Maximal Attractors in Nonideal Hydrodynamic Systems

Aleksandr Shvets and Serhii Donetskyi

Abstract Some nonideal hydrodynamic systems of the type "tank with fluid - source of excitation of oscillation" are considered. New types of limit sets of such systems, so called maximal attractors, have been discovered and described. It was found that the maximal attractors can be both regular and chaotic. Main characteristics of the described maximal attractors are analyzed in details. Transitions to deterministic chaos in such systems are considered. Despite the fact that maximal attractors are not attractors in the traditional sense of this term, it is shown that the transition from regular maximal attractors to chaotic maximal attractors can occur by known before scenarios transition to chaos for "usual" attractors.

Keywords Nonideal hydrodynamic systems · Maximal attractors · Scenarios of transition to chaos

1 Introduction

Many modern machines, mechanisms and technical devices as structural elements contain cylindrical tanks partially filled with fluid. Therefore, the study of oscillations free surface of fluid in cylindrical tanks over the past decades has been attracting close attention [1–4].

Since the end of 70s years of the last century, there have been the so-called "low-dimensional" mathematical models describing such oscillations [5–8]. These models allow us to describe oscillations of the free surface of the liquid in the tank using nonlinear systems of ordinary differential equations instead of partial differential equations that arise when describing the problem in the general setting. "Low-

A. Shvets (✉) · S. Donetskyi
National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institue", Kyiv, Ukraine
e-mail: aleksandrshvetskpi@gmail.com

S. Donetskyi
e-mail: dsvshka@gmail.com

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dimensional" models allow you to get a fairly adequate description of the problem in cases where the power produced by of the source of excitation of oscillations significantly exceeds the power consumed by the oscillating load (cylindrical tank with fluid). These cases are called ideal by Sommerfeld–Kononenko. However, in practice, most often there are cases in which the power source of excitation of oscillations is comparable to the power consumed by the oscillatory load. Such cases are called nonideal. In these cases, it is imperative to take into account the interaction between the source of excitation of oscillations and the oscillatory load, which leads to essential refinement of the mathematical models used in ideal cases. The neglect of the interaction between the excitation source and the oscillatory load leads to gross errors in the description of the dynamics of the studied systems [9–17].

2 Evolution Equation

Consider dynamic system, the layout of which is shown in Fig. 1. The electric motor shaft is connected to the platform through the crank mechanism, on which a rigid cylindrical tank of radius R is fixed, partially filled with liquid.

When the crank a rotates through an angle Ψ, the platform makes a vertical movement of the form $v(t) = a \cos \Psi(t)$. To describe the vibrations of the free surface of a liquid, we introduce a cylindrical coordinate system $Oxr\theta$ with origin in the tank axis, on the undisturbed fluid surface. The relief equation of free surface of the fluid we write down in the form $x = \eta(r, \theta, t)$. Suppose liquid inviscid and incompressible with density ρ and fills a cylindrical tank of cross-section S section to the depth $x = -d$.

We will find function of the relief of surface of liquid in the form of an eigenmode expansion:

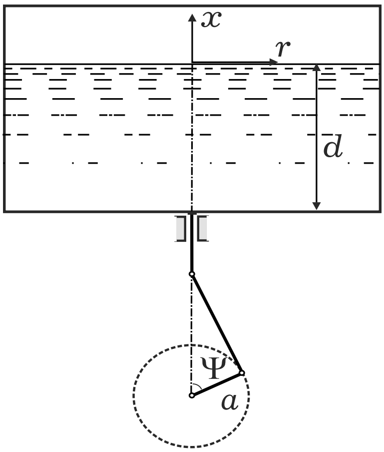


Fig. 1 Scheme of system

$$\eta(r,\theta,t)=\sum_{i,j}[q_{ij}^{c}(t)k_{ij}(r)\cos i\theta+q_{ij}^{s}(t)k_{ij}(r)\sin i\theta]. \tag{1}$$

Then we write the kinetic energy of the total system in the form [15, 16]:

$$T=\frac{1}{2}I\dot{\Psi}^2+\frac{1}{2}m_0\dot{v}^2+\frac{1}{2}\rho S\sum_{i,j,m,n}a_{ijmn}\dot{q}_{ij}^{c,s}\dot{q}_{mn}^{c,s}. \tag{2}$$

Here I is the moment of inertia of the motor shaft; m_0—mass of the liquid tank; a_{ijmn}—nonlinear functions of $q_{ij}^{c,s}(t)$, $q_{mn}^{c,s}(t)$.

