using the Plücker embedding,[3] view $X = Gr(2, \mathbb{C}^4)$, the Grassmannian of 2-planes in $\mathbb{C}^4$, as a subset of $\mathbb{CP}^5$ with coordinates $\{x_{ij};\ 1 \le i < j \le 4\}$, consisting of points satisfying

$$x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0.$$

Consider the subset $\mathfrak{X} \subset \mathbb{CP}^5 \times \mathbb{C}$ consisting of points satisfying

$$x_{12}x_{34} - x_{13}x_{24} + t x_{14}x_{23} = 0,$$

where t denotes the coordinate in $\mathbb{C}$. Let $\pi\colon \mathfrak{X} \to \mathbb{C}$ be the restriction to $\mathfrak{X}$ of the projection onto the second factor. This family constitutes a toric degeneration of $Gr(2, \mathbb{C}^4)$. In fact, $\{x_{ij}\}$ form a SAGBI basis of the homogeneous coordinate ring of X and this ensures the flatness [8, Theorem 15.17]. Clearly $\pi^{-1}(1)$ is $Gr(2, \mathbb{C}^4)$. Moreover, performing a change of coordinates, one can show that $\pi^{-1}(t)$ for $t \neq 0$ is also bihomolomorhpic to $Gr(2, \mathbb{C}^4)$. The central fiber, $\pi^{-1}(0)$, is described by the binomial ideal $\langle x_{12}x_{34} - x_{13}x_{24}\rangle$ and thus is a toric variety.

Harada and Kaveh [16] enriched the construction of Anderson by incorporating a symplectic structure. They start with a smooth projective variety X, of complex

[3] Recall that the Plücker embedding sends a Grassmannian spanned by vectors $v, w \in \mathbb{C}^4$ to a point $[x_{12} : \ldots : x_{34}] \in \mathbb{CP}^5$ with x_{ij} equal to the determinant of the 2×2 minor of $[v^T, w^T]$ spanned by rows i and j.

dimension n, equipped with a very ample line bundle $\mathcal{L}$, with some fixed Hermitian structure. Let $L := \mathrm{H}^0(X, \mathcal{L})$ denote the vector space of holomorphic sections, $\Phi_{\mathcal{L}}\colon X \to \mathbb{P}(L^*)$ the Kodaira embedding and $\omega = \Phi_{\mathcal{L}}^*(\omega_{FS})$ the pull back of the Fubini–Study form, i.e., of the standard symplectic structure on complex projective spaces. Then (X, ω) is a Kähler manifold. With this data they construct (under certain assumptions) not only a flat family $\pi : \mathfrak{X} \to \mathbb{C}$ but also a Kähler structure $\tilde{\omega}$ on (the smooth part of) $\mathfrak{X}$ so that $(\pi^{-1}(1), \tilde{\omega}_{|\pi^{-1}(1)})$ is symplectomorphic to (X, ω). Moreover, the special fiber $X_0 = \pi^{-1}(0)$ obtains a Kähler form, the restriction of $\tilde{\omega}$, defined on its smooth part $U_0 := (X_0)_{\text{smooth}}$, and thus it also obtains a divisor. If X_0 is normal, then the polytope associated to X_0 and this divisor by the usual procedure of toric algebraic geometry (see, for example, [5, Chap. 4]) is the closure of the moment image of the (non-compact) symplectic toric manifold $(U_0, \tilde{\omega}_{|U_0})$. As we will see, this polytope can be computed by analyzing the behaviour of the holomorphic sections of $\mathcal{L}$. Here are more details of this procedure.

Denote by L^m the image of span $\langle f_1 \cdot \ldots \cdot f_m\,;\; f_i \in L\rangle$ in $\mathrm{H}^0(X, \mathcal{L}^{\otimes m})$ and by $R = \mathbb{C}[X] = \oplus_{m \geq 0} L^m$ the homogeneous coordinate ring of X with respect to the embedding $\Phi_{\mathcal{L}}$. An important ingredient of the construction is a choice of a *valuation with one dimensional leaves*, $\nu\colon \mathbb{C}(X) \setminus \{0\} \to \mathbb{Z}^n$, from the ring $\mathbb{C}(X)$ of rational functions on X. A precise definition of a general valuation can be found, for example, in [16, Definition 3.1]. In this paper we only use valuations induced by a flag of subvarieties and a special case of these, called *lowest/highest term valuations associated to a coordinate system.*

Example 4 (*Lowest/highest term valuations* [16, Example 3.2]) Fix a smooth point $p \in X$ and let $(u_1, \ldots, u_n)$ be a system of coordinates in a neighborhood of p, meaning that $u_1, \ldots, u_n$ are regular functions at p, vanishing at p, and such that their differentials $du_1, \ldots, du_n$ are linearly independent at p. Then any regular function at p can be represented as a power series $\sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha u^\alpha$. Here by u^α, with $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, we mean $u_1^{\alpha_1} \cdot \ldots \cdot u_n^{\alpha_n}$. Choose and fix a total order $>$ on $\mathbb{Z}^n$ respecting the addition, for example the lexicographic order. Define a map ν from the set of functions regular at p to $\mathbb{Z}^n$ by

$$\nu\Big(\sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha u^\alpha\Big) = \min\{\alpha;\; c_\alpha \neq 0\},$$

and extend it to $\mathbb{C}(X) \setminus \{0\}$ by setting $\nu(f/g) = \nu(f) - \nu(g)$. Then ν is a valuation with one dimensional leaves, called a *lowest term valuation*. If one uses max instead of min in the definition of ν, one obtains a *highest term valuation*.

Example 5 (*Valuations induced by a flag of subvarieties* [16, Example 3.3]) Take a flag of normal subvarieties (called a Parshin point) of X

$$\{p\} = Y_n \subset \ldots \subset Y_0 = X,$$

with $\dim_{\mathbb{C}}(Y_k) = n - k$ and Y_k non-singular along Y_{k+1}. By the non-singularity assumption there exists a collection of rational functions $u_1, \ldots, u_n$ on X such that $u_{k|Y_{k-1}}$ is a rational function on Y_{k-1} which is not identically zero and which has a zero of first order on Y_k. Then the lowest term valuation with respect to the lexicographic order can alternatively be described in the following way: for any $f \in \mathbb{C}(X)$, $f \neq 0$, the valuation $\nu(f) = (k_1, \dots, k_n)$ where k_1 is the order of vanishing of f on Y_1, k_2 is the order of vanishing of $f_1 := (u_1^{-k_1} f)_{|Y_1}$ on Y_2, etc.

Given such X, $\mathcal{L}$, and ν we form a semigroup $S = S(\nu, \mathcal{L})$, in the following way. Fix a non-zero element $h \in L$ and use it to identify L with a subspace of $\mathbb{C}(X)$ by mapping $f \in L$ to $f/h \in \mathbb{C}(X)$. Similarly identify L^m with a subspace of $\mathbb{C}(X)$ by sending $f \in L^m$ to $f/h^m \in \mathbb{C}(X)$. As any valuation satisfies that $\nu(fg) = \nu(f) + \nu(g)$, the set

$$S = S(\nu, \mathcal{L}) = \bigcup_{m \geq 0} \{(m, \nu(f/h^m)) \mid f \in L^m \setminus \{0\}\}$$

is a semigroup with identity (i.e. a monoid). If S is finitely generated, one can construct a toric degeneration whose special fiber is a toric variety $\operatorname{Proj} \mathbb{C}[S]$ (which is normal if S is saturated). Moreover we obtain an Okounkov body

$$\Delta = \Delta(S) = \overline{\operatorname{conv}\Big(\bigcup_{m>0} \{x/m \mid (m, x) \in S\}\Big)} \subset \mathbb{R}^n.$$

Note that if S is finitely generated, then Δ is a rational convex polytope. The toric variety corresponding to Δ is the normalization of $\operatorname{Proj} \mathbb{C}[S]$.[4]

In the following theorem we rephrase several results from [16].

Theorem 6 ([16]) *Let $\mathcal{L}$ be a very ample Hermitian line bundle on a smooth n-dimensional projective variety X and $\omega = \Phi_{\mathcal{L}}^*(\omega_{FS})$ the induced symplectic form. Let $\nu\colon \mathbb{C}(X) \setminus \{0\} \to \mathbb{Z}^n$ be a valuation with one dimensional leaves, and such that the associated semigroup S is finitely generated. Then*

- *There exists a toric degeneration $\pi : \mathfrak{X} \to \mathbb{C}$ with generic fiber X and special fiber $X_0 := \operatorname{Proj} \mathbb{C}[S]$, and a Kähler structure $\tilde{\omega}$ on (the smooth part of) $\mathfrak{X}$ such that $(\pi^{-1}(1), \tilde{\omega}_{|\pi^{-1}(1)})$ is symplectomorphic to (X, ω) and the closure of the moment image of symplectic toric manifold $(U_0, \tilde{\omega}_{|U_0})$, where $U_0 := (X_0)_{smooth}$, is the Okounkov body $\Delta(S)$. The set U_0 contains the preimage of the interior of $\Delta(S)$.*
- *Moreover, there exists a surjective continuous map $\phi\colon X \to X_0$ that restricts to a symplectomorphism from $(\phi^{-1}(U_0), \omega)$ to $(U_0, \tilde{\omega}_{|U_0})$.*

In particular, if $X_0 = \operatorname{Proj} \mathbb{C}[S]$ built from S is smooth (thus also normal), then $\phi^{-1}(U_0) = X$ and therefore ϕ provides a symplectomorphism between (X, ω) and the symplectic toric manifold $(X_{\Delta(S)}, \omega_{\Delta(S)})$ associated to $\Delta(S)$ via Delzant's construction.

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[4] Recall that for a graded algebra $A = \oplus_{j=0}^{\infty} A_j$ the set Proj A is the set of homogeneous prime ideals in A that do not contain all of $A_+ := \oplus_{j=1}^{\infty} A_j$. The topology on Proj A is defined by setting the closed sets to be $V(I) := \{J;\ J \subset I$ is a homogeneous prime ideal of A not containing all of $A_+\}$, for some homogeneous ideal I of A. For more details see, for example [17, II.2], [9, III.2], and [5, Chap. 7].

