Some Aspects of Rainbows and Black Hole Linked to Mandelbrot Set and Farey Diagram

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Abstract We present in this work some experiments involving the optics of systems where the paraxial optics approximation cannot be applied, as in the case of rainbow formation by raindrops and the case of light deflection by massive objects, such as in the vicinity of black holes. The first experiment is the injection of laser light into a glass cylinder, while the other is a circular billiard formed by a circular mirror and a laser beam. We use the theory of dynamical systems and the Mandelbrot set as an analogy to represent the paths of the light beam, as well as the properties involving the Farey sequence.

Keywords Farey mediant · Optics · Dynamical systems · Mandelbrot set

1 Introduction

This work was developed due to the authors' interest in different aspects of dynamical systems that involve optics [1, 2], gravitation and the aesthetic appeal of fractal forms [3, 4]. In this way we will present some connections between concepts of optics and Mandelbrot sets resulting from recursion formulas, feedback processes or systems in which we have the repeated application of some mapping rule.

How to compare trajectories of light rays with a recursive mapping normally used in dynamic systems? In Fig. 1 we have a luminous ray inside a glass plate that undergoes multiple reflections. A change in the path of this ray can be associated with the series of events shown in Fig. 1b. This series of events is reproduced topologically with the "circle map" of the graphic diagram in Fig. 1c, with the Poincaré section

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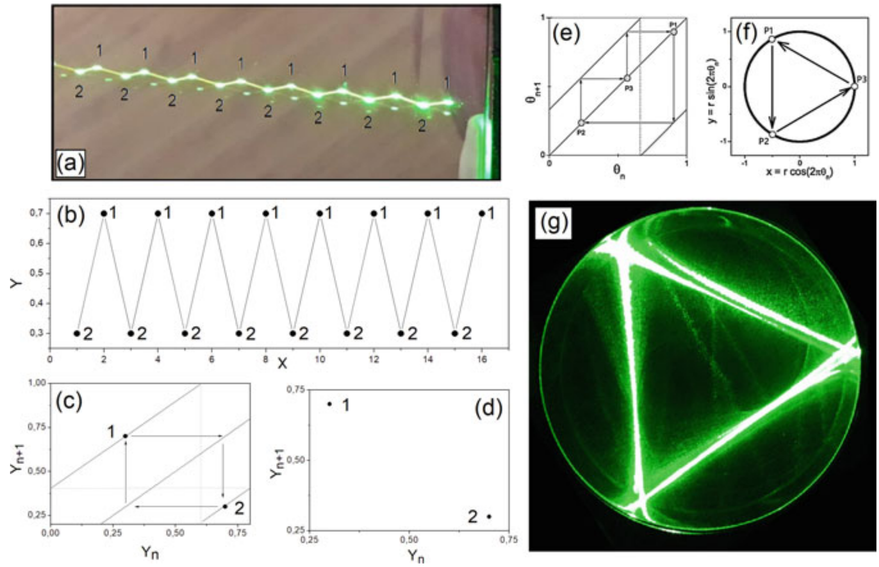

Fig. 1 Multiple reflections of a laser beam in a glass plate in (**a**). **b** Sequence of events. **c** Iterations in the circle map for a period-2. **d** Poincaré section of this period-2. Circle map in a period-3 in a circle map and its respective orbits in a circle in (**f**). Experiment showing multiple reflections in a period-3 in a reflective cylinder in (**g**)

shown in Fig. 1d [1]. We can see the same thing applied to period-3 in Fig. 1e for the case of the circle map, and the same dynamical system in a circular billiard in (f) and (g). For the cases found in nature, we know from optics that light rays in a raindrop of Fig. 2, which can behave like particles in a circular billiard, forming multiple rainbows, through recurring internal reflections. In this case, in addition to the rays of light that hit the droplet wall several times, obeying the law of reflection, we have rays that escape the droplet by refraction forming multiple rainbows, which give us some information about the shape of the droplet, as in the case of twin rainbows, in which we see an ellipsoidal rainbow between two circular rainbows, which appears to be a bifurcation, as it also occurs in other atmospheric optics phenomena, as in the case of the circumscribed halo surrounding the 22° solar halo of Fig. 3.