In turn, the potential energy of movements of the free surface of the liquid is Shvets [15, 16]

$$V=\rho\int\int_S dS\int_0^{\eta}(g+\ddot{v})x\,dx=\frac{1}{2}\rho S(g+\ddot{v})\sum_{i,j}q_{ij}^{c,s}q_{ij}^{c,s}, \tag{3}$$

where g is the acceleration of gravity.

Therefore, the Lagrangian of the system takes the form

$$\begin{aligned}L=&\frac{1}{2}I\dot{\Psi}^2+\frac{1}{2}m_0a^2\dot{\Psi}^2\sin^2\Psi+\frac{1}{2}\rho S\sum_{i,j,m,n}a_{ijmn}\dot{q}_{ij}^{c,s}\dot{q}_{mn}^{c,s}\\&+\frac{1}{2}\rho Sa(\dot{\Psi}^2\cos\Psi+\ddot{\Psi}\sin\Psi)\sum_{i,j}q_{ij}^{c,s}q_{ij}^{c,s}-\frac{1}{2}\rho Sg\sum_{i,j}q_{ij}^{c,s}q_{ij}^{c,s}.\end{aligned} \tag{4}$$

As a result, for $\Psi(t)$ we obtain the following evolution equation

$$\begin{aligned}I\ddot{\Psi}=&-2m_0a^2\dot{\Psi}^2\sin\Psi\cos\Psi-m_0a^2\ddot{\Psi}\sin^2\Psi+a\rho S(\dot{\Psi}^2\sin\Psi\\&-\ddot{\Psi}\cos\Psi)\sum_{i,j}q_{ij}^{c,s}q_{ij}^{c,s}-2a\rho S\dot{\Psi}\cos\Psi\sum_{i,j}q_{ij}^{c,s}\dot{q}_{ij}^{c,s}+\Phi(\Psi)-H(\Psi).\end{aligned} \tag{5}$$

The last two terms on the right side of (5) are the driving moment and the moment internal forces of resistance of the electric motor.

Suppose that the speed of rotation of the shaft $\dot{\Psi}(t)$ in steady state conditions of the engine is close to $2\omega_1$, where ω_1 is natural frequency of main tone of oscillations of the free surface, which corresponds to the modes $q_{11}^{c}(t)k_{11}(r)\cos\theta$ and $q_{11}^{s}(t)k_{11}(r)\sin\theta$.

Let us introduce into consideration a small positive parameter

$$\varepsilon=\omega_1\sqrt{\frac{a}{g}}. \tag{6}$$

Also assume that

$$\dot{\Psi} - 2\omega_1 = \varepsilon^2 \omega_1 \beta. \tag{7}$$

The oscillations of the free surface of the liquid are approximated by oscillations in the main and secondary modes, whose amplitudes are defined as [15, 16]

$$\begin{aligned}
q_{11}^c(t) &= \varepsilon \upsilon \left[p_1(\tau) \cos\frac{\Psi}{2} + q_1(\tau) \sin\frac{\Psi}{2} \right];\\
q_{11}^s(t) &= \varepsilon \upsilon \left[p_2(\tau) \cos\frac{\Psi}{2} + q_2(\tau) \sin\frac{\Psi}{2} \right];\\
q_{01}(t) &= \varepsilon^2 \upsilon \left[A_{01}(\tau) \cos\Psi + B_{01}(\tau) \sin\Psi + C_{01}(\tau) \right];\\
q_{21}^{c,s}(t) &= \varepsilon^2 \upsilon \left[A_{21}^{c,s}(\tau) \cos\Psi + B_{21}^{c,s}(\tau) \sin\Psi + C_{21}^{c,s}(\tau) \right].
\end{aligned} \tag{8}$$

