Approximate Solution of Inverse Problems of Gravity Exploration on Fractals

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Abstract The work is devoted to the approximate methods for solution direct and inverse problems of gravity exploration on bodies with a fractal structure. It is known that in order to construct mathematical models adequate to the geological reality, it is necessary to take into account the orderliness inherent in geological environments. One of the manifestations of orderliness is self-similarity, which remains during the transition from the microlevel to the macrolevel. Scaling of geological media can be traced in petrophysical data and in anomalous fields. It should be noted that in real structures there is no infinite self-similarity and scaling must be considered in a certain range. The work investigates analytical and numerical methods for solving inverse contact problems of the logarithmic and Newtonian potential in the generalized setting. In the case of a Newtonian potential, the problem is formulated as follows. It is required, having three independent functionals of the gravity field above the Earth's surface and additional information on the self-similarity of the disturbing body, to determine the depth, the density and the surface of the perturbing body.

Keywords Self-similarity · Fractional measure · Disturbances of the Earth's gravitational field · The logarithmic potential · The Newtonian potential · Petrophysical data

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C. H. Skiadas and Y. Dimotikalis (eds.), *14th Chaotic Modeling and Simulation International Conference*, Springer Proceedings in Complexity,
https://doi.org/10.1007/978-3-030-96964-6_8

1 Introduction

For the effective solution of direct and inverse problems of gravity prospecting, the methods of modeling bodies that perturb the potential and gravitational fields of the Earth (perturbing bodies) are of great importance. In most works, disturbing bodies are modeled by a set of the simplest geometric bodies (bar, parallelepiped, ball) [1]. In the works [2, 3], modeling is carried out with spheroids. In recent years, a large number of studies have been carried out on the fractality of individual minerals and the entire Earth as a whole [4–7]. Scaling of geological media can be traced in petrophysical data and in anomalous fields [8], etc. On the basis of the apparatus of fractional measure and fractional dimension, the processing of disturbances of the Earth's gravitational field is investigated [9].

Most minerals are porous. There are two types of porosity: the porosity of minerals and the porosity of liquids. Numerous studies have shown that in both mention cases, the porosity has a fractal structure.

In particular, the group of authors argues that sandstones have a fractal structure [4, 5, 10]. Hansen and Skjeltorp [6] investigated the fractal dimension D of a flat sandstone sample and obtained $D = 1,73$. Brakenseik [11] determined the fractal dimension of a two-dimensional oil cut. It is equal to $D = 1,8$. In [12], the fractal dimension of the surfaces of porous ceramic materials is investigated.

In the monograph [7] the Menger's sponge is proposed as a mathematical model of porosity, which is constructed somewhat differently from the standard construction.

In this work, when constructing fractal models of geological environments, the authors proceed not from fractals, but, following [13], from additions to fractals, since areas (volumes) of additions tend to areas (volumes) of the original body.

Taking into account the fractal components of gravitational fields makes it possible to clarify the structure of the disturbing bodies.

Methods for solving contact inverse problems of logarithmic and Newtonian potentials in a generalized setting are analyzed [14]. The problem is formulated as follows. It is required, having three independent functionals of the gravity field over the Earth's surface $z = 0$ and additional information about the self-similarity of the disturbing body, determine the depth H, the density $\sigma(x, y)$ and the surface $H - \varphi(x, y)$ of the disturbing body occupying the region $H \le z(x, y) \le H - \varphi(x, y)$.

Taking into account the fractal components of the gravitational and magnetic fields makes it possible to clarify the structure of the disturbing bodies.

The work is devoted to the approximate solution of direct and inverse problems of gravity prospecting on bodies with a fractal structure.

When solving inverse problems, a continuous method for solving nonlinear operator equations is used, which is presented in the next section.

2 Continuous Operator Method

Let B be a Banach space, $a, z \in B$, K be a linear operator mapping from B to B, $\Lambda(K)$ be the logarithmic norm [15] of the operator K, and I be the identity operator. We shall use the following notation: $B(a,r) = \{z \in B : \|z-a\| \leq r\}$, $S(a,r) = \{z \in B : \|z-a\| = r\}$, $\mathcal{R}eK = K_R = (K+K^*)/2$, $\Lambda(K) = \lim_{h\downarrow 0}(\|I+hK\|-1)/h$.

