Forced van der Pol Oscillator—Synchronization from the Bifurcation Theory Point of View

Jan Ševčík and Lenka Přibylová

Abstract The contribution presents a bifurcation theory point of view to synchronization of a forced van der Pol oscillator, which is coupled to a master oscillator as a system with a stable limit cycle corresponding to harmonic oscillation. We present bifurcation manifolds, 3D sections of the phase space and its Poincaré sections for parameters close to these manifolds providing a clear visualization of the dynamics of the 4D system. Among other things, we present the coexistence of a stable torus and a stable cycle arising from q-fold bifurcation on an Arnold tongue.

Keywords Synchronization · Van der Pol oscillator · Bifurcations of limit cycles · Neimark–Sacker bifurcation · q-fold bifurcation · Arnold tongues

1 Introduction

Synchronization of coupled systems of oscillators is an important phenomenon that touches a large class of nonlinear dynamical systems. Synchronization is ubiquitous and methods of applied nonlinear dynamics can thus help to solve problems and create new technologies in neuroscience [1, 5, 14], chemistry [6], biology [12], superconducting electronics [3, 10], spintronics [9], computing [8], or even particle physics [2]. Since these nonlinear systems exhibit complex and sometimes even counterintuitive dynamics, the most commonly used methods to study synchronizations are simulations.

Although the theory of bifurcations offers a suitable apparatus for the analysis of the systems mentioned above, it is usually not used. The highly abstract thinking and mathematically generalized view of dynamics needed for such an analysis are not

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the only obstacles to using bifurcation analysis methods. Another problem occurs because the phase variables present in such models usually enter as harmonic terms. Due to that, the systems are typically stiff, and standard numerical continuation techniques fail.

Our contribution brings a suitable method for analyzing dynamics of forced oscillators concerning synchronization. We present this method on the forced van der Pol oscillator example. In addition, it also allows excellent visualization of the state space in the neighborhood of bifurcation manifolds that belong to the onset of synchronization. All nonlinear phenomena that are closely related to it, as torus birth, resonances, or complex dynamics near double Hopf bifurcation, can be visualized in 3D space which greatly simplifies their explanation. This approach can be used for much more complex systems of coupled oscillators as you can see in Záthurecký and Přibylová [13].

2 Forced van der Pol Oscillator Representation

Consider the widely known van der Pol oscillator driven by an external harmonic force represented by the equation

$$\ddot{x} - \mu\left(1 - x^2\right)\dot{x} + \omega_0^2 x + A\sin\omega t = 0, \tag{1}$$

where $x \in \mathbb{R}$ is a time-dependent position coordinate, $\mu > 0$ denotes a parameter indicating the nonlinearity (the strength of the damping), and $\omega_0 \in \mathbb{R}$ is the natural frequency. The last term represents the external driving force with amplitude $A > 0$ and frequency $\omega \in \mathbb{R}$.

This second-order differential equation can be expressed in the following form of two-dimensional non-autonomous system

$$\dot{x} = y + \varepsilon\cos\omega t, \tag{2a}$$

$$\dot{y} = \mu\left(1 - x^2\right)y - \omega_0^2 x, \tag{2b}$$

where $\varepsilon = \frac{A}{\omega}$. To obtain an autonomous system, it is usually convenient to rewrite the time-dependent term $\varepsilon\cos\omega t$ in (2a) using a pair of new variables, specifically

$$\dot{x} = y + \varepsilon u, \tag{3a}$$

$$\dot{y} = \mu\left(1 - x^2\right)y - \omega_0^2 x, \tag{3b}$$

$$\dot{u} = -\omega v, \tag{3c}$$

$$\dot{v} = \omega u. \tag{3d}$$

Unfortunately, this system is stiff, and the continuation is impossible since the periodic solution of (3c), (3d) is not asymptotically stable. Therefore, we replace this subsystem with a normal form of supercritical Hopf bifurcation

$$\dot{u} = ru - \omega v - u\left(u^2 + v^2\right),$$
$$\dot{v} = \omega u + rv - v\left(u^2 + v^2\right)$$

with an exponentially stable driving cycle allowing a stable continuation of limit cycles and their bifurcations. Note that the added parameter r provides a possibility to investigate bifurcations connected to the birth of an invariant torus.

