Chapter 12
An Eisenbud–Goto-Type Upper Bound for the Castelnuovo–Mumford Regularity of Fake Weighted Projective Spaces

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Abstract We will give an upper bound for the k-normality of very ample lattice simplices, and then give an Eisenbud–Goto-type bound for some special classes of projective toric varieties.

Keywords k-normality · Toric variety · Very ample lattice simplex · Einsenbud-Goto conjecture · Castelnuovo-Mumford regularity

12.1 Introduction

The study of the Castelnuovo–Mumford regularity for projective varieties has been greatly motivated by the Eisenbud–Goto conjecture [7] which asks for any irreducible and reduced variety X, is it always the case that

$$\operatorname{reg}(X) \leq \deg(X) - \operatorname{codim}(X) + 1?$$

The Eisenbud–Goto conjecture is known to be true for some particular cases. For example, it holds for smooth surfaces in characteristic zero [13], connected reduced curves [8], etc. Inspired by the conjecture, there are also many attempts to give an upper bound for the Castelnuovo–Mumford regularity for various types of algebraic and geometric structures [5, 12, 15, 20].

For toric geometry, suppose that (X, L) is a polarized projective toric varieties such that L is very ample. Then there is a corresponding very ample lattice polytope $P := P_L$ associated to L such that $\Gamma(X, L) = \bigoplus_{m \in P \cap M} \mathbb{C} \cdot \chi^m$ [4, Sect. 5.4]. Therefore, by studying the k-normality of P (cf. Definition 2), we can obtain the k-normality and also the regularity of the original variety (X, L). For the purpose of this article,

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we will focus on the case that X is a fake weighted projective d-space and P_L a d-simplex.

For any fake weighted projective d-space X embedded in $\mathbb{P}^r$ via a very ample line bundle, Ogata [17] gives an upper bound for its k-normality:

$$k_X \leq \dim X + \left\lfloor \frac{\dim X}{2} \right\rfloor - 1.$$

In this article, we will improve Ogata's bound by giving a new upper bound for the k-normality of very ample lattice simplices and show that

$$\operatorname{reg}(X) \leq \deg(X) - \operatorname{codim}(X) + \left\lfloor \frac{\dim X}{2} \right\rfloor. \qquad (12.1)$$

Recently, McCullough and Peeva showed some counterexamples to the Eisenbud–Goto conjecture and that the difference $\operatorname{reg}(X) - \deg(X) + \operatorname{codim}(X)$ can be arbitrary large [14, Counterexample 1.8]. However, for any fake weighted projective space X embedded in $\mathbb{P}^r$ via a very ample line bundle, it follows from (12.1) that $\operatorname{reg}(X) - \deg(X) + \operatorname{codim}(X)$ is bounded above by $\dim(X)/2$. Furthermore, we will show that the Eisenbud–Goto conjecture holds for any projective toric variety corresponding to a very ample Fano simplex.

12.2 Background Material

12.2.1 Toric Varieties and Lattice Simplices

We begin this section by recalling the definition of the Castelnuovo–Mumford regularity:

Definition 1 Let $X \subseteq \mathbb{P}^r$ be an irreducible projective variety and $\mathcal{F}$ a coherent sheaf over X. We say that $\mathcal{F}$ is k-regular if

$$\mathrm{H}^i(X, \mathcal{F}(k-i)) = 0$$

for all $i > 0$. The regularity of $\mathcal{F}$, denoted by $\operatorname{reg}(\mathcal{F})$, is the minimum number k such that $\mathcal{F}$ is k-regular. We say that X is k-regular if the ideal sheaf $\mathcal{I}_X$ of X is k-regular and use $\operatorname{reg}(X)$ to denote the regularity of X (or of $\mathcal{I}_X$).

As the main object of the article is to find an upper bound for k-normality of very ample lattice simplices, it is important for us to revisit the definition of k-normality of lattice polytopes.

Definition 2 A lattice polytope P is k-normal if the map

$$\underbrace{P \cap M + \cdots + P \cap M}_{k \text{ times}} \to kP \cap M$$

is surjective. The k-normality of P, denoted by k_P, is the smallest positive integer k_P such that P is k-normal for all $k \geq k_P$.