Let a complex matrix $A = \{a_{ij}\}$,$i, j = 1, 2, \ldots, n$, be given in $n-$ dimensional space R^n of vectors x with the norms $\|x\|_1 = \sum_{k=1}^n |x_k|$, $\|x\|_2 = \left[\sum_{k=1}^n |x_k|^2\right]^{1/2}$, and $\|x\|_3 = \max_{1\leq k\leq n}|x_k|$.

The corresponding logarithmic norms of the matrix A then read [16]:

$$\Lambda_1(A) = \max_j(\mathcal{R}e(a_{jj}) + \sum_{i=1,i\neq j}^{n} |a_{ij}|),$$
$$\Lambda_2(A) = \lambda_{\max}\big((A+A^T)/2\big),$$
$$\Lambda_3(A) = \max_i(\mathcal{R}e(a_{ii}) + \sum_{j=1,j\neq i}^{n} |a_{ij}|).$$

Here $\lambda_{\max}(A)$ means the largest real part of eigenvalues of the matrix A.

Consider an equation

$$A(x) - f = 0, \tag{2.1}$$

where $A(x)$ is a nonlinear operator mapping from Banach space B to B.

Let x^* be a solution of the (2.1). In [17] the connection between stability of solutions of operator differential equations in Banach spaces and resolving operator equations of the form (2.1) has been established. Here we shall summarize the results on the method.

Let us associate the (2.1) with the following Cauchy problem

$$\frac{dx(t)}{dt} = A(x(t)) - f, \tag{2.2}$$

$$x(0) = x_0. \tag{2.3}$$

Theorem 2.1 [17]. Let the (2.1) has a solution x^* and on any differentiable curve $g(t)$ in Banach space B the inequality is valid

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \Lambda(A'(g(\tau))d\tau \leq -\alpha_g,\ \alpha_g > 0\ . \tag{2.4}$$

Then the solution of the Cauchy problem (2.2), (2.3) converges to the solution x^* of the (2.1) for any initial approximation.

Theorem 2.2 [17]. Let the (2.1) has a solution x^* and for any differentiable curve $g(t)$ in a ball $B(x^*, r)$ the following conditions are satisfied:

(1) for any $t (t > 0)$

$$\int_0^t \Lambda(A'(g(\tau))d\tau \leq 0; \tag{2.5}$$

(2) the inequality (2.4) is valid.

Then the solution of the Cauchy problem (2.2), (2.3) converges to a solution of the (2.1).

Note 1. In the inequality (2.4) it is assumed that the constants $\alpha_g > 0$ can differ for different curves $g(t)$.

Note 2. From inequalities (2.4), (2.5) it follows that the logarithmic norm $\Lambda(A'(g(\tau))$ can be positive for some values of τ; i.e. the Frechet derivative $A'(g(\tau))$ can degenerate into an identically zero operator along the curve.

Note 3. An example in [18] (an approximate solution of a hypersingular integral equation) has demonstrated convergence of an iterative process based on a continuous operator method when the Frechet derivative vanishes at the initial approximation.

Note 4. Logarithmic norm has the property which is very useful for numerical analysis. Let A, B be square matrices of order n with complex elements and $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, $\xi = (\xi_1, \dots, \xi_n)$, $\eta = (\eta_1, \dots, \eta_n)$ are n-dimensional vectors with complex components. Let us consider the following systems of algebraic equations: $Ax = \xi$ and $By = \eta$. The norm of a vector and its subordinate operator norm of the matrix are fixed; the logarithmic norm $\Lambda(A)$ corresponds to the operator norm.

Theorem 2.3 [19]. If $\Lambda(A) < 0$, the matrix A is non-singular and $\left\|A^{-1}\right\| \leq 1/|\Lambda(A)|$.

Theorem 2.4 [19]. Let $Ax = \xi$, $By = \eta$ and $\Lambda(A) < 0$, $\Lambda(B) < 0$. Then

$$\|x - y\| \leq \frac{\|\xi - \eta\|}{|\Lambda(B)|} + \frac{\|A - B\|}{|\Lambda(A)\Lambda(B)|}.$$

Main properties of the logarithmic norm are given in [15].