Here τ is slow time, $\tau = \frac{1}{4}\varepsilon^2\Psi$, $\upsilon = \frac{R}{1.8412}\tanh\left(\frac{1.8412}{R}d\right)$. Having determined the dimensionless amplitudes $A_{ij}^{c,s}(\tau)$, $B_{ij}^{c,s}(\tau)$, $C_{ij}^{c,s}(\tau)$ secondary modes by Miles method [5, 6, 8] through the amplitudes $p_1(\tau)$, $q_1(\tau)$, $p_2(\tau)$, $q_2(\tau)$ and applying the procedure of averaging the Lagrangian over the explicitly entering fast time $\Psi(t)$, for the amplitudes of dominant modes, we obtain the following system of equations [15, 16]:

$$\begin{aligned}
\frac{dp_1}{d\tau} &= \alpha p_1 - \left[\beta + \frac{A}{2}(p_1^2 + q_1^2 + p_2^2 + q_2^2) \right] q_1 + B(p_1 q_2 - p_2 q_1) p_2 + 2q_1;\\
\frac{dq_1}{d\tau} &= \alpha q_1 + \left[\beta + \frac{A}{2}(p_1^2 + q_1^2 + p_2^2 + q_2^2) \right] p_1 + B(p_1 q_2 - p_2 q_1) q_2 + 2p_1;\\
\frac{d\beta}{d\tau} &= N_3 + N_1 \beta + \mu_1 (p_1 q_1 + p_2 q_2);\\
\frac{dp_2}{d\tau} &= \alpha p_2 - \left[\beta + \frac{A}{2}(p_1^2 + q_1^2 + p_2^2 + q_2^2) \right] q_2 - B(p_1 q_2 - p_2 q_1) p_1 + 2q_2;\\
\frac{dq_2}{d\tau} &= \alpha q_2 + \left[\beta + \frac{A}{2}(p_1^2 + q_1^2 + p_2^2 + q_2^2) \right] p_2 - B(p_1 q_2 - p_2 q_1) q_1 + 2p_2.
\end{aligned} \tag{9}$$

In the system of (9), we have the following designations: $\alpha = -\frac{\delta}{\omega_1}$ – reduced factor of damping, N_0, N_1 – constants of linear static characteristic of the electric motor, $N_3 = \frac{1}{\omega_1}\left(N_0 + 2N_1\omega_1\right)$; $\mu_1 = \frac{\rho S \upsilon R^2}{(1.8412)^2(2I + m_0 a^2)\omega_1^2}$; A and B – constants ranging from the diameter of the tank and filling it with liquid. The system of evolutionary equations (9) is used as the main mathematical model for study the dynamics of oscillations of a tank with a liquid, excited by an electric motor of limited power.

The main aim of the research is to study the possible types of limit sets of the system (9). Since this system is a rather complex nonlinear system of equations of the fifth order, then for constructing its limit sets, a whole complex of numerical methods and algorithms were used. The technique for carrying out such numerical calculations for system with limited excitation is described in detail in Shvets [18] and Krasnopolskaya and Shvets [16].

3 Numerical Studies of Steady-State Regimes of Oscillations

Initially, we define the conditions under which the system is dissipative. Let us denote by $\boldsymbol{F}$ the vector field generated by the system of equations (9). Accordingly, by F_1, F_2, F_3, F_4, F_5 we denote the components of this vector field, that is, the right-hand sides of the system of equations (9). Then the divergence of this vector field can be found by the formula

$$\begin{aligned} \operatorname{div}\boldsymbol{F} &= \frac{\partial F_1}{\partial p_1} + \frac{\partial F_2}{\partial q_1} + \frac{\partial F_3}{\partial \beta} + \frac{\partial F_4}{\partial p_2} + \frac{\partial F_5}{\partial q_2} = \alpha - Ap_1q_1 + Bp_2q_2 + \alpha \\ &+ Ap_1q_1 - Bp_2q_2 + N_1 + \alpha - Ap_2q_2 + Bp_1q_1 + \alpha + Ap_2q_2 - Bp_1q_1 \\ &= 4\alpha + N_1. \end{aligned} \tag{10}$$

So the divergence of the vector field $\mathbf{F}$ is constant. The dissipativity condition for the system of equations has the form,

$$4\alpha + N_1 < 0. \tag{11}$$

The quantities included formula (11), α (coefficient of damping) and N_1 (angle of inclination of the static characteristic of electricmotor) are always negative. Therefore, the divergence of the vector field generated by the system of equations (9) will always be negative. Thus, this system will always be dissipative.

