Chapter 7
The Motion of a Body with No Fixed Point

7.1 General Considerations

In previous chapters, we studied problems of motion of a rigid body with one point fixed in space, i.e. in an inertial reference frame. In the present chapter, we study certain problems of motion when the body is not fixed from any point. For the moment, we shall not begin with constructing a Lagrangian for the motion. To keep the applicability of the equations of motion as wide as possible, we assume that the body is subject to a set of forces, which are not necessarily time independent and which may not have a potential.

As in Chap. 3, we write the equation of motion of an element dm of the body mass

$$dm\frac{d^2\mathbf{r}}{dt^2} = d\mathbf{F} + d\mathbf{F}' \tag{7.1}$$

summing over the mass of the body one gets

$$M\frac{d^2\mathbf{r}_0}{dt^2} = \mathbf{F}. \tag{7.2}$$

The centre of mass of the body moves as a particle acted upon by a force equal to the resultant of all external forces acting on the body.

On the other hand, let $\boldsymbol{\rho}$ be the position vector of the mass element dm with respect to the centre of mass, so that

$$\mathbf{r} = \mathbf{r}_0 + \boldsymbol{\rho}. \tag{7.3}$$

Multiplying (7.1) vectorially by $\boldsymbol{\rho}$ and integrating over the body, we obtain the equation for the rotational motion in the form

$$\frac{d\mathbf{G}_c}{dt} = \mathbf{L}_c, \tag{7.4}$$

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where $\mathbf{G}_c = \int \boldsymbol{\rho} \times \frac{d\boldsymbol{\rho}}{dt} dm$ and $\mathbf{L}_c = \int \boldsymbol{\rho} \times d\mathbf{F}$ are, respectively, the angular momentum and the resultant moment of external forces about the centre of mass. The law of rotational motion about the centre of mass of the body resembles that of motion relative to the inertial frame.

Example 1 A rigid body is free to move in vacuo under the action of a uniform gravity field. Describe the motion.

The centre of mass of the body moves as a projectile, while the body performs a torque-free motion about the centre of mass as in Euler's case.

In particular, if the body begins with zero angular velocity, it will continue translational motion of its centre of mass without change in its orientation. If the body begins with a rotation about one of its principal axes of inertia, it will continue rotation with the same angular velocity about the same axis, which keeps fixed orientation in space.

Kinetic energy of the rigid body: To construct the Lagrangian of the motion, one needs to have an expression for the kinetic energy. That is

$$
\begin{aligned}
T &= \frac{1}{2} \int (\mathbf{v}_c + \frac{d\boldsymbol{\rho}}{dt})^2 dm \\
&= \frac{1}{2} M \mathbf{v}_c^2 + \frac{1}{2} \int (\frac{d\boldsymbol{\rho}}{dt})^2 dm.
\end{aligned}
\tag{7.5}
$$

The kinetic energy of the body in a general translational and rotational motion is the sum of two terms: the kinetic energy of a mass M, equal to the mass of the body and moving with its centre of mass, and the kinetic energy of the rotation of the body about its centre of mass. The last formula gives a simple expression of the kinetic energy, very useful in application to many problems of rigid body dynamics. In the next section, we study an example of such application.

Example 2 A heavy magnetized rigid body is free to move in vacuo under the action of two skew uniform gravity and magnetic fields. Describe the motion.

Let (X, Y, Z) be the coordinates of the centre of mass P of the body relative to an inertial frame $OXYZ$ with axis Z directed vertically upwards and let $\mathbf{H} = H\mathbf{e}$ be the magnetic field of fixed magnitude and fixed direction in space determined by the unit vector $\mathbf{e}$. The Lagrangian of the problem can be written as

$$
\begin{aligned}
L = &\frac{1}{2} M(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) - MgZ \\
&+ \frac{1}{2}(A_c p^2 + B_c q^2 + C_c r^2) - \mathbf{m} \cdot \mathbf{H},
\end{aligned}
$$

where M, $\mathbf{m}$ are the mass of the body and its magnetic moment, respectively, and A_c, B_c, C_c are the central principal moments of inertia. One can see at once that the translational motion of the centre of mass and the rotational motion are completely independent. The centre of mass P moves as a projectile on a parabola with its axis vertical and vertex upwards. The equations of the rotational motion are obtained from the Lagrangian

$$L' = \frac{1}{2}(A_c p^2 + B_c q^2 + C_c r^2) - H\mathbf{m} \cdot \mathbf{e}. \tag{7.6}$$