In the context of relativity, massive objects can "bend" light, forming Einstein's rings [1], which behave like lenses, as in the case of the lens in the form of a pseudo-sphere. In the case of a black hole acting as a lens in Fig. 3b, the properties of Minkowsky's space–time are altered by the presence of the mass, causing it to curve around itself, as if it were an optical lens. We also observed the reverse shape of this lens causing light to be distorted in Fig. 3c.

In our previous work on this subject, we studied rainbows and massive objects in the formation of luminous halos, starting from an optical system composed of a cylinder with the injection of a laser beam of Fig. 4, where we use the circular section

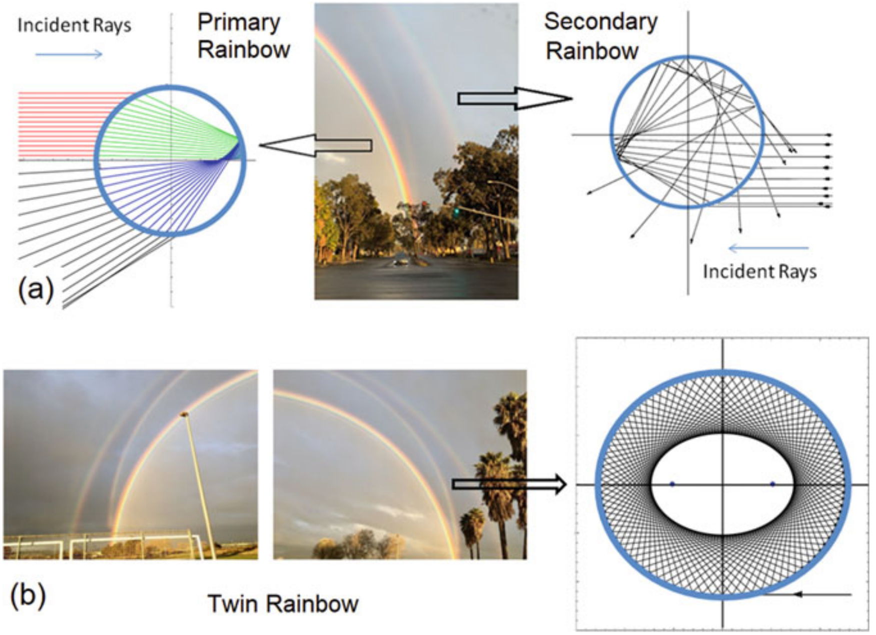

Fig. 2 In (**a**) reflections in a rain drop and primary rainbow, and reflections in a drop and secondary rainbow. In (**b**) the curious case of twin rainbows that could be explained considering elliptical drops

of the cylinder and compare it with the circular section of a drop. The modification of the cylinder topology led to the case of the pseudo-sphere.

This optical system formed by the laser hitting obliquely in the glass cylinder of Fig. 4c, d, behaves like an open billiard, with which we can see the formation of curious patterns associated with the star polygons, which due to the fact of restrictions to the total internal reflection, follow some rules related to a Farey mediant of Fig. 4f. This system is related to the phenomenon known as period locking (mode-locking), which can be observed in patterns projected on a screen of Fig. 5. This mode-locking is also seen in Mandelbrot set systems.

Increasing the inclination of the laser beam in the cylinder, we perceive a process of "optical deformation" of the circular section of the cylinder, for an elliptical billiard. With that, we recognize some interesting effects like the formation of caustics in the rainbow angle, which unfolded in separated branches [1]. This suggested to us the possibility that elliptical rainbows may be related to caustics unfolded from the cylinder, as discussed in the literature for ellipsoidal drops of Fig. 2b.