Checking whether S is finitely generated is a very difficult problem. However, it was observed by Kaveh [20] that even if S is not finitely generated one can still form a (not flat) family with generic fiber X and special fiber $(\mathbb{C}^*)^n$. Even though such a construction provides much less information about X, it still suffices for the purpose of finding lower bounds on the Gromov width. We describe this idea in Sect. 13.3.

13.3 Gromov Width

The *Gromov width* of a $2n$-dimensional symplectic manifold (X, ω) is the supremum of the set of the positive real numbers a such that the ball of capacity a (radius $\sqrt{\frac{a}{\pi}}$),

$$B_a^{2n} = B^{2n}\Big(\sqrt{\frac{a}{\pi}}\Big) = \big\{(x_1, y_1, \dots, x_n, y_n) \in \mathbb{R}^{2n} \;\Big|\; \pi \sum_{i=1}^{n}(x_i^2 + y_i^2) < a\big\} \subset (\mathbb{R}^{2n}, \omega_{st}),$$

can be symplectically embedded in (X, ω). Here $\omega_{st} = \sum_j dx_j \wedge dy_j$ denotes the standard symplectic form on $\mathbb{R}^{2n}$. This question was motivated by the Gromov non-squeezing theorem which states that a ball $B^{2n}(r) \subset (\mathbb{R}^{2n}, \omega_{st})$ cannot be symplectically embedded into $B^2(R) \times \mathbb{R}^{2n-2} \subset (\mathbb{R}^{2n}, \omega_{st})$ unless $r \leq R$.

J-holomorphic curves give obstructions to ball embeddings, while Hamiltonian torus actions can lead to constructions of such embeddings (by extending a Darboux chart using the flow of the vector field induced by the action).

This is why toric degenerations provide a useful tool for finding lower bounds on the Gromov width. Given a toric degeneration of (X, ω), as described in Theorem 6, one can use the toric action on X_0 to construct embeddings of balls into a smooth symplectic toric manifold $(U_0, \tilde{\omega}_{|U_0})$, where $U_0 = (X_0)_{\text{smooth}}$. Postcomposing such embedding with the symplectomorphism ϕ^{-1} produces a symplectic embedding into (X, ω).

Moreover, many embeddings of balls into symplectic toric manifolds can be read off from the associated (by the Delzant classification theorem) polytope. Identify the dual of the Lie algebra of the compact torus T with Euclidean space using the convention that $S^1 = \mathbb{R}/\mathbb{Z}$, i.e. the lattice of $\mathfrak{t}^*$ is mapped to $\mathbb{Z}^{\dim\, T} \subset \mathbb{R}^{\dim\, T}$. With this convention, the moment map for the standard $(S^1)^n$ action on $(\mathbb{R}^{2n}, \omega_{st})$ maps B_a^{2n} onto an n-dimensional simplex of size a, closed on n sides

$$\mathfrak{S}^n(a) := \left\{(x_1, \dots, x_n) \in \mathbb{R}^n \,|\, 0 \leq x_j < a,\; \sum_{j=1}^{n} x_j < a\right\}.$$

Moreover, if the moment image contains an open simplex of size a, then for any $\varepsilon > 0$ a ball of capacity $a - \varepsilon$ can be embedded into the given symplectic toric manifold: see [28, Lemma 5.3.1] and, independently, [26, Propositions 2.1 and 2.4].

Proposition 7 ([24, Proposition 1.3] and [26, Proposition 2.5]) *For any connected, proper (not necessarily compact) symplectic toric manifold U of dimension* $2n$*, with a momentum map* μ*, the Gromov width of U is at least*

$$\sup\{a > 0 \mid \exists\, \Psi \in \mathrm{GL}(n, \mathbb{Z}), x \in \mathbb{R}^n, \text{ such that } \Psi(\mathrm{int}\, \mathfrak{S}^n(a)) + x \subset \mu(U)\}.$$

The appearance of Ψ and x comes from the facts that the identification $\mathfrak{t}^* \cong \mathbb{R}^{\dim\ T}$ depends on a splitting of T into $(\dim\ T)$ circles, and that a translation of a moment map also provides a moment map.

The above results lead to the following method for finding lower bounds on the Gromov width.

Corollary 8 *Let X be a smooth projective variety of complex dimension n,* $\mathcal{L}$ *an ample line bundle on X, and* $\omega = \Phi_{\mathcal{L}}^*(\omega_{FS}) \in \mathrm{H}^2(X, \mathbb{Z})$ *an integral Kähler form obtained using the Kodaira embedding* $\Phi_{\mathcal{L}}\colon X \to \mathbb{P}(L^*)$*. Suppose that there exists a valuation* ν *giving a finitely generated and saturated semigroup* $S = S(\nu, \mathcal{L})$*. Let* Δ *be the associated Okounkov body. The Gromov width of* (X, ω) *is at least*

$$\sup\{a > 0 \mid \exists\, \Psi \in \mathrm{GL}(n, \mathbb{Z}), x \in \mathbb{R}^n, \text{ such that } \Psi(\mathrm{int}\, \mathfrak{S}^n(a)) + x \subset \Delta\}.$$

Proof By the result of [16] cited here as Theorem 6, there exists a toric degeneration of (X, ω) to a normal toric variety $X_0 = \mathrm{Proj}\, \mathbb{C}[S]$, and a surjective continuous map $\phi\colon X \to X_0$ whose appropriate restriction is a symplectomorphism. The subset $U := \phi^{-1}(U_0)$ of X inherits a toric action whose moment image contains int Δ, the interior of Δ (recall that a moment map sends singular points of a toric variety to the boundary of the moment polytope). The corollary follows from Proposition 7. □

In fact one does not need S to be saturated. The same corollary holds even if X_0 is not a normal toric variety. This is because a normalization map for X_0 induces a biholomorphism between $(X_0)_{\mathrm{smooth}}$ and an appropriate subset of the normalization of X_0.

It is, however, necessary that S is finitely generated for a toric degeneration to exist. Otherwise one can still form a family of manifolds, but one cannot guarantee that this family is flat, and thus X and X_0 are no longer so strongly related. As we already mentioned, Kaveh in [20] observed that such a (not necessarily flat) family, with $X_0 = (\mathbb{C}^*)^n$, still provides information about the Gromov width of (X, ω). To state this result we need additional notation. In the notation of Sect. 13.2, for any $m \in \mathbb{Z}_{>0}$ let

$$\mathcal{A}_m := \{\nu(f/h^m) \mid f \in L^m \setminus \{0\}\} \subset \mathbb{Z}^n, \quad \Delta_m = \frac{1}{m} \operatorname{conv}(\mathcal{A}_m).$$

Note that $\Delta = \overline{\cup_{m>0} \Delta_m}$. Fix m and let $r = r_m$ denote the number of elements in $\mathcal{A}_m = \{\beta_1, \ldots, \beta_r\}$. From these data we form a symplectic form, ω_m, on $(\mathbb{C}^*)^n$ using a standard procedure: ω_m is the pull back of the Fubini–Study form on $\mathbb{CP}^{r-1}$ via the map $\Psi_m\colon (\mathbb{C}^*)^n \to \mathbb{CP}^{r-1}, u \mapsto (u^{\beta_1} c_1, \ldots, u^{\beta_r} c_r)$, where $c = [(c_1, \ldots, c_r)]$ is some element in $\mathbb{CP}^{r-1}$ with all $c_i \neq 0$. (In [20] the elements c_i come from coefficients of leading terms of elements in appropriately chosen basis of L^m. One also needs that the differences of elements in $\mathcal{A}_m$ span $\mathbb{Z}^n$, which, by [20, Remark 5.6], is always true for lowest term valuations.)

Kaveh proved that:

1. for every $m > 0$ there exists an open subset $U \subset X$ such that (U, ω) is symplectomorphic to $((\mathbb{C}^*)^n, \frac{1}{m}\omega_m)$ [20, Theorem 10.5];
2. the Gromov width of $((\mathbb{C}^*)^n, \frac{1}{m}\omega_m)$ is at least R_m, where R_m is the size of the largest open simplex that fits in the interior of $\Delta_m = \frac{1}{m}\operatorname{conv}(\mathcal{A}_m)$ [20, Corollary 12.3].

This leads to the following corollary.

Corollary 9 *([20, Corollary 12.4]) Let X be a smooth projective variety of dimension n, $\mathcal{L}$ an ample line bundle on X, and $\omega = \Phi^*_{\mathcal{L}}(\omega_{FS}) \in H^2(X, \mathbb{Z})$ an integral Kähler form. Let v be a lowest term valuation on $\mathbb{C}(X)$, with values in $\mathbb{Z}^n$, and Δ the associated Okounkov body. The Gromov width of (X, ω) is at least R, where R is the size of the largest open simplex that fits in the interior of Δ.*

13.3.1 Results About Coadjoint Orbits

The methods for finding the Gromov width described in Corollaries 8 and 9 have been used in [11, 14] for coadjoint orbits of compact Lie groups.

Recall that given a compact Lie group K each orbit $\mathcal{O} \subset \mathfrak{k}^* := (\operatorname{Lie} K)^*$ of the coadjoint action of K on $\mathfrak{k}^*$ is naturally a symplectic manifold. Namely it can be equipped with the Kostant–Kirillov–Souriau symplectic form ω^{KKS} defined by:

$$\omega^{KKS}_{\xi}(X^\#, Y^\#) = \langle \xi, [X, Y] \rangle, \quad \xi \in \mathcal{O} \subset \mathfrak{k}^*, \; X, Y \in \mathfrak{k},$$

where $X^\#, Y^\#$ are the vector fields on $\mathfrak{k}^*$ induced by $X, Y \in \mathfrak{k}$ via the coadjoint action of K. For more details see, for example, [7, Sect. 21.5, Homeworks 16 and 17]. Coadjoint orbits are in bijection with points in a positive Weyl chamber as every coadjoint orbit intersects such a chamber in a single point. An orbit is called generic (resp. degenerate) if this intersection point is an interior point of the chamber (resp. a boundary point). For example, when $K = \mathrm{U}(n, \mathbb{C})$ is the unitary group, a coadjoint orbit can be identified with the set of Hermitian matrices with a fixed set of eigenvalues. The orbit is generic if all eigenvalues are different, and in this case it is diffeomorphic to the manifold of complete flags in $\mathbb{C}^n$.

