The Lattice Structure of Coverings in an Incomplete Information Table with Value Similarity

Michinori Nakata[1(✉)], Norio Saito[1], Hiroshi Sakai[2], and Takeshi Fujiwara[3]

[1] Faculty of Management and Information Science, Josai International University, 1 Gumyo, Togane, Chiba 283-8555, Japan
nakatam@ieee.org, saitoh_norio@jiu.ac.jp

[2] Department of Mathematics and Computer Aided Sciences, Faculty of Engineering, Kyushu Institute of Technology, Kitakyushu, Tobata 804-8550, Japan
sakai@mns.kyutech.ac.jp

[3] Faculty of Informatics, Tokyo University of Information Sciences, 4-1 Onaridai, Wakaba-ku, Chiba 265-8501, Japan
fujiwara@rsch.tuis.ac.jp

Abstract. Based on Lipski's approach dealing with incomplete information tables, we describe lower and upper approximations using coverings under incomplete information and similarity of values. Lots of coverings, called possible coverings, on a set of attributes are derived in an incomplete information table with similarity of values, although the covering is unique in a complete information table. The family of possible coverings has a lattice structure with the minimum and maximum elements. This is true for the family of maximal descriptions, but is not for the family of minimal descriptions and the family of sets of close friends. As was shown by Lipski, what we can obtain from an information table with incomplete information is the lower and upper bounds of information granules. Using only two coverings: the minimum and maximum possible ones, we obtain the lower and upper bounds of lower and upper approximations. Therefore, there is no difficulty of the computational complexity in our approach.

Keywords: Rough sets · Incomplete information · Possible coverings · Possibly indiscernible classes · Lower and upper approximations

1 Introduction

Rough sets, constructed by Pawlak [1], are based on equality of values characterizing objects. The rough sets are used as an effective method for data mining and so on. The framework is usually used under complete information tables with no similarity of objects and creates significant results in various fields. However, value similarity often appears in the real world. Also, incomplete information ubiquitously occurs in the real world. By dealing with value similarity and

K. Honda et al. (Eds.): IUKM 2022, LNAI 13199, pp. 16–28, 2022.
https://doi.org/10.1007/978-3-030-98018-4_2

incomplete information, we can make better use of information obtained from the real world. Therefore, rough sets need to be extended to deal with incomplete information tables with value similarity.

Lipski showed that we can obtain the lower and upper bounds of the answer set of a query to an information table with incomplete information, although we cannot obtain the precise answer set [2]. This means that when trying to extract information granules from an incomplete information table, what is obtained without information loss is the lower and upper bounds of the information granules. This is true for lower and upper approximations that are the core of rough sets. Therefore, what we can obtain is the lower and upper bounds of these approximations.

It is the process proposed by Kryszkiewicz [3] that most authors use to handle incomplete information. The process a priori gives indiscernibility between an object with incomplete information and another object. Using the given indiscernibility, unique approximations are derived. Clearly, the process produces information loss from Lipski's point of view. As a result, the approach creates poor results [4–6].

We develop an approach using possible coverings without a priori giving indiscernibility between objects. First, we describe a structure of possible coverings. We will show that the lower and upper bounds of lower and upper approximations are obtained without the difficulty of computational complexity under the structure.

Lipski used a possible table as a possible world in possible world semantics. Unfortunately, we cannot use the possible table in an incomplete information table with continuous values. So, we showed a way that does not use the possible table under continuous values [7]. Using a similar way, we deal with categorical values. This means that we can deal with categorical and numerical values in the same framework.

2 Coverings in a Complete Information Table

A complete information table is constructed with $(U, \mathcal{A}, \{V(a) \mid a \in \mathcal{A}\})$, where U is the universe that consists of objects. $\mathcal{A}$ is a non-empty finite set of attributes such that $a : U \rightarrow V(a)$ for every $a \in \mathcal{A}$ where $V(a)$ is the set of values that attribute a takes.

Binary relation R_a^{δ}[1] expressing indiscernibility of objects on attribute $a \in \mathcal{A}$ is called the indiscernibility relation for a under threshold δ_a.

$$R_a^{\delta} = \{(o, o') \in U \times U \mid SIM_a(o, o') \geq \delta_a\}, \tag{1}$$

where $SIM_a(o, o')$ is the similarity degree between objects o and o' for attribute a and δ_a is a threshold fixed for attribute a.

$$SIM_a(o, o') = sim(a(o), a(o')), \tag{2}$$

[1] Unless confused, symbols without subscripts or superscripts are used.

where $sim(a(o), a(o'))$ is the similarity degree between $a(o)$ and $a(o')$. $sim(a(o), a(o'))$ is given whose values are reflexive, symmetric, and not transitive. The indiscernibility relation is a tolerance relation[2].

From indiscernibility relation R_a^δ, the indiscernible class $C(o)_a^\delta$ of object o on a is defined:

$$C_a^\delta(o) = \{o' \mid (o, o') \in R_a^\delta\}. \tag{3}$$

$C_a^\delta(o)$ is not an equivalence class.

Family $\mathcal{C}_a^\delta$ of indiscernible classes on attribute a is:

$$\mathcal{C}_a^\delta = \{C \mid C = C_a^\delta(o) \wedge o \in U\}. \tag{4}$$

Clearly, $\cup_{C \in \mathcal{C}_a^\delta} C = U$. Based on Zakowski [9], $\mathcal{C}_a^\delta$ is a covering, which is unique for a. Under $\mathcal{C}_a^\delta$, minimal description $Md\mathcal{C}_a^\delta(o)$ of object o, formulated by [10], is:

$$Md\mathcal{C}_a^\delta(o) = \{C \in \mathcal{C}_a^\delta \mid o \in C \wedge \forall C' \in \mathcal{C}_a^\delta(o \in C' \wedge C' \subseteq C \Rightarrow C = C')\}. \tag{5}$$

Set $CFriend_{\mathcal{C}_a^\delta}(o)$ of close friends of o with respect to $\mathcal{C}_a^\delta$, proposed by [11], is:

$$CFriend_{\mathcal{C}_a^\delta}(o) = \cup_{C \in Md\mathcal{C}_a^\delta(o)} C. \tag{6}$$

Also, maximal description $MD\mathcal{C}_a^\delta(o)$ of object o, described by [11,12], is:

$$MD\mathcal{C}_a^\delta(o) = \{C \in \mathcal{C}_a^\delta \mid o \in C \wedge \forall C' \in \mathcal{C}_a^\delta(o \in C' \wedge C' \supseteq C \Rightarrow C = C')\}. \tag{7}$$

Using covering $\mathcal{C}_a^\delta$, lower approximation $\underline{apr}_a^\delta(\mathcal{O})$ and upper approximation $\overline{apr}_a^\delta(\mathcal{O})$ for a of set $\mathcal{O}$ of objects are:

$$\underline{apr}_a^\delta(\mathcal{O}) = \{o \in U \mid C_a^\delta(o) \subseteq \mathcal{O} \wedge C_a^\delta(o) \in \mathcal{C}_a^\delta\}, \tag{8}$$

$$\overline{apr}_a^\delta(\mathcal{O}) = \{o \in U \mid C_a^\delta(o) \cap \mathcal{O} \neq \emptyset \wedge C_a^\delta(o) \in \mathcal{C}_a^\delta\}. \tag{9}$$

3 Coverings in an Incomplete Information Table

An incomplete information table has $a : U \rightarrow s_a$ for every $a \in \mathcal{A}$ where s_a is the family of disjunctive sets of values over $V(a)$. So, value $v \in a(o)$ is a possible value that may be the actual one of attribute a in object o.