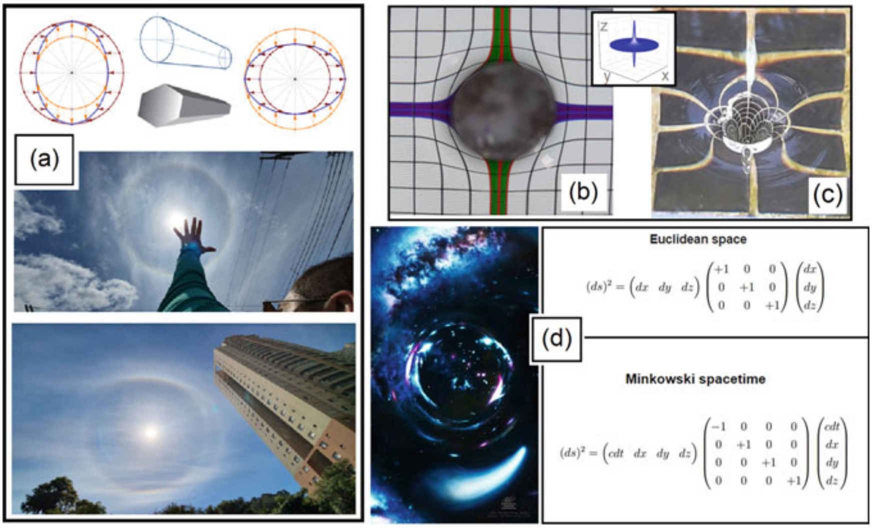

Fig. 3 In **a** cylinders and hexagonal bars can be used as lens with a laser to generate patterns. For the case of cylinder, we can associate elliptical profile with two circles. In 22° halo formed by hexagonal column of ice crystals can be measured with a hand flat. The 22° halo and the circumscribed halo. A pseudosphere lens in (**b**) and inverse pseudosphere lens from a pool drain in (**c**). Comparing the three-dimensional Euclidean space and Minkowski spacetime in (**d**) with a representation of the optical effects of a massive object such as black hole. In the four-dimensional Minkowski spacetime used in general relativity equations, time and space expand in opposite direction, with one solution of these equations being the black hole depicted in (**d**)

2 Mandelbrot Set and Star Polygons

Motivated by the observation of geometric patterns in the laser/cylinder system, we realized that the formation of caustics in cylinders may be related to the recursive rotations of maps in the complex plane, as in the case of the Mandelbrot Set, with the formation of star polygons for different initial conditions. The evolution of the iterations in Mandelbrot set presents a rich dynamics, with some complete routes to chaos, like Feigenbaum route and mode-locking route to chaos, besides intermittency. The Mandelbrot can be used as a toy model to explore the dynamics of the light in the cylinder, in order to help to explore this system and the pseudo-sphere used to simulate black holes or massive objects bending light.

For example, in Fig. 6, we present the existence of star polygons in the Mandelbrot set, with period-2, period-3, period-5 and period-7. For this last case we can compare the dynamics of ray in a billiard forming a period-7 with the Mandelbrot set.

The orbits of the Mandelbrot system are directly related to the Farey mediant, as shown in the diagram in Fig. 7a. The stability of the orbits can be observed with the cobweb diagram for values of the Mandelbrot set with real numbers. We can observe the stability of the system converging to a fixed point in Fig. 7b. When the control parameter is changed in Fig. 7c, we have different behaviors such as intermittency

