1

Bilevel Discrete Optimisation: Computational Complexity and Applications

Yury Kochetov, Alexander Plyasunov, and Arteam Panin

1.1 Introduction

The first bilevel optimisation problem was formulated by the German economist Heinrich Freiherr von Stackelberg as the economic two players game in 1934 [1]. Therefore, bilevel optimisation problems are sometimes called Stackelberg games and their solutions are called the Stackelberg equilibria. In 1973, Jerome Bracken and James T. McGill proposed a modern formulation of the bilevel programming problems (BPPs) [2]. These problems contain two levels of decision-making: upper and lower. Each level has its optimisation problem. The upper-level problem is called the leader's problem. The lower-level problem is the follower's problem. Each problem has its objective function, constraints, and decision vector as variables. The lower-level problem is a parametric optimisation problem with an upper-level decision vector as the parameter. The follower's problem is used to form an

Y. Kochetov (✉) · A. Plyasunov · A. Panin
Sobolev Institute of Mathematics, Novosibirsk, Russia
e-mail: jkochet@math.nsc.ru

A. Plyasunov
e-mail: apljas@math.nsc.ru

A. Panin
e-mail: a.panin@g.nsu.ru

S. Salhi and J. Boylan (eds.), *The Palgrave Handbook of Operations Research*,
https://doi.org/10.1007/978-3-030-96935-6_1

extreme constraint to the leader's problem. This constraint is usually written as a set-theoretic inclusion [3]. It implies that the lower-level variables are the optimal solution to the lower-level problem parameterised by the upper-level variables.

This extreme constraint is the reason why the bilevel problems turned out to be surprisingly difficult to solve. For example, the linear programming problem is polynomially solvable, but its bilevel analogue is NP-hard in the strong sense. Moreover, to check the feasible domain is empty or not is an NP-hard problem in the strong sense too. This property is preserved for very special cases of the bilevel linear programming problems with the follower's variables in the leader's constraints.

Currently, there are no direct or simple modifications of standard approaches to design exact and approximate algorithms for the bilevel cases. The main reason is that the set of feasible solutions usually turns out to be non-convex and disconnected. For the bilevel mixed-integer linear problems, the optimal value of its linear programming relaxation cannot guarantee a lower bound for the integer global minimum. Moreover, if the optimal solution to the relaxed problem is a feasible solution to the original bilevel problem, then it does not follow that it is the optimal solution to the original problem. If we look at one of the standard approaches for solving optimisation problems, namely local search, then even in this case we will not be successful. This is due to the non-polynomiality of even the simplest neighbourhoods since they require the optimal solution to the follower's problem which, as a rule, is NP-hard.

What makes bilevel models highly relevant is that they are typically characterised by very large unexpected effects on the economy and surrounding environment. Given the far-reaching future impacts of the decisions, it is not surprising that the interest in bilevel optimisation has grown strong especially among researchers dealing with large-scale public sector decision-making problems. To date, a huge number of articles have appeared that study the bilevel models from theoretical and empirical sides. Similarly, there is a rapid increase in the number of papers devoted to bilevel programming applications in various fields of science and industry.

This chapter is organised as follows. Section 1.2 presents bilevel discrete optimisation in general formulation, definitions of feasible and optimal solutions for this ill-posed problem, main concepts, and examples. The relationship of the bilevel problems with Stackelberg games and multi-objective problems is discussed here. A short survey of exact and approximate methods, including metaheuristics as the most strong technique for application, is given in Sect. 1.3. Section 1.4 addresses important questions related to the computational complexity of bilevel discrete optimisation and the approximation complexity. Section 1.5 provides an overview of the literature with practical bilevel problems in economics, industry, transport, engineering, facility location, network design, etc. Finally, Sect. 1.6 concludes the paper and shows some future research directions and perspectives.

1.2 Main Definitions and Properties

Let us consider a sequential game where the first player (leader) chooses her solution (vector $x \in X$) and incorporates the optimal reaction (vector $y \in Y$) of the second player (follower) into her optimisation process. This game can be described mathematically as *bilevel programming problem*:

$$\min_{x \in X, y \in Y} F(x, y)$$

$$\text{s.t. } G(x, y) \leq 0,$$

$$y \in opt(x),$$

where $opt(x)$ is the set of optimal solutions to the follower's problem parameterised by the vector x:

$$\min_{y \in Y} f(x, y)$$

$$\text{s.t. } g(x, y) \leq 0.$$

In this formulation, $F(x, y)$ is the leader's objective function and $f(x, y)$ is the follower's objective function. The leader's and the follower's constraints are defined by the vector functions $G(x, y)$ and $g(x, y)$, respectively. In this program, the leader is free, whenever the set $opt(x)$ does not shrink to a singleton, to select an element of $opt(x)$ that suits her best. This case corresponds to the *optimistic* formulation. Alternatively, in the *pessimistic* formulation, we consider the case when the leader protects herself against the worst possible situation. To this end, we change the objective function of the leader as follows:

$$\min_{x \in X} \max_{y \in opt(x)} F(x, y)$$

and must guarantee that the follower cannot violate the upper-level constraints. In other words, all optimal solutions for the follower's problem must satisfy these joint (or coupling) constraints [4]. As a result, the pessimistic bilevel problems are very difficult to solve. Most theoretical and algorithmic contributions relate to the optimistic formulations. Detail discussion of the pessimistic formulations can be found in [4, 5].

In real-world applications, we often know nothing about follower behaviour. In such a case, the optimistic and pessimistic solutions show the lower and upper bounds for the optimal value of the objective function of the leader. We deal with the ill-posed problem indeed in the case of the multiple optimal solutions for the follower's problem. The difference between the optimistic and pessimistic approaches can also be explained from the follower viewpoint. The optimistic solution results from the friendly or cooperative behaviour of the players, while an aggressive follower produces the pessimistic solution.

Program BPP is often called the *upper (first level, outer, leader)* problem. The mathematical program parameterised by the vector x is the *lower (second level, inner, follower)* problem. The set of follower's optimal solutions $opt(x) = \{y \mid y \in \arg\min\{f(x, y') \mid y' \in sol(x)\}\}$ is also called the set of *rational reactions* where $sol(x) = \{y \mid g(x, y) \le 0\}$ is the set of follower's feasible solutions.

The set of *feasible optimistic solutions* for the BPP is defined as $Sol = \{(x, y) \mid G(x, y) \le 0, y \in opt(x)\}$. This set is usually non-convex and might be disconnected. Sometimes, it is called the *inducible region*. The set of *optimal optimistic solutions* for the BPP is the set of the best feasible optimistic solutions for the leader, that is the set $\{(x, y) \mid (x, y) \in \arg\min\{F(x', y') \mid (x', y') \in Sol\}\}$. The *relaxed constraint region* is defined as $S = \{(x, y) \mid G(x, y) \le 0, g(x, y) \le 0\}$.

For a better understanding of the bilevel formulations, let us consider an example of the bilevel knapsack problem [6]. Two players hold their own knapsacks and choose items from a common item set. Firstly, the leader packs some items into her knapsack. Later on, the follower packs some of the remaining items into his knapsack. The objective of the follower is to maximise the total profit of the items in his knapsack. The objective of the hostile leader is to minimise this profit. In other words, we face the following bilevel discrete optimisation problem with n items, p_i is the profit of item i, coefficients a_i and b_i are its weights for the leader and the follower, A and B

are the capacities of the leader's and the follower's knapsacks, respectively.

$$\min_{x,y\in\{0,1\}} \sum_{i=1}^{n} p_i y_i$$

$$\sum_{i=1}^{n} a_i x_i \leq A;$$

$$y \in opt(x),$$

where $opt(x)$ is the set of optimal solutions of the lower-level problem:

$$\max_{y\in\{0,1\}} \sum_{i=1}^{n} p_i y_i$$

$$\sum_{i=1}^{n} b_i y_i \leq B;$$

$$y_i \leq 1 - x_i, \quad 1 \leq i \leq n.$$

Note that we have to solve the NP-hard problem for arbitrary feasible solution x to calculate the value of the leader's objective function. Thus, the finding of the best leader's solution is a very hard problem. Below, we will see that it is harder than NP-complete problem, unless P = NP. This example is also interesting because the optimistic and pessimistic solutions coincide here even if the set $opt(x)$ contains multiple solutions. The upper-level constraint contains only the leader's variables. The follower cannot violate them. It is a simple case of BPPs.

For the pessimistic case, we have to modify the definition of feasible solution. The set of feasible pessimistic solutions is the set of pairs $(x, y) \in Sol$ which satisfy two conditions:

- $G(x, \overline{y}) \leq 0, \quad \forall \overline{y} \in opt(x);$
- $F(x, y) \geq F(x, \overline{y})), \quad \forall \overline{y} \in opt(x).$

For given x, all optimal follower's solutions must satisfy the joint constraints and y is the worst answer of the follower according to the leader objective function [7]. Optimal solution in pessimistic case is the best feasible pessimistic solution for the leader.

Consider some properties of the bilevel discrete optimisation problems. Let us find the minimum of the leader's objective function $F(x, y)$ for $(x, y) \in S$ in the relaxed constraint region. In other words, the leader makes a decision instead of the follower. It is so called *high point relaxation*. In that case, the optimal solution (with rare exceptions) will be infeasible for original bilevel problem. This is due to the non-optimality of the follower's response to the leader's decision. But we have got a lower bound for the global optimum.

The next unusual property deals with the integrality constraints. For the classical discrete optimisation, relaxation of this type of constraints lead to a lower bound. Moreover, if the optimal solution to the relaxed problem is integer, we have got the optimal solution to the initial discrete problem. It is not the case for the bilevel discrete optimisation. Let us consider an illustrative example of the pure integer bilevel linear problem proposed by James T. Moore and Jonathan F. Bard [8]:

$$\min_{x,y} F(x, y) = -x - 10y$$

$$\text{s.t. } x \geq 0, \text{ integer}; \quad y \in opt(x);$$

$$\min_{y} f(x, y) = y$$

$$\text{s.t. } -25x + 20y \leq 30; \quad x + 2y \leq 10;$$

$$2x - y \leq 15; \quad 2x + 10y \geq 15; \quad y \geq 0, \text{ integer}.$$

The solution $(x, y) = (8, 1)$ with the value $F(x, y) = -18$ is optimal to the relaxation and it is integer. But the optimal integer solution is $(x^*, y^*) = (2, 2)$ with the value of -22 (see Fig. 1.1).

The lower green lines show the feasible domain to the relaxation. As we can see, it is a non-convex area for this linear bilevel program. Moreover, the feasible domain of linear BPP, even with continuous variables, may be disconnected in general case [9]. Therefore, the idea of the simplex method does not work here. The red domain is the convex hull for the feasible discrete points. It is a small part of the relaxed constraint region S.