Hence, one can examine the forced van der Pol oscillator (1) as two interacting master-slave oscillators in the form

$$\dot{x} = y + \varepsilon u, \tag{4a}$$
$$\dot{y} = \mu\left(1 - x^2\right) y - \omega_0^2 x, \tag{4b}$$
$$\dot{u} = ru - \omega v - u\left(u^2 + v^2\right), \tag{4c}$$
$$\dot{v} = \omega u + rv - v\left(u^2 + v^2\right). \tag{4d}$$

This step also provides an opportunity to clearly visualize synchronization phenomena of the famous van der Pol oscillator since variables u and v are complementary, and one of them can be omitted in the state space description.

3 Basic Bifurcation Analysis

The studied system (4) is evidently uncoupled for zero coupling, i.e., $\varepsilon = 0$. In this case, one can investigate both subsystems separately. Assuming $\mu = 0$, there is no damping in the van der Pol system, and thus the system exhibits simple conservative harmonic oscillations with frequency ω_0. It is known that the unforced van der Pol oscillator undergoes a supercritical Hopf bifurcation that gives rise to a stable limit cycle while crossing $\mu = 0$ as well as the forcing, master system while crossing $r = 0$. It follows that a double Hopf bifurcation manifold (i.e., parameter subspace $\mu = 0, r = 0, \varepsilon = 0$) can be detected as a transversal intersection of these two Hopf hyperplanes.

4 Torus Birth and Synchronization

Double Hopf bifurcation leads to complex dynamics that is related to other bifurcations for nearby parameters. Generically, two branches of Neimark–Sacker bifurcation of a cycle, resulting in a torus birth, emanate from the double Hopf point.

The system (4) gives birth to the stable invariant torus for positive μ and r near zero obviously since supercritical bifurcations appear at $\mu = 0$ and $r = 0$, respectively. An

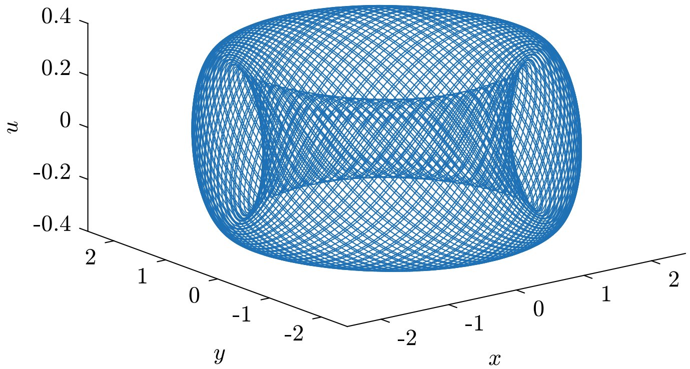


Fig. 1 A segment of a quasiperiodic orbit on a stable invariant torus of system (4) for $\mu = 0.1$, $r = 0.1$, $\varepsilon = 2.5$, $\omega_0 = 1$ and $\omega = \sqrt{5}$

example of a quasiperiodic trajectory densely covering the torus is presented in Fig. 1. The trajectories of the system (4) on the invariant torus can become periodic since the torus is described by a pair of frequencies that can be in a mutually rational proportion. In that moment, the synchronization appears in terms of phase- or frequency-locking. For given external harmonic forcing with nonzero amplitude r and natural frequency ω_0, zero damping μ and zero coupling ε, it is exactly for ratios $\frac{\omega_0}{\omega}$ that are rational. These points are resonances (two-parametric cusp bifurcations of cycles or q-fold bifurcation points in the notation of the bifurcation theory) that correspond to cusp Arnold tongues emanating from Neimark–Sacker bifurcation manifold $\varepsilon = 0$. The Arnold tongues' borders are fold bifurcation manifolds of a stable cycle and a saddle cycle that coincide with each other. The stable cycle persists inside the Arnold tongue and corresponds to the synchronization. Notice that Neimark–Sacker bifurcation, the torus, and fold bifurcation of a cycle manifold continue to positive ε. Since the cusp bifurcation has a typical V-shape, more coupling strength makes the synchronization easier.