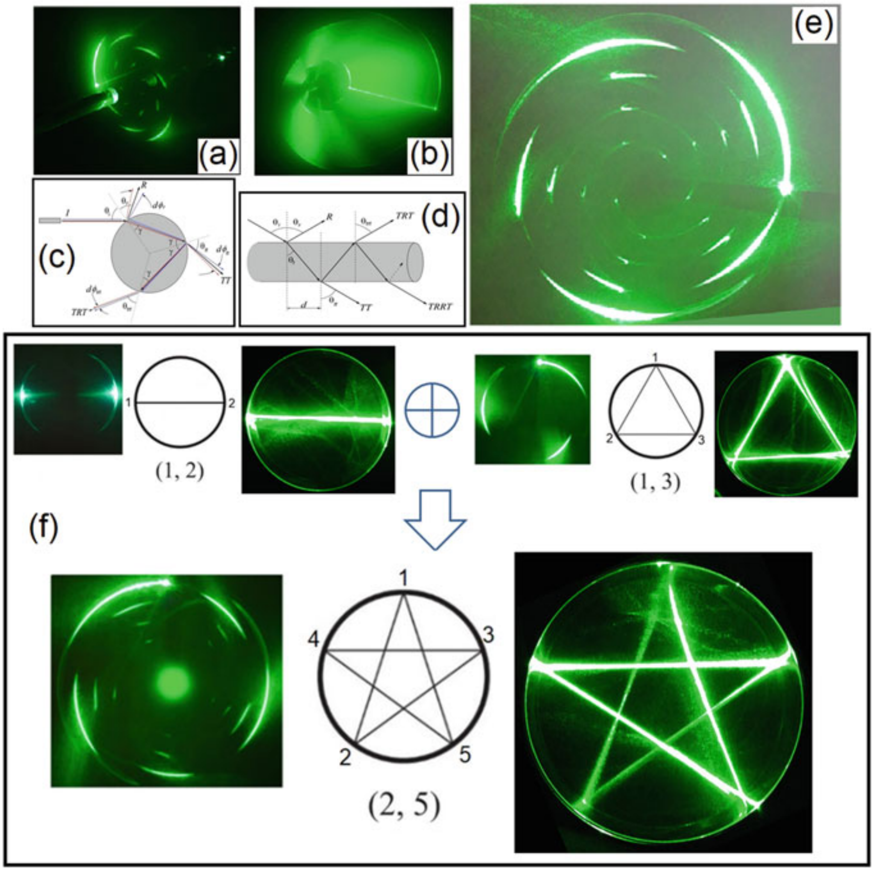

Fig. 4 Images of light scattering in cylinders in (**a**) and (**b**). Diagrams of light rays in the glass cylinder in (**c**) and (**d**). Light spiral in (**e**) obtained from this experiment. Comparison between stellar polygons in a billiard and the case of the glass cylinder, in which we can observe the case of Farey's mediant in (**f**)

in Fig. 7b, or this fixed point becoming a saddle point in which a doubling of period occurs in Fig. 7e. Changing the control parameter further, we have a period-4 in Fig. 7f, followed by chaotic behavior in Fig. 7g.

Another interesting relationship between the Mandelbrot set and the formation of caustics in a circular billiards is shown in Fig. 8. This relationship can be better understood by watching the video cited in [5] along with the papers describing the process of caustic formation [6–8]. The central idea is that these caustics are related to the fractal pattern profile associated with different polynomials of the Mandelbrot set.

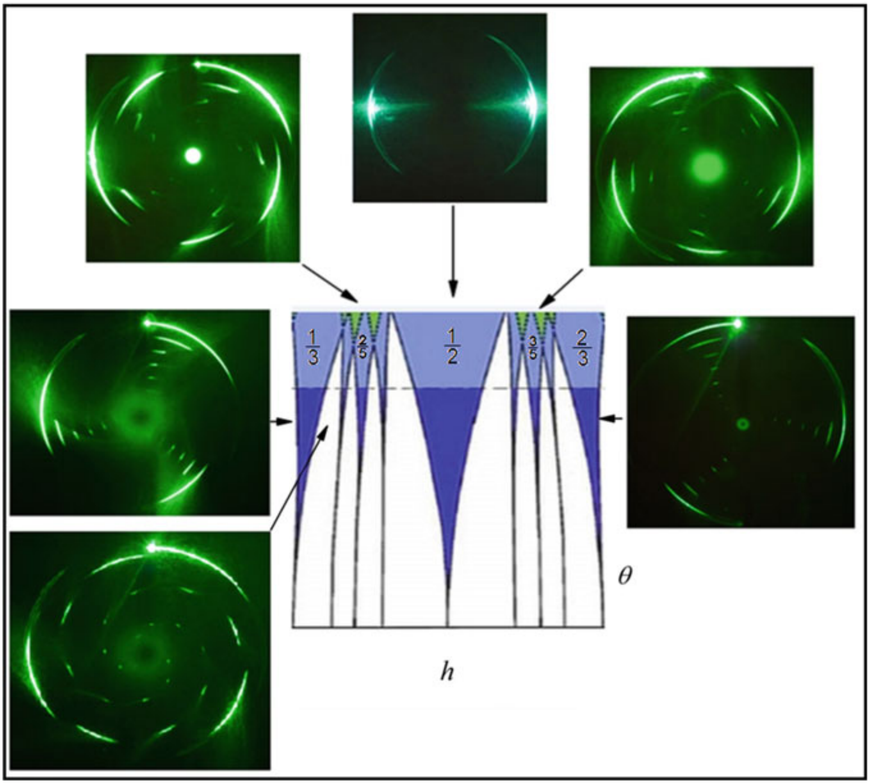

Fig. 5 Mode-locking and Farey sequence in the Arnold tongues with some light patterns

Finally, we show in Fig. 9 two examples of initial conditions in the Mandelbrot set for the case of orbits forming spiral patterns and compare them with the case of the experiment with the laser beam in the glass cylinder. We can see that the rotation orientation of the spiral depends on the position of choice of the initial condition, represented by the blue dot, with respect to the horizontal axis, which makes the spiral clockwise or counterclockwise. The same happens with the experimentally observed dynamical system, with the choice of the starting point that the laser beam touches the glass cylinder with respect to the horizontal axis, with the red spiral rotating clockwise and the green spiral rotating counterclockwise. For more information about these experiments, we recommend this video in [9].