A covering on a is unique in a complete information table, but lots of coverings, called possible coverings, are derived in an incomplete information table [13,14], although some authors deal with only a covering [15–17]. A possible covering is derived from a possible indiscernibility relation. Many possible indiscernibility relations is derived in an incomplete information table. The number of possible indiscernibility relations may grow exponentially as the number of values with incomplete information increases. However, this does not cause any

[2] See [8] for properties of tolerance relations.

difficulties due to computational complexity in obtaining the lower and upper bounds of approximations, as is shown later.

Family FPR_a^δ of possible indiscernibility relations, as is shown in [7,18], is constructed using certain pairs and possible pairs of objects. The certain pair surely has the same characteristic value, while the possible pair may have the same characteristic value. Set SR_a^δ of certain pairs on attribute a is:

$$SR_a^\delta = \{(o, o') \in U \times U \mid (o = o') \vee (\forall u \in a(o) \forall v \in a(o') sim(u, v) \geq \delta_a)\}. \quad (10)$$

Set MPR_a^δ of possible pairs on attribute a is:

$$MPR_a^\delta = \{(o, o') \in U \times U \mid \exists u \in a(o) \exists v \in a(o') sim(u, v) \geq \delta_a\} \backslash SR_a^\delta. \quad (11)$$

Using these two sets, family FPR_a^δ of possible indiscernibility relations is:

$$FPR_a^\delta = \{PR \mid PR = SR_a^\delta \cup e \wedge e \in \mathcal{P}(MPPR_a^\delta)\}, \quad (12)$$

where each element is a possible indiscernibility relation and $\mathcal{P}(MPPR_a^\delta)$ is the power set of $MPPR_a^\delta$ that is:

$$MPPR_a^\delta = \{\{(o', o), (o, o')\} | (o', o) \in MPR_a^\delta\}. \quad (13)$$

Clearly, FPR_a^δ is a lattice for set inclusion. SR_a is the minimum possible indiscernibility relation in FPR_a^δ, whereas $SR_a^\delta \cup MPR_a^\delta$ is the maximum possible indiscernibility relation. All the possible indiscernibility relations do not correspond to the indiscernibility relation derived from a possible table where every attribute value is replaced by a possible value in the original information table. The possible indiscernibility relation without a corresponding possible table is artificial. However, the minimum and the maximum possible indiscernibility relations are equal to the intersection and the union of indiscernibility relations derived from possible tables, respectively. The minimum possible indiscernibility relation contains only the pairs of objects that are surely indiscernible with each other, while the maximum possible indiscernibility relation contains all the pairs that are possibly indiscernible. Only the two possible indiscernibility relations are used to derive the lower and upper bounds of approximations, as is shown later. Artificially possible indiscernibility relations are rather useful to derive the lower and upper bounds of approximations.

Example 1. Let similarity degree $sim(u, v)$ on $V(a_1) = \{a, b, c, d, e, f\}$ and incomplete information table IT be as follows:

$$sim(u, v) = \begin{pmatrix} 1 & 0.9 & 0.9 & 0.6 & 0.2 & 0.4 \\ 0.9 & 1 & 0.8 & 0.8 & 0.1 & 0.5 \\ 0.9 & 0.8 & 1 & 0.3 & 0.2 & 0.4 \\ 0.6 & 0.8 & 0.3 & 1 & 0.9 & 0.6 \\ 0.2 & 0.1 & 0.2 & 0.9 & 1 & 0.7 \\ 0.4 & 0.5 & 0.4 & 0.6 & 0.7 & 1 \end{pmatrix}.$$

IT

U	a_1	a_2
o_1	$< a >$	$< x >$
o_2	$< b, e >$	$< x, y >$
o_3	$< c >$	$< x >$
o_4	$< d >$	$< y >$
o_5	$< e >$	$< z >$
o_6	$< f >$	$< z >$

In incomplete information table IT with $U = \{o_1, o_2, o_3, o_4, o_5, o_6\}$, let threshold δ_{a_1} be 0.75 on attribute a_1. Expression $< b, e >$ of a disjunctive set means that the actual value is b or e. The set of certain pairs of indiscernible objects on a_1 under the above $sim(u, v)$ is $\{(o_1, o_1), (o_1, o_3), (o_2, o_2), (o_2, o_4), (o_3, o_3), (o_3, o_1), (o_4, o_4), (o_4, o_2), (o_4, o_5), (o_5, o_5), (o_5, o_4), (o_6, o_6)\}$. The set of possible pairs of indiscernible objects is $\{(o_1, o_2), (o_2, o_1), (o_2, o_3), (o_3, o_2), (o_2, o_5), (o_5, o_2)\}$. Using formulae (10)–(13), the family of possible indiscernibility relations is obtained: $PR^{0.75}_{a_1} = \{PR_1, \cdots, PR_8\}$, and 8 possible indiscernibility relations are:

$$
\begin{aligned}
PR_1 = \{&(o_1, o_1), (o_1, o_3), (o_2, o_2), (o_2, o_4), (o_3, o_3), (o_3, o_1), (o_4, o_4), (o_4, o_2), (o_4, o_5),\\
&(o_5, o_5), (o_5, o_4), (o_6, o_6)\},\\
PR_2 = \{&(o_1, o_1), (o_1, o_3), (o_2, o_2), (o_2, o_4), (o_3, o_3), (o_3, o_1), (o_4, o_4), (o_4, o_2), (o_4, o_5),\\
&(o_5, o_5), (o_5, o_4), (o_6, o_6), (o_1, o_2), (o_2, o_1)\},\\
PR_3 = \{&(o_1, o_1), (o_1, o_3), (o_2, o_2), (o_2, o_4), (o_3, o_3), (o_3, o_1), (o_4, o_4), (o_4, o_2), (o_4, o_5),\\
&(o_5, o_5), (o_5, o_4), (o_6, o_6), (o_2, o_3), (o_3, o_2)\},\\
PR_4 = \{&(o_1, o_1), (o_1, o_3), (o_2, o_2), (o_2, o_4), (o_3, o_3), (o_3, o_1), (o_4, o_4), (o_4, o_2), (o_4, o_5),\\
&(o_5, o_5), (o_5, o_4), (o_6, o_6), (o_2, o_5), (o_5, o_2)\},\\
PR_5 = \{&(o_1, o_1), (o_1, o_3), (o_2, o_2), (o_2, o_4), (o_3, o_3), (o_3, o_1), (o_4, o_4), (o_4, o_2), (o_4, o_5),\\
&(o_5, o_5), (o_5, o_4), (o_6, o_6), (o_1, o_2), (o_2, o_1), (o_2, o_3), (o_3, o_2)\},\\
PR_6 = \{&(o_1, o_1), (o_1, o_3), (o_2, o_2), (o_2, o_4), (o_3, o_3), (o_3, o_1), (o_4, o_4), (o_4, o_2), (o_4, o_5),\\
&(o_5, o_5), (o_5, o_4), (o_6, o_6), (o_1, o_2), (o_2, o_1), (o_2, o_5), (o_5, o_2)\},\\
PR_7 = \{&(o_1, o_1), (o_1, o_3), (o_2, o_2), (o_2, o_4), (o_3, o_3), (o_3, o_1), (o_4, o_4), (o_4, o_2), (o_4, o_5),\\
&(o_5, o_5), (o_5, o_4), (o_6, o_6), (o_2, o_3), (o_3, o_2), (o_2, o_5), (o_5, o_2)\},\\
PR_8 = \{&(o_1, o_1), (o_1, o_3), (o_2, o_2), (o_2, o_4), (o_3, o_3), (o_3, o_1), (o_4, o_4), (o_4, o_2), (o_4, o_5),\\
&(o_5, o_5), (o_5, o_4), (o_6, o_6), (o_1, o_2), (o_2, o_1), (o_2, o_3), (o_3, o_2), (o_2, o_5), (o_5, o_2)\}.
\end{aligned}
$$

From each possible indiscernibility relation $PR^{\delta}_{a,j}$ in FPR^{δ}_{a}, possible indiscernible class $C(o)^{\delta}_{a,j}$ on attribute a for object o is:

$$C(o)^{\delta}_{a,j} = \{o' \mid (o, o') \in PR^{\delta}_{a,j} \wedge PR^{\delta}_{a,j} \in FPR^{\delta}_{a}\}. \tag{14}$$

Proposition 1. If $PR^{\delta}_{a,k} \subseteq PR^{\delta}_{a,l}$, then $C(o)^{\delta}_{a,k} \subseteq C(o)^{\delta}_{a.l}$.

From this proposition, the family of possible indiscernible classes for an object is a lattice for set inclusion.

Example 2. (continuation from Example 1). For object o_1, $C(o_1)^{0.75}_{a_1,j} = \{o_1, o_3\}\ for j = 1, 3, 4, 7$, $C(o_1)^{0.75}_{a_1,j} = \{o_1, o_2, o_3\}\ for\ j = 2, 5, 6, 8$. For object o_2, $C(o_2)^{0.75}_{a_1,1} = \{o_2, o_4\}$, $C(o_2)^{0.75}_{a_1,2} = \{o_1, o_2, o_4\}$, $C(o_2)^{0.75}_{a_1,3} = \{o_2, o_3, o_4\}$, $C(o_2)^{0.75}_{a_1,4} = \{o_2, o_4, o_5\}$, $C(o_2)^{0.75}_{a_1,5} = \{o_1, o_2, o_3, o_4\}$, $C(o_2)^{0.75}_{a_1,6} = \{o_1, o_2, o_4, o_5\}$, $C(o_2)^{0.75}_{a_1,7} = \{o_2, o_3, o_4, o_5\}$, $C(o_2)^{0.75}_{a_1,8} = \{o_1, o_2, o_3, o_4, o_5\}$. For object o_3, $C(o_3)^{0.75}_{a_1,j} = \{o_1, o_3\}\ for\ j = 1, 2, 4, 6$, $C(o_3)^{0.75}_{a_1,j} = \{o_1, o_2, o_3\}\ for\ j = 3, 5, 7, 8$.

For object o_4, $C(o_4)^{0.75}_{a_1,j} = \{o_2, o_4, o_5\}$ for $j = 1, \ldots, 8$. For object o_5, $C(o_5)^{0.75}_{a_1,j} = \{o_4, o_5\}$ for $j = 1, 2, 3, 5$, $C(o_5)^{0.75}_{a_1,j} = \{o_2, o_4, o_5\}$ for $j = 4, 6, 7, 8$, For object o_6, $C(o_6)^{0.75}_{a_1,j} = \{o_6\}$ for $j = 1, \ldots, 8$.

A possible covering is derived from a possible indiscernibility relation. Possible covering $PC^{\delta}_{a,j}$ obtained from possible indiscernibility relation $PR^{\delta}_{a,j}$ is:

$$P\mathcal{C}^{\delta}_{a,j} = \{e \mid e = C(o)^{\delta}_{a,j} \wedge o \in U\}. \tag{15}$$

One of possible coverings is the actual covering, although we cannot know it without additional information.

Proposition 2. If $PR^{\delta}_{a,k} \subseteq PR^{\delta}_{a,l}$, then $P\mathcal{C}^{\delta}_{a,k} \sqsubseteq P\mathcal{C}^{\delta}_{a,l}$[3].

From Proposition 2 family $FP\mathcal{C}^{\delta}_{a}$ of possible coverings is a lattice for $\sqsubseteq$.

Proposition 3. If $PR^{\delta}_{a,k} \subseteq PR^{\delta}_{a,l}$, then $\forall o \in U$ $MD\mathcal{C}^{\delta}_{a,k}(o) \subseteq MD\mathcal{C}^{\delta}_{a,l}(o)$ where $MD\mathcal{C}^{\delta}_{a,k}(o)$ is the maximal description of o with respect to $P\mathcal{C}^{\delta}_{a}$ in $PR^{\delta}_{k,a}$.

From Proposition 3 family $FMD\mathcal{C}^{\delta}_{a}(o)$ of maximal descriptions is a lattice for $\subseteq$.

Example 3. Possibly indiscernible classes of objects are obtained in each possible indiscernibility relation PR_i with $i = 1, \ldots, 8$ of Example 1.

