# Chapter 1
# Difference Between Families of Weakly and Strongly Maximal Integral Lattice-Free Polytopes

**Gennadiy Averkov**

**Abstract** A $d$-dimensional closed convex set $K$ in $\mathbb{R}^d$ is said to be lattice-free if the interior of $K$ is disjoint with $\mathbb{Z}^d$. We consider the following two families of lattice-free polytopes: the family $\mathcal{L}^d$ of integral lattice-free polytopes in $\mathbb{R}^d$ that are not properly contained in another integral lattice-free polytope and its subfamily $\mathcal{M}^d$ consisting of integral lattice-free polytopes in $\mathbb{R}^d$ which are not properly contained in another lattice-free set. It is known that $\mathcal{M}^d = \mathcal{L}^d$ holds for $d \le 3$ and, for each $d \ge 4$, $\mathcal{M}^d$ is a proper subfamily of $\mathcal{L}^d$. We derive a super-exponential lower bound on the number of polytopes in $\mathcal{L}^d \setminus \mathcal{M}^d$ (with standard identification of integral polytopes up to affine unimodular transformations).

**Keywords** Egyptian fraction · Hollow polytope · Lattice-free set · Lattice polytope · Maximality

## 1.1 Introduction

By $|X|$ we denote the cardinality of a finite set $X$. Let $\mathbb{N}$ be the set of all positive integers and let $d \in \mathbb{N}$ be the dimension. Elements of $\mathbb{Z}^d$ are called *integral points* or *integral vectors*. We call a polyhedron $P \subseteq \mathbb{R}^d$ *integral* if $P$ is the convex hull of $P \cap \mathbb{Z}^d$. Let $\operatorname{Aff}(\mathbb{Z}^d)$ be the group of affine transformations $A : \mathbb{R}^d \to \mathbb{R}^d$ satisfying $A(\mathbb{Z}^d) = \mathbb{Z}^d$. We call elements of $\operatorname{Aff}(\mathbb{Z}^d)$ *affine unimodular transformations*. For a family $\mathcal{X}$ of subsets of $\mathbb{R}^d$, we consider the family of equivalence classes

$$\mathcal{X}/\operatorname{Aff}(\mathbb{Z}^d) := \left\{\left\{A(X) \ : \ A \in \operatorname{Aff}(\mathbb{Z}^d)\right\} \ : \ X \in \mathcal{X}\right\}$$

with respect to identification of the elements of $\mathcal{X}$ up to affine unimodular transformations. A subset $K$ of $\mathbb{R}^d$ is called *lattice-free* if $K$ is closed, convex, $d$-dimensional and

G. Averkov (✉)
Brandenburgische Technische Universität Cottbus-Senftenberg, Fakultät 1, Platz der Deutschen Einheit 1, 03046 Cottbus, Germany
e-mail: averkov@b-tu.de

 
A. M. Kasprzyk and B. Nill (eds.), *Interactions with Lattice Polytopes*,
Springer Proceedings in Mathematics & Statistics 386,
https://doi.org/10.1007/978-3-030-98327-7_1

the interior of $K$ contains no points from $\mathbb{Z}^d$. A set $K$ is called *maximal lattice-free* if $K$ is lattice-free and is not a proper subset of another lattice-free set.

Our objective is to study the relationship between the following two families of integral lattice-free polytopes:

1. The family $\mathcal{L}^d$ of integral lattice-free polytopes $P$ in $\mathbb{R}^d$ such that there exists no integral lattice-free polytope properly containing $P$. We call elements of $\mathcal{L}^d$ *weakly maximal* integral lattice-free polytopes.
2. The family $\mathcal{M}^d$ of integral lattice-free polytopes $P$ in $\mathbb{R}^d$ such that there exists no lattice-free set properly containing $P$. We call the elements of $\mathcal{L}^d$ *strongly maximal* integral lattice-free polytopes.

