








Stress-Strain State and Bearing Capacity of Members Under Biaxial Bending

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Abstract. The features of the stress-strain state of reinforced concrete beams subject to biaxial bending are described. The analysis of various methods for calculating the strength along normal sections of reinforced concrete members under biaxial bending, proposed by both various scientists and standards, is presented. The main prerequisites for calculating each method, similarities and differences are noted. A comparison of the experimental values of the ultimate bending moments perceived by the cross sections of reinforced concrete specimens subjected to biaxial bending with the theoretical values determined by different methods is performed. Engineering methods for strength design of reinforced concrete members under biaxial bending based on the generally accepted prerequisites for calculating such members using a rectangular stress distribution diagram in concrete of a compressed zone, as well as methods based on the properties of isostatic curves, showed sufficient accuracy of the results of determining the load-bearing capacity with the test data of reinforced concrete beams of rectangular cross section.

Keywords: Biaxial bending · Reinforced concrete · Strength along normal sections · Experimental investigations · Reinforced concrete beam

1 Introduction

Structural members of modern buildings and structures are increasingly working under conditions of combined stresses [1–4], including biaxial bending, which is due to the variety of structural forms and conditions of their operation. Therefore, research on such elements remains relevant [5–7]. The extreme precast reinforced concrete girders, end beams of a shell, crane beams, horizontal members of half-timbered outer walls of frame buildings, foundation and framing beams, wall panels, etc. are subject to biaxial bending. In such members, the plane of action of the bending moment does not coincide with any of the planes passing through the principal axes of inertia of the cross section and the longitudinal axis of the member.

Opposed to the case of a plane bending in a member subjected to biaxial bending

- the neutral axis is not perpendicular to the plane of action of the bending moment;

- the shape of the compressive zone is different from the rectangular one, so for a rectangular cross section it can be triangular or trapezoidal (Fig. 1);
- it is recommended to place the centroid of the longitudinal reinforcement closer to the most stretched angle of the cross section, taking into account the design requirements for reinforcing.

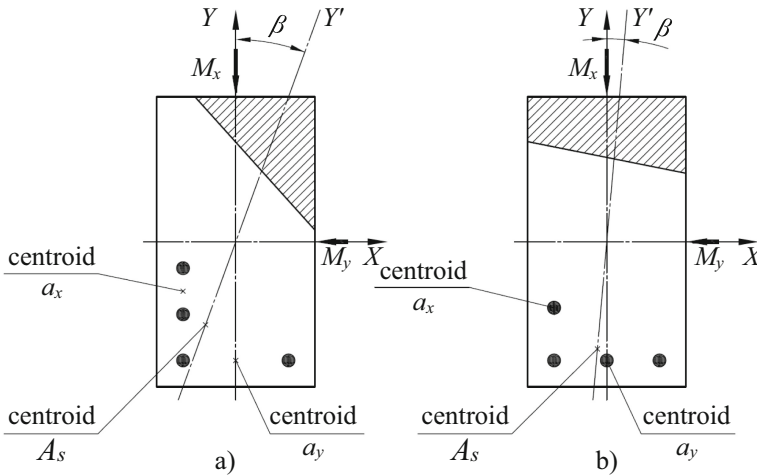


Fig. 1. Types of forms of the compressive zone in a reinforced concrete member under biaxial bending: triangular (a) and trapezoidal (b). The location of the longitudinal reinforcement in the cross section. Y' – plane of action of the bending moment, β – angle of inclination of the plane Y' relative to the Y axis, M_x and M_y – bending moments relative to the X and Y axes, respectively.

In most cases, scientists consider the general case of biaxial bending – biaxial bending with axial load [8–11]. In the work of B Bresler [8], the strength condition of reinforced concrete members subjected to biaxial bending with axial load is based on the approximation of the failure surface, which provides the dependence of the nominal axial load P_n on the eccentricities x and y or the components of the bending moment M_x and M_y . Thus, the failure surface can be represented as “reciprocal” surface $S_1(1/P_n, x, y)$ or surface $S_2(P_n, M_x, M_y)$.

The reciprocal load method is a simplified method for calculating reinforced concrete members subjected to biaxial bending with axial load, based on the approximation of the reciprocal surface S_1 by a plane formed by three points. The equation of the plane is [8, 12]

$$\frac{1}{P_{ni}} = \frac{1}{P_{nx}} + \frac{1}{P_{ny}} - \frac{1}{P_0}, \tag{1}$$

The load contour method is based on the approximation of the surface S_2 by a family of curves corresponding to the constant value of the axial load P_n . The equation of the

curve approximating the load contour in general is [8, 12]

$$\left(\frac{M_{nx}}{M_{n0x}}\right)^\alpha + \left(\frac{M_{ny}}{M_{n0y}}\right)^\beta = 1, \quad (2)$$

Where M_{nx} and M_{ny} are the bending moments caused by the external load, relative to the X and Y axes, respectively; M_{n0x} and M_{n0y} are the bending moments perceived by the cross section under eccentric compression, relative to the X and Y axes, respectively; α and β are the exponent of a power depending on the size of the element, the number and distribution of reinforcement, the mechanical characteristics of the reinforcement and concrete materials, the size of the protective layer and transverse reinforcement.