From now on, we will consider fixed values $\mu = 0.1$, $r = 0.1$ and $\omega_0 = 1$. The following results are independent of the choice in the sense that we can choose any small μ and r to start with quasiperiodic orbit on a torus. The natural frequency ω_0 is taken as normalized, but it can be easily reparametrized. Let us study the effect of parameters ω and ε on the synchronization in system (4). Using numerical continuation methods in MATCONT toolbox by Dhooge et al. [4], one can compute bifurcation curves of (4) in the parameter space (ω, ε).

Since the natural frequency of the van der Pol oscillator is chosen as $\omega_0 = 1$, the Arnold tongues emanate from all rational numbers on the ω-axis, i.e., points $(\omega, \varepsilon) = \left(\frac{p}{q}, 0\right)$ for coprime $p, q \in \mathbb{N}$. Figure 2 shows several Arnold tongues $\mathcal{A}_{p:q}$ in space (ω, ε) representing parameter values, for which the synchronization $p : q$ takes place in the studied system (4) ($p : q$ is a ratio between the two frequencies on the torus, $q : p$ is the period ratio). As usual, most of the Arnold tongues are

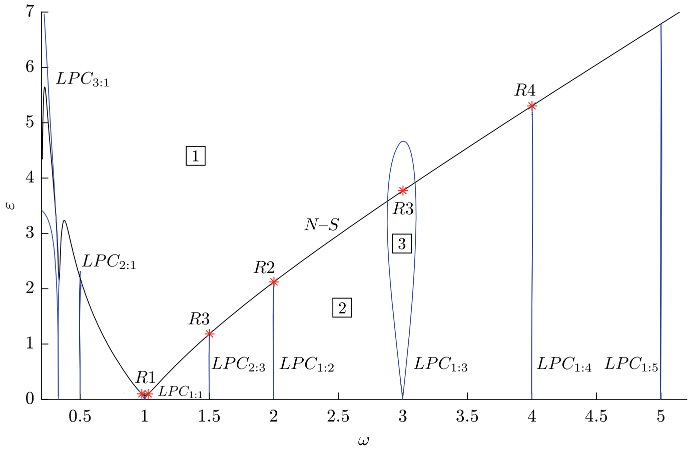


Fig. 2 Bifurcation diagram of system (4) in the parameter space (ω, ε) for $\mu = 0.1$, $r = 1$ and $\omega_0 = 1$

relatively narrow and hence difficult to be manually detected. Notice that we found a non-trivial branch of Neimark–Sacker bifurcation that is different from $\mu = 0, r > 0$ or $r = 0$, $\mu > 0$, respectively, in this parameter space. Dynamics near this branch for $1 : 3$ resonance is shortly mentioned in Sect. 6.

5 Visualizations of the Torus Birth

In addition to the analysis itself, we focused on visualization of dynamics near bifurcation manifolds. One dimension of the 4D state space of the system (4) can be omitted easily as a complement due to harmonic forcing. The 3D invariant torus that appears in the state space for positive r and μ is projected to a two-dimensional torus. Its natural section in a given phase is a Poincaré 2D plane section of a trajectory on the torus. This situation makes it possible to explicitly show qualitative changes in the neighborhood of bifurcation manifolds in the plane and 3D space.