The family $\mathcal{L}^d$ has applications in mixed-integer optimization, algebra and algebraic geometry; see [1, 3, 4, 13], respectively. In [2, 11] it was shown that $\mathcal{L}^d$ is finite up to affine unimodular transformations:

**Theorem 1** ([2, Theorem 2.1], [11, Corollary 1.3]) $\mathcal{L}^d / \operatorname{Aff}(\mathbb{Z}^d)$ *is finite.*

Several groups of researchers are interested in enumeration of $\mathcal{L}^d$, up to affine unimodular transformations, in fixed dimensions. This requires understanding geometric properties of $\mathcal{L}^d$. Currently, no explicit description of $\mathcal{L}^d$ is available for dimensions $d \geq 4$ and, moreover, it is even extremely hard to decide if a given polytope belongs to $\mathcal{L}^d$. A brute-force algorithm based on volume bounds for $\mathcal{L}^d$ (provided in [11]) would have doubly exponential running time in $d$. In contrast to $\mathcal{L}^d$, its subfamily $\mathcal{M}^d$ is easier to deal with. Lovász's characterization [9, Proposition 3.3] of maximal lattice-free sets leads to a straightforward geometric description of polytopes belonging to $\mathcal{M}^d$. This characterization can be used to decide whether a given polytope is an element of $\mathcal{M}^d$ in only exponential time in $d$. Thus, while enumeration of $\mathcal{M}^d$ in fixed dimensions is a hard task, too, enumeration of $\mathcal{L}^d$ is even more challenging.

For a given dimension $d$, it is a priori not clear whether or not $\mathcal{M}^d$ is a proper subset of $\mathcal{L}^d$. Recently, it has been shown that the inequality $\mathcal{M}^d = \mathcal{L}^d$ holds if and only if $d \leq 3$. The equality $\mathcal{M}^d = \mathcal{L}^d$ is rather obvious for $d \in \{1, 2\}$, as it is not hard to enumerate $\mathcal{L}^d$ in these very small dimensions and to check that every element of $\mathcal{L}^d$ belongs to $\mathcal{M}^d$. Starting from dimension three, the problem gets very difficult. Results in [1, 2] establish the equality $\mathcal{M}^3 = \mathcal{L}^3$ and enumerate $\mathcal{L}^3$, up to affine unimodular transformations. As a complement, in [11, Theorem 1.4] it was shown that for all $d \geq 4$ there exists a polytope belonging to $\mathcal{L}^d$ but not to $\mathcal{M}^d$.

While Theorem 1.4 in [11] shows that $\mathcal{L}^d$ and $\mathcal{M}^d$ are two different families, it does not provide information on the number of polytopes in $\mathcal{L}^d$ that do not belong to $\mathcal{M}^d$. Relying on a result of Konyagin [6], we will show that, asymptotically, the gap between $\mathcal{L}^d$ and $\mathcal{M}^d$ is very large.

For $a_1, \ldots, a_d > 0$, we introduce

$$\kappa(a) := \kappa(a_1, \ldots, a_d) = \frac{1}{a_1} + \cdots + \frac{1}{a_d}.$$

Reciprocals of positive integers are sometimes called *Egyptian fractions*. Thus, if $a \in \mathbb{N}^d$, then $\kappa(a)$ is a sum of $d$ Egyptian fractions. We consider the set

$$\mathcal{A}_d := \left\{(a_1, \dots, a_d) \in \mathbb{N}^d \ : \ a_1 \le \cdots \le a_d, \ \kappa(a_1, \dots, a_d) = 1\right\}$$

of all different solutions of the Diophantine equation

$$\kappa(x_1, \dots, x_d) = 1$$

in the unknowns $x_1, \dots, x_d \in \mathbb{N}$. The set $\mathcal{A}_d$ represents possible ways to write 1 as a sum of $d$ Egyptian fractions. It is known that $\mathcal{A}_d$ is finite. Our main result allows is a lower bound on the cardinality of $(\mathcal{L}^d \backslash \mathcal{M}^d)/\operatorname{Aff}(\mathbb{Z}^d)$:

**Theorem 2** $\left|(\mathcal{L}^{d+5} \backslash \mathcal{M}^{d+5})/\operatorname{Aff}(\mathbb{Z}^{d+5})\right| \ge \left|\mathcal{A}_d\right|$.