It has been unofficially conjectured[5] that the Gromov width of $(\mathcal{O}_\lambda, \omega^{KKS})$ of K, through a point λ in a positive Weyl chamber should be given by the following neat

[5] During the work on the project [18], about complex Grassmannians, Karshon and Tolman looked at various examples of other coadjoint orbits and got the impression that the above value might be the Gromov width of all coadjoint orbits. They never formulated this expectation formally as their

formula, expressed entirely in the Lie-theoretical language

$$\min\{\,\left|\left\langle\lambda,\alpha^\vee\right\rangle\right|;\ \alpha^\vee \text{ a coroot and } \left\langle\lambda,\alpha^\vee\right\rangle \neq 0\}.$$

For example, as $\{e_{ii}-e_{jj};\ i\neq j\}$ forms a root system for the unitary group $U(n,\mathbb{C})$, the Gromov width of its coadjoint orbit O_λ passing through a point

$$\lambda = \mathrm{diag}(\lambda_1,\dots,\lambda_n)\in\mathfrak{u}(n)^*,$$

integral up to scaling, is equal to $\min\{|\lambda_i-\lambda_j|;\ i,j\in\{1,\dots,n\},\ \lambda_i\neq\lambda_j\}$. Here we identified $\mathfrak{u}(n)$ and $\mathfrak{u}(n)^*$ with the set of $n\times n$ Hermitian matrices.

This conjecture was motivated by the computation of the Gromov width of complex Grassmannians, i.e. degenerate coadjoint orbits of $U(n,\mathbb{C})$, done by Karshon and Tolman [18], and independently by Lu [23]. Later, using holomorphic techniques, Zoghi [29] showed that the above formula provides an upper bound for the Gromov width for generic indecomposable[6] orbits of $U(n,\mathbb{C})$. This result was generalized to all coadjoint orbits by Caviedes [2]. The fact that this formula also provides a lower bound was proved using explicit Hamiltonian torus actions by several authors: [29] gives a proof for generic indecomposable orbits of $U(n,\mathbb{C})$ using the standard action of the maximal torus, Lane [21] proves this lower bound for generic orbits of the exceptional group G_2, and [26] settled the case of $U(n,\mathbb{C})$, $SO(2n,\mathbb{C})$ and $SO(2n+1,\mathbb{C})$ orbits[7] using the Gelfand–Tsetlin torus action.

13.3.2 A Sketch of the Proof of Theorem 1

The first usage of toric degenerations in Gromov width problems appeared in [14], where the generic orbits of the symplectic group $Sp(n)=U(n,\mathbb{H})$ are considered. Then it was used in [11] to prove that the formula (13.1) is a lower bound for the Gromov width of any coadjoint orbit of any compact connected simple Lie group K, passing through a point in the Weyl chamber, integral up to scaling, i.e. to prove Theorem 1.

The rationality assumption comes from the fact that the toric degeneration method can be applied only to the orbits passing through an integral point λ of a positive Weyl

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conjecture, but they shared this idea with other mathematicians in private communications. This is how this value became to be known as the expected Gromov width for coadjoint orbits.

[6] A coadjoint orbit through a point λ in the interior of a chosen positive Weyl chamber is called indecomposable in [29] if there exists a simple positive root α such that for any positive root α' there exists a positive integer k such that $\langle\lambda,\alpha'\rangle = k\langle\lambda,\alpha\rangle$.

[7] The result about $SO(2n+1,\mathbb{C})$ holds only for orbits satisfying a mild technical condition: the point λ of intersection of the orbit and a chosen positive Weyl chamber should not belong to a certain subset of one wall of the chamber; see [26] for more details. In particular, all generic orbits satisfy this condition.

chamber, i.e., in the language of representation theory language, through a dominant weight.

Let G be a simply connected simple complex algebraic group and $K \subset G$ be its maximal compact subgroup. With a dominant weight λ one can associate an irreducible representation $V(\lambda)$ of G of highest weight λ. Let $\mathbb{C}_{v_\lambda}$ be the highest weight line and $P = P_\lambda$ be the normalizer in G of this line. Then the coadjoint orbit $\mathcal{O}_\lambda$ of K is diffeomorphic to G/P (and to $K/K \cap P$) and there exists a very ample line bundle $\mathcal{L}_\lambda$ on G/P such that the pull back of the Fubini–Study form on the projective space $\mathbb{P}(\mathrm{H}^0(G/P, \mathcal{L}_\lambda)^*) = \mathbb{P}(V(\lambda))$ via the Kodaira embedding $G/P \to \mathbb{P}(\mathrm{H}^0(G/P, \mathcal{L}_\lambda)^*)$ is exactly the Kostant–Kirillov–Souriau symplectic form ω^{KKS} on $\mathcal{O}_\lambda$ (see for example [2, Remark 5.5]). Thus for integral λ's one can try to construct toric degenerations of projective variety G/P with line bundle $\mathcal{L}_\lambda$ and obtain some lower bounds for the Gromov width of the orbit $\mathcal{O}_\lambda$. Rescaling of symplectic forms allows to extend such a result to orbits $\mathcal{O}_{a\lambda}$, for any $a \in \mathbb{R}_{>0}$.

It remains to discuss how one can construct desired toric degenerations.

A great advantage of working with coadjoint orbits of a complex algebraic group G is that a lot of information can be obtained from studying representations of G. This leads to a beautiful interplay between symplectic geometry and representation theory. A reduced decomposition of the longest word in the Weyl group, $\underline{w}_0 = s_{i_{\alpha_1}} \cdot \ldots \cdot s_{i_{\alpha_N}}$ provides the following items (defined below) related in an interesting way:

1. a valuation $\nu_{\underline{w}_0}$;
2. a string parameterization of a crystal basis of V_λ^*.

We continue to denote by λ a dominant weight (i.e. an integral element in a positive Weyl chamber of $\mathfrak{g}^*$) and by V_λ the finite dimensional irreducible representation of G with highest weight λ. Let V_λ^* denote the dual representation. One often seeks for a basis of V_λ^* consisting of elements which behave nicely under the action of Kashiwara operators. A crystal basis is a basis whose elements are permuted under the Kashiwara operators. Given a crystal basis one can form a crystal graph of a given representation: vertices are elements of the crystal basis and $\{0\}$, and edges are labelled by simple roots and correspond to the action of Kashiwara operators. There are (not canonical) ways of embedding such graph into $\mathbb{R}^N$, $N = \dim_{\mathbb{C}} G/P$. A reduced decomposition of the longest word in the Weyl group (into a composition of reflections with respect to simple roots), $\underline{w}_0 = s_{\alpha_1} \cdot \ldots \cdot s_{\alpha_N}$, provides a way of assigning to each vertex of the crystal graph a string of N integers (string parametrization), and thus gives such an embedding. A convex hull of the image of string parametrization is called a string polytope. It depends on λ and also on the chosen decomposition $\underline{w}_0$. String polytopes have been extensively studied in representation theory.

Moreover, a reduced decomposition $\underline{w}_0 = s_{i_{\alpha_1}} \cdot \ldots \cdot s_{i_{\alpha_N}}$ defines a sequence of Schubert subvarieties

$$[P] = Y_N \subset \ldots \subset Y_0 = G/P,$$

where Y_j denotes the Schubert variety corresponding to element $s_{i_{\alpha_{j+1}}} \cdot \ldots \cdot s_{i_{\alpha_N}}$ of the Weyl group. We denote by $\nu_{\underline{w}_0}$ the highest term valuation associated with this flag of subvarieties.

A theorem of Kaveh relates these two objects.

Theorem 10 ([19]) *The string parametrization for* $V_\lambda^* = \mathrm{H}^0(G/P, \mathcal{L}_\lambda)$ *obtained using the reduced decomposition* $\underline{w}_0$ *is the restriction of the valuation* $\nu_{\underline{w}_0}$ *and thus the corresponding string polytope is the Okounkov body* $\Delta(\nu_{\underline{w}_0})$.

Detailed computations for the case of $G = SL(3, \mathbb{C})$ and $\underline{w}_0 = s_1 s_2 s_2$ are presented in [19, Sect. 5].

Explicit descriptions of string polytopes for classical Lie groups and some well-chosen reduced decompositions of the longest words were presented in the work of Littelmann [22]. With a bit of work one can show that the string polytope for V_λ^* with $G = \mathrm{Sp}(2n, \mathbb{C})$ the symplectic group (with maximal compact subgroup $K = \mathrm{Sp}(n) = \mathrm{U}(n, \mathbb{H})$), described in [22], contains a simplex of size prescribed by (13.1). Then, the result of Kaveh, [19], quoted above together with Corollary 8 prove that the Gromov width of $\mathrm{Sp}(n)$ coadjoint orbit $(\mathcal{O}_\lambda, \omega^{KKS})$ is at least equal to the value prescribed by (13.1), i.e. proves Theorem 1 for the case of the symplectic group. This is exactly the argument used in [14].

Similar methods could be applied for other classical Lie groups. However, one would need to consider each type separately, as the descriptions of string polytopes contained in [22] depend on reduced decompositions which are different for different group types.