In PR_1, $C(o_1)_{a_1} = \{o_1, o_3\}$, $C(o_2)_{a_1} = \{o_2, o_4\}$, $C(o_3)_{a_1} = \{o_1, o_3\}$, $C(o_4)_{a_1} = \{o_2, o_4, o_5\}$, $C(o_5)_{a_1} = \{o_4, o_5\}$, and $C(o_6)_{a_1} = \{o_6\}$.
In PR_2, $C(o_1)_{a_1} = \{o_1, o_2, o_3\}$, $C(o_2)_{a_1} = \{o_1, o_2, o_4\}$, $C(o_3)_{a_1} = \{o_1, o_3\}$, $C(o_4)_{a_1} = \{o_2, o_4, o_5\}$, $C(o_5)_{a_1} = \{o_4, o_5\}$, and $C(o_6)_{a_1} = \{o_6\}$.
In PR_3, $C(o_1)_{a_1} = \{o_1, o_3\}$, $C(o_2)_{a_1} = \{o_2, o_3, o_4\}$, $C(o_3)_{a_1} = \{o_1, o_2, o_3\}$, $C(o_4)_{a_1} = \{o_2, o_4, o_5\}$, $C(o_5)_{a_1} = \{o_4, o_5\}$, and $C(o_6)_{a_1} = \{o_6\}$.
In PR_4, $C(o_1)_{a_1} = \{o_1, o_3\}$, $C(o_2)_{a_1} = \{o_2, o_4, o_5\}$, $C(o_3)_{a_1} = \{o_1, o_3\}$, $C(o_4)_{a_1} = \{o_2, o_4, o_5\}$, $C(o_5)_{a_1} = \{o_2, o_4, o_5\}$, and $C(o_6)_{a_1} = \{o_6\}$.
In PR_5, $C(o_1)_{a_1} = \{o_1, o_2, o_3\}$, $C(o_2)_{a_1} = \{o_1, o_2, o_3, o_4\}$, $C(o_3)_{a_1} = \{o_1, o_2, o_3\}$, $C(o_4)_{a_1} = \{o_2, o_4, o_5\}$, $C(o_5)_{a_1} = \{o_4, o_5\}$, and $C(o_6)_{a_1} = \{o_6\}$.
In PR_6, $C(o_1)_{a_1} = \{o_1, o_2, o_3\}$, $C(o_2)_{a_1} = \{o_1, o_2, o_4, o_5\}$, $C(o_3)_{a_1} = \{o_1, o_3\}$, $C(o_4)_{a_1} = \{o_2, o_4, o_5\}$, $C(o_5)_{a_1} = \{o_2, o_4, o_5\}$, and $C(o_6)_{a_1} = \{o_6\}$.
In PR_7, $C(o_1)_{a_1} = \{o_1, o_3\}$, $C(o_2)_{a_1} = \{o_2, o_3, o_4, o_5\}$, $C(o_3)_{a_1} = \{o_1, o_2, o_3\}$, $C(o_4)_{a_1} = \{o_2, o_4, o_5\}$, $C(o_5)_{a_1} = \{o_2, o_4, o_5\}$, and $C(o_6)_{a_1} = \{o_6\}$.
In PR_8, $C(o_1)_{a_1} = \{o_1, o_2, o_3\}$, $C(o_2)_{a_1} = \{o_1, o_2, o_3, o_4, o_5\}$, $C(o_3)_{a_1} = \{o_1, o_2, o_3\}$, $C(o_4)_{a_1} = \{o_2, o_4, o_5\}$, $C(o_5)_{a_1} = \{o_2, o_4, o_5\}$, and $C(o_6)_{a_1} = \{o_6\}$.

[3] $\sqsubseteq$ is defined as $\mathcal{E} \sqsubseteq \mathcal{E}'$ if $\forall E \in \mathcal{E} \exists E' \in \mathcal{E}' \wedge E \subseteq E'$.

Using these possibly indiscernible classes, possible coverings are obtained as follows:

$$
\begin{aligned}
PC_1 &= \{\{o_1, o_3\}, \{o_2, o_4\}, \{o_2, o_4, o_5\}, \{o_4, o_5\}, \{o_6\}\},\\
PC_2 &= \{\{o_1, o_2, o_3\}, \{o_1, o_2, o_4\}, \{o_1, o_3\}, \{o_2, o_4, o_5\}, \{o_4, o_5\}, \{o_6\}\},\\
PC_3 &= \{\{o_1, o_3\}, \{o_2, o_3, o_4\}, \{o_1, o_2, o_3\}, \{o_2, o_4, o_5\}, \{o_4, o_5\}, \{o_6\}\},\\
PC_4 &= \{\{o_1, o_3\}, \{o_2, o_4, o_5\}, \{o_6\}\},\\
PC_5 &= \{\{o_1, o_2, o_3\}, \{o_1, o_2, o_3, o_4\}, \{o_2, o_4, o_5\}, \{o_4, o_5\}, \{o_6\}\},\\
PC_6 &= \{\{o_1, o_2, o_3\}, \{o_1, o_2, o_4, o_5\}, \{o_1, o_3\}, \{o_2, o_4, o_5\}, \{o_6\}\},\\
PC_7 &= \{\{o_1, o_3\}, \{o_2, o_3, o_4, o_5\}, \{o_1, o_2, o_3\}, \{o_2, o_4, o_5\}, \{o_6\}\},\\
PC_8 &= \{\{o_1, o_2, o_3\}, \{o_1, o_2, o_3, o_4, o_5\}, \{o_2, o_4, o_5\}, \{o_6\}\}.
\end{aligned}
$$

Minimal descriptions, sets of close friends, and maximal descriptions are as follows:

$$
\begin{aligned}
\text{For } PC_1, MdC(o_1) &= \{\{o_1, o_3\}\}, MdC(o_2) = \{\{o_2, o_4\}\}, MdC(o_3) = \{\{o_1, o_3\}\},\\
MdC(o_4) &= \{\{o_2, o_4\}, \{o_4, o_5\}\}, MdC(o_5) = \{\{o_4, o_5\}\},\\
MdC(o_6) &= \{\{o_6\}\},\\
CFriend_C(o_1) &= \{o_1, o_3\}, CFriend_C(o_2) = \{o_2, o_4\},\\
CFriend_C(o_3) &= \{o_1, o_3\}, CFriend_C(o_4) = \{o_2, o_4, o_5\},\\
CFriend_C(o_5) &= \{o_4, o_5\}, CFriend_C(o_6) = \{o_6\},\\
MDC(o_1) &= \{\{o_1, o_3\}\}, MDC(o_2) = \{\{o_2, o_4, o_5\}\}, MDC(o_3) = \{\{o_1, o_3\}\},\\
MDC(o_4) &= \{\{o_2, o_4, o_5\}\}, MDC(o_5) = \{\{o_2, o_4, o_5\}\},\\
MDC(o_6) &= \{\{o_6\}\},\\
\text{For } PC_2, MdC(o_1) &= \{\{o_1, o_2, o_4\}, \{o_1, o_3\}\},\\
MdC(o_2) &= \{\{o_1, o_2, o_3\}, \{o_1, o_2, o_4\}, \{o_2, o_4, o_5\}\},\\
MdC(o_3) &= \{\{o_1, o_3\}\}, MdC(o_4) = \{\{o_1, o_2, o_4\}, \{o_4, o_5\}\},\\
MdC(o_5) &= \{\{o_4, o_5\}\}, MdC(o_6) = \{\{o_6\}\},\\
CFriend_C(o_1) &= \{o_1, o_2, o_3, o_4\}\}, CFriend_C(o_2) = \{o_1, o_2, o_3, o_4, o_5\},\\
CFriend_C(o_3) &= \{o_1, o_3\}, CFriend_C(o_4) = \{o_1, o_2, o_4, o_5\},\\
CFriend_C(o_5) &= \{o_4, o_5\}, CFriend_C(o_6) = \{o_6\},\\
MDC(o_1) &= \{\{o_1, o_2, o_4\}, \{o_1, o_2, o_3\}\},\\
MDC(o_2) &= \{\{o_1, o_2, o_3\}, \{o_1, o_2, o_4\}, \{o_2, o_4, o_5\}\},\\
MDC(o_3) &= \{\{o_1, o_2, o_3\}\}, MDC(o_4) = \{\{o_1, o_2, o_4\}, \{o_2, o_4, o_5\}\},\\
MDC(o_5) &= \{\{o_2, o_4, o_5\}\}, MDC(o_6) = \{\{o_6\}\},\\
\text{For } PC_3, MdC(o_1) &= \{\{o_1, o_3\}\}, MdC(o_2) = \{\{o_2, o_3, o_4\}, \{o_1, o_2, o_3\}, \{o_2, o_4, o_5\}\},\\
MdC(o_3) &= \{\{o_1, o_3\}, \{o_2, o_3, o_4\}\}, MdC(o_4) = \{\{o_2, o_3, o_4\}, \{o_4, o_5\}\},\\
MdC(o_5) &= \{\{o_4, o_5\}\}, MdC(o_6) = \{\{o_6\}\},\\
CFriend_C(o_1) &= \{o_1, o_3\}, CFriend_C(o_2) = \{o_1, o_2, o_3, o_4, o_5\},\\
CFriend_C(o_3) &= \{o_1, o_2, o_3, o_4\}, CFriend_C(o_4) = \{o_2, o_3, o_4, o_5\},\\
CFriend_C(o_5) &= \{o_4, o_5\}, CFriend_C(o_6) = \{o_6\},\\
MDC(o_1) &= \{\{o_1, o_2, o_3\}\}, MDC(o_2) = \{\{o_2, o_3, o_4\}, \{o_1, o_2, o_3\}, \{o_2, o_4, o_5\}\},\\
MDC(o_3) &= \{\{o_2, o_3, o_4\}, \{o_1, o_2, o_3\}\},
\end{aligned}
$$

$MD\mathcal{C}(o_4) = \{\{o_2, o_3, o_4\}, \{o_2, o_4, o_5\}\}$,
$MD\mathcal{C}(o_5) = \{\{o_2, o_4, o_5\}\}, MD\mathcal{C}(o_6) = \{\{o_6\}\}$,
For $P\mathcal{C}_4, Md\mathcal{C}(o_1) = \{\{o_1, o_3\}\}, Md\mathcal{C}(o_2) = \{\{o_2, o_4, o_5\}\}, Md\mathcal{C}(o_3) = \{\{o_1, o_3\}\}$,
$Md\mathcal{C}(o_4) = \{\{o_2, o_4, o_5\}\}, Md\mathcal{C}(o_5) = \{\{o_2, o_4, o_5\}\}, Md\mathcal{C}(o_6) = \{\{o_6\}\}$,
$CFriend_{\mathcal{C}}(o_1) = \{o_1, o_3\}, CFriend_{\mathcal{C}}(o_2) = \{o_2, o_4, o_5\}$,
$CFriend_{\mathcal{C}}(o_3) = \{o_1, o_3\}, CFriend_{\mathcal{C}}(o_4) = \{o_2, o_4, o_5\}$,
$CFriend_{\mathcal{C}}(o_5) = \{o_2, o_4, o_5\}, CFriend_{\mathcal{C}}(o_6) = \{o_6\}$,
$MD\mathcal{C}(o_1) = \{\{o_1, o_3\}\}, MD\mathcal{C}(o_2) = \{\{o_2, o_4, o_5\}\}, MD\mathcal{C}(o_3) = \{\{o_1, o_3\}\}$,
$MD\mathcal{C}(o_4) = \{\{o_2, o_4, o_5\}\}, MD\mathcal{C}(o_5) = \{\{o_2, o_4, o_5\}\}, MD\mathcal{C}(o_6) = \{\{o_6\}\}$,
For $P\mathcal{C}_5, Md\mathcal{C}(o_1) = \{\{o_1, o_2, o_3\}\}, Md\mathcal{C}(o_2) = \{\{o_1, o_2, o_3\}, \{o_2, o_4, o_5\}\}$,
$Md\mathcal{C}(o_3) = \{\{o_1, o_2, o_3\}\}, Md\mathcal{C}(o_4) = \{\{o_1, o_2, o_3, o_4\}, \{o_4, o_5\}\}$,
$Md\mathcal{C}(o_5) = \{\{o_4, o_5\}\}, Md\mathcal{C}(o_6) = \{\{o_6\}\}$,
$CFriend_{\mathcal{C}}(o_1) = \{o_1, o_2, o_3\}, CFriend_{\mathcal{C}}(o_2) = \{o_1, o_2, o_3, o_4, o_5\}$,
$CFriend_{\mathcal{C}}(o_3) = \{o_1, o_2, o_3\}, CFriend_{\mathcal{C}}(o_4) = \{o_1, o_2, o_3, o_4, o_5\}$,
$CFriend_{\mathcal{C}}(o_5) = \{o_4, o_5\}, CFriend_{\mathcal{C}}(o_6) = \{o_6\}$,
$MD\mathcal{C}(o_1) = \{\{o_1, o_2, o_3, o_4\}\}, MD\mathcal{C}(o_2) = \{\{o_1, o_2, o_3, o_4\}, \{o_2, o_4, o_5\}\}$,
$MD\mathcal{C}(o_3) = \{\{o_1, o_2, o_3, o_4\}\}, MD\mathcal{C}(o_4) = \{\{o_1, o_2, o_3, o_4\}, \{o_2, o_4, o_5\}\}$,
$MD\mathcal{C}(o_5) = \{\{o_2, o_4, o_5\}\}, MD\mathcal{C}(o_6) = \{\{o_6\}\}$,
For $P\mathcal{C}_6, Md\mathcal{C}(o_1) = \{\{o_1, o_3\}, \{o_1, o_2, o_4, o_5\}\}$,
$Md\mathcal{C}(o_2) = \{\{o_1, o_2, o_3\}, \{o_2, o_4, o_5\}\}, Md\mathcal{C}(o_3) = \{\{o_1, o_3\}\}$,
$Md\mathcal{C}(o_4) = \{\{o_2, o_4, o_5\}\}, Md\mathcal{C}(o_5) = \{\{o_2, o_4, o_5\}\}, Md\mathcal{C}(o_6) = \{\{o_6\}\}$,
$CFriend_{\mathcal{C}}(o_1) = \{o_1, o_2, o_3, o_4, o_5\}, CFriend_{\mathcal{C}}(o_2) = \{o_1, o_2, o_3, o_4, o_5\}$,
$CFriend_{\mathcal{C}}(o_3) = \{o_1, o_3\}, CFriend_{\mathcal{C}}(o_4) = \{o_2, o_4, o_5\}$,
$CFriend_{\mathcal{C}}(o_5) = \{o_2, o_4, o_5\}, CFriend_{\mathcal{C}}(o_6) = \{\{o_6\}\}$,
$MD\mathcal{C}(o_1) = \{\{o_1, o_2, o_3\}, \{o_1, o_2, o_4, o_5\}\}$,
$MD\mathcal{C}(o_2) = \{\{o_1, o_2, o_3\}, \{o_1, o_2, o_4, o_5\}\}, MD\mathcal{C}(o_3) = \{\{o_1, o_2, o_3\}\}$,
$MD\mathcal{C}(o_4) = \{\{o_1, o_2, o_4, o_5\}\}, MD\mathcal{C}(o_5) = \{\{o_1, o_2, o_4, o_5\}\}, MD\mathcal{C}(o_6) = \{\{o_6\}\}$,
For $P\mathcal{C}_7, Md\mathcal{C}(o_1) = \{\{o_1, o_3\}\}, Md\mathcal{C}(o_2) = \{\{o_1, o_2, o_3\}, \{o_2, o_4, o_5\}\}$,
$Md\mathcal{C}(o_3) = \{\{o_1, o_3\}, \{o_2, o_3, o_4, o_5\}\}, Md\mathcal{C}(o_4) = \{\{o_2, o_4, o_5\}\}$,
$Md\mathcal{C}(o_5) = \{\{o_2, o_4, o_5\}\}, Md\mathcal{C}(o_6) = \{\{o_6\}\}$,
$CFriend_{\mathcal{C}}(o_1) = \{o_1, o_3\}, CFriend_{\mathcal{C}}(o_2) = \{o_1, o_2, o_3, o_4, o_5\}$,
$CFriend_{\mathcal{C}}(o_3) = \{o_1, o_2, o_3, o_4, o_5\}, CFriend_{\mathcal{C}}(o_4) = \{o_2, o_4, o_5\}$,
$CFriend_{\mathcal{C}}(o_5) = \{o_2, o_4, o_5\}, CFriend_{\mathcal{C}}(o_6) = \{o_6\}$,
$MD\mathcal{C}(o_1) = \{\{o_1, o_2, o_3\}\}, MD\mathcal{C}(o_2) = \{\{o_1, o_2, o_3\}, \{o_2, o_3, o_4, o_5\}\}$,
$MD\mathcal{C}(o_3) = \{\{o_1, o_2, o_3\}, \{o_2, o_3, o_4, o_5\}\}, MD\mathcal{C}(o_4) = \{\{o_2, o_3, o_4, o_5\}\}$,
$MD\mathcal{C}(o_5) = \{\{o_2, o_3, o_4, o_5\}\}, MD\mathcal{C}(o_6) = \{\{o_6\}\}$,
For $P\mathcal{C}_8, Md\mathcal{C}(o_1) = \{\{o_1, o_2, o_3\}\}, Md\mathcal{C}(o_2) = \{\{o_1, o_2, o_3\}, \{o_2, o_4, o_5\}\}$,
$Md\mathcal{C}(o_3) = \{\{o_1, o_2, o_3\}\}, Md\mathcal{C}(o_4) = \{\{o_2, o_4, o_5\}\}$,
$Md\mathcal{C}(o_5) = \{\{o_2, o_4, o_5\}\}, Md\mathcal{C}(o_6) = \{\{o_6\}\}$.
$CFriend_{\mathcal{C}}(o_1) = \{o_1, o_2, o_3\}, CFriend_{\mathcal{C}}(o_2) = \{o_1, o_2, o_3, o_4, o_5\}$,
$CFriend_{\mathcal{C}}(o_3) = \{o_1, o_2, o_3\}, CFriend_{\mathcal{C}}(o_4) = \{o_2, o_4, o_5\}$,
$CFriend_{\mathcal{C}}(o_5) = \{o_2, o_4, o_5\}, CFriend_{\mathcal{C}}(o_6) = \{o_6\}$.

