Bifurcations "Cycle–Chaos–Hyperchaos" in Some Nonideal Electroelastic Systems

S. V. Donetskyi and **A. Yu. Shvets**

Abstract A non-ideal dynamic system "piezoceramic transducer—LC generator" is considered. Various scenarios of transition to deterministic chaos in such a system are described. For the first time, the implementation of the "chaos-chaos" and "chaos-hyperchaos" transitions according to the scenarios of generalized intermittency has been discovered.

1 Introduction

Any oscillatory system consists of two main elements, namely, a source of excitation of oscillations and the actual oscillatory load. If the power of the oscillation excitation source is comparable to the power consumed by the oscillatory load, then such systems are called nonideal systems or systems with limited excitation. In nonideal nonlinear dynamical systems, the interaction between the source of oscillation excitation and the oscillatory subsystem can lead to completely unexpected steady-state regimes, in particular to the emergence of the deterministic chaos. Especially interesting cases are when the occurrence of chaos is associated exclusively with the interaction between the excitation source and the oscillatory load, and not with the internal properties of the subsystems.

For the first time, studies of limited excitation were started in the works of Arnold Sommerfeld [27, 28]. In the future, such studies were continued by Timoshenko [29], Kononenko 1969, Nayfeh and Mook [19]. Among the works of recent decades, significant contribution was made by Krasnopolskaya [12, 13], Warminski and Balthazar [30], Balthazar et al. [2], Palacios Felix and Balthazar [20] and many others.

The main purpose of this paper is to study new bifurcations of the transition to deterministic chaos in some nonlinear dynamic system "piezoceramic transducer—analog generator of limited power.

S. V. Donetskyi · A. Yu. Shvets (✉)
National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute", Kyiv, Ukraine

2 Mathematical Model of System "piezoceramic Transducer-LC-Generator"

Let us consider a piezoceramic rod transducer, which is loaded on the acoustic medium and to which electrodes the electrical voltage is affixed, raised by the *LC*–generator (Fig. 1). The selection of the generator of such type is caused by the renaissance of its application observable now in the up-to-date technique. This is related to facts that the electrovacuum-tube (analogue) devices ensure higher metrological characteristic to comparison with the numeral devices.

In papers [14, 16] in strict accordance with the theory of the relationship between mechanical and electrical fields in piezoceramic media [1, 4, 9], as well as based on the general principles of the theory of systems with limited excitation (Kononenko 1969), a mathematical model of the "transducer-generator" system was derived. It is proved that such a mathematical model can be written in the form of the following system of ordinary differential equations:

$$\ddot{\varphi} + \omega_0^2 \varphi = a_1 \dot{\varphi} + a_2 \dot{\varphi}^2 - a_3 \dot{\varphi}^3 - a_4 V,$$
$$\ddot{V} + \omega_1^2 V = a_5 \varphi + a_6 \dot{\varphi} - a_7 \dot{V}. \tag{1}$$

Here $V(t)$—voltage in the electrodes of the transducer; t—time; $\varphi(t) = \int_0^t (e_g - E_g) dx$; e_g—tube grid voltage; E_g—the constant component of voltage e_g. A detailed description of all parameters of the mathematical model (1), each of which depends on many electrical and elastic properties of the "transducer-generator" system is given in the paper [16].

Note that the mathematical model (1) was derived for one specific type of dynamical systems. However, as it was subsequently established in this dynamic system, a unique variety of steady-state dynamic regimes is realized. So, in this system, all the main types of regular attractors were discovered, such as equilibrium positions, limit cycles and invariant tori [3, 16, 25]. Chaotic attractors, including hyperchaotic ones, were also found in the system [1, 3, 13, 22]. Transitions to chaos (hyperchaos)
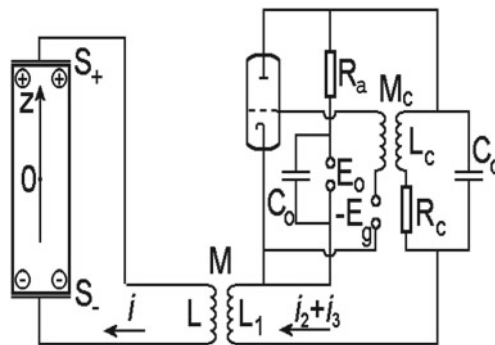

Fig. 1 Scheme of viewed system

through a cascade of period doubling bifurcations [7, 8] and through intermittency [18, 21] were identified. And at last, in papers [25, 26] self-excited, hidden and rare attractors were discovered in this system.