Let us slightly modify this example and include a new upper-level constraint $x + y = 5$. Now only three integer points (2, 3), (3, 2), (4, 1) satisfy all constraints. The first one is the most interesting for the leader with value of -32 but $y = 3$ is not optimal for the follower's problem with $x = 2$. For the second point, we have the same property. Hence, the last point is

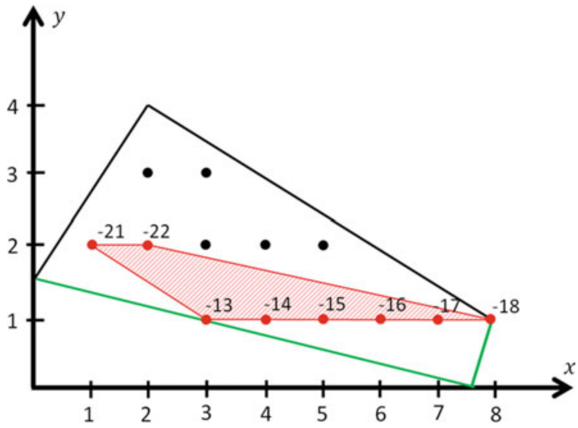


Fig. 1.1 The illustrative example

optimal now with a value of -15. Note that it is not a vertex of the convex hull of the integer feasible discrete points. Moreover, if we move this new constraint to the lower level, we have got another bilevel problem with an optimal value of -32.

Bilevel programming problems can be divided into three categories: (1) all solutions are real-valued vectors; such problems are known as continuous BPPs, (2) all solutions are discrete-valued vectors, discrete BPPs, (3) all solutions are vectors of continuous or discrete components, mixed BPPs. Further in this section, we consider various reformulations and generalisations of bilevel problems and their connections with Stackelberg games and multi-criteria optimisation problems.

The bilevel optimisation problem can be considered as a generalisation of the two-stage Stackelberg game [10]. For the first time, such a game was studied by Von Stackelberg in the context of unbalanced economic markets [1]. Indeed, BPPs are more or less similar to Stackelberg games in game theory [11]. In Stackelberg games, the lower-level problem is an equilibrium problem, while in bilevel optimisation, an optimisation problem arises in the lower level. A Stackelberg game may differ from the BPP when the reaction set of the lower-level decision-maker is not a singleton for some decisions of the leader. As a result, a solution of the static Stackelberg game may not be a solution to the BPP [12].

The optimal solution of the BPP is not necessarily a Pareto optimal solution of the corresponding bi-objective problem composed with the upper-level and the lower-level objectives and vice versa [13, 14]. The relationship between the bilevel problem and bicriteria optimisation is illustrated in [15–18]. In [19–23] the authors propose a methodology in which the

BPP is transformed to an equivalent multi-objective optimisation problem. A specific cone dominance concept is used. An application of these results to solve linear bilevel optimisation problems is given in [24]. The discrete bilevel problems with multiple followers are studied in [25–27]. The bilevel problems with multiple leaders are considered in [28, 29]. Nine different kinds of relationships between followers can be found in [30]. A surprising fact is that the globally optimal solutions to the bilevel optimisation problem need not remain globally optimal if a constraint is added to the lower-level problem which is inactive at the optimal solution [31, 32].

1.3 Computational Methods

The main purpose of this section is to present some state-of-the-art exact methods, metaheuristics, and hybrids of metaheuristics with exact methods to bilevel discrete optimisation. We have to skip the approximation schemes and polynomial-time approximation algorithms with a guaranteed performance. Some negative results in this area for the BPPs establish the inapproximability of various Σ_2^P-hard optimisation problems [33–35], unless P = NP. Nevertheless, the first approximation scheme for a Σ_2^P-hard optimisation problem is designed in [36]. It is excellent result for the bilevel knapsack problem from Sect. 1.2 for the case $b_i = a_i, \forall i$. To the best of our knowledge, it is the first and still unique PTAS for the Σ_2^P-hard BPP.

1.3.1 Exact Methods

This class of methods can be divided into four types. The first one includes reformulation approaches. The BPP is reformulated as a single-level optimisation problem with a large number of variables and (or) constraints. Another approach here is the mathematical decomposition to reduce the original bilevel problem to the single-level problems. The second type includes branch and bound/branch and cut techniques. It is the basis of the mixed-integer bilevel optimisation solvers [37, 38] (see also [39, 40]). The third type includes the parametric programming approaches. The fourth type includes hybrid methods which may have the characteristics of the first three types.

The methods are based on reformulations. An algorithm based on the Benders decomposition method is proposed in [41]. The KKT optimality conditions are used as a reformulation procedure. The only assumption of the proposed algorithm is that although integer variables could appear in both levels, they should be controlled by the upper optimisation problem.

In [42], the linear BPP with binary leader variables and continuous follower variables is considered. The bilevel problem is reduced to a single-level problem using KKT conditions, a suitable linearisation technique, and the Benders decomposition method is applied.

In [43], a decomposition algorithm based on a column-and-constraint generation scheme using a single-level reformulation is proposed to solve the general bilevel mixed-integer linear problem. Another approach for binary lower-level problems is recently proposed in [44].

An algorithm for solving bilevel problems with boolean variables based on optimal value reformulation and the cutting plane technique is proposed in [45]. A similar technique is used in [46]. Some new results in this area are obtained in [47, 48].

The branch and bound/branch and cut methods. The first branch and bound method for discrete bilevel optimisation is developed in [8]. The bilevel problems with the linear programming problem at the lower level and the integer programming problem at the upper level are studied in [49]. A branch and bound method is designed for this special case.

In [50], a class of the BPP is considered where the leader controls continuous and discrete variables and wants to minimise a convex nonlinear objective function. The follower's objective function is a convex quadratic in a continuous decision space. All constraints are linear. A branch and bound algorithm is developed to find the global optima.

A new version of the branch and bound method for the mixed-integer upper- and lower-level problems with joint constraints at the upper level is proposed in [51]. The linking variables are discrete and all discrete variables are bounded. The high point relaxation is used to find the optimal solution in optimistic case.

In [52], two exact algorithms are proposed. The first one is based on the cutting plane technique, and the second one belongs to the class of branch and cut algorithms. In [6, 53], an algorithm based on a branch and cut approach is proposed for the BPP problems without continuous variables and without joint constraints at the upper level. A generalisation of this algorithm that allows a mixed-integer environment at both levels is proposed in [54]. Other efficient cutting plane algorithms can be found in [51, 55–58]. Some valid inequalities and facets for the network pricing problem with connected toll arcs and its variants are designed in [59].

Parametric programming approaches. Two exact algorithms for the integer and mixed-integer bilevel programming problems via multi-parametric programming are proposed in [60]. The first algorithm addresses the integer case of the linear BPP and employs a reformulation linearisation technique to construct a parametric convex hull representation of the inner problem constraint set. The second algorithm addresses the mixed-integer case and employs a similar convexification procedure as the previous algorithm. In contrast to the first algorithm, where a continuous multi-parametric programming approach is used, the second algorithm utilises multi-parametric mixed-integer programming to solve the inner problem.

In [61], exact global optimisation algorithms are presented for two classes of BPP, namely: the mixed-integer linear BPP and the mixed-integer convex quadratic BPP. The proposed algorithms are a result of multi-parametric programming theory. The main idea here is to recast the lower-level problem as a multi-parametric programming problem where the optimisation variables of the upper-level problem (both continuous and integer) are considered as parameters for the lower-level problem. The resulting exact parametric solutions are then substituted into the upper-level problem and it is solved as a set of single-level deterministic mixed-integer programs.

A BPP with only integer decision variables at the lower level and constraints at both levels is studied in [62]. The leader has integer or continuous decision variables. The solution approach is based on the theory of parametric integer programming and runs in polynomial time when the number of decision variables of the follower is fixed.

Non-standard solution methods. In [63], an algorithm based on the concept of *k*-th best solution, first developed for the linear case, is developed for the integer linear fractional BPP. The correctness of this algorithm is shown in [64]. Upper approximations of the optimal objective function value of the lower-level problem are used for solving mixed-integer BPP in [65].

A multi-way branching method is used for solving the mixed-integer bilevel linear program in the case all leader variables are integer and bounded [66]. An exact algorithm for solving the discrete linear bilevel optimisation problems using multi-way disjunction cuts to remove infeasible solutions for the bilevel problem from the search space is presented in [67]. Additional information about the exact methods for solving (mixed-integer) linear bilevel optimisation can be found in [61].

1.3.2 Metaheuristics

The word heuristic (from the Greek word heuriskein) means the art of discovering new strategies to solve problems. The suffix meta (Greek) means upper-level methodology. The term metaheuristic has been introduced by

Fred Glover [68]. Unlike exact methods, metaheuristics allow finding effective approximate solutions to large-scale problem instances in a reasonable time without guarantee to get global optimal solutions or even solutions with small deviation from the optimal value. Metaheuristics became popular in the last 30 years. They demonstrate efficiency and effectiveness to solve large and sophisticated problems in many applications [69–71].

Due to the intrinsic complexity of bilevel models, BPPs have been recognised as one of the most difficult classses to solve. Hence, metaheuristic algorithms have been applied here. We can divide them into two classes [70]: population-based search and single-solution-based search. Single-solution-based algorithms (e.g. local search, variable neighbourhood search, tabu search, simulated annealing) manipulate and transform a single solution during the search while in population-based algorithms (e.g. particle swarm, evolutionary algorithms) a whole population of solutions is evolved. These two families have complementary characteristics: single-solution-based metaheuristics are exploitation-oriented. They have the power to intensify the search in local regions. Population-based metaheuristics are exploration-oriented. They allow better diversification in the whole search space.

Metaheuristics are approximate algorithms. In bilevel problems, there is a limitation that forces us to solve the lower-level problem exactly. This fact makes it impossible to use metaheuristics directly to the bilevel problems. The following classification distinguishes metaheuristics according to the way they work with the lower-level problem: (1) Single-level transformation approach; (2) Nested sequential approach; (3) Multi-objective approach; (4) Co-evolutionary approach.

In the single-level transformation approach, we try to reformulate the BPP into a single-level optimisation problem. Then, the classical metaheuristics can be used to solve the single-level problem.

In the nested sequential approach, the lower-level optimisation problem is solved in nested and sequential ways to evaluate the solutions generated at the upper level of the BPP.

In the multi-objective approach, the BPP is transformed into a multi-objective optimisation problem. Then, any multi-objective metaheuristic can be used to solve the generated problem.

Finally, the co-evolutionary approach is the most general methodology to solve the BPPs. In this case, metaheuristics for different levels of the problem coevolve in parallel and exchange information during the search.

A detailed description of this classification with links to corresponding algorithms can be found in [70]. For solving bilevel linear problems, heuristic approaches such as evolution algorithms, tabu search, simulated annealing, grid search, and other algorithms can be found in [71, 72]. Computational results for the metaheuristics to the $(r|p)$-centroid problem and other bilevel competitive facility location models can be found in [70, 73]. Implementation of genetic and memetic algorithms, ant colony systems, tabu search, local search, particle swarm optimisation, simulated annealing, and other heuristic approaches are presented in [70, 71, 74, 75].

Let us return to the definition of a feasible solution to the BPP. The set of feasible solutions is defined as $Sol = \{(x, y) \mid G(x, y) \leq 0, y \in opt(x)\}$. Hence, we need an optimal solution to the follower's problem to get a feasible solution for given x. As a result, metaheuristics have to include an exact method for this aim. Thus, they are hybrid methods. Now we will consider the case when the follower's problem has multiple solutions. For the optimistic case, we face an additional problem to pick up the best solution for the leader in the set $opt(x)$. To this end, we must solve the following optimisation problem for given x:

$$\min_{y \in Y} F(x, y)$$

$$\text{s.t. } G(x, y) \leq 0,$$

$$f(x, y) = f^*(x),$$

$$g(x, y) \leq 0,$$

where $f^*(x)$ is the optimal value to the follower's problem. If this auxiliary problem is infeasible then we must change the solution for the leader. Note that we need the optimal solution here. Hence, we need an exact method again.

