7

Competitive Facilities Location

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7.1 Introduction

The competitive facilities location problem is the location of one or more facilities among existing competing facilities. The facilities attract demand generated by customers in the area. The objective is to maximize the market share captured by the new facilities. Over the years many ways to estimating the market share captured by each facility were proposed. It is assumed that customers divide their buying power among facilities according to the facilities' attractiveness and their distance relative to other facilities. Once the market share attracted by one or more facilities can be estimated, a procedure for finding the best locations for the new facilities can be constructed. The objective function is not convex. Therefore, only heuristic procedures, that do not guarantee optimality, were proposed for solving most models.

The basic competitive model is to find the location of one or more facilities that maximize the market share captured by the new facilities. Some extensions/variations to the basic model were investigated. These extensions are listed below and are detailed in Sect. 7.4.

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Minimax Regret: Incorporating future market conditions, such as future changes in demand, into the model. Future market conditions are defined by a set of possible scenarios. For each scenario there is an optimal location for a facility yielding the maximum possible market share for that scenario. We find a location for a facility to accommodate the individual scenarios. For each scenario, some market share is lost at a location, compared to the optimal value at the best location for that scenario. The objective is to minimize the maximum loss in market share across all scenarios.

The Threshold Criterion: Rather than the objective of maximizing the total buying power attracted by the chain, there is a minimum buying power threshold to be met. If the chain fails to attract the threshold buying power, the company fails. The proposed objective is minimizing the probability that the company fails to meet the threshold.

The Leader-Follower: (Von Stackelberg equilibrium [121]). We find the best location in anticipation of future competition. The leader locates his facility and the follower (competitor) locates his facility *knowing* the leader's location. The follower has all the necessary information for his location decision. Therefore, the follower's problem is the standard competitive location problem. The leader's problem is more complicated. The objective is to maximize his market share *following* the follower's decision.

Location and Design: A limited budget is available. The improvement in the attractiveness of a facility depends on the budget allocated to it. We find the location of a new facility, its attractiveness, and possibly improving existing facilities subject to the limited budget.

Lost Demand: Customers may choose a substitute product if the product they are looking for is located too far. For example, if potential customers are interested in a Chinese restaurant but the closest one is too far, they may choose a non-Chinese restaurant which is close by, or eat at home.

Cannibalization: Minimize the cannibalization of one's chain facilities when constructing new ones. Cannibalization at the retail chain facilities is important, especially in the case of franchises. Franchisees may lose sales, which may be more than is allowed by the contract they signed with the company.

Recent reviews of competitive facilities location models are Berman et al. [6], Drezner [23, 24], Drezner and Eiselt [48], Eiselt [59], Eiselt et al. [60], Kress and Pesch [87], Lederer [90], Marianov et al. [96], Pelegrín et al. [109].

7.2 Approaches to Estimating Market Share

7.2.1 Deterministic Rules

According to the deterministic rules all customers residing at the same demand point patronize the same facility.

Proximity Rule: Hotelling [79] proposed that competitors compete by charging different mill prices and customers select the facility that provides the lowest mill price plus the cost of travel. This approach led to many papers, for example [50, 64, 69–72, 116, 119], that apply the proximity rule, i.e., customers patronize the closest facility. The proximity rule implies that all facilities charge the same price and thus are equally attractive.

Utility Function: The utility model is an extension of the proximity rule [16, 18]. A list of M quality measures, Q_i, $i = 1, \ldots, M$, each with a weight w_i, is determined. The utility function is $\sum_{i=1}^{M} w_i Q_i - d$, where d is the distance to the facility. A customer selects the facility with the maximum utility.

7.2.2 Probabilistic Rules

By the probabilistic rules, customers residing at the same demand point divide their patronage among several competing facilities. It can be interpreted as "each facility is patronized with a certain probability".

Random Utility: The random utility rule [25, 91] is an extension of the utility rule. The utility function parameters, except for the distance, are randomly distributed. Each customer patronizes the facility with the largest utility according to his assessment of the parameters. Therefore, not all customers residing at the same demand point patronize the same facility.

Gravity Model: The gravity model, sometimes referred to as the "Huff" model, was proposed by Reilly [115] and refined by Huff [80, 81]. According to the gravity model, the probability that a customer patronizes a facility is proportional to its attractiveness and declines according to a distance decay function. The basic gravity model is based on p competing facilities and n demand points that exist in an area [17]. A distance decay function $f(d, \lambda)$ with a parameter λ depending on the retail category is defined. For example, the distance decay function for grocery stores is different from the one for shopping malls. Let:

B_i	be the buying power at demand point i for $i = 1, \ldots, n$,
A_j	be the attractiveness level of facility j for $j = 1, \ldots, p$,
d_{ij}	be the distance between demand point i and facility j,
$f(d, \lambda)$	be the distance decay function,
λ	be the parameter of the distance decay function which depends on the retail category.
M_j	be the expected market share captured by facility j.