The proof of Theorem 2 is constructive. This means that, for every $a \in \mathcal{A}_d$, we generate an element in $P_a \in \mathcal{L}^{d+5} \backslash \mathcal{M}^{d+5}$ such that for two different elements $a$ and $b$ of $\mathcal{A}_d$, the respective polytopes $P_a$ and $P_b$ do not coincide up to affine unimodular transformations. The proof of Theorem 2 is inspired by the construction in [11]. Using lower bounds on $\left|\mathcal{A}_d\right|$ from [6], we obtain the following asymptotic estimate:

**Corollary 3** $\ln\ln\left|\left(\mathcal{L}^d \backslash \mathcal{M}^d\right)/\operatorname{Aff}(\mathbb{Z}^d)\right| = \Omega\left(\frac{d}{\ln d}\right)$*, as* $d \to \infty$.

**Note 4** We view the elements of $\mathbb{R}^d$ as columns. By $o$ we denote the zero vector and by $e_1, \dots, e_d$ the standard basis of $\mathbb{R}^d$. If $x \in \mathbb{R}^d$ and $i \in \{1, \dots, d\}$, then $x_i$ denotes the $i$-th component of $x$. The relation $a \le b$ for $a, b \in \mathbb{R}^d$ means $a_i \le b_i$ for every $i \in \{1, \dots, d\}$. The relations $\ge$, $>$ and $<$ on $\mathbb{R}^d$ are introduced analogously. The abbreviations aff, conv, int and relint stand for the affine hull, convex hull, interior and relative interior, respectively.

## 1.2 An Approach to Construction of Polytopes in $\mathcal{L}^d \backslash \mathcal{M}^d$

We will present a systematic approach to construction of polytopes in $\mathcal{L}^d \backslash \mathcal{M}^d$, but first we discuss general maximal lattice-free sets.

**Definition 5** Let $P$ be a lattice-free polyhedron in $\mathbb{R}^d$. We say that a facet $F$ of $P$ is *blocked* if the relative interior of $F$ contains an integral point.

Maximal lattice-free sets can be characterized as follows:

**Proposition 6** ([9, Proposition 3.3]) *Let $K$ be a $d$-dimensional closed convex subset of $\mathbb{R}^d$. Then the following conditions are equivalent:*

1. *$K$ is maximal lattice-free;*
2. *$K$ is a lattice-free polyhedron such that every facet of $K$ is blocked.*

It can happen that some facets of a maximal lattice-free polyhedron are more than just blocked. We introduce a respective notion. Recall that the *integer hull* $K_I$ of a compact convex set $K$ in $\mathbb{R}^d$ is defined by

$$K_I := \mathrm{conv}(K \cap \mathbb{Z}^d).$$

**Definition 7** Let $P$ be a $d$-dimensional lattice-free polyhedron in $\mathbb{R}^d$. A facet $F$ of $P$ is called *strongly blocked* if $F_I$ is $(d-1)$-dimensional and $\mathbb{Z}^d \cap \mathrm{relint}\, F_I \neq \emptyset$. The polyhedron $P$ is called *strongly blocked* if all facets of $P$ are strongly blocked.

The following proposition extracts the geometric principle behind the construction from [11, Sect. 3]. (Note that arguments in [11, Sect. 3] use an algebraic language.)

**Proposition 8** *Let $P$ be a strongly blocked lattice-free polytope in $\mathbb{R}^d$. Then $P_I \in \mathcal{L}^d$. Furthermore, if $P$ is not integral, then $P_I \notin \mathcal{M}^d$.*

***Proof*** In order to show $P_I \in \mathcal{L}^d$ it suffices to verify that, for every $z \in \mathbb{Z}^d$ such that $\mathrm{conv}(P_I \cup \{z\})$ is lattice-free, one necessarily has $z \in P_I$. If $z \notin P_I$, then $z \notin P$ and so, for some facet $F$ of $P$, the point $z$ and the polytope $P$ lie on different sides of the hyperplane aff $F$. Then $\emptyset \neq \mathbb{Z}^d \cap \mathrm{relint}\, F_I \subseteq \mathrm{int}(\mathrm{conv}(P \cup \{z\}))$, yielding a contradiction to the choice of $z$. Thus, for every facet $F$ of $P$, $z$ and $P$ lie on the same side of aff $F$. It follows $z \in P$. Hence $z \in P \cap \mathbb{Z}^d \subseteq P_I$.