To obtain a unified proof which works for all group types, in [11] we use lowest term valuations ν arising from a system of parameters induced by an enumeration $\{\beta_1, \ldots, \beta_N\}$ of certain positive roots, also in the cases where this enumeration does not come from a reduced decomposition of the longest word. In these cases it might be very difficult to show that the associated semigroup $S(\nu)$ is finitely generated (if it is) and to find an explicit description of the associated Okounkov body. Moreover, on the representation theory side, we do not have a natural way of obtaining a string parametrization of a crystal basis of V_λ^* from such enumerations. However, in [10] the authors managed to give a representation-theoretic description of the associated semigroup $S(\nu)$ in the case when the enumeration is a good ordering in the sense of [10]. Here is the main idea. Given such enumeration one can define for each $\alpha \in \mathbb{Z}^N_{\geq 0}$ subspaces $V(\lambda)_{\leq \alpha}$ and $V(\lambda)_{<\alpha}$ of $V(\lambda)$. An element $\alpha \in \mathbb{Z}^N_{\geq 0}$ is called essential for λ if $\dim V(\lambda)_{\leq\alpha}/V(\lambda)_{<\alpha} = 1$. It was proved in [10] that the set $\{(l, \alpha);\ l \in \mathbb{Z}_{\geq 0},\ \alpha \text{ essential for } l\lambda\}$ is a semigroup which coincides with $S(\nu)$. Moreover, building on other results from [10] concerning essential elements, one can show that the Okounkov body associated to $S(\nu)$ contains a simplex of size prescribed by (13.1). Then, using the result of [20] cited here as Corollary 9 (which does not require the semigroup to be finitely generated), one proves Theorem 1. The details of this argument are presented in [11].

13.4 Cohomological Rigidity

The following section is based on a project joint with Sue Tolman [27].

Cohomological rigidity problems are problems where one tries to determine whether the integral cohomology ring can distinguish between manifolds of certain family, and whether all isomorphisms between integral cohomology rings are induced by maps (homeomorphisms or diffeomorphisms, depending on the setting) between manifolds. The integral cohomology ring is too weak to distinguish a homeomorphism type of a manifold. However, by a result of Freedman, it classifies (up to a homeomorphism) all closed, smooth, simply connected 4-manifolds. Masuda and Suh posed a question of whether the cohomological rigidity holds for the family of symplectic toric manifolds. The question was studied by its authors, Choi, and Panov. No counterexample was found and partial positive results were proved. (Interested reader is encouraged to consult a nice survey [4] and references therein.) Due to the presence of symplectic structure, it seems natural to consider the following symplectic variant of the above question.

Question 11 (*Symplectic cohomological rigidity for symplectic toric manifolds*)

1. (weak) Are two symplectic toric manifolds (M, ω_M) and (N, ω_N) necessarily symplectomorphic whenever there exists an isomorphism $F\colon \mathrm{H}^*(M;\mathbb{Z}) \to \mathrm{H}^*(N;\mathbb{Z})$ sending the class $[\omega_M]$ to the class $[\omega_N]$?
2. (strong) Is any such isomorphism $F\colon \mathrm{H}^*(M;\mathbb{Z}) \to \mathrm{H}^*(N;\mathbb{Z})$ induced by a symplectomorphism?

In [27] it is shown that weak and strong symplectic cohomological rigidity hold for the family of Bott manifolds with rational cohomology ring isomorphic to that of a product of copies of $\mathbb{CP}^1$. Bott manifolds can be viewed as higher dimensional generalizations of Hirzebruch surfaces discussed in the example below. For the definition see Sect. 13.4.2.

Remark 12 Strong (not symplecic) cohomological rigidity, with diffeomorphisms, was already proved for this family by Choi and Masuda [3]. Their diffeomorphisms usually do not preserve the complex structure. If they had, then our result would be an immediate consequence of theirs. Indeed, if $f\colon N \to M$ is a biholomorphism inducing F, then ω_N and $f^*(\omega_M)$ are both Kähler forms on N, defining the same cohomology class in $\mathrm{H}^*(N;\mathbb{Z})$, and thus in this case (N, ω_N) and $(N, f^*(\omega_M))$ are symplectomorphic by the Moser's trick.

Example 13 (*Hirzebruch surfaces*) Hirzebruch surfaces are $\mathbb{CP}^1$ bundles over $\mathbb{CP}^1$. As complex manifolds they are classified by integers (encoding the twisting of the bundle): for each $A \in \mathbb{Z}$ we denote by $\mathcal{H}_{-A}$ the bundle $\mathbb{P}(\mathcal{O}(A) \oplus \mathcal{O}(0)) \to \mathbb{CP}^1$. In particular, $\mathcal{H}_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$. They can be equipped with a symplectic (even Kähler) structure and a toric action. A polytope corresponding to $\mathcal{H}_{-A}$ in Delzant classification is (up to $\mathrm{GL}(2,\mathbb{Z})$ action) a trapezoid with outward normals $(-1,0)$, $(0,-1)$, $(1,0)$, $(A,1)$. The lengths of the edges of this trapezoid depend on the chosen symplectic structure and can be encoded in $\lambda = (\lambda_1, \lambda_2) \in (\mathbb{R}_{>0})^2$. We denote by $(\mathcal{H}_{-A}, \omega_\lambda)$ the symplectic toric manifold corresponding to the trapezoid $\Delta(A,\lambda) := \mathrm{conv}((0,0), (\lambda_1, 0), (\lambda_1, \lambda_2 - A\lambda_1), (0, \lambda_2))$. For example, Fig. 13.1 presents $(\mathcal{H}_0, \omega_{(1,3)})$ and $(\mathcal{H}_{-2}, \omega_{(1,5)})$.

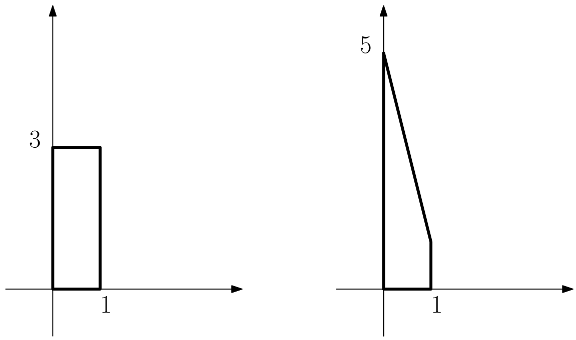

Fig. 13.1 Hirzebruch surfaces $(\mathcal{H}_0, \omega_{(1,3)})$ and $(\mathcal{H}_{-2}, \omega_{(1,5)})$

It was observed by Hirzebruch that $\mathcal{H}_{-A}$ and $\mathcal{H}_{-\widetilde{A}}$ are diffeomorphic if and only if $A \cong \widetilde{A} \mod 2$. Moreover, the symplectic toric manifolds $(\mathcal{H}_{-A}, \omega_\lambda)$ and $(\mathcal{H}_{-\widetilde{A}}, \omega_{\widetilde{\lambda}})$ are (not equivariantly) symplectomorphic if and only if $A \cong \widetilde{A} \mod 2$ and the widths and the areas of the associated polytopes agree, i.e. $\lambda_1 = \widetilde{\lambda}_1$ and $\lambda_2 - \frac{1}{2}A\lambda_1 = \widetilde{\lambda}_2 - \frac{1}{2}\widetilde{A}\widetilde{\lambda}_1$. For example, the manifolds presented on Fig. 13.1 are symplectomorphic. The cohomology ring can be presented as

$$\mathrm{H}^*(\mathcal{H}_{-A}; \mathbb{Z}) = \mathbb{Z}[x_1, x_2]/\langle x_2^2, x_1^2 + Ax_1x_2\rangle,$$

with $[\omega_\lambda] = \lambda_1 x_1 + \lambda_2 x_2$. If $A \cong \widetilde{A} \mod 2$, then the isomorphism $\mathbb{Z}[x_1, x_2] \to \mathbb{Z}[\widetilde{x}_1, \widetilde{x}_2]$ defined by $x_1 \mapsto \widetilde{x}_1 + \frac{1}{2}(\widetilde{A} - A)\widetilde{x}_2$, $x_2 \mapsto \widetilde{x}_2$ descends to an isomorphism between $\mathrm{H}^*(\mathcal{H}_{-A}; \mathbb{Z})$ and $\mathrm{H}^*(\mathcal{H}_{-\widetilde{A}}; \mathbb{Z})$. Note that this isomorphism sends $[\omega_\lambda] = \lambda_1 x_1 + \lambda_2 x_2$ to $\lambda_1 \widetilde{x}_1 + (\lambda_2 + \frac{\widetilde{A}-A}{2}\lambda_1)\,\widetilde{x}_2$ which is equal to $[\omega_{\widetilde{\lambda}}]$ if and only if $\lambda_1 = \widetilde{\lambda}_1$ and $\lambda_2 - \frac{A}{2}\lambda_1 = \widetilde{\lambda}_2 - \frac{\widetilde{A}}{2}\widetilde{\lambda}_1$. Therefore, for Hirzebruch surfaces (weak) symplectic cohomological rigidity holds.

To approach symplectic cohomological rigidity problem one needs a good method of constructing symplectomorphisms. Here is where toric degenerations come into play. By Theorem 6 a toric degeneration whose central fiber Proj $\mathbb{C}[S]$ is smooth produces a symplectomorphism between the symplectic manifold one started with and the central fiber. The main difficulty in this method of constructing symplectomorphisms lies in finding toric degenerations with smooth central fibers.

A great advantage of working with toric manifolds is that the sections of their line bundles are well understood and one can form very concrete constructions of toric degenerations.

13.4.1 Toric Degenerations for Symplectic Toric Manifolds

Let (X_P, ω_P) be a symplectic toric manifold with $\omega_P \in \mathrm{H}^2(M, \mathbb{Z})$, corresponding to a polytope $P \subset \mathbb{R}^n$ via Delzant construction. Then P is an integral polytope (i.e.

with vertices in $\mathbb{Z}^n$) and there exists a very ample line bundle $\mathcal{L}$ over X_P inducing ω_P. In this situation a basis of the space of holomorphic sections of $\mathcal{L}$ can be identified with the integral points of P, ([6], see also [15]). Without loss of generality we can assume that P in a neighborhood of some vertex looks like $(\mathbb{R}_{\geq 0})^n$ in a neighborhood of the origin in $\mathbb{R}^n$. Then we can identify $L = \mathrm{H}^0(X_P, \mathcal{L})$ with a subset of the ring of rational functions, $\mathbb{C}(X_P)$, as described on Sect. 13.2, using the section corresponding to the origin as the fixed element h:

$$f \mapsto \frac{f}{\text{section corresponding to the origin}}.$$

Note 14 For simplicity of notation, given a valuation ν we will write $\nu(L)$ to denote

$$\nu(L) := \{\nu(f/h);\ f \in L \setminus \{0\}\}.$$

Similarly, let $\nu(L^m) := \{\nu(f/h^m);\ f \in L^m \setminus \{0\}\}$ for any $m > 1$.