In the pessimistic case, we have a more sophisticated position. As we have noted in Sect. 1.2, the leader wishes to protect herself against the worst possible answer of the follower and upper-level joint constraints must be satisfied for all possible answers. Hence, we need to solve the following bilevel optimisation problem:

$$\min_{x \in X} \max_{y \in opt(x)} F(x, y),$$

$$\text{s.t. } G(x, y) \leq 0, \forall y \in opt(x), \ x \in X.$$

For given x, we can find the worst follower's answer by solving new auxiliary problem:

$$\max_{y \in Y} F(x, y)$$

$$\text{s.t. } f(x, y) = f^*(x),$$

$$g(x, y) \leq 0.$$

Later on, we must be sure that all optimal solutions to the follower's problem satisfy the joint constraints. If the follower's variables are not presented in the upper-level constraints then this step can be omitted [3]. Otherwise, we need to check all solutions in the set $opt(x)$.

Let $G(x, y) = (G_1(x, y), \ldots, G_m(x, y))$ and we wish to find an optimal solution for the follower which does not satisfy the leader's constraints. Thus, we solve m problems for given x:

$$\alpha_j = \max\{G_j(x, y) | f(x, y) = f^*(x), g(x, y) \leq 0\}, \quad j = 1, \ldots, m.$$

If $\max \alpha_j > 0$ then we have infeasible solution in pessimistic case and the point x must be replaced. The pessimistic case is the most difficult and interesting line for future research.

The following simple example illustrates that the optimistic and pessimistic bilevel problems are similar but their optimal solutions can differ considerably [4].

$$\min_{x \in \{-1,0,1\}} x$$

$$x \geq y, \quad y \in opt(x),$$

and $opt(x)$ is the set of optimal solutions to the follower's problem:

$$\max_{y \in \{-1,0,1\}} y^2$$

The follower's problem is optimised by $y \in \{-1, 1\}$, independent of the leader's decision. The pessimistic bilevel problem requires x to exceed 1, resulting in an optimal objective value of 1. In contrast, the optimistic bilevel problem requires x to exceed -1, which results in an optimal objective value of -1.

1.4 Computational and Approximation Complexity

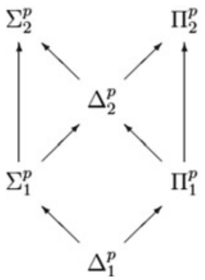
According to the principle of bounded rationality [76], economic agents cannot use excessively large resources to find the optimal solution. In operations research, the interaction of economic agents is described by mathematical models. Therefore, computational resources are the most critical type of resource here. From this point of view, the theory of computational complexity is a natural mathematical tool for describing and investigating the behaviour of economic agents based on the principle of bounded rationality [77]. Thus, the knowledge of the relations of the optimisation problem with complexity classes allows us to estimate what computational resources are needed to find a rational solution.

When we discuss the computational complexity of the single-level problems, we usually refer to the complexity classes P and NP for the decision problems and the classes PO, NPO for the optimisation problems. The peculiarity of bilevel (multi-level) problems is that many of them are outside of these classes. Therefore, we are forced to introduce the concepts of a polynomial and approximation hierarchy of complexity classes for the BPP.

1.4.1 Polynomial Hierarchy

We remind the notations used in computational complexity theory to describe the polynomial hierarchy of complexity classes [78, 79]. The first two main classes of the decision problems P and NP are defined with deterministic and non-deterministic Turing machines [78]. The class P contains the decision problems solvable in polynomial time on deterministic Turing machines. The class NP is defined as the class of the decision problems solvable in polynomial time on non-deterministic Turing machines. It means that we can verify the answer *Yes* in time polynomial in the size of the input data of the problem. The third class co-NP consists of decision problems whose complements belong to the class NP. It means that we can verify the answer *No* in time polynomial in the size of the input data of the problem. These classes form the first level of the polynomial hierarchy. They are denoted as Δ_1^P, Σ_1^P, and Π_1^P, respectively. The second level of the polynomial hierarchy is defined with deterministic and nondeterministic oracle Turing machines [78].

A decision problem belongs to the class Δ_2^P if there exists a deterministic Turing machine with an oracle that solves this problem in polynomial time, using as oracle some language (decision problem) from the class NP. A



$$\Sigma_2^p = NP^{NP}, \quad \Pi_2^p = co-NP^{NP};$$

$$\Delta_2^p = P^{NP} = P^{co-NP};$$

$$\Sigma_1^p = NP, \quad \Pi_1^p = co-NP;$$

$$\Delta_1^p = \Sigma_0^p = \Pi_0^p = P.$$

Fig. 1.2 The first two levels of hierarchy

decision problem belongs to the class Σ_2^P if there exists a non-deterministic Turing machine with an oracle that solves this problem in polynomial time, using as oracle some language from the class NP. The class Π_2^P consists of decision problems whose complements belong to Σ_2^P. The class Δ_2^P is often denoted as P^{NP} and the class Σ_2^P is denoted as NP^{NP}. Figure 1.2 shows all inclusions between hierarchy classes on the first two levels.

It is clear that this hierarchy can be expanded if we take the class Σ_{k-1}^P as the oracle. It is known that if NP $\neq$ co-NP then these inclusions are strict [78]. The notions of completeness and hardness commonly used for the class NP are translated directly to the classes Σ_k^P. In particular, the k-level optimisation problems with binary variables, linear constraints, and linear objective functions are Σ_k^P-hard [80]. We will focus our attention on the class Σ_2^P as the most appropriate for the BPP and difficult from the point of view of optimisation methods. A compendium of Σ_2^P-complete/hard problems can be found in [81], [82], with more recent updates available online.

1.4.2 Σ_2^P-Hard Bilevel Programming Problems

The structure of the bilevel problems makes their standard decision problem a natural candidate for membership in the Σ_2^P class. This is the case if the lower-level parametric problem lies in the class NPO. If the standard decision problem for the lower-level parametric problem is NP-complete, then as a rule, the corresponding bilevel problems turn out to be Σ_2^P-hard.

Three Σ_2^P-hard bilevel knapsack problems can be found in [36]. One of them we have discussed in Sect. 1.2. Let us formulate two remaining problems. In the first one [83], the leader controls the capacity x of the knapsack while the follower controls all items and decides which of them are packed into it. The objective function of the leader depends on the capacity x and the packed items, whereas the objective function of the follower solely depends on the packed items.

$$\min_{x,y}\left\{Ax+\sum_{i=1}^{n}a_i y_i\right\} \text{ s.t. } C \le x \le C', \; y \in opt(x),$$

where $opt(x)$ is the set of optimal solutions of the lower-level problem:

$$\max_{y}\sum_{i=1}^{n}b_i y_i \text{ s.t. } \sum_{i=1}^{n}b_i y_i \le x; \; y_i \in \{0,1\}, \; 1 \le i \le n.$$

In the second problem [84], players pack items into the same knapsack with a prespecified capacity of C. The leader controls part of the item set. The follower controls the rest of this set. The leader starts the game by packing some of her items into the knapsack. Later on, the follower adds some further items from his set. The objective function of the leader depends on all items packed by the leader and the follower, whereas the objective function of the follower solely depends on his items.

$$\min_{x,y}\left\{\sum_{j=1}^{m}a_j x_j+\sum_{i=1}^{n}a_i' y_i\right\} \text{ s.t. } x_j \in \{0,1\}, \; 1 \le j \le m, \; y \in opt(x),$$

where $opt(x)$ is the set of optimal solutions of the lower-level problem:

$$\max_{y}\sum_{i=1}^{n}b_i y_i \text{ s.t. } \sum_{i=1}^{n}c_i' y_i \le C-\sum_{j=1}^{m}c_j x_j, \; y_i \in \{0,1\}, \; 1 \le i \le n.$$

To show the Σ_2^P-hardness of these bilevel knapsack variants, the following Σ_2^P-complete decision problem (Subset-Sum-Interval) is used [85].

Instance: A sequence $q_1, q_2, \ldots, q_k$ *of positive integers; two positive integers* R *and* r *with* $r \le k$.

Question: Does there exist an integer S *with* $R \le S < R + 2^r$ *such that none of the subsets* $I \subseteq \{1, \ldots, k\}$ *satisfies* $\Sigma_{i \in I} q_i = S$?

Reducibilities guarantee Σ_2^P-hardness of all three bilevel knapsack variants under the optimistic and pessimistic scenarios [36].

In the field of bilevel facility location, we have some variants of $(r|p)$-centroid problem which are Σ_2^P-hard [86, 87]. Let us consider a two-dimensional Euclidean plane in which n clients are located. We assume that each client j has a positive demand w_j. Let X be the set of p points where the leader opens her facilities and let Y be the set of r points where the follower opens his facilities. The distances from client j to the closest facility of the leader and the closest facility of the follower are denoted as $d(j, X)$ and $d(j, Y)$, respectively. The client j prefers Y over X if $d(j, Y) < d(j, X)$ and prefers X over Y otherwise. By

$$U(Y \prec X) := \{j \mid d(j, Y) < d(j, X)\}$$

we denote the set of clients preferring Y over X. The total demand captured by the follower by locating his facilities at Y while the leader locates her facilities at X is given by

$$W(Y \prec X) := \sum (w_j \mid j \in U(Y \prec X)).$$

For X given, the follower tries to maximise his own market share. The maximal value $W^*(X)$ is defined to be

$$W^*(X) := \max_{Y, |Y|=r} W(Y \prec X).$$

This maximisation problem will be called the *follower problem*. The leader tries to minimise the market share of the follower. This minimal value $W^*(X^*)$ is defined to be

$$W^*(X^*) := \min_{X, |X|=p} W^*(X).$$

For the best solution X^* of the leader, her market share is $\sum_{j=1}^{n} w_j - W^*(X^*)$. In the $(r|p)$-centroid problem, the goal is to find X^* and $W^*(X^*)$.

The discrete $(r|p)$-centroid problem and the $(r|p)$-centroid problem on a network are Σ_2^P-hard [86]. The hardness proof uses a reduction from the Σ_2^P-complete decision problem $\exists\forall 3SAT$ [86]. It is shown in [87] that the $(r|p)$-centroid problem in the plane is Σ_2^P-hard. The hardness proof uses a reduction from the Σ_2^P-complete decision problem $\exists\forall 3, 4SAT$ [87]. Based on this reducibility, the following results are also obtained:

- the discrete $(r|p)$-centroid problem is Σ_2^P-hard even the clients and facilities are placed in the two-dimensional Euclidean plane;
- the $(r|p)$-centroid problem on a network is Σ_2^P-hard even for planar graphs with vertices in the two-dimensional Euclidean plane and weights of the edges are Euclidean distances between corresponding points.

Another Σ_2^P-hard problem of competitive facility location is considered in [88]. The hardness proof uses a reduction from the Σ_2^P-complete decision problem $\exists_1\forall_3 Sat$ [88].

In [89, 90], there is a leader–follower location and pricing problem that is also Σ_2^P-hard. In [89] the hardness proof uses a reduction from the Σ_2^P-complete standard decision problem of the $(r|p)$-centroid problem. In [90] the hardness proof uses a reduction from the Σ_2^P-complete decision problem $\exists\forall 3, 4SAT$ [87].

The bilevel problem of strategic base station placement in cognitive radio networks is studied in [91]. It is shown that the problem is Σ_2^P-hard. The proof uses a reduction from the Σ_2^P-complete decision problem $\exists\forall 3, 4SAT$ [87].