At first, let's look at the transition between regions $\boxed{1}$ and $\boxed{2}$ (see Fig. 3). The system possesses a stable limit cycle in the region $\boxed{1}$ (see Fig. 4). When crossing the non-trivial Neimark–Sacker curve into region $\boxed{2}$, the corresponding Neimark–Sacker bifurcation of a cycle causes a loss of the cycle's stability. It gives rise to a stable invariant torus in its neighborhood (see Figs. 5 and 6). As these figures show, using Poincaré section determined by zero u-coordinate, for example, one can visualize bifurcations of limit cycles via specific orbit topological change of the discrete dynamical system (see Neimark–Sacker bifurcation of maps in Kuznetsov [7]) on the corresponding Poincaré section.

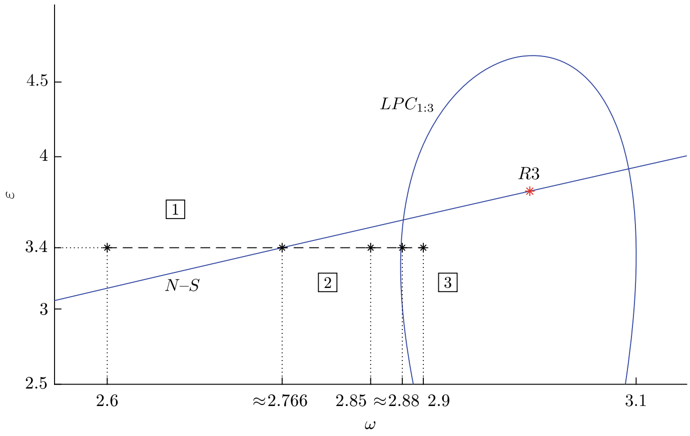


Fig. 3 Considered transitions between regions 1, 2 and 3 in the parameter space (ω, ε) for $\mu = 0.1$, $r = 1$ and $\omega_0 = 1$

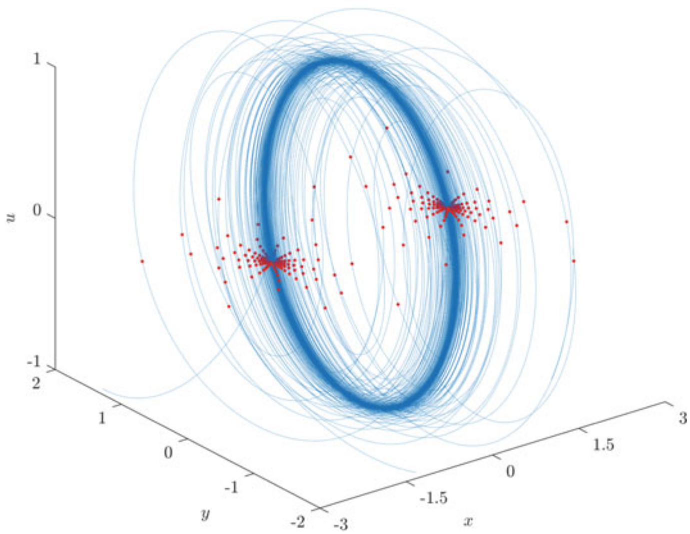


Fig. 4 Poincaré section $\{u = 0\}$ of system (4) for $(\omega, \varepsilon) = (2.6, 3.4)$, region 1

6 Bistability of the Forced van der Pol Oscillator

Finally, let's look closely to qualitative changes of dynamics near 1 : 3 resonance point R3 on the non-trivial Neimark–Sacker branch depicted in Figs. 2 or 3 (for the positive r, μ, ε and ω). Figure 7 shows a typical symmetric dynamic structure near 1 : 3 resonance (see Kuznetsov [7]). It visualizes the transition between regions 2 and 3. As we have just seen, in the region 2 (outside the Arnold tongue), the