The estimated market share captured by facility j according to the gravity model is:

$$M_j = \sum_{i=1}^{n} B_i \frac{A_j f(d_{ij}, \lambda)}{\sum\limits_{k=1}^{p} A_k f(d_{ik}, \lambda)} \quad . \tag{7.1}$$

where the distance decay function $f(d, \lambda)$ is the same for all competing facilities in the same retail category. Note that some models assume a decay function $f(d)$ without a parameter λ.

In the original gravity model [115] it is assumed that the distance decay parallels gravity decay and thus $f(d) = \frac{1}{d^2}$. Huff [80, 81] suggested a decay function $f(d, \lambda) = \frac{1}{d^\lambda}$. Other distance decay functions include: exponential decay $f(d, \lambda) = e^{-\lambda d}$ [77, 132], $f(d) = e^{-1.705 d^{0.409}}$ [5], and a logit function [56]. Based on a real data set, Drezner [20] showed that exponential decay $f(d, \lambda) = e^{-\lambda d}$ fits the data better than a power decay $f(d, \lambda) = \frac{1}{d^\lambda}$. It is well recognized that the decay function varies across retail categories. For example, for the decay function $f(d, \lambda) = \frac{1}{d^\lambda}$ it was found that $\lambda = 3$ for grocery stores [81], $\lambda = 3.191$ for clothing stores [80], $\lambda = 2.723$ for furniture stores [80], and $\lambda = 1.27$ for shopping malls [20].

Flow Interception: Berman and Krass [8] introduced a competitive location model where demand is attracted from customers traveling en route to some destination while passing by a facility ("impulsive" shoppers) and demand originated at nodes of the network (planned purchase trips). Customers may change their mind on the way to the selected facility and stop at a less attractive facility just because they noticed it on the way. The latter issue is discussed in [37] concluding with a recommendation to locate a small retail facility "on the way" to a large shopping mall.

Cover Based Model: Launhardt [89] and Fetter [62] coined the term "Economic law of market areas". This concept was formalized by defining a "radius of influence", which is at the core of central place theory [11, 94]. According to central place theory there is a maximum "range of the good", depending on retail category, that customers are willing to travel to obtain the good. Drezner et al. [38, 39] proposed that each competing facility has a "sphere of influence" determined by its attractiveness level. More attractive facilities have a larger sphere of influence. The buying power spent by a customer in the sphere of influence of several facilities is divided among the competing facilities. The buying power of a customer located outside all the spheres of influence is lost. Drezner et al. [42] refined the model by assuming that patronage does not drop abruptly at the radius of influence, but declines gradually near that radius.

7.2.3 Estimating Attractiveness

The models for estimating the captured market share (except for the proximity rule) rely on a good estimate of the facilities' attractiveness levels. Therefore, estimating the attractiveness of a facility is an important component required for a successful implementation of the models.

Nakanishi and Cooper [101] suggested to determine a list of properties and calculated the attractiveness of a facility as a product of these properties' values, each raised by a power. Many researchers conducted public opinion surveys to determine the attributes affecting the attractiveness of the competing facilities and then establish their attractiveness. Properties that were found by opinion surveys to affect attractiveness include:

Supermarkets: Square footage [81]; price [113]; price, freshness, availability, convenience, quality service, parking [118]; choice range for daily/non-daily goods, price for daily/non-daily goods, parking [127]; store image, layout, appearance, accessibility, service, employee composition [83]; cleanliness, brands I like, better produce, low prices [16]; cost of products [5].

Furniture: Square footage [80].

Clothing: Square footage [80]; parking availability, choice range [126].

Central Business District: price, visual appearance, reputation, range of goods, shopping hours, atmosphere, design, service [15].

Shopping Malls: variety of stores, mall appearance, favorite brand names [49]. They tested 6 more attributes that were not found significant: Mall prices, distance to mall, adequate parking, mall safety, food court/restaurants, movies/entertainment.

There are other approaches to estimating and analyzing attractiveness levels:

- Drezner and Drezner [29] observed that the annual sales of a retail facility are a clear indication of its attractiveness. Higher attractiveness level is reflected in higher sales figures. However, sales figures alone cannot be directly used as the attractiveness measures because sales are also dependent on the affluence (or buying income) of potential customers in a retail facility's trade area. The proposed procedure adjusts the sales figures by the demographic characteristics of the area, thus deriving the attractiveness measures of the retail facilities mainly from these two pieces of data.