Further research by scientists was aimed at clarifying the unknown parameters of the general equation describing the load contour (2). B Bresler accepted $\alpha = \beta = 2$ [8], C-T T Hsu [10, 13] introduced an additional term of sum that takes into account the ratio between the applied load and the internal longitudinal forces, and set the values of the exponents equal to $\alpha = \beta = 1.5$. In the work of A L Parme, J M Nieves, and A Gouwens [14], the following equation approximating the load contour is proposed

$$\left(\frac{M_{nx}}{M_{n0x}}\right)^{\frac{\log 0.5}{\log \beta}} + \left(\frac{M_{ny}}{M_{n0y}}\right)^{\frac{\log 0.5}{\log \beta}} = 1, \quad (3)$$

Where β is the ordinate of the contour at the point where the relative moments are equal.

Load contour method for determining the bearing capacity of reinforced concrete members in conditions of combined stresses is also used in modern works of scientists [15, 16].

The case of pure biaxial bending of a reinforced concrete member is considered in the works of T Au [17], E Cohen [17], C S Whitney [18], etc. The assumption is made that the planes of action of external forces and internal forces coincide. The calculation is performed using the equilibrium equations

$$\sum Z = 0, \quad \sum M_x = 0, \quad \sum M_y = 0 \quad (4)$$

The curved stress distribution diagram in the compressive concrete zone is replaced by an equivalent rectangular one, with an ordinate of $0.85f_c'$, where f_c' is the cylindrical strength of the concrete, and the actual height of the compressive concrete zone h is reduced to a value $a = k_1h$, the coefficient k_1 is assumed to be less than or equal to 0.85 [17].

A H Mattock and L B Kriz conducted experimental and theoretical investigations of reinforced concrete members under biaxial bending of an asymmetric (L-shaped) cross section [19]. It is found that the use of an equivalent rectangular diagram simplifies the calculation and gives a satisfactory agreement with the experimental data.

The current US and European standards ACI 318 [12, 20] and EN 1992-1-1 [21] provide requirements for calculating the strength of reinforced concrete members under biaxial bending with axial load:

- by the reciprocal load method [12, 20];
- by the load contour method in ACI 318 [12, 20] using the Eq. (3), in EN 1992-1-1 [21] according to the Eq. (2), at, where the parameter $\alpha = \beta$ is assumed to be from 1 to 2 in accordance with paragraph 5.8.9;
- according to the general method based on nonlinear calculation [21].

The first research works of reinforced concrete members subjected to biaxial bending began in the 30s of the XX century. Based on the method of allowable stress design, considering the elastic work of reinforced concrete, they did not reflect the actual behavior of such members. After the appearance of limit stage design method, a number of versatile investigations were carried out, and to date, many proposals have been accumulated to determine the strength of normal cross sections of members under biaxial bending, covering all possible types of reinforced concrete structures.

2 Materials and Methods

Further, a number of engineering methods for strength design of reinforced concrete members under biaxial bending are considered and the values of the ultimate bending moment found in different ways are compared with the experimental values obtained during tests.

Calculated Ultimate Bending Moments

The strength design of the normal cross sections of the reinforced concrete members under biaxial bending is carried out from the equilibrium Eq. (4).

In the work of Zhen Bai Yu, the assumption is accepted that the force plane passes through the centroid of the tensile reinforcement and compressive zone, thus, the member is subjected to biaxial bending without torsion. The stress distribution in the compressive zone is assumed to be uniform.

A significant contribution to the investigation of reinforced concrete members subjected to complex deformations was made by M S Toryanik. Under his leadership, P F Vakhnenko, A V Gorik, A N Pavlikov, Yu M Rudenko, L I Serdyuk, B S Chuprina, and others were engaged in the research of biaxial bending members.

The main prerequisites of the calculation method proposed by M S Toryanik:

- the calculation is performed according to the first limit state – strength;
- the stress distribution in the compressive zone of concrete is assumed to be rectangular with the ordinate equal to the compressive strength of concrete;
- the influence of the tensile zone is not taken into account in the calculation;
- the stresses in the reinforcement reach the yield strength;
- the discrete arrangement of the reinforcement is replaced by evenly distributed belts;

- the angle β should be attributed to the area of the horizontal reinforcement with an area of a_x .