The logarithmic norm of the operator K can have different (positive or negative) values in different spaces.

The continuous method for solving nonlinear operator equations admits the following generalization. Let us return to (2.1). Denote by $A'(x_0)$ the Gateaux (Frechet) derivative on the element x_0. We introduce the equation

$$(A'(x_0))^* A(x) - (A'(x_0))^* f = 0. \tag{2.6}$$

Equation (2.6) is associated with the Cauchy problem

$$\frac{dx}{dt} = -(A'(x_0))^* A(x) - (A'(x_0))^* f), \tag{2.7}$$

$$x(0) = x_0. \tag{2.8}$$

If $\Lambda_2(A'(x_0))^* A'(x_0)) > 0$, then in some neighborhood $B(x_0, r)$ of the element x_0 the Euclidean logarithmic norm of the operator—$A'(x_0))^* A(x)$ will be negative and $\|x(t)\| < \|x(0)\|$ for some interval $t \in (t_0, t_1]$, $t_0 = 0$.

Let the inequality $\Lambda_2(A'(x_0))^* A'(x)) > 0$ be satisfied on the segment $t \in [t_0, t_1]$, $t_0 = 0$. (Here $x(t)$ is the solution to the Cauchy problem (2.7), (2.8)).

For $t \geq t_1$, consider the Cauchy problem

$$\frac{dx_1(t)}{dt} = -(A'(x_1)^* A(x) - (A'(x_1))^* f), \tag{2.9}$$

$$x_1(t_1) = x(t_1)(3.10) \tag{2.10}$$

and define the segment $[t_1, t_2]$, in which the inequality $\Lambda_2(A'(x_1))^* A'(x_1)) > 0$ occur.

Taking $x_2(t_2) = x_1(t_2)$ as an initial value when solving the Cauchy problem

$$\frac{dx_2(t)}{dt} = -(A'(x_2))^* A(x) - (A'(x_2))^* f), \tag{2.11}$$

$$x_2(t_2) = x_1(t_2), \tag{2.12}$$

we have $\lim_{t\to\infty} \left\| \frac{dx(t)}{dt} \right\| = 0$ and therefore $\lim_{t\to\infty} x(t) = x^*$.

Assertions follow from this remark.

Theorem 2.5. Suppose that (2.6) has a solution x^* and for any differentiable curve in the Banach space B the inequality

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \Lambda((A'(g(\tau)))^* A'(g(\tau)))d\tau \leq -\alpha_g, \alpha_g > 0 \tag{2.13}$$

occur. Then the solution to the sequence of Cauchy problems ((2.7), (2.8)), ((2.9), (2.10)), ((2.11), (2.12)), etc. converges to the solution x^* of (2.6).

Theorem 2.6. Suppose that (2.6) has a solution x^* and for any differentiable curve in the sphere $B(x^*, r)$ the inequalities

$$\int_0^t \Lambda((A'(g(\tau)))^* A'(g(\tau)))d\tau < 0 \tag{2.14}$$

and (2.13) occur. Then the solution of the sequence of Cauchy problems ((2.7), (2.8)), ((2.9), (2.10)), ((2.11), (2.12)), etc. converges to the solution x^* of (2.6).

If the conditions of Theorems 2.5 and 2.6 are not satisfied, the regularization

$$\frac{dx}{dt} = -\alpha x(t) - ((A'(x_0))^* A(x) - (A'(x_0))^* f), \alpha > 0,$$

is carried out.

3 Direct Tasks

Let us consider a geological deposit represented by the uniform body D of arbitrary form. Assuming that the body has fractal dimension $D_H < 3$, we will approximate it with its complement of the Menger sponge [7]. Let the body D be situated in the cube $\Omega = [-a, a]^3$. Let us construct the n-th order prefractal (n-th iteration of the fractal) for the Menger sponge in the cube Ω. During the construction of the first iteration the cube Ω is divided into 27 equal cubes with sides $r_1 = 2a/3$, and 7 central cubes are dropped.