- Drezner [20] estimated the attractiveness of facilities by asking shoppers about the zip code they came from and determined the attractiveness level by a least squares approach. She found the best fit to the observed distances by defining the attractiveness levels as variables.
- Drezner et al. [46] refined the gravity model by assigning different decay functions to different facilities. Customers patronize a more attractive facility at greater distances. A more attractive facility has a flatter decay function. The multiplicative attractiveness values are replaced by different decay functions. This approach is easier to implement because there is no need for public opinion surveys. It yielded more accurate results on a real data set. They modified the Drezner [20] approach by replacing the multiplicative attractiveness coefficients A_j as variables, with different decay parameter λ_j as variables. There is no multiplicative attractiveness level, i.e., $A_j = 1 \forall j$. Drezner et al. [47] incorporated the attractiveness level A_j in addition to variable decay parameters λ_j into the model.
- Drezner et al. [45] observed that not all customers perceive the same attractiveness level for the same facility. They proposed that the attractiveness level A_j is normally distributed with some mean and variance. Existing models use the mean attractiveness as A_j. Drezner et al. [45] defined an "effective" attractiveness level which is found to be lower than the average attractiveness assumed in gravity models. Greater variances yield lower effective attractiveness because the loss in market share by a given decline in attractiveness is greater than the gain in market share by increasing the attractiveness by the same amount.

7.3 Distance Correction

In most location models it is assumed that demand is generated at demand "points". In reality demand is generated in neighborhoods and not all residents in a neighborhood reside at the same distance from a facility. Demand generated in an area (for non-competitive location models) is investigated in, for example, [55, 122, 131].

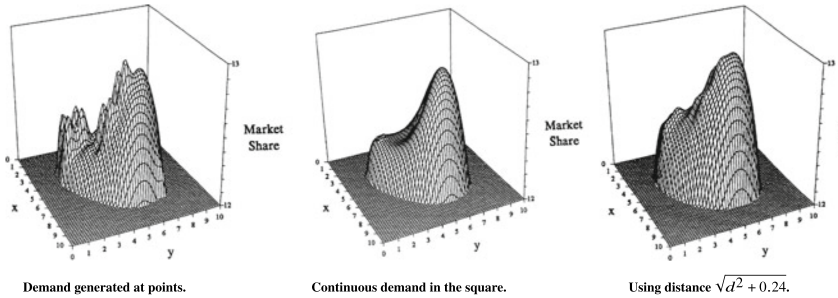

Fig. 7.1 Discrete and continuous market share surfaces

Drezner and Drezner [26] proposed a distance correction to the gravity model. Data may be available by zip codes or census tracts. Listing all individual customers is impractical. The distance correction incorporates these considerations. Drezner and Drezner [26] suggested that if the area of a demand "point" is A, the distance to a facility from the center of the area (the demand point) is d, then the corrected distance to be used in the gravity model is about $\sqrt{d^2 + 0.24A}$.

Drezner and Drezner [26] used an example problem of 100 demand points in a square of size 10 by 10 with seven existing facilities. This example problem was used in many papers, for example [18, 106]. Each demand point has an area of 1. The market share captured by the new facility is plotted in Drezner and Drezner [26], and depicted in Fig. 7.1. On the left, the surface plot of the "standard" gravity model using $f(d) = \frac{1}{d^2}$ as the decay function is depicted. In the middle, the market share captured when demand is continuous in the 10 by 10 square is shown. On the right, the market share surface using a decay function of $f(d) = \frac{1}{d^2+0.24}$ (distance correction) is depicted. When demand is generated at demand "points" there are many local maxima at various locations. The surface on the right is "smooth" and very close to the continuous surface with two local maxima.

7.4 Extensions

7.4.1 Minimax Regret Criterion

Drezner [21] incorporated future market conditions into the gravity model for the retail facility location. Future market conditions were analyzed as a set of possible scenarios. The best location for a new retail facility is at a location where the market share captured at that location is as close to the maximum as possible regardless of which future scenario takes place. Each scenario may also span different time horizons. The objective is the minimax regret which is used in other models of location analysis, for example [4, 13, 114].

Suppose there are K possible scenarios, $k = 1, \ldots, K$. For each scenario we can calculate the market share $M_k(X)$ at location X. The maximum buying power that can be captured according to each scenario, $M_k^* = \max_X\{M_k(X)\}$, is calculated. The minimax regret objective $R(X)$, to be minimized, is then

$$R(X) = \max_{k=1,\ldots,K}\left\{M_k^* - M_k(X)\right\}$$

Drezner [21] applied the multi-start heuristic approach to find M_k^* and minimize $R(X)$ for the location of one facility in the plane.

7.4.2 The Threshold Objective

Drezner et al. [44] suggested a different objective for competitive location models. Rather than the objective of maximizing the total buying power attracted by a chain, there is a minimum buying power threshold T to be met. If the chain fails to attract the buying power T, the company fails. The proposed objective is minimizing the probability that the company fails to meet the threshold. The threshold concept has been employed in financial circles as a form of insurance on a portfolio, either to protect the portfolio or to protect a firm's minimum profit, for example [82, 84, 107].

In competitive facility location, let the buying power at demand point $1 \le i \le n$ be distributed according to some distribution with a mean of μ_i and a standard deviation σ_i. The buying powers of two demand points i and j are correlated with a correlation coefficient r_{ij}. The total buying power attracted by the chain has a mean of μ and a standard deviation σ (see [44] for detailed calculations). The objective function is to minimize $p(X) = P\left(Z \le \frac{T-\mu}{\sigma}\right)$. By the Central Limit Theorem $M(X)$ can be assumed Normal but this is unnecessary because any cumulative distribution is monotonically increasing, thus minimizing $p(X)$ is equivalent to minimizing $\frac{T-\mu}{\sigma}$.