The design equations for the case of biaxial bending are derived in general form from the solution of the system of Eq. (4) for a T-shaped cross section with double reinforcement. When the moment M_y is equal to zero, the equation degenerate into the well-known equations for plane bending. The strength design methods proposed by S I Glazer and M S Toryanik, in fact, differ little from each other. These methods are based on the same prerequisites and assumptions for the derivation of the design equations and are reduced to the task of determining the load-bearing capacity of the member. For the inverse problem – the selection of reinforcement and simplification of the calculation of the load-bearing capacity, the authors have compiled tables. These methods are quite simple to calculate “manually”, since they are reduced to finding one unknown – the size of the compressive zone along the height of the cross section $\xi_1 h$ (or $\eta_1 h$).

I K Nikitin and A S Zalesov proposed a strength design method based on the equilibrium Eq. (4) and the condition of parallelism of the planes of action of external and internal moments Eq. (5). This calculation method is presented in the design handbook to SP 63.13330

$$\operatorname{tg}\beta = \frac{M_y}{M_x} \quad (5)$$

This method considers only one position of the neutral axis, at which the arm of couple of forces will be minimal (triangular compressive zone of concrete). With the trapezoidal shape of the compressive zone, the strength design of the member is proposed to be carried out according to the equations of plane bending, for the action of the moment $M = M_x$. A N Pavlikov and A V Gorik supplemented this method by extending it to the case of a trapezoidal compressive zone.

An original solution for estimation the load-bearing capacity of reinforced concrete members subjected to biaxial bending was proposed by V N Baikov, based on the equivalence of the virtual work of the moment and the total work of its components in two mutually perpendicular directions. In fact, the proposed method is reduced to calculating the effect of two moments separately.

There are a number of proposals for determining the load-bearing capacity of member under biaxial bending based on the properties of isostatic curves – isostatic and isobent curve, in combination with the static equilibrium equation. O N Totsky replaces the equations of curves with approximate curvilinear functional relation. Analytical investigations of Yu M Rudenko showed that the isocurve displacements of the centroid of the compressive zone can be approximated by two straight lines. It is noted that the error in determining the arm of couple of forces under unfavorable conditions does not exceed 10%. Methods based on the properties of isocurves are approximate in nature and do not allow to determine the position of the neutral axis.

Experimental Ultimate Bending Moments

Experimental investigations of members subjected to biaxial bending were carried out in the laboratories of SPbGASU [22]. Reinforced concrete beams with a size of 2460 × 220 × 120 ($l \times h \times b$) mm were manufactured at the precasting factory in steel

pans, a total of 2 series of 2 beams each. The longitudinal reinforcement was placed asymmetrically along the cross section of the member, considering the assumed angle of inclination of the force plane β . The summary of beam detail is presented in Table 1. To prevent the structural failure of the member along the oblique section stirrups with diameters of 5 mm of class B500 with a step of 50 mm was installed. In the constant moment zone, the spacing of the stirrups was 150–200 mm. Handling reinforcement were installed with a diameter of 6 mm of class A240.

Table 1. Summary of beam detail.

Series	Beam no.	Angle $\beta, ^\circ$	Scheme of reinforcement of the cross section of the beam in the constant moment zone	Prism strength, MPa
1	RC1.1	10		16.0
	RC1.2			
2	RC2.1	20		16.0
	RC2.2			

At the same time, concrete cubes (100 × 100 × 100 mm) and prisms (100 × 100 × 400 mm) were produced for testing to determine the mechanical characteristics of concrete. The beams were made of fine grained concrete. The load was applied in accordance with the angle β , using two hydraulic cells. Loading was carried out in steps of 0.1 of the expected ultimate load (P_{ult}). After the loading level of $0.7P_{ult}$, the stages were reduced to $0.05P_{ult}$. The exposure time at each stage was 10–15 min.

For the ultimate bending moment of beams take the moment when the last, less loaded, bar reaches stresses equal to the physical yield strength, or the moment when the strength of the compressive zone is exhausted. Figure 2 shows the failure modes and crack patterns of reinforced concrete beams subjected to biaxial bending.