We will begin the study of the dynamics of the system (9) by finding its equilibrium positions. Obviously, that

$$p_1 = 0, q_1 = 0, \beta = -\frac{N_3}{N_1}, p_2 = 0, q_2 = 0 \tag{12}$$

is one of those equilibrium position. This equilibrium position is isolated one. The conditions of the asymptotic stability of this equilibrium position may be obtained by using Liénard-Chipart theorem [19]).

In addition to the isolated equilibrium position (12), there is an infinite number of non-isolated equilibrium positions. These equilibrium positions form a family of non-isolated equilibrium positions, which exists in a form of some closed line. These equilibrium positions can be found only using numerical methods, for example,

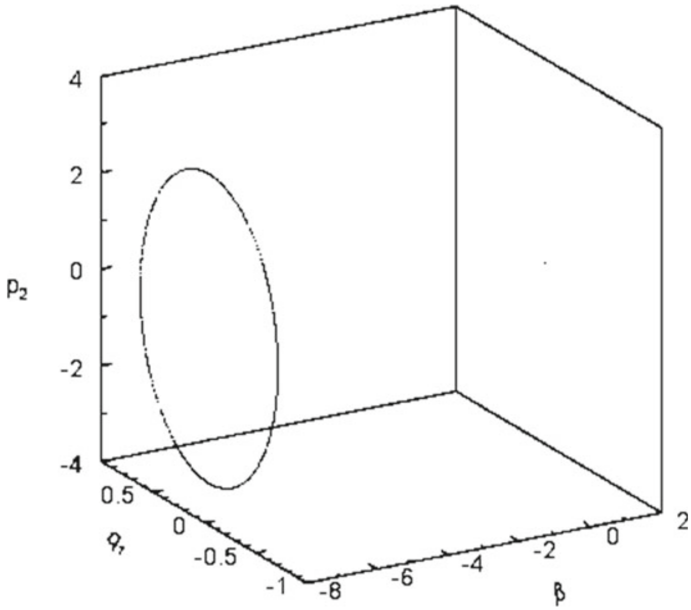


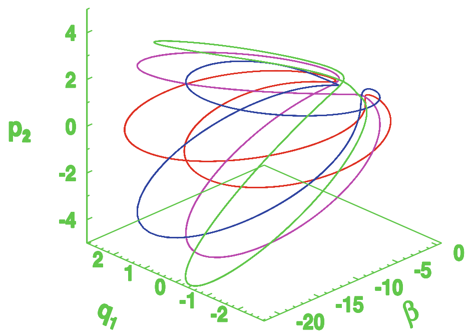
Fig. 2 Family of equilibrium positions

Newton's method. In Fig. 2, an example of such family of equilibrium positions for one concrete values of parameters of the system (9) is shown. Conditions for the stability of such family can be obtained using the Liénard-Chipart theorem. True, these conditions are extremely cumbersome. Their analysis can be carried out in reality only by using computers. Note that all equilibrium positions shown in Fig. 2 may be stable, but can not be asymptotically stable. In the case of stability of these non-isolated equilibrium positions, each of them belongs to the limit set of system (9), but is not an attractor in the traditional sense of this term. We will give a description of the attractive properties of this family below.

There are sufficiently large regions in space of parameters of the system (9), in which all equilibrium positions are unstable. In these areas, extremely interesting limit sets of this system arise, which can be as regular, and as chaotic.

Limit sets of the first type may be periodic. In this case they form family of an infinite number of closed trajectories (cycles), all of which exists simultaneously. Any cycle neighbourhood contains other cycles of the family, that is, they are not isolated. However, such cycles do not have tangency or intersection points. Each such closed trajectory is itself a limit set. This is due to fact that almost any trajectory that starts in some large area of phase space tends to one of the cycles of the family. But none of cycles is an attractor in the traditional sense of this term. So, each of these cycles is not limit cycle. Moreover, every single cycle has same period, same Lyapunov's characteristic exponents and similar Poincare sections. It is worth noting that the cardinality of this family is equal to continuum.

Fig. 3 Four representatives of maximal regular attractor.