This Lagrangian is form-identical with (3.43) of the classical problem of motion of a heavy body about a fixed point and the problem can be put in Lagrangian, Hamiltonian or in the Euler–Poisson form. It follows immediately that the present problem is generally integrable only in the following three cases:

(1) The analog of Euler's case. The case of no magnetic effect $H = 0$ (no magnetic field) or $\mathbf{m} = \mathbf{0}$ (no magnetization in the body)
(2) The analog of Lagrange's case. The central inertia ellipsoid is a spheroid and the magnetic moment is parallel to the symmetry axis.
(3) The analog of Kowalevski's case. $A_c = B_c = 2C_c$ and the magnetic moment lies in the equatorial plane ($m_3 = 0$).

The conditional integrable case of Goryachev and Chaplygin has its analog valid on the level $f = 0$, f being the component of the angular momentum in the direction of the magnetic field ($\boldsymbol{\omega}\mathbf{I}_c \cdot \mathbf{e} = f$). The same applies to all particular solutions of the classical problem, a complete list of which will be provided in Chap. 8.

7.2 Poisson's Top. A Top on a Smooth Horizontal Plane

In our previous study of Lagrange's top (the axi-symmetric top), the assumption was made that the pin of the top serves as a fixed point in the inertial space. A related model was first considered by Poisson [309], but much rarely mentioned in textbooks on the subject. For a somewhat detailed treatment see [221, 222].

In this model, a symmetrical top moves with its apex Q constrained to move without friction on a horizontal plane, so that the reaction R of the plane on the body remains in the vertical direction. Denote by P the centre of mass of the top. From symmetry, P lies at a point on its axis of the top at a distance a (say) from Q. Let (x, y, z) be the coordinates of P relative to an inertial frame $Oxyz$ with axis z directed vertically upwards and the xy-plane is the plane of motion of the apex. In this system, the apex Q has coordinates $(X, Y, 0)$. The rotational motion of the top will be described by the Eulerian angles: ψ between the vertical plane containing QP and the xz-plane, θ between QP and the vertical upwards and finally φ the angle of proper rotation of the top about its axis (See Fig. 7.1).

To derive equations of motion, one may use (7.5) to write down the Lagrangian of the motion as

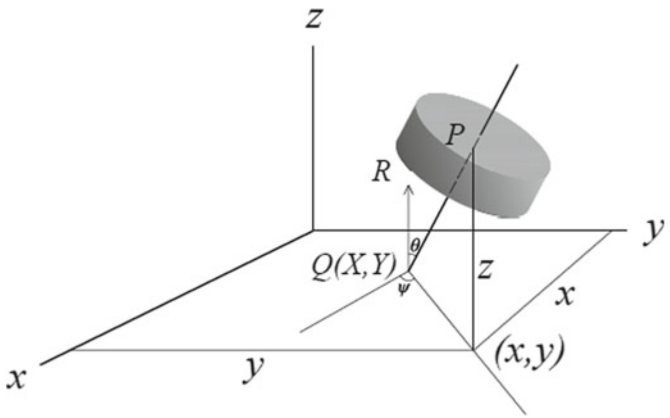

Fig. 7.1 Poisson's top

$$
\begin{aligned}
L = {} & \frac{1}{2}M(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\
& + \frac{1}{2}[A(\dot{\theta}^2 + \sin^2\theta\dot{\psi}^2) + C(\dot{\psi}\cos\theta + \dot{\varphi})^2] \\
& - Mgz
\end{aligned}
$$

where M is the mass of the top, C is its axial moment of inertia, A is the moment of inertia about any axis passing through P and orthogonal to QP and g is the acceleration of gravity. Recalling that $z = a\cos\theta$, we can rewrite the Lagrangian in the form

$$
\begin{aligned}
L = {} & \frac{1}{2}M(\dot{x}^2 + \dot{y}^2 + a^2\sin^2\theta\dot{\theta}^2) \\
& + \frac{1}{2}[A(\dot{\theta}^2 + \sin^2\theta\dot{\psi}^2) + C(\dot{\psi}\cos\theta + \dot{\varphi})^2] \\
& - Mgz.
\end{aligned}
\tag{7.7}
$$