This problem was solved heuristically in [44]. It is possible to solve it optimally using BTST [53], but to the best of our knowledge no such attempt was made.

7.4.3 Leader-Follower Models

Drezner and Drezner [36] provide a review of the leader-follower model. Other papers on the topic are Küçükaydın et al. [112], Plastria and Vanhaverbeke [88].

There are two well researched two players' games: Nash equilibrium [102] and the leader-follower game also termed the von-Stackelberg equilibrium [121] and in voting theory is known as Simpson's problem [120]. In the Nash equilibrium game no player can improve his objective when the other player does not change his strategy. In many cases no equilibrium exists. In the leader-follower game the leader adopts a strategy and the follower adopts his best strategy knowing the leader's strategy. The follower's goal is to maximize his objective function while the leader's goal is to maximize his objective function *following* the follower's action.

Early contributions to Nash equilibrium location problems include Eaton and Lipsey [58], Hotelling [79], Lerner and Singer [92], Wendell and McKelvey [130]. The leader-follower location problem was introduced to competitive location models by Hakimi [69], and published in Hakimi [70–72], for location on network nodes using the Hotelling [79] premise that each customer patronizes the closest facility, see also Hansen and Labbè [73].

Drezner [50] analyzed two competitive location models in the plane. One is the location of a new facility that will attract the most buying power from an existing facility (the follower's problem). The other is the location of a facility that will secure the most buying power against the best location of a competing facility to be set up in the future (the leader's problem). The proximity rule using Euclidean distances is assumed.

Let n demand points be located in the plane. A buying power $b_i > 0$ is associated with demand point i for $i = 1, \ldots, n$. The leader locates his facility at X and the follower locates his facility at Y. Customers will patronize the follower's facility Y if the Euclidean distance between the customer and Y is less than the distance between the customer and X. Two problems are considered:

The follower's problem: Given the location of an existing facility X serving the demand points, find a location for a new facility Y that will attract the most buying power from demand points.

The leader's problem: Find a location for X such that it will retain the most buying power against the best possible location for the follower's facility Y.

For given locations X and Y, the distribution of the buying power can be found by constructing the perpendicular bisector to the segment connecting X and Y. This perpendicular bisector divides the plane into two half planes. All points in the closed half plane which includes X (including points on the perpendicular bisector itself) will patronize X and all the points in the other open half plane which includes Y, will patronize Y. This is a generalization of Hotelling's analysis on a line [79].

It is shown in [50] that one of the optimal locations for Y when X is given is infinitesimally close to X but not on X. It follows that the best location for the follower, Y^*, is "adjacent" (close) to X. The variable yet to be determined is the direction in which Y is "touching" X. In conclusion, finding an optimal location for Y is equivalent to finding the best line through X such that the open half plane defined by this line contains the most buying power for Y. Finding the best line by simple enumeration is detailed in [50].

The algorithm that solves the leader's problem is based on the algorithm used for solving the follower's problem. It can be found whether attracting a certain market share P_0 or higher by Y is possible by finding whether there is a feasible solution to a linear program. The algorithm is based on a bisection on the value of P_0. Complete details are given in [50].

The two problems can be modified by an extra restriction that the follower cannot locate his facility closer than a given distance R from the leader's facility. To solve the modified follower's problem for a given X it can be shown that the best solution for Y is determined by open half planes defined by tangent lines to the circle centered at X with a radius of $\frac{1}{2}R$ rather than lines through X. The details of the algorithms for solving the modified problems are available in [50].

Drezner and Zemel [57] considered the following problem: a large number of customers are spread uniformly over a given region $A \subseteq \mathbb{R}^2$. What configuration of facilities that cover the area will best protect against a future competing facility? The proximity rule is assumed, i.e., each customer patronizes the closest facility. There are three evenly spread configurations that cover the whole $\mathbb{R}^2$ plane with equilateral polygons depicted in Fig. 7.2: a triangular grid; a square grid; and an hexagonal grid (beehive). No other cover of the plane by identical equilateral polygons exists. Drezner and Zemel [57] found that the solution to the problem of covering the whole $\mathbb{R}^2$ plane is the hexagonal pattern. Then they analyzed the finite area problem and found bounds on the difference between the configurations as the number of facilities increases.