The dynamical system [1] has a wider variety of steady-state regimes and scenarios for the transition from one type of regime to another than, for example, the classical systems of Lorenz and Roessler. Such system is the "library" of regular and chaotic dynamics and can be used as a basic one in the study of general theory of dynamical systems.

3 Research Methodology and Numerical Results

The system of Eq. (1) is a nonlinear system of differential equations with a four-dimensional phase space. Therefore, in the general case, a solution to such a system can only be found using numerical or numerical-analytical methods. For the convenience of using such methods, we bring the system of Eq. (1) to a normal form. We introduce new "dimensionless" phase variables and "dimensionless" time by the formulas:

$$\xi = \frac{\varphi \omega_0}{E_g}, \quad \frac{d\xi}{d\tau} = \zeta, \quad \beta = \frac{V}{E_g}, \quad \frac{d\beta}{d\tau} = \gamma, \quad \tau = \omega_0 t. \tag{2}$$

Then the system of Eq. (1) can be written in the form:

$$\frac{d\xi}{d\tau} = \varsigma, \quad \frac{ds}{d\tau} = -\xi + \alpha_1 s + \alpha_2 s^2 - \alpha_3 s^3 + \alpha_4 \beta,$$
$$\frac{d\beta}{d\tau} = \gamma, \quad \frac{d\gamma}{d\tau} = \alpha_5 \xi + \alpha_6 \zeta - \alpha_0 \beta - \alpha_7 \gamma. \tag{3}$$

where the coefficients are equal to

$$\alpha_0 = \omega_1^2/\omega_0^2, \quad \alpha_1 = a_0/\omega_0, \quad \alpha_2 = a_2 E_g/\omega_0, \quad \alpha_3 = a_3 E_g^2/\omega_0,$$
$$\alpha_4 = -a_4/\omega_0, \quad \alpha_5 = a_5/\omega_0^3, \quad \alpha_6 = a_6/\omega_0^2, \quad \alpha_7 = a_7/\omega_0.$$

For construction and study of attractors of system (3) (both regular and chaotic), a whole complex of numerical methods was applied. Such as, the fifth order method of Runge–Kutta with the application of correcting procedure of Dormand–Prince [10], the algorithm of Benettin [6, 5], the method of Henon [11] and some other methods. A detailed methodology for applying the above methods is described in the papers [17, 23].

At studying attractors of dynamical systems, a description of scenarios (a sequence of bifurcations) of transitions from an attractor of one type to an attractor of another type have great interest. In particular, investigation of scenarios of transitions from regular to chaotic attractors, as well as transitions from a chaotic attractor of one type to a chaotic attractor of another type. As noted earlier, transitions to chaos were

Fig. 2 Dependence of two Lyapunov characteristic exponents on the parameter α_2
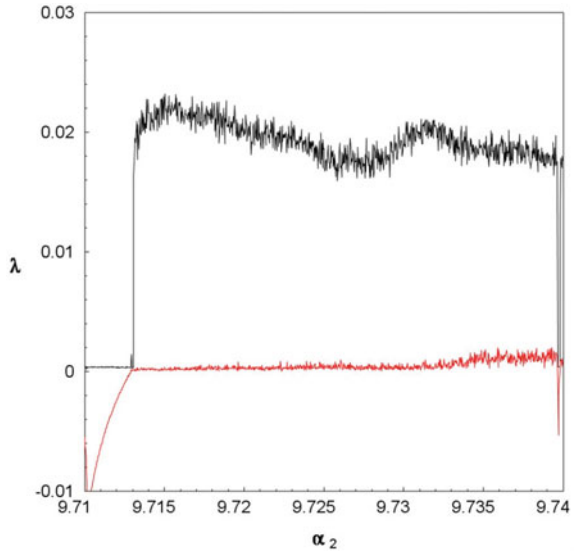
found in the "transducer-generator" system according to the Feigenbaum scenario (a cascade of period doubling bifurcations) and according to the Manneville-Pomeau scenario (through intermittency).