The mechanical system under consideration has five degrees of freedom. It also admits five integrals of motion: the energy integral and four integrals corresponding to the four cyclic coordinates x, y, ψ and φ, so that there is no need to write any of the Lagrange's equations of motion, but only to write the integrals

$$
\begin{aligned}
\dot{x} &= U, \\
\dot{y} &= V, \\
\dot{\psi}\cos\theta + \dot{\varphi} &= r_0, \\
A\sin^2\theta\dot{\psi} + Cr_0\cos\theta &= n, \\
\frac{1}{2}M(U^2 + V^2 + a^2\sin^2\theta\dot{\theta}^2) + \frac{1}{2}[A(\dot{\theta}^2 + \sin^2\theta\dot{\psi}^2) + Cr_0^2] + Mgz &= h,
\end{aligned}
\tag{7.8}
$$

where U, V, r_0, n, h are arbitrary constants of motion.

From the first two integrals, it turns out that the centre of mass of the top moves with uniform horizontal velocity. This agrees with the fact that the body moves under the action of only two vertical forces: its weight and the reaction of the plane. From the last three integrals, one gets

$$\dot{\psi} = \frac{n - Cr_0\cos\theta}{A\sin^2\theta},$$
$$\dot{\varphi} = r_0 - \frac{\cos\theta(n - Cr_0\cos\theta)}{A\sin^2\theta} \tag{7.9}$$

and the final equation for the angle θ

$$(A + Ma^2\sin^2\theta)\dot{\theta}^2 = 2(E - Mga\cos\theta) - \frac{(n - Cr_0\cos\theta)^2}{A\sin^2\theta}, \tag{7.10}$$

where $E = h - \frac{1}{2}M(U^2 + V^2) - \frac{1}{2}Cr_0^2$.

From the analytical point of view, setting $\cos\theta = u$, we transform (7.10) to the form

$$[A + Ma^2(1 - u^2)]\dot{u}^2 = 2(E - Mgau)(1 - u^2) - \frac{1}{A}(n - Cr_0u)^2, \tag{7.11}$$

which leads through separation of variables to the relation

$$t = \int^u \sqrt{\frac{A + Ma^2(1 - u^2)}{2(E - Mgau)(1 - u^2) - \frac{1}{A}(n - Cr_0u)^2}}du$$
$$= \sqrt{\frac{a}{2g}}\int^u \frac{du}{\sqrt{F(u)}}, \tag{7.12}$$

where

$$F(u) = \sqrt{\frac{(u - u_1)(u_2 - u)(u_3 - u)}{u_4^2 - u^2}},$$

and $0 \leq u_1 \leq u_2 \leq 1 \leq u_3$, $u_4 = \sqrt{1 + \frac{A}{Ma^2}} > 1$.

Now, we note that Eq. (7.9) is identical with the ones considered for Lagrange's top in Chap. 3. Also, (7.11) is similar to its corresponding Eq. (4.41) in Lagrange's case, and differs from it only by presence of the term $Ma^2(1 - u^2)$ in the coefficient of $\dot{u}^2$ on its left-hand side. However, this term does not change the sign of the coefficient. Thus, the general qualitative character of the motion of the Poisson top is almost the same as in Lagrange's top and it will not be repeated here.

In general, the last integral is hyper-elliptic, compared to the elliptic integral (4.42) for Lagrange's top. Inverting this relation we express $u = \cos\theta$ in terms of time, and then integrating (7.9) with respect to time we obtain ψ and φ. In Klein's work [221], where the top under consideration is termed the "toy top", explicit expressions of the Cayley–Klein parameters describing the motion as hyper-elliptic integrals in u were given, so that together with (7.12) this gives a parametric representation of the solution. Those results were detailed and refined in [339], where also degenerate cases when hyper-elliptic integral reduce to elliptic were singled out.