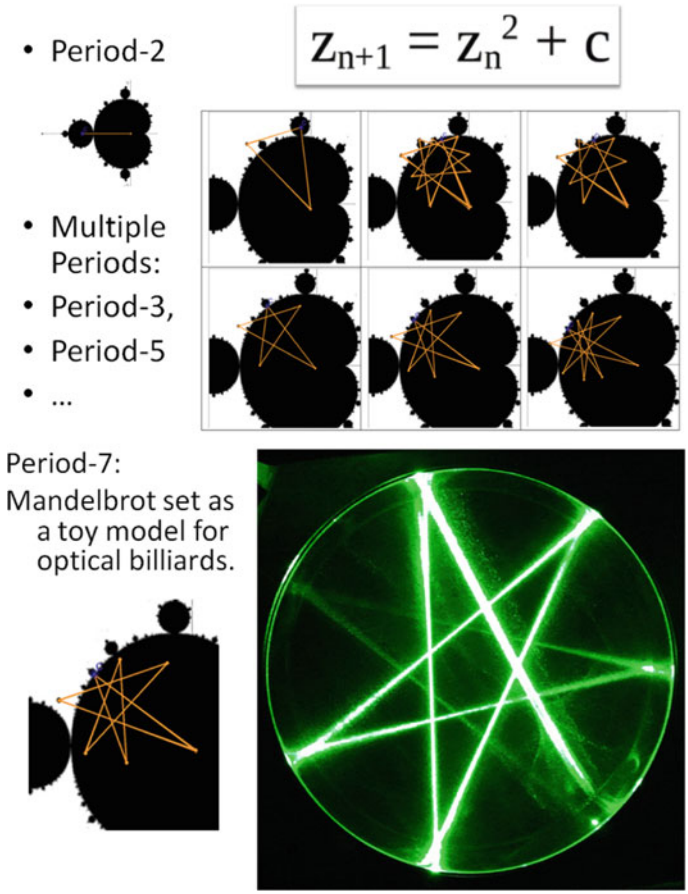

Fig. 6 Some star polygons in the Mandelbrot set. Period-7 in the Mandelbrot set and in the circular billiard

3 Conclusions

These connections between billiard systems, Optics, Mandelbrot set and caustics were explored due to their interesting aesthetic appeal, which facilitates the investigation of the optical system with the well-known system formed by Mandelbrot set. Multiple reflections of a laser inside a cylinder, which escapes by refraction and can projected in a screen, can be compared to the trajectories of the iterations of Mandelbrot map for certain regions in the Argand plane. In systems like the ones discussed in this paper, we can observe mode-locking and the formation of star polygons. These comparisons allow us to extract the essence of the dynamics existing in experimental optical systems that we cannot use the well-known paraxial optics, as happens in real physical systems such as light in raindrops or in a black hole. In