Σ_2^P-hard problems of public–private partnership can be found in [92, 93]. In [92], it is shown that the problem without tax benefits and infrastructure projects still remains Σ_2^P-hard even in the case of a three-year planning horizon and the optimistic and pessimistic setting. The hardness proofs use a reduction from the Σ_2^P-complete decision problem Subset-Sum-Interval [85].

1.4.3 Approximation Hierarchy

Let us consider the computational complexity of finding near optimal solutions for bilevel discrete optimisation problems. In the classical case, we study the computational complexity from the point of view of approximate algorithms with a performance guarantee for the optimisation problems from the class NPO [79]. By definition, this class consists of optimisation problems for which the standard decision problem belongs to class NP. The class PO is the subclass of the NPO problems that admitting an exact polynomial-time algorithm.

Remind some definitions from [79]. Given an optimisation problem Q in NPO, an approximation algorithm A for Q, and a function $r : N \mapsto (1, \infty)$, we say that A is an $r(n)$-approximate algorithm for Q if, for any instance x with a non-empty set of feasible solutions, the performance ratio of the feasible solution $y = A(x)$ with respect to x verifies the following inequality:

$$R(x, A(x)) = \max\left(\frac{f(x, y)}{f^*(x)}, \frac{f^*(x)}{f(x, y)}\right) \leq r(|x|),$$

where $f(x, y)$ is the value of the objective function of the problem on a feasible solution y for x, $f(x^*)$ is the optimal value of the objective function for x.

An algorithm A is said to be a polynomial-time approximation scheme (PTAS) for Q if, for any instance x and for any rational value $r > 1$, A returns an r-approximate solution of x in time polynomial in $|x|$.

An algorithm A is said to be a fully polynomial-time approximation scheme (FPTAS) for Q if, for any instance x and for any rational value $r > 1$, A returns an r-approximate solution of x in time polynomial both in $|x|$ and $1/(r-1)$.

The class FPTAS is the class of NPO problems that admitting a fully polynomial-time approximation scheme, while the class PTAS is the class of NPO problems that admitting a polynomial-time approximation scheme.

Given a class of functions F, F-APX is the class of all NPO problems Q such that, for some function $r \in F$, there exists a polynomial-time $r(n)$-approximate algorithm for Q.

Denote by APX, *Log*-APX, *Poly*-APX, *Exp*-APX classes F-APX with F equal to the set of constant functions, to the set $O(\log n)$ functions, to the set $\cup_{k>0} O(n^k)$ functions, and to the set $\cup_{k>0} O(2^{n^k})$ functions, respectively.

The classes APX, *Log*-APX, *Poly*-APX, and *Exp*-APX consist of the problems for which there exist polynomial approximate algorithms with constant, logarithmic, polynomial, and exponential performance guarantee, respectively. In the last three cases, the values of the above-mentioned functions depend on the length of the problem input data.

To determine the approximation complexity of an optimisation problem, it is enough to find its position in the hierarchy of approximation classes:

$$PO \subseteq FPTAS \subseteq PTAS \subseteq APX \subseteq Log\text{-}APX \subseteq Poly\text{-}APX \\ \subseteq Exp\text{-}APX \subseteq NPO.$$

Assuming $P \neq NP$, these inclusions are proper [79, 94].

This hierarchy is used to describe the properties of optimisation problems from the class NPO. To compare the approximability properties of arbitrary two problems from the class NPO, reducibility that preserves approximability is used [79, 94]. Similar approximation classes for the BPPs can be found in [89]. The definition of each of these new approximation classes is obtained from the original one by replacing the polynomial-time deterministic algorithm by a polynomial-time deterministic algorithm with an oracle from the class NP.

Bilevel knapsack problems are studied in [36, 95]. In [36], a polynomial-time approximation scheme is developed for the bilevel knapsack problem proposed in [6]. It is the first approximation scheme for a Σ_2^P-hard optimisation problem in the history of approximation algorithms.

The question of the complexity of the approximation of optimisation problems associated with pricing processes is presented in [96–100]. In [96], it is shown that the unit demand envy-free pricing problem is APX-hard. APX-hardness of the Stackelberg network pricing problem is shown in [97]. The Stackelberg minimum spanning tree game is also APX-hard [98]. In [99], it is shown that the bilevel problem with the mill pricing belongs to the class *Log*-APX.

The approximability of the bilevel facility location and pricing problems is discussed in [100]. It is shown that such problems with different pricing strategies are *Poly*-APX-hard if we can open a fixed number of facilities. Without the last constraint, such problems are complete in the class *Poly*-APX. The approximability of the discrete bilevel strategic planning model for public–private partnership is discussed in [101]. It is shown that this problem is NPO-hard, and the investor's problem is NPO-complete. As in the previous case, AP-reducibility is used to give these results. Unless P=NP, there cannot exist polynomial-time approximate algorithms with guaranteed perfomance that correspond to the classes *APX*, *Log-APX*, *Poly-APX*, and *Exp-APX*, and there also cannot exist polynomial-time approximation schemes for the investor's problem at the lower level. Hence, we have the same negative result for the bilevel linear problem.

1.5 Applications

The mixed-integer bilevel programming problems have a wide range of applications in various spheres of life. A comprehensive overview of such applications can be found in [9, 71]. A lot of results are related to the chemical industry, network design, environmental problems, military applications, competitive facility location, pricing, interdiction problems, and many others. Moreover, the review [9] contains some studies on the natural gas cash-out problem, the deregulated electricity market equilibrium problem, biofuel problems, a problem of designing coupled energy carrier networks, and so forth. Most part of them deals with the bilevel models with continuous variables. Below we present some applications of integer or mixed-integer bilevel models.

Network design. The term *network design* means any model that involves the variables to define the structure of a graph or a network. In many cases, we discuss a transportation network. Usually, the objective of network design models is to satisfy data communication requirements and minimise the total expenses. Requirement scope can vary widely from one network design project to another based on geographic particularities and the nature of the data requiring transport.

The network design problem is first modelled as a bilevel program by Gao et al. [102]. The leader determines new road links to minimise the system travelling costs. The follower's problem is design to characterise the user equilibrium. Fontaine and Minner [103] reformulate the network design problem as a mixed-integer linear problem using the same approach as Labbé et al. [104]. An improved Benders decomposition technique is devised. In a recent paper, Fontaine and Minner [105] apply this decomposition method to an extended version of the problem. Different vehicle flow patterns are considered in a time-varying fashion. Bagloee et al. [106] study the interaction of two types of vehicles in the follower's problem, and propose a hybrid algorithm that combines a generalised Benders decomposition with the branch and bound approach. In [107] the joint design and pricing problems are considered. These problems are related to designing freight carrying services and determining their associated prices as observed by the shipper firms. In [108] the bilevel model for the discrete network design problem on trains and its solution method based on the genetic algorithm are proposed.

Interdiction games play an important role in military and drug enforcement applications. In both cases, the goal is to disrupt elements of a transportation network to reduce as much as possible the enemy's movements on the network. In [109] a bilevel problem of network interdiction is proposed and studied. In [110], the discrete network design problem is defined as a bilevel optimisation problem. The leader wants to identify the optimal network structure and minimise the network travel time. The follower problem represents the network user's reaction as a static traffic assignment problem under user equilibrium. The bilevel optimisation model for the hazardous materials (hazmat) transportation problem with lane reservation is studied in [74]. The problem lies in selecting lanes to be reserved in the network and planning paths for hazmat transportation tasks. The trade-off among transportation cost, risk, and impact on normal traffic is considered. In the context of vehicle transportation in congested roads, an optimisation framework to integrate the operator decisions on network pricing, regulation, and expansion while accounting for the shipments of hazardous materials is proposed in [75].

Defence and cybersecurity applications. The literature on the Stackelberg security games and the bilevel defender–attacker models is quite extensive. A large part of publications are devoted to applied problems of ensuring the security of objects (see [111–113] and references in them). For example, in security resource optimisation problems, research has led to decision aids for real-world security agencies which need to deploy patrols and checkpoints to protect targets from terrorists and criminals. Stackelberg security games are a powerful tool to study *a competition* between a defender and an adversary. The defender commits to a mixed strategy—a randomised resource allocation specified by a probability distribution over deterministic schedules—which takes into account the adversary's best response to his observation of the mixed strategy.

In [73, 114–120], another line of research deals with the r-interdiction median problems with fortification are studied. The leader protects q objects, and the follower attacks r unprotected objects. The defensive maximal covering location model as a leader–follower attacker–defender game–theoretic model is studied in [73].

Among modern publications, we can also highlight papers [121–123]. In the *attacker–defender* model [121], a more general situation is considered. The attacker and the defender have various means (methods) for attacking and defending objects, respectively. The losses of the attacking side and the result of the attack depend on the means. In such a situation, the attacking side enters into a game interaction with the defence side and take into account the response of the other side which, choosing its solution, aims to inflict maximum losses for the attacker. To find the optimal solution of the bilevel mixed-integer problem, the feasible region is split into subsets and the bilevel problem is reduced to a sequence of bilevel subproblems. Each bilevel subproblem is reformulated as a mixed-integer programming problem. Previously, the same idea was used to design exact polynomial-time algorithms for the bilevel problems with knapsack problem and continuous variables at the lower level [7, 124–127].

In [122], the leader does not know the attack scenario and the follower's priorities for selecting targets for the attack. But, she can consider several possible scenarios that cover the follower's plans. The leader's problem is to select the set of objects for protection to minimise the total costs of protecting the objects and eliminating the consequences of the attack associated with the reassignment of the facilities for customer service. Reformulation of the bilevel problem as some single-level problems is proposed.

In the bilevel optimisation framework, a leader chooses her solution assuming that a follower answers by an optimal reaction according to the lower-level problem. However, the lower-level problems might be nontrivial. In practice, it might be inexactly solved by metaheuristics. Zare et al. [123] study a broad class of bilevel optimisation problems where the follower might not optimally react to the leader's actions. The authors provide algorithmic implementations of a new framework for a class of nonlinear bilevel knapsack problems and illustrate the impact of incorporating this realistic feature in the context of defender-attacker problems.

Recently proposed attacker-defender models for power system vulnerability assessment perform a worst-case analysis considering both natural-occurring events and malicious attacks. The worst-case analysis is crucial for vulnerability assessment and mitigation of critical infrastructure such as power systems. Defence applications including electric grid defence planning and defence models in interdiction problems can be found in [71].

Information protection and cybersecurity discrete bilevel problems can be found in review [9]. In the bilevel formulation from [128], the goal of the destructive agent is to minimise the number of power system components that must be destroyed in order to cause a loss of load greater than or equal to a specified level. The system operator will implement all feasible corrective actions to minimise the level of system load shed. The resulting nonlinear mixed-integer bilevel programming formulation is transformed into an equivalent single-level mixed-integer linear program and solved by commercially available software.

The bilevel model specifically allows one to define different objective functions for the terrorist and the system. Researchers have begun to look into some new ways of addressing the security assessment problem. For example, in [129], a multiagent system is proposed capable of assessing power system vulnerability, monitoring hidden failures of protection devices, and providing adaptive control actions to prevent catastrophic failures and cascading sequences of events.