In Fig. 3 regular periodic limit sets of system (9) are constructed at $\alpha = -0.8$, $A = 1.12$, $B = -1.531$, $N_1 = -1.25$, $N_3 = 2$, $\mu_1 = -5.15$. Each cycle is plotted in different color. There are four cycles in total, each of which is a representative of the infinite family of cycles. We emphasize once again that each of the cycles forming the family is not an attractor in the traditional sense of this concept. In our opinion, the most suitable term for describing such family is the concept of maximal attractor.

A clear definition of the concept of maximal attractor is given by Milnor [20], by Anischenko and Vadivasova [21], as well as by Sharkovsky [22]. Thus, two different families that are shown in Figs. 2 and 3, are essentially two different types of regular maximal attractors.

With an increase in the value of the parameter μ_1, family of chaotic trajectories arises in the system. The arising family includes an infinite number of chaotic trajectories. It is known that the "traditional" chaotic attractor consists of an infinite number of unstable trajectories. The resulting family, at first glance, is a union of an infinite number of chaotic attractors. However, each member of this family is not an attractor in the "traditional" sense. Here, as before, to define such union, the concept of maximal attractor can be proposed. All trajectories of the chaotic maximal attractor have same spectrum of Lyapunov's characteristic exponents, including positive one. The Poincare sections of each of the trajectories of the family are structurally similar chaotic sets consisted of an infinite number of points.

In Fig. 4 for the values $\alpha = -0.8$, $A = 1.12$, $B = -1.531$, $N_1 = -1.25$, $N_3 = 2$, $\mu_1 = -4.6463$ limit sets of second type of the system (9) is constructed. Each represenative of the chaotic maximal attractor is plotted in its own color. In total, there are three chaotic trajectories of the family are presented in Fig. 4.

In Fig. 5 three representatives of another kind of chaotic maximal attractor, constructed at $\mu_1 = -4.6462$, are shown. On the whole, the chaotic maximal attractor of the second kind are characterized by a much denser filling of the localization region with trajectories. This two kinds of chaotic maximal attractor are typical for system (9).

We note one more feature of the constructed maximal attractors, both regular and chaotic. Some trajectories of those families are localized in the three-dimensional subspace of the five-dimensional phase space of system (9). So the trajectories shown

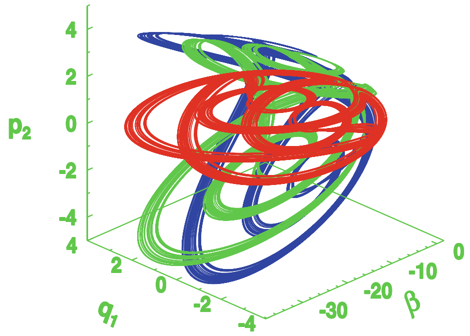


Fig. 4 Three representatives of maximal chaotic attractor of first kind

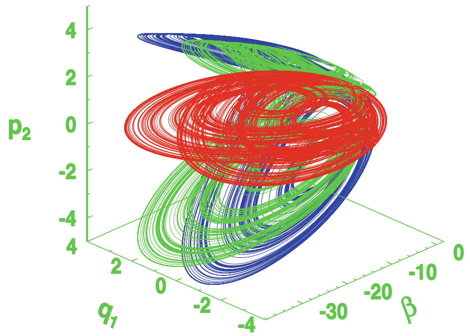


Fig. 5 Three representatives of maximal chaotic attractor of second kind

in red in Figs. 3, 4 and 5 are localized in three-dimensional space. This means that the coordinates p_2 and q_2 of the "red" trajectories are equal to zero. That is, there are no oscillations in the second dominant mode.

Shortly we underscore one more interesting feature of the maximal attractors of the system (9). Although these attractors are not attractors in the traditional sense, the transition from regular to chaotic regimes and the "chaos-chaos" transitions follow the scenarios inherent in such transitions for traditional attractors. Thus transitions according to the Feigenbaum scenario [23, 24], Manneville−Pomeau scenario were found [25, 26] along with various scenarios of generalized intermittency [16, 17, 27, 28].

Let us briefly consider the features of the transition to chaos according to the Feigenbaum and Manneville-Pomeau scenarios for maximal attractors. One of the possible scenarios is the transition from regular regime to a chaotic one is a cascade of bifurcations of period doubling of the cycles. At the same value of the bifurcation parameter the period of all cycles, that form the maximal attractor, is doubled. Then, at the next bifurcation point, the period of all cycles of the maximal attractor is again doubled, and so on. This endless process of period doubling bifurcations ends with the emergence of a chaotic maximal attractor. That is, the transition from a periodic limit set to a chaotic limit set is realized according to the classical Feigenbaum's scenario.