Suppose now that X is a fake weighted projective d-space embedded in $\mathbb{P}^r$ via a very ample line bundle. Then the polytope P corresponding to the embedding is a very ample lattice d-simplex. Furthermore, $\operatorname{codim}(X) = |P \cap M| - (d + 1)$, where M is the ambient lattice, and $\deg(X) = \operatorname{Vol}(P)$, the normalized volume of P.

We have a combinatorial interpretation of $\operatorname{reg}(X)$ in terms of k_P and $\deg(P)$ [21, Proposition 4.1.5] as follows:

$$\operatorname{reg}(X) = \max\{k_P, \deg(P)\} + 1. \tag{12.2}$$

From this, we obtain a combinatorial form of the Eisenbud–Goto conjecture: for very ample lattice polytope $P \subset M_{\mathbb{R}}$, is it always true that

$$\max\{\deg(P), k_P\} \leq \operatorname{Vol}(P) - |P \cap M| + d + 1?$$

The first inequality was confirmed to be true recently [11, Proposition 2.2]; namely,

$$\deg(P) \leq \operatorname{Vol}(P) - |P \cap M| + d + 1. \tag{12.3}$$

Therefore, in order to verify the Eisenbud–Goto conjecture for the polarized toric variety (X, L), it suffices to check if

$$k_P \leq \operatorname{Vol}(P) - |P \cap M| + d + 1. \tag{12.4}$$

12.2.2 Ehrhart Theory

We now recall some basic facts about Ehrhart theory of polytopes and the definition of their degree.

Let P be a lattice polytope of dimension d. We define $\operatorname{ehr}_P(k) = |kP \cap M|$, the number of lattice points in kP. Then from Ehrhart's theory [6, 19],

$$\operatorname{Ehr}_P(t) = \sum_{k=0}^{\infty} \operatorname{ehr}_P(k) t^k = \frac{h_P^*(t)}{(1-t)^{d+1}}$$

for some polynomial $h_P^* \in \mathbb{Z}_{\geq 0}[t]$ of degree less than or equal to d. Let $h_P^*(t) = \sum_{i=0}^{d} h_i^* t^i$. We have

$$h_0^* = 1, \quad h_1^* = |P \cap M| - d - 1, \quad h_d^* = |P^0 \cap M|, \text{ and } \sum_{i=0}^{d} h_i^* = \text{Vol}(P).$$

Definition 3 ([1, Remark 2.6]) Let P be a lattice polytope of dimension d. We define the degree of P, denoted by $\deg(P)$, to be the degree of $h_P^*(t)$. Equivalently,

$$\deg(P) = \begin{cases} d & \text{if } |P^0 \cap M| \neq 0, \\ \min\left\{i \in \mathbb{Z}_{\geq 0} | (kP)^0 \cap M = \emptyset \text{ for all } 1 \leq k \leq d - i\right\} & \text{otherwise.} \end{cases}$$

12.3 *k*-Normality of Very Ample Simplices

The following lemma by Ogata is crucial to the main result of this article:

Lemma 4 ([17, Lemma 2.1]) *Let $P = \text{conv}(v_0, \ldots, v_d)$ be a very ample lattice n-simplex. Suppose that $k \geq 1$ is an integer and $x \in kP \cap M$. For any $i = 0, \ldots, d$, we have*

$$x + (k-1)v_i = \sum_{j=1}^{2k-1} u_j$$

for some $u_j \in P \cap M$.

Using the ideas in [17, Lemma 2.5], we generalize the above lemma as follows.

Lemma 5 *Suppose that $P = \text{conv}(v_0, \ldots, v_d)$ is a very ample d-simplex. Let $k \in \mathbb{N}_{\geq 1}$. Then for any $x \in kP \cap M$, $a_0, \ldots, a_d \in \mathbb{Z}_{\geq 0}$ such that $\sum_{i=0}^{d} a_i = k - 1$, we have*

$$\sum_{i=0}^{d} a_i v_i + x = \sum_{i=1}^{2k-1} u_i$$

for some $u_i \in P \cap M$.