Since customers are attracted to the closest facility, the market share captured by each facility is proportional to the area attracted to the closest

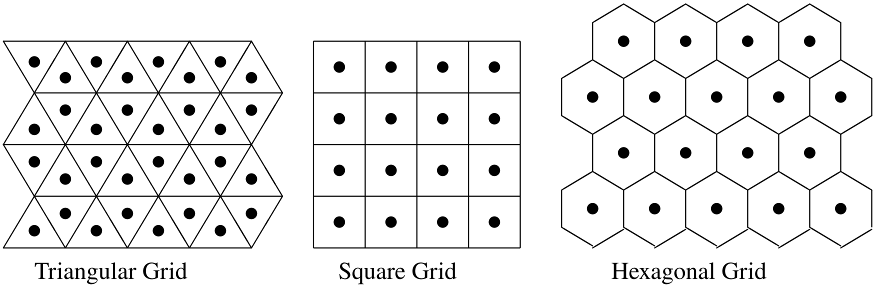

Fig. 7.2 Various configurations

facility. This is similar to the Voronoi diagram concept [3, 104, 124, 129]. In the configurations depicted in Fig. 7.2, the market share attracted by each facility is the area of the polygon. Let A be the area attracted by each facility (the area of the triangle, square, or hexagon). It is shown in [57] that:

- For the triangular grid the competitor's facility can attract a maximum of $\frac{2}{3}A$.
- For a square grid the competitor's facility can attract a maximum of $\frac{9}{16}A = 0.5625A$.
- For an hexagonal grid the competitor's facility can attract a maximum of $0.5127A$.

The hexagonal pattern provides the best protection from a future competitor. It is interesting that for hexagonal and square grids the competitor captures at least half of A at any point in the plane.

Hexagonal pattern is optimal for many location problems with numerous facilities covering a large area. For example:

- packing the largest number of circles in an area [12, 76, 125],
- p-median [105], p-center [122] and p-cover [54],
- p-dispersion [52, 93, 95, 103],
- equalizing the load covered by facilities [123].

It is also the preferred arrangement for a beehive in nature which has developed over the years in the evolutionary process.

Drezner and Drezner [27] proposed three heuristic approaches for finding a good solution to the leader-following model where market share is estimated by the gravity model: brute force, pseudo mathematical programming, and gradient search.

The Brute Force Approach: A grid of potential locations for the leader that cover the area is generated. For each grid location, the market share attracted by the follower is found and the market share attracted by the leader (after the follower locates at his best location) is calculated. If the grid is dense enough, the vicinity of the global maximum can be identified. If a more precise location is sought, a finer grid can be evaluated in that vicinity.

The Pseudo Mathematical Programming Approach: If the market share captured by the leader and the follower were concave, the following mathematical programming formulation (termed the "pseudo" problem) would have solved the problem: Maximize the market share captured by the leader subject to the derivative of the market share captured by the follower equal to zero (indicating a local maximum for the follower).

The Gradient Search Approach: A gradient search that directly finds a local maximum for the leader-follower problem is suggested. It guarantees termination at a local maximum once the optimal market share attracted by the follower can be found. It is recommended that this procedure is repeated many times in order to have a reasonable chance of getting the global optimum.

Drezner et al. [40] investigated a leader-follower competitive location model incorporating facilities' attractiveness (design) subject to limited budgets for both the leader and follower. The competitive model is based on the concept of cover [38, 39]. The leader and the follower, each has a budget to be spent on the expansion of their chains either by improving their existing facilities or by constructing new ones. The objective of the leader is to maximize his market share following the follower's reaction. The follower's problem is identical to the three problems analyzed in [39] because market conditions are fully known to the follower. A branch and bound algorithm as well as a tabu search [65–67] were proposed in [39] for the solution of each of these three strategies. For complete details the reader is referred to [85].

7.4.4 Location and Design

Combining the location decision with the design (treating the attractiveness level of the facility as a variable) was investigated, for example, in [1, 19, 61, 111, 128].

Drezner [19] assumed that the facilities' attractiveness are variables. A budget is available for locating new facilities and for establishing the new facilities' attractiveness levels. The problem is determining the facilities' attractiveness levels within the available budget. It is solved by a gradient search when the budget constraint is kept as equality. Plastria and Vanhaverbeke [112] combined the limited budget model with the leader-follower model.

The analysis in [19] for various assumptions about the attractiveness as a function of the investment in the facility leads to some interesting insights:

1. For firms with a decreasing marginal return on investment curve, the fixed budget allocation solution with equally divided budget among the new facilities is very close to optimality.
2. For firms with a fixed (constant) marginal return on investment the fixed budget allocation solution with equally divided budget works well, and can be used if the computational effort required to obtain the flexible budget allocation solution is prohibitive.
3. For a rapidly increasing marginal return, one should consider opening only one new facility investing all the budget in it.
4. Mildly increasing marginal return leads to a middle ground solution and none of the extreme budget allocation strategies is appropriate. In this case it is recommended to find the best budget allocation by applying the algorithm in [19].