Let us show that in system (3) a transition is realized from a chaotic attractor of one type to a chaotic attractor of another type according to a more complex scenario of generalized intermittency. The scenario of generalized intermittency for nonideal hydrodynamic systems was described in papers [15, 16, 24].

Suppose that the parameters of system (3) are respectively equal to $\alpha_0 = -0.104$; $\alpha_1 = 0.0535$; $\alpha_3 = 9.95$; $\alpha_4 = 0.103$; $\alpha_5 = 0.0604$; $\alpha_6 = 0.12$; $\alpha_7 = 0.01$. We choose α_2 as the bifurcation parameter.

In Fig. 2, the dependences of two Lyapunov characteristic exponents on the parameter α_2 are plotted. The maximal exponent λ_1 is shown in black and the second λ_2 is shown in red.

As can be seen from Fig. 2 for $\alpha_2 < 9.7128$ maximal Lyapunov exponent will be zero, while the second exponent will be negative. This means that the attractor of system (3) for such values will be the limit cycle. At $\alpha_2 > 9.7128$ maximal Lyapunov exponent becomes positive, which indicates the appearance of a chaotic attractor in system (3). Chaos in the system will exist for almost all the values of α_2 considered in Fig. 2, with the exception of a very narrow periodicity window at the right boundary of the interval $9.7128 < \alpha_2 < 9.74$. As for the second Lyapunov exponent, it (up to the error of the Benettin et al. method) will be zero at $9.7128 < \alpha_2 < 9.7348$. At $\alpha_2 > 9.7348$ the second, the Lyapunov exponent becomes positive. The presence of two positive Lyapunov indicators indicates the emergence of a hyperchaotic attractor.

In Fig. 3 the phase-parametric characteristic (bifurcation tree) of the system (3) is shown. Limit cycles correspond to individual branches of this tree, and chaotic

Fig. 3 Phase-parametric
characteristic
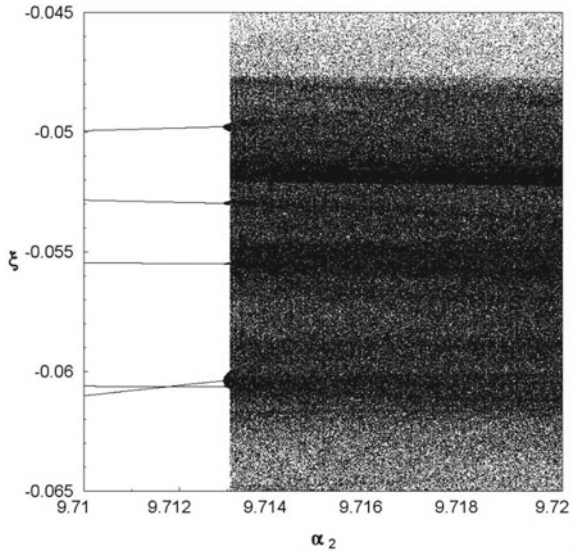
attractors correspond to densely black areas. In Fig. 3, two types of densely black
areas are clearly visible. These densely black areas correspond to different types
of chaotic attractors, which differ noticeably in the size of the attractor localization
region in the phase space.

Figure 4 shows an enlarged fragment of the phase-parametric characteristics of
the system (3). This figure makes it possible to very clearly illustrate the transitions

Fig. 4 Enlarged fragment of
phase-parametric
characteristic
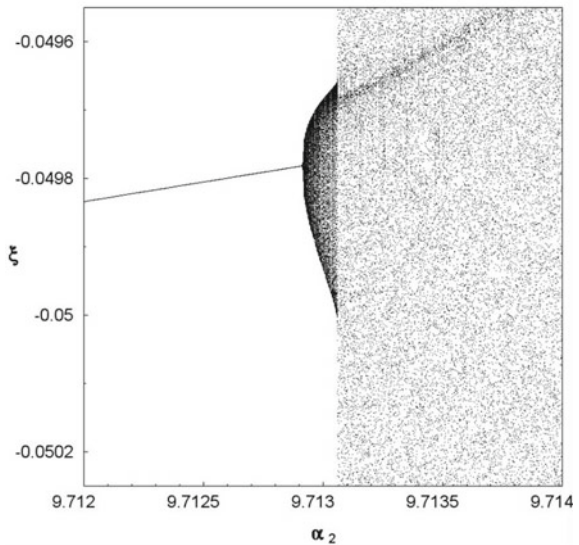
from one type of attractor to another. Thus, with an increase in the value of the parameter α_2 the limit cycle is replaced by a chaotic attractor with a small region of localization in the phase space. In turn, this chaotic attractor is replaced by a chaotic attractor of another type with a much larger area of localization in the phase space. In addition, Fig. 3 and Fig. 4 allow us to make an assumption about the scenario of the transition from a chaotic attractor of one type to a chaotic attractor of another type through generalized intermittency [16, 24]. However, most clear the scenario of such a transition can be revealed when studying the projections of the phase portraits of attractors and the distributions of invariant measures over the phase portraits.