$$MD\mathcal{C}(o_1) = MD\mathcal{C}(o_2) = MD\mathcal{C}(o_3) = MD\mathcal{C}(o_4) = MD\mathcal{C}(o_5)$$
$$= \{\{o_1, o_2, o_3, o_4, o_5\}\}, MD\mathcal{C}(o_6) = \{\{o_6\}\}.$$

The family of possible coverings in Example 3 has the lattice structure for $\sqsubseteq$, which is shown in Fig. 1.

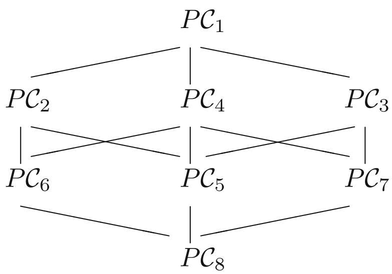

Fig. 1. Lattice structure

$P\mathcal{C}_1$ is the minimum element, whereas $P\mathcal{C}_8$ is the maximum element. On the other hand, the family of minimum descriptions is not a lattice for $\sqsubseteq$; for example, as is clarified for minimum descriptions in $P\mathcal{C}_6$ and $P\mathcal{C}_8$ in Example 3. Also, the family of sets of close friends of an object is not so.

By using possible covering $P\mathcal{C}^{\delta}_{a,j}$, lower and upper approximations of set $\mathcal{O}$ of objects in $PR^{\delta}_{a,j}$ are:

$$\underline{apr}^{\delta}_{a,j}(\mathcal{O}) = \{o \in U \mid C^{\delta}_{a,j}(o) \subseteq \mathcal{O} \wedge C^{\delta}_{a,j}(o) \in P\mathcal{C}^{\delta}_{a,j} \wedge P\mathcal{C}^{\delta}_{a,j} \in FP\mathcal{C}^{\delta}_{a}\}, \quad (16)$$

$$\overline{apr}^{\delta}_{a,j}(\mathcal{O}) = \{o \in U \mid C^{\delta}_{a,j}(o) \cap \mathcal{O} \neq \emptyset \wedge C^{\delta}_{a,j}(o) \in P\mathcal{C}^{\delta}_{a,j} \wedge P\mathcal{C}^{\delta}_{j} \in FP\mathcal{C}^{\delta}_{a}\}. \quad (17)$$

Proposition 4. If $P\mathcal{C}_{a,k} \sqsubseteq P\mathcal{C}^{\delta}_{a,l}$ for possible indiscernibility relations $P\mathcal{C}^{\delta}_{a,k}, P\mathcal{C}^{\delta}_{a,l} \in FP\mathcal{C}^{\delta}_{a}$, then $\underline{apr}^{\delta}_{a,k}(\mathcal{O}) \supseteq \underline{apr}^{\delta}_{a,l}(\mathcal{O})$, and $\overline{apr}^{\delta}_{a,k}(\mathcal{O}) \subseteq \overline{apr}^{\delta}_{a,l}(\mathcal{O})$.