Proof We will use induction in this proof. The case $k = 1$ is trivial. Suppose that the lemma holds for $k = s - 1$. We will now show that it holds for $k = s$; i.e., for any $x \in sP \cap M$, $a_1, \ldots, a_d \in \mathbb{Z}_{\geq 0}$ such that $\sum_{i=0}^{d} a_i = s - 1$, we have

$$\sum_{i=0}^{d} a_i v_i + x = \sum_{i=1}^{2s-1} u_i \tag{12.5}$$

for some $u_i \in P \cap M$. Without loss of generality, we can take a_0 to be positive and move v_0 to the origin. By Lemma 4,

$$(s-1)v_0 + x = \sum_{i=1}^{2s-1} w_i$$

for some $w_i \in P \cap M$. Since $v_0 = 0$, we can write $x = \sum_{i=1}^{2s-1} w_i$. If $w_i + w_j \in P \cap M$ for any $i \neq j$, then we can let $t_i = w_{2i-1} + w_{2i}$ for $i = 1, \ldots, s-1$ and have $x = t_1 + \cdots + t_{s-1} + w_{2s-1}$, which lies in $\sum_{i=1}^{s} P \cap M$. Therefore,

$$\sum_{i=0}^{d} a_i v_i + x = \sum_{i=0}^{d} a_i v_i + \sum_{i=1}^{s-1} t_i + w_{2s-1},$$

which satisfies (12.5). Conversely, without loss of generality, suppose that $w_1 + w_2 \notin P \cap M$. Then since $x = w_1 + w_2 + (w_3 + \cdots + w_{2s-1}) \in sP \cap M$, we have $y := w_3 + \cdots + w_{2s-1} \in (s-1)P \cap M$ and $v_0 + x = w_1 + w_2 + y$. Using the induction hypothesis,

$$\underbrace{(a_0 - 1)v_0 + \sum_{i=1}^{d} a_i v_i}_{a_0 - 1 + \sum_{i=1}^{d} a_i = s-2} + y = \sum_{i=1}^{2(s-1)-1} w_i'$$

for some $w_i' \in P \cap M$. It follows that

$$\begin{aligned}\sum_{i=0}^{d} a_i v_i + x &= v_0 + x + (a_0 - 1)v_0 + \sum_{i=1}^{d} a_i v_i \\ &= w_1 + w_2 + y + (a_0 - 1)v_0 + \sum_{i=1}^{d} a_i v_i \\ &= w_1 + w_2 + \sum_{i=0}^{2(s-1)-1} w_i'.\end{aligned}$$

The conclusion follows. □

Now define the invariants d_P and ν_P as in [21, Definition 2.2.8]:

Definition 6 Let P be a lattice polytope with the set of vertices $\mathcal{V} = \{v_0, \ldots, v_{n-1}\}$. We define d_P to be the smallest positive integer such that for every $k \geq d_P$,

$$(k+1)P \cap M = P \cap M + kP \cap M.$$

We also define ν_P to be the smallest positive integer such that for any $k \geq \nu_P$,

$$(k+1)P \cap M = \mathcal{V} + kP \cap M.$$

Notice that for P an n-simplex, $d_P \leq \nu_P \leq n - 1$.

Proposition 7 *Let $P = \mathrm{conv}(v_0, \ldots, v_d)$ be a very ample d-simplex. Then*

$$k_P \leq \nu_P + d_P - 1.$$

Proof For any $k \geq d_P + \nu_P - 1$ and $p \in kP \cap M$, by the definition of d_P and ν_P, we have

$$p = x + \sum_{i=1}^{\nu_P - d_P} u_i + \sum_{i=0}^{d} a_i v_i \tag{12.6}$$

for some $x \in d_P P \cap M$, $u_i \in P \cap M$, $\sum_{i=0}^{d} a_i = k - \nu_P$. By assumption, $k - \nu_P \geq d_P - 1$, so it follows from Lemma 5 that

$$x + \sum_{i=0}^{d} a_i v_i = \sum_{i=1}^{d_P + k - \nu_P} u_i' \tag{12.7}$$

for some $u_i' \in P \cap M$. Substitute (12.7) into (12.6), we have

$$p = \sum_{i=1}^{\nu_P - d_P} u_i + \sum_{i=1}^{d_P + k - \nu_P} u_i'.$$

The conclusion follows. □

Remark 8 This bound is stronger than [17, Proposition 2.4] since $\nu_P \leq d$ [21, Proposition 2.2] and $d_P \leq d/2$ [17, Proposition 2.2].

12.4 Eisenbud–Goto-Type Upper Bound for Very Ample Simplices

Suppose that P is a very ample simplex. If P is unimodularly equivalent to the standard simplex $\Delta_d = \mathrm{conv}(0, e_1, \ldots, e_d)$ then (12.4) holds. Now consider the case P is not unimodularly equivalent to Δ_d.