Attack tree is another widely used combinatorial model in the cybersecurity analysis [130]. Defence trees have been developed to investigate the effect of defence mechanisms using measures such as attacker's cost and security cost, return on investment and return on the attack. In an attack response tree, an attacker–defender game is used to find an optimal policy from the countermeasures' pool.

Cybersecurity is becoming an area of growing concern in the electric power industry with the development of the smart grid. A false data injection attack has recently attracted the ever wider interest of researchers. A special type of false data injection attack or a Load redistribution (LR) attack is developed in [131]. The damage from LR attacks to power system operations can manifest in an immediate or a delayed fashion. For the immediate attacking goal, the most damaging attack can be identified through a max–min attacker–defender model.

Facility location. Facility location problems arise as real-life applications in both public and private sectors try to determine the optimal location for facilities such as warehouses, plants, distribution centres, shopping malls, hospitals, and post offices. They can have different objectives such as maximisation of the profit obtained from customers and minimisation of the total cost incurred by locating facilities and serving customers. There are many different bilevel settings for the facility location models [71, 73, 132, 133]. For example, mathematical models and computational methods for the classical competitive facility location model so-called $(r|p)$-centroid problem, can be found in [70].

The facility location problems with customer's preferences are studied in [134–141]. In these problems, the upper-level problem describes the facility location process as opposed to the lower-level problem that describes the customer allocation process. In [142], there is the problem of locating differentiated waste collection centres. Another bilevel programming problem of locating waste collection centres can be found in [143]. The model of competitive facility location and pricing is investigated in [90]. The choice of prices is based on the Bertrand model of price competition and the possibility of splitting a client's demand if it is profitable for both players. In [144], a firm wants to open p facilities to enter a market and maximise its market share, where other firms already operate. Customers can patronise any open facility and maximise their utility function. A comprehensive overview of bilevel facility location models can be found in [145].

The mixed-integer bilevel problem of strategic base stations placement in cognitive radio networks is considered in [91]. Two operators want to exploit the unused capacity of the primary network and maximise their profits derived from operating the base stations installed and clients served. The leader is aware of the future arrival of the follower, who is able to capture clients by placing its own base stations. It has also to limit the interference power at some measurement points defined by the primary user. The authors develop a matheuristic where a mixed-integer program derived from the follower's problem is solved by CPLEX software.

An interesting leader–follower facility location model for 5G high-speed networks is presented in [146]. Two mobile operators compete to attract customers with high-speed internet connections. The leader acts first by opening some base stations, anticipating that the follower will react by creating her own base stations and renting some leader's stations. Each customer patronises an operator with the highest speed of connection for him. The leader and the follower maximise their profits at the first and second levels, respectively and the customers move from one operator to another until the Nash equilibrium is reached.

Another line of interesting application is presented by the leader–follower hub location problems. The discrete hub network design model in a competitive environment is investigated in [147] with a flow threshold. Two firms, a leader and a follower, compete with each other to maximise their profits. The level of captured passengers is determined by the logit function and each route contains one hub only.

In [148] the authors consider the bilevel problem which extends the p-hub median problem [149], taking into account intentional hub disruptions. In this formulation, the leader chooses p hubs and additionally the emergency hub to minimise two objectives, one of which corresponds to the total transportation cost in the normal situation while the other corresponds to the total transportation cost in the worst-case after r-interdiction. Two formulations, the multiple allocation hub interdiction problem and hub protection problem are studied in [150]. The first formulation refers to the bilevel programming problem and the second one to the three-level programming problem. For the multiple allocation hub interdiction problem, two reductions to mixed-integer problems are proposed. The first one is based on the dual problem to the linear programming lower-level problem. The second reduction is based on replacing the lower-level problem with an equivalent system of closest assignment constraints. The ideas developed on the multiple allocation hub interdiction problem are used to solve the hub protection problem (see also [151, 152]).

The paper [153] addresses the $(r|p)$-hub centroid problem. The leader locates p hubs, later on, the follower locates r hubs. The customers choose one firm with respect to provided service levels. The goal of each firm is market share maximisation. An exact solution method is proposed.

Three variants of the bilevel hub interdiction problem are presented in [154]: the multiple allocation p-hub median, the p-hub maximal covering, and the p-hub centre problems under intentional disruptions. In these problems, the leader locates p hubs, whereas the follower tries to interdict r such hubs that their loss would diminish the network performance the most. The simulated annealing heuristic is applied as a solution approach.

In [155] the leader–follower hub location problem under fixed markups is introduced deploying an alternating heuristic as a solution approach. In [156] the authors investigate the leader–follower single allocation hub location problem under fixed markups. Two variants of this Stackelberg competition are addressed: deterministic and robust. For the deterministic variant, a mixed-integer linear reformulation of the follower's model is given. For the robust variant, it is shown how to reformulate the follower's program as a mixed-integer conic quadratic one. As a solution approach for the leader, the alternating heuristic is used also.

The $(r|p)$-hub centroid problem under the price war can be found in [157]. It was shown that there is a solution when the objective is profit maximisation, and the customer demand is split according to the logit model. The equilibrium price equations are presented, accompanied by some computational complexity observations. In [158], an extension of this model is presented. The paper provides a theoretical indication of the effect of the price sensitivity parameter on profit. It is shown that the optimal routes under the Bertrand-Nash price equilibrium are among the lowest cost ones. As a solution approach is used the basic variable neighbourhood search algorithm, based on a novel local search stopping rule and objective function estimation.

Supply chain. A bilevel optimisation problem to model the planning of a distribution network is proposed in [159]. In this problem, there are manufacturing plants, depots, and customers. The purpose is to decide which depots should be used and how the product should be distributed from depots to customers and from plants to depots aiming to minimise the total fixed costs and delivery costs. A metaheuristic approach based on evolutionary algorithms is developed.

The paper [160] formulates the joint configuration of a product family and its supply chain. The upper-level problem optimises the selection of modules, module instances, and product variants. The lower-level problem responds to decisions of the upper level in order to determine an optimal supply chain configuration and inventory policies. To solve this nonlinear optimisation model, a bilevel nested genetic algorithm with constraint reasoning is developed and implemented.

The distribution centre problem is represented in [161]. The upper and lower levels are to find the minimum transportation cost of shipping products from plants to distribution centres and from distribution centres to customers, respectively.

Bilevel models associated with timberlands systems are proposed in [162, 163]. The paper [162] investigates the economic impact of a new biorefinery on an established timberlands system. The work [163] studies the supply allocation problem for an established timberlands supply chain with an additional decision of new biorefinery investments. The paper [164] focuses on a multi-product vendor–buyer supply chain considering environmental factors in the product manufacturing process. The model determines the optimal selling prices, advertising expenditures, wholesale prices, vendor's environmental improvements, and ordering policies of the vendor and the buyer.

Scheduling problems. The bilevel flow shop scheduling problem is proposed in [165, 166]. The shop owner (upper level) assigns the jobs to the machines in order to minimise the flow time while the customer is at the lower level and decides on a job schedule in order to minimise the makespan. The authors use the concepts of tolerance membership function at each level to define a fuzzy decision model for generating optimal (satisfactory) solutions for the bilevel flow shop scheduling problem.

The bilevel multi-objective job-shop scheduling problem is considered in [167]. At the upper level, idle time on the system bottleneck is minimised. At the lower level, a decision is made to plan other machines while maintaining the maximum use of the bottleneck and gaining improvements in other performance measures.

The problem of scheduling inbound trucks at the inbound doors of a cross-dock facility under truck arrival time uncertainty is proposed in [168]. A single-level and a bilevel optimisation problem are formulated. A genetic algorithm and its modification are discussed for the single- and bilevel optimisation problems.

Multi-job scheduling problems are considered in [169, 170]. In these problems, the variation of energy consumption with the performance of servers is taken into account for cloud computing. Moreover, task-scheduling strategies depend directly on data placement policies. To solve the bilevel discrete model efficiently, specific-design encoding and decoding methods are designed. Based on these, an effective genetic algorithm is proposed, in which a local search operator is introduced to accelerate the convergent speed and enhance searching ability.

Other application. A new approach to the development of a strategic program for a mineral resource region based on public–private partnership mechanisms is proposed in [92, 93, 101, 171–173]. The government not only carries out general-purpose infrastructural projects but also provides a part of the costs related to compensating for ecological losses caused by the investment projects. As we discussed in Sect. 1.4.2, these bilevel problems are Σ_2^P-hard even in case of a three-year planning horizon.

Evacuation models are proposed in [174, 175]. In [174] to enable an efficient evacuation, a network optimisation model which integrates lane-based reversal design and routing with intersection crossing conflict elimination for evacuation is constructed. The proposed bilevel model minimises the total evacuation time to leave the evacuation zone. A tabu search algorithm is applied to find an optimal lane reversal plan in the upper level. The lower level utilises a simulated annealing algorithm. The paper [175] develops a model to optimise the issuance of evacuation orders with explicit consideration of the highly uncertain evolution of the storm and the complexity of the behavioural reaction to evolving storm conditions. A solution procedure based on progressive hedging is developed. A realistic case study for the eastern portion of the state of North Carolina is presented.

1.6 Conclusion

In this chapter, we have presented discrete bilevel optimisation models, discussed their computational complexity and applications. We have pointed out some lines for future research and wish to discuss below one of them without details. In real-world applications, we often try to choose a solution that is strong (optimal or near optimal) not only for the current input data but also for a wide range of similar instances. Common methods for analysing such situations lead us to solve more computationally difficult problems than the original ones [176]. A new idea in this area had been suggested based on a threshold approach [177, 178]. Let us consider a single-level income maximisation problem with uncertainty in the input data. Now instead of maximising the income, we wish to maximise the norm of possible variations in the input data subject to the income is at least as the given threshold. This elegant approach allows us to significantly improve the quality of the solution obtained in terms of stability with small degradation of the income. In [178], this approach is applied to the facility location problems and analysed as bi-objective optimisation.

Another line of research is related to the uncertainty arising from the rationality of the follower behaviour. As we have mentioned above, economic agents cannot use excessively large resources to find the optimal solution. A new approach which deals with the solution method uncertainty at the lower level is developed in [123]. Earlier we assume that the follower must respond by the optimal solution to the leader's solution. However, the follower has to respond by a feasible solution using some approximate algorithms in case of resource constraints, time or computational resources. Three approaches are presented for this case in [123]. The leader does not know the algorithm that is used by the follower but knows a set of possible follower's solution methods. A similar approach in a simplified form was used in [179, 180]. The optimistic and pessimistic models and the strong-weak approach from

[181–184] of the standard bilevel optimisation also can be viewed as special cases of approach from [123]. We guess that it is a promising area for future research.

Acknowledgements Many thanks to Ivan Davydov and Andrey Melnikov for useful discussions and their efforts in improving the presentation of the paper. The research is supported by the Russian Science Foundation, Project No. 21-41-09017.