Aboolian et al. [1] studied the problem of simultaneously finding the number of facilities, their location, and their design. For the problem with discrete design scenarios the TLA (tangent line approximation) procedure (see Sect. 7.5.3) can be applied. Aboolian et al. [1] assume that A_j, the attractiveness of facility j, is a function of K attraction attributes. These attributes can be continuous (e.g. facility size, product price) or discrete (e.g. number of parking spots, product variety). Moreover, it is assumed that for any potential facility j there is a basic design γ_j and K levels of improvement y_{jk} $j \in \bar{E}$, $k \in \{1, \ldots, K\}$ over the basic design. The attractiveness of facility j can now be expressed as:

$$A_j = I_j \gamma_j \prod_{k=1}^{K} \left(1 + y_{jk}\right)^{Q_k}$$

where Q_k is the sensitivity parameter of the utility function with respect to attribute k, $0 \leq Q_k \leq 1$, and $I_j = 1$ if facility j is open and 0 otherwise. When y_{jk} is continuous, it is assumed to belong to an interval $[0, y_{jk}^{\max}]$, and when it is discrete it is assumed to belong to a certain discrete set (note that $y_{jk}^1 > y_{jk}$ if y_{jk}^1 is preferred to y_{jk}). The utility u_{ij} is now defined as: $u_{ij} = A_j(d_{ij} + 1)^{-\beta}$.

Two heuristics are introduced: an adapted weighted greedy and a steepest descent. The adapted greedy heuristic starts with an empty set of facilities and at each iteration the location design pair that results in the largest improvement per unit cost is added as long as it is within the available budget. The ascent approach starts with a given location set Q_0 and at each iteration either a new facility in the neighborhood of Q_0 is added to Q_0, or a facility is removed from Q_0 or an exchange of a new facility with one of the facilities from Q_0 takes place.

Drezner et al. [41] suggested the model assuming that the market can be partitioned into mutually exclusive sub-markets. For example, expanding a franchise around the world in New York, Paris, Tokyo, Beijing, etc., that customers residing in one sub-market patronize facilities only in that sub-market. Suppose that a procedure for finding the market share at each sub-market for a given budget allocated to the sub-market is available. The problem is then to determine the allocation of the budget among the sub-markets. A constraint that the sum of these individual budgets is equal to the available budget is added.

Three objectives were investigated: (i) maximizing the firm's profit, (ii) maximizing the firm's return on investment, and (iii) maximizing profit subject to a minimum threshold return on investment. Once the market share for a given budget at each individual market can be determined, the allocation of the budget among the markets is found by dynamic programming. For complete details see [41].

7.4.5 Lost Demand

Standard competitive location models assume that the total expenditures of each customer are a constant and are not affected by the location or number of service facilities. Therefore, locating new facilities only changes the allocation of the market share between the existing and new facilities. However, if there are no facilities nearby, customers may choose a substitute facility and the potential demand is lost.

As an example consider Chinese restaurants. There exists a certain buying power in the area which people who "love" Chinese food are willing to spend in Chinese restaurants. If there is at least one Chinese restaurant close to a customer, the available buying power will likely be completely spent at Chinese restaurants. If the closest Chinese restaurant is quite far from the customer, this customer may patronize a closer non-Chinese restaurant or eat at home rather than travel a long distance, and thus the buying power will be lost by the chain of Chinese restaurants.

Drezner and Drezner [33] suggested the following approach to estimating lost demand. The buying power at a demand point will be completely spent at a facility which is at distance "0" from the community. Furthermore, let k retail facilities be located at distances $d_1, \ldots, d_k$ from a community with buying power B. We assume that the probability that a customer located at the community will *not* spend his/her buying power at facility j is $1 - e^{-\lambda_j d_j}$ for some constant λ_j. The probability that a customer will not spend his buying power at any of the facilities is $\prod_{j=1}^{k} [1 - e^{-\lambda_j d_j}]$. The total buying power spent at all competing facilities is therefore:

$$B\left\{1 - \prod_{j=1}^{k}\left[1 - e^{-\lambda_j d_{ij}}\right]\right\} \tag{7.2}$$

and the buying power lost is

$$B\prod_{j=1}^{k}\left[1 - e^{-\lambda_j d_{ij}}\right] \tag{7.3}$$

Drezner and Drezner [33] suggested to adjust the buying power at each community by Eq. (7.2).

Another approach to modeling lost demand is proposed in [35]. They assume that there is a maximum distance D that customers are willing to drive to a facility. A "dummy" competitor is created at an imaginary location which is at a distance D from all demand points. This dummy competitor attracts the lost demand. By the proximity rule, if the distance to the closest facility is greater than D, demand is lost. By the gravity model, the dummy competitor attracts the lost demand and the total market share is that attracted by "real" facilities. By Eq. (7.1) $\sum_{j=1}^{p} M_j = \sum_{i=1}^{n} B_i$. However, not including the dummy facility in the sum, $\sum_{j=1}^{p} M_j < \sum_{i=1}^{n} B_i$ and the difference is the lost market share.

7.4.6 Cannibalization

Marketers commonly use a definition of cannibalization that focuses on a company eating into its own market by introducing a new product to an existing product line or an established brand (product line extension and multi-brand strategies) at the expense of the old brand. For example, if Coca

Cola introduces Coke2, and customers buy Coke2 instead of the original Coke, sales may be up for the new product, but these sales may be eating into Coke's sales of the original Coke. In such cases, overall company sales may not be increasing. This form of cannibalization is well recognized and well researched in the marketing literature. See for example [10, 22, 97, 98, 100].