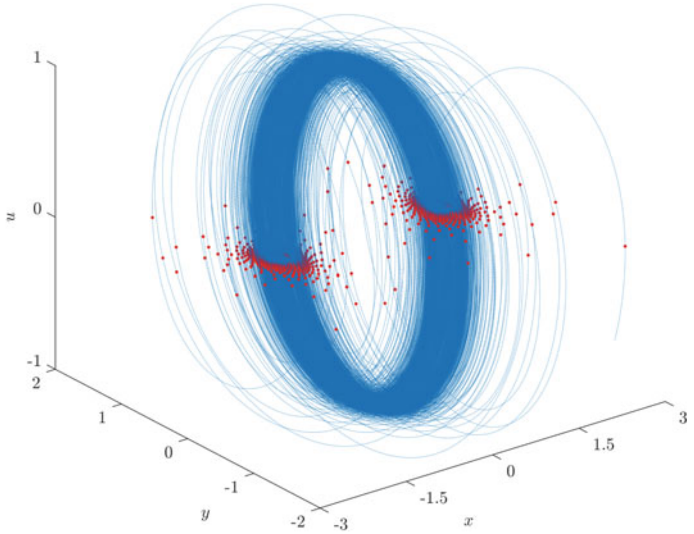


Fig. 5 Poincaré section $\{u = 0\}$ of system (4) $(\omega, \varepsilon) = (2.766, 3.4)$, near N–S manifold

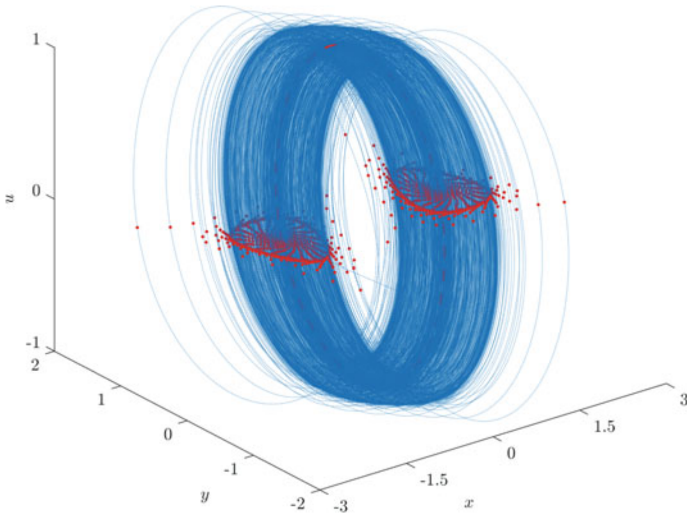
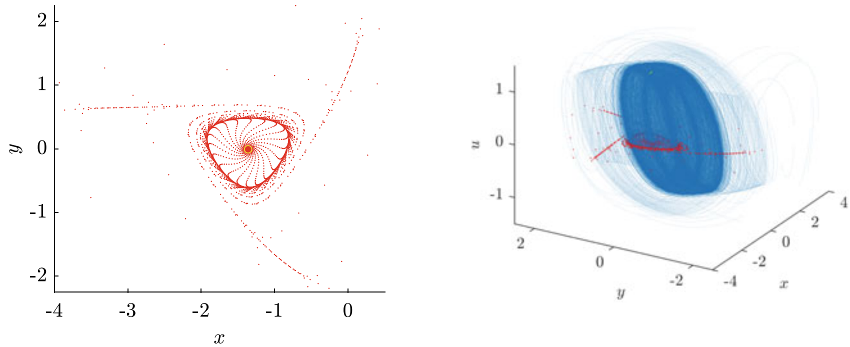
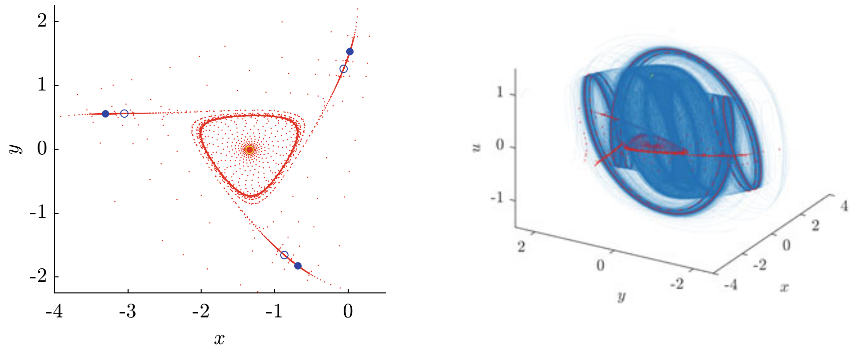