We denote by $f_j \in \mathbb{C}(X_P)$ the rational function coming from the section corresponding to the j-th basis vector, $j = 1, \ldots, n$. Note that $f_1, \ldots, f_n$ form a coordinate system around the fixed point of X_P corresponding to the origin via the moment map. To see this, one can, for example, use the description of X_P and f_j's from [15].

Choose and fix a non-negative integer c and two elements $k < l \in \{1, \ldots, n\}$. Then

$$\{u_1 = f_1, \ldots, u_{k-1} = f_{k-1}, u_k = f_k - f_l^c, u_{k+1} = f_{k+1}, \ldots, u_n = f_n\}$$

also gives a coordinate system. Let ν be the associated lowest term valuation (as in Example 4). The image $\nu(L)$ can be obtain by using a "sliding" operator $\mathcal{F}_{-e_k+ce_l}$, defined as follows. For each affine line ℓ in $\mathbb{R}^n$ in the direction of $-e_k + ce_l$, with $P \cap \ell \cap \mathbb{Z}^n \neq \emptyset$, translate the set $\{P \cap \ell \cap \mathbb{Z}^n\}$ by $a(-e_k + ce_l)$ with $a \geq 0$ maximal non-negative number for which $a(-e_k + ce_l) + \{P \cap \ell \cap \mathbb{Z}^n\} \subset (\mathbb{R}_{\geq 0})^n$.

Lemma 15 *One obtains $\nu(L)$ by sliding the integral points of P in the direction $-e_k + ce_l$, inside $(\mathbb{R}_{\geq 0})^n$, i.e.*

$$\nu(L) = \mathcal{F}_{-e_k+ce_l}(P \cap \mathbb{Z}^n).$$

Instead of the proof, which can be found in [27], we give the following example which ilustrates the main idea behind the proof.

Example 16 Let (X_P, ω_P) be the symplectic toric manifold corresponding to the polytope $P = \operatorname{conv}\{(0,0), (1,0), (1,3), (0,3)\} \subset \mathbb{R}^2$. That is, X_P is diffeomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$ with product symplectic structure (with different rescaling of the Fubini–Study symplectic form on each factor). Let ν be the lowest term valuation associated to the coordinate system

$$u_1 = f_1 - f_2^2,\ u_2 = f_2.$$

Line $\{(0,2)+t(1,-2);\ t\in\mathbb{R}\}$ intersects P in two integral points: $(1,0)$ and $(0,2)$. The corresponding functions are f_1 and f_2^2, and one can easily calculate that

$$\nu(f_1)=\nu(f_2^2)=(0,2)\quad\text{and}\quad \nu(f_1-f_2^2)=(1,0).$$

Similarly, using the integral points on the line $\{(0,3)+t(1,-2);\ t\in\mathbb{R}\}$ we obtain

$$\nu(f_1f_2)=\nu(f_2^3)=(0,3)\quad\text{and}\quad \nu(f_1f_2-f_2^3)=\nu((f_1-f_2^2)f_2)=(1,1).$$

More generally, if the integral points $(a,b),(a,b)+(1,-2),\dots,(a,b)+m(1,-2)$ are in P (implying that $b-2m>0$), then one can use the corresponding functions to construct functions with valuations $(0,b+2a),(0,b+2a)+(1,-2),\dots,(0,b+2a)+m(1,-2)=(m,2a+b-2m)$. Precisely, for any $l=0,\dots,m$

$$f_1^af_2^{b-2l}(f_1-f_2^2)^l=\sum_{j=0}^{l}(-1)^{l-j}f_1^{a+j}f_2^{b-2j}\quad\text{and}$$
$$\nu(f_1^af_2^{b-2l}(f_1-f_2^2)^l)=(l,2a+b-2l).$$

This proves that $\nu(L)\supset\mathcal{F}_{(-1,2)}(P\cap\mathbb{Z}^2)$. By [16, Proposition 3.4] the cardinality of $\nu(L)$ is the dimension of L, that is, the number of integral points in P. Therefore

$$\nu(L)=\mathcal{F}_{(-1,2)}(P\cap\mathbb{Z}^2).$$

The polytopes P and $\operatorname{conv}(\nu(L))$ are presented in Fig. 13.1.

Understanding $\nu(L)$ is not enough for constructing and understanding a toric degeneration. First of all, to construct a flat family with toric fiber $\pi^{-1}(0)$ one needs the associated semigroup $S=S(\nu)$ to be finitely generated. Additionally, this toric fiber $\pi^{-1}(0)=\operatorname{Proj}\mathbb{C}[S]$ is the toric variety associated to the Okounkov body Δ if $\operatorname{Proj}\mathbb{C}[S]$ is normal, that is, if S is saturated. Moreover, to describe the Okounkov body one also needs to find $\nu(L^m)$ for $m>1$. Note that in general L^m differs from $\mathrm{H}^0(X,\mathcal{L}^{\otimes m})$. The following proposition describes an especially nice situation where all these conditions simplify.

Proposition 17 *Let $(X,\omega=\Phi_{\mathcal{L}}^*(\omega_{FS}))$ be a $2n$ dimensional projective symplectic toric manifold associated to a smooth polytope P, with the projective embedding induced by a very ample line bundle $\mathcal{L}$. Let ν be a lowest term valuation associated to a coordinate system of the type presented on Sect. 13.4.1, and S the induced semigroup. Assume that there exists a smooth integral polytope $\Delta\subset\mathbb{R}^n$ such that*

$$S=(\operatorname{cone}(\{1\}\times\Delta))\cap(\mathbb{Z}\times\mathbb{Z}^n).$$

Then (X,ω) is symplectomorphic to the symplectic toric manifold (X_Δ,ω_Δ) associated to Δ via Delzant construction.

Here cone $(\{1\} \times \Delta)$ denotes the set $\{(t, tx);\, x \in \Delta, t \in \mathbb{R}_+\} \subset \mathbb{R}^{n+1}$.

Proof *(sketch)* The assumptions imply that the semigroup S is saturated and (by Gordan's Lemma) finitely generated. Therefore there is a toric degeneration $(\mathfrak{X}, \tilde{\omega})$ with generic fiber (X, ω) and the special fiber $\pi^{-1}(0) = \operatorname{Proj} \mathbb{C}[S]$ which is a normal toric variety. Moreover, the Okounkov body associated to the semigroup S is precisely Δ and therefore $\operatorname{Proj} \mathbb{C}[S]$ equipped with the restriction of $\tilde{\omega}$, is the symplectic toric manifold $(X_\Delta, \omega_\Delta)$ associated to Δ via Delzant construction. □

Note that S=(cone $\Delta) \cap (\mathbb{Z} \times \mathbb{Z}^n)$ imply, in particular, that $\nu(L^m)$ contains "enough" of integral points, namely that

$$\forall\, m \geq 1 \;\; \nu(L^m) = m\,\Delta \cap \mathbb{Z}^n = \operatorname{conv}(\nu(L^m)) \cap \mathbb{Z}^n.$$

To understand better the requirement $\operatorname{conv}(\nu(L^m)) \cap \mathbb{Z}^n = \nu(L^m)$, consider the following example.

Example 18 (*"Enough" of integral points and saturation*) Let (X_P, ω_P) be the symplectic toric manifold corresponding to the polytope

$$P = \operatorname{conv}\{(0,0), (2,0), (2,2), (0,2)\} \subset \mathbb{R}^2,$$

that is, X_P is diffeomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$ as in the previous example, but the symplectic form is different. As before, let ν be the lowest term valuation associated to the coordinate system

$$u_1 = f_1 - f_2^2,\; u_2 = f_2.$$

Then

$$\begin{aligned}\nu(L) &= \mathcal{F}_{(-1,2)}(P \cap \mathbb{Z}^2)\\ &= \{(0, j);\, j = 0, \dots, 6\} \cup \{(1,0), (1,2)\} \subsetneq \operatorname{conv}(\nu(L)) \cap \mathbb{Z}^2.\end{aligned}$$

In fact $\operatorname{conv}(\nu(L))$ is exactly the associated Okounkov body $\Delta(S(\nu))$. Indeed, $\Delta(S(\nu))$ must contain $\operatorname{conv}(\nu(L))$. Moreover, $2!\, vol_2(\Delta(S(\nu)))$ is the degree of the Kodaira embedding $\Phi_{\mathcal{L}}\colon X_P \to \mathbb{P}(L^*)$ induced by the line bundle $\mathcal{L}$ corresponding to ω_P [16, Theorem 3.9]. Thus the area of $\Delta(S(\nu))$ must be equal to the area of P, which in this case is also the area of $\operatorname{conv}(\nu(L))$. Therefore, in our example, $\nu(L)$ is "missing" the point $(1,1)$ in a sense that $\nu(L) = \Delta(S(\nu)) \cap \mathbb{Z}^2 \setminus \{(1,1)\}$, and thus $(1,1,1) \notin S(\nu)$. However, the line $\{t(1,1,1);\; t \in \mathbb{R}_+\}$ intersects $S(\nu)$: $(2,2,2) = \big(2, \nu(f_1(f_1 - f_2^2) \cdot (f_1 - f_2^2))\big) \in \{2\} \times \nu(L^2)$. Therefore the semigroup $S(\nu)$ is not saturated.