References

1. von Stackelberg, H.: Marktform und Gleichgewicht. Springer, Vienna (1934)
2. Bracken, J., McGill, J.: Mathematical programs with optimization problems in the constraints. Oper. Res. **21**, 37–44 (1973)
3. Dempe, S.: Foundations of Bilevel Programming. Nonconvex Optimization and Its Applications (Book seriesL NOIA, volume 61). Springer (2002)
4. Wiesemann, W., Tsoukalas, A., Kleniati, P.-M., Rustem, B.: Pessimistic bilevel optimization. SIAM J. Optim. **23(1)**, 353–380 (2013)
5. Liu, J., Fan, Y., Chen, Z., Zheng, Y.: Pessimistic bilevel optimization: A survey. Int. J. Comput. Intell. Syst. **11(1)**, 725–736 (2018)
6. DeNegre, S.: Interdiction and discrete bilevel linear programming. Ph.D. dissertation, Lehigh University (2011)
7. Kochetov, Y.A., Plyasunov, A.V.: A polynomial solvable class of two-level linear programming problems. Diskret. Anal. Issled. Oper. **4(2)**, 23–33 (1997) (in Russian)
8. Moore, J.T., Bard, J.: The mixed integer linear bilevel programming problem. Oper. Res. **38(5)**, 911–921 (1990)
9. Kalashnikov, V.V., Dempe, S., Pérez-Valdés, G.A., Kalashnykova, N.I., Camacho-Vallejo, J-F.: Bilevel programming and applications. Math. Probl. Eng. **2015**, Article ID 310301 (2015)
10. Ungureanu, V.: Pareto-Nash-Stackelberg Game and Control Theory. Springer (2018)
11. Fudenberg, D., Tirole, J.: Game Theory. MIT Press (1993)
12. Shi, C.: Linear bilevel programming technology—Models and algorithms. PhD thesis, University of Technology, Sydney, Australia (2005)
13. Soismaa, M.: A note on efficient solutions for the linear bilevel programming problem. Eur. J. Oper. Res. **112**, 427–431 (1999)
14. Dempe, S.: Comment to “interactive fuzzy goal programming approach for bilevel programming problem" by S.R. Arora and R. Gupta. Eur. J. Oper. Res. **212(2)**, 429–431 (2011)
15. Candler, W.: A linear bilevel programming algorithm: A comment. Comp. Oper. Res. **15**, 297–298 (1988)

16. Marcotte, P., Savard, G.: A note on the Pareto optimality of solutions to the linear bilevel programming problem. Comp. Oper. Res. **18**, 355–359 (1991)
17. Migdalas, A.: When is Stackelberg equilibrium Pareto optimum? In: Pardalos, P. et al. (eds.) Advances in Multicriteria Analysis. Kluwer Academic Publishers, Dordrecht (1995)
18. Wen, U., Hsu, S.: A note on a linear bilevel programming algorithm based on bicriteria programming. Comp. Oper. Res. **16**, 79–83 (1989)
19. Fliege, J., Vicente, L.N.: Multicriteria approach to bilevel optimization. J. Optim. Theory. Appl. **131**, 209–225 (2006)
20. F*ü*l*ö*p, J.: On the equivalence between a linear bilevel programming problem and linear optimization over the efficient set. Technical Report WP 93-1, Laboratory of Operations Research and Decision Systems, Computer and Automation Institute, Hungarian Academy of Sciences, Budapest, Hungary (1993)
21. Ivanenko, D.S., Plyasunov, A.V.: Reducibility of bilevel programming problems to vector optimization problems. J. Appl. Ind. Math. **2**, 179–195 (2008)
22. Pieume, C.O., Fotso, L.P., Siarry, P.: Solving bilevel programming problems with multicriteria optimization techniques. OPSEARCH. **46**, 169–183 (2009)
23. Ruuska, S., Miettinen, K., Wiecek, M.M.: Connections between single-level and bilevel multiobjective optimization. J. Optim. Theory Appl. **153**, 60–74 (2012)
24. Glackin, J., Ecker, J.G., Kupferschmid, M.: Solving bilevel linear programs using multiple objective linear programming. J. Optim. Theory Appl. **140**, 197–212 (2009)
25. Jing Y., Zhang S.: The solution to a kind of stackelberg game systems with multi-follower: Coordinative and incentive. In: Bensoussan A., Lions J.L. (eds.) Analysis and Optimization of Systems. Lecture Notes in Control and Information Sciences, **111** Springer, Berlin, Heidelberg, pp. 593–602 (1988)
26. Zhang, G., Lu, J.: Fuzzy bilevel programming with multiple objectives and cooperative multiple followers. J. Glob. Optim. **47**, 403–419 (2010)
27. Alekseeva, E., Kochetov, Y., Talbi, E.-G.: A matheuristic for the discrete bilevel problem with multiple objectives at the lower level. I. Trans. Oper. Res. **24(5)**, 959–981 (2017)
28. Sherali, H.: A multiple leader Stackelberg model and analysis. Oper. Res. **32**, 390–404 (1984)
29. DeMiguel, V., Xu, H.: A stochastic multiple-leader Stackelberg model: Analysis, computation, and application. Oper. Res. **57(5)**, 1220–1235 (2009)
30. Lu, J., Shi, C., Zhang, G.: On bilevel multi-follower decision making: General framework and solutions. Inf. Sci. **176(11)**, 1607–1627 (2006)
31. Dempe, S.: Dependence of bilevel programming on irrelevant data. Technical Report 2011-01, TU Bergakademie Freiberg, Department of Mathematics and Computer Science. www.optimization-online.org (2011)

32. Macal, C.M., Hurter, A.P.: Dependence of bilevel mathematical programs on irrelevant constraints, Comput. Oper. Res. **24**, 1129–1140 (1997)
33. Ko, K., Lin, C.L.: On the complexity of min-max optimization problems and their approximation. In Du, D.-Z. and Pardalos, P.M. (eds.) Minimax and Applications. Kluwer Academic Publishers, Dordrecht, 219–239 (1995)
34. Umans, C.: Hardness of approximating Σ_2^P minimization problems. In Proceedings of the 40th Annual Symposium on Foundations of Computer Science (FOCS'1999), Los Alamitos, CA, 465–474 (1999)
35. Umans, C.: Optimization problems in the polynomial-time hierarchy. In Proceedings of the 3rd International Conference on Theory and Applications of Models of Computation (TAMC'2006). Lect. Notes in Comput. Sci. **3959**, 345–355 (2006)
36. Caprara, A., Carvalho, M., Lodi, A., Woeginger, G.J.: A study on the computational complexity of the bilevel knapsack problem. SIAM J. Optim. **24(2)**, 823–838 (2014)
37. Solvebilevel. https://yalmip.github.io/command/solvebilevel/
38. MibS. https://github.com/coin-or/MibS
39. Pyomo. https://pyomo.readthedocs.io/en/stable/modeling_extensions/bilevel.html
40. GAMS. https://www.gams.com/latest/docs/UG_EMP_Bilevel.html
41. Saharidis, G.K., Ierapetritou, M.G.: Resolution method for mixed integer bilevel linear problems based on decomposition technique. J. Global Optim. **44(1)**, 29–51 (2009)
42. Fontaine, P., Minner, S.: Benders decomposition for discrete-continuous linear bilevel problems with application to traffic network design. Transp. Res. Part B: Methodol. **70**, 163–172 (2014)
43. Zeng, B., An, Y.: Solving bilevel mixed integer program by reformulations and decomposition. Optimization online (2014)
44. Shi, X., Prokopyev,O., Ralphs, T.K.: Mixed integer bilevel optimization with k-optimal follower: A hierarchy of bounds. COR@L Technical Report 20T-012. http://www.optimization-online.org/DB_HTML/2020/06/7874.html (2020)
45. Dempe, S., Kue, F.M.: Solving discrete linear bilevel optimization problems using the optimal value reformulation. J. Global Optim. **68(2)**, 255–277 (2017)
46. Lozano, L., Smith, J.: A value-function-based exact approach for the bilevel mixed-integer programming problem. Oper. Res. **65(3)**, 768–786 (2017)
47. Bolusani, S., Coniglio, S. Ralphs, T.K., Tahernejad, S.: A unified framework for multistage mixed integer linear optimization. In: Dempe, S. and Zemkoho, A. (eds.) Bilevel Optimization: Advances and Next Challenges. Springer, Cham, pp. 513–560 (2020)
48. Bolusani, S., Ralphs, T.K.: A framework for generalized Benders' decomposition and its application to multilevel optimization. COR@L Technical

Report 20T-004. http://www.optimization-online.org/DB_HTML/2020/04/7755.html (2020)
49. Wen, U., Yang, Y.: Algorithms for solving the mixed integer two-level linear programming problem. Comput. Oper. Res. **17**, 133–142 (1990)
50. Edmunds, T., Bard, J.F.: An algorithm for the mixed-integer nonlinear bilevel programming problem. Ann. Oper. Res. **34**, 149–162 (1992)
51. Fischetti, M., Ljubic, I., Monaci, M., Sinnl, M.: On the use of intersection cuts for bilevel optimization. Math. Program. **172(1)**, 77–103 (2018)
52. Caramia,M., Mari, R.: Enhanced exact algorithms for discrete bilevel linear problems. Optim. Lett. **9(7)**, 1447–1468 (2015)
53. DeNegre, S.T., Ralphs,T.K.: A branch-and-cut algorithm for integer bilevel linear programs. Oper. Res./Comput. Sci. Interfaces Ser. **47**, 65–78 (2009)
54. Tahernejad, S., Ralphs, T.K., DeNegre, S.T.: A branch-and-cut algorithm for mixed integer bilevel linear optimization problems and its implementation. Math. Program. Comput. **12**, 529–568 (2020)
55. Fischetti, M., Ljubic, I., Monaci, M., Sinnl, M.: A new general-purpose algorithm for mixed-integer bilevel linear programs. Oper. Res. **65(6)**, 1615–1637 (2017)
56. Fischetti, M., Monaci, M.: A branch-and-cut algorithm for mixed-integer bilinear programming. Eur. J. Oper. Res. **282**, 506–514 (2020)
57. Beresnev, V., Melnikov, A.: Approximation of the competitive facility location problem with MIPs. Comput. Oper. Res. **104**, 139–148 (2019)
58. Hemmati, M., Smith, J.C.: A mixed-integer bilevel programming approach for a competitive prioritized set covering problem. Discr. Optimiz. **20**, 105–134 (2016)
59. Heilporn, G., Labbé, M., Marcotte, P., Savard, G.: A polyhedral study of the network pricing problem with connected toll arcs. Networks. **55(3)**, 234–246 (2010)
60. Domínguez, L.F., Pistikopoulos, E.N.: Multiparametric programming based algorithms for pure integer and mixed-integer bilevel programming problems. Comput. Chem. Eng. **34(12)**, 2097–2106 (2010)
61. Avraamidoua, S., Pistikopoulos, E.N.: A multi-parametric optimization approach for bilevel mixed-integer linear and quadratic programming problems. Comput. Chem. Eng. **125**, 98–113 (2019)
62. Köppe, M., Queyranne, M., Ryan, C.T.: Parametric integer programming algorithm for bilevel mixed integer programs. J. Optim. Theory Appl. **146(1)**, 137–150 (2010)
63. Thirwani, D., Arora, S.R.: An algorithm for the integer linear fractional bilevel programming problem. Optim. **39(1)**, 53–67 (1997)
64. Calvete, H.I., Gal, C.: A note on bilevel linear fractional programming problem. Europ. J. Open Res. **152(1)**, 296–299 (2004)
65. Mitsos, A.: Global solution of nonlinear mixed-integer bilevel programs. J. Global Optim. **47(4)**, 557–582 (2010)