If $P$ is not integral, then $P_I \notin \mathcal{M}^d$ since $P_I \subsetneq P$ and $P$ is lattice-free. □

## 1.3 Lattice-Free Axis-Aligned Simplices

For $a \in \mathbb{R}^d_{>0}$, the $d$-dimensional simplex

$$T(a) := \mathrm{conv}\{o, a_1 e_1, \dots, a_d e_d\}.$$

is called *axis-aligned*. The proof of the following proposition is straightforward.

**Proposition 9** *For $a \in \mathbb{R}^d_{>0}$, the following statements hold:*

1. *the simplex $T(a)$ is a lattice-free set if and only if $\kappa(a) \geq 1$;*
2. *the simplex $T(a)$ is a maximal lattice-free set if and only if $\kappa(a) = 1$.*

We introduce transformations which preserve the values of $\kappa$. The transformations arise from the following trivial identities for $t > 0$:

$$\frac{1}{t} = \frac{1}{t+1} + \frac{1}{t(t+1)}, \tag{1.1}$$

$$\frac{1}{t} = \frac{1}{t+2} + \frac{1}{t(t+2)} + \frac{1}{t(t+2)}, \tag{1.2}$$

$$\frac{1}{t} = \frac{2}{3t} + \frac{1}{3t}. \tag{1.3}$$

Consider a vector $a \in \mathbb{R}^d_{>0}$. By (1.1), if $t$ is a component of $a$, we can replace this component with two new components $t + 1$ and $t(t + 1)$ to generate a vector $b \in \mathbb{R}^{d+1}_{>0}$ satisfying $\kappa(b) = \kappa(a)$. Identities (1.2) and (1.3) can be applied in a similar fashion. For every $d \in \mathbb{N}$, with the help of (1.1)–(1.3), we introduce the following maps:

$$\phi_d : \mathbb{R}^d_{>0} \to \mathbb{R}^{d+1}_{>0}, \qquad \phi_d(a) := \begin{pmatrix} a_1 \\ \vdots \\ a_{d-1} \\ a_d + 1 \\ a_d(a_d + 1) \end{pmatrix}, \tag{1.4}$$

$$\psi_d : \mathbb{R}^d_{>0} \to \mathbb{R}^{d+3}_{>0}, \qquad \psi_d(a) := \begin{pmatrix} a_1 \\ \vdots \\ a_{d-1} \\ a_d + 3 \\ a_d(a_d + 1) \\ (a_d + 1)(a_d + 3) \\ (a_d + 1)(a_d + 3) \end{pmatrix},$$

$$\xi_d : \mathbb{R}^d_{>0} \to \mathbb{R}^{d+1}_{>0} \qquad \xi_d(a) := \begin{pmatrix} a_1 \\ \vdots \\ a_{d-1} \\ \frac{3}{2}a_d \\ 3a_d \end{pmatrix}. \tag{1.5}$$

The map $\phi_d$ replaces the component $a_d$ by two other components based on (1.1), while $\xi_d$ replaces $a_d$ based on (1.3). The map $\psi_d$ acts by replacing the component $a_d$ based on (1.1) and then replacing the component $a_d + 1$ based on (1.2). Identities (1.1)–(1.3) imply

$$\kappa(\phi_d(a)) = \kappa(\psi_d(a)) = \kappa(\xi_d(a))) = \kappa(a). \tag{1.6}$$

**Lemma 10** *Let $P = T(\xi_d(a))$, where $a \in \mathcal{A}_d$ and $d \geq 2$. Then $P$ is a strongly blocked lattice-free $(d + 1)$-dimensional polytope. Furthermore, if $a_d$ is odd, $P$ is not integral.*

***Proof*** In this proof, we use the *all-ones vector*

$$\mathbb{1}_d := \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^d.$$

For the sake of brevity we introduce the notation $t := a_d$. One has $1 = \kappa(a) = \sum_{i=1}^{d} \frac{1}{a_i} \ge \sum_{i=1}^{d} \frac{1}{t} = \frac{d}{t}$, which implies $t \ge d \ge 2$. By (1.6), one has $\kappa(\xi_d(a)) = 1$ and so, by Proposition 9, $P$ is maximal lattice-free.

If $t$ is even, the polytope $P$ is integral and hence every facet of $P$ is integral, too. In view of Proposition 6, integral maximal lattice-free polytopes are strongly blocked, and so we conclude that $P$ is strongly blocked.