This proposition shows that the families of lower and upper approximations under possible coverings are also lattices for set inclusion, respectively. Unfortunately this does not hold in approximations using minimal descriptions and sets of close friends, although various types of covering-based approximation are proposed [11,19–21].

We aggregate the lower and upper approximations under possible coverings. Certain lower approximation $S\underline{apr}^{\delta}_{a}(\mathcal{O})$ of set $\mathcal{O}$ of objects, the lower bound of the lower approximation, is:

$$S\underline{apr}^{\delta}_{a}(\mathcal{O}) = \{o \in U \mid \forall P\mathcal{C}^{\delta}_{a,j} \in FP\mathcal{C}^{\delta}_{a}\ o \in \underline{apr}^{\delta}_{a,j}(\mathcal{O})\}. \quad (18)$$

Possible lower approximation $P\underline{apr}^{\delta}_{a}(\mathcal{O})$, the upper bound of the lower approximation, is:

$$P\underline{apr}^{\delta}_{a}(\mathcal{O}) = \{o \in U \mid \exists P\mathcal{C}^{\delta}_{a,j} \in FP\mathcal{C}^{\delta}_{a}\ o \in \underline{apr}^{\delta}_{a,j}(\mathcal{O})\}. \quad (19)$$

Certain upper approximations $S\overline{apr}_a^{\delta}(\mathcal{O})$, the lower bound of the upper approximation, is:

$$S\overline{apr}_a^{\delta}(\mathcal{O}) = \{o \in U \mid \forall P\mathcal{C}_{a,j}^{\delta} \in FP\mathcal{C}_a^{\delta}\ o \in \overline{apr}_{a,j}^{\delta}(\mathcal{O})\}. \tag{20}$$

Possible upper approximation $P\overline{apr}_a^{\delta}(\mathcal{O})$, the upper bound of the upper approximation, is:

$$P\overline{apr}_a^{\delta}(\mathcal{O}) = \{o \in U \mid \exists P\mathcal{C}_{a,j}^{\delta} \in FP\mathcal{C}_a^{\delta}\ o \in \overline{apr}_{a,j}^{\delta}(\mathcal{O})\}. \tag{21}$$

Using Proposition 4, these approximations are transformed into the following formulae:

$$S\underline{apr}_a^{\delta}(\mathcal{O}) = \underline{apr}_{a,max}^{\delta}(\mathcal{O}),\ P\underline{apr}_a^{\delta}(\mathcal{O}) = \underline{apr}_{a,min}^{\delta}(\mathcal{O}), \tag{22}$$

$$S\overline{apr}_a^{\delta}(\mathcal{O}) = \overline{apr}_{a,min}^{\delta}(\mathcal{O}),\ P\overline{apr}_a^{\delta}(\mathcal{O}) = \overline{apr}_{a,max}^{\delta}(\mathcal{O}), \tag{23}$$

where $\underline{apr}_{a,max}^{\delta}(\mathcal{O})$ is the lower approximations under the maximum possible covering deriving from the maximum indiscernibility relation and $\overline{apr}_{a,min}^{\delta}(\mathcal{O})$ is the upper approximations under the minimum possible covering deriving from the minimum indiscernibility relation. These formulae show that we can obtain the lower and upper bounds of approximations without computational complexity, no matter how many possible coverings.

Example 4. We go back to Example 3. Let set $\mathcal{O}$ of objects be $\{o_1, o_3\}$. Using formulae (16) and (17), lower and upper approximations are obtained in each possible covering. For $P\mathcal{C}_1$, $\underline{apr}_{a_1,1}^{0.75}(\mathcal{O}) = \{o_1, o_3\}$, $\overline{apr}_{a_1,1}^{0.75}(\mathcal{O}) = \{o_1, o_3\}$.

For $P\mathcal{C}_2$, $\underline{apr}_{a_1,2}^{0.75}(\mathcal{O}) = \{o_3\}$, $\overline{apr}_{a_1,2}^{0.75}(\mathcal{O}) = \{o_1, o_2, o_3\}$.
For $P\mathcal{C}_3$, $\underline{apr}_{a_1,3}^{0.75}(\mathcal{O}) = \{o_1\}$, $\overline{apr}_{a_1,3}^{0.75}(\mathcal{O}) = \{o_1, o_2, o_3\}$.
For $P\mathcal{C}_4$, $\underline{apr}_{a_1,4}^{0.75}(\mathcal{O}) = \{o_1, o_3\}$, $\overline{apr}_{a_1,4}^{0.75}(\mathcal{O}) = \{o_1, o_3\}$.
For $P\mathcal{C}_5$, $\underline{apr}_{a_1,5}^{0.75}(\mathcal{O}) = \emptyset$, $\overline{apr}_{a_1,5}^{0.75}(\mathcal{O}) = \{o_1, o_2, o_3\}$.
For $P\mathcal{C}_6$, $\underline{apr}_{a_1,6}^{0.75}(\mathcal{O}) = \{o_3\}$, $\overline{apr}_{a_1,6}^{0.75}(\mathcal{O}) = \{o_1, o_2, o_3\}$.
For $P\mathcal{C}_7$, $\underline{apr}_{a_1,7}^{0.75}(\mathcal{O}) = \{o_1\}$, $\overline{apr}_{a_1,7}^{0.75}(\mathcal{O}) = \{o_1, o_2, o_3\}$.
For $P\mathcal{C}_8$, $\underline{apr}_{a_1,8}^{0.75}(\mathcal{O}) = \emptyset$, $\overline{apr}_{a_1,8}^{0.75}(\mathcal{O}) = \{o_1, o_2, o_3\}$.

By using formulae (22) and (23), $S\underline{apr}_{a_1}^{0.75}(\mathcal{O}) = \emptyset$, $P\underline{apr}_{a_1}^{0.75}(\mathcal{O}) = \{o_1, o_3\}$, $S\overline{apr}_{a_1}^{0.75}(\mathcal{O}) = \{o_1, o_3\}$, $P\overline{apr}_{a_1}^{0.75}(\mathcal{O}) = \{o_1, o_2, o_3\}$.

Using the lower and upper bounds of approximations denoted by formulae (22) and (23), lower and upper approximations are expressed in interval sets. Certain and possible approximations are the lower and upper bounds of the actual approximation.

Furthermore, the following proposition is valid from formulae (22) and (23).