66. Xu, P., Wang, L.: An exact algorithm for the bilevel mixed integer linear programming problem under three simplifying assumptions. Comput. Oper. Res. **41**, 309–318 (2014)
67. Wang, L., Xu, P.: The watermelon algorithm for the bilevel integer linear programming problem. SIAM J. Optim. **27(3)**, 1403–1430 (2017)
68. Glover, F.: Future paths for integer programming and links to artificial intelligence. Comput. Ops. Res. 13(5), 533–549 (1986)
69. Talbi, E-G. (Ed.): Metaheuristics—From Design to Implementation. Computer Science (2009)
70. Talbi, E-G. (Ed.): Metaheuristics for bi-level optimization. Stud. Comput. Intell. **482** (2013)
71. Dempe, S., Zemkoho, A.: Bilevel Optimization Advances and Next Challenges. Springer Optimization and Its Applications (Book Series: SOIA, **161**) (2020)
72. Bard, J.F.: Practical Bilevel Optimization: Algorithms and Applications. Part of the Nonconvex Optimization and Its Applications. Springer (1998)
73. Mallozzi, L., D'Amato, Eg., Pardalos, P. (Eds.): Spatial Interaction Models Facility Location Using Game Theory. Springer Optimization and Its Applications (2017)
74. Zhang, S., Hui, Q., Bai, X., Sun, R.: Bilevel optimization for the Hazmat transportation problem with lane reservation. J. Adv. Transp. **2020(34)**, 1–14 (2020)
75. López-Ramosa, F., Nasinib, S., Guarnaschelli, A.: Road network pricing and design for ordinary and Hazmat vehicles: Integrated model and specialized local search. Comput. Oper. Res. **109**, 170–187 (2019)
76. Simon, H.: Theories of bounded rationality. In: Radner, R., (ed.) Decision and Organization. North-Holland, New York (1972)
77. Deng, X., Papadimitriou, C.H.: On the complexity of cooperative solution concepts. Math. Oper. Res. **19(2)**, 257–266 (1994)
78. Attallah, M.: Algorithms and Theory of Computation Handbook. CRC Press, Boca Raton (1999)
79. Ausiello, G., Crescenzi, P., Gambosi, G., et al.: Complexity and Approximation: Combinatorial Optimization Problems and Their Approximability Properties. Springer-Verlag, Berlin (1999)
80. Jeroslow, R.G.: The polynomial hierarchy and a simple model for competitive analysis. Math. Program. **32(2)**, 146–164 (1985)
81. Schaefer, M., Umans, C.: Completeness in the polynomial-time hierarchy: A compendium. SIGACT News. **33(3)**, 32–49 (2002)
82. Garey, M.R., Johnson, D.S.: Computers and Intractability: A Guide to the Theory of NP-Completeness, San Francisco (1978)
83. Dempe, S., Richter, K.: Bilevel programming with Knapsack constraints. Centr. Eur. J. Oper. Res. **8**, 93–107 (2000)
84. Mansi, R., Alves, C., de Carvalho, J.M.V., Hanafi, S.: An exact algorithm for bilevel 0–1 knapsack problems. Math. Probl. Eng., Article ID 504713 (2012)

85. Eggermont, C., Woeginger, G.J.: Motion planning with pulley, rope, and baskets. In Proceedings of the 29th International Symposium on Theoretical Aspects of Computer Science (STACS–2012), 374–383 (2012)
86. Noltemeier, H., Spoerhase, J., Wirth, H.: Multiple voting location and single voting location on trees. Europ. J. Oper. Res. **181**, 654–667 (2007)
87. Davydov, I., Kochetov, Y., Plyasunov, A.: On the complexity of the $(r|p)$-centroid problem on the plane. TOP. **22(2)**, 614–623 (2014)
88. Melnikov, A.: Computational complexity of the discrete competitive facility location problem. J. Appl. Indust. Math. **8**, 557–567 (2014)
89. Panin, A.A., Pashchenko, M.G., Plyasunov, A.V.: Bilevel competitive facility location and pricing problems. Autom. Remote Control. **75(4)**, 715–727 (2014)
90. Kononov, A.V., Panin, A.A., Plyasunov, A.V.: A bilevel competitive location and pricing model with nonuniform split of demand. J. Appl. Indust. Math. **13(3)**, 500–510 (2019)
91. Iellamo, S., Alekseeva, E., Chen, L., Coupechoux, M., Kochetov, Yu.: Competitive location in cognitive radio networks. 4OR. **13(1)**, 81–110 (2015)
92. Lavlinskii, S., Panin, A., Plyasunov, A.: The Stackelberg model in territorial planning. Automat. Remote Control. **80(2)**, 286–296 (2019)
93. Lavlinskii S., Panin A., Plyasunov A.: Bilevel models for investment policy in resource-rich regions. Commun. Comput. Inf. Sci. **1275**, 36–50 (2020)
94. Bazgan, C., Escoffer, B., Paschos, V.Th.: Completeness in standard and differential approximation classes: Poly-(D)APX- and (D)PTAS-completeness. Theor. Comput. Sci. **339**, 272–292 (2005)
95. Qiu, X., Kern, W.: Improved approximation algorithms for a bi-level knapsack problem. Theor Comput Sci. **595**, 120–129 (2015)
96. Guruswami, V., Hartline, J.D., Karlin, A.R., Kempe, D., Kenyon, C., McSherry, F.: On profit-maximizing envy-free pricing. In Proceedings of the Annual ACM-SIAM Symposium on Discrete Algorithms, 1164–1173 (2005)
97. Joret, G.: Stackelberg network pricing is hard to approximate. Networks. **57(2)**, 117–120 (2011)
98. Cardinal, J., Demaine, E.D., Fiorini, S., Joret, G., Langerman, S., Newman, I., Weimann, O.: The Stackelberg minimum spanning tree game. Algorithmica. **59(2)**, 129–144 (2011)
99. Plyasunov, A.V., Panin, A.A.: The pricing problem. Part 2. The computational complexity. J. Appl. Indust. Math. **7(3)**, 420–430 (2013)
100. Panin, A., Plyasunov, A.: On complexity of the bilevel location and pricing problems. J. Appl. Indust. Math. **8(4)**, 574–581 (2014)
101. Lavlinskii, S., Panin, A., Plyasunov, A.: A bilevel planning model for public-private partnership. Autom. Remote Control. **76(11)**, 1976–1987 (2015)
102. Gao, Z., Wu, J., Sun, H.: Solution algorithm for the bi-level discrete network design problem. Transp. Res. Part B. **39(6)**, 479–495 (2005)

103. Fontaine, P., Minner, S.: Benders decomposition for discrete–continuous linear bilevel problems with application to traffic network design. Transp. Res. Part B. **70**, 163–172 (2014)
104. Labbé, M., Marcotte P., Savard, G.: A bilevel model of taxation and its application to optimal highway pricing. Manage. Sci. **44**, 1608–1622 (1998)
105. Fontaine, P., Minner, S.: A dynamic discrete network design problem for maintenance planning in traffic networks. Ann. Oper. Res. **253**, 757–772 (2017)
106. Bagloee, S.A., Sarvi, M., Patriksson, M.: A hybrid branch-and-bound and benders decomposition algorithm for the network design problem. Comput.-Aided Civ. Infrastruct. Eng. **32**, 1–25 (2016)
107. Tawfik, C., Limbourg, S.: A bilevel model for network design and pricing based on a level-of-service assessment. Transp. Sci. **53(6)**, 1609–1626 (2019)
108. Tian, Y., Dong, H.-H., Jia, L.-M., Qin, Y., Li, S.-Y.: The bilevel design problem for communication networks on trains: Model, algorithm, and verification. Mathematical Problems in Engineering. N 840619 (2014)
109. Labbé, M., Marcotte, P.: Bilevel Network Design (hal-01937014). https://hal.inria.fr/hal-01937014 (2019)
110. Rey, D.: Computational benchmarking of exact methods for the bilevel discrete network design problem. Transp Res Procedia. **47**, 11–18 (2020)
111. Delle Fave, F.M., Jiang, A.X., Yin, Z., Zhang, C., Tambe, M., Kraus, S., Sullivan, J.P.: Game-theoretic security patrolling with dynamic execution uncertainty and a case study on a real transit system. J. Artif. Intell. Res. **50**, 321–367 (2014).
112. Jain, M., Tsai, J., Pita, J., Kiekintveld, C., Rathi, S., Tambe, M., Ordónez, F.: Software arshal service. Interfaces. **40(4)**, 267–290 (2010)
113. Jiang, A.X., Yin, Z., Zhang, C., Tambe, M., Kraus, S.: Game-theoretic randomization for security patrolling with dynamic execution uncertainty. In Proceedings of the 12th International Conference on Autonomous Agents Multiagent Systems, Saint Paul, MN, USA, May 6–10, 2013 (Int. Found. Autonomous Agents Multiagent Syst., Richland, SC, 2013), 207–214 (2013)
114. Church, R.L., Scaparra, M.P.: Protecting critical assets: The r-interdiction median problem with fortification. Geogr. Anal. **39(2)**, 129–146 (2007)
115. Scaparra, M.P., Church, R.L.: A bilevel mixed integer program for critical infra-structure protection planning. Comput. Oper. Res. **35(6)**, 1905–1923 (2008)
116. Scaparra, M.P., Church, R.L.: An exact solution approach for the interdictionmedian problem with fortification. Europ. J. Open Res. **189(1)**, 76–92 (2008)
117. Scaparra, M.P., Church, R.L.: A bilevel mixed-integer program for critical infrastructure protection planning. Comput. Oper. Res. **35**, 1905–1923 (2008)

118. Aksen, D., Piyade, N., Aras, N.: The budget constrained r-interdiction median problem with capacity expansion. Central Europ. J. Oper. Res. **18(3)**, 269–291 (2010)
119. Zhu, Y., Zheng, Z., Zhang, X., Cai, K.Y.: The r-interdiction median problem with probabilistic protection and its solution algorithm. Comput. Oper. Res. **40**, 451–462 (2013)
120. Aksen, D., Aras, N.: A bilevel fixed charge location model for facilities under imminent attack. Comput. Oper. Res. **39(1)**, 1364–1381 (2012)
121. Beresnev, V., Melnikov, A.: A bilevel attacker-defender model to choosing the composition of attack means. J. Appl. Indust. Math. **13(4)**, 612–622 (2019)
122. Beresnev, V., Davydov, I., Kononova, P., Melnikov, A.: Bilevel defender–attacker model with multiple attack scenarios. J. Appl. Indust. Math. **12(3)**, 417–425 (2018)
123. Zare, M.H., Prokopyev, O.A., Sauré, D.: On bilevel optimization with inexact follower. Decis. Anal. **17(1)**, 1–22 (2020)
124. Kochetov,Y.A., Pljasunov, A.V.: Efficient algorithm for a class of bilevel linear programming problems. In Operations Research Proceedings 1996. Berlin, Springer, 10–13 (1997)
125. Kochetov, Y.A., Plyasunov, A.V.: The problem of selecting a number of products with partial exterior financing. Diskret. Anal. Issled. Oper., Serija 2. **9(2)**, 78–96 (2002) (in Russian)
126. Plyasunov, A.V.: A polynomially solvable class of bilevel nonlinear programming problems. Diskret. Anal. Issled. Oper., Seriya 2. **7**, 89–113 (2000) (in Russian)
127. Plyasunov, A.V.: A bilevel linear programming problem with a multivariant knapsack at the lower level. Diskret. Anal. Issled. Oper. **10(1)**, 44–52 (2003) (in Russian)
128. Arroyo, J.M., Galiana, F.D.: On the solution of the bilevel programming formulation of the terrorist threat problem. IEEE Trans. Power Syst. **20(2)**, 789–797 (2005)
129. Liu, C.-C., Jung, J. Heydt, G.T., Vittal, V., Phadke, A.G.: The strategic power infrastructure defense (SPID) system. A conceptual design. IEEE Control. Syst. Mag. **20(4)**, 40–52 (2000)
130. Roy,A., Kim, D.S., Trivedi, K.S.: Cyber security analysis using attack countermeasure trees. In Proceedings of the 6th Annual Workshop on Cyber Security and Information Intelligence Research (2010)
131. Yuan,Y.-L., Li,Z.-I., Ren, K.-I.: Quantitative analysis of load redistribution attacks in power systems. IEEE Transactions on Parallel and Distributed Systems. **23(9)**, 1731–1738 (2012)
132. Karakitsiou, A.: Modeling Discrete Competitive Facility Location. Springer, Cham, Heidelberg (2015)
133. Kononov, A.V., Kochetov, Yu.A., Plyasunov, A.V.: Competitive facility location models. Comput. Math. Math. Phys. **49(6)**, 994–1009 (2009)