Let us analyse why in the above example the point $(1,1)$ is missing. Observe that the parallel lines $\ell_1 := \{(0,2) + t(-1,2);\; t \in \mathbb{R}\}$, $\ell_2 := \{(0,3) + t(-1,2);\; t \in \mathbb{R}\}$ and $\ell_3 := \{(0,4) + t(-1,2);\; t \in \mathbb{R}\}$ intersect P at intervals of the same length but with, respectively, 2, 1 and 2 integral points. Therefore the intersections of ℓ_1, ℓ_2

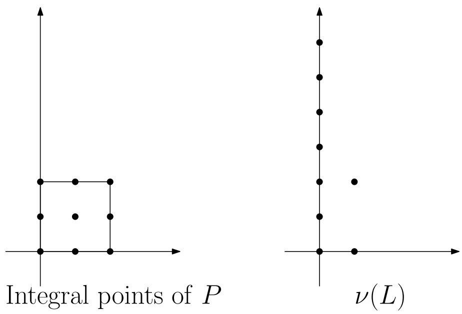

Fig. 13.2 Illustration of Example 18

and ℓ_3 with $\nu(L) = \mathcal{F}_{(-1,2)}(P \cap \mathbb{Z}^2)$ also contain, respectively, 2, 1 and 2 integral points. As a result, the points $(1, 0)$ and $(1, 2)$ are in $\nu(L)$, but $(1, 1)$ is not. The following condition is sufficient, though not necessary, to guarantee that we do not encounter that problem and have enough of integral points (Fig. 13.2).

Lemma 19 *Let* $\lambda_1, \lambda_2, c \in \mathbb{Z}_{>0}$ *and*

$$\Delta = \left\{p \in \mathbb{R}^2 \;\middle|\; 0 \le \langle p, e_1\rangle \le \lambda_1,\ 0 \le \langle p, e_2\rangle \text{ and } \left\langle p, e_2 + Ae_1\right\rangle \le \lambda_2\right\}.$$

Assume that

$$\lambda_2 - c\lambda_1 > 0.$$

Then the polytope conv $\mathcal{F}_{(-1,c)}(\Delta \cap \mathbb{Z}^2)$ *is a trapezoid of the same area as* Δ *and*

$$(\text{conv } \mathcal{F}_{(-1,c)}(\Delta \cap \mathbb{Z}^2)) \cap \mathbb{Z}^2 = \mathcal{F}_{(-1,c)}(\Delta \cap \mathbb{Z}^2).$$

If $c \le A$ then conv $\mathcal{F}_{(-1,c)}(\Delta \cap \mathbb{Z}^2)$ is simply Δ, and if $c > A$ then it is

$$\left\{p \in \mathbb{R}^2 \;\middle|\; 0 \le \langle p, e_1\rangle \le \lambda_1,\ 0 \le \langle p, e_2\rangle \text{ and } \left\langle p, e_2 + (2c - A)e_1\right\rangle \le \lambda_2 + (c - A)\lambda_1)\right\}.$$

An example is illustrated in Fig. 13.3.

13.4.2 Cohomological Rigidity for Bott Manifolds

A Bott manifold is a manifold obtained as a total space of a tower of iterated bundles with fiber $\mathbb{CP}^1$ and the first base space $\mathbb{CP}^1$. Such a manifold naturally carry an algebraic torus action, and can be viewed as a toric manifold. Note that 4-dimensional Bott manifolds are exactly the Hirzebruch surfaces discussed in Example 13. For more information about Bott manifolds see, for example, [13].

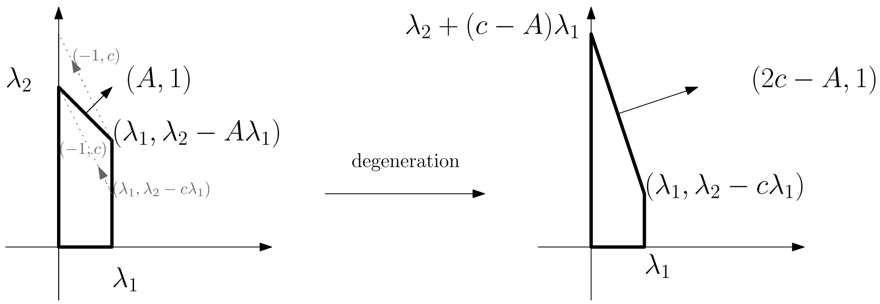

Fig. 13.3 Toric degeneration of a Hirzebruch surface

The simplest example of an $2n$-dimensional Bott manifold is the product of n copies of $\mathbb{CP}^1$'s. Equipped with a product symplectic structure $\omega = \pi_1^*(a_1\omega_{FS}) + \ldots + \pi_n^*(a_n\omega_{FS})$, for some $a_j \in \mathbb{R}_{>0}$, and the standard toric action[8] it becomes a symplectic toric manifold, whose Delzant polytope is a product of intervals, with lengths depending on a_j's. Here $\pi_j \colon \mathbb{CP}^1 \times \ldots \times \mathbb{CP}^1 \to \mathbb{CP}^1$ denotes the projection onto the j-th factor, and ω_{FS} stands for the Fubini–Study symplectic form. In particular, if all a_j's are equal, then the moment image is a hypercube.

A moment image for a general $2n$-dimensional Bott manifold is combinatorially an n-dimensional hypercube. By applying a translation and a $\mathrm{GL}(n, \mathbb{Z})$ transformation one can always arrange that the moment image is a polytope of the form

$$\Delta = \Delta(A, \lambda) = \Big\{p \in \mathbb{R}^n \;\Big|\; \langle p, e_j\rangle \geq 0 \text{ and } \big\langle p, e_j + \sum_i A^i_j e_i\big\rangle \leq \lambda_j \;\forall\, 1 \leq j \leq n\Big\},$$

where $A \in M_n(\mathbb{Z})$ is an $n \times n$ strictly upper-triangular integral matrix, that is $A^i_j = 0$ unless $i < j$, and $\lambda \in (\mathbb{R}_{>0})^n$. Certain relation between A and λ must be satisfied in order for $\Delta(A, \lambda)$ to have 2^n facets and be combinatorially equivalent to a hypercube (see [27].) In that case we say that (A, λ) *defines a symplectic toric Bott manifold* (M_A, ω_λ) corresponding to the Delzant polytope $\Delta(A, \lambda)$. The matrix A encodes the twisting of consecutive $\mathbb{CP}^1$ bundles, and thus determines a diffeomorphism type of M_A, while λ determines the symplectic structure. By a classical result of Danilov [6]

$$\mathrm{H}^*(M_A; \mathbb{Z}) = \mathbb{Z}[x_1, \ldots, x_n]/\big(x_i^2 + \sum_j A^i_j x_j x_i\big), \tag{13.2}$$

with $[\omega_\lambda] = \sum_i \lambda_i x_i \in \mathrm{H}^*(M_A; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$. If all coefficients λ_i are integral then $[\omega_\lambda]$ is an integral symplectic. Note that this particular presentation of $\mathrm{H}^*(M_A; \mathbb{Z})$ depends

[8] In the standard action of $(S^1)^n$ on $(\mathbb{CP}^1)^n$ each S^1 in $(S^1)^n$ acts on the respective copy of $\mathbb{CP}^1$ by $e^{it} \cdot [(z_0, z_1)] = [(z_0, e^{it} z_1)]$.

on A. (The element x_j is the Poincaré dual to the preimage of facet $\Delta(A, \lambda) \cap \{\langle p, e_j + \sum_i A^i_j e_i \rangle = \lambda_j\}$.)

We say that a Bott manifold is $\mathbb{Q}$-*trivial* if $\mathrm{H}^*(M; \mathbb{Q}) \simeq \mathrm{H}^*((\mathbb{CP}^1)^n; \mathbb{Q})$. For example, observe that all Hirzebruch surfaces are $\mathbb{Q}$-trivial Bott manifolds.

Recall that we want to prove Theorem 2 which says that for $\mathbb{Q}$-trivial Bott manifolds (N, ω_N) and (M, ω_M), and any ring isomorphism $F\colon \mathrm{H}^*(M; \mathbb{Z}) \to \mathrm{H}^*(N; \mathbb{Z})$, with $F([\omega_M]) = [\omega_N]$, there exists a symplectomorphism $f\colon (N, \omega_N) \to (M, \omega_M)$ inducing F. The key ingredient of the proof of Theorem 2 is the following construction of symplectomorphisms, which uses toric degenerations.

Proposition 20 ([27]) *Let (M, ω) and $(\widetilde{M}, \widetilde{\omega})$ be symplectic Bott manifolds associated to strictly upper triangular A and $\widetilde{A}$ in $M_n(\mathbb{Z})$ and λ and $\widetilde{\lambda}$ in $\mathbb{Z}^n$, respectively. Assume that there exist integers $1 \le k < \ell \le n$ so that A^k_ℓ and $\widetilde{A}^k_\ell$ are of the same parity and the isomorphism from $\mathbb{Z}[x_1, \dots, x_n]$ to $\mathbb{Z}[\widetilde{x}_1, \dots, \widetilde{x}_n]$ that sends x_k to $\widetilde{x}_k + \frac{\widetilde{A}^k_\ell - A^k_\ell}{2}\, \widetilde{x}_\ell$ and x_i to $\widetilde{x}_i$ for all $i \neq k$ descends to an isomorphism from $\mathrm{H}^*(M; \mathbb{Z})$ to $\mathrm{H}^*(\widetilde{M}; \mathbb{Z})$ and takes $\sum \lambda_i x_i$ to $\sum \widetilde{\lambda}_i \widetilde{x}_i$. If $A^k_\ell + \widetilde{A}^k_\ell \ge 0$, then M and $\widetilde{M}$ are symplectomorphic.*

Proof *(sketch)* Without loss of generality we can assume that the polytope $\Delta(A, \lambda)$ associated to (A, λ) is *normal*, that is, any integral point of $m\, \Delta(A, \lambda)$ can be expressed as a sum of m integral points of $\Delta(A, \lambda)$. Indeed, if $\Delta(A, \lambda)$ is not a normal polytope, replace (M, ω) and $(\widetilde{M}, \widetilde{\omega})$ by $(M, (n-1)\, \omega)$ and $(\widetilde{M}, (n-1)\, \widetilde{\omega})$. This dilates the corresponding polytopes by $(n-1)$. For any integral polytope $P \subset \mathbb{R}^n$ its dialate mP with $m \ge n-1$ is normal (see, for example, [5, Theorem 2.2.12]). Obviously if $(M, (n-1)\, \omega)$ and $(\widetilde{M}, (n-1)\, \widetilde{\omega})$ are symplectomorphic, then so are (M, ω) and $(\widetilde{M}, \widetilde{\omega})$. As usually, let $\mathcal{L}$ denote the very ample line bundle over M corresponding to ω and L the space of its holomorphic sections. Note that normality implies that L^m can be identified with $H^0(M, \mathcal{L}^{\otimes m})$ because a basis for both of these vector spaces is given by the integral points $m\, \Delta(A, \lambda) \cap \mathbb{Z}^n$.