Another form of cannibalization occurs at the retail level of chain facilities, especially in the case of franchises. In this form of cannibalization, opening a new retail outlet in close proximity to an existing outlet, the new outlet cannibalizes the sales of the existing one. Though not a franchise, this applies to Starbucks coffee and other chain retailers. Unlike cannibalization in new product development and introduction that is well researched, cannibalization at the retail level has been overlooked for the most part. With the growth of franchise operations, this emerges as an important and timely issue. For as long as companies wish to grow and expand, managers will be faced with the strategic decision of optimally locating new, additional facilities so that cannibalization of existing chain members is minimized.

Schneider et al. [117] report cases of lawsuits regarding cannibalization in fast food franchise systems such as Arby's, Burger King, KFC, McDonald's, Subway, and Taco Bell. This phenomenon is referred to as encroachment. A similar problem is observed and documented in the hospitality/lodging industry for such franchise systems as Holiday Inn, Days Inn, Howard Johnson, Ramada, Comfort Inn, and Quality Inn [108]. Many franchisees believe they have lost business as a result of cannibalization from new units in the same chain, a phenomenon referred to in the lodging industry as "impact".

When a retail chain plans an expansion in a market by building additional outlets, two not necessarily compatible objectives should be considered: (1) Maximize the market share captured by the expanding chain (if the expansion cost is fixed, profit is an increasing function of the market share captured and therefore maximizing market share is equivalent to maximizing profit). (2) Minimize cannibalization of existing chain outlets so as not to gain market share at the expense of member outlets. Cannibalizing chain members may render them nonprofitable and may result in channel conflict (both horizontal and vertical conflict). This consideration is especially critical when the outlets are franchised and gain in market share at the expense of member franchisees may be damaging to the profitability of the whole chain.

Drezner [22] formulated and solved the problem of maximizing the market share captured by the chain facilities subject to a given limit of cannibalization. The market share captured, and consequently the cannibalized portion of it, was calculated using the gravity model [80, 81].

Plastria [110] applied the utility function model [17] in which the optimal solution to maximizing market share usually is not unique but there is an area in the plane such that a facility located at any point in that area attracts the same (maximal) market share. Plastria [110] suggested to locate the facility at the point in that region that minimizes cannibalization, thus maintaining the maximum market share. When the gravity model is used, there is only one optimal solution point that maximizes chain's market share and the planner must accept the cannibalization at that point if he or she does not wish to consider sub-optimal location solutions regarding the market share captured.

Zeller et al. [133] considered the market share captured by an expanding chain. The franchisor attempts to maximize the total market share of the chain (thus implicitly considers the cannibalization of existing outlets) while the new franchisee considers the market share captured by his new outlet. They conclude that the franchisee of a new store may choose a different location for his store than the franchisor. In reality, the franchisee has no input into the location decision and thus his objective is ignored.

Ghosh and Craig [63] developed the FRANSYS model for franchise system growth. Firms seeking to expand franchise distribution systems have to balance two incompatible goals, maximizing system revenues and minimizing the cannibalization of sales of existing outlets. The model uses two constraint types: (1) constraints that disallow new unit locations that do not provide a minimum revenue threshold for the new unit, (2) disallow new units that fail to either protect current revenue for existing units as a group, or, protect current revenue for each existing unit. The first, weaker constraint is not very satisfying to existing franchisees because individual units may lose revenue. The "best" location in terms of maximizing system revenues while protecting current revenue for all existing units, results in a mediocre new location that barely meets its minimum revenue threshold [117]. Application of the FRANSYS model to the hospitality industry would require modification of data input to conform to the market and product characteristics of the hospitality industry [108].

Fernández et al. [61] proposed a related model. Their model is a bi-objective model of maximizing profit while minimizing cannibalization. They consider the location of the new facility along with its attractiveness as a decision variable. The construction cost of the new facility is included in the profit function. In addition, they added constraints forbidding the location of a facility in the vicinity of demand points. All of these components lead to a complicated model that requires extensive data collection and relies on many modeling assumptions.

7.5 Solution Methods

7.5.1 Single Facility

Drezner and Drezner [30] optimally solved the single facility gravity-based competitive location problem by applying the Big Triangle Small Triangle (BTST) global optimization algorithm [53]. The procedure BTST requires effective upper and lower bounds on the market share captured when the facility is located anywhere in a triangle. The procedure is very efficient and finds the optimal solution for 10,000 demand points in less than six minutes of computer time. The generalized Weiszfeld algorithm [51] repeated from 1,000 different starting solutions required about the same time for all 1,000 runs but does not guarantee optimality. However, it found the optimal solution at least 17 times out of 1,000 on a set of test problems.

7.5.2 Multiple Facilities

Drezner et al. [43] proposed five heuristic procedures for the maximization of the market share by locating p new facilities with given attractiveness levels using the gravity rule. The most successful heuristic was found to be:

1. Applying a simulated annealing approach [86] for locations restricted to grid points.
2. Finding a good location for each facility, one at a time, by the generalized Weiszfeld algorithm [51].
3. The two steps are repeated 100 times from randomly generated starting solutions and the best one is selected.