During the construction of the second iteration every cube from the remaining 20 cubes is divided into 27 equal cubes with the sides $2a/9$. As the result we have 729 cubes including 400 central cubes (for every initial cube with the side $2a/3$) that are dropped. Repeating the described operations n times we get the n-th Menger prefractal. As noted in the work [13], not classical fractals but their complements with respect to the initial domain should be used as the model for geological bodies. Consequently geological deposits are modeled with the set of cubes with different lengths of edges (and with different sizes).

When modeling granular and liquid media it seems that it is more efficient to model them with reduced copies of the first iteration of the Menger sponge. In that case we can construct the model using not only classical fractal but also complement to it.

Let us introduce the Cartesian three-dimensional rectangular coordinate system with down-directed z-axis and with the origin of coordinates placed at the Earth surface. Assume that the body D occurs at sufficiently great depth z = H under the Earth surface.

As the parameter H we fix the distance from the Earth surface to the average point (in vertical direction) of gravitating body.

In the introduced coordinate system the domain Ω, which the body D belongs to, rewrites as:

$$\Omega = \{(x, y, z) : -a \le x \le a, -a \le y \le a, H - a \le z \le H + a\}.$$

Let $\Delta_{i,j,k} = [x_i, x_{i+1}; y_j, y_{j+1}; z_k, z_{k+1}]$, $x_i = -a + ai/n, \quad i = 0, 1, \ldots, 2n,$
$y_j = -a + aj/n, \quad j = 0, 1, \ldots, 2n,$
$z_k = H - a + ak/n, \quad k = 0, 1, \ldots, 2n.$

We refer to as marked the cubes Δ_{ijk} that have nonempty intersection with the domain D. In the marked cubes we locate the first iteration of the Menger sponge fractal with the edge length a/n. Suppose that the body is modeled by the first iteration of the fractal. Denote the constructed model of the body by D_n. For computation of the perturbed field it is sufficient to compute the vertical component of the gravity field generated by the cell Δ_{ijk} at the point $(x, y, 0)$.

The cell Δ_{ijk} consists of 20 cubes with edges having the length $a/3n$. Assuming n and H being sufficiently big we may treat $\cos(\Theta(x', y', z'))$, where $(x', y', z') \in \Delta_{ijk}$ as constant within the limits of the cell. Here $\Theta(x', y', z')$ is the angle between the radius-vector $M'P$ ($M' = (x', y', z'), P(x, y, 0)$) and the z-axis.

Let us denote by o_{ijk} the center of the cell Δ_{ijk}. Obviously,

$$o_{ijk} = \left(-a + a(i + 1/2)/n, -a + a(j + 1/2)/n, H - a + a(k + 1/2)/n\right).$$

Let us also denote by θ_{ijk} the angle between the vector $o_{ijk}P$ and the z-axis.

Thus the vertical component of the gravity force generated by the cell Δ_{ijk} at the point $P(x, y, 0)$ equals to $dV_z(i, j, k) = 20\gamma\rho(a/3n)^3 \cos(\theta_{ijk})/\left(r(o_{ijk}, P)\right)^2 = \frac{20\gamma\rho a^3}{27n^3} \cdot \frac{z_k+z_{k+1}}{2}/\left(r(o_{ijk}, P)\right)^3$.

Here γ is the gravitational constant, ρ is a density of body. Therefore the vertical component of the gravity force generated by the disturbing body D at the point $(x, y, 0)$ equals to $V_z(x, y, 0) = \sum_{i,j,k=0}^{2n-1} 20\gamma\rho_{ijk}a^3(\frac{z_k+z_{k+1}}{2})/27n^3\left(r\left(o_{ijk}, P\right)\right)^3$, ρ_{ijk}.is a density of cell. Consider the example.

Let us se t the following parameter values: $H = 5, \quad a = 1/4, \quad n = 10.$

We perform calculations using the formula $dV_z(i, j, k) = \frac{20\gamma\rho a^3}{27n^3} \cdot \frac{z_k+z_{k+1}}{2}/\left(r(o_{ijk}, P)\right)^3$.