Assume that $t$ is odd, then the polytope $P$ has one non-integral vertex. In this case, we need to look at facets of $P$ more closely, to verify that $P$ is strongly blocked. We consider all facets of $P$.

1. The facet $F = \operatorname{conv}\{o, a_1 e_1, \dots, a_{d-1} e_{d-1}, 3t e_{d+1}\}$ is a $d$-dimensional integral integral axis-aligned simplex. Since

$$\kappa(a_1, \dots, a_{d-1}, 3t) < 1,$$

the integral point $e_1 + \dots + e_{d-1} + e_{d+1}$ is in the relative interior of $F$. Hence, $F$ is strongly blocked.

2. The facet $F = \operatorname{conv}\left\{o, a_1 e_1, \dots, a_{d-1} e_{d-1}, \frac{3}{2} t e_d\right\}$ contains the $d$-dimensional integral axis-aligned simplex

$$G := \operatorname{conv}\left\{o, a_1 e_1, \dots, a_{d-1} e_{d-1}, \frac{3t-1}{2} e_d\right\},$$

as a subset. In view of $t \ge 2$, we have

$$\kappa\left(a_1, \dots, a_{d-1}, \frac{3t-1}{2}\right) < 1,$$

which implies that the integral point $e_1 + \dots + e_d$ is in the relative interior of $G$. It follows that $F$ is strongly blocked.

3. The facet $F := \operatorname{conv}\left\{a_1 e_1, \dots, a_{d-1} e_{d-1}, \frac{3}{2} t e_d, 3t e_{d+1}\right\}$ contains the integral $d$-dimensional simplex

$$G := \operatorname{conv}\left\{a_1 e_1, \dots, a_{d-1} e_{d-1}, \frac{3t-1}{2} e_d + e_{d+1}, 3t e_{d+1}\right\}.$$

as a subset. It turns out that $\mathbb{1}_{d+1}$ is the relative interior of $G$, because $\mathbb{1}_{d+1}$ is a convex combination of the vertices of relint $G$, with positive coefficients. Indeed, the equality

$$\mathbb{1}_{d+1} = \sum_{i=1}^{d-1} \frac{1}{a_i}\big(a_i e_i\big) + \lambda\Big(\frac{3t-1}{2} e_d + e_{d+1}\Big) + \mu\big(3t e_{d+1}\big)$$

holds for $\lambda = \frac{2}{3t-1}$ and $\mu = \frac{t-1}{t(3t-1)}$, where

$$\sum_{i=1}^{d-1} \frac{1}{a_i} + \lambda + \mu = 1.$$

4. It remains to consider faces $F$ with the vertex set

$$\left\{o, a_1 e_1, \ldots, a_d e_d, \frac{3}{2} t e_d, 3t e_{d+1}\right\} \setminus \{a_i e_i\},$$

where $i \in \{1, \ldots, d+1\}$. Without loss of generality, let $i = 1$ so that

$$F = \text{conv}\left\{o, a_2 e_2, \ldots, \frac{3}{2} t e_d, 3t e_{d+1}\right\}.$$

This facet contains the integral $d$-dimensional simplex

$$G := \text{conv}\left\{o, a_2 a_2, \ldots, a_{d-1} e_{d-1}, \frac{3t-1}{2} e_d + e_{d+1}, 3t e_{d+1}\right\}.$$

Similarly to the previous case, one can check that $e_2 + \cdots + e_{d+1}$ is an integral point in the relative interior of $G$. Consequently, $F$ is strongly blocked. □

## 1.4 Proof of the Main Result

For $d \geq 4$, Nill and Ziegler [7] construct one vector $a \in \mathbb{R}^d_{>0}$ with $T(a)_I \in \mathcal{L}^d \setminus \mathcal{M}^d$. We generalize this construction and provide many further vectors $a$ with the above properties. We will also need to verify that for different choices of $a$, we get essentially different polytopes $T(a)_I$.