7.5.3 The TLA Method

The TLA (tangent line approximation) method [2] can find an optimal solution to the gravity model within a given accuracy. For its implementation, the objective function should be a concave function, twice differentiable and non-decreasing of a linear function. These conditions hold for the gravity model. The idea is to replace the objective function by a piece-wise linear function. The feasible range is divided into segments and a tangent line is constructed in each segment touching the objective function at the segment's center. The objective function is formulated by adding a binary variable for each segment and maintaining the original constraints. Optimal solutions of the modified problem are then found by non-linear solvers. The number of segments is determined by the pre-specified accuracy. For details see [2].

7.6 Applying the Gravity Rule to Other Objectives

The gravity rule can be applied to other commonly used non-competitive location objectives. Rather than assuming that a user gets services from the closest facility, he chooses a facility according to the gravity rule. The probability of patronizing a facility is proportional to the facility's attractiveness and to some decay function of the distance.

7.6.1 Gravity *p*-Median

In the standard p-median model [14] it is assumed that each user travels to the closest facility. This implicitly implies that facility choice is centrally controlled or that all facilities charge the same price for the service. Drezner and Drezner [31, 32] proposed the gravity p-median model. It is assumed that users choose from among the facilities providing services according to the gravity rule rather than from the closest facility. Users consider facilities' attractiveness in their choice. Similar to the standard p-median problem, the objective is to minimize the sum of the expected weighted distances.

Brimberg et al. [9] suggested a similar p-median model based on this idea that customers do not necessarily patronize the closest facility. A list of probabilities $P_1 \geq P_2, \ldots, \geq P_p$ that add up to 1 is constructed. The probability that a customer patronizes the closest facility is P_1. The probability he patronizes the second closest facility is P_2, and so on.

7.6.2 Gravity Hub Location

Drezner and Drezner [28] applied the gravity rule to the hub location problem. A traveler needs to fly from one airport to another. Several potential hubs are available. If the origin or the destination is a hub airport, the traveler chooses a non-stop flight. Otherwise, the probability that a certain hub is selected is proportional to the hub's attractiveness (price, walking distance from the arrival gate to the connecting one, chance of inclement weather, etc.) and to a distance decay function such as the total travel distance (or time) raised to a given inverse power. Such a model can be generalized to selecting a sequence of two or more hubs.

7.6.3 Gravity Multiple Server

Drezner and Drezner [34] considered the gravity rule version of the multiple server location problem [7]. Total service time consists of travel time to the facility, waiting time in line, and service time. There is a given number of servers to be distributed among the facilities. Each facility acts as an M/M/k queuing system. In [34] customers select a server with a probability proportional to its attractiveness and to a decay function of the distance, not necessarily the closest one. Two models are proposed: a stationary one and an interactive one. In the stationary model it is assumed that customers do not consider the expected waiting time in line and service time at the facility in their facility selection decision simply because they do not know these values. In the interactive model it is assumed that customers know the expected waiting time in line and service time at the facility and do consider them in their facility selection decision.

7.7 Summary and Suggestions for Future Research

In this chapter we reviewed competitive location models which are part of the field of facility location. Facility location models investigate the location of one or more facilities to achieve a certain objective. In competitive location models the objective is to attract as much buying power as possible from competitors' facilities by constructing new facilities and/or improving existing ones. A main component of such models is the estimation of how customers select the facility to patronize. Demand attracted by a facility depends on its attractiveness, on the buying power customers are planning to spend, and on the distance customers need to travel to get to the facility. What distinguishes different models is the assessment of the relationship between these factors and the market share captured. It is clear that higher attractiveness and buying power lead to higher market share, and a greater distance lowers the expected market share captured.

The gravity model [80, 81, 115] estimates the probability of patronizing a facility by these three components. Other approaches include the proximity rule (customer patronize the closest facility), utility and random utility models, cover-based models, and the flow interception model (all discussed in Sects. 7.2.1 and 7.2.2). One important implementation issue is the assessment of these components, especially the attractiveness level of a facility.

Many extensions to the basic models were investigated. For example, anticipating future changes in the market, considering lost demand due to long distances, cannibalization of one's chain facilities. Optimal location of one facility can be found by branch and bound algorithms such as Big Square Small Square [75], or Big Triangle Small Triangle [53]. Location of multiple facilities is usually solved heuristically by various approaches tailored to the specific model, or metaheuristic methods such as tabu search [65–67], simulated annealing [86], genetic algorithms [68, 78], variable neighborhood search [74, 99] and others.

There are many opportunities for future research. Improving and fitting the models better to real circumstances; obtaining better estimates for attractiveness of facilities. There are many solution methods for multiple facilities location models and constrained models that can be improved by designing more efficient heuristic algorithms that will enable practitioners to solve larger problems.

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