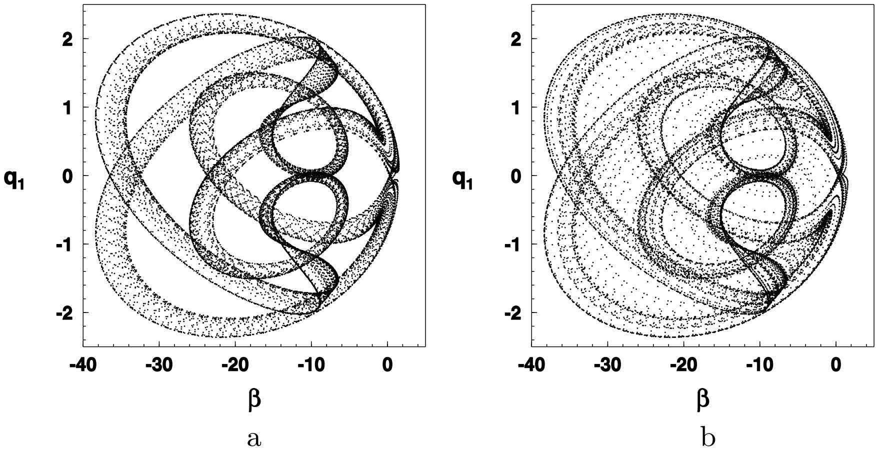


Fig. 6 Distribution of invariant measure over projections of maximal attractors

The transition to chaos through intermittency (the Manneville-Pomeau scenario) for the maximal attractors occurs as follows. The system has a maximal attractor consisting of an infinite set of simultaneously existing cycles. Moreover, all the trajectories of the family have the same period. When passing through the bifurcation point, all cycles of the family disappear and a chaotic maximal attractor arises in the system. The movement along the trajectories of all representatives of this maximum attractor consists of two phases - laminar and turbulent. That is, for all representatives of the family of cycles, there is a simultaneous transition to chaos, through one rigid bifurcation.

In conclusion, let us illustrate the transition from the maximal chaotic attractor of one type to the maximal chaotic attractor of another type through generalized intermittency. In Fig. 6a the distribution of the invariant measure over the phase portrait projection of the representative of the chaotic maximal attractors of the system (9) constructed at $\alpha = -0.8$, $A = 1.12$, $B = -1.531$, $N_1 = -1.25$, $N_3 = 2$, $\mu_1 = -4.6463$ is shown. At $\mu_1 = -4.6462$ (other parameters unchanged) maximal attractor disappears and chaotic maximal attractor of new type is born in the system. The distribution of the invariant measure over the projection of the phase portrait of the representative of the new chaotic maximal attractor is shown in Fig. 6b. The transition from one type of chaotic maximal attractor to the chaotic maximal attractor of another type occurs according to the scenario of generalized intermittency, which was described for attractors in the traditional sense of this term. At such transition, the scenario of generalized intermittency is simultaneously fulfilled for all representatives of chaotic maximal attractor that presented in Fig. 4. For each representative of the new chaotic maximal attractor, the motion along the trajectory consists of two alternating phases, namely rough-laminar phase and turbulent phase. In the rough-laminar phase, the trajectory makes chaotic movements in the neighborhood of the trajectories of the representative of the disappeared chaotic maximal attractor. Then, at an unpredictable moment of time, the trajectory leaves the localization region of the representative of the disappeared maximal attractor and moves to distant regions of the phase space. Rough-laminar phase corresponds to the much blacker areas in fig in Fig. 6. These areas in Fig. 6a are nearly the same as the distribution of the invariant measure from in Fig. 6b. In turn, turbulent phase corresponds to much less darkened areas in Fig. 6b. After some time, the movement of the trajectory returns to the rough-laminar phase again. Then, trajectories switch to turbulent phase again. Such transitions are repeated an infinite number of times. Note that the duration of both rough-laminar and turbulent phases is unpredictable as are the moments of times of transition from one phase to another.

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