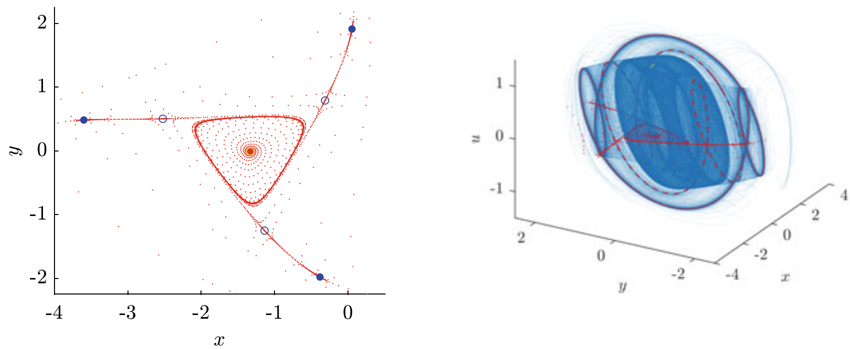
Fig. 6 Poincaré section $\{u = 0\}$ of system (4) for $(\omega, \varepsilon) = (2.85, 3.4)$, region 2



(a) Region 2: $(\omega, \varepsilon) = (2.85, 3.4)$, before crossing $LPC_{1:3}$



(b) Region 3: $(\omega, \varepsilon) = (2.88, 3.4)$, right after crossing $LPC_{1:3}$



(c) Region 3: $(\omega, \varepsilon) = (2.90, 3.4)$, after crossing $LPC_{1:3}$

Fig. 7 The onset of synchronization 1 : 3 in system (4) visualized using Poincaré section $\{u = 0, v = -1\}$ in the state space for parameters (ω, ε) from regions 2 and 3, i.e., for the crossing of the $LPC_{1:3}$ curve corresponding to fold bifurcation of limit cycles (see Fig. 3) system possesses a stable quasiperiodic invariant torus. When crossing the LPC curve (entering the Arnold tongue, $\boxed{3}$), the fold bifurcation gives rise to a pair of limit cycles—stable and saddle, respectively. The forced van der Pol oscillator evince bistable behavior for these parameters inside and close to the Arnold tongue border since there are two stable attractors—an outside stable limit cycle and a stable invariant torus. The torus may be destructed via a heteroclinic bifurcation. In the Poincaré section depicted red in Fig. 7b, c, you can see a symmetric triplet of saddles approaching the invariant loop that belongs to the inside torus. The coincidence of the saddle cycle with the loop destroys the stable torus.

7 Discussion and Conclusions

To summarize and outline the possible research connected to synchronizations of forced oscillators, we mention the topics we would like to focus on.

There is much more to study since Arnold tongues interfere with each other, and symmetries near resonances give birth to various types of synchronizations. Also, there is usually a period-doubling cascade inside the Arnold tongues, and this route to a chaotic attractor is possible and likely. Study of all these phenomena is allowed only using a suitable representation (4) of forced van der Pol oscillator (1). The proper transformation of the original system and using Poincaré sections give possibility to use continuation methods of bifurcation theory, and also visualize in 3D the hidden phenomena behind synchronizations of limit cycles.

We plan to continue with an analysis of bifurcation manifolds near the mentioned double Hopf bifurcation, as well as near resonances. Very interesting dynamics could be found near the torus break on the heteroclinic orbit for parameters inside the Arnold tongues. We would like to focus also on bistability in the case of coupled oscillators. We are convinced that this phenomenon is closely related to chimera-like dynamics, as well as to routes to complexity and chaos.

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