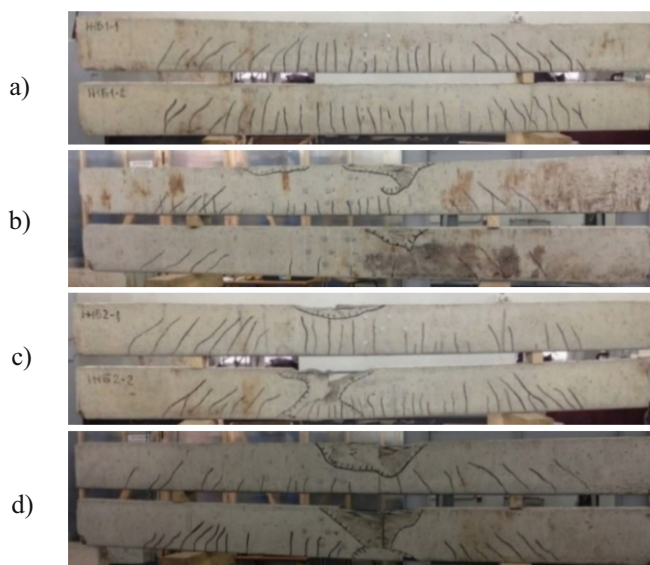


Fig. 2. Failure modes and crack patterns of reinforced concrete beams subjected to biaxial bending of the series RC1: a) face 1; b) face 2; and RC2: c) face 1; d) face 2.

3 Results and Discussion

According to the engineering methods for strength design noted above, which allow “manually” determining the load-bearing capacity of a reinforced concrete member under biaxial bending, the values of the ultimate bending moments presented in Table 2 were found. Based on the results of tests experimental values of bending moments were obtained, presented in Table 2.

Table 2. Values of calculated and experimental moments, kN•m.

Author	Beam	
	RC1	RC2
O N Totsky	25.33	21.59
Yu M Rudenko	24.73	21.47
S I Glazer	25.28	21.47
V N Baikov	25.10	22.78
Zhen Bai Yu	25.28	21.47
M S Toryanik	25.96	23.04
A N Pavlikov, A V Gorik	25.28	24.83
I K Nikitin, A S Zalesov	26.29	20.17
Experimental moment [22]	28.57	25.09
	26.53	24.58

The divergence of values of the theoretical moments and the experimental ones was 7.7% for the angle β equal to 10° , and 11.1% for 20° in the safety margin. With an increase in the angle of inclination of the force plane β , the load-bearing capacity of the reinforced concrete member under biaxial bending decreases by an average of 13% based on theoretical calculations, and by 8% based on the results of tests.

The methods proposed by M S Toryanik and S I Glazer, when one of the moments is equal to zero, degenerate into well-known equations for the case of plane bending, which is a special case of biaxial bending, which is incomparably their advantage. Even though the methods based on the properties of isostatic curves are considered approximate, the results of determining the load-bearing capacity using these methods are identical to the others.

To this date, in Russian Federation, in accordance with the standards that establish requirements for the design of reinforced concrete structures SP 63.13330, the strength design of the normal cross sections of reinforced concrete members under biaxial bending should be performed on the basis of a nonlinear deformation model. The design handbook to SP 63.13330 also specifies a practical method proposed by I K Nikitin and A S Zalesov. It was noted that, provided that all the stretched rods achieve deformations of at least R_s/E_s , the use of a rectangular stress diagram in the compressive zone of concrete when calculating the strength of such members will lead to an excess of the load-bearing capacity by no more than 2.4% compared to the calculation using a nonlinear deformation model.

4 Conclusions

Engineering methods for strength design of reinforced concrete members under biaxial bending based on the generally accepted prerequisites for calculating such members using a rectangular stress distribution diagram in concrete of a compressed zone, as well as methods based on the properties of isostatic curves, showed sufficient accuracy of the results of determining the load-bearing capacity with the test data of reinforced concrete beams of rectangular cross section.

Engineering methods for strength design of reinforced concrete members under biaxial bending allow to quickly assess the load-bearing capacity of such members. However, the main disadvantage for most of them is the inability to solve the direct problem – the selection of reinforcement, but only to check the load-bearing capacity of such members.

With an increase in the angle of inclination of the force plane β , the load-bearing capacity of the reinforced concrete member under biaxial bending decreases by an average of 13% based on theoretical calculations, and by 8% based on the results of tests. First, this is due to a decrease in the shoulder of the arm of couple of forces, so not counting biaxial bending or reduction to the case of pure bending will lead to an underestimation of the load-bearing capacity and, as a result, to premature failure of the reinforced concrete members under biaxial bending.

The decrease in the shoulder of the arm of couple of forces, and as a result of the load-bearing capacity of the members subjected to biaxial bending with an increase in the angle of inclination of the plane of action of the bending moment, allows to conclude

that unsymmetrical reinforcement for such members is the most rational both in terms of using the strength properties of the reinforcement, and in order to increase the load-bearing capacity by increasing the shoulder of the arm of couple of forces in comparison with traditional symmetrical reinforcement.

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