**Lemma 11** *Let $P$ and $Q$ be $d$-dimensional strongly blocked lattice-free polytopes such that for their integral hulls the equality $Q_I = A(P_I)$ holds for some $A \in$* $\text{Aff}(\mathbb{Z}^d)$. *Then $Q = A(P)$.*

***Proof*** Since $A$ is an affine transformation, we have

$$A(P_I) = A(\text{conv}(P \cap \mathbb{Z}^d)) = \text{conv}\, A(P \cap \mathbb{Z}^d).$$

Using $A \in \text{Aff}(\mathbb{Z}^d)$, it is straightforward to check the equality $A(P \cap \mathbb{Z}^d) = A(P) \cap \mathbb{Z}^d$. We thus conclude that $A(P_I) = A(P)_I$. The assumption $Q_I = A(P_I)$ yields $Q_I = A(P)_I$. Since $P$ is strongly blocked lattice-free, $A(P)$ too is strongly blocked lattice-free. We thus have the equality $Q_I = A(P)_I$ for strongly blocked lattice-free polytopes $Q$ and $A(P)$. To verify the assertion, it suffices to show that a strongly blocked lattice-free polytope $Q$ is uniquely determined by the knowledge of its integer hull $Q_I$. This is quite easy to see. For every strongly blocked facet $G$ of $Q_I$, the affine hull of $G$ contains a facet of $Q$. Conversely, if $F$ is an arbitrary facet of $Q$, then $G = F_I$ is a strongly blocked facet of $Q_I$. Thus, the knowledge of $Q_I$ allows to determine affine hulls of all facets of $Q$. In other words, $Q_I$ uniquely determines a hyperplane description of $Q$. □

**Lemma 12** *Let $a, b \in \mathbb{R}^d_{>0}$ be such that the equality $T(b) = A(T(a))$ holds for some $A \in \mathrm{Aff}(\mathbb{Z}^d)$. Then $a$ and $b$ coincide up to permutation of components.*

***Proof*** We use induction on $d$. For $d = 1$, the assertion is trivial. Let $d \geq 2$. One of the $d$ facets of $T(a)$ containing $o$ is mapped by $A$ to a facet of $T(b)$ that contains $o$. Without loss of generality we can assume that the facet $T(a_1, \ldots, a_{d-1}) \times \{0\}$ of $T(a)$ is mapped to the facet $T(b_1, \ldots, b_{d-1}) \times \{0\}$ of $T(b)$. By the inductive assumption, $(a_1, \ldots, a_{d-1})$ and $(b_1, \ldots, b_{d-1})$ coincide up to permutation of components. Since unimodular transformations preserve the volume, $T(a)$ and $T(b)$ have the same volume. This means, $\prod_{i=1}^d a_i = \prod_{i=1}^d b_i$. Consequently, $a_d = b_d$ and we conclude that $a$ and $b$ coincide up to permutation of components.

***Proof*** (Proof of Theorem 2) For every $a \in \mathcal{A}_d$, we introduce the $(d+5)$-dimensional integral lattice-free polytope

$$P_a := T(\eta(a))_I,$$

where

$$\eta(x) := \xi_{d+4}(\psi_{d+1}(\phi_d(x)))$$

and the functions $\xi_{d+4}$, $\psi_{d+1}$ and $\phi_d$ are defined by (1.4)–(1.5).

By (1.6) for each $a \in \mathcal{A}_d$, we have $\kappa(\eta(a)) = 1$. For $a \in \mathcal{A}_d$ the last component of $\phi_d(a)$ is even. This implies that the last component of $\psi_{d+1}(\phi_d(a))$ is odd. Thus, by Lemma 10, $T(\eta(a))$ is strongly blocked lattice-free polytope which is not integral.

Let $a, b \in \mathcal{A}_d$ be such that the polytopes $P_a$ and $P_b$ coincide up to affine unimodular transformations. Then, by Lemma 11, $T(\eta(a))$ and $T(\eta(b))$ coincide up to affine unimodular transformations. But then, by Lemma 12, $\eta(a)$ and $\eta(b)$ coincide up to permutations. Since the components of $a$ and $b$ are sorted in the ascending order, the components of $\eta(a)$ and $\beta(b)$ too are sorted in the ascending order. Thus, we arrive at the equality $\eta(a) = \eta(b)$, which implies $a = b$.