The following lemma is a rewording of [9, Proposition IV.10] to fit our purpose. We provide a proof for the sake of completeness.

Lemma 9 *Let* $\mathcal{V} = \{v_0, \ldots, v_d\}$ *and suppose that* $P = \mathrm{conv}(\mathcal{V})$ *is a lattice simplex not unimodularly equivalent to* Δ_d. *Then* $\deg(P) \geq \nu_P$.

Proof Since $\nu_P \leq d$, it suffices to show that for any $d \geq k \geq \deg(P)$,

$$\mathcal{V} + kP \cap M \twoheadrightarrow (k+1)P \cap M.$$

Indeed, any $x \in (k+1)P \cap M$ can be written as $x = \sum_{i=0}^{d} a_i v_i$ such that $a_i \geq 0$ and $\sum_{i=0}^{d} a_i = k + 1$. If $a_i < 1$ for all i, then $d > k$ and the point $\sum_{i=0}^{d}(1 - a_i)v_i$ is an interior lattice point of $(d-k)P$, a contradiction since $d - k \leq d - \deg(P)$. Hence, $a_i \geq 1$ for some i, say $a_0 \geq 1$. Then

$$x = v_0 + (a_0 - 1)v_0 + \sum_{i=1}^{d} a_i v_i = v_0 + \left((a_0 - 1)v_0 + \sum_{i=1}^{d} a_i v_i\right) \in \mathcal{V} + kP \cap M.$$

Hence, $k \geq \nu_P$. The conclusion follows. □

Proposition 10 *Let* $P = \mathrm{conv}(v_0, \ldots, v_d)$ *be a very ample simplex. Then*

$$k_P \leq \mathrm{Vol}(P) - |P \cap M| + d + \left\lfloor \frac{d}{2} \right\rfloor.$$

Proof From Proposition 7, (12.3), and Lemma 9,

$$\begin{aligned} k_P &\leq d_P + \nu_P - 1 \leq d_P + \deg(P) - 1 \\ &\leq d_P + \mathrm{Vol}(P) - |P \cap M| + d. \end{aligned}$$

By [17, Proposition 2.2], $d_P \leq \frac{d}{2}$. Therefore, since k_P, $\mathrm{Vol}(P)$, and $|P \cap M|$ are all integers,

$$k_P \leq \mathrm{Vol}(P) - |P \cap M| + d + \left\lfloor \frac{d}{2} \right\rfloor.$$

□

Remark 11 We show some cases that the result of Proposition 10 is stronger than [17, Proposition 2.4]:

1. $\mathrm{Vol}(P) \leq |P \cap M| + 2$. In this case,

$$\mathrm{Vol}(P) - |P \cap M| + d + \left\lfloor \frac{d}{2} \right\rfloor \leq d + \left\lfloor \frac{d}{2} \right\rfloor - 2.$$

Example 12 Let Δ_d be the standard d-simplex. Then

$$\mathrm{Vol}(\Delta_d) - |\Delta_d \cap M| + d + \left\lfloor \frac{d}{2} \right\rfloor = 1 - (d+1) + d + \left\lfloor \frac{d}{2} \right\rfloor = \left\lfloor \frac{d}{2} \right\rfloor.$$

This is clearly a better bound compared to $d + \left\lfloor \frac{d}{2} \right\rfloor - 1$.

2. $P^0 \cap M = \emptyset$ or equivalently $\deg(P) \leq d - 1$. Indeed, in this case,

$$k_P \leq d_P + \deg(P) - 1 \leq \left\lfloor \frac{d}{2} \right\rfloor + d - 2.$$

We will show in next section that this is the only case that we still need to consider in order to verify the Eisenbud–Goto conjecture for very ample simplices.