Also without loss of generality we can assume that $\widetilde{A}^k_\ell \ge A^k_\ell$. Let $c = \frac{1}{2}(A^k_\ell + \widetilde{A}^k_\ell) \ge 0$. We will work with a lowest term valuation ν associated to the following coordinate system

$$\{u_1 = f_1, \dots, u_{k-1} = f_{k-1}, u_k = f_k - f_l^c, u_{k+1} = f_{k+1}, \dots, u_n = f_n.\}$$

From Lemma 15 and the normality assumption, for all $m \ge 1$ we have that

$$\nu(L^m) = \mathcal{F}_{-e_k + ce_l}(m\, \Delta(A, \lambda) \cap \mathbb{Z}^n).$$

To understand $\mathcal{F}_{-e_k + ce_l}(m\, \Delta(A, \lambda) \cap \mathbb{Z}^n)$ consider the action of $\mathcal{F}_{-e_k + ce_l}$ on 2-dimensional "slices", that is, the intersections of $m\, \Delta(A, \lambda)$ with affine subspaces which are translations of (e_k, e_l)-planes. Such slices are either empty or are trapezoids like in Example 16 and Corollary 19, possibly with a cut. A bit tedious computation shows that

$$\mathcal{F}_{-e_k + ce_l}(m\, \Delta(A, \lambda) \cap \mathbb{Z}^n) = m\, \Delta(\widetilde{A}, \widetilde{\lambda}) \cap \mathbb{Z}^n.$$

For that computation one uses relations between A, λ, $\widetilde{A}$ and $\widetilde{\lambda}$ which are implied by the facts that $\Delta(A,\lambda)$ and $\Delta(\widetilde{A},\widetilde{\lambda})$ are combinatorially hypercubes, and by the existence of the isomorphism described in the statement of the proposition. In particular, these relations also allow to generalize Corollary 19 (precisely: to show that the equivalent of condition $\lambda_2 - c\lambda_1 > 0$ holds). Therefore the semigroup S associated to the valuation ν of (M,ω) is exactly $S = (\text{cone }\Delta(\widetilde{A},\widetilde{\lambda})) \cap (\mathbb{Z}\times\mathbb{Z}^n)$. Then the claim follows from Proposition 17. □

Using Proposition 20 we show below (Corollary 23) that each $\mathbb{Q}$-trivial Bott manifold is associated to a matrix A of a particularly easy form. To explain this idea we need few more definitions. Recall the presentation of the cohomology of symplectic Bott manifold M_A given in (13.2). We define the following special elements

$$\alpha_k = -\sum_j A^k_j x_j \in \mathrm{H}^*(M_A;\mathbb{Z}), \quad y_k = x_k - \frac{1}{2}\alpha_k \in \mathrm{H}^*(M_A;\mathbb{Q})$$

for all k. We say x_k is of *even (odd) exceptional type* if $\alpha_k = c y_l$ for some l, where c is an even (respectively, odd) integer. In "coordinates", this means that $A^k_j = 0$ for $j < l$ and $A^k_j = \frac{1}{2}A^k_l A^l_j$ for $j > l$. Note that if x_k is even (resp. odd) exceptional, say $\alpha_k = m y_l$, then one can construct an isomorphism of Proposition 20 from $\mathrm{H}^*(M_A;\mathbb{Z})$ to $\mathrm{H}^*(M_{\widetilde{A}};\mathbb{Z})$ for some $\widetilde{A}$ with $\widetilde{A}^k_l$ equal to 0 (resp. -1). For example if x_k is of even exceptional type, i.e. $\alpha_k = 2m y_l$ for some m and l, implying that $A^k_l = -2m$ and $A^k_j = -mA^l_j$ for $j \neq l$, then one should put $\widetilde{A}^k_l = 0$, $\widetilde{A}^i_j = A^i_j$ for all i and all $j \neq l$, and $\widetilde{A}^i_l = A^i_l + mA^i_k$ for all $i \neq k$. Therefore, consecutive applications of the above proposition lead to simplifying the description of a given Bott manifold.

Corollary 21 *Any symplectic toric Bott manifold, with integral symplectic form is symplectomorphic to one for which $A^k_l = 0$ (resp. $A^k_l = -1$) whenever x_k has even (resp. odd) exceptional type and $\alpha_k = m y_l$.*

In the case of $\mathbb{Q}$-trivial Bott manifolds all x_i have exceptional type, [3, Proposition 3.1]. Therefore, any $\mathbb{Q}$-trivial symplectic toric Bott manifold with integral symplectic form must be a product of the following standard models of $\mathbb{Q}$-trivial Bott manifolds.

Example 22 ($\mathbb{Q}$*-trivial Bott manifold*) Take $n \in \mathbb{Z}_{>0}$. Let $A^i_n = -1$ for all $1 \leq i < n$, and $A^i_j = 0$ otherwise. For such upper triangular matrix $A = [A^i_j]$ and any $\lambda \in (\mathbb{R}_{>0})^n$, the polytope $\Delta(A,\lambda)$ is combinatorially a hypercube, thus it defines a symplectic toric Bott manifold, which we will denote by $\mathcal{H} = \mathcal{H}(\lambda_1,\ldots,\lambda_n)$. Observe that

$$\mathrm{H}^*(\mathcal{H};\mathbb{Z}) = \mathbb{Z}[x_1,\ldots,x_n]/\big(x_1^2 - x_1x_n,\ldots,x_{n-1}^2 - x_{n-1}x_n, x_n^2\big).$$

Consider elements $y_i = x_i - \frac{1}{2}x_n \in \mathrm{H}^*(\mathcal{H};\mathbb{Q})$ for all $i < n$, and $y_n = x_n$, and note that they form a basis for $\mathrm{H}^*(\mathcal{H};\mathbb{Q})$. Moreover, as $y_i^2 = 0$ for all i, we get that $\mathrm{H}^*(\mathcal{H};\mathbb{Q}) \simeq \mathbb{Q}[y_1,\ldots,y_n]/\big(y_1^2,\ldots,y_n^2\big)$, that is, $\mathcal{H}$ is $\mathbb{Q}$-trivial.

More generally, any partition of n, $\sum_{i=1}^{m} l_i = n$ together with $\lambda \in (\mathbb{R}_{>0})^n$, define a $\mathbb{Q}$-trivial Bott manifold

$$\mathcal{H}(\lambda_1, \ldots, \lambda_{l_1}) \times \ldots \times \mathcal{H}(\lambda_{n-l_m+1}, \ldots, \lambda_n).$$

Corollary 23 *Each $2n$-dimensional $\mathbb{Q}$-trivial Bott manifold M with integral symplectic form is symplectomorphic to*

$$\mathcal{H}(\lambda_1, \ldots, \lambda_{l_1}) \times \cdots \times \mathcal{H}(\lambda_{n-l_m+1}, \ldots, \lambda_n),$$

for some partition $n = \sum_{i=1}^{m} l_i$ of n and some $\lambda_1, \ldots, \lambda_n \in \mathbb{Z}_{>0}$.

The above standard model is easy enough, so that one can understand all possible ring isomorphisms between cohomology rings and prove that they are induced by maps on manifolds.

Lemma 24 *Fix $n \in \mathbb{Z}_{>0}$. Let $\sum_{i=1}^{m} l_i = \sum_{i=1}^{\widetilde{m}} \widetilde{l}_i = n$ be partitions of n, and let $\lambda, \widetilde{\lambda} \in (\mathbb{R}_{>0})^n$. Consider symplectic Bott manifolds*

$$(M, \omega) = \mathcal{H}(\lambda_1, \ldots, \lambda_{l_1}) \times \cdots \times \mathcal{H}(\lambda_{n-l_m+1}, \ldots, \lambda_n),$$
$$(\widetilde{M}, \widetilde{\omega}) = \mathcal{H}(\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_{\widetilde{l}_1}) \times \cdots \times \mathcal{H}(\widetilde{\lambda}_{n-\widetilde{l}_{\widetilde{m}}+1}, \ldots, \widetilde{\lambda}_n).$$

Given a ring isomorphism $F\colon \mathrm{H}^*(M; \mathbb{Z}) \to \mathrm{H}^*(\widetilde{M}; \mathbb{Z})$ such that $F[\omega] = [\widetilde{\omega}]$, there exists a symplectomorphism f from $(\widetilde{M}, \widetilde{\omega})$ to (M, ω) so that $\mathrm{H}^*(f) = F$.