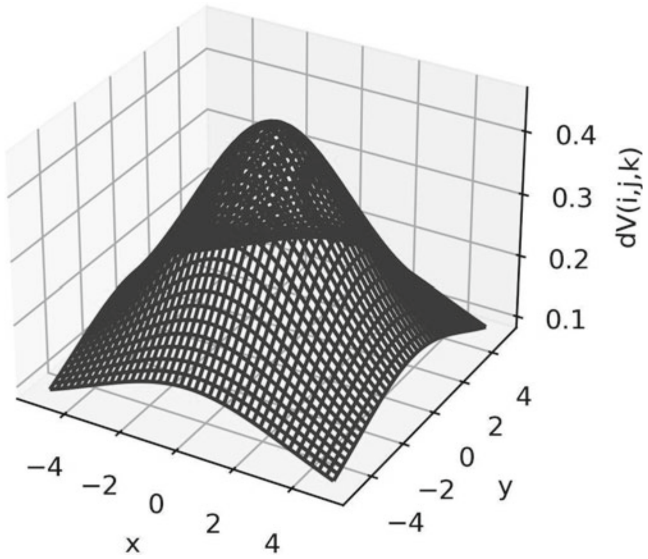
Let us fix $i = j = k = n$, that corresponds to the central cell Δ_{ijk} in the domain Ω For illustrative purposes the product of the constants γ and ρ we set to 10^6.

The field $dV(i,j,k)$ of the vertical component of anomalous gravity force generated by the described cell at the Earth surface is shown in the figure (Picture 1).

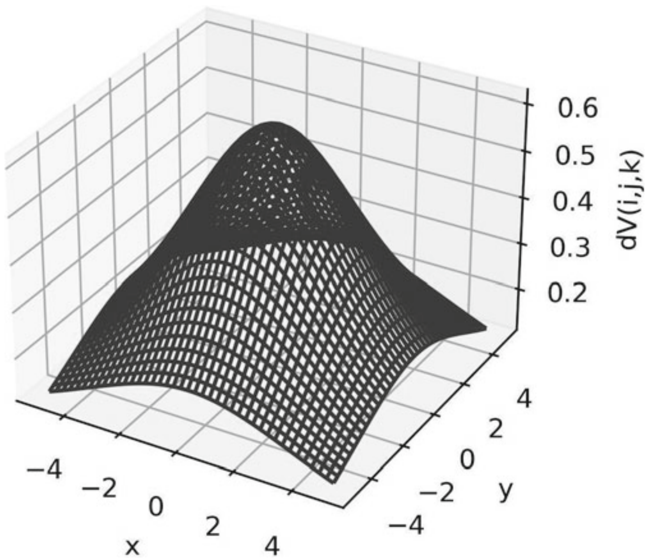
For comparison we also introduce the plot of the vertical component of the anomalous gravity field generated by the continuous body occupying the domain.

The computed field is depicted in the following figure (Picture 2).

From the comparison of the computed fields it is obvious that the solution of the direct problem is essentially dependent on the chosen model for representation of the elementary cell.



Picture 1 The vertical component of anomalous gravity force generated by the cell of the Menger sponge first order prefractal



Picture 2 The vertical component of anomalous gravity force generated by the elementary cell

4 Inverce Tasks

This section examines the influence of the chosen model on the accuracy of the interpretation of the results.

Let in the domain $D\{D : -l \le x \le l, -l \le y \le l_1, H \le z \le H - \varphi(x, y)\}$ are distributed with density $\sigma(x, y, z)$ sources disturbing gravitational field of the Earth. The gravity field above the Earth's surface is determined by the equation

$$G \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{H-\varphi(\zeta,\eta)}^{H} \frac{\sigma(\zeta, \eta, \xi)(\xi - z)\, d\zeta\, d\eta\, d\xi}{((x-\zeta)^2 + (y-\eta)^2 + (\xi - z)^2)^{3/2}} = f(x, y, z), \tag{4.1}$$

where $f(x, y, z)$ is the experimentally determined value, $G-$ gravitational constant, which for the convenience of further calculations will be set equal to $G = 1/2\pi$.

To describe the force of gravity on the Earth's surface in (4.1), one should set $z = 0$.