Example 13 Consider $P = 2\Delta_d$ for $d \geq 4$, where Δ_d is the standard d-simplex. Then $\deg(P) = 2$ and by Proposition 7,

$$k_P \le d_P + 1 \le \left\lfloor \frac{d}{2} \right\rfloor + 1 < \left\lfloor \frac{d}{2} \right\rfloor + d - 1.$$

Theorem 14 *Suppose that X is a fake weighted projective space embedded in $\mathbb{P}^r$ via a very ample line bundle. Then*

$$\operatorname{reg}(X) \le \deg(X) - \operatorname{codim}(X) + \left\lfloor \frac{\dim(X)}{2} \right\rfloor.$$

Proof Let P be the corresponding polytope of the embedding. From (12.2), (12.3), and Proposition 10, it follows that

$$\operatorname{reg}(X) \le \operatorname{Vol}(P) - |P \cap M| + d + \left\lfloor \frac{d}{2} \right\rfloor + 1 = \deg(X) - \operatorname{codim}(X) + \left\lfloor \frac{d}{2} \right\rfloor.$$

12.5 Eisenbud–Goto Conjecture for Non-hollow Very Ample Simplices

In this section, we will improve the bound of k-normality for non-hollow very ample simplices.

Definition 15 A lattice polytope is hollow if it has no interior lattice points.

We now show that the inequality (12.4) holds for non-hollow very ample simplices.

Proposition 16 *Let $P \subseteq M_{\mathbb{R}}$ be a non-hollow very ample lattice d-simplex. Then*

$$k_P \le \operatorname{Vol}(P) - |P \cap M| + d + 1.$$

Proof We will consider two cases, namely $|P \cap M| = d + 2$ and $|P \cap M| \ge d + 3$. For the first case, we have the following lemma:

Lemma 17 *Suppose that $P = \operatorname{conv}(v_0, \ldots, v_d)$ is a very ample lattice d-simplex with u is the only lattice point beside the vertices. Then P is normal.* □

Proof Assume that $d_P \ge 2$. Then there exists a point $p \in d_P P \cap M$ such that p cannot be written as $p = x + w$ for some $x \in (d_P - 1)P \cap M$ and $w \in P \cap M$. Since P is a simplex, u and p can be uniquely written as

$$p = \sum_{i=0}^{d} \lambda_i v_i, \qquad \sum_{i=0}^{d} \lambda_i = d_P$$

and

$$u = \sum_{i=0}^{d} \lambda_i^* v_i, \qquad \sum_{i=0}^{d} \lambda_i^* = 1,$$

respectively. It follows from the condition of p that $\lambda_i < 1$ for all $0 \le i \le d$ and there exists $0 \le i \le d$ such that $\lambda_i < \lambda_i^*$, say $i = 0$. By Lemma 4,

$$p + (d_P - 1)v_1 = \sum_{i=0}^{d} a_i v_i + bu$$

for some $a_i, b \in \mathbb{Z}_{\ge 0}$ such that $b + \sum_{i=0}^{d} a_i = 2d_P - 1$. Replacing p by $\sum_{i=0}^{d} \lambda_i v_i$ and u by $\sum_{i=0}^{d} \lambda_i^* v_i$ yields

$$\begin{aligned} \lambda_0 &= a_0 + b\lambda_0^* \\ \lambda_1 + d_P - 1 &= a_1 + b\lambda_1^* \\ \lambda_2 &= a_2 + b\lambda_2^* \\ &\vdots \\ \lambda_d &= a_d + b\lambda_d^*. \end{aligned}$$

Since $\lambda_0 < \lambda_0^*$ and $\lambda_i < 1$ for all $0 \le i \le d$, it follows that $a_0 = a_2 = \cdots = a_d = 0$ and $b = 0$. Then $p = d_P v_1$, a contradiction to the choice of p. Therefore, P is normal. □

As a consequence, $1 = k_P \le \mathrm{Vol}(P) - |P \cap M| + d + 1 = \mathrm{Vol}(P) - 1$. Now we consider the case $|P \cap M| \ge d + 3$. By the hypothesis, $|P \cap M| - (d + 1) \ge 2$. Consider the Ehrhart vector $h^* = (h_0^*, \cdots, h_d^*)$ of P. We have

$$\begin{aligned} h_0^* &= 1 \\ h_1^* &= |P \cap M| - d - 1 \ge 2 \\ h_d^* &= |P^0 \cap M| \ge 2. \end{aligned}$$

By [10, Theorem 1.1], $2 \le h_1^* \le h_i^*$ for all $1 \le i < d$. Therefore,

$$\mathrm{Vol}(P) - |P \cap M| + d + 1 = h_0^* + h_2^* + \cdots + h_d^* \ge 1 + 2(d - 1) = 2d - 1.$$

By [17, Proposition 2.4],

$$k_P \le \left\lfloor \frac{d}{2} \right\rfloor + d - 1 \le 2d - 1 \le \mathrm{Vol}(P) - |P \cap M| + d + 1$$

for all $d \ge 3$. The conclusion follows. □

Let us now recall the definition of Fano polytopes:

Definition 18 A Fano polytope is a convex lattice polytope $P \subseteq M_{\mathbb{R}}$ such that $P^0 \cap M = \{0\}$ and each vertex v of P is a primitive point of M.