To find the equation of the trajectory of the tip of the top on the plane, we write

$$\begin{aligned} X &= x - a\sin\theta\cos\psi = x_0 + Ut - a\sin\theta\cos\psi, \\ Y &= y - a\sin\theta\sin\psi = y_0 + Vt - a\sin\theta\sin\psi. \end{aligned} \tag{7.13}$$

The reaction of the plane on the apex can be found from the equation of motion of the centre of mass in the z direction

$$M\ddot{z} = R - Mg,$$

so that

$$\begin{aligned} R &= M(g + \ddot{z}) \\ &= M(g + a\ddot{u}). \end{aligned} \tag{7.14}$$

Remark: Without any effect on the rotational motion of the top, it can be assumed that $U = V = x_0 = y_0 = 0$, so that $x = y = 0$. This fixes the choice of the inertial frame as the one whose Z-axis passes through the initial position of the centre of mass P of the top. In this frame, P moves only vertically up and down the Z-axis.

7.2.1 Regular Precession of Poisson's Top

As it was in the case of Lagrange's top, regular precession corresponds to the nutation angle θ taking a constant value θ^* (say), and then from (7.9) we find that the other two Eulerian angles φ, ψ change with time in constant rates. This occurs in two qualitatively different ways:

1- When $u_1 = u_2 = u^*$, $0 < u^* \leq 1$. This can happen at arbitrary inclination of the body axis to the vertical, including the standing gyroscope positions ($u = 1$). Regular precessions of this type correspond to Fig. 7.2a for the function F. They are all stable, since a slight perturbation of the motion causes splitting of the two roots in a small neighbourhood of u^*. This leads to a small periodic change in the nutation angle θ and consequently small wobbling in the rates $\dot{\psi}$ and $\dot{\varphi}$.

2- When $u_1 < 1$, $u_3 = u_2 = 1$. This gives a different standing position, corresponding to Fig. 7.2b. On perturbation, the equal roots split into $u_3 > 1$, $u_2 < 1$. The figure axis begins a finite periodic motion, in which it goes to position u_1 before it returns to u_2. This standing position is unstable.

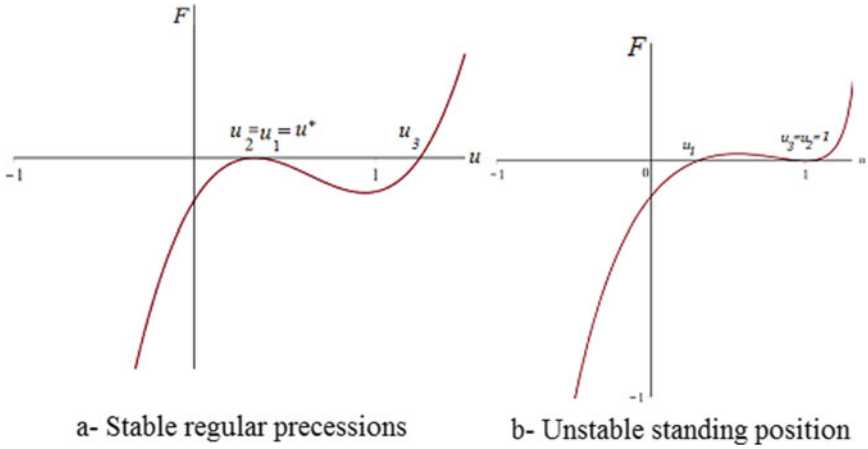

Fig. 7.2 Stability of regular precession for Poisson's top

7.3 Exercises

1. Show that the resulting problem of example 2 in Sect. 7.1 above can be generalized to the case of a body-gyrostat.

2. The Poisson top is given the upper equilibrium position. Show that this position, which is unstable when $r_0 = 0$, can be stabilized by giving the top an angular speed r_0 about its axis of symmetry provided

$$|r_0| > 2\frac{\sqrt{MgaA}}{C}.$$

3. A rigid body moves freely in a gravitational field with homogeneous quadratic potential

$$V_g = \frac{1}{2}(a\xi^2 + b\eta^2 + c\zeta^2)$$

in the inertial frame. Show that the equations of translational motion of the centre of mass and the rotational motion about the centre of mass are completely separate. For more details see Chap. 14 Sect. 14.3.1.