134. Hansen, P., Kochetov, Y., Mladenovic, N.: Lower bounds for the uncapacitated facility location problem with user preferences. Technical Report, Les Cahiers du GERAD, G-2004-24 (2004)
135. Vasilyev, I.L., Klimentova, K.B., Kochetov, Yu.A.: New lower bounds for the facility location problem with clients' preferences. Comput. Math. Math. Phys. **49(6)**, 1010–1020 (2009)
136. Vasilyev, I.V., Klimentova, K.B., Boccia, M.: Polyhedral study of simple plant location problem with order. Oper. Res. Lett. **41(2)**, 153–158 (2013)
137. Cánovas, L., García, S., Labbé, M., Marín, A.: A strengthened formulation for the simple plant location problem with order. Operat. Res. Letts. **35(2)**, 141–150 (2007)
138. Casas-Ramírez, M.-S., Camacho-Vallejo, J.-F., Martínez-Salazar, I.-A.: Approximating solutions to a bilevel capacitated facility location problem with customer's patronization toward a list of preferences. Appl. Math. Comput. **319**, 369–386 (2018)
139. Calvete, H.I., Galé, C., Iranzo, J.A., Camacho-Vallejo, J.-F., Casas-Ramírez, M.-S.: A matheuristic for solving the bilevel approach of the facility location problem with cardinality constraints and preferences. Comput. Oper. Res. **124**, N 105066 (2020)
140. Beresnev, V.L., Melnikov, A.A.: The branch-and-bound algorithm for a competitive facility location problem with the prescribed choice of suppliers. J. Appl. Ind. Math. **8**, 177–189 (2014)
141. Beresnev, V.L., Melnikov, A.A.: Exact method for the capacitated competitive facility location problem. Comp. Oper. Res. **95**, 73–82 (2018)
142. Caramia, M., Giordani, S.: Location of differentiated waste collection centers with user cooperation: A bilevel optimization approach. Optim. Lett. **14**, 85–99 (2020)
143. Caramia, M., Dalla Costa, M.: An application of bilevel optimisation to the waste collection centres location problem. Int. J. Math. Oper. Res. **16(1)**, 118–137 (2020)
144. Casas-Ramírez, M.-S., Camacho-Vallejo, J.-F., Díaz, J.A., Luna, D.E.: A bilevel maximal covering location problem. Oper. Res. **20**, 827–855 (2020)
145. Drezner, T., Drezner, Z.: Leader-follower models in facility location. In: Mallozzi, L., D'Amato, E., and Pardalos, P.M. (eds.) Spatial Interaction Models Facility Location Using Game Theory. Springer, pp. 73–104 (2017)
146. Davydov, I., Kochetov, Y., Dempe, S.: Local search approach for the competitive facility location problem in mobile networks. Int. J. Artif. Intell. **16(1)**, 130–143 (2018)
147. Sasaki, M.: Hub network design model in a competitive environment with flow threshold. J. Oper. Res. Soc. Japan. **48**, 158–171 (2005)
148. Parvaresh, F., Moattar Husseini, S.M., Hashemi Golpayegany, S.A., Karimi, B.: Hub network design problem in the presence of disruptions. J. Intell. Manuf. **25**, 755–774 (2014)

149. Hamacher, H.W., Labbé, M., Nickel, S., Sonneborn, T.: Adapting polyhedral properties from facility to hub location Problems. Discret. Appl. Math. **145(1)**, 104–116 (2004)
150. Ramamoorthy, P., Jayaswal, S., Sinha, A., Vidyarthi, N.: Multiple allocation hub interdiction and protection problems: Model formulations and solution approaches. Eur. J. Oper. Res. **270(1)**, 230–245 (2018)
151. Lei, T.L.: Identifying critical facilities in hub-and-spoke networks: A hub interdiction median problem. Geogr. Anal. **45(2)**, 105–122 (2013)
152. Ghaffarinasab N., Atayi R.: An implicit enumeration algorithm for the hub interdiction median problem with fortification. Eur. J. Oper. Res. **267(1)**, 23–39 (2018)
153. Mahmutogullari, I., Kara, B.: Hub location under competition. Eur. J. Oper. Res. **250(1)**, 214–225 (2016)
154. Ghaffarinasab, N., Motallebzadeh, A.: Hub interdiction problem variants: Models and metaheuristic solution algorithms. Eur. J. Oper. Res. **267(2)**, 496–512 (2018)
155. Ćvokić, D.D., Kochetov,Y.A., Plyasunov, A.V.: A leader-follower hub location and pricing problem under fixed markups. Lect. Notes Comput. Sci. **9869**, 350–363 (2016)
156. Ćvokić, D.D.: A leader-follower single allocation hub location problem under fixed markups. Filomat. **34(8)**, 2463–2484 (2020).
157. Ćvokić, D.D., Kochetov, Y.A., Plyasunov, A.V.: The competitive hub location under the price war. Lect. Notes Comput. Sci. **11548**, 133–146 (2019)
158. Ćvokić, D.D., Kochetov, Y.A., Plyasunov, A.V., Savić, A.: A variable neighborhood search algorithm for the $(r|p)$ hub-centroid problem under the price war. J. Glob. Optim. https://doi.org/10.1007/s10898-021-01036-9 (2021)
159. Calvete, H.I., Galé, C., Iranzo, J.A.: Planning of a decentralized distribution network using bilevel optimization. Omega. **49**, 30–41 (2014)
160. Yang, D., Jiao, J., Ji, Y., Du, G., Helo, P., Valente, A.: Joint optimization for coordinated configuration of product families and supply chains by a leader-follower Stackelberg game. Eur. J. Oper. Res. **246(1)**, 263–280 (2015)
161. Saranwong, S., Likasiri, C.: Bi-level programming model for solving distribution center problem: a case study in Northern Thailand's sugarcane management. Comput. Ind. Eng. **103**, 26–39 (2017)
162. Yeh, K., Realff, M.J., Lee, J.H., Whittaker, C.: Analysis and comparison of single period single level and bilevel programming representations of a pre-existing timberlands supply chain with a new biorefinery facility. Comput. Chem. Eng. **68**, 242–254 (2014)
163. Yeh, K., Whittaker, C., Realff, M.J., Lee, J.H.: Two stage stochastic bilevel programming model of a pre-established timberlands supply chain with biorefinery investment interests. Comput. Chem. Eng. **73**, 141–153 (2015)
164. Wang, M., Zhang, R., Zhu, X.: A bi-level programming approach to the decision problems in a vendor-buyer eco-friendly supply chain. Comput. Ind. Eng. **105**, 299–312 (2017)

165. Karlof, J.K., Wang, W.: Bilevel programming applied to the flow shop scheduling problem. Comput. Oper. Res. **23**, 443–451 (1996)
166. Abass, S.A.: Bilevel programming approach applied to the flow shop scheduling problem under fuzziness. Comput. Manag. Sci. **2(4)**, 279–293 (2005)
167. Kasemset, C., Kachitvichyanukul, V.: A PSO-based procedure for a bi-level multi-objective TOC-based job-shop scheduling problem. Int. J. Oper. Res. **14(1)**, 50–69 (2012)
168. Konur, D., Golias, M.M.: Analysis of different approaches to cross-dock truck scheduling with truck arrival time uncertainty. Comput. Ind. Eng. **65(4)**, 663–672 (2013)
169. Wang, X., Wang, Y., Cui, Y.: A new multi-objective bi-level programming model for energy and locality aware multi-job scheduling in cloud computing. Future Gener. Comput. Syst. **36**, 91–101 (2014)
170. Wang, X., Wang, Y., Cui, Y.: An energy-aware bi-level optimization model for multi-job scheduling problems under cloud computing. Soft Comput. **20(1)**, 303–317 (2016)
171. Lavlinskii, S., Panin, A., Plyasunov, A.: Stackelberg model and public-private partnerships in the natural resources sector of Russia. Lect. Notes Comput. Sci. **11548**, 158–171 (2019)
172. Lavlinskii, S., Panin, A., Plyasunov, A. Public-private partnership models with tax incentives: Numerical analysis of solutions. Commun. Comput. Inf. Sci. **871**, 220–234 (2018)
173. Lavlinskii, S., Panin, A., Plyasunov, A.: Comparison of models of planning public-private partnership. J. Appl. Indust. Math. **10(3)**, 356–369 (2016)
174. Zhao, X., Feng, Z.-Y., Li, Y., Bernard, A.: Evacuation network optimization model with lanebased reversal and routing. Math. Prob. Eng. **13**, N 1273508 (2016)
175. Yi, W., Nozick, L., Davidson, R., Blanton, B., Colle, B.: Optimization of the issuance of evacuation orders under evolving hurricane conditions. Transp. Res. B Methodol. **95**, 285–304 (2017)
176. Beresnev, V.L., Melnikov, A.A.: ε-Constraint method for bi-objective competitive facility location problem with uncertain demand scenario. Eur. J. Comput. Optim. **8**, 33–59 (2020)
177. Carrizosa, E., Nickel, S.: Robust facility location. Math. Meth. Oper. Res. **58**, 331–349 (2003)
178. Carrizosa, E., Ushakov, A., Vasilyev, I.: Threshold robustness in discrete facility location problems: A bi-objective approach. Optim. Lett. **9**, 1297–1314 (2015)
179. Smith, J.C., Lim, C., Sudargho, F.: Survivable network design under optimal and heuristic interdiction scenarios. J. Global Optim. **38(2)**, 181–199 (2007)
180. Shan, X., Zhuang, J.: Hybrid defensive resource allocations in the face of partially strategic attackers in a sequential defender attacker game. Eur. J. Oper. Res. **228(1)**, 262–272 (2013)

181. Aboussoror, A., Loridan, P.: Strong-weak Stackelberg problems in finite dimensional spaces. Serdica Math. J. **21(2)**, 151–170 (1995)
182. Cao, D., Leung, L.: A partial cooperation model for non-unique linear two-level decision problems. Eur. J. Oper. Res. **140(1)**, 134–141 (2002)
183. Zheng, Y., Wan, Z., Jia, S., Wang, G.: A new method for strong-weak linear bilevel programming problem. J. Indust. Management Optim. **11(2)**, 529–547 (2015)
184. Zare M.H., Özaltin O.Y., Prokopyev O.A.: On a class of bilevel linear mixed-integer programs in adversarial settings. J. Global Optim. **71(1)**, 91–113 (2018)