Having calculated the integral on the left-hand side of (4.1) by parts and assuming that the density does not depend on ξ, we have

$$\begin{aligned} &\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma(\zeta, \eta)\big[((x-\zeta)^2 + (y-\eta)^2 + (H - z - \varphi(\zeta, \eta))^2)^{-1/2} \\ &-((x-\zeta)^2 + (y-\eta)^2 + (H-z)^2)^{-1/2}\big] d\zeta\, d\eta = f(x, y, z). \end{aligned} \tag{4.2}$$

We represent (4.2) in the form

$$\begin{aligned} &\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma(\zeta, \eta)\big[((x-\zeta)^2 + (y-\eta)^2 + (H-z)^2)^{-1/2}(1+u)^{-1/2} \\ &-((x-\zeta)^2 + (y-\eta)^2 + (H-z)^2)^{-1/2}\big] d\zeta\, d\eta = f(x, y, z), \end{aligned} \tag{4.3}$$

where $u = \frac{\varphi^2(\zeta,\eta) - 2(H-z)\varphi(\zeta,\eta)}{(x-\zeta)^2 + (y-\eta)^2 + (H-z)^2}$. Under the assumption that $|u| < 1$, the function $\frac{1}{(1+u)^{1/2}}$ is expanded in the series

$$\frac{1}{(1+u)^{1/2}} = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)!!}{2^n n!} u^n. \tag{4.4}$$

Substituting (4.4) into (4.3) and using the uniform convergence of series (4.4), we have

$$\frac{1}{2\pi}\sum_{n=1}^{\infty}(-1)^n\frac{(2n-1)!!}{2^n n!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sigma(\zeta,\eta)\frac{(\varphi^2(\zeta,\eta)-2(H-z)\varphi(\zeta,\eta))^n\,d\zeta\,d\eta}{((x-\zeta)^2+(y-\eta)^2+(H-z)^2)^{n+1/2}}$$
$$= f(x,y,z). \tag{4.5}$$

Let us approximate (4.5), limiting ourselves to one term on the left-hand side. As a result, we obtain the equation [14]

$$-\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sigma(\zeta,\eta)\frac{(\varphi^2(\zeta,\eta)-2(H-z)\varphi(\zeta,\eta))\,d\zeta\,d\eta}{((x-\zeta)^2+(y-\eta)^2+(H-z)^2)^{3/2}} = f(x,y,z). \tag{4.6}$$

Equation (4.6) contains three unknowns: the depth of the gravitating body H, the density of the body $\sigma(x,y)$ and the shape of the surface $H-\varphi(x,y)$. To find these unknowns, it is necessary, in addition to values of the gravity field on some surface, to have two more linearly independent sources of information. As these functionals, one can use values of the gravity field at three different levels, a combination of the values of the gravity field and its derivatives in different directions, etc.

Note. Having values of the gravity field at the same level, it is possible to restore values of the gravity field at several levels using the Poisson formula.

In the work [14], analytical and numerical methods are proposed for the simultaneous determination of the depth of the disturbing body, its density and the surface equation in contact problems of the logarithmic and Newtonian potential. In [14], the disturbing body was assumed to be solid.

Compared with iterative methods for solving (4.6), studied in [14], the preferable is the continuous operator method described in Sect. 3. In both cases, the density is interpreted as a constant function within the unit cell, which simulates the gravitating body. In the case of modeling a gravitating body with fractals, the density in elementary cells is not constant. It is of interest to study the influence of fractals chosen for modeling disturbing bodies on the accuracy of determining their densities.

In [14] the following example was analytically solved.

Let in the domain $\Omega = \{5 \le z(x,y) \le 5-\varphi(x,y), -\infty < x, y < \infty\}$, there is a perturbing body with density $\sigma(x,y)$. Let the gravity force and its first two derivatives be known on the surface $z=0$:

$$f_0(x,y,0) = \frac{24\pi}{(x^2+y^2+36)^{3/2}} - \frac{7\pi}{5(x^2+y^2+49)^{3/2}},$$
$$f_1(x,y,0) = \left.\frac{\partial f(x,y,z)}{\partial z}\right|_{z=0} = \frac{432\pi}{(x^2+y^2+36)^{5/2}} - \frac{4\pi}{(x^2+y^2+36)^{3/2}} - \frac{147\pi/5}{(x^2+y^2+49)^{5/2}} - \frac{2\pi/25}{(x^2+y^2+49)^{3/2}},$$
$$f_2(x,y,0) = \left.\frac{\partial^2 f(x,y,z)}{\partial z^2}\right|_{z=0} = \frac{12960\pi}{(x^2+y^2+36)^{7/2}} - \frac{1029\pi}{(x^2+y^2+49)^{7/2}} -$$
$$-\frac{216\pi}{(x^2+y^2+36)^{5/2}} + \frac{21\pi/25}{(x^2+y^2+49)^{5/2}} - \frac{4\pi/125}{(x^2+y^2+49)^{3/2}}.$$

It is necessary to find a depth of the gravitating body H, a density of the body $\sigma(x,y)$ and a shape of the surface $H-\varphi(x,y)$. To solve this problem, in addition to (4.6), two more equations are added

$$\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \frac{2Hw_1(\xi,\eta) - w_2(\xi,\eta)}{((x-\xi)^2 + (y-\eta)^2 + H^2)^{3/2}} d\xi\, d\eta = f_0(x, y),$$

$$\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \left\{ \frac{6H^2 w_1(\xi,\eta) - 3Hw_2(\zeta,\eta)}{((x-\xi)^2 + (y-\eta)^2 + H^2)^{5/2}} - \frac{2w_1(\xi,\eta)}{((x-\xi)^2 + (y-\eta)^2 + H^2)^{3/2}} \right\} d\xi\, d\eta$$
$$= f_1(x, y),$$

$$\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \left\{ \frac{3w_2(\zeta,\eta) - 18Hw_1(\xi,\eta)}{((x-\xi)^2 + (y-\eta)^2 + H^2)^{5/2}} + \frac{30H^3 w_1(\xi,\eta) - 15H^2 w_2(\xi,\eta)}{((x-\xi)^2 + (y-\eta)^2 + H^2)^{7/2}} \right\} d\xi\, d\eta$$
$$= f_2(x, y). \tag{4.7}$$

When obtaining system (4.7), the following formulas were used $w_1(x, y) = \sigma(x, y)\varphi(x, y), w_2(x, y) = \sigma(x, y)\varphi^2(x, y)$.

Its exact solution was obtained: $H = 5$, $\varphi(x, y) = \left(\frac{x^2+y^2+1}{x^2+y^2+4}\right)^{3/2}$, $\sigma(x, y) = \frac{(x^2+y^2+4)^{3/2}}{(x^2+y^2+1)^3}$.

When solving the system of (4.7) by the spline-collocation method with zero-order splines, an error is equal to $O(N^{-1})$, where $h = N^{-1}$ is a step of the computational scheme by coordinates x, y. Hence it follows that the results of the approximate solution can be interpreted as follows. In area $(x, y) : \{\frac{(x^2+y^2+4)^{3/2}}{(x^2+y^2+1)^3} \le 1/N\}$ let us put $\sigma(x, y) = 0$. Domain G defined by the inequality $\{\frac{(x^2+y^2+4)^{3/2}}{(x^2+y^2+1)^3} \ge 1/N\}$ we will cover with elementary cells (cubes) with edges of length d/N, where d is area diameter G. Place the first-order prefractal of the Mergel sponge in the elementary cells. Then, depending on the mineral filling the addition of the Margel sponge to the unit cell, the density of the body varies from $\sigma(x, y)$ to $27\sigma(x, y)/20$. Thus, when solving inverse problems on fractals, an additional problem arises of choosing an appropriate model (fractal, multifractal) for a gravitating body.

5 Conclusions

In this work by the example of the Menger sponge approximate methods for solution of direct and inverse problems of gravity exploration using fractals are investigated. As far as inverse geophysical problems belong to the class of ill-posed problems for their solution in this work we propose the generalization of continuous operator method for solution of nonlinear equations. The proposed method allows to obtain stable solution for inverse problems which are modeled with nonlinear convolutional equations. At the core of the method there are criteria for asymptotic stability of solutions of systems of ordinary differential equations. The method can be used for solution of numerous equations of mathematical physics. In solving direct and inverse problems using fractals we show the problem of dependency of interpretation of computational results on the chosen model.

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