In view of Proposition 8, each $P_a$ with $a \in \mathcal{A}_d$ belongs to $\mathcal{L}^d$ but not to $\mathcal{M}^d$. Thus, the equivalence classes of the polytopes $P_a$ with $a \in \mathcal{A}_d$ with respect to identification up to affine unimodular transformations form a subset of $(\mathcal{L}^{d+5} \setminus \mathcal{M}^{d+5}) / \mathrm{Aff}(\mathbb{Z}^{d+5})$ of cardinality $|\mathcal{A}_d|$. This yields the desired assertion. □

***Proof*** (Proof of Corollary 3) The assertion is a direct consequence of Theorem 2 and the asymptotic estimate

$$\ln \ln |\mathcal{A}_d| = \Omega\left(\frac{d}{\ln d}\right)$$

of Konyagin [6, Theorem 1] (see also [5, Corollary 1.2]). □

**Remark 13** In view of the asymptotic upper bound $\ln \ln |\mathcal{A}_d| = O(d)$, determined with different degrees of precision in [8, 10] and [12, Theorem 2], the lower bound of Konyagin is optimal up to the logarithmic factor in the denominator.

Since all known elements of $\mathcal{L}^d$ are of the form $P_I$, for some strongly blocked lattice-free polytope $P$, we ask the following:

**Question 14** Do there exist polytopes $L \in \mathcal{L}^d$ which cannot be represented as $L = P_I$ for any strongly blocked lattice-free polytope $P$?

If there is a gap between the families $\mathcal{L}^d$ and the family

$$\left\{P_I \; : \; P \subseteq \mathbb{R}^d \text{ strongly blocked lattice-free polytope}\right\},$$

then it would be interesting to understand how irregular the polytopes from this gap can be. For example, one can ask the following:

**Question 15** Do there exist polytopes $L \in \mathcal{L}^d$ with the property that no facet of $L$ is blocked?

**Acknowledgements** I would like to thank Christian Wagner for valuable comments and Christian Elsholtz for pointing to [5, 8, 10].

## References

1. Averkov, G., Krümpelmann, J., Weltge, S.: Notions of maximality for integral lattice-free polyhedra: the case of dimension three. Math. Oper. Res. **42**(4), 1035–1062 (2017)
2. Averkov, G., Wagner, C., Weismantel, R.: Maximal lattice-free polyhedra: finiteness and an explicit description in dimension three. Math. Oper. Res. **36**(4), 721–742 (2011)
3. Blanco, M., Haase, C., Hofmann, J., Santos, F.: The finiteness threshold width of lattice polytopes. Trans. Am. Math. Soc. Ser. B **8**, 399–419 (2021)
4. Del Pia, A., Weismantel, R.: Relaxations of mixed integer sets from lattice-free polyhedra. Ann. Oper. Res. **240**(1), 95–117 (2016)
5. Elsholtz, C.: Egyptian fractions with odd denominators. Q. J. Math. **67**(3), 425–430 (2016)
6. Konyagin, S.V.: Double exponential lower bound for the number of representations of unity by Egyptian fractions. Math. Notes **95**(1–2), 277–281 (2014). Translation of Mat. Zametki **95** (2014), no. 2, 312–316
7. Lagarias, J.C., Ziegler, G.M.: Bounds for lattice polytopes containing a fixed number of interior points in a sublattice. Canad. J. Math. **43**(5), 1022–1035 (1991)
8. Landau, E.: Über die Klassenzahl der binären quadratischen Formen von negativer Discriminante. Math. Ann. **56**(4), 671–676 (1903)
9. Lovász, L.: Geometry of numbers and integer programming. In: Mathematical Programming, Tokyo (1988). Mathematics Applied (Japanese Ser.), vol. 6, pp. 177–201. SCIPRESS, Tokyo (1989)
10. Newman, M.: A bound for the number of conjugacy classes in a group. J. London Math. Soc. **43**, 108–110 (1968)
11. Nill, B., Ziegler, G.M.: Projecting lattice polytopes without interior lattice points. Math. Oper. Res. **36**(3), 462–467 (2011)

12. Sándor, C.: On the number of solutions of the Diophantine equation $\sum_{i=1}^{n} \frac{1}{x_i} = 1$. Period. Math. Hungar. **47**(1–2), 215–219 (2003)
13. Treutlein, J.: 3-dimensional lattice polytopes without interior lattice points. Ph.D. thesis, Eberhart Karls Universität Tübingen (2010)