From Proposition 16, we obtain the following corollary:

Corollary 19 *The Eisenbud–Goto conjecture holds for any projective toric variety corresponding to a very ample Fano simplex.*

12.6 Final Remarks

We start with a remark that Proposition 7 fails in general.

Example 20 ([3]) Consider the polytope P which is the convex hull of the vertices given by the columns of the following matrix

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & s & s+1 \end{pmatrix}$$

with $s \geq 4$. Then $d_P = \nu_P = 2$, and by [2, Theorem 3.3], $k_P = s - 1$. It is clear that $k_P > d_P + \nu_P - 1$ for all $s \geq 6$.

Furthermore, it can be shown that P cannot be covered by very ample simplicies [21, Proposition 4.3.3]; hence, it is very unlikely that we can apply Proposition 7 to find a bound of the k-normality of generic very ample polytopes.

12.6.1 Hollow Very Ample Simplices

Finally, we would love to see a classification of hollow very ample lattice simplices. For dimension 2, Rabinotwiz [18, Theorem 1] showed that any such simplex is unimodularly equivalent to either $T_{p,1} := \operatorname{conv}(0, (p, 0), (0, 1))$ for some $p \in \mathbb{N}$ or $T_{2,2} = \operatorname{conv}(0, (2, 0), (0, 2))$. Now we will show a way to obtain some hollow very ample simplices in any dimension with arbitrary volume.

We recall the definition of lattice pyramids as in [16]:

Definition 21 Let $B \subseteq \mathbb{R}^k$ be a lattice polytope with respect to $\mathbb{Z}^k$. Then $\operatorname{conv}(0, B \times \{1\}) \subseteq \mathbb{R}^{k+1}$ is a lattice polytope with respect to $\mathbb{Z}^{k+1}$, called the (1-fold) standard pyramid over B. Recursively, we define for $l \in \mathbb{N}_{\geq 1}$ in this way the l-fold standard pyramid over B. As a convention, the 0-fold standard pyramid over B is B itself.

Proposition 22 *Let P be a lattice polytope. Then the 1-fold pyramid over P is very ample if and only if P is normal.*

Proof Let $Q = \mathrm{conv}(0, P \times \{1\})$ be the 1-fold pyramid over P. Then it is easy to see that if P is normal then so is Q. Now suppose that Q is very ample. We have $k_Q \geq k_P$ [21, Lemma 4.2.2] and each lattice point of $k_Q Q \cap M$ sits in $(tP \cap M) \times \{t\}$ for some $0 \leq t \leq k_Q$. In particular, suppose that $(x, t) \in (tP \cap M) \times \{t\} \subseteq k_Q Q \cap M$. Then

$$(x, t) = \sum_{i=1}^{t} (u_i, 1) + (k_Q - t)0$$

for some $u_i \in P \cap M$. It follows that $x = \sum_{i=1}^{t} u_i$. Hence, P is t-normal for all $k_Q \geq t \geq 1$. Since $k_Q \geq k_P$, it follows that P is normal. The conclusion follows. □

From Proposition 22, if we take any $(d-2)$-fold pyramid over either $T_{p,1}$ with $p \in \mathbb{Z}_{\geq 1}$ or $T_{2,2}$, which are all normal, then we obtain a hollow normal (hence very ample) d-simplex with normalized volume p. The Eisenbud–Goto conjecture holds for these.

Example 23 We give here an example to demonstrate the case that if Q is very ample but not normal then the 1-fold pyramid over Q is not very ample. Let Q be the convex polytope given by taking $s = 4$ in Example 20. Then Q is very ample; however, the 1-fold pyramid of Q, which is given by the convex hull of

$$\begin{pmatrix} 0&0&1&0&0&1&0&1&1 \\ 0&0&0&1&0&0&1&1&1 \\ 0&0&0&0&1&1&1&4&5 \\ 0&1&1&1&1&1&1&1&1 \end{pmatrix},$$

is not very ample.

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