Let us consider the dynamic behavior of system (3) with increasing parameter α_2. In Fig. 5a), a projection of the phase portrait of the limit cycle at $\alpha_2 = 9.7125$ is shown. As the parameter α_2 increases up to $\alpha_2 \approx 9.7128$, the limit cycle disappears and a chaotic attractor arises in the system. The projection of the phase portrait of a chaotic attractor constructed at $\alpha_2 \approx 9.71305$ is shown in Fig. 5b). The transition from a limit cycle to a chaotic attractor occurs through intermittency trough one rigid bifurcation [18]. Despite the fact that the chaotic attractor is very similar in shape
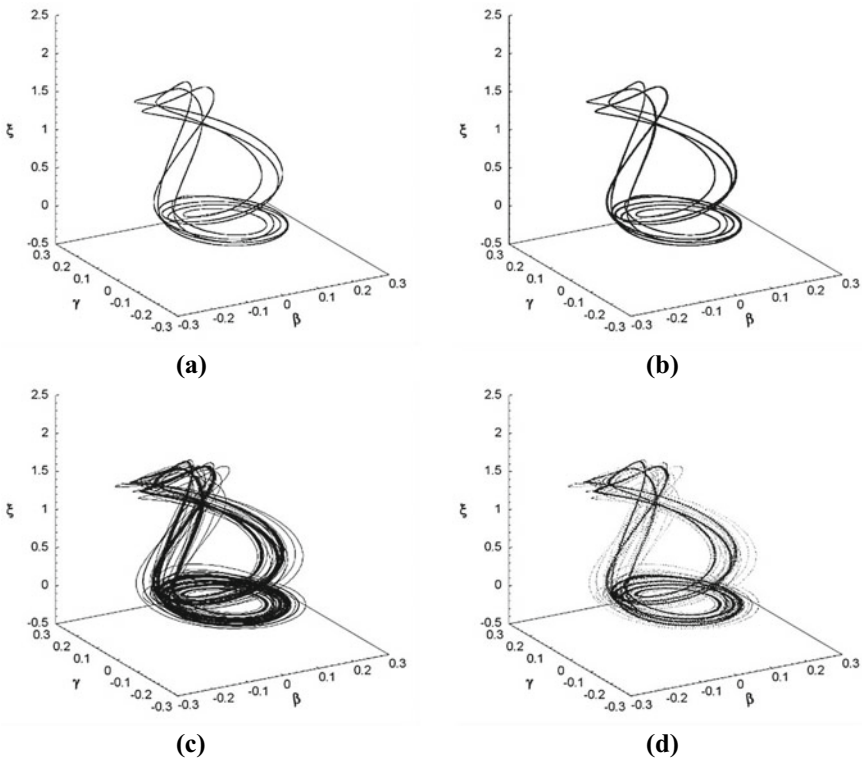

(a)   **(b)**

(c)   **(d)**

Fig. 5 Limit cycle at $\alpha_2 = 9.7125$ (**a**); Chaotic attractor at $\alpha_2 = 9.7125$ (**b**); Chaotic attractor at $\alpha_2 = 9.7131$ (**c**); Distribution of invariant measure at $\alpha_2 = 9.713$ (**d**)

to the disappeared limit cycle, there are fundamental differences between them. The limit cycle consists of one orbitally stable trajectory along which the movement is strictly periodic. A chaotic attractor consists of an infinite set of arbitrarily close to each other open trajectories along which the motion is unpredictable.