Proof *(sketch)* First consider the situation when

$$(M, \omega) = \mathcal{H}(\lambda_1, \ldots, \lambda_n) \quad \text{and} \quad (\widetilde{M}, \widetilde{\omega}) = \mathcal{H}(\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_n).$$

The $\mathbb{Q}$-triviality assumption implies that there are exactly $2n$ primitive classes in $\mathrm{H}^2(M; \mathbb{Z})$ which square to 0. A short computation shows that these are $\pm z_1, \ldots, \pm z_n$, where $z_n = x_n$ and $z_i = 2x_i - x_n$ for all $i < n$. Similarly for $\widetilde{M}$. As the cohomology of a symplectic toric manifold is generated in degree 2, any ring isomorphism between $\mathrm{H}^*(M; \mathbb{Z})$ and $\mathrm{H}^*(\widetilde{M}; \mathbb{Z})$ restricts to a bijection on the set of such elements, that is, there exists $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{-1, 1\}^n$ and a permutation $\sigma \in \mathcal{S}_n$ such that $F(z_j) = \epsilon_j \widetilde{z}_{\sigma(j)}$. Moreover, presenting $[\omega]$ (resp. $[\widetilde{\omega}]$) in $\mathbb{R}$-basis $\{z_1, \ldots, z_n\}$ of $\mathrm{H}^*(M; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ (resp. $\{\widetilde{z}_1, \ldots, \widetilde{z}_n\}$) and recalling that the isomorphism F is to map $[\omega]$ to $[\widetilde{\omega}]$, one can deduce that F acts by a permutation: $F(z_j) = \widetilde{z}_{\sigma(j)}$ for some permutation $\sigma \in \mathcal{S}_n$ with $\sigma(n) = n$, and that $\lambda_j = \widetilde{\lambda}_{\sigma(j)}$. Moreover F takes x_i to $x_{\sigma(i)}$ and it holds that $A^i_j = \widetilde{A}^{\sigma(i)}_{\sigma(j)}$ for all i, j. If $\Lambda \in \mathrm{GL}(n, \mathbb{Z})$ denotes the unimodular matrix taking e_i to $e_{\sigma(i)}$, then $\Lambda^T(\Delta(\widetilde{A}, \widetilde{\lambda})) = \Delta(A, \lambda)$; Therefore, by the Delzant theorem, the manifolds (M, ω) and $(\widetilde{M}, \widetilde{\omega})$ are (equivariantly) symplectomorphic, by some symplectomorphism f. Moreover, as Λ^T maps the facet $\{\langle p, e_{\sigma(j)}\rangle = 0\} \cap \Delta(\widetilde{A}, \widetilde{\lambda})$ to the facet $\{\langle p, e_j\rangle = 0\} \cap \Delta(A, \lambda)$, and $\{\langle p, e_{\sigma(j)} + \sum_i \widetilde{A}^i_{\sigma(j)} e_i\rangle = \widetilde{\lambda}_{\sigma(j)}\} \cap \Delta(\widetilde{A}, \widetilde{\lambda})$ to $\{\langle p, e_j + \sum_i A^i_j e_i\rangle = \lambda_j\} \cap \Delta(A, \lambda)$, the map $\mathrm{H}^*(f)$ induced by f on cohomology must map the Poincaré duals of preimages of these facets accordingly. That is, $H^*(f) = F$.

In a general case, denote by λ^{l_s} the l_s-tuple of numbers $(\lambda_{l_1+\cdots+l_{s-1}+1}, \ldots, \lambda_{l_1+\cdots+l_s})$, and define $\widetilde{\lambda}^{l_s}$ similarly. Again, we look at primitive elements with trivial squares. In $H^*(M; \mathbb{Z})$ these are precisely

$$\pm x_{l_s} \text{ and } \pm(2x_i - x_{l_s}) \text{ for } s = 1, \ldots, m \text{ and } i_{s-1} < i < i_s.$$

Note that each such element is contained in some subring $H^*(\mathcal{H}(\lambda^{l_s}); \mathbb{Z}) \subseteq H^*(M; \mathbb{Z})$, and that all primitive square zero elements in $H^*(\mathcal{H}(\lambda^{l_s}); \mathbb{Z})$ are equal modulo 2. Therefore F must restrict to an isomorphism from $H^*(\mathcal{H}(\lambda^{l_s}); \mathbb{Z})$ to some $H^*(\mathcal{H}(\widetilde{\lambda}^{\widetilde{l}_r}); \mathbb{Z})$ with $l_s = \widetilde{l}_r$. This implies that both partitions of n are equal, up to permutation of factors. Repeating the arguments of the previous paragraph one can construct a symplectomorphism inducing the ring isomorphism F. □

Proof (*Proof of Theorem 2*) Let (M, ω), $(\widetilde{M}, \widetilde{\omega})$ be two $\mathbb{Q}$-trivial Bott manifolds with symplectic forms integral up to scaling and let $F\colon H^*(M; \mathbb{Z}) \to H^*(\widetilde{M}; \mathbb{Z})$ be a ring isomorphism such that $F[\omega] = [\widetilde{\omega}]$. Rescaling the symplectic forms if necessary we can assume that both ω and $\widetilde{\omega}$ are integral. As the cohomology of a symplectic toric manifold is generated in degree 2, the isomorphism F must map $H^2(M; \mathbb{Z})$ to $H^2(\widetilde{M}; \mathbb{Z})$. Using (13.2) we see that $\dim H^2(M; \mathbb{Z}) = \frac{1}{2} \dim M$, and similarly $\dim H^2(\widetilde{M}; \mathbb{Z}) = \frac{1}{2} \dim \widetilde{M}$. Therefore $\dim M = \dim \widetilde{M}$. We will denote this dimension by $2n$. By Corollary 23 and the assumption that the symplectic forms are integral we have that

$$(M, \omega) = \mathcal{H}(\lambda_1, \ldots, \lambda_{l_1}) \times \cdots \times \mathcal{H}(\lambda_{n-l_m+1}, \ldots, \lambda_n),$$
$$(\widetilde{M}, \widetilde{\omega}) = \mathcal{H}(\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_{\widetilde{l}_1}) \times \cdots \times \mathcal{H}(\widetilde{\lambda}_{n-\widetilde{l}_{\widetilde{m}}+1}, \ldots, \widetilde{\lambda}_n).$$

for some $\sum_{i=1}^{m} l_i = \sum_{i=1}^{\widetilde{m}} \widetilde{l}_i = n$ partitions of n, and some $\lambda, \widetilde{\lambda} \in (\mathbb{Z}_{>0})^n$. Now Lemma 24 gives that there exist a symplectomorphism f from $(\widetilde{M}, \widetilde{\omega})$ to (M, ω) so that $H^*(f) = F$. □

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References

1. Anderson, D.: Okounkov bodies and toric degenerations. Math. Ann. **356**(3), 1183–1202 (2013)
2. Caviedes Castro, A.: Upper bound for the Gromov width of coadjoint orbits of compact Lie groups. J. Lie Theory **26**(3), 821–860 (2016)
3. Choi, S., Masuda, M.: Classification of $\mathbb{Q}$-trivial Bott manifolds. J. Symplectic Geom. **10**(3), 447–461 (2012)
4. Choi, S., Masuda, M., Suh, D.Y.: Rigidity problems in toric topology: a survey. Tr. Mat. Inst. Steklova **275**(Klassicheskaya i Sovremennaya Matematika v Pole Deyatel'nosti Borisa Nikolaevicha Delone), 188–201 (2011)
5. Cox, D.A., Little, J.B., Schenck, H.K.: Toric varieties, Graduate Studies in Mathematics, vol. 124. American Mathematical Society, Providence, RI (2011)
6. Danilov, V.I.: The geometry of toric varieties. Uspekhi Mat. Nauk **33**(2(200)), 85–134, 247 (1978)
7. da Silva, A.C.: Lectures on symplectic geometry. Lecture Notes in Mathematics, vol. 1764. Springer, Berlin (2001)
8. Eisenbud, D.: Commutative algebra. Graduate Texts in Mathematics, vol. 150. Springer, New York (1995). With a view toward algebraic geometry
9. Eisenbud, D., Harris, J.: The geometry of schemes. In: Graduate Texts in Mathematics, vol. 197. Springer, New York (2000)
10. Fang, X., Fourier, G., Littelmann, P.: Essential bases and toric degenerations arising from birational sequences. Adv. Math. **312**, 107–149 (2017)
11. Fang, X., Littelmann, P., Pabiniak, M.: Simplices in Newton-Okounkov bodies and the Gromov width of coadjoint orbits. Bull. Lond. Math. Soc. **50**(2), 202–218 (2018)
12. Fulton, W.: Introduction to toric varieties. Annals of Mathematics Studies, vol. 131. Princeton University Press, Princeton, NJ (1993). The William H. Roever Lectures in Geometry
13. Grossberg, M., Karshon, Y.: Bott towers, complete integrability, and the extended character of representations. Duke Math. J. **76**(1), 23–58 (1994)
14. Halacheva, I., Pabiniak, M.: The Gromov width of coadjoint orbits of the symplectic group. Pacific J. Math. **295**(2), 403–420 (2018)
15. Hamilton, M.D.: The quantization of a toric manifold is given by the integer lattice points in the moment polytope. In: Toric topology, Contemporary Mathematics, vol. 460, pp. 131–140. American Mathematical Society, Providence, RI (2008)
16. Harada, M., Kaveh, K.: Integrable systems, toric degenerations and Okounkov bodies. Invent. Math. **202**(3), 927–985 (2015)
17. Hartshorne, R.: Algebraic Geometry. Springer, New York, Heidelberg (1977). Graduate Texts in Mathematics, No. 52
18. Karshon, Y., Tolman, S.: The Gromov width of complex Grassmannians. Algebr. Geom. Topol. **5**, 911–922 (2005)
19. Kaveh, K.: Crystal bases and Newton-Okounkov bodies. Duke Math. J. **164**(13), 2461–2506 (2015)
20. Kaveh, K.: Toric degenerations and symplectic geometry of smooth projective varieties. J. Lond. Math. Soc. (2) **99**(2), 377–402 (2019)
21. Lane, J.: A Completely Integrable System on g_2 Coadjoint Orbits (2016). arXiv:1605.01676
22. Littelmann, P.: Cones, crystals, and patterns. Transform. Groups **3**(2), 145–179 (1998)
23. Lu, G.: Gromov-Witten invariants and pseudo symplectic capacities. Israel J. Math. **156**, 1–63 (2006)
24. Lu, G.: Symplectic capacities of toric manifolds and related results. Nagoya Math. J. **181**, 149–184 (2006)
25. Nishinou, T., Nohara, Y., Ueda, K.: Toric degenerations of Gelfand-Cetlin systems and potential functions. Adv. Math. **224**(2), 648–706 (2010)
26. Pabiniak, M.: Gromov width of non-regular coadjoint orbits of $U(n)$, $SO(2n)$ and $SO(2n+1)$. Math. Res. Lett. **21**(1), 187–205 (2014)

27. Pabiniak, M., Tolman, S.: Symplectic cohomological rigidity via toric degnerations. In preparation
28. Schlenk, F.: Embedding problems in symplectic geometry. In: De Gruyter Expositions in Mathematics, vol. 40. Walter de Gruyter GmbH & Co. KG, Berlin (2005)
29. Zoghi, M.: The Gromov Width of Coadjoint Orbits of Compact Lie Groups. ProQuest LLC, Ann Arbor, MI (2010). Thesis (Ph.D.)–University of Toronto (Canada)