Proposition 5.

$$S\underline{apr}_a^{\delta}(\mathcal{O}) = \{o \mid C(o)_{a,max}^{\delta} \subseteq \mathcal{O}\},\ P\underline{apr}_a^{\delta}(\mathcal{O}) = \{o \mid C(o)_{a,min}^{\delta} \subseteq \mathcal{O}\},$$
$$S\overline{apr}_a^{\delta}(\mathcal{O}) = \{o \mid C(o)_{a,min}^{\delta} \cap \mathcal{O} \neq \emptyset\},\ P\overline{apr}_a^{\delta}(\mathcal{O}) = \{o \mid C(o)_{a,max}^{\delta} \cap \mathcal{O} \neq \emptyset\},$$

where $C(o)_{a,min}^{\delta}$ and $C(o)_{a,max}^{\delta}$ are the minimum and the maximum possibly indiscernible classes of object o on a which are derived from applying formula (14) to minimum and maximum possible indiscernibility relations $PR_{a,min}^{\delta}$ and $PR_{a,max}^{\delta}$, respectively.

From this proposition, if the minimum and the maximum possibly indiscernible classes of each object are derived, then the lower and upper bounds of approximations can be obtained. And, $C(o)_{a,min}^{\delta}$ and $C(o)_{a,max}^{\delta}$ can be directly derived from the following formula:

$$C(o)_{a,min}^{\delta} = \{o' \in U \mid (o = o') \vee \forall u \in a(o) \forall v \in a(o') sim(u,v) \geq \delta_a\},$$
$$C(o)_{a,max}^{\delta} = \{o' \in U \mid \exists u \in a(o) \exists v \in a(o') sim(u,v) \geq \delta_a\}.$$

As a result, this justifies directly using minimum and maximum possibly indiscernible classes from the viewpoint of possible world semantics.[4]

4 Conclusions

We have described the structure of possible coverings under possible world semantics in incomplete information tables with similarity of values. Lots of coverings are derived in an incomplete information table, whereas the covering that is unique is derived in a complete information table. The number of possible coverings may grow exponentially as the number of objects with incomplete information grows. This seems to present some difficulties due to computational complexity of deriving rough sets, but it is not, because the family of possible coverings is a lattice with the minimum and maximum elements. This is also true for the family of maximal descriptions, but is not so for the family of minimal descriptions and the family of sets of close friends.

As Lipski derived the lower and upper bounds of an answer set of a query, we have obtained the lower and upper bounds of approximations. Lower and upper approximations can be derived from only the minimum and maximum coverings by the lattice structure of the family of possible coverings. Therefore, there are no difficulties regarding computational complexity due to the number of incompletely informative objects.

[4] This type of justification was first introduced by [22] in extending rough sets to deal with incomplete information.

References

1. Pawlak, Z.: Rough Sets: Theoretical Aspects of Reasoning about Data. Kluwer Academic Publishers, Dordrecht (1991). https://doi.org/10.1007/978-94-011-3534-4
2. Lipski, W.: On semantics issues connected with incomplete information databases. ACM Trans. Database Syst. **4**, 262–296 (1979)
3. Kryszkiewicz, M.: Rules in incomplete information systems. Inf. Sci. **113**, 271–292 (1999)
4. Nakata, M., Sakai, H.: Applying rough sets to information tables containing missing values. In: Proceedings of 39th International Symposium on Multiple-Valued Logic, pp. 286–291. IEEE Press (2009). https://doi.org/10.1109/ISMVL.2009.1
5. Stefanowski, J., Tsoukiàs, A.: Incomplete information tables and rough classification. Comput. Intell. **17**, 545–566 (2001)
6. Yang, T., Li, Q., Zhou, B.: Related family: a new method for attribute reduction of covering information systems. Inf. Sci. **228**, 175–191 (2013)
7. Nakata, M., Sakai, H., Hara, K.: Rough sets based on possibly indiscernible classes in incomplete information tables with continuous values. In: Hassanien, A.E., Shaalan, K., Tolba, M.F. (eds.) AISI 2019. AISC, vol. 1058, pp. 13–23. Springer, Cham (2020). https://doi.org/10.1007/978-3-030-31129-2_2
8. Skowron, A., Stepaniuk, J.: Tolerance approximation spaces. Fund. Inform. **27**, 245–253 (1996)
9. Zakowski, W.: Approximations in the Space (U, Π), Demonstratio Mathematica, **XVI**(3), 761–769 (1983)
10. Bonikowski, Z., Bryniarski, E., Wybraniec-Skardowska, U.: Extensions and intensions in the rough set theory. Inf. Sci. **107**, 149–167 (1998)
11. Zhu, W., Wang, F.: On three types of covering-based rough sets. IEEE Trans. Knowl. Data Eng. **19**(8), 1131–1144 (2007)
12. Yao, Y., Yao, B.: Covering based rough set approximations. Inf. Sci. **200**, 91–107 (2012)
13. Couso, I., Dubois, D.: Rough sets, coverings and incomplete information. Fund. Inform. **108**(3–4), 223–347 (2011)
14. Lin, G., Liang, J., Qian, Y.: Multigranulation rough sets: from partition to covering. Inf. Sci. **241**, 101–118 (2013) http://dx.doi.org/10.1016/j.ins.2013.03.046
15. Chen, D., Wang, C., Hu, Q.: A new approach to attribute reduction of consistent and inconsistent covering decision systems with covering rough sets. Inf. Sci. **177**, 3500–3518 (2007). https://doi.org/10.1016/j.ins.2007.02.041
16. Chen, D., Li, W., Zhang, Z., Kwong, S.: Evidence-theory-based numerical algorithms of attribute reduction with neighborhood-covering rough sets. Int. J. Approximate Reasoning, **55**, 908–923 (2014). http://dx.doi.org/10.1016/j.ijar.2013.10.003
17. Zhang, X., Mei, C.L., Chen, D.G., Li, J.: Multi-confidence rule acquisition oriented attribute reduction of covering decision systems via combinatorial optimization. Knowl.-Based Syst. **50**, 187–197 (2013). https://doi.org/10.1016/j.knosys.2013.06.012
18. Nakata, M., Sakai, H., Hara, K.: Rough sets and rule induction from indiscernibility relations based on possible world semantics in incomplete information systems with continuous domains. In: Hassanien, A.E., Darwish, A. (eds.) Machine Learning and Big Data Analytics Paradigms: Analysis, Applications and Challenges. SBD, vol. 77, pp. 3–23. Springer, Cham (2021). https://doi.org/10.1007/978-3-030-59338-4_1

19. Safari, S., Hooshmandasl, M.R.: On twelve types of covering-based rough sets. Springerplus **5**, 1003 (2016). https://doi.org/10.1186/s40064-016-2670-y
20. Zhu, W.: Relationship among basic concepts in covering-based rough sets. Inf. Sci. **179**, 2478–2486 (2009). https://doi.org/10.1016/j.ins.2009.02.013
21. Liu, Y., Zhu, W.: On three types of covering-based rough sets via definable sets. In: 2014 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), pp. 1226–1233 (2014)
22. Nakata, M., Sakai, H.: Twofold rough approximations under incomplete information. Int. J. Gen. Syst. **42**, 546–571 (2013). https://doi.org/10.1080/17451000.2013.798898