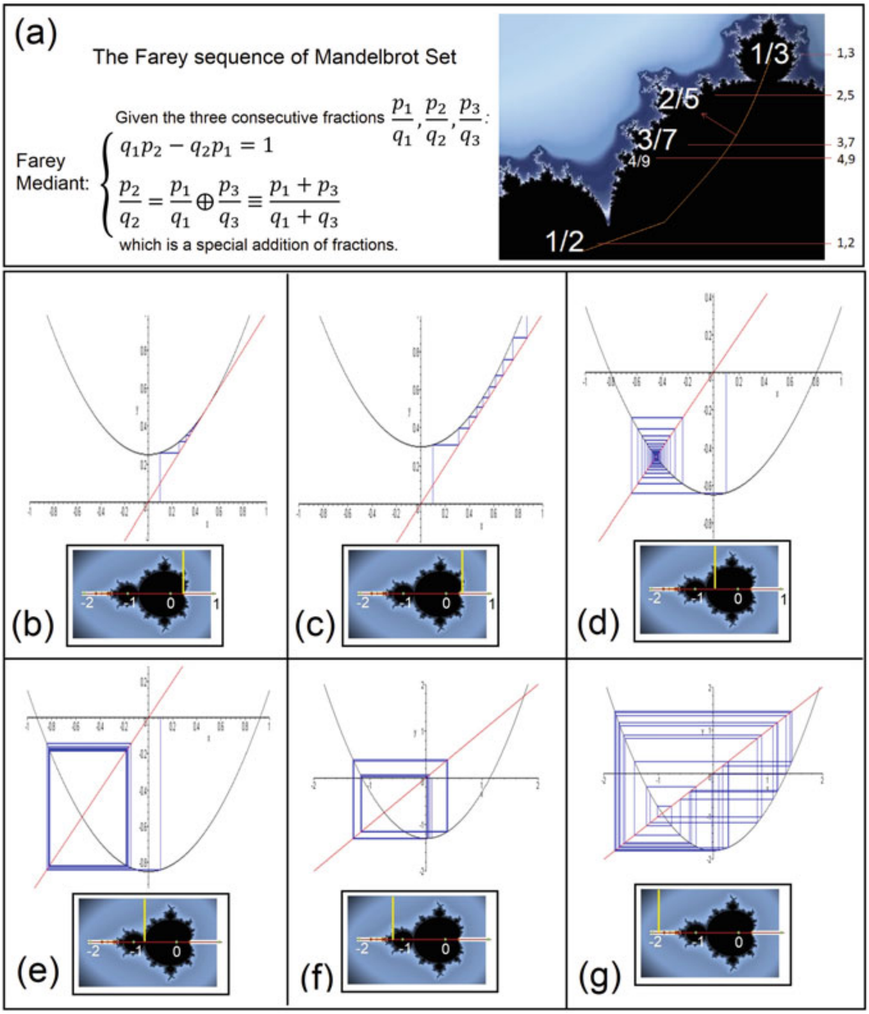

Fig. 7 Example of Farey sequence in the Mandelbrot set and the stability for some initial conditions in the Mandelbrot set

the case of massive object optics, such as the bending of light around black holes, matrix operations involving Minkowski space can generate rotations that cause the doubling of light rays from distant stars, as well as the formation of halos, such as Einstein rings, which were discussed in the case of light patterns in the laser/glass cylinder system.

Nonlinear dynamical systems that involve some kind of rotation as the case of the map of the circle presented in our previous works, or the Mandelbrot set, or matrix operations for the case of general relativity, can lead to certain results that can be observed experimentally with a non-paraxial optic that involves multiple