Note that such a chaotic attractor exists on a very small interval of variation of the parameter α_2, and small increase in parameter α_2, namely $\alpha_2 \approx 9.7131$ leads to next rigid bifurcation after which chaotic attractor of another type arises. The existing chaotic attractor disappears and a chaotic attractor of a different type appears in the system. The projection of the phase portrait of the new chaotic attractor is shown in Fig. 5c). The distribution of the Krylov-Bogolyubov measure over the projection of the phase portrait of the new chaotic attractor is shown in Fig. 5d).

The scenario of such a transition from a chaotic attractor of one type to a chaotic attractor of another type is called generalized intermittency [16, 24]. In this scenario, after passing the bifurcation point, the chaotic attractor disappears and a chaotic attractor of a new type appears. Motion along the trajectory new chaotic attractor consists of two alternating phases, namely rough-laminar phase and turbulent phase. In the rough-laminar phase, the trajectory makes chaotic movements in a neighborhood of the trajectories of the disappeared chaotic attractor. Then, at an unpredictable moment of time, the trajectory leaves the localization region of the disappeared attractor and moves to more distant regions of the phase space. Rough-laminar phase corresponds to the much blacker areas in Fig. 5c, d. In turn, turbulent phase corresponds to much less darkened areas in Fig. 5c, d. After some time, the movement of the trajectory returns to the rough-laminar phase again. Then, trajectories switch to turbulent phase again. Such transitions are repeated an infinite number of times. From Fig. 5b, d it is especially clearly seen that the contours of the disappeared chaotic attractor are essentially rough laminar phase of a new chaotic attractor. Note that the duration of both rough-laminar and turbulent phases is unpredictable as are the moments of times of transition from one phase to another.

With a further increase in the value of the parameter α_2, the second Lyapunov characteristic exponent also becomes positive. A chaotic attractor turns into a hyperchaotic one. Hyperchaotic attractors have two directions in the phase space along which the trajectories of the hyperchaotic attractor run away.

Finally, let us consider on one more interesting feature of the "transducer-generator" system. Figure 6 shows the phase-parametric characteristic for another interval of variation of the parameter α_2. Here, as before, the separated "branches" of the bifurcation tree correspond to the limit cycles, and the densely black areas correspond to chaotic attractors. Clearly, it can be identified the transitions from densely black areas to densely black areas with noticeably greater size. At such transitions, the chaotic attractor of one type is replaced by a chaotic attractor of another type. As before, such transition is carried out according to the scenario of generalized intermittency. However, there is difference in sequence of bifurcations with such transitions. Therefore, on transitions corresponding to the Fig. 3, following bifurcation sequence occurs: limit cycle–intermittency–chaotic attractor of one type–generalized intermittency–chaotic attractor of another type. In turn, Fig. 6 corresponds to the following sequence of bifurcations: limit cycle–cascade of bifurcations of

Fig. 6 Phase-parametric
characteristic
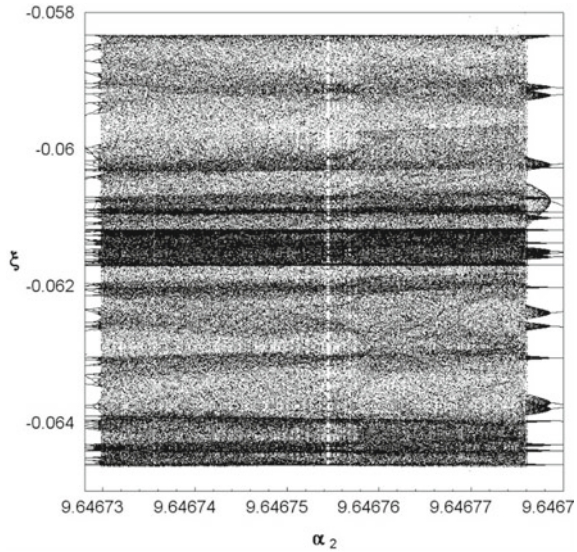
period doubling–intermittency–chaotic attractor of one type–generalized intermit-
tency–chaotic attractor of another type. Thus, first transition to chaos corresponding
to Fig. 3 is carried out through one rigid bifurcation, and the first transition to chaos
corresponding Fig. 6 is carried out through an infinite number of soft bifurcations.

In conclusion, we emphasize that implementation of the scenario of generalized
intermittency for the "transducer-generator" system was found for the first time.

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