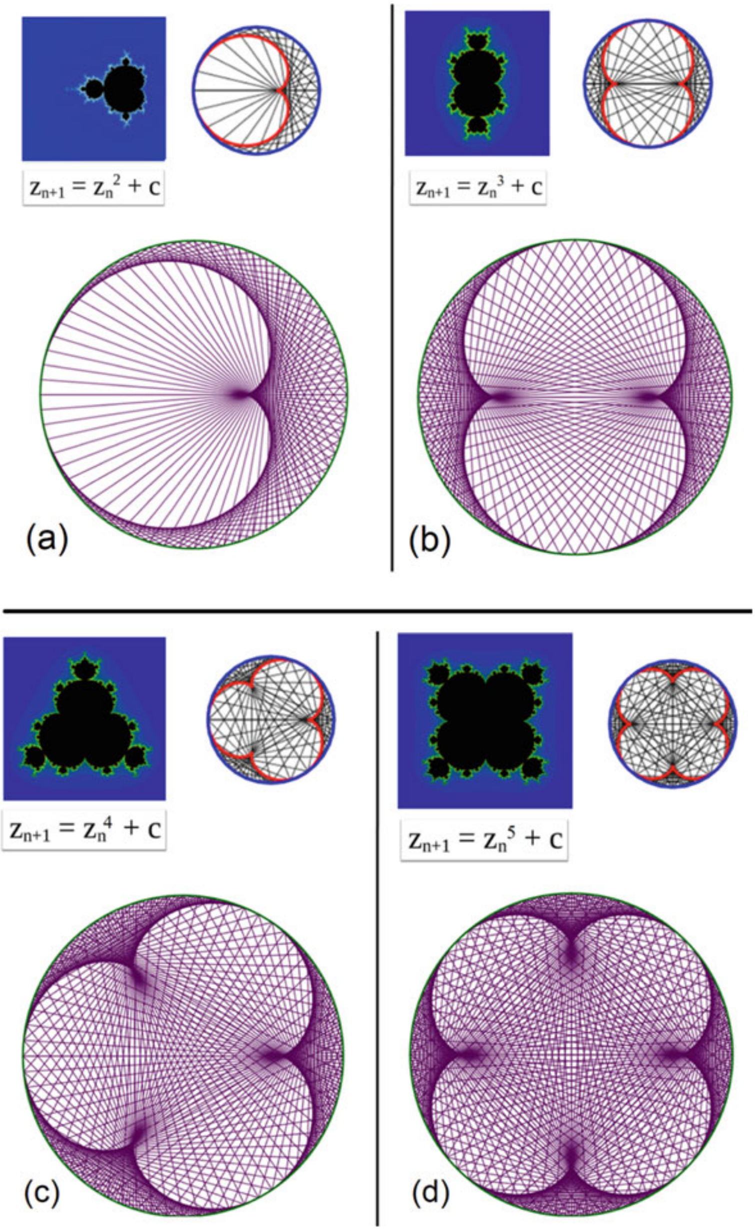

Fig. 8 Mandelbrot set and some caustics obtained with multiplication table applied to ray dynamics in a circle: in **a** the cardioid, in **b** the nephroid, with three lobes in (**c**) and with four lobes in (**d**)

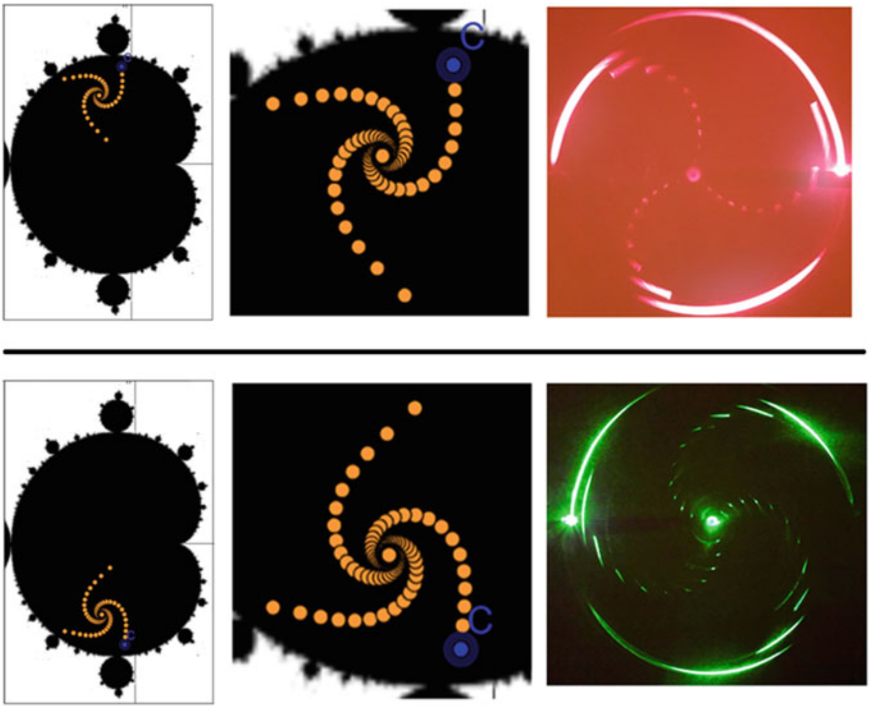

Fig. 9 Dynamics of spiral formation in the Mandelbrot set compared to the laser in the glass cylinder

reflections as in the case of the cylinder/laser and circular billiards. In this work, we emphasize how the use of Mandelbrot systems helps us to explore the different behaviors in a practical way, observing the different types of orbits based on the choice of initial conditions, which allows us to analyze the stability behavior of these orbits. Furthermore, we reproduce some interesting results of the formation of caustics in circular billiards through the multiplication rules, which are directly connected with Mandelbrot polynomials and their remarkable fractal representation. Overall, we believe that a lot can still be explored in the context of these analogies.

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